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# A NÉRON–OGG–SHAFAREVICH CRITERION FOR K3 SURFACES

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ABSTRACT. The naive analogue of the Néron–Ogg–Shafarevich criterion is false for K3 surfaces, that is, there exist K3 surfaces over Henselian, discretely valued fields  $K$ , with unramified  $\ell$ -adic étale cohomology groups, but which do not admit good reduction over  $K$ . Assuming potential semi-stable reduction, we show how to correct this by proving that a K3 surface has good reduction if and only if  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified, and the associated Galois representation over the residue field coincides with the second cohomology of a certain “canonical reduction” of  $X$ . We also prove the corresponding results for  $p$ -adic étale cohomology.

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## 1. INTRODUCTION

Let  $\mathcal{O}_K$  be a Henselian DVR, whose residue field  $k$  is perfect of characteristic  $p \geq 0$ , and whose fraction field  $K$  is of characteristic 0. Let  $G_K = \text{Gal}(\overline{K}/K)$  be the absolute Galois group of  $K$ . If  $p > 0$ , let  $W = W(k)$  be the Witt ring and  $K_0 = \text{Frac}(W)$  its fraction field, thus we have  $K_0 \subset \widehat{K}$ .

**1.1. Good reduction and Galois-representations.** Given a variety  $X$  that is smooth and proper over  $K$ , one can ask whether  $X$  has *good reduction*, that is, whether there exists a smooth and proper model of  $X$  over  $\text{Spec } \mathcal{O}_K$ . If it exists, then

(1) the  $G_K$ -representation on  $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified for all  $n$  and all primes  $\ell \neq p$ , and

(2) if  $p > 0$ , the  $G_K$ -representation  $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$  is crystalline for all  $n$ .

This yields necessary conditions for good reduction of  $X$  in terms of  $G_K$ -representations, and we refer to Section 2 for details.

**1.2. Abelian varieties.** If  $A$  is an Abelian variety over  $K$ , then a famous theorem of Serre and Tate [ST68], extending previous work of Néron, Ogg, and Shafarevich on elliptic curves, states that  $X$  has good reduction if and only if the  $G_K$ -representation on  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified for one (resp. all) primes  $\ell \neq p$ . If  $p > 0$ , then good reduction of  $A$  is also equivalent to the  $G_K$ -representation on  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)$  being crystalline, which was proven in special cases by Fontaine [Fon79] and Mokrane [Mok93] and by Coleman and Iovita [CI99] in general.

**1.3. Curves.** If  $C/K$  is a curve (smooth, projective and geometrically connected), then results of Oda [Oda95, Theorem 3.2] show that  $C$  has good reduction if and only if the outer  $G_K$ -action on its  $\mathbb{Q}_\ell$ -unipotent fundamental group  $\pi_1^{\text{ét}}(C_{\overline{K}})_{\mathbb{Q}_\ell}$  is unramified for one (resp. all) primes  $\ell \neq p$ . The  $p$ -adic version of this result was proven by Andreatta, Iovita, and Kim [AIK15].

**1.4. Models of K3 surfaces.** An interesting, rich, and accessible class of algebraic varieties beyond curves and Abelian varieties are K3 surfaces. We refer to Section 5 for definitions. To establish criteria of good reduction, it is oftentimes helpful to have good models of these varieties at hand and sometimes, it even suffices to have these models only after base change to a finite extension of  $K$ . For example, for curves, these are semi-stable models and for Abelian varieties, these are Néron models (maybe even with semi-Abelian reduction).

For K3 surfaces, such good models were introduced by Kulikov [Kul77] and later studied by Pinkham–Persson [PP81] and Nakajima [Nak00]. More precisely, if  $X$  is a K3 surface over  $K$ , then a *Kulikov model* is a flat and proper algebraic space

$$\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$$

with generic fiber  $X$ , whose special fiber is a strict normal crossing divisor, and whose relative canonical divisor is trivial. Following the discussion in [LM14, Section 3.1], we make the following assumption.

**Assumption ( $\star$ ).** A K3 surface  $X$  over  $K$  satisfies ( $\star$ ) if there exists a finite field extension  $L/K$  such that  $X_L$  admits a Kulikov model.

Roughly speaking, ( $\star$ ) would follow from potential semi-stable reduction of K3 surfaces, which is not known yet. At the moment, ( $\star$ ) is known to hold in equal characteristic zero or in mixed characteristic  $(0, p)$  if the K3 surface admits an ample line bundle of self-intersection strictly less than  $(p - 4)$ . We also note that Kulikov models usually only exist as algebraic spaces and

that they are usually not unique - even in the case of good reduction. We refer to [LM14] for precise statements, additional results, and discussion.

### 1.5. A $p$ -adic criterion for potential good reduction of K3 surfaces.

In this article, we study criteria for good reduction of a K3 surface  $X$  over  $K$  in terms of the  $G_K$ -representation on  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ . If  $\ell \neq p$ , then such a criterion was already established in [LM14]. Our first result is the following extension to the  $p$ -adic case.

**Theorem 1.1.** *Suppose that  $p > 0$ , and let  $X$  be a K3 surface over  $K$  that satisfies  $(\star)$ . Then, the following are equivalent:*

- (1)  $X$  has good reduction after a finite and unramified extension of  $K$ .
- (2) The  $G_K$ -representation on  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified for one  $\ell \neq p$ .
- (3) The  $G_K$ -representation on  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified for all  $\ell \neq p$ .
- (4) The  $G_K$ -representation on  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$  is crystalline.

**Remark 1.2.** Since there is a canonical isomorphism  $G_K \cong G_{\widehat{K}}$  between the absolute Galois group of  $K$  and that of its completion, it makes sense to speak of  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$  being crystalline even if  $K$  is not necessarily complete.

The equivalences (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) were already established in [LM14] and Matsumoto [Mat15] showed that when  $K$  is complete, (4) implies good reduction after some finite, but possibly ramified, extension of  $K$ .

**Remark 1.3.** Using the examples constructed in [LM14], we see that the implication (4)  $\Rightarrow$  (1) requires in general a *non-trivial* unramified extension: more precisely, for every prime  $p \geq 5$ , there exists a K3 surface over  $K = \mathbb{Q}_p$  that satisfies conditions (2), (3), and (4) of Theorem 1.1 but that does *not* have good reduction over  $K$ , see Theorem 5.2.

**1.6. RDP models and the canonical reduction.** Thus, although we usually do not have good reduction over  $K$  in the situation of Theorem 1.1, we may appeal to [LM14] to construct certain “RDP models” of K3 surfaces over  $K$ , which are models that have reasonably mild singularities.

More precisely, let  $X$  be a K3 surface over  $K$  that satisfies  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, there exists a projective and flat model

$$\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$$

for  $X$  such that the special fiber  $\mathcal{X}_k$  has at worst canonical singularities, and its minimal resolution is a K3 surface over  $k$ , see Theorem 6.1. We refer to Definition 3.1 for the notion of canonical singularities.

It follows from a theorem of Matsusaka and Mumford [MM64] that the minimal resolution of singularities

$$Y \rightarrow \mathcal{X}_k$$

of the special fiber is unique up to unique isomorphism. In particular, it does not depend on the choice of RDP model  $\mathcal{X}$ . If  $X$  does in fact have

good reduction over  $K$ , then  $Y$  is simply the special fiber of any smooth model. We will refer to  $Y$  as the *canonical reduction* of  $X$ .

It is important to stress that the models  $\mathcal{X}$  asserted by Theorem 1.1 and Theorem 6.1 are usually not unique. However, once a polarization, that is an ample line bundle  $\mathcal{L}$  on  $X$ , is chosen, there is a canonical choice of an RDP model

$$P(X, \mathcal{L}) \rightarrow \mathrm{Spec} \mathcal{O}_K$$

that we call the *canonical RDP model* of the pair  $(X, \mathcal{L})$ , see the discussion in Section 6.

**1.7. The Weyl group of a polarized K3 surface.** The existence of examples as in Remark 1.3 raises the question of how to distinguish between good reduction, and good reduction over an unramified extension. To do so, we introduce the Weyl group of a polarized K3 surface  $(X, \mathcal{L})$  over  $K$  that has good reduction over an unramified extension.

Let  $P(X, \mathcal{L}) \rightarrow \mathrm{Spec} \mathcal{O}_K$  be the canonical RDP model of the pair  $(X, \mathcal{L})$ . Then, its special fiber has canonical singularities, and the minimal resolution of singularities

$$Y \rightarrow P(X, \mathcal{L})_k$$

is given by the canonical reduction of  $X$ . Denote by  $E_{X, \mathcal{L}} \subset Y$  the exceptional locus.

Over  $\bar{k}$  the divisor  $E_{X, \mathcal{L}}$  splits into a union of smooth and rational curves, which have self-intersection number  $-2$ . Then, the *Weyl group* of  $(X, \mathcal{L})$  is the subgroup

$$\mathcal{W}_{X, \mathcal{L}}^{\mathrm{nr}} \leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y_{\bar{k}}))$$

that is generated by the reflections in these  $(-2)$ -curves of  $E_{X, \mathcal{L}, \bar{k}}$ , we refer to Section 3 for details. The superscript “nr” refers to the fact that we are working over  $\bar{k}$ , or, equivalently over  $K^{\mathrm{nr}}$ , and there are similar versions over any finite unramified extension  $L/K$ . We note that the absolute Galois group  $G_k = \mathrm{Gal}(\bar{k}/k)$  naturally acts on  $\mathcal{W}_{X, \mathcal{L}}^{\mathrm{nr}}$  via its action on  $\mathrm{Pic}(Y_{\bar{k}})$ , or, equivalently on  $E_{X, \mathcal{L}, \bar{k}}$ .

We now come to our first improvement of Theorem 1.1, which yields a necessary and sufficient criterion for good reduction of a K3 surface over  $K$ . This result also “explains” the counter-examples from [LM14].

**Theorem 1.4.** *Let  $X$  be a K3 surface over  $K$  that satisfies  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, for any choice  $\mathcal{L}$  of polarization on  $X$ , there exists a non-abelian cohomology class*

$$\alpha_{X, \mathcal{L}}^{\mathrm{nr}} \in H^1(G_k, \mathcal{W}_{X, \mathcal{L}}^{\mathrm{nr}})$$

*depending only on  $(X, \mathcal{L})$ , which is trivial if and only if  $X$  has good reduction over  $K$ .*

**1.8. Another interpretation of the cohomology class.** As well as controlling the failure of  $X$  to have good reduction over  $K$  itself, the cohomology class  $\alpha_{X,\mathcal{L}}^{\text{nr}}$  also describes the difference between the  $G_k$ -modules

$$H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell), \quad \text{and} \quad H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell)$$

for all primes  $\ell \neq p$ , as well as the difference between the  $F$ -isocrystals

$$\mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)) \quad \text{and} \quad H_{\text{cris}}^2(Y/K_0)$$

when  $p > 0$ , where  $Y$  denotes the canonical reduction of  $X$  (again, we are using the fact that  $G_K \cong G_{\widehat{K}}$  to define  $\mathbb{D}_{\text{cris}}$  in the latter case). More precisely, there are  $G_k$ -equivariant homomorphisms

$$\begin{aligned} \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{\mathbb{Q}_\ell} \left( H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell) \right) & (\ell \neq p) \\ \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{K_0^{\text{nr}},F} \left( H_{\text{cris}}^2(Y/K_0) \otimes K_0^{\text{nr}} \right) & (p > 0) \end{aligned}$$

where  $K_0^{\text{nr}}$  is the maximal unramified extension of  $K_0$ , and the subscript “ $F$ ” in the refers to the fact that we are considering Frobenius equivariant automorphisms. Using these homomorphisms and the induced map on cohomology, we obtain  $\ell$ -adic and crystalline realizations

$$\begin{aligned} \beta_{X,\mathcal{L},\ell}^{\text{nr}} &\in H^1(G_k, \text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell))) & (\ell \neq p) \\ \beta_{X,\mathcal{L},p}^{\text{nr}} &\in H^1(G_k, \text{Aut}_{K_0^{\text{nr}},F}(H_{\text{cris}}^2(Y/K_0) \otimes K_0^{\text{nr}})) & (p > 0) \end{aligned}$$

of the cohomology class  $\alpha_{X,\mathcal{L}}^{\text{nr}}$  from Theorem 1.4.

By the general theory of descent, we can then “twist”  $H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell)$  and  $H_{\text{cris}}^2(Y/K_0)$  by these cohomology classes to obtain new  $G_k$ -modules and, when  $p > 0$ ,  $F$ -isocrystals over  $K_0$ ,

$$H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell)^{\beta_{X,\mathcal{L},\ell}^{\text{nr}}} \quad \text{and} \quad H_{\text{cris}}^2(Y/K_0)^{\beta_{X,\mathcal{L},p}^{\text{nr}}},$$

respectively. After these preparations, we can now state our next result.

**Theorem 1.5.** *Under the assumptions and notations of Theorem 1.4, there are natural isomorphisms*

$$\begin{aligned} H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell) &\cong H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell)^{\beta_{X,\mathcal{L},\ell}^{\text{nr}}} & (\ell \neq p) \\ \mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)) &\cong H_{\text{cris}}^2(Y/K_0)^{\beta_{X,\mathcal{L},p}^{\text{nr}}} & (p > 0) \end{aligned}$$

of  $G_k$ -modules and  $F$ -isocrystals over  $K_0$ , respectively.

**1.9. A Néron–Ogg–Shafarevich criterion for K3 surfaces.** Our main result is then the following analogue of the Néron–Ogg–Shafarevich criterion for the good reduction of K3 surfaces over  $K$ .

**Theorem 1.6.** *Let  $X$  be a K3 surface over  $K$  satisfying  $(\star)$ . Then, the following are equivalent.*

- (1)  $X$  has good reduction over  $K$ .
- (2) There exists a prime  $\ell \neq p$  such that (resp. for all primes  $\ell \neq p$ ) the  $G_K$ -representation on  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified and there exists an isomorphism

$$H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell)$$

of  $G_k$ -modules.

(3) If  $p > 0$ ,  $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$  is crystalline and there exists an isomorphism

$$\mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)) \cong H_{\text{cris}}^2(Y/K_0)$$

of  $F$ -isocrystals over  $K_0$ .

Here,  $Y$  denotes the canonical reduction of  $X$ .

By Theorem 1.4 and Theorem 1.5, the proof of this amounts to showing that a certain map in non-abelian group cohomology has trivial kernel. The details of this reduction are spelled out in Section 9, and the group cohomology calculation is performed in Section 4.

**1.10. Integral  $p$ -adic Hodge theory.** In fact, these results hold integrally as well, although the statement of this in the  $p$ -adic case needs some set-up. Indeed, if  $p > 0$ , if

$$\rho : G_K \rightarrow \text{GK}(V)$$

is a crystalline  $G_K$ -representation, and if  $\Lambda \subseteq V$  is a  $G_K$ -stable  $\mathbb{Z}_p$ -lattice, then Kisin [Kis06] associated to it a Breuil–Kisin module  $\text{BK}_{\mathcal{O}_K}(\Lambda)$ , which is a  $\mathfrak{S} := W[[u]]$ -module together with some extra data (we refer to Section 2 for more details). In particular, we may specialize such an object along the homomorphism  $\mathfrak{S} \rightarrow W$  given by the composite of the “ $u = 0$ ” map and the Frobenius on  $W$ , to obtain an  $F$ -crystal

$$\text{BK}_{\mathcal{O}_K}(\Lambda) \otimes_{\mathfrak{S}, \sigma} W$$

over  $W$  (the reason for using this map rather than simply the “ $u = 0$ ” map is explained in Section 2).

If  $X$  is a smooth and proper variety over  $K$  with good reduction, say via some smooth and proper model  $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$  with special fiber  $\mathcal{X}_k$ . Let us assume moreover that  $\mathcal{X}_k$  is a scheme and that  $H_{\text{cris}}^*(\mathcal{X}_k/W)$  is torsion-free. Then, the  $G_K$ -representations on  $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$  are crystalline, and after results of Bhatt, Morrow, and Scholze [BMS15, BMS16], there exists an isomorphism of  $F$ -crystals

$$\text{BK}_{\mathcal{O}_K}(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S}, \sigma} W \cong H_{\text{cris}}^n(\mathcal{X}_k/W).$$

which is compatible with the usual (rational) crystalline comparison theorem. We also refer to Section 2 for details.

Now, suppose that  $X$  is a K3 surface over any Henselian  $K$  as above (i.e. allowing  $p = 0$ ), that satisfies  $(\star)$  and the equivalent conditions of Theorem 1.1. Then we obtain the same relationship between

$$H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Z}_\ell) \text{ and } H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Z}_\ell),$$

as well as between

$$\text{BK}_{\mathcal{O}_K}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S}, \sigma} W \text{ and } H_{\text{cris}}^2(Y/W),$$

as we saw in Theorem 1.5 in the rational case. Namely, there are  $G_k$ -equivariant homomorphisms

$$\begin{aligned} \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{\mathbb{Z}_\ell} (H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell)) & (\ell \neq p) \\ \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{W^{\text{nr}},F} (H_{\text{cris}}^2(Y/W) \otimes W^{\text{nr}}) & (p > 0) \end{aligned}$$

where  $W^{\text{nr}}$  is the ring of integers in  $K_0^{\text{nr}}$ . We therefore obtain integral realisations of the class  $\alpha_{X,\mathcal{L}}^{\text{nr}}$

$$\begin{aligned} \beta_{X,\mathcal{L},\ell}^{\text{nr}} &\in H^1(G_k, \text{Aut}_{\mathbb{Z}_\ell}(H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell))) & (\ell \neq p) \\ \beta_{X,\mathcal{L},p}^{\text{nr}} &\in H^1(G_k, \text{Aut}_{K_0^{\text{nr}},F}(H_{\text{cris}}^2(Y/W) \otimes W^{\text{nr}})) & (p > 0) \end{aligned}$$

as before.

**Theorem 1.7.** *Let  $X$  be a K3 surface over  $K$  that satisfies  $(\star)$  and the equivalent conditions of Theorem 1.1.*

(1) *There are natural isomorphisms*

$$\begin{aligned} H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Z}_\ell) &\cong H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell)^{\beta_{X,\mathcal{L},\ell}^{\text{nr}}} & (\ell \neq p) \\ \text{BK}_{\mathcal{O}_K}(H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S},\sigma} W &\cong H_{\text{cris}}^2(Y/W)^{\beta_{X,\mathcal{L},p}^{\text{nr}}} & (p > 0) \end{aligned}$$

*of  $G_k$ -modules and  $F$ -crystals over  $W$ , respectively.*

(2)  *$X$  has good reduction over  $K$  if and only if there exists an isomorphism*

$$H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Z}_\ell) \cong H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell)$$

*of  $G_k$ -modules for some  $\ell \neq p$  (resp. for all  $\ell \neq p$ ). When  $p > 0$ ,  $X$  has good reduction over  $K$  if and only if there exists an isomorphism*

$$\text{BK}_{\mathcal{O}_K}(H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S},\sigma} W \cong H_{\text{cris}}^2(Y/W)$$

*of  $F$ -crystals over  $W$ .*

**1.11. Equal and positive characteristic.** One might ask whether all these results also hold if  $\mathcal{O}_K$  is an excellent and Henselian DVR of equal characteristic  $p > 0$ . The main problem is Assumption  $(\star)$ : if we knew that the extension  $L/K$  there can always be chosen *separable*, then the statements and proofs of [LM14] and this article would most likely go through verbatim. At the moment, however, we do not know whether it is reasonable to expect this separability in general.

