Canonical models of surfaces with $K^2 = 7$ and $p_g = 4$

by

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Declarations

I declare that the content of this thesis is my own original work except where clearly stated. I confirm that this thesis has not been submitted anywhere else for any degree.

Abstract

Geometrically, the main goal of this thesis is to refine the classification of minimal surfaces S with $K_S^2 = 7$ and $p_g = 4$ due to Ingrid Bauer and published in her monograph *Surfaces with* $K_S^2 = 7$ *and* $p_g = 4$ (cf. [Bauer]). She found that they belong to 10 families according to the behaviour of the canonical map φ_{K_S}. The 10 families form 3 irreducible components of moduli, but the details of how this happens remained unknown except for a few particular cases.

Our treatment consists in studying the abstract canonical model Proj $R(S, K_S)$, where $R(S, K_S) := \bigoplus_{n\geq 0} H^0(S, \mathcal{O}_S(nK_S))$ is the pluricanonical ring. Except when $|K_S|$ is base point free, these rings are Gorenstein of codimension ≥ 4. We show that the only previously known deformation family of such rings (constructed by Bauer, Catanese and Pignatelli in [Bauer et al]) relating the 2 families with φ_{K_S} birational can be recovered using basic arguments about halfcanonical curves. Our techniques also allow us to construct new explicit flat families for cases on which φ_{K_S} is not birational. In particular, we construct a 1-parameter flat family of Gorenstein rings with general fibre of codimension 4 and special fibre of codimension 6. At the end we discuss possible applications of our methods to the cases on which $|K_S|$ defines a 2-to-1 map to a quadratic surface. We conjecture that the moduli space of surfaces with $K_S^2 = 7$ and $p_g = 4$ is connected.

Chapter 1

Introduction and Preliminaries

This chapter introduces the notation and well known results used in the rest of the thesis. It also discusses the motivation leading to the geometric problems we are interested in.

1.1 Notation and conventions

The base field will always be the field of complex numbers $\mathbb{C}$, most of the times denoted simply by k.

A variety is an integral separated scheme of finite type over k; a curve is a variety of dimension 1, a surface is a variety of dimension 2, etc. Usually I will use the letters C and S for referring to a curve and a surface respectively.

I will write $\mathbb{P}^n$ and $\mathbb{A}^n$ for the $n-$dimensional projective and affine spaces over k. In this work however, the varieties will be more conveniently embedded in a weighted projective space (w.p.s.); the notation $\mathbb{P}^n(w_1^{i_1}, \dots, w_s^{i_s})$, where $\sum_{j=1}^s i_j = n+1$, stands for the w.p.s. corresponding to the Proj of the ring $k[x_0, \dots, x_n]$ with its grading induced by the $k^\times-$action on $k^{n+1} \smallsetminus \{0\}$ given by

$$\lambda \cdot (a_0, \dots, a_n) \mapsto (\lambda^{w_1} a_0, \dots, \lambda^{w_1} a_{i_1}, \dots, \lambda^{w_s} a_{n+1-i_s}, \dots, \lambda^{w_s} a_n).$$

Now, let X be a nonsingular variety, $Y \subset X$ a nonsingular hypersurface, D, D_1, D_2 divisors on X and write $\mathcal{O}_X(\cdot)$ for the corresponding invertible sheaf. Then I write:

- $D_1 \sim D_2$ for linear equivalence of divisors.

- $H^0(n_1D_1 + n_2D_2)$ where $n_1, n_2 \in \mathbb{Z}$, as a short for

$$H^0(X, \mathcal{O}_X(D_1)^{\otimes n_1} \otimes \mathcal{O}_X(D_2)^{\otimes n_2}),$$

whenever X is clear from the context.

- $D|_Y$ or sometimes simply D_Y for the restriction of the divisor D to Y.

- In case $X = S$ is a surface, D_1D_2 denotes the intersection number.

- In case $X = C$ is a curve, $\deg(D)$ denotes de degree of the divisor D.

- $h^i(D)$ stands for the dimension of $H^0(X, \mathcal{O}_X(D))$ as a k–vector space.

- $\chi(D)$ for the Euler characteristic of the sheaf $\mathcal{O}_X(D)$, that is, $\sum_{i=0}^n (-1)^i h^i(D)$.

- $|D|$ for the linear system in which D moves.

- $r(D)$ or simply r if there it is clear by the context, for the dimension of $|D|$, that is, $h^0(D) - 1$.

- φ_D for the rational map $X \overset{\varphi_D}{\dashrightarrow} \mathbb{P}^r$, in case $r \geq 1$.

- g^r_d for a linear system on a curve, of degree d and dimension r.

- K_X for a canonical divisor of X.

- If $X = S$ is a surface, I write $p_g = p_g(S) = h^0(K_S)$ for its geometric genus and $q = q(S) = h^1(K_S)$ for its irregularity. If $X = C$ is a curve, usually $g = g(C) = h^0(K_C)$ will denote its genus. Notice that Serre duality allows us to define in an equivalent way $p_g = h^2(\mathcal{O}_S)$, $q = h^1(\mathcal{O}_S)$ and similarly for curves.

- $R(X, D)$ stands for the full graded ring of sections of the divisorial sheaves $\mathcal{O}_X(nD)$:

$$R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).$$

In particular $R(X, K_X)$ and $R(X, -K_X)$ are called the canonical and anti-canonical rings respectively. If D is a divisor satisfying $2D \sim K_X$ (D is a theta characteristic, semicanonical/halfcanonical divisor, etc.), the ring $R(X, D)$ is called a halfcanonical ring on X.

- The notation $S^a(x_1, \ldots, x_n)$ stands for the set of monomials of degree a on the x_is.

1.2 Basic facts and formulas

Here I state the results that will be used more frequently during the thesis. Since all of them are very well known, I omit giving any specific reference for their proofs. The reason for citing them is to make things clear afterwards, when I will simply write *by Clifford's theorem*, etc.

Theorem 1.2.1. (Riemann-Roch for curves) *Let C be a nonsingular projective curve and D a divisor on it. Then*

$$\chi(D) = 1 - g + \deg(D).$$

Remark The equality from previous theorem can be rewritten using Serre's duality as:

$$h^0(D) - h^0(K_C - D) = 1 - g + \deg(D).$$

Theorem 1.2.2. (Riemann-Roch for surfaces) *Let S be a nonsingular projective surface and D a divisor on it. Then*

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}D(D - K_S).$$

I will write simply R-R for referring to a Riemann-Roch theorem.

Theorem 1.2.3. (Adjunction formula) *Let $X \subset Y$ be a nonsingular hypersurface on a nonsingular variety. Then*

$$K_X = (K_Y + X)|_X.$$

More generally, if Y is a Cohen-Macaulay variety and D is an effective Cartier divisor on Y, then

$$\omega_D = \omega_Y(D) \otimes \mathcal{O}_D$$

is a dualizing sheaf for D.

Corollary 1.2.4. *If $C \subset S$ is a nonsingular curve contained in a nonsingular surface, then:*

$$2g - 2 = (K_S + C)C$$

and the same is true for any curve C if we replace g for the arithmetic genus $p_a(C)$.

Theorem 1.2.5. (Bertini's theorem) *Let X be a nonsingular projective variety and $|D|$ a linear system on X with no fixed part. Then the general member of $|D|$ can only have singularities at the base locus.*

Theorem 1.2.6. (Castelnuovo's base point free pencil trick) *Let C be a smooth projective curve, L an invertible sheaf on C and F a torsion free $\mathcal{O}_C$−module. Suppose that $\{s_1, s_2\}$ is a linearly independent set of sections of L and denote by V the subspace of $H^0(C, L)$ it spans. Then the kernel of the cup-product map*

$$V \otimes H^0(C, F) \longrightarrow H^0(C, F \otimes L)$$

is isomorphic to $H^0(C, F \otimes L^{-1}(B))$, where B is the base locus of the pencil spanned by s_1 and s_2.

Theorem 1.2.7. (Clifford's theorem) *Let C be a smooth projective genus g curve with an effective divisor D of degree d with $d \leq 2g - 1$. Then*

$$r(D) \leq d/2.$$

Moreover, if the equality holds, then either D is zero, D is a canonical divisor or C is hyperelliptic and D is linearly equivalent to a multiple of a hyperelliptic divisor.

Remark A curve is called hyperelliptic if it admits a hyperelliptic divisor, that is, a divisor D with $\deg(D) = h^0(D) = 2$.

Theorem 1.2.8. (Max Noether's theorem) *If C is not hyperelliptic, then the morphisms*

$$Sym^{\ell} H^0(K_C) \longrightarrow H^0(\ell K_C)$$

are surjective for $\ell \geq 1$.

1.3 Classical Surface Theory background

Let S be a projective surface. Recall that the *Kodaira dimension* of S, denoted by $\kappa(S)$, is defined to be

$$\kappa(S) := \begin{cases} \mathrm{Tr}_k[R(S, K_S)] - 1 & \text{if } R(S, K_S) \ncong k \\ -\infty & \text{if } R(S, K_S) \cong k, \end{cases}$$

where $\mathrm{Tr}_k[R(S, K_S)]$ is the transcendence degree (over $\mathbb{C}$) of the canonical ring.

The starting point of the Kodaira-Enriques classification of algebraic surfaces is noticing the possible values for $\kappa(S)$ are $-\infty, 0, 1$ and 2; surfaces corresponding to the classes defined by the first 3 values are nowadays well understood, whereas there is still a huge number of very hard open questions about surfaces with $\kappa(S) = 2$.

Definition S is said to be a surface of general type if $\kappa(S) = 2$.

The next step in the classification is given by the following classic result:

Theorem 1.3.1. (Castenuovo's contraction theorem) *Let $C \subset S$ be a curve on a projective surface, such that $C \cong \mathbb{P}^1$ and $C^2 = -1$. Then there exists a nonsingular surface S_0 and a morphism*

$$f : S \longrightarrow S_0,$$

satisfying:

1. *$f(C) = P$ a point in S_0.*
2. *$f : S \setminus \{C\} \longrightarrow S \setminus \{P\}$ is an isomorphism.*

So, the following definitions make sense:

Definition Keeping the notation from previous theorem:

1. C is said to be an *exceptional curve of the first kind.*
2. If S is smooth and contains no exceptional curves of the first kind, then is said to be a *minimal model.*

The fact that every surface of general type can be obtained from a minimal model after blowing up a finite subset of smooth points, and moreover such minimal model is unique up to isomorphism, can be consulted in a number of well known references, (such as Chapter III, sections 4.4-4.6 of [BHPV]), and is one of the main guiding results on which the celebrated Minimal Model Program for varieties of higher dimensions is inspired.

Thus, classifying surfaces of general type leads to the study of the unique minimal model contained in each birational class. The most important numerical data associated to a minimal surface S is the triplet of integers formed by its geometrical genus p_g, the irregularity q, and the self-intersection number K_S^2. With the exception of K_S^2, all of these are birational invariants.

Definition Let S be a minimal surface of general type. The *numerical type* of S is the triplet (K_S^2, p_g, q).

Remark The numerical type determines every other classical numerical invariant of S:

1. The Euler characteristic of the structure sheaf is by definition:
$$\chi(\mathcal{O}_S) = 1 - q + p_g.$$
2. The topological Euler characteristic is (by a classical theorem of M. Noether):
$$e(S) = 12\chi(\mathcal{O}_S) - K_S^2.$$
3. The plurigenera $P_m(S) := h^0(mK_S)$, $m \geq 2$ is (by R-R and Mumford's vanishing theorem):
$$P_m(S) = \chi(\mathcal{O}_S) + \binom{m}{2} K_S^2.$$

For every minimal surface of general type S, there exists a unique normal surface $X \subset \mathbb{P}^{P_5(S)-1}$ birational to S that is obtained contracting to points all the $(-2)-$curves of S (that is, curves $E \subset S$ such that $E \cong \mathbb{P}^1$ and $E^2 = -2$). Such an X is called the *canonical model* of S (cf. [Bombieri]) and in this context we know the following result:

Theorem 1.3.2. (cf.[Gieseker]) *There exists a quasi projective coarse moduli scheme for canonical models of surfaces of general type S with fixed K_S^2 and $e(S)$.*

In practice, the approach is to fix a numerical type (K_S^2, p_g, q) and consider the subscheme $\mathcal{M}_{K_S^2,p_g,q}$ of the scheme whose existence is given by previous theorem. $\mathcal{M}_{K_S^2,p_g,q}$ is thus, a quasi projective scheme, in particular, it has finitely many irreducible components. The ultimate goal in surface theory is to completely describe $\mathcal{M}_{K_S^2,p_g,q}$ for as many numerical types as possible.

Remark *(cf.[Debarre82],[Debarre83],[Miyaoka] and [Yau])* Of course there are restrictions for the possible numerical type of a surface of general type; some well known inequalities involving these numbers are the following:

- $K_S^2 \geq 1$.
- *(Noether)* $K_S^2 \geq 2p_g - 4$.
- *(Debarre)* if $q \geq 1$, $K_S^2 \geq 2p_g$.
- *(Miyaoka-Yau)* $K_S^2 \leq 9\chi(\mathcal{O}_S)$.

1.4 Previous work and statement of the problem

The case we want to attack were first seriously studied by Federigo Enriques in his famous book on algebraic surfaces [Enriques]. There he focuses on the effective construction of surfaces S, in particular, those with $p_g = 4$ and whose canonical map φ_{K_S} is a birational morphism onto a singular surface $\Sigma \subset \mathbb{P}^3$. The first open case, corresponding to $K_S^2 = 7$, has attracted the attention of several other mathematicians besides Enriques himself; Franchetta ([Franchetta]), Maxwell ([Maxwell]) and Kodaira ([Kodaira]) to mention some.

Remark By Debarre's inequality , it follows that if S is a minimal surface of general type with $p_g = 4$ and $K_S^2 = 7$, then S is regular (that is, $q = 0$). Thus in the subsequent, we will write simply $\mathfrak{M}_{7,4} := \mathcal{M}_{K_S^2=7,p_g=4,q=0}$ for the corresponding moduli space.

However, it was only until the beginnings of the 2000s when Ingrid Bauer, in her monograph [Bauer], gave a complete classification of surfaces in $\mathfrak{M}_{7,4}$, separating them into 10 families as stated in the following theorem (for a detailed version, consult Theorem 5.1, page 51 of [Bauer]):

Theorem 1.4.1. *Let S be a smooth minimal surface with $K_S^2 = 7$ and $p_g = 4$. Then S belongs to exactly one of the following families:*

- ***Family (0)**: $|K_S|$ is base point free and the canonical map φ_{K_S} is a birational morphism from S onto a surface of degree 7 in $\mathbb{P}^3$. These surfaces form an open unirational, irreducible set of dimension 36 in the moduli space $\mathfrak{M}_{7,4}$.*
- ***Family (I)**: $|K_S|$ has exactly one simple base point P_1. Let $\pi_1 : \widetilde{S} \longrightarrow S$ be the blowup of S at P_1 and let $|\widetilde{H}|$ be the movable part of $|K_{\widetilde{S}}|$. Then we have the following subfamilies:*
  - ***Family (I.1)**: φ_{K_S} is a birational map. Then $\Sigma := \varphi_{\widetilde{H}}(\widetilde{S}) \subset \mathbb{P}^3$ is a surface of degree 6 with the following properties:*
    - *(a) The double curve $\Gamma \subset \Sigma$ is a plane conic.*
    - *(b) If $\gamma \subset \mathbb{P}^3$ is the plane containing Γ, then Σ has a generalised tacnode $o \in \gamma \setminus \Gamma$ with tacnodal plane $\alpha \neq \gamma$.*
    - *(c) $\varphi_{\widetilde{H}}(E) = \alpha \cap \gamma$, where E is the exceptional divisor of the blowup of $P_1 \in S$.*

    Apart from rational double points, Σ does not have other singularities. The surfaces of subfamily (I.1) form an irreducible unirational set of dimension 35 in $\mathfrak{M}_{7,4}$.

- *Family (I.2)*: φ_{K_S} *is a map of degree 2 onto a cubic in* $\mathbb{P}^3$ *that has only isolated singularities. These surfaces form an irreducible unirational set of dimension 33 in* $\mathfrak{M}_{7,4}$.
- *Family (I.3)*: φ_{K_S} *is a map of degree three onto a quadric in* $\mathbb{P}^3$. *These surfaces also form an irreducible unirational set of dimension 35 in* $\mathfrak{M}_{7,4}$.

• ***Family (III)****:* $|K_S|$ *has exactly three pairwise different simple base points* $P_1, P_2, P_3 \in S$. *Let* $\pi' : \widetilde{S}' \longrightarrow S$ *be the blowup of* S *at* P_1, P_2, P_3, *let* $|\widetilde{H}'|$ *be the movable part of* $|K_{\widetilde{S}'}|$ *and let* $E_i := \pi'^{-1}(P_i)$ *be the exceptional divisors. Then we have the following subfamilies:*

  - ***Family (III.α)****:* $\varphi_{\widetilde{H}'}(\widetilde{S}') = \mathbb{P}^1 \times \mathbb{P}^1$, *(embedded in* $\mathbb{P}^3$ *by the linear system* $|\Gamma_1 + \Gamma_2|$*; where* $\Gamma_1^2 = \Gamma_2^2 = 0$ *and* $\Gamma_1\Gamma_2 = 1$*), and* $\varphi_{\widetilde{H}'}(E_1)$, $\varphi_{\widetilde{H}'}(E_2)$, $\varphi_{\widetilde{H}'}(E_3)$ *are pairwise disjoint lines. These surfaces form an irreducible unirational set of dimension 36 in* $\mathfrak{M}_{7,4}$.
  - ***Family (III.β)****:* $\varphi_{\widetilde{H}'}(\widetilde{S}') = \mathbb{P}^1 \times \mathbb{P}^1$ *and without restriction* $\varphi_{\widetilde{H}'}(E_1), \varphi_{\widetilde{H}'}(E_2) \equiv \Gamma_1$, $\varphi_{\widetilde{H}'}(E_3) \equiv \Gamma_2$. *These surfaces form an irreducible unirational set of dimension 38 in* $\mathfrak{M}_{7,4}$.
  - ***Family (III.γ)****:* $\varphi_{\widetilde{H}'}(\widetilde{S}')$ *is the quadric cone. These surfaces form an irreducible unirational set of dimension 35 in* $\mathfrak{M}_{7,4}$ *and they are obtained as degenerations of surfaces of family (III.α).*

• ***Family (F)****:* $|K_S|$ *has a non trivial fixed part* F, *with* $F^2 = -2$ *and* $K_S F = 0$. *Then, if* $|H|$ *is the movable part of* $|K_S|$, $|H|$ *has exactly one simple base point* X. *Let* $\pi : \widetilde{S} \longrightarrow S$ *be the blowup of* S *at* X *and let* $|\widetilde{H}|$ *be the movable part of* $|K_{\widetilde{S}}|$. *Then* $\varphi_{\widetilde{H}} : \widetilde{S} \longrightarrow Q$ *is a morphism of degree 2 onto a quadric and* $\varphi_{\widetilde{H}}(F)$ *is a line. We have the following 2 subfamilies:*

  - ***Family (F.1)****:* $\varphi_{\widetilde{H}}(\widetilde{S}) = \mathbb{P}^1 \times \mathbb{P}^1$. *These surfaces form an irreducible unirational set of dimension 35 in* $\mathfrak{M}_{7,4}$.
  - ***Family (F.2)****:* $\varphi_{\widetilde{H}}(\widetilde{S})$ *is the quadric cone. These surfaces form an irreducible unirational set of dimension 34 in* $\mathfrak{M}_{7,4}$ *and they are obtained as degenerations of surfaces of family (F.1).*

• ***Family (F')****:* $|K_S| = |H| + F'$ *and here the fixed part satisfies* $F'^2 = -1$, $K_S F' = 1$ *and* $HF' = 2$. *In this case* $|H|$ *is base point free and* $\varphi_H(S) \subset \mathbb{P}^3$ *is the quadric cone. These surfaces form an irreducible unirational set of dimension 37 in* $\mathfrak{M}_{7,4}$ *and they are obtained as degenerations of surfaces of family (III.β).*

Remark In the above Theorem the roman digits indicate the number of base points of the canonical system $|K_S|$, whereas the arabic numbers in the case (I) indicate the degree of the canonical map.

By Kuranishi's theorem (cf. [Kuranishi]), the dimension of the moduli space of all surfaces with $K^2 = 7$ and $p_g = 4$ in a generic point is at least $10\chi - 2K^2 = 36$. Whence, the moduli space $\mathfrak{M}_{7,4}$ has 3 irreducible components. There are two components of dimension 36 that Bauer denotes by $\overline{\mathfrak{M}_{(0)}}$ and $\overline{\mathfrak{M}_{(III.\alpha)}}$. As the notation suggests, the components are the Zariski closure of the sets of surfaces of types (0) and $(III.\alpha)$ respectively. The third component, $\overline{\mathfrak{M}_{(III.\beta)}}$, has dimension 38 and is the Zariski closure of the set of surfaces of type $(III.\beta)$.

The main result of Bauer concerning the moduli space $\mathfrak{M}_{7,4}$ is the following:

Theorem 1.4.2. *1. The decomposition of the moduli space $\mathfrak{M}_{7,4}$ in irreducible components is:*

$$\mathfrak{M}_{7,4} = \overline{\mathfrak{M}_{(0)}} \cup \overline{\mathfrak{M}_{(III.\alpha)}} \cup \overline{\mathfrak{M}_{(III.\beta)}}.$$

2. The surfaces of type $(III.\gamma)$ are contained in $\overline{\mathfrak{M}_{(0)}} \cap \overline{\mathfrak{M}_{(III.\alpha)}}$.

3. The two irreducible components $\overline{\mathfrak{M}_{(III.\alpha)}}$ and $\overline{\mathfrak{M}_{(III.\beta)}}$ have empty intersection.

Remark It follows that $\mathfrak{M}_{7,4}$ has at most 2 connected components.

The next breakthrough came with the joint work of Bauer, Fabrizio Catanese and Roberto Pignatelli. By the results obtained previously by Bauer, it follows that is possible to find small deformations of surfaces in (I.1) falling into some of the bigger pieces of moduli. In their paper [Bauer et al], they show how a surface in family (I.1) deforms into one in family (0). The main tool used to find such a deformation is the antisymmetric-extrasymmetric format, first discovered by Duncan Dicks and Miles Reid, when they studied surfaces with $K^2 = 4$ and $p_g = 3$. This format is useful for the presentation of certain Gorenstein rings of codimension 4 and has been used ever since many times to construct explicit deformation families of algebraic varieties. The format allows to deform the ring into one with codimension 3, which in our situation corresponds to a canonical surface of family (0). From the graded ring perspective, the difficulties in getting any other deformation and consequently a clearer picture of $\mathfrak{M}_{7,4}$ are the following:

1. Techniques for constructing the pluricanonical ring $R(S, K_S)$ such as the Eisenbud-Buchsbaum structure theorem for codimension 3 Gorenstein rings

(useful for family (0) of theorem 1.4.1) or the one given in [Bauer et al] (useful for family (I.1)) cannot be applied in any of the remaining cases.

2. Even if one manages to construct the pluricanonical ring, there are no formats that allow us to deform it if its codimension is greater than 4.