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## NOTATIONS AND CONVENTIONS

Throughout the whole article, we fix the following notations

$\mathcal{O}_K$	a Henselian DVR
$\pi$	a uniformizer of $\mathcal{O}_K$
$K$	the field of fractions of $\mathcal{O}_K$ , which we assume to be of characteristic 0
$k$	the residue field of $\mathcal{O}_K$ , which we assume to be perfect of characteristic $p \geq 0$
$\widehat{K}$	the completion of $K$
$W = W(k)$	if $p > 0$ , the ring of Witt vectors of $k$
$K_0$	if $p > 0$ , the field of fractions of $W$
$K_0^{\text{nr}}, K^{\text{nr}}$	the maximal unramified extension of $K_0$ and $K$ respectively
$\overline{K}, \overline{k}$	fixed algebraic closures of $K$ and $k$ , respectively
$G_K, G_k$	the absolute Galois groups of $K$ and $k$ , respectively

If  $L/K$  is a field extension, and  $X$  is a scheme over  $K$ , we abbreviate the base-change  $X \times_{\text{Spec } K} \text{Spec } L$  by  $X_L$ . If  $p > 0$ , and  $Y$  is a scheme over  $k$ , we will write

$$H_{\text{cris}}^n(Y/K_0) := H_{\text{cris}}^n(Y/W) \otimes_W K_0$$

and similarly over any finite extension of  $k$ .

## 2. GENERALITIES

In this section, we recall a couple of general facts on models of varieties, crystalline Galois representations, the functors of Fontaine and Kisin, and their relation to good reduction.

**2.1. Models.** We start with the definition of various types of models.

**Definition 2.1.** Let  $X$  be a smooth and proper variety over  $K$ .

- (1) A *model of  $X$  over  $\mathcal{O}_K$*  is an algebraic space that is flat and proper over  $\text{Spec } \mathcal{O}_K$  and whose generic fiber is isomorphic to  $X$ .
- (2) We say that  $X$  has *good reduction* if there exists a model of  $X$  that is smooth over  $\mathcal{O}_K$ .

We stress that we have to allow algebraic space models when dealing with K3 surfaces, which is different from the situation for curves and Abelian varieties. In fact, we refer the interested reader to [Mat15, Section 5.2] for

examples of K3 surfaces with good reduction, where no smooth model exists in the category of schemes.

**2.2. Inertia and monodromy.** The  $G_K$ -action on  $\overline{K}$  induces an action on  $\mathcal{O}_{\overline{K}}$  and by reduction, an action on  $\overline{k}$ . This gives rise to a continuous and surjective homomorphism  $G_K \rightarrow G_k$  of profinite groups. Thus, we obtain a short exact sequence

$$1 \rightarrow I_K \rightarrow G_K \rightarrow G_k \rightarrow 1,$$

whose kernel  $I_K$  is called the *inertia group*. In fact,  $I_K$  is the absolute Galois group of the maximal unramified extension  $K^{\text{nr}}$  of  $K$ . Then, an  $\ell$ -adic representation of  $G_K$  consists of a finite-dimensional  $\mathbb{Q}_\ell$ -vector space  $V$  together with a continuous group homomorphism  $\rho : G_K \rightarrow \text{GL}(V)$ . The representation  $\rho$  is called *unramified* if  $\rho(I_K) = \{\text{id}_V\}$ .

A relation between good reduction and unramified Galois representations on  $\ell$ -adic cohomology groups is given by the following well-known result, which follows from the proper smooth base change theorem. For schemes, it is stated in [SGA4.3, Théorème XII.5.1], and in case the model is an algebraic space, we refer to [LZ14, Theorem 0.1.1] or [Art73, Chapitre VII].

**Theorem 2.2.** *Let  $X$  be a smooth and proper variety over  $K$  with good reduction. Then, the  $G_K$ -representation on  $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$  is unramified for all  $n$  and for all  $\ell \neq p$ .*

**2.3.  $p$ -adic Galois representations.** Now let us assume that  $p > 0$ . In [Fon82, Fon94a], Fontaine introduced his famous period rings

$$B_{\text{cris}} \subseteq B_{\text{st}} \subseteq B_{\text{dR}},$$

which are (in particular)  $\mathbb{Q}_p$ -algebras equipped with compatible actions of  $G_K$ . One usually assume that  $K$  complete, but since completion does not change the absolute Galois group, there is no need to do so. Given a  $p$ -adic representation  $\rho : G_K \rightarrow \text{GL}(V)$ , that is,  $V$  is a finite dimensional  $\mathbb{Q}_p$ -vector space and  $\rho$  is a continuous group homomorphism, we define

$$\begin{aligned} \mathbb{D}_{\text{cris}}(V) &:= (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K} \\ \mathbb{D}_{\text{st}}(V) &:= (V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{G_K} \\ \mathbb{D}_{\text{dR}}(V) &:= (V \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_K} \\ \mathbb{D}_{\text{pst}}(V) &:= \varinjlim_{K'/K} (V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{G_{K'}}, \end{aligned}$$

where the limit in the last definition is taken over all finite extensions of  $K$  (within our fixed algebraic closure  $\overline{K}$ ), and where all  $G_K$ -invariants are taken with respect to the diagonal action given by  $\rho$  on the first factor and the natural  $G_K$ -action on the period rings. Then:

- (1)  $\mathbb{D}_{\text{cris}}(V)$  and  $\mathbb{D}_{\text{st}}(V)$  are finite-dimensional  $K_0$ -vector spaces that come equipped with a semi-linear Frobenius, that is, they are  $F$ -isocrystals,

- (2)  $\mathbb{D}_{\mathrm{dR}}(V)$  is a finite dimensional  $\widehat{K}$ -vector space,
- (3)  $\mathbb{D}_{\mathrm{pst}}(V)$  is a finite-dimensional  $K_0^{\mathrm{nr}}$ -vector space equipped with a semi-linear Frobenius and a semi-linear action of  $G_K$ .

Here,  $G_K$ -acts on  $K_0^{\mathrm{nr}}$  via its quotient  $G_k \cong \mathrm{Gal}(K_0^{\mathrm{nr}}/K_0)$  and in particular, the induced  $I_K$ -action is linear. (Of course, this does not exhaust the extra structures these vector spaces have, but we will not need to explicitly use either the Hodge filtration or the monodromy operator.) By [Fon94b, Propositions 1.4.2 and 5.1.2], we have inequalities

$$\begin{aligned} \dim_{K_0} \mathbb{D}_{\mathrm{cris}}(V) &\leq \dim_{K_0} \mathbb{D}_{\mathrm{st}}(V) \leq \dim_{K_0^{\mathrm{nr}}} \mathbb{D}_{\mathrm{pst}}(V) \\ &\leq \dim_{\widehat{K}} \mathbb{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V. \end{aligned}$$

The representation  $\rho$  is called *crystalline* (resp. *semi-stable*, resp. *potentially semi-stable*, resp. *de Rham*) if we have equality everywhere (resp. the last three, two, one inequalities are equalities).

To obtain results for *integral*  $p$ -adic cohomology, we will need to recall the notion of a Breuil–Kisin module, originally introduced by Kisin in [Kis06]. For  $\mathfrak{S} := W[[u]]$ , we have two natural and surjective maps  $\mathfrak{S} \rightarrow W$  and  $\mathfrak{S} \rightarrow \mathcal{O}_{\widehat{K}}$  defined by  $u \mapsto 0$  and  $u \mapsto \pi$ , respectively. Let  $E(u)$  be the monic Eisenstein polynomial that generates the kernel of  $\mathfrak{S} \rightarrow \mathcal{O}_{\widehat{K}}$ . There is a Frobenius map  $\sigma : \mathfrak{S} \rightarrow \mathfrak{S}$  that is the absolute Frobenius on  $W$  and sends  $u \mapsto u^p$ .

**Definition 2.3.** A *Breuil–Kisin module* over  $\mathcal{O}_K$  is a finite free  $\mathfrak{S}$ -module  $M$  together with a morphism of  $\mathfrak{S}$ -modules

$$\varphi : M \otimes_{(\mathfrak{S}, \sigma)} \mathfrak{S} \rightarrow M$$

that becomes an isomorphism after inverting  $E(u)$ .

Again, by definition this only depends on  $\widehat{K}$ . By specializing along  $u = 0$ , there is a functor from Breuil–Kisin modules over  $\mathcal{O}_K$  to  $F$ -crystals over  $W$ . Alternatively, we can consider the closely related functor which specializes along the composite map

$$\mathfrak{S} \xrightarrow{u=0} W \xrightarrow{\sigma} W,$$

where  $\sigma$  is the Frobenius map on  $W$ . Then, Kisin showed how to construct a functor  $\mathrm{BK}_{\mathcal{O}_K}$  from  $G_K$ -stable lattices in crystalline  $G_K$ -representations to Breuil–Kisin modules, for a precise statement we refer to the reader to [Kis10, Theorem 1.2.1]. In particular, by specialization, we obtain  $F$ -crystals  $\mathrm{BK}_{\mathcal{O}_K}(\Lambda) \otimes_{\mathfrak{S}} W$  and  $\mathrm{BK}_{\mathcal{O}_K}(\Lambda) \otimes_{\mathfrak{S}, \sigma} W$  associated to such a lattice  $\Lambda \subset V$ . It follows from [Kis10, Theorem 1.2.1 (1)] that there is an isomorphism  $\mathrm{BK}_{\mathcal{O}_K}(\Lambda) \otimes_{\mathfrak{S}} K_0 \cong \mathbb{D}_{\mathrm{cris}}(V)$  of  $F$ -isocrystals over  $K_0$ .

**2.4. The connection to geometry.** Let us continue to suppose that  $p > 0$ . If  $X$  is a smooth and proper variety over  $K$ , then it is well-known (see, for example, [Ber97, Section 6.3.3]), that the  $G_K$ -representations  $H_{\mathrm{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$  are potentially semi-stable for all  $n$ . The  $p$ -adic analog of Theorem 2.2 is

the following theorem, essentially due to Colmez–Niziol [CN17] and Bhatt–Morrow–Scholze [BMS16]. For models of K3 surfaces, which is the case we are interested in, a part of it has already been proven by Matsumoto [Mat15, Section 2].

**Theorem 2.4.** *Let  $X$  be a smooth and proper variety over  $K$  and assume that there exists a smooth and proper algebraic space*

$$\mathcal{X} \rightarrow \mathrm{Spec} \mathcal{O}_K,$$

whose generic fiber is  $X$  and whose special fiber  $\mathcal{X}_k$  is a scheme. Then,

- (1) the  $G_K$ -representation on  $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$  is crystalline for all  $n$ .
- (2) For all  $n$ , there exist isomorphisms

$$\mathbb{D}_{\text{cris}}(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)) \cong H_{\text{cris}}^n(\mathcal{X}_k/K_0)$$

of  $F$ -isocrystals over  $K_0$ .

- (3) Assume that  $H_{\text{cris}}^*(\mathcal{X}_k/W)$  is torsion-free. Then, also  $H_{\text{ét}}^*(X_{\overline{K}}, \mathbb{Z}_p)$  is torsion-free, and for all  $n$  there exist isomorphisms

$$\mathrm{BK}_{\mathcal{O}_K}(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S}, \sigma} W \cong H_{\text{cris}}^n(\mathcal{X}_k/W)$$

of  $F$ -crystals over  $W$ .

PROOF. We may assume that  $K = \widehat{K}$ . We first claim that the formal completion  $\mathfrak{X}$  of  $\mathcal{X}$  along the special fiber is in fact a formal scheme. Indeed, for every  $m \geq 1$ , the reduction of the algebraic space  $\mathcal{X}_m := \mathcal{X} \times \mathrm{Spec} \mathcal{O}_K/(\pi^{m+1})$  is equal to  $\mathcal{X}_k$ , which is a scheme. It follows from [Knu71, Chapter 3, Corollary 3.6], that  $\mathcal{X}_m$  is also a scheme, hence  $\mathfrak{X} = \varinjlim_m \mathcal{X}_m$  is a formal scheme.

Let  $\mathfrak{X}_K$  denote its generic fiber as a rigid space over  $K$ . Next, we claim that there is an isomorphism  $\mathfrak{X}_K \cong X^{\text{an}}$ . (This is well-known for schemes, and is probably also well-known to the experts for algebraic spaces.) Choose a presentation

$$\mathcal{X}^1 \rightrightarrows \mathcal{X}^0 \rightarrow \mathcal{X}$$

of  $\mathcal{X}$  as the coequalizer of an étale equivalence relation. Let  $\mathfrak{X}^i$  denote the formal completion of  $\mathcal{X}^i$  and let  $X^i$  be the generic fiber. By [Ber96, Proposition 0.3.5] we obtain a diagram

$$\begin{array}{ccccc} \mathfrak{X}_K^1 & \rightrightarrows & \mathfrak{X}_K^0 & \longrightarrow & \mathfrak{X}_K \\ \downarrow & & \downarrow & & \\ X^{1,\text{an}} & \rightrightarrows & X^{0,\text{an}} & \longrightarrow & X^{\text{an}} \end{array}$$

with vertical maps open immersions. In particular, we get a map  $\mathfrak{X}_K \rightarrow X^{\text{an}}$  by the universal property of coequalizers. This moreover fits into a

commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_K^0 & \longrightarrow & \mathfrak{X}_K \\ \downarrow & & \downarrow \\ X^{0,\text{an}} & \longrightarrow & X^{\text{an}} \end{array}$$

of rigid analytic spaces over  $K$ , in which the horizontal arrows are étale (of the same degree) and the left hand vertical map is an open immersion. In particular,  $\mathfrak{X}_K \rightarrow X^{\text{an}}$  is étale and since both  $\mathfrak{X}_K$  and  $X^{\text{an}}$  are proper, it must be finite. Using again the commutativity of the diagram, we see that it must be of degree one, and hence, an isomorphism.

Thus applying [Hub96, Theorem 3.8.1] we obtain a  $G_K$ -equivariant isomorphism

$$H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Z}_p) \cong H_{\text{ét}}^n(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p),$$

for all  $n \geq 0$ , and the same is also true for  $\mathbb{Q}_p$ -valued étale cohomology. Claims (1) and (2) now follow from [CN17, Corollary 1.10], and in fact all three follow from [BMS16, Theorem 1.1].  $\square$

**Remark 2.5.** (1) Note that the integral comparison result in [BMS16] is stated without a Frobenius pull-back, however, it was pointed out to us by T. Koshikawa and P. Scholze that this is not quite correct. The problem is that the map  $\mathfrak{S} \rightarrow A_{\text{inf}}$  defined on p.33 of [BMS16] does not make the resulting diagram commute: one needs to compose with the Frobenius map on  $W(k)$ . Since the comparison isomorphism provided by [BMS16, Theorem 1.1] is defined by first base changing to  $A_{\text{inf}}$ , to get the correct statement over  $\mathfrak{S}$  one needs to include this Frobenius pull-back.

(2) As already mentioned in Section 2.1, one has to work with models that are algebraic spaces. The assumption in Theorem 2.4 on the special fiber of a smooth and proper model being a scheme is unnatural and it is probably not needed, but we had to impose it in order to use the results that are currently available. We note that smooth and proper algebraic spaces of dimension  $\leq 2$  over  $k$ , as well as smooth and proper algebraic spaces over  $k$  that are group spaces, are schemes by [Knu71, II.6.7 and V.4.10]. Thus, if  $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$  is a smooth and proper algebraic space model of some variety  $X$ , such that  $X$  is of dimension  $\leq 2$  or such that  $X$  is an Abelian variety over  $K$ , then the special fiber  $\mathcal{X}_k$  is a scheme.

**2.5. A lemma on crystalline representations.** In the proof of Theorem 1.1 below, we will need some elementary facts about potentially crystalline and potentially semi-stable  $p$ -adic Galois representations, which are certainly well-known to the experts. We continue to assume that  $p > 0$ .

**Lemma 2.6.** *Let  $\rho : G_K \rightarrow \text{GL}(V)$  be a  $p$ -adic Galois representation. Let  $K \subseteq K'$  be a finite field extension and denote by  $\rho' : G_{K'} \rightarrow \text{GL}(V)$  the restriction of  $\rho$  to  $G_{K'}$ .*

- (1) Assume that  $K \subseteq K'$  is unramified. Then,  $\rho$  is a crystalline  $G_K$ -representation if and only if  $\rho'$  is a crystalline  $G_{K'}$ -representation.
- (2) Assume that  $K \subseteq K'$  is totally ramified and Galois, say with Galois group  $H = \text{Gal}(K'/K)$ . Then, the following are equivalent:
  - (a)  $\rho$  is crystalline,
  - (b)  $\rho'$  is crystalline and the induced  $H$ -action on

$$(V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}}$$

is trivial.

- (3) The following are equivalent:
  - (a)  $\rho$  is crystalline,
  - (b)  $\rho$  is potentially crystalline and the  $I_K$ -action on  $\mathbb{D}_{\text{pst}}(V)$  is trivial.

PROOF. We may assume that  $K = \widehat{K}$ . First, suppose that  $K'/K$  is unramified. If  $\rho$  is crystalline, then so is  $\rho'$  and thus, it remains to show the converse direction. After replacing the extension  $K \subseteq K'$  by its Galois closure, we may assume that  $K'$  is Galois over  $K$ , say with group  $H$ . Then, we have the following invariants with respect to the induced  $H$ -actions

$$(1) \quad \left( (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}} \right)^H \cong (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K}.$$

Since  $\rho'$  is crystalline,  $(V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}}$  is a  $K'_0$ -vector space of dimension  $d := \dim_{\mathbb{Q}_p}(V)$ . Since  $K'$  is unramified over  $K$ , also  $K_0 \subseteq K'_0$  is Galois with group  $H$ . Thus, the  $H$ -invariants in (1) are  $K_0$ -vector spaces of dimension  $d$ . This proves that  $\rho$  is crystalline and establishes claim (1).

Next, assume that  $K'/K$  is totally ramified. Then, we have  $K'_0 = K_0$  and since  $G_{K'}$  is contained in  $G_K$ , we find

$$\dim_{K_0} (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K} \leq \dim_{K_0} (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}} \leq \dim_{\mathbb{Q}_p} V.$$

We have equality on the right if and only if  $\rho'$  is crystalline, equality on the left if and only if  $H$  acts trivially on  $(V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}}$ , and equality everywhere if and only if  $\rho$  is crystalline. This establishes claim (2).

For claim (3), we note that since crystalline implies potentially crystalline, we may assume that  $\rho$  is potentially crystalline. Next, we choose a finite Galois extension  $K'/K$  such that  $\rho' := \rho|_{G_{K'}}$  is crystalline. Applying part (1) and replacing  $K$  by its maximal unramified extension inside  $K'$ , we may assume that  $K'/K$  is totally ramified. Denote by  $H$  its Galois group. Given part (2), it now suffices to observe that in this situation we have

$$\mathbb{D}_{\text{pst}}(V) \cong (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}} \otimes_{K_0} K_0^{\text{nr}},$$

and the  $I_K$ -action on  $\mathbb{D}_{\text{pst}}(V)$  is simply the extension of scalars of the natural action of  $I_K$  on

$$(V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_{K'}}$$

via the surjective map  $I_K \rightarrow H$ .  $\square$

**2.6. Forms of  $F$ -isocrystals and non-abelian cohomology.** In this section we recall some material on  $L$ -forms and descent for  $F$ -isocrystals. Here, we will assume that  $p > 0$ , and that  $K = K_0$  is complete and absolutely unramified.

Let  $L/K$  be a finite and unramified Galois extension with Galois group  $G$ , and let  $V$  be an  $F$ -isocrystal over  $K$ .

**Definition 2.7.** (1) An  $L$ -form of  $V$  is an  $F$ -isocrystal  $W/K$  such that there exists an isomorphism  $V \otimes L \cong W \otimes L$  as  $F$ -isocrystals over  $L$ .  
 (2) Two  $L$ -forms of  $V$  are said to be *equivalent* if they are isomorphic as  $F$ -isocrystals over  $K$ .

We denote the set of equivalence classes of  $L$ -forms by  $E(L/K, V)$ .

We note that  $E(L/K, V)$  is a pointed set, the distinguished element being the class of  $V$  itself. Given a form  $W \in E(L/K, V)$  then by definition there is an isomorphism

$$f : V \otimes_K L \xrightarrow{\sim} W \otimes_K L$$

of  $F$ -isocrystals over  $L$ . Moreover, both sides carry a natural  $G$ -action coming from the  $G$ -action on  $L$ , but in general,  $f$  will *not* be equivariant for these actions. In fact, we can define a function

$$\begin{aligned} \alpha_f : G &\rightarrow \text{Aut}_{L,F}(V \otimes_K L) \\ \alpha_f(\sigma) &= f^{-1} \circ \sigma(f) \end{aligned}$$

that measures the failure of  $f$  to be  $G$ -equivariant. We note that  $\alpha_f$  is a 1-cocycle for the natural action of  $G$  on  $\text{Aut}_{L,F}(V \otimes L)$ . Then, we have the following standard result.

**Proposition 2.8.** *This map induces a bijection*

$$E(L/K, V) \rightarrow H^1(G, \text{Aut}_{L,F}(V \otimes_K L))$$

*of pointed sets.*

It is worth recalling how to construct the inverse functor, via a slightly different perspective on descent. That is, as well as the category  $F\text{-Isoc}(K)$  of  $F$ -isocrystals over  $K$ , we can consider the category  $G\text{-}F\text{-Isoc}(L)$  of  $F$ -isocrystals over  $L$  together with a compatible, semi-linear  $G$ -action. Again, the proof of the following result is straightforward.

**Proposition 2.9.** *The functors*

$$\begin{array}{lcl} F\text{-Isoc}(K) & \rightarrow & G\text{-}F\text{-Isoc}(L) : V \mapsto V \otimes_K L \\ G\text{-}F\text{-Isoc}(L) & \rightarrow & F\text{-Isoc}(K) : W \mapsto W^G \end{array}$$

*are mutual inverse equivalences of categories.*

Using this proposition, the inverse map in Proposition 2.8 is then described as follows: given a 1-cocycle  $\alpha : G \rightarrow \text{Aut}_{L,F}(V \otimes L)$ , we can define

a new *action* of  $G$  on  $V \otimes L$  via the *homomorphism*

$$\begin{aligned}\rho^\alpha : G &\rightarrow \text{Aut}_{L,F}(V \otimes_K L) \\ \rho^\alpha(\sigma)(v) &= \alpha(\sigma)(\sigma(v))\end{aligned}$$

where  $\sigma(v)$  is the “standard” action. By Proposition 2.9 we obtain an  $F$ -isocrystal

$$V^\alpha := (V \otimes L)^{\rho^\alpha}$$

over  $K$ , which is the form of  $V$  giving rise to  $\alpha$ . Note in particular, that  $V$  and  $V^\alpha$  are isomorphic over  $K$  if and only if there exists an automorphism

$$V \otimes_K L \rightarrow V \otimes_K L$$

of  $F$ -isocrystals over  $L$  which intertwines the ‘natural’  $G$ -action on the left hand side with the ‘ $\alpha$ -twisted’ action on the right hand side.

There is also similar discussion when  $L$  is replaced by the maximal unramified extension  $K^{\text{nr}}$ , as well as for  $F$ -crystals over  $W$ . We leave the details to the reader.

**2.7. Forms of  $G_k$ -modules and non-abelian cohomology.** There is a similar theory for  $\ell$ -adic Galois representations of the absolute Galois group  $G_k$ , allowing  $p \geq 0$  and  $\ell$  to be any prime, including possibly  $p$ .

If we let  $k'/k$  be a finite Galois extension, with Galois group  $G$ , then we have an exact sequence

$$1 \rightarrow G_{k'} \rightarrow G_k \rightarrow G \rightarrow 1,$$

where  $G_{k'}$  is the absolute Galois group of  $k'$ .

**Definition 2.10.** Let  $V$  be a finite dimensional  $\mathbb{Q}_\ell$  vector space and  $\rho : G_k \rightarrow \text{GL}(V)$  be an  $\ell$ -adic Galois representation.

- (1) A  $k'$ -form of  $V$  is an  $\ell$ -adic Galois representation  $\psi : G_k \rightarrow \text{GL}(W)$  such that there exists a  $G_{k'}$ -equivariant and  $\mathbb{Q}_\ell$ -linear isomorphism  $V|_{G_{k'}} \cong W|_{G_{k'}}$ .
- (2) Two  $k'$ -forms of  $V$  are said to be *equivalent* if they are isomorphic as  $G_k$ -representations.

We denote the set of equivalence classes of  $k'$ -forms by  $E(k'/k, V)$ .

As in the previous section,  $E(k'/k, V)$  is a pointed set. There is a natural  $G_k$  action on  $\text{Aut}_{G_{k'}}(V)$  via conjugation, and by definition  $G_{k'}$  acts trivially. Thus, this action factors through  $G$ . Given a  $k'$ -form  $W \in E(k'/k, V)$ , we can choose a  $G_{k'}$ -equivariant isomorphism

$$f : V \rightarrow W$$

and we obtain a 1-cocycle

$$\begin{aligned}\alpha_f : G &\rightarrow \text{Aut}_{G_{k'}}(V) \\ \alpha_f(\sigma) &= f^{-1} \circ \sigma(f)\end{aligned}$$

as before. Then, we have the following proposition, whose proof we leave to the reader.



**Proposition 2.11.** *This map induces a bijection*

$$E(k'/k, V) \rightarrow H^1(G, \text{Aut}_{G_{k'}}(V))$$

*of pointed sets.*

We can describe the inverse functor relatively easily as follows. Namely, if we have a  $G_k$ -representation  $\rho : G_k \rightarrow \text{GL}(V)$  and a 1-cocycle

$$\alpha : G \rightarrow \text{Aut}_{G_{k'}}(V),$$

then we can construct a “new”  $G_k$ -action on  $V$  by defining

$$\rho^\alpha(g)(v) := \alpha(g)(\rho(g)(v))$$

Then,  $\rho^\alpha : G_k \rightarrow \text{GL}(V)$  is the  $k'$ -form of  $V$  giving rise to  $\alpha$ . All of this works integrally as well, that is, replacing  $\mathbb{Q}_\ell$  by  $\mathbb{Z}_\ell$ , and there is a version replacing  $k'/k$  with  $\bar{k}/k$ . We leave all these details up to the reader.

### 3. CANONICAL SURFACE SINGULARITIES AND WEYL GROUPS

We will need to fix a couple of definitions concerning surfaces and their singularities, as well as make a few group-theoretic calculations involving Dynkin diagrams and Weyl groups.

**Definition 3.1.** Let  $F$  be a field.

- (1) A *surface* over  $F$  is a separated, geometrically integral scheme  $S$  of finite type and dimension 2 over  $\text{Spec } F$ .
- (2) A surface  $S$  over  $F$  has *(at worst) canonical singularities* if it is geometrically normal, Gorenstein, and if the minimal resolution of singularities  $f : \tilde{S} \rightarrow S$  satisfies  $f^*\omega_S \cong \omega_{\tilde{S}}$ . Here,  $\omega_S$  and  $\omega_{\tilde{S}}$  denote the dualizing sheaves of  $S$  and  $\tilde{S}$ , respectively.
- (3) A *K3 surface* over  $F$  is smooth and proper surface  $S$  over  $F$  with  $\omega_S \cong \mathcal{O}_S$  and  $H^1(S, \mathcal{O}_S) = 0$ .

We note that canonical surface singularities are rational singularities, and that canonical surface singularities are characterized as being those rational singularities that are Gorenstein. If the ground field  $F$  is algebraically closed, then canonical surface singularities are also known as *rational double point singularities*, *du Val singularities*, or *Kleinian singularities*, and we refer to Artin’s articles [Art62, Art66] for details and proofs.

We also note here that an algebraic space that is two-dimensional, smooth, and proper over a field is a projective scheme [Knu71, V.4.10]. By [Art62, Theorem 2.3], the same is true for two-dimensional and proper algebraic spaces with rational singularities, which includes canonical surface singularities.

**3.1. Classification of canonical singularities.** If  $F$  is a field, then canonical surface singularities over  $F$  are classified by the finite Dynkin diagrams, as explained in [Lip69, §24]. Let us recall here a little about how this works, as well as fixing some notations that will be used later on.

Let  $S/F$  be a surface with canonical singularities and let  $\tilde{S} \rightarrow S$  be its minimal resolution. Then, the possible configurations of the exceptional divisor  $E_P \subset \tilde{S}$  over some singular point  $P \in S$  are described by the Dynkin diagrams

$$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$$

(whenever we say ‘‘Dynkin diagram’’, we will always mean a *finite, reduced, irreducible* Dynkin diagram, that is, one from the above list). The geometric interpretation of a diagram as follows. Nodes of the diagram correspond to irreducible (but not necessarily geometrically irreducible) components  $E_{P,i} \subset E_P$ , and we have

$$n_{P,i} = h^0(E_{P,i}) := \dim_{F(P)} H^0(E_{P,i}, \mathcal{O}_{E_{P,i}}),$$

where  $n_{P,i}$  is the integer labelling the node corresponding to  $E_{P,i}$ , and where  $F(P)$  is the residue field at  $P$ . The existence of an edge between  $E_{P,i}$  and  $E_{P,j}$  means that  $E_{P,i} \cap E_{P,j} \neq \emptyset$ . This then completely determines the intersection matrix of the  $E_{P,i}$  by

$$E_{P,i} \cdot E_{P,j} = \begin{cases} -2n_{P,i} & \text{if } i = j \\ \max(n_{P,i}, n_{P,j}) & \text{if } i \neq j \text{ and } E_{P,i} \cap E_{P,j} \neq \emptyset \\ 0 & \text{if } E_{P,i} \cap E_{P,j} = \emptyset. \end{cases}$$

Of course, we have  $E_{P,i} \cdot E_{Q,j} = 0$  if  $P \neq Q$ .

We will call the subgroup  $\Lambda_S \subseteq \text{Pic}(\tilde{S})$  that is (freely) generated by the integral components of all  $E_P$  (as  $P$  ranges over the singular points of  $S$ ), together with its natural intersection pairing, the *root lattice* of  $S$ , and its Weyl group

$$\mathcal{W}_S \leq \text{Aut}_{\mathbb{Z}}(\Lambda_S)$$

the *Weyl group* of  $S$ . Concretely,  $\mathcal{W}_S$  is generated by the maps

$$\begin{aligned} s_{P,i} : \Lambda_S &\rightarrow \Lambda_S \\ D &\mapsto D + \frac{1}{n_{P,i}}(D \cdot E_{P,i}) \cdot E_{P,i} \end{aligned}$$

for all integral components  $E_{P,i}$  of all  $E_P$ . Using the same formula, we can view  $\mathcal{W}_S$  as a subgroup of  $\text{Aut}_{\mathbb{Z}}(\text{Pic}(\tilde{S}))$ .

Note that the natural ‘‘geometric’’ sign convention we are using is the opposite of that usually found when discussing root systems, for us therefore Cartan matrices will be negative definite rather than positive definite. Also note that the root lattice and the Weyl group break up into direct products of the root lattices and Weyl groups associated to each individual singular point of  $S$ .

It is important to remember that these definitions are not stable under a base field extension  $F'/F$ , *even if all the singularities are  $F$ -rational and of*

*type ADE*. For example, an  $F$ -rational singularity of type  $A_1$  could become a singularity of type  $A_2$  geometrically.

Nonetheless, we can describe their behaviour under such extensions as follows, at least when the extension  $F'/F$  is Galois, say with group  $G$ . Indeed, in this case there are natural and compatible  $G$ -actions on both  $\Lambda_{S_{F'}}$  and  $\mathcal{W}_{S_{F'}}$ , and we clearly have  $\Lambda_S = \Lambda_{S_{F'}}^G$ . Moreover, we can construct a homomorphism

$$\mathcal{W}_S \rightarrow \mathcal{W}_{S_{F'}}^G$$

from  $\mathcal{W}_S$  into the  $G$ -invariants of  $\mathcal{W}_{S_{F'}}$ , as follows. Take a reflection  $s_\alpha$  in a simple root  $\alpha$  of  $\Lambda_S$ , and suppose that  $\alpha$  splits into simple roots  $\{\alpha_1, \dots, \alpha_n\}$  of  $\Lambda_{S_{F'}}$ . Then,

- (1) either  $(\alpha_i \cdot \alpha_j) = 0$  for all  $i \neq j$ ,
- (2) or else  $n$  is even and after possibly re-ordering we have  $(\alpha_{2i-1} \cdot \alpha_{2i}) = 1$  for all  $i$  and  $(\alpha_i \cdot \alpha_j) = 0$  for all other  $i \neq j$ ,

where we argue as in the proof of [LM14, Lemma 4.3] to see that these are the only cases. In the first case, we simply map  $s_\alpha$  to the product  $\prod_i s_{\alpha_i}$  of the reflection in each of the  $\alpha_i$  (in any order, since they all commute). In the second case, we map  $s_\alpha$  to

$$\prod_{i=1}^{\frac{n}{2}} s_{\alpha_{2i-1}} s_{\alpha_{2i}} s_{\alpha_{2i-1}} = \prod_{i=1}^{\frac{n}{2}} s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}.$$

We wish to prove that this induces an isomorphism

$$\mathcal{W}_{S_{F'}}^G \xrightarrow{\sim} \mathcal{W}_S,$$

and this amounts to a lemma on invariants of Weyl groups under folding, that surely must be well-known to the experts.

So suppose that we have a Dynkin diagram  $T$ , and a group  $G \leq \text{Aut}(T)$  of automorphisms of  $T$ . As before, we can see the root lattice  $\Lambda_{T/G}$  of  $T/G$  as the  $G$ -invariants  $\Lambda_T^G$  of the root lattice of  $T$ , and exactly as above we may construct a homomorphism

$$\mathcal{W}_{T/G} \rightarrow \mathcal{W}_T^G$$

from the Weyl group of the quotient diagram  $T/G$  to the  $G$ -invariants of the Weyl group of  $T$ .

**Lemma 3.2.** *This homomorphism  $\mathcal{W}_{T/G} \rightarrow \mathcal{W}_T^G$  is an isomorphism.*

PROOF. We split into two cases, depending on whether each orbit of  $G$  on the simple roots in  $\Lambda_T$  only contains pairwise orthogonal roots or not.

If the simple roots in each orbit are pairwise orthogonal, then we use an argument we learned from [Ste08]. Indeed, it is easy to verify that if  $w \in \mathcal{W}_{T/G}$  maps to  $\bar{w} \in \mathcal{W}_T$ , then  $w$  and  $\bar{w}$  agree as endomorphisms of  $\Lambda_{T/G} = \Lambda_T^G \subset \Lambda_T$ , thus the given map is injective. To see surjectivity, that is, that every  $w \in \mathcal{W}_T^G$  is in fact an element of  $\mathcal{W}_{T/G}$  we induct on the *length* of  $w$  in the sense of [Hum90, §1.6]. Indeed, if  $\ell(w) = 0$ , then there is nothing

to prove, and if  $\ell(w) > 0$  then we can find a basic reflection  $s_\alpha$  in some simple root  $\alpha \in \Lambda_T$  such that  $\ell(ws_\alpha) < \ell(w)$ . Thus  $\ell(ws_\alpha) = \ell(w) - 1$  and  $w(\alpha) < 0$  by [Hum90, Lemma 1.6, Corollary 1.7].

Let  $O(\alpha)$  be the orbit of  $\alpha$  under the  $G$ -action, and  $s := \prod_{\beta \in O(\alpha)} s_\beta$ . Since any automorphism of  $T$  is length preserving we see that we also have  $\ell(ws_\beta) = \ell(w) - 1$  and  $w(\beta) < 0$  for any  $\beta \in O(\alpha)$ . Now let  $w' = ws$ , we claim that  $\ell(w') < \ell(w's_\beta)$  for all  $\beta \in O(\alpha)$ . Indeed, since the  $\beta$  are pairwise orthogonal, we find that  $w'(\beta) = w(-\beta) > 0$  and thus again that  $\ell(w') = \ell(w's_\beta) - 1$ .

Therefore  $w'$  is a minimal coset representative for the parabolic subgroup generated by the  $\{s_\beta \mid \beta \in O(\alpha)\}$  in the sense of [Hum90, §1.10]. Since  $w = w's$  we can therefore deduce that  $\ell(w) = \ell(w') + \ell(s)$ , and again since the  $\beta$  are pairwise orthogonal that  $\ell(s) = |O(\alpha)|$ . Thus,  $\ell(w') < \ell(w)$ , and since  $s$  is clearly  $G$ -invariant, so is  $w'$ . We may therefore apply the induction hypothesis to conclude.

In the second case, the only possibility is that  $T = A_{2n}$ ,  $G = \mathbb{Z}/2\mathbb{Z}$  and  $T/G = C_n$ . If we label the vertices of  $A_{2n}$  in the obvious way, then we have  $\mathcal{W}_T \cong S_{2n+1}$ , mapping the reflection  $s_{\alpha_i}$  in the  $i$ th basis element to the transposition  $(i, i+1)$ , see [Hum90, §2.10]. One can verify exactly as above that the map

$$\mathcal{W}_{T/G} \rightarrow \mathcal{W}_T^G,$$

is injective, hence it suffices to calculate the orders of both groups. The order of the former is  $2^n n!$  by [Hum90, §2.10]. Moreover, the action of  $\mathbb{Z}/2\mathbb{Z}$  is to send  $s_{\alpha_i}$  to  $s_{\alpha_{2n+1-i}}$ , which, via the isomorphism  $\mathcal{W}_T \cong S_{2n+1}$ , corresponds to conjugation by the element

$$\sigma := (1, 2n+1)(2, 2n)(3, 2n-1) \dots (n, n+2).$$

The order of  $\mathcal{W}_T^G$  is therefore  $(2n+1)!$  divided by the size of the conjugacy class of  $\sigma$ . Since  $\sigma$  is a product of  $n$  disjoint 2-cycles in  $S_{2n+1}$ , we can calculate the size of its conjugacy class as

$$\frac{1}{n!} \prod_{i=0}^{n-1} \binom{2n-2i+1}{2} = \frac{(2n+1)!}{2^n n!},$$

which gives the required order of  $\mathcal{W}_T^G$ .  $\square$

**Corollary 3.3.** *If  $S$  is a surface over  $F$  that has canonical singularities and  $F'/F$  is a finite Galois extension with group  $G$ , then we have an isomorphism*

$$\mathcal{W}_{S_{F'}}^G \xrightarrow{\sim} \mathcal{W}_S.$$

**Remark 3.4.** It is important to note that the extensions of elements of  $\mathcal{W}_S$  to  $\mathcal{W}_{S_{F'}}$  are *not* generally defined by the naive extension of the formula for the basic reflections. As we saw, if  $E_{P,i}$  is an integral component of the exceptional locus of  $\tilde{S} \rightarrow S$  such that  $E_{P,i}$  breaks into two disconnected components  $E_{Q_1,i}$  and  $E_{Q_2,i}$  over  $F'$ , then the extension of  $s_{P,i}$  to  $\text{Aut}_{\mathbb{Z}}(\Lambda_{S_{F'}})$

provided by Corollary 3.3 is given by  $s_{Q_1,i}s_{Q_2,i}$ . This only agrees with the “naive” extension

$$D \mapsto D + \frac{1}{n_{P,i}}(D \cdot E_{P,i}) \cdot E_{P,i}$$

on the subspace of  $\Lambda_{S_{F'}}$  fixed by  $G$ , that is,  $\Lambda_S$ . More generally, these two formulae agree on the subspace  $\text{Pic}(\tilde{S}) \subseteq \text{Pic}(\tilde{S}_{F'})$ .

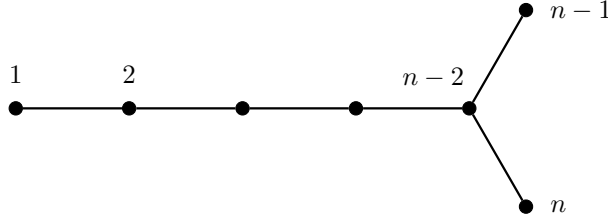
We will also need the following rather ugly lemma.