Thus, the main problem of this thesis is:

Problem 1.4.1. *Can you obtain any other explicit deformation family of rings relevant to surfaces belonging to the cases described in Theorem 1.4.1?*

1.5 Contents of the thesis

1.5.1 Main results

The main original contributions of this thesis are the following:

- (cf. Theorem 5.1.1). Consider the graded ring $R := k[x_0, x_1, x_2, x_3, y_1, y_2, z]/I$, where $\deg x_i = 1$, $\deg y_j = 2$, $\deg z = 3$ and I is the homogeneous ideal generated by 9 elements defined as follows:

$$\text{Let } \mathcal{A} := \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 + a_1 x_0 x_2 + a_2 x_0^2 & \mathcal{Q} & y_1 & z \end{pmatrix},$$

$$\text{with } \mathcal{Q} := x_3^2 + a_3 x_1 x_2 + a_4 x_1^2 + (a_5 x_1 + a_6 x_2 + a_7 x_3) x_0 + a_8 x_0^2$$

$$\text{and let } \mathcal{M} := \begin{pmatrix} \mathcal{Q}_1 & \frac{1}{2} a_{19} x_0^2 & \frac{1}{2}(a_{20} x_0 x_2 + a_{21} x_0^2) & \frac{1}{2} a_{22} x_0 \\ \frac{1}{2} a_{19} x_0^2 & \mathcal{Q}_2 & \frac{1}{2} a_{27} x_0^2 & \frac{1}{2} a_{28} x_0 \\ \frac{1}{2}(a_{20} x_0 x_2 + a_{21} x_0^2) & \frac{1}{2} a_{27} x_0^2 & \mathcal{Q}_3 & \frac{1}{2} a_{35} x_0 \\ \frac{1}{2} a_{22} x_0 & \frac{1}{2} a_{28} x_0 & \frac{1}{2} a_{35} x_0 & -1 \end{pmatrix},$$

$$\text{with } \begin{cases} \mathcal{Q}_1 := & a_9 x_1^2 + a_{10} x_1 x_2 + a_{11} x_1 x_3 + a_{12} x_2^2 + a_{13} x_2 x_3 + a_{14} x_3^2 \\ & + (a_{15} x_1 + a_{16} x_2 + a_{17} x_3) x_0 + a_{18} x_0^2, \\ \mathcal{Q}_2 := & (a_{23} x_1 + a_{24} x_2 + a_{25} x_3) x_0 + a_{26} x_0^2, \\ \mathcal{Q}_3 := & a_{29} x_1 x_2 + a_{30} y_1 + (a_{31} x_1 + a_{32} x_2 + a_{33} x_3) x_0 + a_{34} x_0^2. \end{cases}$$

  The first 6 generators are the 2×2 minors of $\mathcal{A}$ and the last 3 are the distinct entries of the symmetric 2×2 matrix $\mathcal{A}\mathcal{M}({}^T\mathcal{A})$. Then, for a general choice of parameters $a_i \in \mathbb{C}$, $1 \le i \le 35$, $R = R(S, K_S)$ where S is a surface of general type with $p_g = 4$ and $K^2 = 7$ whose canonical map is 2-to-1 onto a cubic surface in $\mathbb{P}^3$, that is, S is a surface belonging to subfamily (I.2) of Theorem 1.4.1.

- In Theorem 5.1.3 we prove that the rings described in the above result are degenerations of canonical rings of surfaces in the stratum (I.1).
- In Section 5.2 we give a new and simpler proof of the main result obtained by Bauer, Catanese and Pignatelli in their paper [Bauer et al]. That is, every surface in stratum (I.1) can be obtained as a degeneration of a surface in stratum (0).
- Let $\widetilde{(I.3)}$ be the stratum formed by surfaces defined as Proj R where R is a ring of the form:

  $$R := k[x_0, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2]/I,$$

  with x_i, y_j and z_ℓ of degrees 1, 2 and 3 respectively. $\overline{P}$ is the degree 4 homogeneous form:

  $$\overline{P} := a_1 x_0^3 x_2 + x_0^2 A_0 + x_1^2 A_1 + x_2^2 A_2 + x_3^2 A_3 + x_1 x_2 B_1 + x_2 x_3 B_2, \tag{1.1}$$

  where

  $$\begin{aligned}
  A_0 &:= a_2 x_1^2 + a_3 x_1 x_2 + a_4 x_1 x_3 + a_5 x_2^2 + a_6 x_2 x_3 + a_7 x_3^2,\\
  A_1 &:= a_8 x_0 x_1 + a_9 x_0 x_2 + a_{10} x_1^2 + a_{11} x_1 x_2 + a_{12} x_2^2 + a_{13} y_1 + a_{14} y_2 + a_{15} y_3,\\
  A_2 &:= a_{16} x_0 x_1 + a_{17} x_0 x_2 + a_{18} x_0 x_3 + a_{19} x_1 x_2 + a_{20} x_1 x_3 + a_{21} x_2 x_3 + a_{22} x_3^2 + a_{23} y_3,\\
  A_3 &:= a_{24} x_0 x_2 + a_{25} x_0 x_3 + a_{26} x_2 x_3 + a_{27} x_3^2 + a_{28} y_3\\
  B_1 &:= a_{29} y_2 + a_{30} y_3,\\
  B_2 &:= a_{31} y_3,
  \end{aligned}$$

  $a_1, \dots, a_{31} \in \mathbb{C}$ and I is generated by the 2×2 minors of

  $$\begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & \overline{P} \end{pmatrix}.$$

  Then every surface in $\widetilde{(I.3)}$ has a small deformation to either a surface in $(I.3)$ or $(I.1)$ (cf. §6.3 and Theorem 6.3.2).

If we make the following definition (cf. §2.3):

Definition Let $(\star)$ and $(\bullet)$ be two families of surfaces according to the classification of Theorem 1.4.1.

1. If I write $(\star) \longrightarrow (\bullet)$, I mean that there exists a flat family with base a small disc $\Delta_t \subset \mathbb{C}$ whose special fibre is of type $\bullet$, whose general fibre is of type $\star$ and this family exists for every point of a stratum $\bullet$.

2. If I write $(\star) \dashrightarrow (\bullet)$, I mean that there exists a flat family with base a small disc $\Delta_t \subset \mathbb{C}$ whose special fibre is of type $\bullet$, whose general fibre is of type $\star$ and this family exists only for some points of a stratum $\bullet$.

Our main results together with the conjectures made in the last chapter of this thesis lead to the following diagram:

$$\begin{tikzcd}
 & & (I.2) & \widetilde{(I.3)} & (I.3) \arrow[l] \\
 & (III.\gamma) & & (I.1) \arrow[ul] \arrow[u] & \\
 & & (III.\alpha) \arrow[ul] \arrow[dl] & (0) \arrow[u] \arrow[l, dashed] \arrow[r, dashed] \arrow[ull, bend right] & (III.\beta) \arrow[d] \\
(F.2) & (F.1) \arrow[l] & & & (F')
\end{tikzcd}$$

Here, the purple arrows are conjectured to exist, whereas the 3 different colours of names of the strata indicate the irreducible component of the moduli space they belong to (cf. §7.1).

1.5.2 Brief description of the chapters

- Chapter 2 describes Reid's approach to the problem. In short terms, Reid's philosophy is that studying surfaces is, because of the hyperplane section principle, closely related to the study curves and threefolds. If one is to find the deformations mentioned in Problem 1.4.1, one should study the canonical ring $R(S, K_S)$, whose algebraic structure is closely related with that of the halfcanonical ring $R(C, K_S|_C)$, where $C \in |K_S|$ is a canonical curve.

- In Chapter 3, I start constructing the halfcanonical ring of a canonical curve of a surface corresponding to the families of surfaces of theorem 1.4.1 whose canonical system has no fixed part. It turns out that the rings corresponding to the families (0) and (I), (with the exception of the subfamily (I.3)) have codimension 3 or 4.

- In Chapter 4, I deal with the cases (I.3) and (III), whose corresponding Gorenstein rings have codimension 6 and 8, respectively. The algebra of these rings is much more subtle than that of those studied in previous chapter, but surprisingly, the geometry of the curves/surfaces is somewhat simpler and allows us to compute the rings by restricting sections of certain toric key varieties.

- Chapter 5 discusses explicit deformations of surfaces of types (I.1) and (I.2); both subfamilies can be studied as Proj of Gorenstein codimension 4 rings that extend the formats found in the curve case. In particular, we see that the result of Bauer, Catanese and Pignatelli can be obtained starting from a halfcanonical ring of a curve and then using the extension algorithm.

- In Chapter 6, it is shown that all the formats obtained in Chapter 4 for the halfcanonical rings of curves in the canonical system of surfaces of subfamily (I.3) extend to the surface case. These rings are Gorenstein codimension 6, related by 20 equations yoked together in 64 syzygies and are rather complicated.

- In the final Chapter we discuss a plausible strategy for treating the cases on which the canonical system defines a 2-to-1 map to a quadratic surface. The method is analogous to that used in previous chapter to deform the codimension 6 rings. The rings involved here have codimension 8 and although working explicitly with them is naturally much more complicated, we believe that we will be able to answer important questions about the connectedness of the moduli space of surfaces with $K^2 = 7$ and $p_g = 4$ in the near future. In fact we conjecture that the moduli space is connected and provide some experimental evidence suggesting this.

- **Appendix** This thesis contains a series of computer algebra codes that are sanity checks needed in several parts of the text. All of them can be run in Magma Online Calculator (cf. [BCP-Magma]):

  `http://magma.maths.usyd.edu.au/calc`

  In addition they can be found in my personal website:

  `https://sites.google.com/view/juan-garza/nigromante/magma-codes`

Chapter 2

The Graded Ring Program

Reid's suggestion to study problems on minimal surfaces of general type, is to look at the deformations of their canonical rings. His program was carried out by Duncan Dicks in his PhD thesis [Dicks] for surfaces with $K_S^2 = 4$ and $p_g = 3$ and shortly later explained in detail in his article [Reid D-E], where he applied it to study deformations of Horikawa quintics and also set the challenge of applying his methods to cases corresponding to higher values of K_S^2 and p_g.

This fits in the framework of the more general and ambitious Graded Ring Program; starting with a polarised variety X, L, one follows Zariski's standard construction of the graded ring $R(X, L) = \bigoplus_{n\geq 0} H^0(X, nL)$ which in many interesting cases, turns out to be a Cohen-Macaulay, or even better, a Gorenstein ring. Knowing how to construct a ring $R(X, L)$ by giving a presentation (that is by generators and relations), can be achieved by combining algebro-geometric techniques depending on the particular choice of the pair (X, L) and gives precise answers to questions not only on embedding $X \hookrightarrow \mathbb{P}^n$ and determining the equations of the image but also, if the presentation obtained is good enough, on the deformation families of such rings/varieties. The worked example 2.4 illustrates many features of the program in an elementary case.

A couple of general conventions about the graded rings R that we will be working with are the following:

1. Graded rings will be $\mathbb{N}$–graded, that is, $R = \bigoplus_{n\geq 0} R_n$.
2. The base of the ring will be k, that is, $R_0 = \mathbb{C}$.

3. We do not require R to be generated over k by R_1. Geometrically this implies that in general, our varieties will be embedded in a weighted projective space, algebraically this implies that the codimension (and hence the number of equations defining the variety) remains small.

2.1 Gorenstein rings

For the purposes of this thesis, I can use the next definition of Gorenstein rings (for a more detailed discussion see [Reid 4], §1.1):

Definition Let $I \subset \mathcal{O}$ be a graded ideal in a regular graded ring. Let $R = \mathcal{O}/I$. Consider a minimal free resolution with graded $\mathcal{O}$–modules:

$$R \leftarrow R_0 \leftarrow R_1 \leftarrow \cdots \leftarrow R_{c-1} \leftarrow R_c \leftarrow 0 \tag{2.1}$$

Then R is a *Cohen-Macaulay* ring if $c = \operatorname{codim}_{\mathcal{O}} I$. If R is Cohen-Macaulay and $R_c \cong \mathcal{O}(-\ell)$ for some $\ell \in \mathbb{Z}$, then R is a *Gorenstein* ring.

Remark If R is Gorenstein, in particular one has a pairing on the resolution (2.1), $R_{c-i}^{\vee}(-\ell) \cong R_i$ coming from Serre duality.

In our context, R is of the form $R(X, L)$ with X a variety and L and ample Cartier divisor and the following result gives a characterisation of Cohen-Macaulay and Gorenstein rings that can be taken as an alternative definition (cf. [Hartshorne DT], Proposition 8.6).

Proposition 2.1.1. *Consider the graded ring $R(X, L) = \bigoplus_{n \geq 0} H^0(X, nL)$, with X a projective variety and L an ample Cartier divisor. Then:*

1. *R is Cohen-Macaulay if and only if:*
   (a) *$h^i(X, nL) = 0$ for all n and every i with $0 < i < \dim X$,*
   (b) *$h^0(X, nL) = 0$ for all $n < 0$,*
   (c) *$h^{\dim X}(X, nL) = 0$ for $n \gg 0$.*
2. *R is Gorenstein if and only if it is Cohen-Macaulay and $K_X = \ell L$ for some $\ell \in \mathbb{Z}$.*

The following is an immediate consequence of the above criterion, the remark in section § 1.4 and Kodaira vanishing theorem:

Corollary 2.1.2. *Let S be a minimal surface of general type with invariants $K_S^2 = 7$ and $p_g = 4$. Then $R(S, K_S)$ is a Gorenstein ring.*

2.1.1 Structure theorems

There are general structure theorems for Gorenstein rings in codimensions 2, 3 and 4 (by Serre, Buchsbaum-Eisenbud and Reid respectively) that we briefly recall next.

- In codim 2, Serre proved every Gorenstein ring is a complete intersection (see [Serre]).
- Codimension 3 is the easiest case we will find in this thesis, and the structure theorem is given by Buchsbaum and Eisenbud in [Buchsbaum-Eisenbud]. The result states that $R = \mathcal{O}/I$ is a Gorenstein ring of codim 3, if and only if it has a minimal free resolution of the form

$$R \leftarrow \mathcal{O} \leftarrow \mathcal{O}^{2m+1} \xleftarrow{\phi} \mathcal{O}^{2m+1} \leftarrow \mathcal{O} \leftarrow 0, \tag{2.2}$$

with ϕ given by a skew $(2m+1) \times (2m+1)$ matrix whose $2m \times 2m$ Pfaffians generate I. In most of the cases occurring in practice (and in particular in the ones we will find in this thesis) it turns out that $m = 2$ works. For this and further purposes, it is convenient to introduce some notation for 4×4 Pfaffians and skew matrices:

We will always omit the diagonal of zeroes and lower triangular antisymmetric block of any skew matrix appearing in this thesis, so for example, a skew 5×5 matrix will be written as

$$\phi = \begin{pmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{23} & a_{24} & a_{25} \\ & & a_{34} & a_{35} \\ & & & a_{45} \end{pmatrix} \tag{2.3}$$

And we will write $\mathrm{Pf}_i(\phi)$ for the Pfaffian of the 4×4 submatrix obtained after deleting the ith row and column of ϕ. So, Buchsbaum-Eisenbud result says that in case $m = 2$, following above notation, the generators of I are (modulo plus/minus signs):

$$\begin{aligned} \mathrm{Pf}_1(\phi) &= a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}, \\ \mathrm{Pf}_2(\phi) &= a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34}, \\ \mathrm{Pf}_3(\phi) &= a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24}, \\ \mathrm{Pf}_4(\phi) &= a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23}, \\ \mathrm{Pf}_5(\phi) &= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}. \end{aligned} \tag{2.4}$$

More generally, if we take M, a skew $m \times m$ matrix with $m \geq 6$ and entries $M_{r,s}$ we will denote simply by $ij.k\ell$, where $i < j < k < \ell$, the Pfaffian of the 4×4 matrix obtained by picking 4 rows and the corresponding columns according to the index selection. More precisely:

$$ij.k\ell = M_{ij}M_{k\ell} - M_{ik}M_{j\ell} + M_{i\ell}M_{jk}. \tag{2.5}$$

- Reid's codimension 4 structure theorem given in [Reid 4], §2.5, is an analogous result to that of Buchsbaum-Eisenbud in codimension 3, however, as stated there, in terms of certain key varieties called Spin-Hom varieties, is still far from being applicable in practice. As a substitute, we will be using certain ad-hoc methods for computing each of the codim 4 or higher Gorenstein rings that we find during the thesis.

2.1.2 Rolling factors

Throughout the thesis, we will come up with rings admitting a presentation that we will say to be in **rolling factors format**. Although we will emphasise on this later on, we include the definitions here for convenience of the reader.

Let I be an ideal in a polynomial ring $\mathcal{O}$. Suppose that there exists a matrix $A \in \mathrm{Mat}_{2,n}(\mathcal{O})$,

$$A = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \end{pmatrix}$$

such that its $\binom{n}{2}$ two by two minors are in I. We say that an element $r_{\binom{n}{2}+1} \in I$ can be rolled with respect to the matrix A if it is in the ideal generated by the elements of its first row. So $r_{\binom{n}{2}+1}$ is of the form

$$r_{\binom{n}{2}+1} = \sum_{i=1}^{n} a_i f_{1i} \quad \text{for some } a_i \in \mathcal{O}.$$

We call

$$r_{\binom{n}{2}+2} := \sum_{i=1}^{n} a_i f_{2i}$$

a rolling of $r_{\binom{n}{2}+1}$ with respect to A. If $r_{\binom{n}{2}+2} \in I$ and it can be rolled again with respect to A, we say that $r_{\binom{n}{2}+1}$ can be rolled twice, etc. Finally if I is finitely generated and its remaining generators $r_{\binom{n}{2}+2}, \dots, r_{\binom{n}{2}+m}$ are successive rollings of $r_{\binom{n}{2}+1}$ with respect to A, we say that $\mathcal{O}/I$ is in **rolling factors format**.

2.1.3 The hyperplane section theorem

Gorenstein rings are well behaved under taking hyperplane sections. The general philosophy as mentioned before is that the algebro-geometric properties of a ring/variety are closely related to those of its hyperplane sections. The main ingredient formalising these ideas is the following result:

Theorem 2.1.3. (The hyperplane section principle) *Let R be a graded ring, let R_d be the degree d component ($g \geq 1$), and let $x_0 \in R_d$ be a non zero divisor. There exists an exact sequence*

$$0 \to R(-d) \xrightarrow{x_0} R \xrightarrow{\pi} \overline{R} := R/(x_0) \to 0. \tag{2.6}$$

And we have:

1. *If $\overline{x_1}, \dots, \overline{x_n} \in \overline{R}$ generate $\overline{R}$ and $x_1, \dots, x_n \in R$ are such that $\pi(x_i) = \overline{x_i}$ for $1 \leq i \leq n$. Then $x_0, x_1, \dots, x_n$ generate R.*

2. *Suppose*
$$\overline{R} \cong k[\overline{x_1}, \dots, \overline{x_n}]/(f_1, \dots, f_m). \tag{2.7}$$
Then there exists relations $F_1, \dots, F_m$ holding between $x_0, x_1, \dots, x_n$, such that $\pi(F_i) = f_i$ and F_i generate the ideal $\ker ev$*, where*
$$ev : k[x_0, \dots, x_n] \to R. \tag{2.8}$$

3. *Similarly for syzygies, that is, if we have a syzygy $\sigma_i : \sum_{j=1}^{m} l_j f_j \equiv 0 \in k[\overline{x_1}, \dots, \overline{x_n}]$. Then there are $L_j \in k[x_0, \dots, x_n]$ such that the following syzygy holds:*
$$\Sigma_i : \sum_{j=1}^{m} L_j F_j \equiv 0 \in k[x_0, \dots, x_n], \tag{2.9}$$
and L_j reduces to l_j modulo x_0.

Proof. Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{J} \arrow[d] & \mathcal{I} \arrow[d] & \\
0 \arrow[r] & (x_0) \arrow[r] \arrow[d, dashed] & k[x_0,\dots,x_n] \arrow[r] \arrow[d] & k[\overline{x_1},\dots,\overline{x_n}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(-d) \arrow[r, "x_0"] & R \arrow[r, "\pi"] & \overline{R} \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} \tag{2.10}$$

where the two horizontal sequences are exact and the dashed arrow is the unique morphism making the diagram commute. Let $fx_0 \in (x_0)$. Under the dashed morphism, fx_0 maps to $f \in R_{\deg(f)} = R(-d)_{\deg(fx_0)}$. This morphism is well defined because x_0 is a non zero divisor (i.e., $(x_0) \subset R$ is a free R–module). Moreover it is an isomorphism. Then it follows from the snake lemma that

$$k[x_0,\dots,x_n] \to R \tag{2.11}$$

is surjective. In other words, R is generated by x_0 and the preimages of $\overline{x_1},\dots,\overline{x_n}$ under π. It follows also that $\mathcal{I} \cong \mathcal{J}$ as modules over $k[x_0,\dots,x_n]$. Suppose that $f_1,\dots,f_m$ are generators of $\mathcal{I}$ over $k[\overline{x_1},\dots,\overline{x_n}]$, then $f_1,\dots,f_m$ are generators of $\mathcal{I}$ over $k[x_0,\dots,x_n]$. Therefore there exist $F_1,\dots,F_m$ that generate $\mathcal{J}$ over $k[x_0,\dots,x_n]$. Now, notice that the isomorphism $\mathcal{J} \xrightarrow{\cong} \mathcal{I}$ is simply the restriction of $k[x_0,\dots,x_n] \to k[\overline{x_1},\dots,\overline{x_n}]$, which is uniquely determined by $x_i \mapsto \overline{x_i}$ for $1 \le i \le n$, and $x_0 \mapsto 0$.

Finally, let $F_j = f_j + x_0 g_j$ for some $g_j \in k[x_0,\dots,x_n]$. Suppose $\sigma_i : \sum_{j=1}^m l_j f_j \equiv 0$. Then

$$\sum_{j=1}^m l_j F_j \equiv x_0 \sum_{j=1}^m l_j g_j. \tag{2.12}$$

Since $\sum_{j=1}^m l_j F_j \in \mathcal{J}$ and $x_0 \sum_{j=1}^m l_j g_j \in (x_0)$ which, because x_0 is a non zero divisor, implies $\sum_{j=1}^m l_j g_j \in \mathcal{J}$, we can write $\sum_{j=1}^m l_j g_j = \sum_{j=1}^m h_j F_j$. Thus $\Sigma_i : \sum_{j=1}^m L_j F_j \equiv 0$, where $L_j := l_j - x_0 h_j$. Q.E.D.

2.2 Extension algorithm

This section goes trough Reid's extension theory that is clearly explained in [Reid D-E] and Dicks' practical approach to it.

Let S be a surface in $\mathfrak{M}_{7,4}$. Roughly speaking, the hyperplane section theorem says that the generators, relations and syzygies of $R(S, K_S)$ occur in the same degrees of those of $R(C, D)$, where $C \in |K_S|$ and $D = K_S|_C$. Because of the geometry of curves tends to be more tractable than that of surfaces, we will start computing $R(C, D)$. Suppose we find a presentation:

$$R(C,D) \cong k[x_1,\dots,x_n]/\mathcal{I}; \quad \text{where } \mathcal{I} = (f_1,\dots,f_m) \tag{2.13}$$

Together with syzygies

$$\sigma_i : \sum_{j=1}^{m} l_j f_j \equiv 0. \tag{2.14}$$

Then we should aim to find

$$F_j = f_j + x_0 f_j' + x_0 f_j'' + \cdots + x_0^{\ell} f_j^{(\ell)} + \cdots \tag{2.15}$$

such that $f_j^{(\ell)} \in k[x_1,\dots,x_n]$, and the F_j satisfy the syzygies Σ_j of theorem 2.1.3, part 3, so we produce the sequence

$$\{R(C,D)\}, \{R(2C, D^{(2)})\}, \dots, \{R(dC, D^{(d)})\}, \{R(S, K_S)\} \tag{2.16}$$

by calculating F_j in stages allowing successively higher powers of x_0. Each $R((\ell + 1)C, D^{(\ell+1)})$ depends in a linear way on $R(\ell C, D^{(\ell)})$ and we have

$$\{R((\ell+1)C, D^{(\ell+1)})/(x_0^{\ell})\} \subset \{R(\ell C, D^{(\ell)})\}. \tag{2.17}$$

The following definitions are given to show in greater detail how this gets done:

Definition Suppose we have

$$R(C,D) := R^{(1)} \cong k[x_1,\dots,x_n]/\mathcal{I} \tag{2.18}$$

and

$$R^{(\ell)} = k[x_0,\dots,x_n]/(\mathcal{I}^{(\ell)}, x_0^{\ell}) \tag{2.19}$$

such that $R^{(\ell)}$ modulo x_0 is $R^{(1)}$.