**Lemma 3.5.** *Let  $A$  be a (negative definite) Cartan matrix of ADE type, and  $C$  the permutation matrix coming from an automorphism of the associated Dynkin diagram. If  $C \neq I$ , then the matrix  $(C - I)A^{-1}$  does not have all integral entries.*

PROOF. We proceed on a case by case basis, in cases  $E_7$  and  $E_8$  there is nothing to prove. For  $A_n$ , we order the vertices in the obvious way. Then, the only non-trivial possibility for  $C$  is the matrix with 1’s along the anti-diagonal. Therefore, the top row of  $(C - I)$  is  $(-1, 0, 0, \dots, 0, 1)$  and hence the top left hand entry is

$$((C - I)A^{-1})_{1,1} = (A^{-1})_{n,1} - (A^{-1})_{1,1}.$$

The inverses of the Cartan matrices are computed in [Ros97, p.95], and hence, by our conventions, we have  $(A^{-1})_{n,1} = -1/(n + 1)$  and  $(A^{-1})_{1,1} = -n/(n + 1)$ . So  $((C - I)A^{-1})_{1,1}$  is not integral (unless  $n = 1$ ). Similarly, in the  $D_n$  case,  $n \geq 5$ , we order the vertices as follows.



Then the only non-trivial possibility for  $C$  is the matrix

$$\begin{pmatrix} I_{n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

and hence,

$$((C - I)A^{-1})_{n,n} = (A^{-1})_{n-1,n} - (A^{-1})_{n,n}.$$

We again appeal to [Ros97, p.95] (remembering the change in sign convention) to deduce that the top left entry  $((C - I)A^{-1})_{n,n} = 1/2$  is not integral.

Again, in the  $D_4$  and  $E_6$  cases the claim can be verified by just computing all possibilities for  $C$ , once more using the calculation of the inverses of Cartan matrices in [Ros97, p.95].  $\square$

**Remark 3.6.** We will refer to the ordering of the vertices of  $D_n$  described during the proof as the “usual” ordering.

#### 4. SOME CALCULATIONS IN GROUP COHOMOLOGY

The proof of our Néron–Ogg–Shafarevich criterion for K3 surfaces (Theorem 1.6) will essentially depend on showing that a certain map in non-abelian cohomology has trivial kernel. This can be reduced to a completely group theoretic calculation involving the Weyl groups of Dynkin diagrams, and the purpose of this section is to perform this calculation.

So let  $\mathbf{T} = (T_1, \dots, T_d)$  be a finite collection of Dynkin diagrams (that is, each  $T_i$  is finite, reduced and irreducible), with corresponding root lattice  $\Lambda_{\mathbf{T}} = \bigoplus_{i=1}^d \Lambda_{T_i}$  and Weyl group  $\mathcal{W}_{\mathbf{T}} = \prod_{i=1}^d \mathcal{W}_{T_i}$ . Let  $G$  be a finite group acting on  $\mathbf{T}$ , and let  $F$  be a field of characteristic 0.

**Theorem 4.1.** *The induced map*

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}, F}))$$

*in non-abelian cohomology has trivial kernel.*

**Remark 4.2.** Since the  $G$ -modules under consideration are non-abelian,  $H^1$  is just a pointed set, not a group. But it still makes sense to speak of the kernel of a map of pointed sets  $(A, *) \rightarrow (B, *)$ , that is, the fiber over the distinguished point of  $B$ . Since this kernel always contains the distinguished point of  $A$ , triviality then means that it *only* contains this point. Note that this is a weaker claim than the map being injective, and in fact in the situation of Theorem 4.1, the map

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}, F}))$$

will *not* be injective in general.

We start with a special case, that turns out to be the hardest part of the proof. The general case will then follow via an induction argument.

**Proposition 4.3.** *In the situation of Theorem 4.1 assume that  $G$  is cyclic,  $\mathbf{T} = T$  is a single Dynkin diagram, and that the action is faithful, that is,  $G \rightarrow \mathrm{Aut}(T)$  is injective. Let  $\Lambda$  be the root lattice of  $T$  and  $\mathcal{W}$  the Weyl group. Then, the induced map*

$$H^1(G, \mathcal{W}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_F))$$

*has trivial kernel.*

**PROOF.** We note that if  $G$  is non-trivial, then  $T$  is simply-laced, that is, it is of type ADE.

First, assume that  $G = \mathbb{Z}/2\mathbb{Z}$  and  $T$  is of type  $A_n$  with  $n \geq 2$ , of type  $D_n$  with  $n \geq 5$  odd, or of type  $E_6$ . Let  $g$  denote the unique non-trivial element of  $G = \text{Aut}(T)$ . Applying [KM83, Theorem 1] we know that  $-I \in \text{O}(\Lambda) = \mathcal{W} \rtimes G$ , and since  $-I \notin \mathcal{W}$  we can conclude that  $g = -w_0$  for some  $w_0 \in \mathcal{W}$ .

In particular, the action of  $g \in \mathbb{Z}/2\mathbb{Z}$  on both  $\mathcal{W}$  and  $\text{GL}(\Lambda_F)$  is via conjugation by  $w_0$ . We therefore have two bijections

$$\begin{aligned} H^1(\mathbb{Z}/2\mathbb{Z}, \mathcal{W}) &\rightarrow \mathcal{W}[2]/\sim, & \alpha &\mapsto \alpha(g)w_0 \\ H^1(\mathbb{Z}/2\mathbb{Z}, \text{GL}(\Lambda_F)) &\rightarrow \text{GL}(\Lambda_F)[2]/\sim, & \alpha &\mapsto \alpha(g)w_0 \end{aligned}$$

between the relevant cohomology sets and the sets of conjugacy classes of elements of order 2. We thus need to prove that if  $w \in \mathcal{W}[2]$  is such that  $w$  and  $w_0$  are conjugate inside  $\text{GL}(\Lambda_{F_p})$ , then they are already conjugate inside  $\mathcal{W}$ .

The conjugacy classes of involutions in Weyl groups are classified in [Ric82, §4, Theorem A]: they correspond to  $\mathcal{W}$ -orbits of sub-diagrams  $T' \subseteq T$  (where  $T'$  is no longer assumed to be irreducible) that contain  $-I$  in their Weyl group. Moreover, the size of  $T'$  determines the conjugacy class inside  $\text{GL}(\Lambda_F)$ , since it determines the multiplicity of  $-1$  as an eigenvalue. Hence, it suffices to prove that there is a unique orbit of maximal size, and that  $w_0$  is an element of the corresponding conjugacy class.

For  $A_n$ , a maximal such sub-diagram  $T'$  is given by starting at one end and choosing alternating vertices. There is a unique such  $T'$  if  $n$  is odd, and two possible choices when  $n$  is even. Since  $w_0 = -g$ , we see that these two choices are conjugate under  $\mathcal{W}$ . For  $D_n$ , with  $n \geq 5$  odd, a maximal such diagram is provided by  $D_{n-1} \subset D_n$ , and for  $E_6$  by  $D_4 \subset E_6$ . In both cases, there is a unique choice for  $T'$ . To conclude, then, we can simply calculate the eigenvalues of  $w_0 = -g$  and observe that we get the maximal possible multiplicity of  $-1$  in all cases.

The next case we consider is  $\mathbb{Z}/2\mathbb{Z}$  acting on  $D_n$  for  $n \geq 4$  even (although we don't really need to assume  $n$  even in the proof). In this case by choosing a suitable basis for both  $\Lambda_{D_n, F}$  and  $\Lambda_{B_n, F}$  we can use the description of Weyl groups given in [Hum90, §2.10] to construct a commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n & \xrightarrow{\cong} & \mathcal{W}_{D_n} & \longrightarrow & \text{GL}_n(F) \\ \downarrow & & \downarrow & & \parallel \\ (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n & \xrightarrow{\cong} & \mathcal{W}_{B_n} & \longrightarrow & \text{GL}_n(F) \end{array}$$

such that the non-trivial element of  $\mathbb{Z}/2\mathbb{Z}$  acts on both  $\mathcal{W}_{D_n}$  and  $\text{GL}_n(F)$  as conjugation by the element  $w_0 \in \mathcal{W}_{B_n}$  corresponding to the element  $\delta_n = (0, 0, \dots, 0, 1) \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ . Note that the inclusion

$$(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$$

on the first factor is as the subspace of elements summing to zero. We first claim that the induced map

$$H^1(\mathbb{Z}/2\mathbb{Z}, \mathcal{W}_{D_n}) \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \mathcal{W}_{B_n})$$

has trivial kernel. As above, this boils down to showing that if we take an element  $x \in (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$  such that  $x\delta_n$  and  $\delta_n$  are conjugate inside  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ , then they are conjugate *by an element of*  $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ . This follows from the formula

$$(\delta, \sigma)^{-1}(\delta_n, e)(\delta, \sigma) = (\sigma^{-1}(\delta_n), e)$$

for any  $(\delta, \sigma) \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ , since the RHS does not depend on  $\delta$ .

Given this claim, it therefore suffices to show that

$$H^1(\mathbb{Z}/2\mathbb{Z}, \mathcal{W}_{B_n}) \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{GL}(\Lambda_{B_n, F}))$$

also has trivial kernel. Since the action is conjugation by  $w_0$ , as before this amounts to showing that if  $w \in \mathcal{W}_{B_n}$  is an element of order 2 that is conjugate to  $w_0$  inside  $\mathrm{GL}(\Lambda_{B_n, F})$ , then it is already conjugate to  $w_0$  inside  $\mathcal{W}_{B_n}$ . Again, since the element  $w_0$  has  $-1$  as an eigenvalue with multiplicity one and all other eigenvalues  $+1$ , it suffices to observe by [Ric82, §4, Theorem A'] that all sub-diagrams of  $B_n$  of size 1 are conjugate under  $\mathcal{W}_{B_n}$  (see for example the discussion in §4.1 of *loc. cit.*).

Finally, if we look at the action of  $\mathbb{Z}/3\mathbb{Z}$  on  $D_4$ , then it seems more difficult to give such a straightforward description of the action. Instead we will calculate the kernel of

$$H^1(\mathbb{Z}/3\mathbb{Z}, \mathcal{W}) \rightarrow H^1(\mathbb{Z}/3\mathbb{Z}, \mathrm{GL}(\Lambda_F))$$

by brute force as follows: we order the nodes of  $D_4$  in the “usual” way (Remark 3.6) and denote the reflection in the  $i$ th node by  $s_i$ . Let  $\alpha_i$  be the corresponding basis elements for  $\Lambda$ . Then, a generator  $g$  for  $\mathbb{Z}/3\mathbb{Z}$  acts via

$$\begin{aligned} \alpha_1 &\mapsto \alpha_3, & \alpha_2 &\mapsto \alpha_2 \\ \alpha_3 &\mapsto \alpha_4, & \alpha_4 &\mapsto \alpha_1. \end{aligned}$$

A 1-cocycle  $\mathbb{Z}/3\mathbb{Z} \rightarrow \mathcal{W}$  corresponds to an element  $w \in \mathcal{W}$  such that  $(wg)^3 = \mathrm{id}$  and two such elements  $w, w'$  represent the same cohomology class if and only if  $(wg)$  and  $(w'g)$  are conjugate by an element of  $\mathcal{W}$ . Moreover, an element  $w$  represents a class in the kernel of

$$H^1(G, \mathcal{W}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_F))$$

if and only if  $wg$  and  $g$  are conjugate by an element of  $\mathrm{GL}(\Lambda_F)$ . The [SAGE] code reproduced in Appendix A calculates an upper bound for the number of elements in this kernel. It does so by first listing all elements of  $\mathcal{W}$  in matrix form. It then calculates which elements  $w$  give rise to cocycles by calculating  $(wg)^3$ . It then describes a full list of representatives of  $H^1(\mathbb{Z}/3\mathbb{Z}, \mathcal{W})$  by simply calculating  $r^{-1}(wg)r$  for every  $r \in \mathcal{W}$  and every  $w$  such that  $(wg)^3 = \mathrm{id}$ .



It turns out that  $H^1(\mathbb{Z}/3\mathbb{Z}, \mathcal{W})$  has exactly 2 elements, with the non-trivial element being represented by taking

$$w = \begin{pmatrix} -1 & & & \\ -1 & 1 & -1 & \\ -1 & 1 & & -1 \\ -1 & & & 1 \end{pmatrix}$$

in matrix form. It therefore suffices to check that in this case  $wg$  and  $g$  are not conjugate, which we can do by observing that  $g$  has 1 as an eigenvalue, but  $wg$  does not, as they are all non-trivial cube roots of unity. We therefore find that

$$H^1(\mathbb{Z}/3\mathbb{Z}, \mathcal{W}) \rightarrow H^1(\mathbb{Z}/3\mathbb{Z}, \mathrm{GL}(\Lambda_F))$$

has trivial kernel as claimed.  $\square$

An important part of the induction step will be inflation-restriction in non-abelian cohomology. If we let  $N \triangleleft G$  be a normal subgroup of  $G$ , then we can consider the quotient  $\mathbf{T}/N$ , which is again a finite collection of finite Dynkin diagrams, on which  $G/N$  acts. Moreover, we have, by Lemma 3.2, isomorphisms

$$\Lambda_{\mathbf{T}/N} \cong \Lambda_{\mathbf{T}}^N, \quad \mathcal{W}_{\mathbf{T}/N} \cong \mathcal{W}_{\mathbf{T}}^N,$$

as well as a natural map

$$\mathrm{GL}(\Lambda_{\mathbf{T},F})^N \rightarrow \mathrm{GL}(\Lambda_{\mathbf{T}/N,F}^N)$$

which is  $G/N$ -equivariant.

**Lemma 4.4.** *If both maps*

$$\begin{aligned} H^1(N, \mathcal{W}_{\mathbf{T}}) &\rightarrow H^1(N, \mathrm{GL}(\Lambda_{\mathbf{T},F})) \\ H^1(G/N, \mathcal{W}_{\mathbf{T}/N}) &\rightarrow H^1(G/N, \mathrm{GL}(\Lambda_{\mathbf{T}/N,F}^N)) \end{aligned}$$

*have trivial kernel, then so does*

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T},F})).$$

PROOF. By inflation-restriction [Ser94, Ch. I, §5.8 (a)], we have a commutative diagram of pointed sets

$$\begin{array}{ccccc} H^1(G/N, \mathcal{W}_{\mathbf{T}/N}) & \longrightarrow & H^1(G, \mathcal{W}_{\mathbf{T}}) & \longrightarrow & H^1(N, \mathcal{W}_{\mathbf{T}}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(G/N, \mathrm{GL}(\Lambda_{\mathbf{T},F})^N) & \longrightarrow & H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T},F})) & \longrightarrow & H^1(N, \mathrm{GL}(\Lambda_{\mathbf{T},F})) \\ \downarrow & & & & \\ H^1(G/N, \mathrm{GL}(\Lambda_{\mathbf{T}/N,F}^N)) & & & & \end{array}$$

with exact rows and injective left hand horizontal maps. The claim now follows from a simple diagram chase.  $\square$

The other key component of the induction step is the following. Suppose  $G$  acts on  $\mathbf{T} = (T_1 \dots, T_d)$  in such a way that the action on the irreducible

components  $T_i$  is transitive. Let  $H_i \leq G$  be the stabiliser of some  $T_i$ , thus  $H_i$  acts on  $T_i$  and therefore on  $\mathcal{W}_{T_i}$ .

**Lemma 4.5.** *There is an isomorphism*

$$\mathcal{W}_{\mathbf{T}} \cong \text{Ind}_{H_i}^G \mathcal{W}_{T_i}$$

of  $G$ -modules.

PROOF. Let  $\mathbf{T}' = (T_1, \dots, \widehat{T_i}, \dots, T_d)$ , so that we have  $\mathcal{W}_{\mathbf{T}} \cong \mathcal{W}_{T_i} \times \mathcal{W}_{\mathbf{T}'}$  as  $H_i$ -modules. For all  $1 \leq j \leq d$  let  $p_j : \mathcal{W}_{\mathbf{T}} \rightarrow \mathcal{W}_{T_j}$  denote the  $j$ th projection. Then, there is a natural map

$$\mathcal{W}_{\mathbf{T}} \rightarrow \text{Ind}_{H_i}^G \mathcal{W}_{T_i}$$

of  $G$ -modules defined by

$$w \mapsto (f_w(x) := p_i(x(w))).$$

These two  $G$ -modules have the same order, it therefore suffices to show that the map is injective. Since the action of  $G$  on the irreducible components is transitive,  $f_w(x) = 1$  for all  $x$  implies that  $p_j(w) = 1$  for all  $j$  and thus  $w = 1$ .  $\square$

We can now complete the proof of Theorem 4.1.

PROOF OF THEOREM 4.1. We induct on the order of  $G$ , if  $|G| = 1$  then there is nothing to prove. For the induction step we break up  $\mathbf{T} = (\mathbf{T}_1, \dots, \mathbf{T}_r)$  into orbits for the action of  $G$  on the irreducible components  $T_i$ . Then, we have corresponding decompositions

$$\mathcal{W}_{\mathbf{T}} \cong \prod_{j=1}^r \mathcal{W}_{\mathbf{T}_j} \quad \text{and} \quad \Lambda_{\mathbf{T}} \cong \bigoplus_{j=1}^r \Lambda_{\mathbf{T}_j}$$

and thus,

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \cong \prod_{j=1}^r H^1(G, \mathcal{W}_{\mathbf{T}_j}).$$

It therefore suffices to show that each

$$H^1(G, \mathcal{W}_{\mathbf{T}_j}) \rightarrow H^1(G, \text{GL}(\Lambda_{\mathbf{T},F}))$$

has trivial kernel (note that at this point we really mean  $\mathbf{T}$  on the RHS, not  $\mathbf{T}_j$ ). The decomposition

$$\Lambda_{\mathbf{T},F} \cong \bigoplus_j \Lambda_{\mathbf{T}_j,F}$$

is invariant under both  $G$  and  $\mathcal{W}_{\mathbf{T}_j}$ , and moreover, any element of  $\mathcal{W}_{\mathbf{T}_j}$  acts trivially on  $\bigoplus_{j' \neq j} \Lambda_{\mathbf{T}_{j'},F}$ . By general theory,  $\alpha$  mapping to the trivial class in  $H^1(G, \text{GL}(\Lambda_{\mathbf{T},F}))$  is equivalent to the existence of an isomorphism

$$\bigoplus_j \Lambda_{\mathbf{T}_j,F} \cong \bigoplus_j \Lambda_{\mathbf{T}_j,F}^\alpha$$

of  $G$ -representations, where on the RHS we have “twisted” the  $G$ -action by  $\alpha$ . Similarly,  $\alpha$  mapping to the trivial class in  $H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}_j, F}))$  is equivalent to the existence of an isomorphism

$$\Lambda_{\mathbf{T}_j, F} \cong \Lambda_{\mathbf{T}_j, F}^\alpha.$$

Since  $\mathrm{char}(F) = 0$ , the category of  $G$ -representations in finite dimensional  $F$ -vector spaces is semi-simple, and as

$$\Lambda_{\mathbf{T}_{j'}, F} \cong \Lambda_{\mathbf{T}_{j'}, F}^\alpha$$

for  $j \neq j'$ , we see that

$$\Lambda_{\mathbf{T}_j, F} \cong \Lambda_{\mathbf{T}_j, F}^\alpha$$

if and only if

$$\bigoplus_j \Lambda_{\mathbf{T}_j, F} \cong \bigoplus_j \Lambda_{\mathbf{T}_j, F}^\alpha.$$

From this we deduce that  $\alpha$  is trivial in  $H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}, F}))$  if and only if it is so in  $H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}_j, F}))$ . Thus, it suffices to show that each map

$$H^1(G, \mathcal{W}_{\mathbf{T}_j}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}_j, F}))$$

has trivial kernel. Hence, we can assume that the action on the irreducible components of  $\mathbf{T} = (T_1, \dots, T_d)$  is transitive.

If  $d > 1$ , then we use Lemma 4.5. The stabilizer  $H_i \leq G$  of some  $T_i$  is a *proper* subgroup of  $G$ , and by [Ser94, Ch. I, §5.8 (b)] we know that

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \xrightarrow{\sim} H^1(H_i, \mathcal{W}_{T_i}).$$

In particular, if we let  $\mathbf{T}' = (T_1, \dots, \widehat{T_i}, \dots, T_d)$ , we see that the pull-back map

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \rightarrow H^1(H_i, \mathcal{W}_{\mathbf{T}}) \xrightarrow{\sim} H^1(H_i, \mathcal{W}_{T_i}) \times H^1(H_i, \mathcal{W}_{\mathbf{T}'})$$

has trivial kernel. By the induction hypothesis we know that

$$H^1(H_i, \mathcal{W}_{\mathbf{T}}) \rightarrow H^1(H_i, \mathrm{GL}(\Lambda_{\mathbf{T}, F}))$$

has trivial kernel, thus so does

$$H^1(G, \mathcal{W}_{\mathbf{T}}) \rightarrow H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}, F}))$$

by a simple diagram chase.

If  $d = 1$ , then  $\mathbf{T}$  has a unique irreducible component. By inspecting the kernel of the  $G$ -action and using Lemma 4.4, we may consider separately the cases when the  $G$ -action is faithful and when it is trivial. If the  $G$ -action is trivial, we can identify  $H^1(G, \mathcal{W}_{\mathbf{T}})$  with the set of conjugacy classes of homomorphisms  $G \rightarrow \mathcal{W}_{\mathbf{T}}$ , and  $H^1(G, \mathrm{GL}(\Lambda_{\mathbf{T}, F}))$  with the set of conjugacy classes of homomorphisms  $G \rightarrow \mathrm{GL}(\Lambda_{\mathbf{T}, F})$ . But since  $\mathcal{W}_{\mathbf{T}} \rightarrow \mathrm{GL}(\Lambda_{\mathbf{T}, F})$  is injective, we may therefore conclude by observing that the only homomorphism  $G \rightarrow \mathrm{GL}(\Lambda_{\mathbf{T}, F})$  conjugate to the trivial homomorphism is the trivial homomorphism itself.

If the  $G$ -action is faithful, then  $G$  is solvable, as automorphism groups of Dynkin diagrams are solvable. Once more applying Lemma 4.4 we can

reduce either to Proposition 4.3, or to the already handled case when the  $G$ -action is trivial.  $\square$

## 5. CRYSTALLINE GALOIS REPRESENTATIONS AND K3 SURFACES

In this section, we establish a  $p$ -adic criterion for potential good reduction of K3 surfaces (Theorem 1.1), as well as the counter-examples (Remark 1.3) from the introduction.

### 5.1. A $p$ -adic criterion for potential good reduction of K3 surfaces.

We start with the proof of Theorem 1.1.

PROOF. The equivalences (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) have already been established in [LM14, Theorem 1.3 and Corollary 1.4]. For every prime  $\ell$  (including  $\ell = p$ ), we set  $V_\ell := H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$  and let  $\rho_\ell : G_K \rightarrow \text{GL}(V_\ell)$  be the  $\ell$ -adic Galois representation arising from geometry. The  $\mathbb{Q}_\ell$ -dimension of  $V_\ell$  is independent of  $\ell$  and equal to the second Betti number  $b_2 := b_2(X)$ .

Next, assume that  $X$  has good reduction after some finite and unramified extension  $K \subseteq K'$ . By Theorem 2.4, the restriction of  $\rho_p$  to  $G_{K'}$  is a crystalline  $G_{K'}$ -representation. But then,  $\rho_p$  is a crystalline  $G_K$ -representation by Lemma 2.6 (1). This establishes (1)  $\Rightarrow$  (4).

Finally, assume that  $\rho_p$  is crystalline. Since the conditions (3) and (4) are equivalent over  $K$  and  $\widehat{K}$ , we may assume that  $K$  is complete. By Lemma 2.6 (3), the induced  $I_K$ -action on  $\mathbb{D}_{\text{pst}}(V_p)$  is trivial. In particular, for every  $g \in I_K$ , the trace of the  $K_0^{\text{nr}}$ -linear map  $g^*$  on  $\mathbb{D}_{\text{pst}}(V_p)$  is equal to  $b_2$ . But then, also for every prime  $\ell \neq p$ , the trace of the induced  $\mathbb{Q}_\ell$ -linear map  $g^*$  on  $V_\ell$  is equal to  $b_2$  by [Och99, Corollary 2.5]. Since  $\rho_p$  is crystalline, it follows from [Mat15, Theorem 1.1] that there exists a finite and possibly ramified extension  $L/K$  such that  $X_L$  has good reduction over  $\mathcal{O}_L$ . Thus, for every  $\ell \neq p$  the image of  $I_K$  in  $\text{GL}(V_\ell)$  is finite. Since  $\mathbb{Q}_\ell$  is of characteristic zero, a linear automorphism of finite order on  $V_\ell$  is trivial if and only if its trace is equal to  $b_2$ . This proves that  $I_K$  acts trivially on  $V_\ell$  for all  $\ell \neq p$  and establishes (4)  $\Rightarrow$  (3).  $\square$

**Remark 5.1.** Let us comment on Theorem 1.1 and sketch some alternative strategies to prove (parts of it):

- (1) It seems plausible that one can prove the equivalence (1)  $\Leftrightarrow$  (4) by adapting the  $\ell$ -adic arguments in the proof of [LM14, Theorem 6.1] to the  $p$ -adic situation.
- (2) If  $A$  is an Abelian variety over  $K$ , then the analogous equivalences (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) for  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$  follow from work of Serre and Tate [ST68] in case  $\ell \neq p$  and from work of Fontaine [Fon79], Mokrane [Mok93], and Coleman and Iovita [CI99] for  $\ell = p$ .

It seems plausible that one might be able to use this result together with the Kuga–Satake correspondence [KS67, And96, Riz10] to give another proof of the equivalences (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) for K3 surfaces.

- (3) Quite generally, the equivalences (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) are expected to hold for every smooth and proper variety over  $K$ .

**5.2. Counter-examples.** Next, as already noted in Remark 1.3, we show that the unramified extension stated in Theorem 1.1.(1) may be *non-trivial*. More precisely, the examples from [LM14, Section 7] provide the desired counter-examples.

**Theorem 5.2.** *For every prime  $p \geq 5$ , there exists a K3 surface  $X(p)$  over  $\mathbb{Q}_p$  such that*

- (1) *the  $G_{\mathbb{Q}_p}$ -representation on  $H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$  is unramified for all  $\ell \neq p$ ,*
- (2) *the  $G_{\mathbb{Q}_p}$ -representation on  $H_{\text{ét}}^2(X(p)_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$  is crystalline,*
- (3)  *$X(p)$  has good reduction over the unramified extension  $\mathbb{Q}_{p^2}$ , but*
- (4)  *$X(p)$  does not have good reduction over  $\mathbb{Q}_p$ .*

PROOF. Let  $X(p)$  be one of the examples of [LM14, Example 7.1]. By [LM14, Theorem 7.2], it satisfies claims (1), (3) and (4). Since it has good reduction over  $\mathbb{Q}_{p^2}$ , the  $G_{\mathbb{Q}_{p^2}}$ -representation on  $V_p := H_{\text{ét}}^2(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$  is crystalline. By Lemma 2.6 (1), also the  $G_{\mathbb{Q}_p}$ -representation on  $V_p$  is crystalline, which establishes claim (2) for  $X(p)$ .  $\square$

**5.3. Enriques surfaces.** By definition, an *Enriques surface* over a field  $F$  is a smooth and proper surface  $S$  over  $F$  with  $\omega_S \equiv \mathcal{O}_S$  and  $b_2(S) = 10$ , where  $\equiv$  denotes numerical equivalence. If  $\text{char}(F) \neq 2$ , then there exists a canonical and geometrically connected étale cover  $\tilde{S} \rightarrow S$  of degree 2. In fact,  $\tilde{S}$  is a K3 surface over  $F$  and it is called the *K3 double cover* of  $S$ .

The following result is analogous to [LM14, Lemma 3.4], which is why we leave its proof to the reader.

**Lemma 5.3.** *Assume that  $p > 0$ , and let  $Y$  be an Enriques surface over  $K$ . Then, there exists a finite extension  $L/K$  such that the  $G_L$ -representations on  $H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$  are crystalline for all  $n$ .*

Moreover, combining Theorem 1.1 and [LM14, Theorem 3.6], we obtain the following result.

**Theorem 5.4.** *For every prime  $p \geq 5$ , there exists an Enriques surface  $Y(p)$  over  $\mathbb{Q}_p$ , such that*

- (1) *the K3 double cover  $X(p)$  of  $Y(p)$  has good reduction,*
- (2) *the  $G_{\mathbb{Q}_p}$ -representation on  $H_{\text{ét}}^2(X(p)_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$  is crystalline,*
- (3)  *$Y(p)$  has semi-stable reduction of flower pot type, but*
- (4)  *$Y(p)$  does not have potential good reduction.*

In particular, there is no analog of Theorem 1.1 for Enriques surfaces, not even in terms of their K3 double covers.

## 6. CANONICAL REDUCTIONS AND RDP MODELS OF PAIRS

If  $X$  is a K3 surface over  $K$ , satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1, then we may appeal to [LM14, Theorem 1.3] to deduce the existence of “reasonably nice” models of  $X$  over  $\mathcal{O}_K$ .

**Theorem 6.1** (Liedtke, Matsumoto). *Let  $X$  be a K3 surface over  $K$  that satisfies  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, there exists a projective and flat model*

$$\mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}_K$$

for  $X$  such that:

- (1) the special fiber  $\mathcal{X}_k$  has at worst canonical singularities, and
- (2) its minimal resolution is a K3 surface over  $k$ .

While the “RDP model” provided by this theorem is not unique, it is possible to construct, once we fix a polarization  $\mathcal{L}$  on  $X$ , a completely canonical such model  $P(X, \mathcal{L})$  as follows: first of all, one takes a finite, unramified extension  $L/K$ , Galois with group  $G$ , such that  $X$  has good reduction over  $L$ . Let  $k_L$  denote the residue field of  $L$ . Then, for any smooth model  $\mathcal{Y} \rightarrow \operatorname{Spec} \mathcal{O}_L$  for  $X_L$  we can find a birational map  $\mathcal{Y} \dashrightarrow \mathcal{Y}^+$ , such that the specialization  $\mathcal{L}_{k_L}^+$  of  $\mathcal{L}_L$  on the special fiber  $\mathcal{Y}_{k_L}^+$  is big and nef, [LM14, Proposition 4.5]. The projective scheme

$$P(X_L, \mathcal{L}_L) := \operatorname{Proj} \left( \bigoplus_{n \geq 0} H^0(\mathcal{Y}^+, \mathcal{L}^{+, \otimes n}) \right)$$

only depends on  $X_L$  and  $\mathcal{L}_L$  up to a canonical isomorphism by the proof of [LM14, Proposition 4.7]. Moreover, the rational  $G$ -action on  $X_L$  extends to a regular  $G$ -action on  $P(X_L, \mathcal{L}_L)$  by [LM14, Proposition 5.1], and we define

$$P(X, \mathcal{L}) = P(X_L, \mathcal{L}_L)/G$$

to be the quotient. This is then a flat, projective  $\mathcal{O}_K$ -scheme whose special fibre  $P(X, \mathcal{L})_k$  has canonical singularities. The model  $P(X, \mathcal{L})$  depends only on  $X$  and  $\mathcal{L}$ , again up to a canonical isomorphism. Moreover, the minimal resolution of singularities

$$Y \rightarrow P(X, \mathcal{L})_k$$

of the special fibre is a K3 surface over  $k$  by [LM14, Proposition 4.6]. This K3 surface  $Y$  is unique up to canonical isomorphism and depends only on  $X$  by a theorem of Matsusaka and Mumford [MM64] and the theory of minimal models, see also [LM14, Proposition 4.7].

**Definition 6.2.** We call  $Y$  the *canonical reduction* of  $X$  and  $P(X, \mathcal{L})$  the *(canonical) RDP model* of the pair  $(X, \mathcal{L})$ .

We note that all of this is compatible with base change, in that we have

$$P(X_L, \mathcal{L}_L) \cong P(X, \mathcal{L}) \otimes_{\mathcal{O}_K} \mathcal{O}_L$$

for *any* finite extension  $L/K$ . This then implies that the canonical reduction of  $X_L$  is just the base change of the canonical reduction of  $X$ . Moreover, when  $X$  *does* have good reduction, then the canonical reduction  $Y$  is the special fibre of *any* smooth model of  $X$ , again by the results of [MM64].

## 7. THE WEYL GROUP OF A K3 SURFACE WITH POTENTIALLY GOOD REDUCTION

Let  $X/K$  be a K3 surface, satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1. Thus,  $X$  admits good reduction over a finite and unramified Galois extension  $L/K$ , that is, there exists a smooth and proper model  $\mathcal{Y} \rightarrow \mathcal{O}_L$  for  $X_L$ . If we let  $G$  denote the Galois group of  $L/K$ , then there is a natural semi-linear  $G$ -action on  $X_L$ , and hence a *rational*  $G$ -action on  $\mathcal{Y}$ . The following is an analogue of [LM14, Corollary 5.12] in the unramified case.

**Proposition 7.1.** *Assume that the rational and semi-linear  $G$ -action on  $\mathcal{Y}$  is in fact regular. Then, the quotient  $\mathcal{X} := \mathcal{Y}/G$  exists as a smooth and proper algebraic space over  $K$ , and we have a  $G$ -equivariant isomorphism  $\mathcal{Y} \cong \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ .*

PROOF. The  $G$ -action is equivalent to descent data for  $\mathcal{Y}$  via the finite étale cover  $\mathrm{Spec} \mathcal{O}_L \rightarrow \mathrm{Spec} \mathcal{O}_K$ , and for algebraic spaces all such descent data are effective. Moreover, smoothness and properness can both be checked after taking such a cover.  $\square$

In particular,  $X$  has good reduction over  $K$  itself if and only if such a  $\mathcal{Y}$  can be chosen for which the rational  $G$ -action is in fact regular. Thus, a natural question arises of whether or not we can give conditions under which this will happen. This will involve a detailed study of the action of “flops” of arithmetic 3-folds on the Picard group of the special fibre of  $\mathcal{Y}$ , similar to that carried out in [LM14] for  $\ell$ -adic cohomology. We will then use this to provide a cohomological obstruction for the regularity of the  $G$ -action (Proposition 7.12 below).

**7.1. Flops and the Weyl group.** We start by examining the action of flops on smooth models of a fixed, polarized K3 surface  $(X, \mathcal{L})$  over  $K$ , again assuming  $(\star)$  and the conditions of Theorem 1.1. In this case, we have the canonical RDP model  $P(X, \mathcal{L})$  over  $\mathcal{O}_K$ , as well as the canonical reduction  $Y$ , which is the minimal resolution of singularities

$$Y \rightarrow P(X, \mathcal{L})_k$$

of the special fiber of  $P(X, \mathcal{L})$ . We will denote by

$$\begin{aligned} E_{X, \mathcal{L}} &\subset Y \\ \Lambda_{X, \mathcal{L}} &\subset \mathrm{Pic}(Y) \\ \mathcal{W}_{X, \mathcal{L}} &\leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y)) \end{aligned}$$

respectively the exceptional locus of  $Y \rightarrow P(X, \mathcal{L})_k$ , and the associated root lattice and Weyl group as in Section 3. We will call  $\Lambda_{X, \mathcal{L}}$  and  $\mathcal{W}_{X, \mathcal{L}}$  the *root lattice* and the *Weyl group* of the pair  $(X, \mathcal{L})$ .

**Definition 7.2.** We say that a smooth model  $\mathcal{X}$  of  $X$  is  $\mathcal{L}$ -terminal, or sometimes that  $(\mathcal{X}, \mathcal{L})$  is terminal, or even that  $\mathcal{X}$  is a terminal model of  $(X, \mathcal{L})$ , if the specialization  $\mathcal{L}_k$  on the special fibre  $\mathcal{X}_k$  is big and nef.

If we have a birational map  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  of smooth models of K3 surfaces, with  $(\mathcal{X}, \mathcal{L})$  terminal, then we can consider the push-forward  $\mathcal{L}^+$  of  $\mathcal{L}$  on the generic fiber of  $\mathcal{X}^+$ .

**Definition 7.3.** We say that  $f$  is a *terminal* birational map if  $(\mathcal{X}^+, \mathcal{L}^+)$  is also terminal.

If  $X$  has good reduction over  $K$ , that is, if *some* smooth model  $\mathcal{X}$  exists, then an  $\mathcal{L}$ -terminal model exists by [LM14, Proposition 4.5]. We note that if  $\mathcal{X}$  is an  $\mathcal{L}$ -terminal model for  $X$ , then by [LM14, Proposition 4.6(2)] we can identify  $E_{X, \mathcal{L}}$  with exactly those curves  $E \subset \mathcal{X}_k$  on the special fibre that satisfy  $\mathcal{L}_k \cdot E = 0$ .

Given a model  $\mathcal{X}$  of  $X$ , and a birational map  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  to some other smooth model of a K3 surface  $X^+/K$ , we may consider the graph of  $f$

$$\Gamma_f \subset \mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}^+,$$

as well as its special fibre  $\Gamma_{f,k} \subset Y \times_k \mathcal{X}_k^+$ . Note that the generic fiber  $\Gamma_{f_K}$  is simply the graph of the induced isomorphism  $f_K : X \xrightarrow{\sim} X^+$ , but the same is not necessarily true for  $\Gamma_{f,k}$ . This latter cycle induces a homomorphism

$$\begin{aligned} \tilde{s}_f : \text{Pic}(\mathcal{X}_k^+) &\rightarrow \text{Pic}(Y) \\ D &\mapsto p_{1*}(\Gamma_{f,k} \cap p_2^* D) \end{aligned}$$

on Picard groups, where  $p_i : Y \times_k \mathcal{X}_k^+ \rightrightarrows Y, \mathcal{X}_k^+$  are the two projections. Composing with the push-forward via the induced isomorphism  $f_k : Y \rightarrow \mathcal{X}_k^+$ , we obtain an endomorphism

$$s_f : \text{Pic}(Y) \rightarrow \text{Pic}(Y),$$

which can be described as the map induced by the pull-back cycle  $\Gamma \subset Y \times_k Y$  via the same formula.

**Proposition 7.4.** (1) *The map  $s_f$  preserves the intersection pairing, that is,*

$$s_f(D_1) \cdot s_f(D_2) = D_1 \cdot D_2$$

for  $D_i \in \text{Pic}(Y)$ .

(2) *If  $f, g$  are composable birational maps, then we have*

$$s_{g \circ f} = s_f \circ f_k^* \circ s_g \circ (f_k^*)^{-1}.$$

(3) *The map  $s_f$  is invertible, and we have*

$$(s_f)^{-1} = f_k^* \circ s_{f^{-1}} \circ (f_k^*)^{-1}.$$



PROOF. Since clearly  $s_{\text{id}} = \text{id}$ , we see that (2)  $\Rightarrow$  (3), and to prove (1) and (2) it suffices to work instead with  $\tilde{s}_f$ , that is, to show that  $\tilde{s}_f$  preserves the intersection pairing, and that  $\tilde{s}_{g \circ f} = \tilde{s}_f \circ \tilde{s}_g$ .

To prove this, we will anticipate somewhat a later argument and use cohomology to give another interpretation of  $\tilde{s}_f$ . Fix a prime  $\ell \neq p$ , and extend  $\tilde{s}_f$  to an endomorphism on  $\ell$ -adic cohomology

$$\begin{array}{ccc} \tilde{s}_{f,\ell} : H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell) & \rightarrow & H_{\text{ét}}^2(Y_k, \mathbb{Q}_\ell) \\ \alpha & \mapsto & p_{1*}([\Gamma_{f,k}] \cup p_2^* \alpha) \end{array}$$

using the same formula. It therefore suffices to show that  $\tilde{s}_{f,\ell}$  is compatible with composition and preserves the Poincaré pairing. This can be seen by applying [LM14, Lemma 5.6], which implies that there is a commutative diagram

$$\begin{array}{ccc} H_{\text{ét}}^2(X_{\overline{K}}^+, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell) \\ f_K^* \downarrow & & \downarrow \tilde{s}_{f,\ell} \\ H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H_{\text{ét}}^2(Y_k, \mathbb{Q}_\ell) \end{array}$$

where the horizontal arrows come from smooth and proper base change, and the left hand vertical arrow is simply pull-back along the map  $f_K : X \rightarrow X^+$  on generic fibers.  $\square$

The main result we want to prove is that when  $f$  is terminal,  $s_f$  is an element of the Weyl group

$$\mathcal{W}_{X,\mathcal{L}} \leq \text{Aut}_{\mathbb{Z}}(\text{Pic}(Y)).$$

First we will need to show that we can *construct* elements of the Weyl group via this method.