1. Define S_R to be the graded $k[x_1,\dots,x_n]$–module with grading $[S_R]_\delta = R^{(1)}_{\delta-1}$ and multiplication defined by

$$fg = 0 \quad \text{for all } f, g \in S_R. \tag{2.20}$$

2. Define $T_{R^{(\ell)}}$ to be the graded $k[x_0,\dots,x_n]$–module with grading $[T_{R^{(\ell)}}]_\delta = [R^{(\ell)}]_\delta$ and multiplication defined by

$$\left(\sum_{i=0}^{\ell-1} x_0^i p_i\right)\left(\sum_{i=0}^{\ell-1} x_0^i q_i\right) = x_0 \sum_{i=0}^{\ell-2} x_0^i \sum_{j+s=1} p_j q_s, \tag{2.21}$$

for all $\sum_{i=0}^{\ell-1} x_0^i p_i, \sum_{i=0}^{\ell-1} x_0^i q_i \in T_{R^{(\ell)}}$.

Then we have the following result (cf.[Lichtenbaum-Schlessinger], §4.2.5):

Proposition 2.2.1. *With $R^{(1)}$ and $R^{(\ell)}$ as above, $R^{(\ell+1)}$ exists if and only if there exists a degree preserving module homomorphism β_ℓ as shown in the following diagram, with $\beta_\ell \circ \iota'_\ell = \alpha'_\ell$.*

$$\begin{tikzcd}
0 \arrow[r] & \Sigma \arrow[r, "\iota'_\ell"] \arrow[d, "\alpha'_\ell"] & (f_1,\dots,f_m) \arrow[r] \arrow[d, "\alpha_\ell"] \arrow[dl, "\beta_\ell"'] & k[x_1,\dots,x_n] \arrow[r] \arrow[d] & R^{(1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_R \arrow[r, "\iota_\ell"] & T_{R^{(\ell)}} \arrow[r, "j_\ell"] & R^{(\ell)} \arrow[r] & R^{(1)} \arrow[r] & 0
\end{tikzcd} \tag{2.22}$$

The rows of 2.22 are exact. Σ is the $k[x_1,\dots,x_n]$–module generated by the syzygies $\sigma_1,\dots,\sigma_t$ such that the top row is exact. The maps ι_ℓ and j_ℓ are given by

$$\begin{aligned} \iota_\ell(p) &= x_0^{\ell-1} p, \\ j_\ell\left(\sum_{i=0}^{\ell-1} x_0^i p_i\right) &= x_0 \sum_{i=0}^{\ell-2} x_0^i p_i. \end{aligned} \tag{2.23}$$

The maps α_ℓ and α'_ℓ of previous diagram are defined inductively, they depend on the map $\beta_{\ell-1}$ defined by an analogous diagram as we will see:

Consider the first step, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Sigma \arrow[r, "\iota'"] \arrow[d, "\alpha_1'"] & (f_1,\dots,f_m) \arrow[r] \arrow[d, "\alpha_1"] \arrow[dl, "\beta_1"'] & k[x_1,\dots,x_n] \arrow[r] \arrow[d] & R^{(1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_R \arrow[r, "\iota"] & T_{R^{(1)}} \arrow[r, "j"] & R^{(1)} \arrow[r] & R^{(1)} \arrow[r] & 0
\end{tikzcd} \tag{2.24}$$

Where the maps α_1, α_1' and j are all zero, and ι is the identity map. We must find

$$\beta_1 : \mathcal{I} = (f_1, \dots, f_m) \longrightarrow S_R, \tag{2.25}$$

such that $\deg \beta_1(f_i) = \deg f_i - 1$ and $\sum_{i=1}^{m} l_i \beta_1(f_i) = 0$ whenever $\iota'(\sigma_j) = \sum_{i=1}^{m} l_i f_i$ for some σ_j. The fact this problem always has a solution (namely the zero map), corresponds to the fact that first order extensions are always *unobstructed*. The construction of β_1 allows us to define

$$R^{(2)}(\beta_1) = k[x_0, \dots, x_n]/(\mathcal{I}^{(2)}, x_0^2), \tag{2.26}$$

where $\mathcal{I}^{(2)}$ is generated by $F_1, \dots, F_m$ given by $F_i = f_i + x_0 \beta_1(f_i)$.

To go from ring $R^{(2)}$ to $R^{(3)}$, take any map $\beta : (f_1, \dots, f_m) \to S_R$ such that $\beta(f_1), \dots, \beta(f_m)$ are generic polynomials making the above conditions hold. Then this map can be lifted to

$$\tilde{\beta} : \mathcal{I} \longrightarrow k[x_1, \dots, x_n], \tag{2.27}$$

where $\sum_{i=1}^{m} l_i \tilde{\beta}(f_i) \in \mathcal{I}$ whenever $\iota'(\sigma_j) = \sum_{i=1}^{m} l_i f_i$ for some σ_j.

Then $\alpha_2 : \mathcal{I} \to T_{R^{(2)}}$ is given by $\alpha_2(f_i) = \beta(f_i)$. Consider σ_j such that $\iota'(\sigma_j) = \sum_{i=1}^{m} l_i f_i$, then the first step has given the expression

$$\sum_{i=1}^{m} l_i \tilde{\beta}(f_i) = \sum_{s=1}^{m} p_s f_s \in \mathcal{I}. \tag{2.28}$$

We define $\alpha_2'(\sigma_j) := \sum_{s=1}^{m} p_s \beta(f_s)$, and then β_2 in the diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \Sigma \arrow[r, "\iota_2'"] \arrow[d, "\alpha_2'"] & (f_1, \dots, f_m) \arrow[r] \arrow[d, "\alpha_2"] \arrow[dl, "\beta_2"'] & k[x_1, \dots, x_n] \arrow[r] \arrow[d] & R^{(1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_R \arrow[r, "\iota_2"] & T_{R^{(2)}} \arrow[r, "j_2"] & R^{(2)} \arrow[r] & R^{(1)} \arrow[r] & 0
\end{tikzcd}
\tag{2.29}
$$

must satisfy $\sum_{i=1}^{m} l_i \beta_2(f_i) = \sum_{s=1}^{m} p_s \beta(f_s)$ and as in the first step, the construction of β_2 allows us to put

$$R^{(3)}(\beta_2) = k[x_0, \dots, x_n]/(\mathcal{I}^{(3)}, x_0^3), \tag{2.30}$$

where $\mathcal{I}^{(3)}$ is generated by $\{F_1, \dots, F_m\}$ given by

$$F_i = f_i + x_0\beta(f_i) + x_0^2\beta_2(f_i). \tag{2.31}$$

If there are no maps β_2 making diagram 2.29 commute, we say the extension is *obstructed*; when this happens, we will be forced to impose further conditions on the general β to make the diagram commute.

The above procedure continues inductively when getting from $R^{(\ell)}$ to $R^{(\ell+1)}$, the presence of obstructions makes clear why in general

$$\{R((\ell+1)C, D^{(\ell+1)})/(x_0^\ell)\} \subsetneqq \{R(\ell C, D^{(\ell)})\}. \tag{2.32}$$

Remark In practice, it is often possible to save a lot of time and effort by doing some simplifications at each step and using the concept of *flexible formats* to write the generators of $\mathcal{I}$; this is explained later on in Chapter 5. Besides the worked example at the end of this chapter, the reader is also encouraged to consult Pinkham's example, that can be found in [Reid D-E], §2.1.

2.3 The Main Set up

Let S be a minimal surface with $K_S^2 = 7$ and $p_g = 4$. Let $|H|$ be the movable part of $|K_S|$. For convenience of the reader, I summarise the classification given in Theorem 1.4.1 in the following table:

Family	Base locus of $\lvert K_S \rvert$	Brief description of φ_H
(0)	None	Birational to a surface of degree 7.
$(I.1)$	One base point	Birational to a surface of degree 6.
$(I.2)$	One base point	2-to-1 to a cubic surface.
$(I.3)$	One base point	3-to-1 to the quadric cone.
$(III.\alpha)$	Three base points	2-to-1 to $\mathbb{P}^1 \times \mathbb{P}^1$. S admits a genus 2 pencil.
$(III.\beta)$	Three base points	2-to-1 to $\mathbb{P}^1 \times \mathbb{P}^1$. S does not admit a genus 2 pencil.
$(III.\gamma)$	Three base points	2-to-1 to the quadric cone.
$(F.1)$	F with $F^2 = -2$ and $K_S F = 0$	2-to-1 to $\mathbb{P}^1 \times \mathbb{P}^1$.
$(F.2)$	F with $F^2 = -2$ and $K_S F = 0$	2-to-1 to the quadric cone.
(F')	F' with $F'^2 = -1$, $K_S F' = 1$ and $HF' = 2$	2-to-1 to the quadric cone.

Definition Let $(\star)$ and $(\bullet)$ be two families of surfaces according to the classification of Theorem 1.4.1.

1. If I write $(\star) \longrightarrow (\bullet)$, I mean that there exists a flat family with base a small disc $\Delta_t \subset \mathbb{C}$ whose special fibre is of type $\bullet$, whose general fibre is of type $\star$ and this family exists for every point of a stratum $\bullet$.

2. If I write $(\star) \dashrightarrow (\bullet)$, I mean that there exists a flat family with base a small disc $\Delta_t \subset \mathbb{C}$ whose special fibre is of type $\bullet$, whose general fibre is of type $\star$ and this family exists only for some points of a stratum $\bullet$.

For example, the only known situation coming from explicit deformation of canonical rings was obtained in the work of Bauer, Catanese and Pignatelli [Bauer et al] and is:

$$(0) \longrightarrow (I.1).$$

However, by easy geometric arguments, the structure of the moduli space for the cases in which $|K_S|$ has a non-trivial fixed part is almost completely understood. We have the following picture:

$$\begin{tikzcd} & (III.\alpha) \arrow[dl] \arrow[dr] & & (III.\beta) \arrow[d] \\ (III.\gamma) & & (F.1) \arrow[d] & (F') \\ & & (F.2) & \end{tikzcd} \tag{2.33}$$

The above considerations suggest us to focus in the cases in which the canonical system has no fixed part. Guided also by the hyperplane section principle, we propose the following Main Setup:

- In the subsequent, unless otherwise stated, S will denote a minimal surface of general type with $K_S^2 = 7$, $p_g = 4$ and such that $|K_S|$ has no fixed part.
- $C \in |K_S|$ will be a general canonical curve that can be assumed to be nonsingular; because of Bertini's theorem and because it is known that $|K_S|$ can only have simple base points.
  - Let g be the genus of C. By the adjunction formula, $2g - 2 = 2K_S^2$, thus $g = 8$.
  - D will denote the restriction $K_S|_C$. So, again by adjunction, $K_C = 2K_S|_C$, therefore D is a degree 7 halfcanonical divisor.
  - Since $p_g = 4$ and the surfaces we are considering are regular, it follows that $h^0(D) = 3$, so D moves in a g_7^2.

As expected, there are exactly 5 general possibilities for a divisor $D \in g_7^2$, namely:

1. $|D|$ has no base points.
2. $|D|$ has exactly one base point P. Let $D' := D - P$, then $\varphi_{D'} : C \longrightarrow \mathbb{P}^2$ is a degree 6 morphism and one has 3 sub cases:
   (a) $\varphi_{D'}(C)$ is a sextic with 2 double points.
   (b) $\varphi_{D'}(C)$ is an elliptic curve.
   (c) $\varphi_{D'}(C)$ is a plane conic
3. $|D|$ has exactly 3 base points P_1, P_2, P_3. Let $\widetilde{D} := D - P_1 - P_2 - P_3$ so $\varphi_{\widetilde{D}} : C \longrightarrow \mathbb{P}^2$ is a degree 4 morphism. In this case the only possibility for $\varphi_{\widetilde{D}}(C)$ is to be a conic.

Remark • The above possibilities are obtained as easy numeric consequences, following from the factorisations $6 = 2 \times 3$, $4 = 2 \times 2$ and the degree-genus formula (or adjunction formula).

- It can be shown that the sextic in the general case corresponding to 2.(a) is tangent to the line joining its two singular points.
- In case 2.(c), because of the rationality of the conic, it follows that C is a trigonal curve (that is, is non hyperelliptic curve admitting a g_3^1).
- Similarly it follows that any curve in case 3 is a hyperelliptic curve.

Our task during the following two chapters will be to compute $R(C,D)$ for a polarised curve falling in each of the cases above.

2.4 Worked Example

Let C be a nonsingular projective curve of genus $g(C) = h^1(\mathcal{O}_C) = 2$. Assuming Riemann-Roch for curves holds, my aim is then to study C as an abstractly given object, by means of the graded rings $R(C,D)$ with respect to certain polarising ample divisor D.

2.4.1 Canonical and bicanonical rings on a genus 2 curve

First consider the canonical class K_C. By R-R and Serre duality:

$$h^0(K_C) = \deg(K_C).$$

This is the well known fact that every genus 2 curve is hyperelliptic, and $|K_C| = g_2^1$. It defines a morphism

$$\pi := \varphi_{K_C} : C \xrightarrow{\text{2-to-1}} \mathbb{P}^1,$$

branched, by Riemann-Hurwitz formula, at 6 (necessarily distinct, since C is nonsingular) points $B_i \in \mathbb{P}^1$, $1 \le i \le 4$. This leads to the model of C as a hypersurface in weighted projective space (w.p.s.)

$$\mathbb{P}(1,1,3) = \text{Proj } k[t_1,t_2,u];$$

we have

$$R(C,K_C) \cong k[t_1,t_2,u]/(u^2 - f_6(t_1,t_2)), \tag{2.34}$$

where f_6 is a homogeneous degree 6 form in t_1, t_2 whose zeroes define the branch locus. For illustrative purposes, I take

$$f_6(t_1, t_2) = t_1^6 - t_1 t_2^5. \tag{2.35}$$

This is an example of a codimension 1 Gorenstein ring; you can deduce the presentation (2.34) in any number of ways, perhaps the easiest being simply to follow the corresponding Riemann-Roch table (which is explained just below):

Space	Dimension	Generators	Relations
$H^0(C, K_C)$	2	t_1, t_2	None
$H^0(C, 2K_C)$	$3 = 2\deg(K_C) - 1$	$S^2(t_1, t_2)$	None
$H^0(C, 3K_C)$	$5 = 3\deg(K_C) - 1$	$S^3(t_1, t_2)$, u	None
$H^0(C, 4K_C)$	$7 = 4\deg(K_C) - 1$	$S^4(t_1, t_2)$, $S^1(t_1, t_2) \otimes u$	None
$H^0(C, 5K_C)$	$9 = 5\deg(K_C) - 1$	$S^5(t_1, t_2)$, $S^2(t_1, t_2) \otimes u$	None
$H^0(C, 6K_C)$	$11 = 6\deg(K_C) - 1$	$S^6(t_1, t_2)$, $S^3(t_1, t_2) \otimes u$, u^2	$u^2 - t_1^6 - t_2^6$
$H^0(C, nK_C)$	$2n - 1$	$S^n(t_1, t_2)$, $S^{n-3}(t_1, t_2) \otimes u$	None

Let $\{t_1, t_2\}$ be a basis of $H^0(K_C)$. Then $S^n(t_1, t_2)$, $n \geq 2$ gives us $n + 1$ linearly independent sections of $H^0(nK_C)$, because any relation would imply that $\varphi_{K_C}(C) \subset \mathbb{P}^1$ is reducible. Now, by Riemann-Roch I only need the 3 elements of $S^2(t_1, t_2)$ to get a basis of $H^0(2K_C)$, however in degree 3, I need a new generator, u, because $h^0(3K_C) = 5$ and I only have the 4 linearly independent elements from $S^3(t_1, t_2)$.

Next, observe that there is no relation holding between elements of $S^4(t_1, t_2)$ and $u \otimes S^1(t_1, t_2)$, because they belong to different factors of the decomposition of $\pi_* \mathcal{O}_C$ in $\pm$ eigensheaves: $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. Therefore the 7 elements in $S^4(t_1, t_2) \cup u \otimes S^1(t_1, t_2)$ form a basis of $H^0(4K_C)$. For similar reasons, it follows that there are no relations between elements of $S^n(t_1, t_2)$ and $u \otimes S^{n-3}(t_1, t_2)$ for any $n \geq 4$. Thus, the only relation occurs in degree 6, and the coefficient of u^2 must be non-zero. After completing the square and possibly changing coordinates, we get the desired presentation.

Previous computation gives for free a presentation of the *bicanonical* subring

$$R(C, 2K_C) = \bigoplus_{n \geq 0} H^0(C, 2nK_C). \tag{2.36}$$

This is a codimension 3 Cohen-Macaulay ring and the following statements are easy consequences of the presentation we obtained of the canonical ring $R(C,D)$:

1. There is a map $\varphi_{2K_C} : C \xrightarrow{\text{2-to-1}} \overline{C} \subset \mathbb{P}^2$, where $\overline{C} \cong \mathbb{P}^1$ is a nonsingular conic.
2. $R(C,D)$ is generated in degree 2 and related in degree 4; in detail:

$$R(C, 2K_C) \cong k[x_1, x_2, x_3, y_1, y_2]/I,$$

where $\deg(x_i) = 1$, $\deg(y_j) = 2$ for $1 \le i \le 3$, $j = 1,2$ and the ideal I is generated by the following 6 elements (relations):

$$\begin{aligned} r_1 &:= x_1x_3 - x_2^2 \\ r_2 &:= x_1y_2 - x_2y_1 \\ r_3 &:= x_2y_2 - x_3y_1 \\ r_4 &:= y_1^2 - x_1^4 + x_2^3x_3 \\ r_5 &:= y_1y_2 - x_1^3x_2 + x_2^2x_3^2 \\ r_6 &:= y_2^2 - x_1^2x_2^2 + x_2x_3^3 \end{aligned} \tag{2.37}$$

Observe that the *toric* relations r_i, $1 \le i \le 3$, are consequences of the following choice of generators:

$$x_1 := t_1^2, \quad x_2 := t_1t_2, \quad x_3 := t_2^2$$

$$y_1 := t_1u, \quad y_2 := t_2u$$

and can be more conveniently presented as the 2×2 minors of the following matrix:

$$M := \begin{pmatrix} x_1 & x_2 & y_1 \\ x_2 & x_3 & y_2 \end{pmatrix}, \tag{2.38}$$

whereas the *rolling factors*[1] relations r_j, $4 \le j \le 6$ are consequences of the unique relation, $u^2 - t_1^6 + t_1t_2^5$, holding between the generators of the canonical ring.

Remark We can replace relation r_5 (or r_4, or r_6) for $y_1y_2 - x_1^3x_2 + x_1x_3^3$ and generate exactly the same ideal, simply by adding $-x_2^2x_3^2 + x_1x_3^3 \in I$, which is a multiple of relation r_1. We call either of the monomials $x_2^2x_3^2$ and $x_1x_3^3$ *renderings* of the monomial $t_1^2t_2^6$ of the bigger ring and we often write:

$$[t_1^2t_2^6] = x_2^2x_3^2 = x_1x_3^3, \text{ etc.}$$

This choice is sometimes important when we consider deformations of the ring.

[1] This term was coined by D. Dicks and will be explained later on (cf. theorem 3.2.4).

2.4.2 Degree 4 divisors defining birational maps

The most common situation when considering a degree 4 effective divisor D on a nonsingular genus 2 curve C in terms of the linear system $|D|$ is that

$$\varphi_D : C \longrightarrow \overline{C} \subset \mathbb{P}^2,$$

is a birational map onto a plane degree 4 singular curve, in fact, the only case in which this fails is precisely when $D \sim 2K_C$. The other 2 cases are

a) $D_1 \sim K_C + P + Q$,

b) $D_2 \sim K_C + 2P$,

where P and Q are not ramification points of the 2-to-1 cover of $\mathbb{P}^1$ defined by $|K_C|$. Let $\overline{C}_i := \varphi_{D_i}(C) \subset \mathbb{P}^2$ for $i = 1, 2$. In case a), it follows from Riemann-Roch that $|D_1|$ separates any two points of C with the exception of P and Q that get mapped into a simple node of $\overline{C}$, whereas in case b), the linear system separates points but not tangent vectors, and $\overline{C}$ has a cusp in $\varphi_{D_2}(P)$.

For illustrative purposes, suppose that $\{x_1, x_2, x_3\}$ and $\{\overline{x}_1, \overline{x}_2, \overline{x}_3\}$ are basis of $H^0(C, D_1)$ and $H^0(C, D_2)$ respectively and let

$$\overline{C}_1 := V(x_1 x_2 x_3^2 - x_1^4 - x_2^4), \quad \overline{C}_2 := V(\overline{x}_1^2 \overline{x}_3^2 + \overline{x}_2^3 \overline{x}_3 - \overline{x}_1^4 + \overline{x}_2^4). \tag{2.39}$$

In particular, I choose the node (resp. cusp) to be the point with coordinates $(0:0:1)$.

By Riemann-Roch and since neither of the $\overline{C}_i$ are conics, it follows that either ring requires only one extra generator in degree two, that I call y and $\overline{y}$ respectively.

In degree 3, we have $h^0(3D_1) = h^0(3D_2) = 11$. The 10 monomials of $S^3(x_1, x_2, x_3)$ (or $S^3(\overline{x}_1, \overline{x}_2, \overline{x}_3)$) are clearly linearly independent, thus, I only need an extra generator to get a basis, but we already own another 3 degree 3 monomials;

$$S^1(x_1, x_2, x_3) \otimes y, \quad S^1(\overline{x}_1, \overline{x}_2, \overline{x}_3) \otimes \overline{y}.$$

Therefore, there are at least 2 relations, and it is easy to see they are of the form

$$\begin{array}{rlcrl} R_1 := & x_1 y - F_3 & & \widetilde{R}_1 := & \overline{x}_1\overline{y} - \widetilde{F}_3 \\ R_2 := & x_2 y - G_3 & , & \widetilde{R}_2 := & \overline{x}_2\overline{y} - \widetilde{G}_3 \end{array} , \tag{2.40}$$

where F_3, G_3 and $\widetilde{F}_3, \widetilde{G}_3$ are homogeneous forms of degree 3 not involving y nor x_3^3 and $\overline{y}$ nor $\overline{x}_3^3$, respectively. In fact, it follows that the equations cutting the plane quartics are ought to be $x_2F_3 - x_1G_3$ and $\overline{x}_2\widetilde{F}_3 - \overline{x}_1\widetilde{G}_3$. Thus:

$$\begin{array}{rlcrl} R_1 := & x_1 y - x_2^3 & & \widetilde{R}_1 := & \overline{x}_1\overline{y} + \overline{x}_2^3 + \overline{x}_2^2\overline{x}_3 \\ R_2 := & x_2 y + x_1^3 - x_2 x_3^2 & , & \widetilde{R}_2 := & \overline{x}_2\overline{y} + \overline{x}_1^3 - \overline{x}_1\overline{x}_3^2 \end{array} . \tag{2.41}$$

From previous pairs of equations, I can obtain a relation involving it, as follows:
R_2 says: $x_2(y - x_3^2) + x_1^3 \in I$, then $x_2 y(y - x_3^2) + x_1^3 y \in I$, but by R_1, $x_1 y - x_2^3 \in I$. Therefore $x_2 y(y - x_3^2) + x_1^2 x_2^3 \in I$ and since x_2 is not a zero divisor, I get a new relation:

$$R_3 := \; y(y - x_3^2) + x_1^2 x_2^2. \tag{2.42}$$

Analogously, one gets $\widetilde{R}_3 := \overline{y}^2 + (\overline{x}_3^2 - \overline{x}_1^2)(\overline{x}_2^2 + \overline{x}_2\overline{x}_3)$. The fact these generators/relations are sufficient to present the rings is left as an exercise to the reader. This is of course an example illustrating the Hilbert-Burch theorem on the resolution of codimension 2 Cohen-Macaulay rings. Let $\mathcal{O} := k[x_1, x_2, x_3, y]$, $I := (R_1, R_2, R_3)$ and $R := R(C, D_1) \cong \mathcal{O}/I$. There is a minimal free resolution

$$0 \leftarrow R \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-3)^{\oplus 2} \bigoplus \mathcal{O}(-4) \xleftarrow{{}^{T}A} \mathcal{O}(-5)^{\oplus 2} \leftarrow 0, \tag{2.43}$$

where A is the 2×3 matrix of first syzygies, whose 2×2 minors generate I. We have:

$$A := \begin{pmatrix} x_3^2 - y & x_2^2 & x_1 \\ x_1^2 & y & x_2 \end{pmatrix}, \tag{2.44}$$

and similar for the other ring, $R(C, D_2)$. In fact, the upshot is that we managed to present the ring in a way that allows to treat both cases at once in a flat family; let $t \in k$ be an affine parameter and consider the family of rings given by

$$R_t := k[x_1, x_2, x_3, y]/I_t, \tag{2.45}$$

where $\deg x_i = 1$ for $1 \le i \le 3$, $\deg y = 2$ and $I_t = (\bigwedge^2 A_t)$, that is, the ideal of relations is generated by the 2×2 minors of the following matrix:

$$A_t := (1-t)\begin{pmatrix} x_3^2 - y & x_2^2 & x_1 \\ x_1^2 & y & x_2 \end{pmatrix} + t\begin{pmatrix} y & x_2^2 + x_2x_3 & -x_1 \\ x_1^2 - x_3^2 & y & x_2 \end{pmatrix}. \tag{2.46}$$

Observe that for $t = 0$ we get the ring corresponding to case a), whereas $t = 1$ gives the ring of the cuspidal curve of case b). Moreover, A_t carries its own syzygies; to obtain them we simply need to clone one of its rows and then write the identity: determinant of a 3×3 matrix with 2 identical rows is zero; of course one can do this in 2 different ways, giving the 2 syzygies. Since the syzygies of ring R_t reduce modulo t to those of the central fibre ring $R(C, D_1)$, one obtains a flat family of rings. When this happens, we say that the presentation of the ring is given by a *flexible format*. The existence of such formats is a very subtle phenomenon and will play a major role not only in finding the deformations we are looking for in this thesis, but also the extensions (cf. extension algorithm discussed in this chapter).