**Proposition 7.5.** *Let  $(\mathcal{X}, \mathcal{L})$  be terminal and assume that all irreducible components of  $E_{X,\mathcal{L}}$  are geometrically irreducible. Then, for any element  $w \in \mathcal{W}_{X,\mathcal{L}}$ , there exists a terminal birational map*

$$f : \mathcal{X} \dashrightarrow \mathcal{X}^+$$

such that  $s_f = w$  as automorphisms of  $\text{Pic}(Y)$ .

PROOF. By Proposition 7.4(2), it suffices to treat the case when  $w = s_E$  is the reflection in some irreducible component  $E \subset E_{X,\mathcal{L}}$ . In this case, we follow the proof of [LM14, Proposition 4.2].

First of all, there exists a proper and birational morphism  $\mathcal{X} \rightarrow \mathcal{X}'$  that contracts  $E$  and nothing else. Let  $y \in \mathcal{X}'$  denote the image of  $E$ ,  $\widehat{\mathcal{X}}'_y = \text{Spf}(R)$  the formal completion at  $y$ , and  $\widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}'_y$  the formal fiber. Again, following [LM14, Proposition 4.2], we can find an automorphism  $t : \widehat{\mathcal{X}}'_y \rightarrow \widehat{\mathcal{X}}'_y$  which induces  $-1$  on the Picard group  $\text{Pic}(R)$ . Then, letting  $\widehat{\mathcal{X}}^+ \rightarrow \widehat{\mathcal{X}}'_y$  denote the composite  $\widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}'_y \xrightarrow{t} \widehat{\mathcal{X}}'_y$  we can algebraize to obtain a proper,

birational morphism  $\mathcal{X}^+ \rightarrow \mathcal{X}'$  and hence a birational map  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ . We claim that  $s_f = s_E$ .

Indeed, since  $f$  induces an isomorphism

$$\mathcal{X} \setminus E \xrightarrow{\sim} \mathcal{X}^+ \setminus f_k(E),$$

we note that the special fiber of the graph of  $f$ , considered as a cycle on  $Y \times_k \mathcal{X}_k^+$ , can be written as

$$\Gamma_{f,k} := \Gamma_{f_k} + b(E \times f_k(E))$$

for some  $b \in \mathbb{Z}_{\geq 0}$ . Thus, pulling back to  $Y$  we can see that  $s_f$  is given by the formula

$$s_f(D) = D + b(E \cdot D)E.$$

By Proposition 7.4(1), we must have that  $b \in \{0, 1\}$ . If  $b = 0$ , then  $f$  is an isomorphism and if  $b = 1$ , then  $s_f = s_E$ . It therefore suffices to show that

$$f : \mathcal{X} \dashrightarrow \mathcal{X}^+$$

is *not* biregular, and that  $(\mathcal{X}^+, \mathcal{L}^+)$  is terminal. Since all these constructions are compatible with base change, to prove this we may therefore assume that  $k = \bar{k}$  is algebraically closed (this is only to avoid any issues concerning the difference between geometric and closed points during the proof).

To see that  $f$  is not biregular, it suffices to show that the dashed arrow in the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{X}} & \longrightarrow & \mathrm{Spf}(R) \\ \uparrow & & \uparrow t \\ \widehat{\mathcal{X}} & \longrightarrow & \mathrm{Spf}(R) \end{array}$$

is not regular. Indeed, if it were, then we would have a commutative diagram

$$\begin{array}{ccc} (\mathbf{R}^1 g_* \mathcal{O}_{\widehat{\mathcal{X}}}^*)_y & \longrightarrow & \mathrm{Pic}(R) \\ \downarrow & & \downarrow t^* \\ (\mathbf{R}^1 g_* \mathcal{O}_{\widehat{\mathcal{X}}}^*)_y & \longrightarrow & \mathrm{Pic}(R) \end{array}$$

with  $(\mathbf{R}^1 g_* \mathcal{O}_{\widehat{\mathcal{X}}}^*)_y$  generated by the class of the exceptional curve  $E$ . In particular, the left hand map would have to be the identity. This contradicts  $t$  inducing  $-1$  on  $\mathrm{Pic}(R)$ .

Finally, to see that  $(\mathcal{X}^+, \mathcal{L}^+)$  is terminal, we simply note that since  $\mathcal{L}_k \cdot E = 0$ , we can find a divisor representing  $\mathcal{L}$  which doesn't meet  $E$ . Therefore, the specialization  $\mathcal{L}_k^+$  is simply the push-forward  $f_{k*}(\mathcal{L}_k)$  of  $\mathcal{L}_k$  via the induced isomorphism  $f_k : Y \rightarrow \mathcal{X}_k^+$ .  $\square$

With this in place, we can now prove that  $s_f$  is an element of the Weyl group.

**Theorem 7.6.** *Let  $\mathcal{X}$  be a terminal model of a polarized K3 surface  $(X, \mathcal{L})$  over  $K$  and  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  a terminal birational map.*

- (1)  $s_f \in \mathcal{W}_{X, \mathcal{L}} \leq \text{Aut}_{\mathbb{Z}}(\text{Pic}(Y))$ .
- (2)  $f$  is an isomorphism if and only if  $s_f = \text{id}$ .

PROOF. If  $L/K$  is a finite and unramified Galois extension with Galois group  $G$ , then  $\mathcal{W}_{X, \mathcal{L}} \cong \mathcal{W}_{X_L, \mathcal{L}_L}^G$  by Corollary 3.3, and  $f$  being an isomorphism can be detected over  $L$ . Hence, after possibly enlarging  $K$ , we can assume that all irreducible components  $E_i \subset E_{X, \mathcal{L}}$  are geometrically irreducible. In particular, all singularities of  $P(X, \mathcal{L})_k$  are rational double points of type ADE.

As in the proof of Proposition 7.5 above, we know that since  $f$  is terminal, it induces an isomorphism

$$\mathcal{X} \setminus E_{X, \mathcal{L}} \xrightarrow{\sim} \mathcal{X}^+ \setminus E_{X^+, \mathcal{L}^+},$$

thus, we can describe the cycle  $\Gamma \subset Y \times_k Y$  defining  $s_f$  as

$$\Gamma := \Delta_Y + \sum_{i, j} b_{ij} (E_i \times E_j)$$

for some  $b_{ij} \in \mathbb{Z}_{\geq 0}$ . Let  $B = (b_{ij}) \in M_n(\mathbb{Z})$  denote the corresponding matrix. We therefore have

$$s_f(D) = D + \sum_{i, j} b_{ij} (D \cdot E_j) \cdot E_i$$

from which we see that  $s_f$  preserves the root lattice

$$\Lambda_{X, \mathcal{L}} \subseteq \text{Pic}(Y).$$

Moreover, since  $f$  induces an isomorphism  $P(X, \mathcal{L}) \xrightarrow{\sim} P(X^+, \mathcal{L}^+)$ , we deduce that  $b_{ij} \neq 0$  only if  $E_i$  and  $E_j$  are in the same connected component of  $E_{X, \mathcal{L}}$ . Thus,  $s_f$  preserves the sub-lattice  $\Lambda_l \subset \Lambda_{X, \mathcal{L}}$  corresponding to each connected component of  $E_{X, \mathcal{L}}$ . Using the formula  $(s_f)^{-1} = f_k^* \circ s_{f^{-1}} \circ (f_k^*)^{-1}$ , this is also the case for  $(s_f)^{-1}$ .

Hence, applying Proposition 7.4(1) we can write

$$s_f|_{\Lambda_{X, \mathcal{L}}} = \prod_l \tilde{w}_l$$

as a product of elements  $\tilde{w}_l \in O(\Lambda_l)$  in the orthogonal group of each individual  $\Lambda_l$ . Moreover, by [KM83, Theorem 1], we can write  $\tilde{w}_l = \alpha_l w_l$  with  $w_l$  in the Weyl group of  $\Lambda_l$ , and  $\alpha_l$  coming from an automorphism of the Dynkin diagram of  $\Lambda_l$ . Now, let  $w = \prod_l w_l \in \mathcal{W}_{X, \mathcal{L}}$ . By Proposition 7.5, there exists some birational map

$$g : \mathcal{X}^+ \dashrightarrow \mathcal{X}^{+2}$$

with  $(\mathcal{X}^{+2}, \mathcal{L}^{+2})$  terminal such that  $s_g = (f_k^*)^{-1} \circ w^{-1} \circ f_k^*$ . Replacing  $f$  by  $g \circ f$  and applying Proposition 7.4(2), we may therefore assume that each

$w_l = \text{id}$ , that is, that

$$s_f|_{\Lambda_{X,\mathcal{L}}} = \prod_l \alpha_l$$

is a product of elements coming from automorphisms of the relevant Dynkin diagrams. We need to show that  $s_f = \text{id}$ .

To see this, we note that the matrix of  $s_f|_{\Lambda_{X,\mathcal{L}}}$  on a suitable basis of  $\Lambda_{X,\mathcal{L}}$  is  $I + BA$ , where  $A$  is the intersection matrix of  $E_{X,\mathcal{L}}$  and  $B$  is as above. Since both  $s_f$  and  $(s_f)^{-1}$  preserve each summand  $\Lambda_l$ , it follows that  $B$  has to be a block sum matrix over the connected components of  $E_{X,\mathcal{L}}$ , the same is also true for  $A$ . Hence, we may apply Lemma 3.5 to conclude that in fact each  $\alpha_l = \text{id}$ , that is,  $s_f|_{\Lambda_{X,\mathcal{L}}} = \text{id}$ . This then implies that  $BA = 0$ , and since  $A$  is negative definite, it therefore follows that  $B = 0$ . Thus, we have  $\Gamma = \Delta_Y$  and so  $s_f = \text{id}$ .

The proof of (2) is then entirely similar, since  $s_f = \text{id}$  implies that  $BA = 0$ , whereas  $f$  being regular is equivalent to having  $B = 0$ . Negative definiteness of  $A$  implies that  $B = 0 \Leftrightarrow AB = 0$ .  $\square$

This gives us a way of characterising terminal models of a given polarized K3 surface  $(X, \mathcal{L})$  over  $K$ .

**Corollary 7.7.** *Let  $(X, \mathcal{L})$  be a polarized K3 surface and assume that  $X$  has good reduction over  $K$ . Then, the set*

$$\{\mathcal{L}\text{-terminal models of } X\} / \sim$$

*of  $\mathcal{L}$ -terminal models up to equivalence is a torsor under the Weyl group  $\mathcal{W}_{X,\mathcal{L}}$ .*

**Remark 7.8.** Here, two such models  $\mathcal{X}_1, \mathcal{X}_2$  of  $X$  are said to be *equivalent*, if there exists an isomorphism  $\mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$  that induces the identity on the generic fiber  $X$ . Put differently, this says that the rational map  $\mathcal{X}_1 \dashrightarrow \mathcal{X}_2$  defined by the identity on the generic fiber is in fact regular.

PROOF. Fix an  $\mathcal{L}$ -terminal model  $\mathcal{X}$  for  $X$ . Then, we obtain a function

$$\{\mathcal{L}\text{-terminal models of } X\} / \sim \rightarrow \mathcal{W}_{X,\mathcal{L}}$$

as follows: given another  $\mathcal{L}$ -terminal model  $\mathcal{X}^+$ , we obtain a rational map

$$f_{\mathcal{X}^+} : \mathcal{X} \dashrightarrow \mathcal{X}^+$$

by taking the identity on generic fibers, and hence an element

$$s_{\mathcal{X}^+} := s_{f_{\mathcal{X}^+}}$$

of the Weyl group. Injectivity of this map follows from Theorem 7.6(2) above, we would like to show surjectivity.

To do so, we will choose a finite and unramified Galois extension  $L/K$ , with Galois group  $G$ , such that all irreducible components of  $E_{X_L, \mathcal{L}_L}$  are geometrically irreducible. If we have  $w \in \mathcal{W}_{X,\mathcal{L}}$ , then applying Proposition 7.5 we know that there exists an  $\mathcal{L}$ -terminal model  $\mathcal{Y} \rightarrow \mathcal{O}_L$  of  $X_L$  and a

birational map  $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L \dashrightarrow \mathcal{Y}$ , such that  $s_{\mathcal{Y}} = w \in \mathcal{W}_{X_L, \mathcal{L}_L}$ . But now, since  $w$  is fixed by  $G$ , it follows from the already proved injectivity of

$$\{\mathcal{L}_L\text{-terminal models of } X_L\} / \sim \rightarrow \mathcal{W}_{X_L, \mathcal{L}_L}$$

that so is  $\mathcal{Y}$ , for the natural  $G$  action on the left hand side. Concretely, this means that the rational  $G$ -action on  $\mathcal{Y}$  is regular, and thus, by Proposition 7.1, we know that  $\mathcal{Y}$  descends to  $\mathcal{O}_K$ .  $\square$

Since the Weyl group is finite, we obtain the following rather modest corollary.

**Corollary 7.9.** *Any polarized K3 surface over  $K$  with good reduction has finitely many terminal models over  $\mathcal{O}_K$ .*

**7.2. Obstructions to good reduction.** We will now use these results to study the relationship between good reduction and potential good reduction of a K3 surface over  $K$ . We will therefore take a polarized K3 surface  $(X, \mathcal{L})$  over  $K$ , as always satisfying  $(\star)$ , such that the equivalent conditions of Theorem 1.1 hold.

We thus know that  $X$  admits good reduction over a finite and unramified Galois extension  $L/K$ , say with Galois group  $G$  and residue field extension  $k_L/k$ . Let  $P(X, \mathcal{L})$  denote the canonical RDP model over  $\mathcal{O}_K$  and  $Y$  the canonical reduction of  $X$  as in §6. We therefore have

$$\mathcal{W}_{X_L, \mathcal{L}_L} \leq \text{Aut}_{\mathbb{Z}}(\text{Pic}(Y_{k_L})),$$

which is invariant under the natural  $G$ -action on the latter. For any object over  $L$  or  $k_L$ , and any  $\sigma \in G$ , we will denote by  $(-)^{\sigma}$  the base change by  $\sigma$ , thus we have  $(-)^{\sigma\tau} = ((-)^{\tau})^{\sigma}$ .

For any  $\mathcal{L}_L$ -terminal model  $\mathcal{Y}$  of  $X_L$  over  $\mathcal{O}_L$ , we obtain a rational and semilinear  $G$ -action on the pair  $(\mathcal{Y}, \mathcal{L}_L)$ . Hence, by the results of Section 7.1 we can define a function

$$\alpha_{\mathcal{Y}} : G \rightarrow \mathcal{W}_{X_L, \mathcal{L}_L}$$

as follows: for any  $\sigma \in G$  we base change  $\mathcal{Y}$  by  $\sigma$  to obtain  $\mathcal{Y}^{\sigma}$ . The regular  $G$ -action on the generic fibre  $X_L$  provides a terminal (with respect to  $\mathcal{L}_L$ ) birational map

$$f_{\sigma} : \mathcal{Y} \dashrightarrow \mathcal{Y}^{\sigma},$$

which is  $\mathcal{O}_L$ -linear. Then the induced rational map

$$f_{\sigma, k_L} : Y_{k_L} \rightarrow Y_{k_L}^{\sigma}$$

on the special fiber is an isomorphism, and we can concretely describe the  $G$ -action on  $\mathcal{W}_{X_L, \mathcal{L}_L}$  by the formula

$$\sigma(s) = f_{\sigma, k_L}^* \circ s^{\sigma} \circ (f_{\sigma, k_L}^*)^{-1}.$$

By Theorem 7.6 we may define an element  $s_{f_{\sigma}} \in \mathcal{W}_{X_L, \mathcal{L}_L}$  of the Weyl group associated to  $f_{\sigma}$ , and we define a map

$$\begin{aligned} \alpha_{\mathcal{Y}} : G &\rightarrow \mathcal{W}_{X_L, \mathcal{L}_L} \\ \sigma &\mapsto s_{f_{\sigma}}. \end{aligned}$$

Then, we have the following crucial observation.

**Proposition 7.10.** *The map*

$$\alpha_{\mathcal{Y}} : G \rightarrow \mathcal{W}_{X_L, \mathcal{L}_L}$$

*is a 1-cocycle for the  $G$ -action on  $\mathcal{W}_{X_L, \mathcal{L}_L}$ . Moreover, the  $G$ -action on  $\mathcal{Y}$  is regular if and only if this cocycle is trivial, that is, satisfies  $\alpha_{\mathcal{Y}}(\sigma) = 1$  for all  $\sigma \in G$ .*

PROOF. For the first claim, we use the fact that  $f_{\sigma\tau} = f_{\tau}^{\sigma} \circ f_{\sigma}$ , and simply calculate

$$\begin{aligned} \alpha_{\mathcal{Y}}(\sigma\tau) &= s_{f_{\sigma\tau}} = s_{f_{\tau}^{\sigma} \circ f_{\sigma}} = s_{f_{\sigma}} \circ f_{\sigma, k_L}^* \circ s_{f_{\tau}^{\sigma}} \circ (f_{\sigma, k_L}^*)^{-1} \\ &= s_{f_{\sigma}} \circ f_{\sigma, k_L}^* \circ s_{f_{\tau}^{\sigma}} \circ (f_{\sigma, k_L}^*)^{-1} = s_{f_{\sigma}} \circ \sigma(s_{f_{\tau}}) \\ &= \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(\alpha_{\mathcal{Y}}(\tau)) \end{aligned}$$

using Proposition 7.4(2). The second claim is an immediate consequence of Theorem 7.6.  $\square$

The next obvious question to ask is how the cocycle  $\alpha_{\mathcal{Y}}$  changes as the  $\mathcal{L}_L$ -terminal model  $\mathcal{Y}$  changes. So, let us suppose that we have another  $\mathcal{L}_L$ -terminal model  $\mathcal{Y}^+$  for  $X_L$ , and let  $f : \mathcal{Y} \dashrightarrow \mathcal{Y}^+$  be the rational map given by the identity on the generic fibers. Let  $\alpha_{\mathcal{Y}^+} : G \rightarrow \mathcal{W}_{X_L, \mathcal{L}_L}$  denote the cocycle associated to the rational  $G$ -action on  $\mathcal{Y}^+$ , and let  $s_f \in \mathcal{W}_{X_L, \mathcal{L}_L}$  be the element of the Weyl group associated to  $f$ , as provided by Theorem 7.6.