2.4.3 Fun in $\mathbb{Z}/2\mathbb{Z}$

To finish this example, I give a format for the bicanonical ring we obtained in §2.4.1 that is useful to find explicit flat deformations. To keep calculations short and since this is only for illustrative purposes, I make a unique exception on this subsection and I will work over the finite field with 2 elements $k \cong \mathbb{Z}/2\mathbb{Z}$. Consider the following two 5×5 skew matrices:

$$M_1 := \begin{pmatrix} 0 & x_1 & x_2 & y_1 \\ & x_2 & x_3 & y_2 \\ & & y_1 & -x_2^2x_3 \\ & & & -x_1^3 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 0 & x_1 & x_2 & y_1 \\ & x_2 & x_3 & y_2 \\ & & y_2 & -x_2x_3^2 \\ & & & -x_1^2x_2 \end{pmatrix}, \tag{2.47}$$

both of weights $\begin{pmatrix} 0 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 2 & 3 \\ & & & 3 \end{pmatrix}$. Write $\mathrm{Pf}\,{}^{i}_{j}$; $i = 1, 2$, $1 \le j \le 5$ for the jth diagonal 4×4 Pfaffian of matrix M_i. We have the following identities (cf. relations in (2.37)):

$$\begin{aligned} \mathrm{Pf}\,{}^{1}_{3} = \mathrm{Pf}\,{}^{2}_{3} &= r_3, \\ \mathrm{Pf}\,{}^{1}_{4} = \mathrm{Pf}\,{}^{2}_{4} &= r_2, \\ \mathrm{Pf}\,{}^{1}_{5} = \mathrm{Pf}\,{}^{2}_{5} &= r_1, \\ \mathrm{Pf}\,{}^{1}_{1} = \mathrm{Pf}\,{}^{2}_{2} &= r_5, \\ \mathrm{Pf}\,{}^{1}_{2} &= r_4, \\ \mathrm{Pf}\,{}^{2}_{1} &= r_6. \end{aligned} \tag{2.48}$$

Moreover, both matrices imply the 5 syzygies following from $M_i \overline{\mathrm{Pf}\,_i} \equiv 0$, where $\overline{\mathrm{Pf}\,_i}$ is the vector whose jth entry is $(-1)^j \mathrm{Pf}\,{}^{i}_{j}$. Obviously there are a couple of repetitions, the lists are:

$$\begin{array}{llll} \sigma_1 := & -x_1r_3 + x_2r_2 - y_1r_1 & \sigma_1 := & -x_1r_3 + x_2r_2 - y_1r_1 \\ \sigma_2 := & -x_2r_3 + x_3r_2 - y_2r_1 & \sigma_2 := & -x_2r_3 + x_3r_2 - y_2r_1 \\ \sigma_3 := & x_1r_5 - x_2r_4 + y_1r_2 + x_2^2x_3r_1 & \sigma_6 := & x_1r_6 - x_2r_5 + y_2r_2 + x_2x_3^2r_1 \\ \sigma_4 := & x_2r_5 - x_3r_4 + y_1r_3 + x_1^3r_1 & \sigma_7 := & x_2r_6 - x_3r_5 + y_2r_3 + x_1^2x_2r_1 \\ \sigma_5 := & y_1r_5 - y_2r_4 - x_2^2x_3r_3 + x_1^3r_2 & \sigma_8 := & y_1r_6 - y_2r_5 - x_2x_3^2r_3 + x_1^2x_2r_2 \end{array} \tag{2.49}$$

The best way to check that these eight σ_i generate the module of syzygies is by using a computer algebra program (the relevant Magma code is in the appendix A.0 of this thesis).

Although these matrices certainly carry with the syzygies of the bicanonical ring, it turns out that deforming them to get one of the rings of the other 2 families is rather delicate. The procedure serves as a guide for the more complicated deformation calculations that we need to perform on the halfcanonical rings on curves and their extensions.

Both M_1 and M_2 have a zero of degree zero in their $(1, 2)$ entry, this suggests to replace it by an affine parameter $t \in k$ so the fifth diagonal Pfaffian allows us to express either y_1 or y_2 in terms of the degree 1 variables. One hopes that, after doing this and deforming the rest of the entries wisely, we get a codimension 2 Cohen-Macaulay ring corresponding to one of the cases considered previously. This calculation is not always possible, for example if one chooses to write a relation using a non convenient rendering. Also, we obviously are not allowed to consider the

Pfaffians of both matrices simultaneously if we decide to deform one of the degree zero entries. Fortunately, the presentation has been purposely constructed so we can find a flat family deforming a bicanonical ring into a ring corresponding to an arithmetic genus 2 curve polarized by a degree 4 divisor defining a map onto a plane quartic with a cusp:

Let $t \in k$. I choose M_1 to decrease the codimension of the ring, thus I must truncate M_2 eliminating its first row and column. I will make also adjustments to the rest of the entries so for $t \neq 0$ I get a ring isomorphic to $k[x_1, x_2, x_3, y]/I_1$ where I_1 is an ideal generated by the 2×2 minors of a matrix of the form A_1 of equation (2.46).

$$\text{Consider} \quad M_1(t) := \begin{pmatrix} t & x_1 & x_2 & y_1 \\ & x_2 & x_3 & y_2 \\ & & y_1 & -x_2^2 x_3 - t(x_2^3 + x_2 y_1) \\ & & & -x_1^3 + t(x_1 x_3^2 - x_3 y_1) \end{pmatrix}$$

$$\text{and} \quad \widetilde{M_2}(t) := \begin{pmatrix} x_2 & x_3 & y_2 \\ & y_2 & -x_2 x_3^2 - t(x_2^2 x_3) \\ & & -x_1^2 x_2 - t x_1^2 x_3 \end{pmatrix}.$$

Obviously the 4×4 Pfaffians of these matrices generate a bicanonical ideal when $t = 0$. I claim that for $t \neq 0$ they generate the same ideal as $y_1 - x_1 x_3 + x_2^2$ together with the 2×2 minors of $\begin{pmatrix} y_2 & x_2^2 + x_2 x_3 & -x_1 \\ x_1^2 - x_3^2 & y_2 & x_2 \end{pmatrix}$. I also claim this defines a flat family of rings. To check this is an easy but fun exercise for the reader.

Chapter 3

Halfcanonical curves: Low codimension cases

This chapter contains the calculation of a halfcanonical graded ring of a canonical curve C in the 3 first cases listed in Theorem 1.4.1; that is, $D = \frac{1}{2}K_C$ moves in a g^2_7 that defines either a birational map to a septic, a sextic or a 2-to-1 cover of an elliptic curve. As we will see, in each case the ring has codimension at most 4.

I start by stating some basic properties, common to any of these rings (except when clearly stated). Take $R(C,D)$ to be a ring as in the Main Set Up, that is, $2D \sim K_C$ and $|D| = g^2_7$.

Proposition 3.0.1. *R is a Gorenstein ring of codimension ≥ 3.*

Proof. That R is Gorenstein is a consequence of Proposition 2.1.1. Now, we have $h^0(D) = 3$, so let $\{x_1, x_2, x_3\}$ be a basis of $H^0(D)$. Since $S^2(x_1, x_2, x_3)$ has 6 elements we need at least two more independent generators to span $H^0(2D)$. It follows that codim $R(C,D) \geq 3$. Q.E.D.

R-R says that nD is nonspecial for $n \geq 3$. The following table contains the values of the Hilbert function of $R(C,D)$:

Space	Dimension
$H^0(C,\mathcal{O}_C)$	1
$H^0(C,D)$	3
$H^0(C,2D)$	8
$H^0(C,3D)$	14
$H^0(C,4D)$	21
$H^0(C,5D)$	28
$H^0(C,6D)$	35
$H^0(C,nD), \quad n \geq 3$	$7(n-1)$

It follows after a simple calculation, that the Hilbert series is

$$P_{R(C,D)}(t) := \sum_{n=0}^{\infty} h^0(nD)t^n = \frac{1+t+3t^2+t^3+t^4}{(1-t)^2}. \tag{3.1}$$

Proposition 3.0.2. *1. The canonical linear system $|K_C|$ is base point free.*

2. Assume C is not hyperelliptic. Then $|K_C|$ is also very ample.

Proof. 1. Let $P \in C$ be any point, by R-R:

$$\begin{aligned} h^0(C, K_C - P) = & \ 1 - 8 + \deg(K_C - P) + h^0(C,P) \\ = & \ 6 + h^0(C,P), \end{aligned} \tag{3.2}$$

$h^0(C,P) = 1$ for otherwise, C would be rational. The result follows.

2. Let $P, Q \in C$ be any two points. By R-R:

$$h^0(C, P+Q) = h^0(C, K_C - P - Q) - 5. \tag{3.3}$$

By (1) $h^0(C, K_C - P - Q) \leq 7$, so the only possibility is $h^0(C, P+Q) \in \{1,2\}$. But $h^0(C, P+Q) = 2$ implies $|P+Q|$ is a g^1_2, a contradiction. Therefore $H^0(C, K_C - P - Q)$ has codimension 2 in $H^0(C, K_C)$, so functions in $|K_C|$ distinguish between P and Q.

Q.E.D.

Finally, because of a theorem of Reid (cf. [Reid D-E], § 3.4), we know that R is always generated in degree at most 3, and the relations holding between its generators happen at degree at most 6.

3.1 The base point free family

In this section $|D|$ is a base point free g^2_7 on a nonsingular curve C and such that $2D = K_C$. The rings $R(C, D)$ are well known because of previous work of Ide and Mukai (cf. [Ide-Mukai]) among others. I will state the results and references for convenience of the reader.

I start with the following Proposition which will be used afterwards a couple of times:

Proposition 3.1.1. *Let $|D|$ be a base point free complete linear system on a nonsingular curve C of genus 8. Suppose that $r = h^0(D) \geq 3$ and that*

$$\varphi_D : C \longrightarrow \mathbb{P}^{r-1}$$

is a birational morphism. Then the natural map

$$\bigoplus_{n\geq 0} Sym^n H^0(D) \otimes H^0(K_C) \longrightarrow \bigoplus_{n\geq 0} H^0(K_C + nD)$$

is surjective.

Proof. See [Arbarello-Sernesi], pp. 102-103. Q.E.D.

Corollary 3.1.2. *The ring $R(C, D)$ has codimension 3.*

Proof. Apply previous proposition with $r = 3$, $n = 1$. Q.E.D.

It follows from the Buchsbaum-Eisenbud theorem that $R(C, D)$ is a Pfaffian ring. In fact one may observe that the Hilbert series

$$P_{R(C,D)}(t) = \frac{1 + t + 3t^2 + t^3 + t^4}{(1-t)^2} = \frac{1 - 2t^3 - 3t^4 + 3t^5 + 2t^6 - t^9}{(1-t)^3(1-t^2)^2} \tag{3.4}$$

suggest a free resolution of the form:

$$0 \leftarrow R(C, D) \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-3)^{\oplus 2} \bigoplus \mathcal{O}(-4)^{\oplus 3} \leftarrow \mathcal{O}(-5)^{\oplus 3} \bigoplus \mathcal{O}(-6)^{\oplus 2} \leftarrow \mathcal{O}(-9) \leftarrow 0, \tag{3.5}$$

where $\mathcal{O} = k[x_1, x_2, x_3, y_1, y_2]$; $\deg x_i = 1$, $\deg y_j = 2$. The symmetrizer trick allows to write resolution (3.5) as follows:

$$0 \longleftarrow R(C, D) \longleftarrow \mathcal{O} \xleftarrow{\text{Pf}} P_1 \xleftarrow{M} P_1^\vee \otimes \mathcal{O}(-9) \longleftarrow \mathcal{O}(-9) \longleftarrow 0 \tag{3.6}$$

where $P_1 := \mathcal{O}(-3)^{\oplus 2} \oplus \mathcal{O}(-4)^{\oplus 3}$ and M is a skew 5×5 matrix whose 4×4 Pfaffians are two cubic and three quartic homogeneous forms generating an ideal I such that $R(C, D) \cong \mathcal{O}/I$.

The result that makes precise our previous observations is the following:

Theorem 3.1.3. *Let R be the halfcanonical ring $R = R(C, D)$, where $|D| = g^2_7$ is base point free. Then $\varphi_D(C) \subset \mathbb{P}^2$ is a septic with singular locus of degree 7 contained in a conic and R is a codimension 3 Gorenstein ring isomorphic to:*

$$k[x_1, x_2, x_3, y_1, y_2]/I,$$

where the x_i and y_j have degrees 1 and 2 respectively and the ideal I is minimally generated by 2 cubics and 3 quartics that are the 5 diagonal Pfaffians of a skew matrix of the form

$$M := \begin{pmatrix} f_3 & q_1 & q_3 & q_5 \\ & q_2 & q_4 & q_6 \\ & & x_1 & x_2 \\ & & & x_3 \end{pmatrix},$$

where the q_i are generic quadratic homogeneous forms and f_3 is a generic cubic homogeneous form in x_i, y_j.

Proof. See [Ide-Mukai], pp. 9-13. Q.E.D.

A similar analysis of the surface case was made by Fabrizio Catanese. It turns out that the canonical rings of surfaces S with $K^2 = 7$, $p_g = 4$ and base point free canonical system can be presented in exactly the same format as the halfcanonical rings of the corresponding canonical curves found by Ide and Mukai. More precisely, one has the following result:

Theorem 3.1.4. *Let S be a nonsingular surface with $K^2 = 7$, $p_g = 4$ such that the canonical system is base point free. Let $R = R(S, K_S)$ be the canonical ring and let $\{x_0, x_1, x_2, x_3\}$ be a basis of $H^0(S, \mathcal{O}_S(K_S))$. We set $\mathcal{A} := k[x_0, x_1, x_2, x_3]$. Then:*

1. *R has a minimal resolution as $\mathcal{A}$–module given by the matrix*

$$\alpha := \begin{pmatrix} x_0 d_1 d_2 + x_1(d_3 d_4 + d_2^2) + x_2(d_2 d_3 + d_1 d_4) & x_1 d_4 & x_0 d_1 + x_1 d_2 + x_2 d_3 \\ x_1 d_4 & x_0 & x_2 \\ x_0 d_1 + x_1 d_2 + x_2 d_3 & x_2 & x_1 \end{pmatrix}$$

where d_1, d_2, d_3, d_4 are arbitrary quadratic forms in x_i.

2. *α satisfies the rank condition $\bigwedge^2 \alpha = \bigwedge^2 \alpha'$, where α' is obtained by deleting the first row of α, and therefore induces a unique ring structure on R as quotient of $\mathcal{O} := \mathcal{A}[y_1, y_2]$ by the three relations given by*

$$\alpha \begin{pmatrix} 1 \\ y_1 \\ y_2 \end{pmatrix} = 0$$

and three more relations expressing $y_1^2, y_1 y_2, y_2^2$ as linear combinations of the other monomials whose coefficients are determined by the adjoint matrix of α.

3. *R is Gorenstein of codimension 3. In particular, the ideal generated by the aforementioned 6 relations is minimally generated by only five of them, which can be written as the 4×4 Pfaffians of a matrix of the form*

$$\begin{pmatrix} f_3 & q_1 & q_3 & q_5 \\ & q_2 & q_4 & q_6 \\ & & x_1 & x_2 \\ & & & x_3 \end{pmatrix} \tag{3.7}$$

where the q_i are quadratics and f_3 is a cubic form, all of them generic.

4. *Conversely, let $\mathcal{O} := k[x_0, x_1, x_2, x_3, y_1, y_2]$ where the x_i and y_j are indeterminates of degrees 1 and 2 respectively and let $R := \mathcal{O}/I$ where I is the ideal generated by the 5 diagonal Pfaffians of a skew matrix of the form (3.7). Then under suitable open condition R is the canonical ring of a surface of type (0) of Theorem 1.4.1.*

Proof. See [Bauer et al] §4 and the references therein. Q.E.D.

3.2 Family (I.1): the curve case

In this section I compute graded rings relevant to the study of surfaces S of type (I.1) of Theorem 1.4.1. I am interested in a general member of the family. Therefore I make some assumptions on the halfcanonical curve (C, D) to avoid multiple cases.

3.2.1 Assumptions and notation

Let (C, D) is a nonsingular curve polarised by an effective divisor D subject to the following conditions:

1. $2D = K_C$
2. $D = \widetilde{P} + D'$, where $\widetilde{P}$ is the only base point of $|D|$ and $|D'| = g^2_6$ defines a birational morphism

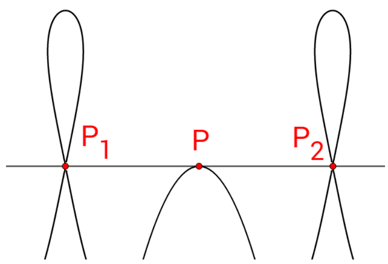
$$\varphi_{D'} : C \longrightarrow \overline{C} \subset \mathbb{P}^2$$

with the following properties:

   (a) $\overline{C}$ has 2 simple nodes; $P_1, P_2 \in \mathbb{P}^2$.

   (b) The line ℓ joining the nodes P_1 and P_2 is the tangent of $\overline{C}$ at the point $P = \varphi_{D'}(\widetilde{P})$.

Figure 3.1: A plane sextic with two nodes



The calculation of the halfcanonical ring is organised in the following 3 propositions, all of which are summarised in Theorem 3.2.4.

Proposition 3.2.1. *Let (C, D) be a polarised curve with the aforementioned properties. Then the full sections ring*

$$R := R(C, D) = \bigoplus_{n \geq 0} H^0(C, nD)$$

has codimension 4; it is minimally generated by 6 variables: x_1, x_2, x_3 of degree 1, y_1, y_2 of degree 2 and z of degree 3.

Proof. We have $h^0(D) = 3$. Let $\{x_1, x_2, x_3\}$ be a basis of $H^0(C, D)$. Since $\overline{C}$ is not contained in any quadratic, the six elements of $S^2(x_1, x_2, x_3)$:

$$\begin{array}{ccccc} & & x_1^2 & & \\ & x_1x_2 & & x_1x_3 & \\ x_2^2 & & x_2x_3 & & x_3^2 \end{array} \tag{3.8}$$

are linearly independent, so we need 2 extra generators to extend to a basis of the 8-dimensional space $H^0(C, K_C) = H^0(C, 2D)$, call them y_1, y_2.

Proposition 3.1.1 implies that the natural map:

$$H^0(C, g_6^2) \otimes H^0(C, K_C) \longrightarrow H^0(C, 3D - \widetilde{P})$$

is surjective. It follows that

$$H^0(C, D) \otimes H^0(C, K_C) \longrightarrow H^0(C, 3D)$$

also surjects onto $H^0(C, 3D - \widetilde{P})$. By Riemann-Roch $h^0(3D - \tilde{P}) = 13$. Therefore there are 3 degree 3 relations r_1, r_2, r_3, holding between the 16 monomials in

$$S^3(x_1, x_2, x_3) \ \oplus \ S^1(x_1, x_2, x_3) \otimes S^1(y_1, y_2), \tag{3.9}$$

I can choose 13 elements from (3.9) forming a linearly independent set that extends to a basis of $H^0(C, 3D)$ by adding only one new degree 3 generator, z. The result now follows from Reid's theorem (see [Reid D-E], § 3.4).

Q.E.D.

Proposition 3.2.2. *Consider the map* $\varphi_{g_6^2} : C \longrightarrow \overline{C} \subset \mathbb{P}^2$. *Choose coordinates so the two nodes of* $\overline{C}$ *are*

$$P_1 := (0:1:0) \quad \textit{and} \quad P_2 := (0:0:1). \tag{3.10}$$

Then we can choose bases for $H^0(D)$ *and* $H^0(2D)$ *so that the degree 3 relations* r_1, r_2, r_3 *of proposition 3.2.1 are the* 2×2 *minors of the following* 2×3 *matrix:*

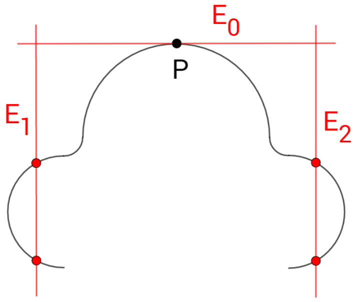
$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 x_3 & y_1 & y_2 \end{pmatrix} \tag{3.11}$$

Proof. Let

$$\pi : dP_7 \xrightarrow{\pi_2} \mathbb{F}_1 \xrightarrow{\pi_1} \mathbb{P}^2. \tag{3.12}$$

be the composition of π_1 and π_2, the blowups of $\mathbb{P}^2$ at the nodes P_1 and P_2 respectively. Write E_1, E_2 for the corresponding exceptional divisors, L for the pullback of a line class of $\mathbb{P}^2$ and E_0 for the strict transform of the line joining P_1 and P_2:

Figure 3.2: Blowing up the nodes



Now, dP_7 is a degree 7 del Pezzo surface, that is,

$$K_{dP_7} = \pi^* K_{\mathbb{P}^2} + E_1 + E_2 \sim -3L + E_1 + E_2 \tag{3.13}$$

is such that $-K_{dP_7}$ is very ample and $K_{dP_7}^2 = 7$. Moreover, by the adjunction formula:

$$2D = K_C = (K_{dP_7} + C)|_C \tag{3.14}$$

and since $C \sim 6L - 2(E_1 + E_2)$, we have

$$K_C = -K_{dP_7}|_C. \tag{3.15}$$

Now $H^0(dP_7, -K_{dP_7})$ is isomorphic to

$$\{s \in H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}^{-1}) \mid s(P_1) = s(P_2) = 0\} \tag{3.16}$$

(cf. [Manin], §24 Theorem 24.5).