**Proposition 7.11.** *For all  $\sigma \in G$  we have*

$$\alpha_{\mathcal{Y}^+}(\sigma) = s_f^{-1} \circ \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(s_f).$$

PROOF. If we view  $\alpha_{\mathcal{Y}}$  as a function

$$\alpha_{\mathcal{Y}} : G \rightarrow \text{Aut}_{\mathbb{Z}}(\text{Pic}(\mathcal{Y}_{k_L})),$$

and  $\alpha_{\mathcal{Y}^+}$  as a function

$$\alpha_{\mathcal{Y}^+} : G \rightarrow \text{Aut}_{\mathbb{Z}}(\text{Pic}(\mathcal{Y}_{k_L}^+)),$$

then what we need to show is that

$$f_{k_L}^* \circ \alpha_{\mathcal{Y}^+}(\sigma) \circ (f_{k_L}^*)^{-1} = s_f^{-1} \circ \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(s_f).$$

Since the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f_{\sigma}} & \mathcal{Y}^{\sigma} \\ \downarrow f & & \downarrow f^{\sigma} \\ \mathcal{Y}^+ & \xrightarrow{f_{\sigma}^+} & \mathcal{Y}^{+, \sigma} \end{array}$$

commutes for all  $\sigma \in G$ , we may again use Proposition 7.4(2) to calculate

$$\begin{aligned} \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(s_f) &= s_{f_\sigma} \circ f_{\sigma, k_L}^* \circ s_f^\sigma \circ (f_{\sigma, k_L}^*)^{-1} \\ &= s_{f_\sigma} \circ f_{\sigma, k_L}^* \circ s_{f_\sigma} \circ (f_{\sigma, k_L}^*)^{-1} \\ &= s_{f_\sigma \circ f_\sigma} = s_{f_\sigma^+ \circ f} \\ &= s_f \circ f_{k_L}^* \circ \alpha_{\mathcal{Y}^+}(\sigma) \circ (f_{k_L}^*)^{-1} \end{aligned}$$

as required.  $\square$

In particular, under the assumption that  $X$  has good reduction over  $L$ , the associated cohomology class

$$\alpha_{X, \mathcal{L}}^L := [\alpha_{\mathcal{Y}}] \in H^1(G, \mathcal{W}_{X_L, \mathcal{L}_L})$$

is independent of the chosen  $\mathcal{L}_L$ -terminal model  $\mathcal{Y}$ .

**Proposition 7.12.** *There exists an  $\mathcal{L}_L$ -terminal model of  $X_L$  over  $\mathcal{O}_L$  for which the  $G$ -action is regular, if and only if the cohomology class*

$$\alpha_{X, \mathcal{L}}^L \in H^1(G, \mathcal{W}_{X_L, \mathcal{L}_L})$$

*is trivial. In particular,  $X$  has good reduction over  $K$  if and only if  $\alpha_{X, \mathcal{L}}^L$  is trivial.*

PROOF. For the first statement, one direction is clear, so suppose that the cohomology class  $\alpha_{X, \mathcal{L}}^L$  is trivial. Fix some  $\mathcal{L}_L$ -terminal model  $\mathcal{Y}$ , with associated cocycle  $\alpha$ . By assumption, there exists some  $w \in \mathcal{W}_{X_L, \mathcal{L}_L}$  such that  $\alpha(\sigma) = w^{-1}\sigma(w)$  for all  $\sigma \in G$ . We also know from Corollary 7.7 that there exists some other terminal model  $f : \mathcal{Y} \dashrightarrow \mathcal{Y}^+$  such that  $s_f = w^{-1}$ . Let

$$\alpha^+ : G \rightarrow \mathcal{W}_{X_L, \mathcal{L}_L}$$

be the cocycle associated to the model  $\mathcal{Y}^+$ . Then, using Proposition 7.11 we can see that  $\alpha^+$  is trivial. Therefore, by Proposition 7.10 we know that the rational  $G$ -action on  $\mathcal{Y}^+$  is in fact regular.

For the second statement, the if direction follows from the first statement combined with Proposition 7.1. Conversely, suppose that  $X$  has good reduction over  $K$ , say with model  $\mathcal{X}$ . Then, by [LM14, Proposition 4.5] we may assume that  $\mathcal{X}$  is  $\mathcal{L}$ -terminal, and hence  $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$  is an  $\mathcal{L}_L$ -terminal model of  $X_L$  to which the  $G$ -action extends. Thus, the cohomology class  $\alpha_{X, \mathcal{L}}^L$  is trivial.  $\square$

To give a condition that does not depend on the choice of unramified field extension  $L/K$ , we simply pass to the limit. So let

$$\mathcal{W}_{X, \mathcal{L}}^{\text{nr}} := \text{colim}_{K \subseteq L \subseteq K^{\text{nr}}} \mathcal{W}_{X_L, \mathcal{L}_L} \cong \mathcal{W}_{X_{K^{\text{nr}}}, \mathcal{L}_{K^{\text{nr}}}}$$

be the colimit over all finite and unramified extensions of  $K$  inside the fixed algebraic closure  $\overline{K}$ . Then, there is a natural  $G_k \cong \text{Gal}(K^{\text{un}}/K)$ -action on  $\mathcal{W}_{\mathcal{L}}^{\text{nr}}$ , and the above cohomology classes

$$\alpha_{X, \mathcal{L}}^L \in H^1(\text{Gal}(L/K), \mathcal{W}_{X_L, \mathcal{L}_L})$$

for finite and unramified Galois extensions  $L/K$  give rise to a well-defined class

$$\alpha_{X,\mathcal{L}}^{\text{nr}} \in H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\text{nr}})$$

in *continuous* cohomology for the pro-finite group  $G_k$ . We can then rephrase Proposition 7.12 as follows.

**Corollary 7.13.** *Let  $X/K$  be a K3 surface satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, the following are equivalent.*

- (1)  $X$  has good reduction over  $K$ .
- (2) The cohomology class  $\alpha_{X,\mathcal{L}}^{\text{nr}} \in H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\text{nr}})$  is trivial for all ample line bundles  $\mathcal{L}$  on  $X$ .
- (3) The cohomology class  $\alpha_{X,\mathcal{L}}^{\text{nr}} \in H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\text{nr}})$  is trivial for some ample line bundle  $\mathcal{L}$  on  $X$ .

PROOF. Since we have

$$H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\text{nr}}) = \text{colim}_L H^1(\text{Gal}(L/K), \mathcal{W}_{X_L, \mathcal{L}_L}),$$

where the colimit is taken over all finite and unramified Galois sub-extensions  $K \subseteq L \subset K^{\text{nr}}$ , this follows from Proposition 7.12.  $\square$

## 8. DESCRIPTION OF $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ AND $\mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p))$

The purpose of this section is to prove Theorem 1.5 from the introduction, which, for a K3 surface  $X$  over  $K$  satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1, describes the unramified  $G_K$ -representation

$$H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell),$$

and, when  $p > 0$ , the  $F$ -isocrystal

$$\mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)),$$

reasonably explicitly in terms of the  $G_k$ -representation  $H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell)$  and the  $F$ -isocrystal  $H_{\text{cris}}^2(Y/K_0)$  associated to the canonical reduction  $Y$  of  $X$ . To achieve this, we will first need a result on compatibility of comparison isomorphisms with cycle class maps, which briefly appeared earlier during the proof of Proposition 7.4.

**8.1. Rational maps and crystalline comparison.** Suppose that we have a birational map  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  between smooth models of K3 surfaces  $X$  and  $X^+$ , respectively. Then, we may consider the graph of  $f$

$$\Gamma_f \subset \mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}^+$$

as well as its generic and special fibers

$$\Gamma_{f_K} \subset X \times_K X^+ \quad \text{and} \quad \Gamma_{f,k} \subset \mathcal{X}_k \times_k \mathcal{X}_k^+,$$



respectively. Note that  $\Gamma_{f_K}$  is genuinely the graph of the induced *isomorphism*  $f_K : X \rightarrow X^+$ , but the same is not true of  $\Gamma_{f,k}$  in general. Since the cycle  $\Gamma_{f,k}$  is purely of dimension 2, we obtain maps on cohomology

$$\begin{aligned} \Gamma_{f,k}^* : H_{\text{ét}}^n(\mathcal{X}_k^+, \mathbb{Q}_\ell) &\rightarrow H_{\text{ét}}^n(\mathcal{X}_k, \mathbb{Q}_\ell) \\ \alpha &\mapsto p_{1*}([\Gamma_{f,k}] \cup p_2^*(\alpha)) \end{aligned}$$

for  $\ell \neq p$  and

$$\begin{aligned} \Gamma_{f,k}^* : H_{\text{cris}}^n(\mathcal{X}_k^+/K_0) &\rightarrow H_{\text{cris}}^n(\mathcal{X}_k/K_0) \\ \alpha &\mapsto p_{1*}([\Gamma_{f,k}] \cup p_2^*(\alpha)) \end{aligned}$$

when  $p > 0$ . The result we require is the following.

**Lemma 8.1.** *In the above situation, the diagrams*

$$\begin{array}{ccc} H_{\text{ét}}^n(X_{\overline{K}}^+, \mathbb{Q}_\ell) &\longrightarrow & H_{\text{ét}}^n(\mathcal{X}_k^+, \mathbb{Q}_\ell) \\ f_K^* \downarrow & & \downarrow \Gamma_{f,k}^* \\ H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_\ell) &\longrightarrow & H_{\text{ét}}^n(\mathcal{X}_k, \mathbb{Q}_\ell) \end{array}$$

for  $\ell \neq p$ , and

$$\begin{array}{ccc} \mathbb{D}_{\text{cris}}\left(H_{\text{ét}}^n(X_{\overline{K}}^+, \mathbb{Q}_p)\right) &\longrightarrow & H_{\text{cris}}^n(\mathcal{X}_k^+/K_0) \\ f_K^* \downarrow & & \downarrow \Gamma_{f,k}^* \\ \mathbb{D}_{\text{cris}}\left(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)\right) &\longrightarrow & H_{\text{cris}}^n(\mathcal{X}_k/K_0) \end{array}$$

when  $p > 0$ , commute. Here, the horizontal arrows are the isomorphisms provided by the smooth and proper base change theorem and the crystalline comparison theorem, respectively.

**Remark 8.2.** Since  $H_{\text{ét}}^n(X_{\widehat{K}}, \mathbb{Q}_p) \cong H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)$  as  $G_{\widehat{K}} \cong G_K$  representations, there exists such a crystalline comparison theorem without necessarily assuming  $K$  to be complete.

PROOF. Since the  $\ell$ -adic case for  $\ell \neq p$  was handled in [LM14, Lemma 5.6], we will only consider the  $p$ -adic case. As usual, we may assume that  $K = \widehat{K}$  is complete. Since  $\mathcal{X}$  and  $\mathcal{X}^+$  have schematic fibers, the completions  $\mathfrak{X}$  and  $\mathfrak{X}^+$  along their special fibers are therefore smooth and proper formal schemes (see the proof of Theorem 2.4). Hence, we have Berthelot–Ogus comparison isomorphisms [BO83, Theorem 2.4]

$$\begin{aligned} H_{\text{cris}}^n(\mathcal{X}_k/W) \otimes_W K &\cong H_{\text{dR}}^n(X/K) \\ H_{\text{cris}}^n(\mathcal{X}_k^+/W) \otimes_W K &\cong H_{\text{dR}}^n(X^+/K) \end{aligned}$$

such that the diagram

$$\begin{array}{ccc} \mathbb{D}_{\text{cris}}(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)) \otimes_{K_0} K & \xrightarrow{\cong} & H_{\text{cris}}^n(\mathcal{X}_k/W) \otimes_W K \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{D}_{\text{dR}}(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)) & \xrightarrow{\cong} & H_{\text{dR}}^n(X/K) \end{array}$$

commutes (and similarly for  $\mathcal{X}^+$ ), see for example [CN17, Corollary 5.26]. Since the induced map  $f_K : X \dashrightarrow X^+$  is regular, it follows that the diagram

$$\begin{array}{ccc} \mathbb{D}_{\text{dR}}(H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p)) & \xrightarrow{\cong} & H_{\text{dR}}^n(X/K) \\ \mathbb{D}_{\text{dR}}(f_K^*) \downarrow & & \downarrow f_K^* \\ \mathbb{D}_{\text{dR}}(H_{\text{ét}}^n(X_{\overline{K}}^+, \mathbb{Q}_p)) & \xrightarrow{\cong} & H_{\text{dR}}^n(X^+/K) \end{array}$$

commutes. It therefore suffices to prove that the Berthelot–Ogus comparison isomorphisms are compatible with  $\Gamma_f^*$ , in other words that the diagram

$$\begin{array}{ccc} H_{\text{cris}}^n(\mathcal{X}_k/W) \otimes_W K & \xrightarrow{\cong} & H_{\text{dR}}^n(X/K) \\ \Gamma_{f,k}^* \downarrow & & \downarrow f_K^* \\ H_{\text{cris}}^n(\mathcal{X}_k^+/W) \otimes_W K & \xrightarrow{\cong} & H_{\text{dR}}^n(X^+/K) \end{array}$$

commutes. Since the horizontal isomorphisms are compatible with cup products (and hence Poincaré duality), it suffices to show that they are also compatible with cycle classes. In other words, we are given a smooth and proper algebraic space  $\mathcal{Z}$  over  $\mathcal{O}_K$ , whose generic fiber  $Z$  and whose special fiber  $\mathcal{Z}_k$  are both schemes, and a closed and integral subspace  $\mathcal{T} \subset \mathcal{Z}$  that is flat and of relative codimension  $c$  over  $\mathcal{O}_K$ , with generic fiber  $T$  and special fiber  $\mathcal{T}_k$ . We must show that the isomorphism

$$H_{\text{cris}}^{2c}(\mathcal{Z}_k/W) \otimes_W K \cong H_{\text{dR}}^{2c}(Z/K)$$

identifies  $\text{cl}(\mathcal{T}_k) \otimes 1$  with  $\text{cl}(T)$ . Since rational crystalline cohomology and algebraic de Rham cohomology both satisfy étale descent, this follows by pulling back to an étale hypercover  $\mathcal{Z}_\bullet \rightarrow \mathcal{Z}$  of  $\mathcal{Z}$  by smooth  $\mathcal{O}_K$ -schemes and applying [CCM13, Corollary 1.5.1].  $\square$

## 8.2. Extending elements of $\mathcal{W}$ to automorphisms of cohomology.

We can now use Lemma 8.1 to give a different interpretation of the element  $s_f \in \mathcal{W}_{X,\mathcal{L}}$  associated to a birational map  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  between terminal models of polarized K3 surfaces in Section 7. This interpretation already appeared during the proof of Proposition 7.4, but it will be helpful to spell it out more clearly, including the  $p$ -adic case.

First of all, we will need to show how to extend elements of the Weyl group to automorphisms on cohomology. So, let  $(X, \mathcal{L})$  be a polarized K3

surface over  $K$ , and  $\mathcal{X}$  an  $\mathcal{L}$ -terminal model for  $X$ . We therefore have the Weyl group

$$\mathcal{W}_{X,\mathcal{L}} \leq \text{Aut}_{\mathbb{Z}}(\text{Pic}(Y))$$

acting on the Picard group of the special fiber  $Y$  of  $\mathcal{X}$ . Moreover, after base changing to  $K^{\text{nr}}$  we also have the corresponding  $G_k$ -equivariant version

$$\mathcal{W}_{X,\mathcal{L}}^{\text{nr}} \leq \text{Aut}_{\mathbb{Z}}(\text{Pic}(Y_{\bar{k}}))$$

over  $\bar{k}$ .

**Lemma 8.3.** *There are  $G_k$ -equivariant and injective homomorphisms*

$$\begin{aligned} i_{\ell} : \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{\mathbb{Q}_{\ell}} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}(1)) \right) & (\ell \neq p) \\ i_p : \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{K_0^{\text{nr}},F} \left( H_{\text{cris}}^2(Y/K_0)(1) \otimes_{K_0} K_0^{\text{nr}} \right) & (p > 0). \end{aligned}$$

PROOF. If  $E \subset E_{X,\mathcal{L},\bar{k}}$  is an irreducible component of the geometric exceptional locus, then, for  $\ell \neq p$ , we simply take the reflection  $s_E$  to the map

$$\begin{aligned} H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}(1)) &\rightarrow H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}(1)) \\ \alpha &\mapsto \alpha + ([E] \cup \alpha)[E] \end{aligned}$$

via the canonical identification  $H_{\text{ét}}^4(Y_{\bar{k}}, \mathbb{Q}_{\ell}(2)) \cong \mathbb{Q}_{\ell}$ . When  $p > 0$ , we choose a finite unramified extension  $L/K$  with residue field  $k_L$  over which  $E$  is defined and send  $s_E$  to the scalar extension of the map

$$\begin{aligned} H_{\text{cris}}^2(Y_{k_L}/L_0)(1) &\rightarrow H_{\text{cris}}^2(Y_{k_L}/L_0)(1) \\ \alpha &\mapsto \alpha + ([E] \cup \alpha)[E] \end{aligned}$$

where  $L_0 = W(k_L)[1/p]$ , again via  $H_{\text{cris}}^4(Y_{k_L}/L_0)(2) \cong L_0$ .  $\square$

In particular, by taking  $G_k$ -invariants we obtain injective homomorphisms

$$\begin{aligned} \mathcal{W}_{X,\mathcal{L}} &\rightarrow \text{Aut}_{G_k} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}(1)) \right) & (\ell \neq p) \\ \mathcal{W}_{X,\mathcal{L}} &\rightarrow \text{Aut}_{K_0,F} \left( H_{\text{cris}}^2(Y/K_0)(1) \right) & (p > 0). \end{aligned}$$

If we have a terminal birational map  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  to some other smooth model of a K3 surface  $X^+/K$ , then by [LM14, Proposition 4.7]  $f$  is defined away from a finite collection of curves on the special fibers, and by minimality the induced rational maps

$$\begin{aligned} f_K : X &\dashrightarrow X^+ \\ f_k : Y &\dashrightarrow \mathcal{X}_k^+ \end{aligned}$$

on the generic and special fibers are isomorphisms. As well as the obvious pull-back maps

$$\begin{aligned} f_{k,\ell}^* : H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_{\ell}(1)) &\rightarrow H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}(1)) & (\ell \neq p) \\ f_{k,p}^* : H_{\text{cris}}^2(\mathcal{X}_k^+/K_0)(1) &\rightarrow H_{\text{cris}}^2(Y/K_0)(1) & (p > 0) \end{aligned}$$

we also have “generic fiber” pull-back maps

$$\begin{aligned} f_{K,\ell}^* : H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_{\ell}(1)) &\rightarrow H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_{\ell}(1)) & (\ell \neq p) \\ f_{K,p}^* : H_{\text{cris}}^2(\mathcal{X}_k^+/K_0)(1) &\rightarrow H_{\text{cris}}^2(Y/K_0)(1) & (p > 0) \end{aligned}$$

which are defined by the commutative diagrams

$$\begin{array}{ccc} H_{\text{ét}}^2(X_{\overline{K}}^+, \mathbb{Q}_\ell(1)) & \xrightarrow{\cong} & H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) \\ \downarrow & & \downarrow f_{K,\ell}^* \\ H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p(1)) & \xrightarrow{\cong} & H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \end{array}$$

for  $\ell \neq p$ , and

$$\begin{array}{ccc} \mathbb{D}_{\text{cris}} \left( H_{\text{ét}}^2(X_{\overline{K}}^+, \mathbb{Q}_p(1)) \right) & \xrightarrow{\cong} & H_{\text{cris}}^2(\mathcal{X}_k^+/K_0)(1) \\ \downarrow & & \downarrow f_{K,p}^* \\ \mathbb{D}_{\text{cris}} \left( H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p(1)) \right) & \xrightarrow{\cong} & H_{\text{cris}}^2(Y/K_0)(1), \end{array}$$

when  $p > 0$ . Here, the horizontal arrows are the appropriate comparison theorems, and the left hand vertical arrows simply pull-back on the generic fiber. We have the following corollary of Lemma 8.1.

**Theorem 8.4.** *Let  $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$  be a terminal birational map between terminal models of K3 surfaces over  $K$ . Then, we have*

$$\begin{aligned} i_\ell(s_f) = f_{K,\ell}^* \circ (f_{k,\ell}^*)^{-1} : H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) &\rightarrow H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \quad (\ell \neq p) \\ i_p(s_f) = f_{K,p}^* \circ (f_{k,p}^*)^{-1} : H_{\text{cris}}^2(Y/K_0)(1) &\rightarrow H_{\text{cris}}^2(Y/K_0)(1) \quad (p > 0). \end{aligned}$$

PROOF. Let  $\tilde{s}_f$  be the map  $\text{Pic}(\mathcal{X}_k^+) \rightarrow \text{Pic}(Y)$  induced by the special fibre  $\Gamma_{f,k}$  of the graph of  $f$ , and for all primes  $\ell$ , including  $\ell = p$  when  $p > 0$ , let  $\tilde{s}_{f,\ell}$  be the map

$$\begin{aligned} \Gamma_{f,k}^* : H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) &\rightarrow H_{\text{ét}}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \quad (\ell \neq p) \\ \Gamma_{f,k}^* : H_{\text{cris}}^2(\mathcal{X}_k^+/K_0)(1) &\rightarrow H_{\text{cris}}^2(Y/K_0)(1) \quad (p > 0) \end{aligned}$$

induced in cohomology by the same cycle. Then, Lemma 8.1 tells us that we have  $f_{K,\ell}^* = \tilde{s}_{f,\ell}$ , and since  $s_f = \tilde{s}_f \circ (f_k^*)^{-1}$ , the result follows.  $\square$

**8.3. Cohomological realisations of  $\alpha_{X,\mathcal{L}}^L$  and  $\alpha_{X,\mathcal{L}}^{\text{nr}}$ .** Now, let us suppose that  $(X, \mathcal{L})$  is a polarized K3 surface over  $K$ , satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1. Let  $Y$  denote the canonical reduction of  $X$ .