Let $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = \langle\{v_1, v_2, v_3\}\rangle$. Because of $\omega_{\mathbb{P}^2}^{-1} \cong \mathcal{O}_{\mathbb{P}^2}(3)$ and from our choice of coordinates for P_1, P_2 given in (3.10), it follows that the anticanonical embedding of dP_7 is defined by the sections arranged in the following Newton polygon:

$$\begin{array}{ccccccc} & & & v_1^3 & & & \\ & & v_1^2 v_2 & & v_1^2 v_3 & & \\ & v_1 v_2^2 & & v_1 v_2 v_3 & & v_1 v_3^2 & \\ * & & v_2^2 v_3 & & v_2 v_3^2 & & * \end{array} \tag{3.17}$$

Let $H^0(C, D) := \langle\{x_1, x_2, x_3\}\rangle$. By abuse of notation I will use the same letters for sections in dP_7 and for their restrictions to C. Thus by (3.15) I can write:

$$\begin{array}{ccccccc} & & v_1^3 & & \\ & v_1^2v_2 & & v_1^2v_3 & \\ v_1v_2^2 & & v_1v_2v_3 & & v_1v_3^2 \\ * & v_2^2v_3 & & v_2v_3^2 & * \end{array} \quad = \quad \begin{array}{ccccccc} & & x_1^2 & & \\ & x_1x_2 & & x_1x_3 & \\ x_2^2 & & x_2x_3 & & x_3^2 \\ * & y_1 & & y_2 & * \end{array} \tag{3.18}$$

in particular if $u : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(\widetilde{P})$ is the constant section we have

$$H^0(C,D) \cong \langle\{uv_1, uv_2, uv_3\}\rangle \tag{3.19}$$

and $u^2 : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(2P) = \mathcal{O}_{dP_7}(L - E_1 - E_2)|_C = \mathcal{O}_{dP_7}(E_0)|_C$. Thus $u^2 = v_1$.

From 3.18 it follows that we have the following relations between x_1, x_2, x_3 and y_1, y_2:

$$\begin{aligned} r_1 &:= x_1y_1 - x_2^2x_3, \\ r_2 &:= x_1y_2 - x_2x_3^2, \\ r_3 &:= x_2y_2 - x_3y_1. \end{aligned} \tag{3.20}$$

That is,

$$\bigwedge^2 \widetilde{M} = 0, \quad \text{where} \quad \widetilde{M} := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2x_3 & y_1 & y_2 \end{pmatrix}. \tag{3.21}$$

Q.E.D.

Proposition 3.2.3. *Let (C, D) be a halfcanonical curve as in Proposition 3.2.2 and assume also without loss of generality that the point of tangency of $\overline{C} = \varphi_D(C)$ with the line joining P_1 and P_2 is $(0 : 1 : -1)$. Then the canonical model of C in $\mathbb{P}^7 = \mathbb{P}(H^0(C, \omega_C))$ is isomorphic to the curve in the w.p.s. $\mathbb{P}^4(1^3, 2^2)$ defined by the following 4 equations:*

$$\bigwedge^2 \widetilde{M} = 0 \quad \textit{and} \quad (y_1 + y_2)^2 + x_2x_3L(y_1, y_2) + x_1^2C_1 + x_2^2C_2 + x_3^2C_3 = 0,$$

where $L(y_1, y_2)$ is a nonzero linear form and the C_i are nonzero homogeneous quadratic forms in x_i, y_j, $1 \le i \le 3$, $j = 1, 2$.

Proof. It follows from proposition 3.2.2 that $\bigwedge^2 \widetilde{M} = 0$ cuts out a surface in $\mathbb{P}^4(1^3, 2^2)$ that is isomorphic to dP_7. dP_7 is embedded in (strictly speaking a different) $\mathbb{P}^7$ by its anticanonical linear system, but because of

$$C \sim 6L - 2(E_1 + E_2) \sim -2K_{dP_7} \quad \text{and} \quad K_C \sim -K_{dP_7}|_C,$$

it follows that the canonical model of C is defined by a quadratic in the monomials of the Newton polygon (3.17):

$$\begin{array}{ccccccc} & & & v_1^3 & & & \\ & & v_1^2v_2 & & v_1^2v_3 & & \\ & v_1v_2^2 & & v_1v_2v_3 & & v_1v_3^2 & \\ * & & v_2^2v_3 & & v_2v_3^2 & & * \end{array}$$

which is of course a relation of degree 4 in the ideal of relations of $R(C,D)$. Because of the choice of the tangency point, it is clear that this relation is of the form

$$v_2^2v_3^2(v_2+v_3)^2 + v_1v_2v_3A_3 + v_1^2B_4; \tag{3.22}$$

this is just prescribing the desired intersection of the canonical model of C with the exceptional curve $(v_1 = 0) = E_0 \subset dP_7$. It is necessary though, that the degree 3 form A_3 does not include further terms in v_1 (such terms can be included in the third summand of the relation).

By definition:

$$[v_2^2v_3^2(v_2+v_3)^2] = (y_1+y_2)^2, \quad [v_1v_2v_3] = x_2x_2$$

and it is clear that any expression of the form $a_1v_1v_2^4v_3 + a_2v_1v_2v_3^4 + v_1^2B_4$ with $a_1, a_2 \in \mathbb{C}$, can be rendered as a expression of the form $x_1^2C_1 + x_2^2C_2 + x_3^2C_3$; where the C_i are quadratics in the x_i and y_j. Q.E.D.

Theorem 3.2.4. *Let C be a nonsingular genus 8 curve admitting a linear system $|D|$ with only one base point $\widetilde{P}$ and satisfying the following properties:*

1. $|D| = \widetilde{P} + g_6^2$.
2. $2D = K_C$.
3. $\varphi_{g_6^2} : C \xrightarrow{birational} \overline{C} \subset \mathbb{P}^2$, *where $\overline{C}$ is a sextic with two nodes, P_1, P_2.*

Then the halfcanonical ring $R := R(C,D) = \bigoplus_{n\geq 0} H^0(C,nD)$ is isomorphic to:

$$k[x_1,x_2,x_3,y_1,y_2,z]/I,$$

where the degrees of the generators are $1,1,1,2,2,3$ respectively and the ideal I is *minimally generated by:*

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_1+y_2 \\ x_2x_3 & y_1 & y_2 & z \end{pmatrix},$$

plus 3 rolling factors equations:

$$(y_1+y_2)^2 + x_2x_3L(y_1,y_2) + x_1^2C_1 + x_2^2C_2 + x_3^2C_3 \tag{3.23a}$$

$$(y_1+y_2)z + y_1x_3L(y_1,y_2) + x_1x_2x_3C_1 + x_2y_1C_2 + x_3y_2C_3 \tag{3.23b}$$

$$z^2 + y_1y_2L(y_1,y_2) + x_2^2x_3^2C_1 + y_1^2C_2 + y_2^2C_3 \tag{3.23c}$$

where $L(y_1,y_2)$ *is a nonzero homogeneous linear form not equal to* y_1+y_2 *and the* C_i *are nonzero homogeneous quadratic forms in* x_i, y_j, $1 \le i \le 3$, $j = 1,2$.

Proof. Choose coordinates so that

$$P := (0:1:-1), \quad P_1 := (0:1:0), \quad P_2 := (0:0:1); \tag{3.24}$$

where $P = \varphi_D(\widetilde{P}) := \varphi_{g_6^2}(\widetilde{P})$. By Propositions 3.2.1-3.2.3 I only need to show that, keeping the same choices for the bases of $H^0(C,D)$ and $H^0(C,2D)$, I can take the last generator $z \in H^0(C,3D)$ as stated.

The 13 linearly independent elements of $H^0(3D)$ we already own are the following:

$$\begin{array}{ccccccccc} & & & & x_1^3 & & & & \\ & & & x_1^2x_2 & & x_1^2x_3 & & & \\ & & x_1x_2^2 & & x_1x_2x_3 & & x_1x_3^2 & & \\ & x_2^3 & & x_2^2x_3 & & x_2x_3^2 & & x_3^2 & \\ * & & x_2y_1 & & x_2y_2 & & x_3y_2 & & * \end{array} \tag{3.25}$$

In terms of $H^0(C,\widetilde{P}) \otimes H^0(C,g_6^2)^{\otimes 4}$ (and using the notation of Proposition 3.2.3), (3.25) is:

$$\begin{array}{ccccccccc} & & & & uv_1^4 & & & & \\ & & & uv_1^3v_2 & & uv_1^3v_3 & & & \\ & & uv_1^2v_2^2 & & uv_1^2v_2v_3 & & uv_1^2v_3^2 & & \\ & uv_1v_2^3 & & uv_1v_2^2v_3 & & uv_1v_2v_3^2 & & uv_1v_3^3 & \\ * & & uv_2^3v_3 & & uv_2^2v_3^2 & & uv_2v_3^3 & & * \end{array} \tag{3.26}$$

All these sections vanish at least once at $\widetilde{P}$ and $3D$ is very ample, so z must not vanish at $\widetilde{P}$. However by M. Noether theorem, the natural map

$$H^0(K_C) \otimes H^0(K_C) \longrightarrow H^0(4D)$$

is a surjection. This imposes conditions on the choice of z that is to be taken so that $x_j z = uv_j z$ is a sextic in the v_ℓs that vanishes 3 times at P_i for $i = 1, 2$ and $j = 2, 3$ respectively. It also vanishes once at $P = \varphi_D(\widetilde{P})$. Thus, I we can choose z so that

$$uz = v_2^2 v_3^2 (v_2 + v_3)$$

and it follows that

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_1 + y_2 \\ x_2 x_3 & y_1 & y_2 & z \end{pmatrix} \subset I. \tag{3.27}$$

The seventh relation,

$$(y_1 + y_2)^2 + x_2 x_3 L(y_1, y_2) + x_1^2 C_1 + x_2^2 C_2 + x_3^2 C_3, \tag{3.28}$$

was obtained already in Proposition 3.2.3, whereas the remaining 2 relations, (3.23b) and (3.23c), hold as a consequence of (3.28) and those coming from (3.27); this is the *rolling factors* format of Dicks (compare [Dicks]) and it is related to the defining equations of a divisor in a (possibly weighted) scroll. We can think on equations (3.27) as a way of saying that the ratio $(u : v_2 v_3)$ between the entries of any given column of the matrix

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & y_1 + y_2 \\ x_2 x_3 & y_1 & y_2 & z \end{pmatrix} \tag{3.29}$$

is preserved. Write M_{ij} for the corresponding entry of M. Then equation (3.28) says:

$$M_{14}(y_1 + y_2) = M_{12} x_3 L(y_1, y_2) + M_{11} x_1 C_1 + M_{12} x_2 C_2 + M_{13} x_3 C_3$$

so one deduces (3.23b) observing that

$$u M_{1j} = v_2 v_3 M_{2j}, \quad \text{for} \quad 1 \le j \le 4,$$

that is, one simply substitutes one entry of the matrix appearing as a factor in a term of the original relation by the second entry in the same column. The last relation of the theorem, $z^2 + y_1 y_2 L(y_1, y_2) + x_2^2 x_3^2 C_1 + y_1^2 C_2 + y_2^2 C_3$, is obtained in the same way. Finally, to show that no more relations are needed is an exercise on counting dimensions and Riemann-Roch. Q.E.D.

The next result gives an alternative presentation of the ring $R(C, D)$ of Theorem 3.2.4 which will be useful later on to study deformation families. Such a presentation is a slight variation of the *symmetric-extrasymmetric* format of Reid (cf. Reid's in [Brown et al], §9). See also the corresponding Magma code at the appendix A.2.

Theorem 3.2.5. *Let $R(C, D) = k[x_1, x_2, x_3, y_1, y_2, z]/I$ be the halfcanonical ring of Theorem 3.2.4. Consider the following 6×6 skew matrix:*

$$\mathcal{M} := \begin{pmatrix} 0 & y_2 & Q_1 & y_1 + y_2 & z \\ & x_3 & x_1 & x_2 + x_3 & y_1 + y_2 \\ & & z + F_3 & x_1 Q_2 & Q_1 Q_2 \\ & & & x_2 Q_3 + x_3 Q_4 & y_1 Q_3 + y_2 Q_4 \\ & & & & 0 \end{pmatrix},$$

where the Q_i are quadratic homogeneous forms such that Q_3 depends only on y_1, y_2 and F_3 is a homogeneous form of degree 3 not involving terms of the form $y_j x_1$, $j = 1, 2$ nor any term with powers greater than 1 in x_1 or x_3. Then if $Q_1 := x_2 x_3$, the 4×4 Pfaffians of $\mathcal{M}$ generate I.

Proof. Once we know what matrix we should take, this is an easy but beautiful calculation. I write simply $ij.kl$ for the diagonal Pfaffian of the 4×4 skew matrix obtained from $\mathcal{M}$ by picking the indicated 4 rows and corresponding columns. Concretely, $1 \le i < j < k < l \le 6$, and

$$ij.kl = \mathcal{M}_{ij}\mathcal{M}_{kl} - \mathcal{M}_{ik}\mathcal{M}_{jl} + \mathcal{M}_{il}\mathcal{M}_{jk}.$$

The matrix I want to consider is

$$\mathcal{M}' := \begin{pmatrix} 0 & y_2 & x_2 x_3 & y_1 + y_2 & z \\ & x_3 & x_1 & x_2 + x_3 & y_1 + y_2 \\ & & z + F_3 & x_1 Q_2 & x_2 x_3 Q_2 \\ & & & x_2 L + x_3 Q_4 & y_1 L + y_2 Q_4 \\ & & & & 0 \end{pmatrix}, \quad \text{of weights} \quad \begin{pmatrix} 0 & 2 & 2 & 2 & 3 \\ & 1 & 1 & 1 & 2 \\ & & 3 & 3 & 4 \\ & & & 3 & 4 \\ & & & & 4 \end{pmatrix}.$$

Where I substituted Q_3 for the $L = L(y_1, y_2)$ of Theorem 3.2.4. Before computing its Pfaffians, let us put some names on the 9 generators of I that we know from

Theorem 3.2.4:

$$\begin{aligned}
r_1 &:= x_1y_1 - x_2^2x_3\\
r_2 &:= x_1y_2 - x_2x_3^2\\
r_3 &:= x_1z - x_2x_2(y_1+y_2)\\
r_4 &:= x_2y_2 - x_3y_1\\
r_5 &:= x_2z - y_1(y_1+y_2)\\
r_6 &:= x_3z - y_2(y_1+y_2)\\
\\
r_7 &:= (y_1+y_2)^2 + x_2x_3L(y_1,y_2) + x_1^2C_1 + x_2^2C_2 + x_3^2C_3\\
r_8 &:= (y_1+y_2)z + y_1x_3L(y_1,y_2) + x_1x_2x_3C_1 + x_2y_1C_2 + x_3y_2C_3\\
r_9 &:= z^2 + y_1y_2L(y_1,y_2) + x_2^2x_3^2C_1 + y_1^2C_2 + y_2^2C_3
\end{aligned}$$

The key observation is that using r_5 and r_6, we have $(y_1+y_2)^2 = z(x_2+x_3)$. Now the 4×4 Pfaffians of $\mathcal{M}'$ are:

$$\begin{aligned}
12.34 &= x_2x_3^2 - x_1y_2 = -r_2\\
12.45 &= x_1(y_1+y_2) - x_2x_3(x_2+x_3) = r_1 + 12.34\\
12.46 &= x_1z - x_2x_3(y_1+y_2) = r_3\\
12.35 &= x_3(y_1+y_2) - (x_2+x_3)y_2 = x_3y_1 - x_2y_2 = -r_4\\
12.36 &= x_3z - y_2(y_1+y_2) = r_6\\
12.56 &= (x_2+x_3)z - (y_1+y_2)^2 = r_5 + 12.36\\
\\
23.45 &= x_3(x_2L + x_3Q_4) - x_1^2Q_2 + (z+F_3)(x_2+x_3)\\
13.45 &= y_2(x_2L + x_3Q_4) - x_1x_2x_3Q_2 + (y_1+y_2)(z+F_3)\\
13.46 &= y_2(y_1L + y_2Q_4) - x_2^2x_3^2Q_2 + z(z+F_3)\\
\\
13.56 &= x_1Q_2z - x_2x_3(y_1+y_2)Q_2 = Q_2(12.46)\\
14.56 &= z(x_2L + x_3Q_4) - (y_1+y_2)(y_1L + y_2Q_4) = L(12.56) + (Q_4 - L)(12.36)\\
23.46 &= x_3(y_1L + y_2Q_4) - x_1x_2x_3Q_2 + (y_1+y_2)(z+F_3) = 13.45 \text{ (using 12.35)}\\
23.56 &= x_1Q_2(y_1+y_2) - (x_2+x_3)x_2x_3Q_2 = Q_2(12.45)\\
24.56 &= (x_2L + x_3Q_4)(y_1+y_2) - (x_2+x_3)(y_1L + y_2Q_4) = (12.35)(Q_4 - L)\\
34.56 &= x_2x_3Q_2(x_2L + x_3Q_4) - x_1Q_2(y_1L + y_2Q_4) = Q_2[L(12.45) + (Q_4 - L)(12.34)]
\end{aligned}$$

Clearly, the first 6 elements generate the sub-ideal of determinantal relations of I (that is, the same ideal as the 2×2 minors of the matrix M of Theorem 3.2.4, equation (3.29)) and the last 6 Pfaffians of the list are redundant (this is a consequence of the extra-symmetry of the matrix $\mathcal{M}'$). Now the Pfaffian 23.45 is

$$z(x_2 + x_3) + x_2x_3L - x_1^2Q_2 + (x_2 + x_3)F_3 + x_3^2Q_4,$$

which, after the different choice of rendering for $z(x_2 + x_3)$, is of the form

$$(y_1 + y_2)^2 + x_2x_3L(y_1, y_2) + x_1^2C_1 + x_2^2C_2 + x_3^2C_3,$$

we can see here that we can take F_3 as stated in the theorem. Notice that any term of the form x_1y_j can be rendered as a term involving only x_2 and x_3. Finally, it is clear that Pfaffians 13.45 and 13.46 can be obtained from 23.45 by rolling factors with respect to the matrix $\begin{pmatrix} x_1 & x_2 & x_3 & y_1 + y_2 \\ x_2x_3 & y_1 & y_2 & z \end{pmatrix}$. Therefore, after naming the corresponding quadratic forms, $r_8 = 13.45$ and $r_9 = 13.46$. Q.E.D.

3.3 Bielliptic curves

In this section we study the second possibility in which the linear system $|D| = g_7^2$ has only one base point $\widetilde{P}$. D will denote an effective divisor on C, a nonsingular curve of genus 8 that moves in the canonical linear system $|K_X|$ of a surface of general type of family (I.2) of Theorem 1.4.1. In this, as well as in the remaining cases, the corresponding map

$$\varphi_D : C \longrightarrow \mathbb{P}^2$$

will no longer be birational but a finite cover onto a normal plane curve. Assuming one base point, the first case is thus $D = \widetilde{P} + \widetilde{D}$, where $\widetilde{D}$ is a degree 6 base point free divisor such that $|\widetilde{D}|$ defines a 2-to-1 cover:

$$\pi : C \longrightarrow E \subset \mathbb{P}^2,$$

onto a plane elliptic curve E. The main result is Theorem 3.3.5, which gives a presentation of the halfcanonical ring $R(C, D)$. It turns out that rings in this family are closely related to the rings computed in the previous section. The calculation is based on some general theory of double covers and a careful study of the geometry of the canonical curve $\overline{C} := \varphi_{K_C}(C) \subset \mathbb{P}^7$.

3.3.1 Double covers

In this subsection we gather some well known facts about *finite double covers.* Given a scheme X, such a cover is an X–scheme

$$Y \xrightarrow{\pi} X$$

such that the $\mathcal{O}_X$–algebra $\pi_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$–module of rank 2.

Proposition 3.3.1. *Suppose $2 \in \mathcal{O}_X$ is invertible. Let $Y \xrightarrow{\pi} X$ be a finite double cover. Then*

$$\pi_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{L}^{-1},$$

where $\mathcal{L}^{-1}$ is the cokernel of the structure map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$.

Proof. Let $U \subset X$ be open and take $y \in \mathcal{O}_Y(\pi^{-1}(U))$. Product by y defines a morphism

$$\pi_*\mathcal{O}_Y(U) \xrightarrow{y} \pi_*\mathcal{O}_Y(U),$$

that, after choosing a basis for $\pi_*\mathcal{O}_Y(U)$, is represented by a matrix

$$M_y \in \mathrm{Mat}_{2\times 2}(\mathcal{O}_X(U)).$$

Because of $2 \in \mathcal{O}_X$ being invertible, we have a well-defined map

$$\pi_*\mathcal{O}_Y(U) \xrightarrow{\psi} \mathcal{O}_X(U), \quad y \mapsto \frac{1}{2}\mathrm{Trace}(M_y).$$

This map gives, by construction, a splitting for the exact sequence

$$0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_Y \to \mathcal{L}^{-1} \to 0.$$

Q.E.D.

Previous result holds for any $d-$ cover as long as d is invertible in $\mathcal{O}_X$, in particular for $\mathbb{C}$–varieties. The cases of our interest however (double covers of nonsingular complex curves/surfaces), allow further interpretations. We have a natural involution

$$\iota : Y \longrightarrow Y,$$

(an automorphism of degree 2) such that $\pi \circ \iota = \pi$. Then

$$\pi_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{L}^{-1}$$

comes from the decomposition of the eigenspace of ι. The algebra structure on $\mathcal{O}_X \oplus \mathcal{L}^{-1}$ is given by a multiplication map

$$\mathcal{L}^{-1} \otimes \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X,$$

that is, a section of the line bundle $\mathcal{L}^{\otimes 2}$. A divisor $B \subset X$ such that $\mathcal{O}_X(B) \cong \mathcal{L}^{\otimes 2}$ must be then, the image under π of the invariant locus of ι, that is, the branch divisor. In particular, $\mathcal{O}_X(B)$ is a perfect square in $\mathrm{Pic}(X)$.

Remark Suppose $X = E$ is an elliptic curve and $Y = C$ is our genus 8 curve. We have $\omega_E \cong \mathcal{O}_E$ and by Riemann-Hurwitz:

$$\omega_C \cong \mathcal{O}_C(R),$$

where $R \subset C$ is the ramification divisor of π. In particular π is ramified in 14 different points. Moreover:

$$\begin{aligned} \pi_*\omega_C &\cong \mathcal{H}om(\pi_*\mathcal{O}_C, \omega_E) \\ &\cong \mathcal{H}om(\mathcal{O}_E \oplus \mathcal{L}^{-1}, \mathcal{O}_E) \\ &= \mathcal{O}_E \oplus \mathcal{L}. \end{aligned}$$

This will allow us the describe the cohomology spaces $H^0(nD)$ in terms of cohomology spaces of divisors on E, that are rather easy to calculate with.

3.3.2 The geometry of the canonical curve

The canonical model of C, that I denote as $\overline{C} = \varphi_{K_C} \subset \mathbb{P}^7$, allows to define geometrically the double covering map $C \xrightarrow{\pi} E$ and hints into the similarity between the halfcanonical rings of this and section 3.2.

We have

$$\pi_{\text{geom}} : \overline{C} \longrightarrow E_7,$$

a 2-to-1 map onto an elliptic curve of degree $\dfrac{\deg(\overline{C})}{2} = 7$ in $\mathbb{P}^6$.

Proposition 3.3.2. *There is a point $P_0 \in \mathbb{P}^7 \setminus \overline{C}$ such that π_{geom} is the restriction to $\overline{C}$ of the projection from P_0.*

Proof. For any degree 2 divisor $Q+R \in \mathrm{Div}(E_7)$, let $\pi^*_{\text{geom}}(Q+R) := Q'_1+Q'_2+R'_1+R'_2$.

Since E_7 is elliptic, $h^0(Q+R)=2$. Thus

$$h^0(Q_1'+Q_2'+R_1'+R_2')=2;$$

then, by the geometric version of Riemann-Roch (cf. [Reid PC], §3.2), the four points Q_1', Q_2', R_1', R_2' are all in the same 2−dimensional plane of $\mathbb{P}^7$. The two lines, $\ell(Q_1', Q_2')$ and $\ell(R_1', R_2')$, joining the corresponding points, meet at a point P_0. This point must be in the intersection of all the planes constructed in this way also by the geometric version of Riemann-Roch (simply use it fixing Q or R and taking a different point in E). Finally, it is clear by construction that $P_0 \notin \overline{C}$. Q.E.D.

The elliptic curve $E_7 = \pi_{\mathrm{geom}}(\overline{C}) \subset \mathbb{P}^6$ of previous proposition has a smooth extension to a degree 7 del Pezzo surface $dP_7 \subset \widetilde{\mathbb{P}^7}$ (do not confuse this space with previous $\mathbb{P}^7$, which was the projective space of $H^0(K_C)$). dP_7 is of course, the blowup of a projective plane at two points, E_7 is the pullback of a plane nonsingular cubic passing through both points and one has

$$E_7 \in |-K_{dP_7}|.$$

If H and H' denote the hyperplane divisor classes of $\mathbb{P}^7$ and $\widetilde{\mathbb{P}^7}$ respectively, we have:

$$2H'|_{E_7} \sim \pi_{\mathrm{geom}*} H|_{\overline{C}},$$

but $H|_{\overline{C}} = K_C = R$ (the ramification divisor). If B is the corresponding branch divisor, it follows that

$$B \sim (-2K_{dP_7})|_{E_7}.$$

3.3.3 Calculation of the ring

I will write again $D = \widetilde{P} + \widetilde{D}$ for the halfcanonical divisor, $\pi = \varphi_{\widetilde{D}} : C \longrightarrow E$ for the double cover and

$$\pi_* \mathcal{O}_C = \mathcal{O}_E \oplus \mathcal{L}^{-1}.$$

Proposition 3.3.3. *$\widetilde{P}$ is a ramification point of π.*

Proof. Because of $\omega_C = \pi^*(\mathcal{L})$ with $\mathcal{L} \in \mathrm{Pic}(E)$, it follows that $\mathcal{O}_C(2\widetilde{P})$ is also of the form $\pi^*(\mathcal{E})$ for some $\mathcal{E} \in \mathrm{Pic}(E)$. Thus the divisor $2\widetilde{P}$ is invariant under the involution ι and it follows that $\widetilde{P}$ is a ramification point of π. Q.E.D.

Let

$$R = \widetilde{P} + P_1 + \cdots + P_{13}$$

be the ramification divisor, write

$$u : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(\widetilde{P}) \quad \text{and} \quad v : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1 + \cdots + P_{13})$$

for the constant sections and let $P \in E$ be the branch point corresponding to $\widetilde{P}$, that is, $\pi^*(P) = 2\widetilde{P}$.

Write $\mathcal{O}_E(1) = \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^2}(1)$. Modulo changing coordinates, I can assume

$$\mathcal{O}_E(1) = \mathcal{O}_E(3P),$$

thus

$$H^0(C, D) \cong u \cdot H^0(E, 3P).$$

The following is a classical exercise using Riemann-Roch on an elliptic curve:

Proposition 3.3.4. *1. The graded ring $R(E,P)$ is isomorphic to $k[a,b,c]/(f_6)$, where the degrees of a,b,c are $1,2,3$ respectively and f_6 is a degree 6 homogeneous polynomial of the form*

$$c^2 - b^3 - \alpha a^4 b - \beta a^6, \ \textit{for some } \alpha, \beta \in k.$$

2. The subring $R(E,3P)$ is

$$k[\hat{x}_1, \hat{x}_2, \hat{x}_3]/(f_3),$$

where $\hat{x}_1 := a^3$, $\hat{x}_2 := ab$, $\hat{x}_3 := c$ and f_3 is just the corresponding rendering of f_6, that is:

$$f_3 = \hat{x}_1\hat{x}_3^2 - \hat{x}_2^3 - \alpha\hat{x}_1^2\hat{x}_2 - \beta\hat{x}_1^3.$$

I keep this notation and write

$$H^0(C, D) \cong \langle x_1 := ua^3, x_2 := uab, x_3 := uc\rangle.$$

Since $2\widetilde{P} = \pi^* P$, we have $u^2 = a$. Moreover $\pi_*\omega_C = \mathcal{O}_E \oplus \mathcal{L}$, where $\mathcal{L} = \mathcal{O}_E(D_7)$ and D_7 is a degree 7 divisor invariant under the involution. Because of my choice of basis I have $D_7 \sim 7P$ and it follows that

$$H^0(C, 2D) = H^0(C, K_C) \cong H^0(E, 7P) \oplus \langle y_2\rangle,$$

where $y_2 = uv$, so $y_2^2 \in \operatorname{Sym}^2 H^0(E, 7P)$. Therefore, we own the following basis of

$H^0(C, K_C)$:

$$\begin{array}{ccccc} x_1^2 = a^7 & x_1x_2 = a^5b & x_1x_3 = a^4c & & \\ & x_2^2 = a^3b^2 & x_2x_3 = a^2bc & & y_1 := b^2c \\ & & x_3^2 = ac^2 & & y_2 \end{array},$$

by construction, the only degree 3 relations holding between x_1, x_2, x_3 and y_1, y_2 are:

$$\begin{aligned} r_1 &:= x_1x_3^2 - x_2^3 - \alpha x_1^2x_2 - \beta x_1^3 \\ r_2 &:= x_1y_1 - x_2^2x_3 \\ r_3 &:= x_2y_1 + \beta x_1^2x_3 + \alpha x_1x_2x_3 - x_3^3 \end{aligned},$$

these are the 2 by 2 minors of the following 2×3 matrix:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2^2 & x_3^2 - \alpha x_1x_2 - \beta x_1^2 & y_1 \end{pmatrix}.$$

By Riemann-Roch, we only need one more generator z in degree 3 that can be chosen to be a section vanishing at $P_1, \ldots, P_{13}$ and with a pole of order 2 at $\widetilde{P}$. Thus, I can take $z := vb^2$. Finally, observe that in order for the branch points to be different and since P is one of them, y_2^2 must be of the form

$$y_2^2 = x_3^2y_1 + S_4(x_1, x_2, x_3), \tag{3.30}$$

therefore

$$R(C, D) \cong k[x_1, x_2, x_3, y_1, y_2, z]/I,$$

with generators of degrees $1, 1, 1, 2, 2, 3$ respectively and I generated by

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 & x_3^2 - \alpha x_1x_2 - \beta x_1^2 & y_1 & z \end{pmatrix},$$

plus the 3 rolling factors equations coming from equation (3.30). Rolling is possible because y_1 appears only in one summand and multiplied by x_3^2. Concretely:

$$\begin{aligned} r_7 &:= y_2^2 - x_1^2Q_1 - x_2^2Q_2 - x_3^2(Q_3 + y_1) \\ r_8 &:= y_2z - x_1x_2^2Q_1 - x_2(x_3^2 - \alpha x_1x_2 - \beta x_1^2)Q_2 - x_3y_1(Q_3 + y_1) \\ r_9 &:= z^2 - x_2^4Q_1 - (x_3^2 - \alpha x_1x_2 - \beta x_1^2)^2Q_2 - y_1^2(Q_3 + y_1) \end{aligned}$$

where the Q_i are homogeneous quadratic forms depending on x_i, for $1 \leq i \leq 3$. The next theorem summarises these results and gives a determinantal presentation of the ring:

Theorem 3.3.5. *Let C be a nonsingular genus 8 curve admitting a linear system $|D|$ with only one base point $\widetilde{P}$ and satisfying the following properties:*

1. $|D| = \widetilde{P} + g_6^2$.
2. $2D = K_C$.
3. $\varphi_{g_6^2} : C \xrightarrow{\text{2-to-1}} E \subset \mathbb{P}^2$, *where E is an elliptic curve.*

Then the halfcanonical ring $R := R(C, D) = \bigoplus_{n \geq 0} H^0(C, nD)$ *is isomorphic to:*

$$k[x_1, x_2, x_3, y_1, y_2, z]/I,$$

with generators of degrees $1, 1, 1, 2, 2, 3$ *respectively and the ideal I is minimally generated by 9 homogeneous forms r_i,* $1 \leq i \leq 9$*, obtained as follows:*

Let

$$A := \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 & x_3^2 - \alpha x_1 x_2 - \beta x_1^2 & y_1 & z \end{pmatrix}, \quad M := \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 + y_1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then r_i, $1 \leq i \leq 6$*, are the* 2×2 *minors of A, whereas r_7, r_8, r_9 are given by*

$$AM({}^T\!A) = 0.$$

Proof. It only remains to check that the matrix equality $AM({}^T\!A) = 0$ indeed gives the rolling factors relations. This is left as an exercise to the reader (see also the Magma code at the appendix, § A.3). Q.E.D.

Remark Halfcanonical rings of family (I.1), (cf. Theorem 3.2.4), can also be presented in this way. Simply take

$$A := \begin{pmatrix} x_1 & x_2 & x_3 & y_1 + y_2 \\ x_2 x_3 & y_1 & y_2 & z \end{pmatrix}, \quad M := \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & \frac{1}{2}L(y_1, y_2) & 0 \\ 0 & \frac{1}{2}L(y_1, y_2) & C_3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This is (modulo slight variations) one of the two *flexible formats* appearing in Reid's treatment of Horikawa quintics (cf. [Reid D-E]).

Chapter 4

The trigonal and hyperelliptic cases

In this chapter we study halfcanonical curves moving in the canonical linear system of a surface belonging to the family (I.3) and the families (III) of Theorem 1.4.1. The corresponding graded rings turn out to have codimension 6 and 8 respectively and the calculations become longer. However, all such curves can be studied also as Cartier divisors in a rational normal scroll, making things easier from a geometric point of view.

4.1 Rational normal scrolls

Before I compute the rings, I collect together some well known facts about rational normal scrolls and I introduce some useful notation. For a detailed exposition, see Chapter 2 of [Reid PC]. This section is based on it.

Intrinsically, a rational normal scroll is a $\mathbb{P}^{n-1}$-bundle over a rational normal curve (that I assume to be $\mathbb{P}^1$):

$$F \longrightarrow \mathbb{P}^1,$$

this can be written as the projectivisation of a rank n vector bundle over the projective line:

$$\mathbb{F}(a_1, \dots, a_n) := \mathbb{P}\left(\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)\right), \quad \text{for some } a_i \in \mathbb{Z}.$$

Points in $\mathbb{F}(a_1, \dots, a_n)$ are in one-to-one correspondence with orbits of the following $\mathbb{G}_m \times \mathbb{G}_m$- action on $\mathbb{A}^2_k \times \mathbb{A}^n_k$:

$$(\lambda, \mu) \cdot (t_1, t_2; u_1, \dots, u_n) \mapsto (\lambda t_1, \lambda t_2; \lambda^{-a_i} \mu u_1, \dots \lambda^{-a_n} \mu u_n).$$

Rational functions on the scroll, are quotients of homogeneous polynomials in $t_1, t_2, u_1, \dots, u_n$ of the same *bi-degree*; a couple of examples:

1. The only homogeneous monomials of bi-degree $(1, 0)$ up to scalar multiplication are t_1 and t_2; the ratio $(t_1 : t_2)$ defines the morphism

$$\pi : \mathbb{F}(a_1, \dots, a_n) \longrightarrow \mathbb{P}^1$$

   that gives the scroll the structure of a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$; the structure group is the diagonal subgroup of $\mathrm{PGL}(n)$.

2. If $a_1, \dots, a_n \geq 0$, for the bi-degree $(0, 1)$, such polynomials are the $\sum_{i=1}^n (a_i + 1)$-dimensional k-vector space based by

$$\bigoplus_{i=1}^n S^{a_i}(t_1, t_2) \otimes u_i.$$

   In particular, if we consider the surface case with $a_1 = 0$, $a := a_2 > 0$, the ratio $(u_1 : t_1^a u_2 : \dots : t_2^a u_2)$ defines a morphism

$$\mathbb{F}_a := \mathbb{F}(0, a) \longrightarrow \mathbb{P}^{a+1},$$

   so $\mathbb{F}_a$ is the blowup of the cone over the rational normal curve of degree a.

Either from its definition as the projectivisation of a line bundle over $\mathbb{P}^1$ or guessing from previous examples, we see that

$$\mathrm{Pic}\ (\mathbb{F}(a_1, \dots, a_n)) \cong \mathbb{Z}L \oplus \mathbb{Z}M,$$

where L is the divisor class of a fibre of the projection $\pi : \mathbb{F}(a_1, \dots, a_n) \to \mathbb{P}^1$, that is, the divisor class of any linear form in t_1, t_2 and M is the divisor class of any monomial of the form $t_1^b t_2^c u_i$ with $b + c = a_i$. In particular, if we take $t_1^{a_i} u_i$ we get

$$M \sim a_i L + F_i,$$

where F_i is the subscroll $(u_i = 0) \cong \mathbb{F}(a_1, \dots, \widehat{a_i}, \dots, a_n) \subset \mathbb{F}(a_1, \dots, a_n)$. It is also clear that

$$-K_{\mathbb{F}(a_1, \dots, a_n)} \sim 2L + \sum_{i=1}^n F_i \sim (2 - \sum_{i=1}^n a_i)L + nM.$$

Example Let $a \geq 0$. Consider the surface scroll $\mathbb{F}_a := \mathbb{F}(0,a)$. I will always choose as generators of Pic $(\mathbb{F}_a)$ the divisor class of a fibre that I shall denote by A and

$$B := F_2 = \text{div } (u_2).$$

So, using previous notation, $bL + cM \sim (ac+b)A + cB$.

One can deduce the intersection pairing in $\mathbb{F}_a$ considering the linear system $|D| := |aA + B|$, which defines the birational morphism that we mentioned earlier,

$$(t_1 : t_2; u_1 : u_2) \mapsto (u_1 : t_1^a u_2 : \cdots : t_2^a u_2)$$

from the surface scroll to the cone over the rational normal curve of degree a. Such a morphism contracts B to the vertex of the cone, and maps the curve $(u_1 = 0) = F_1 \subset \mathbb{F}_a$ to a hyperplane section of the cone that obviously does not pass through its vertex, thus $B(aA+B) = 0$, giving $B^2 = -a$, because $AB = 1$ is obvious. It is also clear that $A^2 = 0$.

4.2 The trigonal family

Let us consider now a nonsingular curve C admitting a linear system $|D| = g^2_7$ such that:

1. $2D = K_C$, in particular $g(C) = 8$.
2. $|D| = P + |D'|$ has only one base point P and

$$\varphi_{D'} : C \xrightarrow{\text{3-to-1}} \overline{C} \cong \mathbb{P}^1 \subset \mathbb{P}^2.$$

Thus, C is a trigonal curve, (that is, admits a g^1_3, in fact $|D| = P + 2g^1_3$). Our aim is to compute $R(C,D)$, let us start with the following observation:

Proposition 4.2.1. *C cannot be hyperelliptic.*

Proof. This follows from the base point free pencil trick; let D_2, D_3 be effective divisors such that $|D_i| = g^1_i$ for $i = 2,3$. Let $\{t_1, t_2\}$ be a basis of $H^0(D_3)$. There is an exact sequence of sheaves

$$0 \to \mathcal{O}_C(D_2) \otimes \mathcal{O}_C(-D_3) \to \mathcal{O}_C(D_2) \oplus \mathcal{O}_C(D_2) \xrightarrow{t_1,t_2} \mathcal{O}_C(D_2) \otimes \mathcal{O}_C(D_3) \to 0$$

where the first map sends a section s of $\mathcal{O}_C(D_2)\otimes\mathcal{O}_C(-D_3)$ to the pair $(st_2,-st_1)$. Taking cohomology gives that the kernel of the natural map

$$t_1H^0(D_2)\oplus t_2H^0(D_2)\longrightarrow H^0(D_2+D_3)$$

is isomorphic to $H^0(D_2-D_3)=\{0\}$. It follows that

$$h^0(D_2+D_3)\geq h^0(D_2)+h^0(D_3)=4.$$

Since the genus of C is 8, then D_2+D_3 is a degree five special divisor and previous inequality together with $2{\cdot}8{-}1>5$ contradicts Clifford's theorem. Therefore such a D_2 cannot exist, (and conversely, no hyperelliptic curve of genus 8 can be trigonal).

Q.E.D.

Using previous result we can argue on the canonical model of C $\varphi_{K_C}(C)\subset\mathbb{P}^7$, to show that it is contained in a surface scroll:

Proposition 4.2.2. *$\varphi_{K_C}(C)\subset\mathbb{P}^7$ is a Cartier divisor in a rational surface scroll $\mathbb{F}_a$.*

Proof. This is a well known fact from classical curve theory (cf. Proposition 3.1, Ch. III § 3 of [ACGH]). However we can explicitly exhibit the inclusion $\widetilde{C}\subset\mathbb{F}_a$ noticing that, by the geometric version of Riemann-Roch, 3 points move in a g^1_3 on $\widetilde{C}$ if and only if they are collinear. This is the restriction of the pencil $|A|$ to the canonical model of C. Consider the scroll $\mathbb{F}_2\cong\mathbb{F}(2,4)$ embedded in $\mathbb{P}^7$ by $|4A+B|$. Because of the intersection pairing, we must have $C\sim bA+3B$ for some $b\in\mathbb{N}$. Now $K_{\mathbb{F}_2}=-4A-2B$, so the adjunction formula implies:

$$\begin{aligned} K_C &= (bA+3B-4A-2B)|_C \\ &= ((b-4)A+B)|_C \end{aligned}$$

Thus $14=\deg\,(K_C)=((b-4)A+B)(bA+3B)=4b-18$, which finally gives $\widetilde{C}\sim 8A{+}3B$. An anologous analysis can be done to show that the general case $\mathbb{F}(3,3)\cong\mathbb{F}_0$ is not possible for $|D|=P+2g^1_3$ and $2D=K_C$. One would get $C\sim 5A+3B$ and $K_C=(3A+B)|_C$ which is nonsense. Q.E.D.

In particular, $C \cdot B = (8A + 3B)B = 2$. Now, by assumption $K_C = 4g^1_3 + 2P$ and on the other hand, by the adjunction formula:

$$\begin{aligned} K_C &= (8A + 3B - 4A - 2B)|_C \\ &= (4A + B)|_C \end{aligned}$$

and since the g^1_3 is precisely $|A|$ restricted to C, it follows that $|2P| = |B|_C$ so C is tangent to B. Now computing $R(C, D)$ is more or less automatic:

4.2.1 Computation of $R(C, D)$

Take coordinates $(\bar{t}_1, \bar{t}_2; \overline{u}_1, \overline{u}_2)$ on $\mathbb{F}_2$. Whenever I write sections of sheaves on $\mathbb{F}_2$ without the over lines I actually mean sections of the restrictions of such sheaves to $C \subset \mathbb{F}_2$. For instance, if I write $u : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P)$ for the constant section then by previous observations $u^2 = u_2$, etc.

Let us start the calculation; in degree 1, we have $H^0(C, D) = u \otimes H^0(C, 2g^1_3) = \langle ut_1^2, ut_1t_2, ut_2^2 \rangle$. Let $x_1 := ut_1^2$, $x_2 := ut_1t_2$, $x_3 := ut_2^2$.

In degree 2 we already get one relation, namely $x_1x_3 - x_2^2$ which defines the image $\varphi_D(C) \subset \mathbb{P}^2$. It follows that $\dim_{\mathbb{C}} \operatorname{Sym}^2(x_1, x_2, x_3) = 5$, so I need 3 new degree 2 generators that I can take from

$$\mathcal{O}_C(2D) = \mathcal{O}_C(K_C) \cong \mathcal{O}_{\mathbb{F}_2}(4A + B)|_C.$$

We have

$$\mathcal{O}_{\mathbb{F}_2}(4A + B) \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4),$$

$$H^0(\mathbb{F}_2, 4A + B) = \langle \overline{u}_1 \cdot S^2(\bar{t}_1, \bar{t}_2) \rangle \oplus \langle \overline{u}_2 \cdot S^4(\bar{t}_1, \bar{t}_2) \rangle$$

and $\langle u_2 \cdot S^4(t_1, t_2) \rangle = \operatorname{Sym}^2 H^0(C, D)$, thus I can take $y_1 := u_1t_1^2$, $y_2 := u_1t_1t_2$ and $y_3 := u_1t_2^2$ to extend to a basis of $H^0(C, K_C)$. This gives us already 4 new relations in degree 3 and one in degree 4:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & y_1 & y_2 \\ x_2 & x_3 & y_2 & y_3 \end{pmatrix}.$$

It follows that, with my choice of bases/generators so far, I still need 2 new degree 3 generators z_1, z_2. This however is independent of the choice:

Proposition 4.2.3. $R(C, D)$ *is minimally generated by 3 generators in degree 1* *plus 3 and 2 on degrees 2 and 3 respectively. Thus* $C \subset \mathbb{P}(1^3, 2^3, 3^2)$ *so* $R(C, D)$ *is a codimension 6 Gorenstein ring.*

Proof. Write D' for a divisor on C such that $|D'| = g_3^1$ and $\frac{1}{2}K_C = D = 2D' + P$ where P is the only base point of $|D|$. Consider the natural map

$$\varphi : H^0(C, D') \otimes H^0(C, K_C) \longrightarrow H^0(C, 5D' + 2P).$$

By Castelnuovo's free pencil trick and Riemann-Roch, $\ker\varphi$ has dimension

$$h^0(3D' + 2P) = 1 - 8 + 11 + h^0(D') = 6.$$

Hence $\dim_{\mathbb{C}} \operatorname{Im} \varphi = 10$, that is, φ is onto and it follows that the image of $H^0(C, D) \otimes H^0(C, K_C) \to H^0(C, 3D)$ is the same as the image of

$$\psi : H^0(C, D') \otimes H^0(C, 5D' + 2P) \longrightarrow H^0(C, 6D' + 2P),$$

and this, again by the pencil trick, has dimension $20 - h^0(4D' + 2P) = 12$. Therefore we always need 2 extra generators to get a basis for the 14-dimensional space $H^0(C, 3D)$. Q.E.D.