We will let  $L/K$  denote a finite and unramified Galois extension over which  $X$  has good reduction. Write  $G$  for the Galois group of  $L/K$  and  $k_L/k$  for the residue field extension. We will write  $L_0 = W(k_L)[1/p] \subset \widehat{L}$  and  $G_{k_L}$  for the absolute Galois group of  $k_L$ . We therefore have the exact sequence

$$1 \rightarrow G_{k_L} \rightarrow G_k \rightarrow G \rightarrow 1.$$

Taking  $G_{k_L}$ -invariants in Lemma 8.3 and twisting, we obtain  $G$ -equivariant homomorphisms

$$\begin{aligned} \mathcal{W}_{X_L, \mathcal{L}_L} &\rightarrow \text{Aut}_{G_{k_L}} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell(1)) \right) & (\ell \neq p) \\ \mathcal{W}_{X_L, \mathcal{L}_L} &\rightarrow \text{Aut}_{L_0, F} \left( H_{\text{cris}}^2(Y_{k_L}/L_0)(1) \right) & (p > 0). \end{aligned}$$

**Definition 8.5.** We define the  $\ell$ -adic and  $p$ -adic realisations of  $\alpha_{X, \mathcal{L}}^L$  to be

$$\begin{aligned} \beta_{X, \mathcal{L}, \ell}^L &:= i_\ell(\alpha_{X, \mathcal{L}}^L) \in H^1 \left( G, \text{Aut}_{G_{k_L}} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell) \right) \right) & (\ell \neq p) \\ \beta_{X, \mathcal{L}, p}^L &:= i_p(\alpha_{X, \mathcal{L}}^L) \in H^1 \left( G, \text{Aut}_{F, L_0} \left( H_{\text{cris}}^2(Y_{k_L}/L_0) \right) \right) & (p > 0) \end{aligned}$$

respectively.

We can now twist the cohomology groups  $H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)$  and  $H_{\text{cris}}^2(Y/K_0)$  by the classes  $\beta_{X, \mathcal{L}, \ell}^L$  and  $\beta_{X, \mathcal{L}, p}^L$ , respectively, via the general procedure outlined in Section 2.6 and Section 2.7. We thus obtain new  $G_k$ -modules and  $F$ -isocrystals

$$H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)^{\beta_{X, \mathcal{L}, \ell}^L} \quad \text{and} \quad H_{\text{cris}}^2(Y/K_0)^{\beta_{X, \mathcal{L}, p}^L},$$

respectively.

**Theorem 8.6.** *Let  $(X, \mathcal{L})$  be a polarized K3 surface over  $K$  satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, there are natural isomorphisms*

$$\begin{aligned} H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell) &\cong H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)^{\beta_{X, \mathcal{L}, \ell}^L} & (\ell \neq p) \\ \mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_p)) &\cong H_{\text{cris}}^2(Y/K_0)^{\beta_{X, \mathcal{L}, p}^L} & (p > 0) \end{aligned}$$

of  $G_k$ -modules and  $F$ -isocrystals over  $K_0$  respectively.

PROOF. This is in fact reasonably straightforward. Suppose that we have a smooth,  $\mathcal{L}_L$ -terminal model  $\mathcal{Y}$  for  $X_L$  over  $\mathcal{O}_L$ . This gives rise to a comparison isomorphism

$$\text{comp}_{\mathcal{Y}, \ell} : H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell),$$

which is  $G_{k_L}$ -equivariant, although not  $G_k$ -equivariant in general. We can therefore define a function

$$\beta_{\mathcal{Y}, \ell} : G_k \rightarrow \text{Aut}_{\mathbb{Q}_\ell} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell) \right)$$

measuring the “difference” of the two actions. Concretely, we have

$$\beta_{\mathcal{Y}, \ell}(\sigma) = \text{comp}_{\mathcal{Y}, \ell} \circ \sigma_g^* \circ \text{comp}_{\mathcal{Y}, \ell}^{-1} \circ (\sigma_s^*)^{-1},$$

where  $\sigma_g^*$  is the action on  $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)$  and  $\sigma_s^*$  that on  $H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)$ . Since  $\text{comp}_{\mathcal{Y}, \ell}$  is  $G_{k_L}$ -equivariant, this map factors through  $G$ . Moreover, by Theorem 8.4 it coincides with the  $\ell$ -adic realization  $i_\ell \circ \alpha_{\mathcal{Y}}$  of the cocycle

$$\alpha_{\mathcal{Y}} : G \rightarrow \mathcal{W}_{X_L, \mathcal{L}_L}$$

constructed in Section 7.2. In other words,  $\text{comp}_{\mathcal{Y}, \ell}$  becomes  $G_k$ -equivariant when we equip the left hand side with its natural action, and the right hand

side with the “twist” of the natural action via the cocycle  $i_\ell \circ \alpha_Y = \beta_{Y,\ell}$ , via the procedure described in Section 2.7.

In the  $p$ -adic case we argue similarly, using Theorem 8.4 to show that the crystalline comparison isomorphism

$$(H_{\text{ét}}^2(X_{\bar{L}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_L} \xrightarrow{\sim} H_{\text{cris}}^2(Y_{k_L}/L_0)$$

over  $L$  becomes  $G$ -equivariant when we endow the left hand side with its natural  $G$ -action and the right hand side with the twist of the natural  $G$ -action via the cocycle  $i_p \circ \alpha_Y$ . We can therefore apply Proposition 2.9.  $\square$

We could equally well work over  $\bar{k}$  as follows: using the  $G_k$ -equivariant homomorphisms

$$\begin{aligned} \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{\mathbb{Q}_\ell} (H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell(1))) & (\ell \neq p) \\ \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{K_0^{\text{nr}},F} (H_{\text{cris}}^2(Y/K_0)(1) \otimes_{K_0} K_0^{\text{nr}}) & (p > 0). \end{aligned}$$

and taking a Tate twist, we obtain cohomology classes

$$\begin{aligned} \beta_{X,\mathcal{L},\ell}^{\text{nr}} &:= i_\ell(\alpha_{X,\mathcal{L}}^{\text{nr}}) \in H^1(G_k, \text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell))) & (\ell \neq p) \\ \beta_{X,\mathcal{L},p}^{\text{nr}} &:= i_p(\alpha_{X,\mathcal{L}}^{\text{nr}}) \in H^1(G_k, \text{Aut}_{F,K_0^{\text{nr}}}(H_{\text{cris}}^2(Y/K_0) \otimes_{K_0} K_0^{\text{nr}})) & (p > 0) \end{aligned}$$

that we can twist  $H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)$  and  $H_{\text{cris}}^2(Y/K_0)$  by, respectively.

**Corollary 8.7.** *Let  $(X, \mathcal{L})$  be a polarized K3 surface over  $K$  satisfying  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, there are natural isomorphisms*

$$\begin{aligned} H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell) &\cong H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)^{\beta_{X,\mathcal{L},\ell}^{\text{nr}}} & (\ell \neq p) \\ \mathbb{D}_{\text{cris}}(H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_p)) &\cong H_{\text{cris}}^2(Y/K_0)^{\beta_{X,\mathcal{L},p}^{\text{nr}}} & (p > 0) \end{aligned}$$

of  $G_k$ -modules and  $F$ -isocrystals, respectively.

## 9. A NÉRON–OGG–SHAFAREVICH CRITERION FOR K3 SURFACES

We now come to the proof of our main result, namely Theorem 1.6 from the introduction. Of course, the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are just the smooth and proper base change and crystalline comparison theorems respectively, the real content is that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). Given Corollary 7.13 and Corollary 8.7, this amounts to showing that, given some ample line bundle  $\mathcal{L}$  on  $X$ , the induced maps

$$i_\ell : H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\text{nr}}) \rightarrow H^1(G_k, \text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell(1))))$$

for  $\ell \neq p$ , and

$$i_p : H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\text{nr}}) \rightarrow H^1(G_k, \text{Aut}_{K_0^{\text{nr}},F}(H_{\text{cris}}^2(Y/K_0)(1) \otimes_{K_0} K_0^{\text{nr}}))$$

when  $p > 0$ , have trivial kernel. Equivalently, if we have some finite and unramified Galois extension  $L/K$  with Galois group  $G$  and residue field extension  $k_L/k$ , over which  $X$  attains good reduction, it amounts to showing that the maps

$$\begin{aligned} i_\ell &: H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L}) \rightarrow H^1(G, \text{Aut}_{G_{k_L}}(H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell(1)))) & (\ell \neq p) \\ i_p &: H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L}) \rightarrow H^1(G, \text{Aut}_{L_0,F}(H_{\text{cris}}^2(Y_{k_L}/L_0)(1))) & (p > 0) \end{aligned}$$

have trivial kernel. Let

$$\begin{aligned} V_\ell &\subseteq H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell(1)) & (\ell \neq p) \\ V_p &\subseteq H_{\text{cris}}^2(Y_{k_L}/L_0)(1) & (p > 0) \end{aligned}$$

denote the subspaces fixed by  $G_{k_L}$  and Frobenius respectively. These are therefore  $G$ -representations in finite dimensional  $\mathbb{Q}_\ell$  (resp.  $\mathbb{Q}_p$ ) vector spaces, and there are natural  $G$ -equivariant maps

$$\begin{aligned} \text{Aut}_{G_{k_L}}(H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell(1))) &\rightarrow \text{GL}(V_\ell) & (\ell \neq p) \\ \text{Aut}_{L_0, F}(H_{\text{cris}}^2(Y_{k_L}/L_0)(1)) &\rightarrow \text{GL}(V_p) & (p > 0). \end{aligned}$$

It therefore suffices to show that the induced map

$$i_\ell : H^1(G, \mathcal{W}_{X_L, \mathcal{L}_L}) \rightarrow H^1(G, \text{GL}(V_\ell))$$

has trivial kernel for all  $\ell$  (including  $\ell = p$  when  $p > 0$ ).

**Proposition 9.1.** *For all primes  $\ell$  (including  $\ell = p$  when  $p > 0$ ), if  $\alpha \in H^1(G, \mathcal{W}_{X_L, \mathcal{L}_L})$  maps to the trivial class in*

$$H^1(G, \text{GL}(V_\ell)),$$

*then it maps to the trivial class in*

$$H^1(G, \text{GL}(\Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell})).$$

**Remark 9.2.** Note that there is no “naturally defined map” between  $\text{GL}(V_\ell)$  and  $\text{GL}(\Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell})$  in either direction, but the Weyl group  $\mathcal{W}_{X_L, \mathcal{L}_L}$  maps into both.

PROOF. This is entirely similar to the “reduction to the transitive case” part of the proof of Theorem 4.1. Consider the exact sequence

$$0 \rightarrow \Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell} \rightarrow V_\ell \rightarrow T_\ell \rightarrow 0$$

of  $G$ -representations. This is also compatible with the action of  $\mathcal{W}_{X_L, \mathcal{L}_L}$  and the induced action of the Weyl group on  $T_\ell$  is trivial. Using the results of Section 2.7, we know that  $\alpha$  being trivial in  $H^1(G, \text{GL}(V_\ell))$  is equivalent to the existence of an isomorphism  $V_\ell^\alpha \cong V_\ell$  as  $G$ -representations, whereas  $\alpha$  being trivial in  $H^1(G, \text{GL}(\Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell}))$  is equivalent to the existence of an isomorphism  $\Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell}^\alpha \cong \Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell}$  as  $G$ -representations.

Since the Weyl group acts trivially on  $T_\ell$ , we have  $T_\ell^\alpha \cong T_\ell$ , and therefore by semi-simplicity of the category of representations of the finite group  $G$ , we can conclude that  $V_\ell^\alpha \cong V_\ell$  if and only if  $\Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell}^\alpha \cong \Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell}$ .  $\square$

Thus, Theorem 1.6 amounts to showing that

$$H^1(G, \mathcal{W}_{X_L, \mathcal{L}_L}) \rightarrow H^1(G, \text{GL}(\Lambda_{X_L, \mathcal{L}_L, \mathbb{Q}_\ell}))$$

has trivial kernel. Since  $G$  is finite, this is exactly the content of Theorem 4.1.

10. INTEGRAL  $p$ -ADIC HODGE THEORY

So far we have worked with the rational étale cohomology of a K3 surface  $X$  over  $K$  with potentially good reduction. In this last section, we will explain how all the results carry over to integral étale cohomology, both  $\ell$ -adically and  $p$ -adically.

Assume therefore that  $X$  satisfies  $(\star)$  and the equivalent conditions of Theorem 1.1. Then, we have maps

$$\begin{aligned} \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{\mathbb{Q}_\ell} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell) \right) & (\ell \neq p) \\ \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{K_0^{\text{nr}},F} \left( H_{\text{cris}}^2(Y/K_0) \otimes_{K_0} K_0^{\text{nr}} \right) & (p > 0). \end{aligned}$$

and it follows from their definition (Lemma 8.3) that they factor through maps

$$\begin{aligned} i_\ell : \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{\mathbb{Z}_\ell} \left( H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell) \right) & (\ell \neq p) \\ i_p : \mathcal{W}_{X,\mathcal{L}}^{\text{nr}} &\rightarrow \text{Aut}_{W^{\text{nr}},F} \left( H_{\text{cris}}^2(Y/W) \otimes_W W^{\text{nr}} \right) & (p > 0), \end{aligned}$$

where  $W^{\text{nr}}$  denote the ring of integers of  $K_0^{\text{nr}}$ . Thus, we could equally well define cohomology classes

$$\begin{aligned} \beta_{X,\mathcal{L},\ell}^{\text{nr}} := i_\ell(\alpha_{X,\mathcal{L}}^{\text{nr}}) &\in H^1(G_k, \text{Aut}_{\mathbb{Z}_\ell}(H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell))) & (\ell \neq p) \\ \beta_{X,\mathcal{L},p}^{\text{nr}} := i_p(\alpha_{X,\mathcal{L}}^{\text{nr}}) &\in H^1(G_k, \text{Aut}_{F,W^{\text{nr}}}(H_{\text{cris}}^2(Y/W) \otimes_W W^{\text{nr}})) & (p > 0) \end{aligned}$$

on the integral level, and we can twist

$$H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell) \quad \text{and} \quad H_{\text{cris}}^2(Y/W)$$

by these classes to obtain new  $G_k$ -modules and  $F$ -crystals

$$H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell)^{\beta_{X,\mathcal{L},\ell}^{\text{nr}}} \quad \text{and} \quad H_{\text{cris}}^2(Y/W)^{\beta_{X,\mathcal{L},p}^{\text{nr}}},$$

respectively.

**PROOF OF THEOREM 1.7.** The second statement is a trivial consequence of Theorem 1.6 and the first follows easily from the proof of Theorem 8.6 as follows: we need to show that if we choose an unramified extension and Galois extension  $L/K$  with Galois group  $G$  and residue field  $k_L$ , large enough so that  $X$  has good reduction over  $L$ , then the induced comparison isomorphisms

$$\begin{aligned} H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Z}_\ell) &\cong H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Z}_\ell) & (\ell \neq p) \\ \text{BK}_{O_L}(H_{\text{ét}}^2(X_{\bar{L}}, \mathbb{Z}_p) \otimes W(k_L)) &\cong H_{\text{cris}}^2(Y_{k_L}/W(k_L)) & (p > 0) \end{aligned}$$

provided by the smooth and proper base change theorem and Theorem 2.4, respectively, identify the natural  $G_k$ -action (resp.  $G$ -action) on the left hand side with the “ $\beta$ -twisted”  $G_k$ -action (resp.  $G$ -action) on the right hand side. But we can embed the left hand side equivariantly in  $H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell)$  (resp.  $(H_{\text{ét}}^2(X_{\bar{L}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_L}$ ), and the right hand side equivariantly in  $H_{\text{ét}}^2(Y_{\bar{k}}, \mathbb{Q}_\ell)$  (resp.  $H_{\text{cris}}^2(Y_{k_L}/L_0)$ ), at least after taking a Frobenius pull-back in the  $p$ -adic case. The result therefore follows from the proof of Theorem 8.6.  $\square$



APPENDIX A. CALCULATING THE KERNEL OF  
 $H^1(\mathbb{Z}/3, \mathcal{W}_{D_4}) \rightarrow H^1(\mathbb{Z}/3, \mathrm{GL}_4(F))$

Here, we reproduce some [\[SAGE\]](#) code that calculates an upper bound for the size of the kernel of the natural map

$$H^1(\mathbb{Z}/3, \mathcal{W}_{D_4}) \rightarrow H^1(\mathbb{Z}/3, \mathrm{GL}_4(F)),$$

for the  $\mathbb{Z}/3$ -action on  $D_4$  and  $F$  a field of characteristic 0. It is explained during the proof of Proposition 4.3.

```
# define the basic reflections s1, s2, s3, s4
s = [];
s.append(matrix([[ -1, 1, 0, 0 ], [0, 1, 0, 0 ],
                 [ 0, 0, 1, 0 ], [0, 0, 0, 1 ]]));
s.append(matrix([[ 1, 0, 0, 0 ], [1, -1, 1, 1 ],
                 [ 0, 0, 1, 0 ], [0, 0, 0, 1 ]]));
s.append(matrix([[ 1, 0, 0, 0 ], [0, 1, 0, 0 ],
                 [ 0, 1, -1, 0 ], [0, 0, 0, 1 ]]));
s.append(matrix([[ 1, 0, 0, 0 ], [0, 1, 0, 0 ],
                 [ 0, 0, 1, 0 ], [0, 1, 0, -1 ]]));

# create the Weyl group as a set of matrices
W = [matrix.identity(4)];
while len(W) < 192:
    for i in range(4):
        for w in W:
            if w*s[i] not in W:
                W.append(w*s[i]);
            if len(W) == 192:
                break;
        if len(W) == 192:
            break;

# check the Weyl group is actually a group
for i in range(4):
    for w in W:
        assert w*s[i] in W;
for w in W:
    assert w.inverse() in W;
assert matrix.identity(4) in W;

# define the matrix giving the action of a generator
# of Z/3 on the root lattice
g = matrix([[0, 0, 0, 1 ], [0, 1, 0, 0 ], [1, 0, 0, 0 ], [0, 0, 1, 0 ]]);
```

```

# calculate the set of cocycles
cocycle = [];
for w in W:
    if w*g*w*g*w*g == matrix.identity(4):
        cocycle.append(w);

# calculate a representing set of cohomology classes
cohom = [];
for w in cocycle:
    i=0
    for r in W:
        i=i+1;
        M = r.inverse()*w*g*r*g.inverse();
        if M in cohom:
            break;
    if i == len(W):
        cohom.append(w);

# check that 1 is an eigenvalue of g
assert 1 in g.eigenvalues();

# calculate an upper bound for the kernel by
# throwing out all w such that w*g does not
# have 1 as an eigenvalue
kernel=[];
for w in cohom:
    N = w*g;
    if 1 in N.eigenvalues():
        kernel.append(w);

print('#_kernel_<=_%d' % len(kernel));

# prints the non-trivial cohomology classes
print('non-trivial_cohomology_class_representatives:')
for w in cohom:
    if w != matrix.identity(4):
        w;

```

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