Without loss of generality, I can assume that the base point has coordinates $(0, 1; 1, 0)$ in $\mathbb{F}_2$ thus $\varphi_D(P) = (0 : 0 : 1)$. Recall that $C \in |8A + 3B|$. We have

$$\mathcal{O}_{F_2}(8A + 3B) \cong \mathcal{O}(2)_{\mathbb{P}^1} \oplus \mathcal{O}(4)_{\mathbb{P}^1} \oplus \mathcal{O}(6)_{\mathbb{P}^1} \oplus \mathcal{O}(8)_{\mathbb{P}^1}$$

so, C is defined by a section of the form

$$\overline{u}_1^3 f_2 + \overline{u}_1^2 \overline{u}_2 f_4 + \overline{u}_1 \overline{u}_2^2 f_6 + \overline{u}_2^3 f_8;$$

where the f_ℓ are homogeneous forms of degree ℓ in $\overline{t}_1, \overline{t}_2$. Because of my choice of coordinates for the base point and because $B \cdot C = 2$, I can assume that $f_2 := \overline{t}_1^2$. Therefore I will write

$$(C \subset \mathbb{F}_2) \;:=\; V(\; \overline{t}_1^2 \overline{u}_1^3 + f_4(\overline{t}_1, \overline{t}_2)\overline{u}_1^2 \overline{u}_2 + f_6(\overline{t}_1, \overline{t}_2)\overline{u}_1 \overline{u}_2^2 + f_8(\overline{t}_1, \overline{t}_2)\overline{u}_2^3 \;). \tag{4.1}$$

Let z_1, z_2 be degree 3 generators completing a basis of $H^0(C, 3D)$. By Max Noether's theorem, $H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, 4D)$ is a surjection. Hence

$x_iz_j \in H^0(C,K_C) \otimes H^0(C,K_C)$ for $1 \le i \le 3$ and $j = 1,2$. Now

$$H^0(C,K_C) \otimes H^0(C,K_C) = \langle u_1^2 \cdot S^4(t_1,t_2)\rangle \oplus \langle u_1u_2 \cdot S^6(t_1,t_2)\rangle \oplus \langle u_2^2 \cdot S^8(t_1,t_2)\rangle. \tag{4.2}$$

Let $\mathcal{I}_n$, $1 \le n \le 3$ be the direct summands of (4.2), respectively. It is clear that $x_iz_j \notin \mathcal{I}_n$ for any i,j and $n \neq 1$, because otherwise z_j would vanish at least once at the base point P, which is impossible because $|3D|$ is very ample and all the sections in $S^3(x_1,x_2,x_3)$ and $S^1(x_1,x_2,x_2) \otimes S^1(y_1,y_2,y_3)$ vanish at least once at P. Therefore I can choose $z_1/u, z_2/u$ to be one of the following monomials:

$$t_1^2u_1^2,\ t_1t_2u_1^2,\ t_2^2u_1^2.$$

Finally, it is easy to see repeating M. Noether's argument but going to degree 6, that the last one is impossible. Thus:

$$uz_1 = t_1^2u_1^2,\ uz_2 = t_1t_2u_1^2$$

and by Reid's theorem, I need no further generators. The following $10 = \binom{5}{2}$ relations are simple consequences of the choice of generators, we may read them as: *the ratio* $(t_1 : t_2)$ *is preserved*:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \end{pmatrix}, \tag{4.3}$$

notice that none of the above though, allows us to write either x_1z_1 or x_3z_2 in terms of $\mathrm{Sym}^2(H^0(C,K_C))$. This leads to a couple of rolling factors equations that are also a consequence of the expressions defining the generators of the ring, or equivalently, of the key variety from which we will cut C:

$$\begin{matrix} x_1z_1 - y_1^2 \\ x_1z_2 - y_1y_2 \end{matrix},\quad \begin{matrix} x_2z_2 - y_2^2 \\ x_3z_2 - y_2y_3 \end{matrix}. \tag{4.4}$$

Choosing different renderings for x_1z_2 and x_2z_2, we can write the same equations (with a couple of repetitions of relations already listed in (4.3)) as:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & y_1 \\ y_1 & y_2 & z_1 \end{pmatrix},\quad \bigwedge^2 \begin{pmatrix} x_3 & y_2 & y_3 \\ y_2 & z_1 & z_2 \end{pmatrix}. \tag{4.5}$$

Observe that we can extend the first matrix to get the 4 new relations all at once, together with 6 repeated relations as follows:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 \\ y_1 & y_2 & y_3 & z_1 & z_2 \end{pmatrix}, \tag{4.6}$$

whereas the second matrix can only be extended to give 3 of the 4 new relations (it misses $x_1z_1 - y_1^2$):

$$\bigwedge^2 \begin{pmatrix} x_2 & x_3 & y_2 & y_3 \\ y_1 & y_2 & z_1 & z_2 \end{pmatrix}. \tag{4.7}$$

Finally, if follows from (4.1) that:

$$uy_1z_1 = t_1^2u_2(u_1^2f_4(t_1,t_2) + u_1u_2f_6(t_1,t_2) + u_2^2f_8(t_1,t_2)); \tag{4.8}$$

it is clear that we can always render $u_1^2f_4(t_1,t_2)$, $u_1u_2f_6(t_1,t_2)$ and $u_2^2f_8(t_1,t_2)$ as linear combinations of monomials in $S^2(y_1,y_2,y_3)$, $S^1(y_1,y_2,y_3)\otimes S^2(x_1,x_2,x_3)$ and $S^4(x_1,x_2,x_3)$ respectively. Therefore:

$$uy_1z_1 = ux_1P, \tag{4.9}$$

where $P = P(x_i,y_j)$; $i,j \in \{1,2,3\}$ is a homogeneous form of degree 4. This gives the following relation:

$$y_1z_1 - x_1P, \tag{4.10}$$

moreover, I can roll factors once using the matrix from (4.6) to get:

$$z_1^2 - y_1P \tag{4.11}$$

and finally I can roll factors in both, (4.10) and (4.11), only that this time using the matrix from (4.3) to get a total of 6 independent relations cutting the curve C:

$$\begin{matrix} y_1z_1 - x_1P \\ y_2z_1 - x_2P \\ y_3z_1 - x_3P \end{matrix}\ , \quad \begin{matrix} z_1^2 - y_1P \\ z_1z_2 - y_2P \\ z_2^2 - y_3P \end{matrix} \tag{4.12}$$

this completes the picture of our matrices from (4.6) and (4.7), giving:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & z_1 \\ y_1 & y_2 & y_3 & z_1 & z_2 & P \end{pmatrix}, \quad \bigwedge^2 \begin{pmatrix} x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & P \end{pmatrix}. \tag{4.13}$$

It follows by comparing Hilbert series that these relations (20 independent ones) form a minimal generating set for the ideal of relations of $R(C,D)$. The following theorem summarises these results and gives another format to present all the relations more concisely as Pfaffians:

Theorem 4.2.4. *Let C be a nonsingular genus 8 curve admitting a linear system $|D|$ with only one base point $\widetilde{P}$ and satisfying the following properties:*

1. $|D| = \widetilde{P} + 2g_3^1$.

2. $2D = K_C$.

3. $\varphi_{g_6^2} : C \xrightarrow{\text{3-to-1}} \mathcal{C} \subset \mathbb{P}^2$, *where $\mathcal{C}$ is a nonsingular conic.*

Then the halfcanonical ring $R := R(C,D) = \bigoplus_{n\geq 0} H^0(C, nD)$ *is isomorphic to:*

$$k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2]/I,$$

with generators of degrees $1,1,1,2,2,2,3,3$ respectively and the ideal I is minimally generated by the 20 different homogeneous forms given by the 2×2 minors of the following 3 matrices:

$$A := \begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \end{pmatrix},$$

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & z_1 \\ y_1 & y_2 & y_3 & z_1 & z_2 & P \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & P \end{pmatrix}$$

Or equivalently, by the 4×4 Pfaffians of the following 8×8 skew matrix:

$$O := \begin{pmatrix} 0 & x_1 & x_2 & y_1 & 0 & 0 & 0 \\ & x_2 & x_3 & y_2 & 0 & 0 & 0 \\ & & x_2 & y_1 & y_1 & y_2 & z_1 \\ & & & 0 & y_2 & y_3 & z_2 \\ & & & & z_1 & z_2 & P \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \quad \text{of weights} \quad \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ & 1 & 1 & 2 & 2 & 2 & 3 \\ & & 1 & 2 & 2 & 2 & 3 \\ & & & 2 & 2 & 2 & 3 \\ & & & & 3 & 3 & 4 \\ & & & & & 3 & 4 \\ & & & & & & 4 \end{pmatrix},$$

where $P = P(x_i, y_j)$ is a generic homogeneous degree 4 form.

The Magma codes for these two presentations can be found at A.4-A.6.

Remark I will discuss later on (see Chapter 6), the conveniences and inconveniences of the presentations of $R(C,D)$ given in previous theorem. The main problem is that they are not useful to deform the ring in any way that allows us to get one of the rings we already know from previous chapter. Fortunately, $R(C,D)$ can also be presented as follows (cf. computer code at A.7):

Let $R(C,D) \cong k[x_1,x_2,x_2,y_1,y_2,z_1,z_2]/I$ be the halfcanonical ring from Theorem 4.2.4. Then the ideal of relations I is generated by the 4×4 Pfaffians of the following two 7×7 matrices:

$$T_1 := \begin{pmatrix} 0 & z_1 & x_1 & x_2 & y_1 & y_2 \\ & z_2 & x_2 & x_3 & y_2 & y_3 \\ & & 0 & P & 0 & 0 \\ & & & y_1 & 0 & 0 \\ & & & & -z_1 & -z_2 \\ & & & & & 0 \end{pmatrix} \quad \text{and} \quad T_2 := \begin{pmatrix} 0 & z_1 & x_2 & x_1 & y_2 & y_1 \\ & z_2 & x_3 & x_2 & y_3 & y_2 \\ & & 0 & P & 0 & 0 \\ & & & y_2 & 0 & 0 \\ & & & & -z_2 & -z_1 \\ & & & & & 0 \end{pmatrix},$$

both of weights:

$$\begin{pmatrix} 0 & 3 & 1 & 1 & 2 & 2 \\ & 3 & 1 & 1 & 2 & 2 \\ & & 4 & 4 & 5 & 5 \\ & & & 2 & 3 & 3 \\ & & & & 3 & 3 \\ & & & & & 4 \end{pmatrix}.$$

As it is, this presentation has no genuine 4×4 Pfaffians since any of them has one or every term equal to zero. Nevertheless, the position of the degree 0 entry suggests a way to deform the equations (see Theorem 6.3.2).

It is also important to notice here that the 2×2 minors of matrices A, M and N of Theorem 4.2.4 can be obtained in a much simpler way by taking the 2×2 minors of the following *doubly symmetric* matrix:

$$\begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & P \end{pmatrix}.$$

4.3 The hyperelliptic family

The halfcanonical curves (C,D) corresponding to the surfaces of family (III) of Theorem 1.4.1, are nonsingular genus 8 hyperelliptic curves, because $|D|$ having 3 base points implies that $\varphi_D : C \to \mathbb{P}^2$ is a 2-to-1 map onto a nonsingular conic. Thus

$$D = P_1 + P_2 + P_3 + D',$$

where $\varphi_D = \varphi_{D'}$ and $|D'| = 2g_2^1$. That is, $\frac{1}{2}D'$ is a hyperelliptic divisor on C. In this section I give a presentation of the ring $R(C, D)$ for a polarised curve in this situation. There is a clear overlap between what is done here and what I did in the bielliptic and the trigonal case, since on one hand we have again a double cover (but the base curve is much simpler now), and on the other hand these curves are again contained in a rational normal scroll (namely, $\mathbb{F}_9$).

4.3.1 Geometry of hyperelliptic curves

In this subsection I collect together some well known facts about hyerelliptic curves that will be needed when I compute the halfcanonical ring (cf. [Reid D-E], §4).

Consider a double cover $\pi : C \longrightarrow \mathbb{P}^1$. I will write B and R for the branch and ramification divisors respectively and $\iota : C \longrightarrow C$ for the hyperelliptic involution. Since $h^0(K_C) = 8$, the Riemann-Hurwitz formula gives

$$2 \cdot 8 - 2 = \deg(R) - 2 \cdot 2,$$

that is, π is branched at 18 points $Q_1, \dots, Q_{18} \in \mathbb{P}^1$ (necessarily distinct, since C is nonsingular). These points lift to 18 ramification points $P_1, \dots, P_{18} \in C$ that are precisely the points P such that $h^0(2P) = 2$, that is, $|2P_i| = g_2^1$ (the Weierstrass points of the hyperelliptic curve C).

Proposition 4.3.1. *1. The ramification divisor of π, $R := \sum_{i=1}^{18} P_i$, satisfies $|R| = 9g_2^1$.*

2. The canonical linear system $|K_C|$ is $(g-1)g_2^1 = 7g_2^1$.

Proof. An affine equation of C is of the form $y^2 = \overline{f}_{18}(t)$, where $\overline{f}_{18}$ is a degree 18 polynomial with distinct roots, it is also harmless to assume that it has no constant term. In such an affine patch, the rational function t^9/y has divisor of zeroes equal to 9 times a Weierstrass point counted twice and divisor of poles equal to the 18 Weierstrass points all together, hence the first part.
Using the same affine plane model, it is clear that any 1-form on C can be written as $p(t)dt/y$, where $p(t)$ is a polynomial of degree at most 7 and the second part follows. Moreover, a basis for the space of 1-forms is $\{dt/y, tdt/y, \dots, t^7 dt/y\}$. It

follows that the canonical map $\varphi_{K_C} : C \longrightarrow \mathbb{P}^7$ is not an embedding any more, but the hyperelliptic map π followed by the $7th$ Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^7$. Q.E.D.

Proposition 4.3.2. *Let $P \in C$ be a Weierstrass point. Then*

$$R(C, 2P) \cong k[t_1, t_2, y]/(y^2 - f_{18}(t_1, t_2)),$$

where $\deg(t_1) = \deg(t_2) = 1$, $\deg(y) = 9$ *and f_{18} is a homogeneous form of degree 18 defining the branch locus $Q_j \in \mathbb{P}^1 = \mathbb{P}(H^0(C, 2P))$; $1 \le j \le 18$.*

Proof. It follows from proposition 4.3.1 and the general theory of double covers (cf. proposition 3.3.1), that the decomposition of $\pi_* \mathcal{O}_C$ into the ± 1−eigensheaves of ι is:

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-9);$$

the algebra structure is given by the polynomial f_{18}. This defines a multiplication map

$$f_{18} : \mathcal{O}_{\mathbb{P}^1}(-18) \longrightarrow \mathcal{O}_{\mathbb{P}^1}.$$

The calculation follows at once; we have for any $n \ge 1$:

$$H^0(C, 2nP) \cong H^0(\mathbb{P}^1, \mathcal{O}(n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(n-9)).$$

Therefore, all the generators of R in degree n for $1 \le n \le 8$, come from $S^n(t_1, t_2)$, where $\{t_1, t_2\}$ is a basis of $H^0(C, g^1_2) \cong H^0(\mathbb{P}^1, \mathcal{O}(1))$ and the last generator needed is of degree 9: y, which is in the -1 eigensheaf of the involution and satisfies $y^2 = f_{18}(t_1, t_2)$. Q.E.D.

Remark Previous proposition gives the model of the hyperelliptic curve as a hypersurface in the weighted projective plane $\mathbb{P}^2(1, 1, 9)$, which is isomorphic to $\overline{\mathbb{F}_9}$, that is, to the image of the rational surface scroll under the linear system $|9A + B|$; C is linearly equivalent to $18A + 2B$. $|A|$ restricted to C is the hyperelliptic class. Finally, the canonical class of $\mathbb{F}_9$ is $-11A - 2B$. Thus $K_C = (7A)|_C$, giving another proof of the second part of proposition 4.3.1.

4.3.2 Halfcanonical rings

The main result of this subsection is the presentation of the halfcanonical ring $R(C, D)$ in Theorem 4.3.3. It is well known (cf. [ACGH], p.288) that an effective divisor like in our situation: $D = P_1 + P_2 + P_3 + D'$ where the P_i are base points of $|D|$

and $|D'| = 2g_2^1$, is made out of Weierstrass points. Recall the notation $R = \sum_{i=1}^{18} P_i$ for the ramification divisor of the double cover π and let

$$u : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1 + P_2 + P_3), \quad v : \mathcal{O}_C \hookrightarrow \mathcal{O}_C\left(\sum_{i=4}^{18} P_i\right)$$

be the constant sections.

Since any 3 different points in $\mathbb{P}^1$ can be mapped to any different 3 points, it is harmless to choose a basis $\{t_1, t_2\}$ of $H^0(\mathbb{P}^1, \mathcal{O}(1)) \cong H^0(C, \frac{1}{2}D')$ such that the degree 3 form that defines the branch points $\pi(P_i)$, $1 \leq i \leq 3$ is given by

$$u^2 = t_1^2 t_2 - t_1 t_2^2.$$

Thus I can assume:

$$\varphi_D(P_1) = (1:0), \quad \varphi_D(P_2) = (1:1), \quad \varphi_D(P_3) = (0:1)$$

and since all the branch points are required to be different, the degree 15 form defining the rest of the branch locus, $v^2 = g_{15}(t_1, t_2)$, must have nonzero coefficients multiplying t_1^{15} and t_2^{15} (so it does not vanish at $(1:0)$ nor $(0:1)$). Thus in general I can write

$$g_{15} = t_1^{15} + \sum_{i=1}^{15} a_i t_1^{15-i} t_2^i,$$

where $a_{15} \neq 0$ and $\sum_{i=1}^{15} a_i \neq -1$ (so g_{15} does not vanish at $(1:1)$). Of course I must further require g_{15} to have distinct roots. This however will not play a role when giving useful presentations of the halfcanonical ring $R(C, D)$.

The ring requires 3 generators in degree 1. We have

$$H^0(C, D) \cong uH^0(\mathbb{P}^1, \mathcal{O}(2)),$$

so I will choose the following basis:

$$x_1 := ut_1^2, \quad x_2 := ut_1t_2, \quad x_3 := ut_2^2.$$

In degree 2, $2D = K_C$ and $|K_C| \sim 7g_2^1$. Thus I can construct a basis of $H^0(C, 2D)$ from any basis of $H^0(\mathbb{P}^1, \mathcal{O}(7))$. Choosing $S^7(t_1, t_2)$ has the advantage of simplifying the way we can write the trivial relations holding between the generators of $R(C,D)$. Let

$$\begin{array}{rcl} w_1 &:=& t_1^7, \\ w_2 &:=& t_1^6 t_2, \\ w_3 &:=& t_1^5 t_2^2, \\ w_4 &:=& t_1^4 t_2^3, \\ w_5 &:=& t_1^3 t_2^4, \\ w_6 &:=& t_1^2 t_2^5, \\ w_7 &:=& t_1 t_2^6, \\ w_8 &:=& t_2^7. \end{array}$$

Because of my choice of the images of the base points, I must choose as new generators $y_1 := t_1^7$ and $y_3 := t_2^7$. It is clear also that any choice of the form $y_2 := t_1^i t_2^{7-i}$ for $1 \le i \le 6$ will suffice to get a basis. I will choose $y_2 := t_1^3 t_2^4$. Thus I have:

$$\begin{array}{rcl} w_1 &=& t_1^7 = y_1, \\ w_2 &=& t_1^6 t_2 = w_3 + x_1^2, \\ w_3 &=& t_1^5 t_2^2 = w_4 + x_1 x_2, \\ w_4 &=& t_1^4 t_2^3 = w_5 + x_1 x_3 = w_5 + x_2^2, \\ w_5 &=& t_1^3 t_2^4 = y_2, \\ w_6 &=& t_1^2 t_2^5 = w_5 - x_2 x_3, \\ w_7 &=& t_1 t_2^6 = w_6 - x_3^2, \\ w_8 &=& t_2^7 = y_3. \end{array}$$

It reemains to choose generators for $H^0(C,3D)$. Consider the divisor $\widetilde{D} := \sum_{i=4}^{18} P_i - 2D'$. Then $\mathcal{O}(D) \cong \mathcal{O}(\widetilde{D})$, because the rational function uv/t_1^9 has divisor of zeroes equal to $\sum_{i=4}^{18} P_i$ and divisor of poles equal to $P_1 + P_2 + P_3 + 3D'$. Thus $D \sim \widetilde{D}$ and multiplication by it, induces the isomorphism. Clearly uv/t_1^9 is a section of the -1-eigensheaf of the involution. Thus the isomorphism is interchanging the positive and negative eigensheaves. It follows that:

$$\pi_*(\mathcal{O}_C(D)) \cong u\mathcal{O}_{\mathbb{P}^1}(2) \oplus v\mathcal{O}_{\mathbb{P}^1}(-4),$$

then, tensoring with $\mathcal{O}_{\mathbb{P}^1}(7) \cong \pi_*(\mathcal{O}_C(K_C))$ and taking cohomology we get:

$$H^0(C,\mathcal{O}_C(3D)) \cong u \otimes \langle S^9(t_1,t_2)\rangle \oplus v \otimes \langle S^3(t_1,t_2)\rangle. \tag{4.14}$$

We already have generators for the first direct summand of (4.14). Therefore I only need 4 new generators in degree 3 to form a basis (and generate $R(C,D)$). I will choose them as follows:

$$z_1 := vt_1^3, \quad z_2 := vt_1^2t_2, \quad z_3 := vt_1t_2^2, \quad z_4 := vt_2^3.$$

Finally, it is clear that the ideal of relations is generated by the obvious relations preserving the ratio $(t_1 : t_2)$ plus the rolling factors relations deduced from the identities

$$u^2 = t_1^2t_2 - t_1t_2^2 \quad \text{and} \quad v^2 = t_1^{15} + \sum_{i=1}^{15} a_i t_1^{15-i} t_2^i.$$

All previous results are summarized in the following theorem:

Theorem 4.3.3. *Let C be a nonsingular genus 8 curve admitting a linear system $|D|$ with three distinct base points, P_1, P_2, P_3 and satisfying the following properties:*

1. $|D| = P_1 + P_2 + P_3 + 2g_2^1$.
2. $2D = K_C$.

Then the halfcanonical ring $R := R(C, D) = \bigoplus_{n\geq 0} H^0(C, nD)$ is isomorphic to:

$$k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, z_4]/I,$$

with generators of degrees $1, 1, 1, 2, 2, 2, 3, 3, 3, 3$ respectively and the ideal I is generated by the homogeneous forms obtained by taking the 2×2 minors of the 2×12 matrix A, where

$$A := \begin{pmatrix} x_1 & x_2 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & z_1 & z_2 & z_3 \\ x_2 & x_3 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & z_2 & z_3 & z_4 \end{pmatrix},$$

with $w_1 := y_1, w_5 := y_2, w_8 := y_3$ and the w_i for $i \in \{2, 3, 4, 6, 7\}$ are defined recursively:

$$\begin{aligned} w_4 &:= w_5 + x_1x_3 = w_5 + x_2^2, \\ w_3 &:= w_4 + x_1x_2, \\ w_2 &= w_3 + x_1^2; \end{aligned} \qquad \begin{aligned} w_6 &:= w_5 - x_2x_3, \\ w_7 &:= w_6 - x_3^2. \end{aligned}$$

Plus 7 rolling factors relations of the form:

$$z_1^2 - y_1^2 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1} \right) - y_2^2 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j \right),$$

$$z_1z_2 - y_1w_2 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1} \right) - y_2w_6 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j \right),$$

$$z_1z_3 - y_1w_3 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_2w_7 \cdot \left(\sum_{j=1}^{8} a_{j+7}w_j\right),$$

$$z_1z_4 - y_1w_4 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_2y_3 \cdot \left(\sum_{j=1}^{8} a_{j+7}w_j\right),$$

$$z_2z_4 - y_1y_2 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - w_6y_3 \cdot \left(\sum_{j=1}^{8} a_{j+7}w_j\right),$$

$$z_3z_4 - y_1w_6 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - w_7y_3 \cdot \left(\sum_{j=1}^{8} a_{j+7}w_j\right),$$

$$z_4^2 - y_1w_7 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_3^2 \cdot \left(\sum_{j=1}^{8} a_{j+7}w_j\right),$$

or any other rendering where possible. Moreover, the constants $a_1, \dots a_{15} \in k$ *are required to satisfy:*

1. $a_{15} \neq 0$,
2. $\sum_{i=1}^{15} a_i \neq -1$,
3. *The polynomial* $1 + \sum_{i=1}^{15} a_i t^i$ *has* 15 *distinct roots.*

It turns out that only 28 of the 66 elements of $\bigwedge^2 A$ suffice to generate I, together with its rolling factors relations. In the last chapter, I will give a format to write these 35 equations as Pfaffians of three types of matrices. The reader can find the Magma code for this family of rings in the appendix A.8.

Chapter 5

Codimension 4 Surfaces

In this chapter I use the extension algorithm of §2.2 to calculate the canonical rings of the surfaces of families (I.1) and (I.2) of Theorem 1.4.1. I start with the family (I.2) because the canonical rings of surfaces of family (I.1) were already studied in detail by Bauer, Catanese and Pignatelli in [Bauer et al]. Their main result however, was obtained relying heavily in their deep understanding of the geometry of the surfaces whereas in this chapter I show that it can be recovered using the much simpler geometry of the halfcanonical curves of Theorem 3.2.4 and the hyperplane section principle.

5.1 Superelliptic Rings

For convenience of the reader, I recall Theorem 3.3.5:

Theorem. *Let C be a nonsingular genus 8 curve admitting a linear system $|D|$ with only one base point $\widetilde{P}$ and satisfying the following properties:*

1. $|D| = \widetilde{P} + g^2_6$.
2. $2D = K_C$.
3. $\varphi_{g^2_6} : C \xrightarrow{\text{2-to-1}} E \subset \mathbb{P}^2$, *where E is an elliptic curve.*

Then the halfcanonical ring $R \coloneqq R(C,D) = \bigoplus_{n\geq 0} H^0(C, nD)$ is isomorphic to:

$$k[x_1, x_2, x_3, y_1, y_2, z]/I,$$

with generators of degrees $1,1,1,2,2,3$ respectively and the ideal I is minimally generated by 9 homogeneous forms r_i, $1 \leq i \leq 9$, obtained as follows:

Let

$$A := \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 & x_3^2 - \alpha x_1 x_2 - \beta x_1^2 & y_1 & z \end{pmatrix}, \quad M := \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 + y_1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then r_i, $1 \le i \le 6$, are the 2×2 minors of A, whereas r_7, r_8, r_9 are given by

$$AM({}^T\!A) = 0.$$

Let $Q := x_3^2 - \alpha x_1 x_2 - \beta x_1^2$. Consider the relation

$$x_1^2 Q_1 + x_2^2 Q_2 + x_3^2 (Q_3 + y_1) - y_2^2. \tag{5.1}$$

Before starting the extension calculations it is convenient to notice that relation $x_1 Q - x_2^3$ allows to render any degree 4 monomial in the x_i's involving powers of x_2 greater than or equal to 2 as one involving a power greater than or equal to 2 in x_1 or x_3 except possibly for $x_1 x_2^2 x_3$. However, if this term appeared in 5.1, relation $x_1 y_1 - x_2^2 x_3$ would imply that y_1 appears multiplied by x_1^2, which is impossible since it was shown that y_1 must be multiplying exclusively x_3^2 in order for the curve to have the right genus. Therefore I can assume $Q_2 = 0$ and for similar reasons it follows that Q_3 can not include any powers of x_3 at all. Thus:

$$\begin{aligned} Q_1 &:= d_1 x_1^2 + d_2 x_1 x_2 + d_3 x_1 x_3 + d_4 x_2^2 + d_5 x_2 x_3 + d_6 x_3^2, \\ Q_3 &:= e x_1 x_2, \end{aligned}$$

for some $d_1, \dots, d_6, e \in \mathbb{C}$. Thus in the rest of the section I write:

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 & Q & y_1 & z \end{pmatrix}, \quad M := \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e x_1 x_2 + y_1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The 9 relations generating the Gorenstein codimension 4 ideal I with their corresponding degree will be named as shown in table 5.1.

5.1.1 Syzygies from the $AM({}^T\!A)$ format

In order to construct the canonical ring of the surfaces of family (I.2), I must lift the relations $r_1, \dots, r_9$ allowing successive powers of the new degree 1 variable x_0 in

Table 5.1: Generators of I

Name	Relation	Degree
r_1	$x_1Q - x_2^3$	3
r_2	$x_1y_1 - x_2^2x_3$	3
r_3	$x_1z - x_2^2y_2$	4
r_4	$x_2y_1 - x_3Q$	3
r_5	$x_2z - y_2Q$	4
r_6	$x_3z - y_1y_2$	4
r_7	$x_1^2Q_1 + x_3^2(ex_1x_2 + y_1) - y_2^2$	4
r_8	$x_1x_2^2Q_1 + x_3y_1(ex_1x_2 + y_1) - y_2z$	5
r_9	$x_2^4Q_1 + y_1^2(ex_1x_2 + y_1) - z^2$	6

such a way that the corresponding syzygies also lift. One can show using computer algebra that the syzygy module of our codimension 4 ideals is minimally generated by 16 elements. The $AM(^TA)$ format is called *flexible* because these syzygies are automatically implied by the format itself as explained below.

For each of the four 2×3 submatrices of A, I can obtain 2 linearly independent syzygies by cloning either row and taking the determinant of the resulting 3×3 matrix that vanishes by construction. This gives table 5.2

Table 5.2: First set of syzygies

Name	Syzygy	Degree
σ_1	$x_1r_4 - x_2r_2 + x_3r_1$	4
σ_2	$x_2^2r_4 - Qr_2 + y_1r_1$	5
σ_3	$x_1r_5 - x_2r_3 + y_2r_1$	5
σ_4	$x_2^2r_5 - Qr_3 + zr_1$	6
σ_5	$x_1r_6 - x_3r_3 + y_2r_2$	5
σ_6	$x_2^2r_6 - y_1r_3 + zr_2$	6
σ_7	$x_2r_6 - x_3r_5 + y_2r_4$	5
σ_8	$Qr_6 - y_1r_5 + zr_4$	6

Now consider the following matrix:

$$A^* := \begin{pmatrix} -x_2^2 & x_1 \\ -Q & x_2 \\ -y_1 & x_3 \\ -z & y_2 \end{pmatrix}.$$

Then

$$A^*A = \begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ -r_1 & 0 & r_4 & r_5 \\ -r_2 & -r_4 & 0 & r_6 \\ -r_3 & -r_5 & -r_6 & 0 \end{pmatrix}.$$

Therefore the identity $A^*(AM({}^TA)) = (A^*A)(M({}^TA))$ gives 8 syzygies (the left hand side is a matrix whose entries are combinations of the relations r_7, r_8 and r_9, whereas the right hand side is a matrix whose entries are combinations of $r_1, \ldots, r_6$). Explicitly:

$$A^*(AM({}^TA)) = \begin{pmatrix} -x_2^2r_7 + x_1r_8 & -x_2^2r_8 + x_1r_9 \\ -Qr_7 + x_2r_8 & -Qr_8 + x_2r_9 \\ -y_1r_7 + x_3r_8 & -y_1r_8 + x_3r_9 \\ -zr_7 + y_2r_8 & -zr_8 + y_2r_9 \end{pmatrix},$$

$$(A^*A)(M({}^TA)) = \begin{pmatrix} x_3(ex_1x_2 + y_1)r_2 - y_2r_3 & y_1(ex_1x_2 + y_1)r_2 - zr_3 \\ -x_1Q_1r_1 + x_3(ex_1x_2 + y_1)r_4 - y_2r_5 & -x_2^2Q_1r_1 + y_1(ex_1x_2 + y_1)r_4 - zr_5 \\ -x_1Q_1r_2 - y_2r_6 & -x_2^2Q_1r_2 - zr_6 \\ -x_1Q_1r_3 - x_3(ex_1x_2 + y_1)r_6 & -x_2^2Q_1r_3 - y_1(ex_1x_2 + y_1)r_6 \end{pmatrix},$$

this gives us 8 new linearly independent syzygies that extend the set of 8 syzygies of table 5.2 into a basis of the corresponding module. They are listed in table 5.3.

Remark A final observation before starting the extension algorithm that helps shorten the calculations enormously consists in listing some second order syzygies (we actually know the module of such higher syzygies has rank 9, because of the Gorenstein palindromic free resolution of our codimension 4 ring). One verifies that

Table 5.3: Second set of syzygies

Name	Syzygy	Degree
σ_9	$x_1r_8 - x_2^2r_7 - x_3(ex_1x_2 + y_1)r_2 + y_2r_3$	6
σ_{10}	$x_2r_8 - Qr_7 + x_1Q_1r_1 - x_3(ex_1x_2 + y_1)r_4 + y_2r_5$	6
σ_{11}	$x_3r_8 - y_1r_7 + x_1Q_1r_2 + y_2r_6$	6
σ_{12}	$y_2r_8 - zr_7 + x_1Q_1r_3 + x_3(ex_1x_2 + y_1)r_6$	7
σ_{13}	$x_1r_9 - x_2^2r_8 - y_1(ex_1x_2 + y_1)r_2 + zr_3$	7
σ_{14}	$x_2r_9 - Qr_8 + x_2^2Q_1r_1 - y_1(ex_1x_2 + y_1)r_4 + zr_5$	7
σ_{15}	$x_3r_9 - y_1r_8 + x_2^2Q_1r_2 + zr_6$	7
σ_{16}	$y_2r_9 - zr_8 + x_2^2Q_1r_3 + y_1(ex_1x_2 + y_1)r_6$	8

the following identities hold:

$$\begin{aligned} x_2\sigma_2 &\equiv Q\sigma_1 \\ x_2\sigma_4 &\equiv Q\sigma_3 \\ x_2\sigma_6 &\equiv Q\sigma_5 \\ x_2\sigma_8 &\equiv Q\sigma_7 \end{aligned}$$

$$x_2\sigma_5 - x_3\sigma_3 + y_2\sigma_1 \equiv x_1\sigma_7$$

$$\begin{aligned} x_2^2\sigma_{10} &\equiv Q\sigma_9 \\ x_2^2\sigma_{11} &\equiv y_1\sigma_9 \\ x_2^2\sigma_{12} &\equiv z\sigma_9 \end{aligned} \quad , \qquad \begin{aligned} x_1\sigma_{14} &\equiv x_2\sigma_{13} \\ x_1\sigma_{15} &\equiv x_3\sigma_{13} \\ x_1\sigma_{16} &\equiv y_2\sigma_{13} \end{aligned} \quad .$$

In words, these identities say that every syzygy has a monomial multiple that is a quasi-homogeneous linear combination of only 5 syzygies, namely: $\{\sigma_1, \sigma_3, \sigma_5, \sigma_9, \sigma_{13}\}$.

5.1.2 Extending the ring

Let S be a surface of type (I.2) of Theorem 1.4.1. By the extension algorithm of §2.2, to construct $R(S, K_S)$ I must start by computing the ring

$$R(2C, D^{(2)}) = k[x_0, x_1, x_2, x_3, y_1, y_2, z]/(I^{(2)}, x_0^2),$$

where $I^{(2)}$ is generated by 9 relations $r_1^{(2)}, \dots, r_9^{(2)}$ that reduce modulo x_0 to relations $r_1, \dots, r_9$ of table 5.1 and that satisfy the syzygies $\sigma_1, \dots, \sigma_{16}$. However, the above remark implies that I only need to do perform the calculations for the syzygies σ_i

with $i \in \{1, 3, 5, 9, 13\}$. We have:

$$\begin{aligned}
r_1^{(2)} &:= r_1 + x_0[a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 + a_{1,4}x_2^2 + a_{1,5}x_2x_3 + a_{1,6}x_3^2 + a_{1,7}y_1 + a_{1,8}y_2] \\
r_2^{(2)} &:= r_2 + x_0[a_{2,1}x_1^2 + \cdots + a_{2,8}y_2] \\
r_4^{(2)} &:= r_4 + x_0[a_{4,1}x_1^2 + \cdots + a_{4,8}y_2],
\end{aligned}$$

for some $a_{i,j} \in \mathbb{C}$, $i = 1, 2, 4$, $1 \le j \le 8$. As σ_1 must lift to a syzygy $\sigma_1^{(2)}$ with $\sigma_1 \equiv \sigma_1^{(2)}($ mod $x_0)$ in $R(C, D)$, I need to impose:

$$x_1[a_{4,1}x_1^2 + \cdots + a_{4,8}y_2] \equiv x_2[a_{2,1}x_1^2 + \cdots + a_{2,8}y_2] - x_3[a_{1,1}x_1^2 + \cdots + a_{1,8}y_2] \mod I.$$

This leads to: $a_{1,6} = a_{1,7} = a_{1,8} = a_{2,4} = a_{2,7} = a_{2,8} = a_{3,1} = a_{3,8} = 0$, $a_{3,2} = a_{2,1}$, $a_{3,3} = -a_{1,1}$, $a_{3,4} = a_{2,2}$, $a_{3,5} = a_{2,3} - a_{1,2}$, $a_{3,6} = -a_{1,3}$, $a_{3,7} = a_{2,5} - a_{1,4}$, $a_{2,6} = a_{1,5}$. Therefore, so far we have:

$$\begin{aligned}
r_1^{(2)} &= r_1 + x_0[a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 + a_{1,4}x_2^2 + a_{1,5}x_2x_3], \\
r_2^{(2)} &= r_2 + x_0[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3 + a_{2,5}x_2x_3 + a_{1,5}x_3^2], \\
r_4^{(2)} &= r_4 + x_0[a_{2,1}x_1x_2 - a_{1,1}x_1x_3 + a_{2,2}x_2^2 + (a_{2,3} - a_{1,2})x_2x_3 - a_{1,3}x_3^2 + (a_{2,5} - a_{1,4})y_1].
\end{aligned}$$

The next syzygy in my list of 5, σ_3, involves the degree 4 relations r_3 and r_5, so I need a basis for the Riemann-Roch space $H^0(3D)$ (which is 14-dimensional) to write general extension polynomials of the form x_0p, with $p \in H^0(3D)$. Similarly, when I work out the effect of the remaining syzygies I will require bases for $H^0(nD)$ for $n = 4, 5$. Using the relations it is easy to see that the choice of monomials shown in table 5.4 actually works.

Table 5.4: Bases of $H^0(C, \mathcal{O}_C(nD))$

Space	Dimension	Elements defining a basis
$H^0(3D)$	14	$x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^2x_3,$ $x_2x_3^2, x_3^2, x_1y_2, x_2y_2, x_3y_1, x_3y_2, z$
$H^0(4D)$	21	$x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_3^2, x_1x_2^2x_3,$ $x_1x_2x_3^2, x_1x_3^3, x_2^2x_3^2, x_2x_3^3, x_3^4, y_1^2, y_1y_2,$ $x_3^2y_1, x_1^2y_2, x_1x_2y_2, x_1x_3y_2, x_2^2y_2, x_2x_3y_2, x_3^2y_2$
$H^0(5D)$	28	$x_1^5, x_1^4x_2, x_1^4x_3, x_1^3x_2^2, x_1^3x_2x_3, x_1^3x_3^2, x_1^2x_2^2x_3,$ $x_1^2x_2x_3^2, x_1^2x_3^3, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_3^4, x_2^2x_3^3, x_2x_3^4,$ $x_3^5, x_1^3y_2, x_1^2x_2y_2, x_1^2x_3y_2, x_1x_2^2y_2, x_1x_2x_3y_2, x_1x_3^2y_2$ $x_2^2x_3y_2, x_2x_3^2y_2, x_3^3y_2, x_3^3y_1, x_3y_1^2, x_3y_1y_2, y_1z$

If I write

$$
\begin{aligned}
r_3^{(2)} &:= r_3 + x_0[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,4}x_1x_2^2 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2 + a_{3,7}x_2^2x_3 \\
&\quad + a_{3,8}x_2x_3^2 + a_{3,9}x_3^2 + a_{3,10}x_1y_2 + a_{3,11}x_2y_2 + a_{3,12}x_3y_1 + a_{3,13}x_3y_2 + a_{3,14}z], \\
r_5^{(2)} &:= r_5 + x_0[a_{5,1}x_1^3 + \cdots + a_{5,14}z]
\end{aligned}
$$

and I plug these in σ_3 together with $r_1^{(2)}$ I get:

$$
\begin{aligned}
x_1[a_{5,1}x_1^3 + \cdots + a_{5,14}z] &\equiv x_2[a_{3,1}x_1^3 + \cdots + a_{3,14}z] \\
&\quad - y_2[a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 + a_{1,4}x_2^2 + a_{1,5}x_2x_3] \mod I.
\end{aligned}
$$

In order for previous identity to hold, the following coefficients are ought to be zero: $a_{3,4}$, $a_{3,7}$, $a_{3,9}$, $a_{3,12}$, $a_{3,14}$, $a_{5,1}$, $a_{5,3}$, $a_{5,6}$, $a_{5,9}$. Furthermore, one gets also these identities: $a_{3,13} = a_{1,5}$, $a_{5,2} = a_{3,1}$, $a_{5,4} = a_{3,2}$, $a_{5,5} = a_{3,3}$, $a_{5,7} = a_{3,5}$, $a_{5,8} = a_{3,6}$, $a_{5,10} = -a_{1,1}$, $a_{5,11} = a_{3,10} - a_{1,2}$, $a_{5,12} = a_{3,8}$, $a_{5,13} = -a_{1,3}$, $a_{5,14} = a_{3,11} - a_{1,4}$.

We can update our lifted relations as follows:

$$
\begin{aligned}
r_1^{(2)} &= r_1 + x_0[a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 + a_{1,4}x_2^2 + a_{1,5}x_2x_3], \\
r_2^{(2)} &= r_2 + x_0[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3 + a_{2,5}x_2x_3 + a_{1,5}x_3^2], \\
r_3^{(2)} &= r_3 + x_0[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2 + a_{3,8}x_2x_3^2 + a_{3,10}x_1y_2 \\
&\quad + a_{3,11}x_2y_2 + a_{1,5}x_3y_2], \\
r_4^{(2)} &= r_4 + x_0[a_{2,1}x_1x_2 - a_{1,1}x_1x_3 + a_{2,2}x_2^2 + (a_{2,3} - a_{1,2})x_2x_3 - a_{1,3}x_3^2 + (a_{2,5} - a_{1,4})y_1], \\
r_5^{(2)} &= r_5 + x_0[a_{3,1}x_1^2x_2 + a_{3,2}x_1x_2^2 + a_{3,3}x_1x_2x_3 + a_{3,5}x_2^2x_3 + a_{3,6}x_2x_3^2 - a_{1,1}x_1y_2 \\
&\quad + (a_{3,10} - a_{1,2})x_2y_2 + a_{3,8}x_3y_1 - a_{1,3}x_3y_2 + (a_{3,11} - a_{1,4})z].
\end{aligned}
$$

Repeating the process using the syzygy σ_5 is completely analogous, I need to introduce new coefficients $a_{6,i} \in \mathbb{C}$, $i = 1, \ldots, 14$ to write:

$$
r_6^{(2)} := r_6 + x_0[a_{6,1}x_1^3 + \cdots, a_{6,14}z],
$$

then, I demand $x_1[a_{6,1}x_1^3 + \cdots + a_{6,14}z]$ to be congruent to

$$
\begin{aligned}
&x_3[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2 + a_{3,8}x_2x_3^2 + a_{3,10}x_1y_2 \\
&+ a_{3,11}x_2y_2 + a_{1,5}x_3y_2] - y_2[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3 + a_{2,5}x_2x_3 + a_{1,5}x_3^2] \mod I.
\end{aligned}
$$

The result is that $a_{3,8}$, $a_{6,1}$, $a_{6,2}$, $a_{6,4}$, $a_{6,7}$, $a_{6,12}$, $a_{6,14}$ all vanish, plus the following extra conditions: $a_{3,11} = a_{2,5}$, $a_{6,3} = a_{3,1}$, $a_{6,5} = a_{3,2}$, $a_{6,6} = a_{3,3}$, $a_{6,8} = a_{3,5}$, $a_{6,9} = a_{3,6}$, $a_{6,10} = -a_{2,1}$, $a_{6,11} = -a_{2,2}$, $a_{6,13} = a_{3,10} - a_{2,3}$.

The calculations get much longer to write them down in full when one con-

siders the syzygies σ_9 and σ_{13}, but still is a completely mechanical bookkeeping exercise, so I will write only what I got at the end of both steps.
The current list of liftings is:

$$
\begin{aligned}
r_1^{(2)} &= r_1 + x_0[a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 + a_{1,4}x_2^2 + a_{1,5}x_2x_3],\\
r_2^{(2)} &= r_2 + x_0[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3 + a_{2,5}x_2x_3 + a_{1,5}x_3^2],\\
r_3^{(2)} &= r_3 + x_0[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2 + a_{3,10}x_1y_2\\
&\quad + a_{2,5}x_2y_2 + a_{1,5}x_3y_2],\\
r_4^{(2)} &= r_4 + x_0[a_{2,1}x_1x_2 - a_{1,1}x_1x_3 + a_{2,2}x_2^2 + (a_{2,3} - a_{1,2})x_2x_3 - a_{1,3}x_3^2 + (a_{2,5} - a_{1,4})y_1],\\
r_5^{(2)} &= r_5 + x_0[a_{3,1}x_1^2x_2 + a_{3,2}x_1x_2^2 + a_{3,3}x_1x_2x_3 + a_{3,5}x_2^2x_3 + a_{3,6}x_2x_3^2 - a_{1,1}x_1y_2\\
&\quad + (a_{3,10} - a_{1,2})x_2y_2 - a_{1,3}x_3y_2 + (a_{2,5} - a_{1,4})z].\\
r_6^{(2)} &= r_6 + x_0[a_{3,1}x_1^2x_3 + a_{3,2}x_1x_2x_3 + a_{3,3}x_1x_3^2 + a_{3,5}x_2x_3^2 + a_{3,6}x_3^3 - a_{2,1}x_1y_2\\
&\quad - a_{2,2}x_2y_2 + (a_{3,10} - a_{2,3})x_3y_2]
\end{aligned}
$$

Let $r_8^{(2)} = r_8 + x_0r_8'$, where $r_8' := a_{8,1}x_1^4 + \cdots + a_{8,21}x_3^2y_2$ and the $a_{8,i}$ are labeled respecting the order of the basis of $H^0(4D)$ given in table 5.4. Analogously, let $r_7^{(2)} := r_7 + x_0r_7'$ and $r_9^{(2)} := r_9 + x_0r_9'$ with $r_7' := a_{7,1}x_1^3 + \cdots + a_{7,14}z$ and $r_9' := a_{9,1}x_1^5 + \cdots + a_{9,28}y_1z$. After considering the congruences modulo I given by σ_9 and σ_{13}:

$$
\begin{aligned}
&x_1r_8' \equiv x_2^2r_7' + x_3(ex_1x_2 + y_1)[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3 + a_{2,5}x_2x_3 + a_{1,5}x_3^2]\\
&-y_2[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2 + a_{3,10}x_1y_2 + a_{2,5}x_2y_2 + a_{1,5}x_3y_2],
\end{aligned}
$$

$$
\begin{aligned}
&x_1r_9' \equiv x_2^2r_8' + y_1(ex_1x_2 + y_1)[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3 + a_{2,5}x_2x_3 + a_{1,5}x_3^2]\\
&-z[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2 + a_{3,10}x_1y_2 + a_{2,5}x_2y_2 + a_{1,5}x_3y_2].
\end{aligned}
$$

It is found that the following coefficients must be zero: $a_{1,5}$, $a_{2,5}$, $a_{3,10}$, whereas the terms $a_{7,12}x_0x_3y_1$ and $a_{7,14}x_0z$ do not die, but can be omitted using a different rendering given by relations $r_4^{(2)}$ and $r_5^{(2)}$ respectively; of course we also get necessary conditions holding between the surviving constants, the result is that the ideal $I^{(2)}$ is minimally generated by 9 relations that are more conveniently listed in 2 groups as explained next:

$$\begin{aligned}
r_1^{(2)} &= r_1 + x_0[a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{1,3}x_1x_3 + a_{1,4}x_2^2],\\
r_2^{(2)} &= r_2 + x_0[a_{2,1}x_1^2 + a_{2,2}x_1x_2 + a_{2,3}x_1x_3],\\
r_3^{(2)} &= r_3 + x_0[a_{3,1}x_1^3 + a_{3,2}x_1^2x_2 + a_{3,3}x_1^2x_3 + a_{3,5}x_1x_2x_3 + a_{3,6}x_1x_3^2],\\
r_4^{(2)} &= r_4 + x_0[a_{2,1}x_1x_2 - a_{1,1}x_1x_3 + a_{2,2}x_2^2 + (a_{2,3} - a_{1,2})x_2x_3 - a_{1,3}x_3^2 - a_{1,4}y_1],\\
r_5^{(2)} &= r_5 + x_0[a_{3,1}x_1^2x_2 + a_{3,2}x_1x_2^2 + a_{3,3}x_1x_2x_3 + a_{3,5}x_2^2x_3 + a_{3,6}x_2x_3^2 - a_{1,1}x_1y_2\\
&\quad -a_{1,2}x_2y_2 - a_{1,3}x_3y_2 - a_{1,4}z],\\
r_6^{(2)} &= r_6 + x_0[a_{3,1}x_1^2x_3 + a_{3,2}x_1x_2x_3 + a_{3,3}x_1x_3^2 + a_{3,5}x_2x_3^2 + a_{3,6}x_3^3 - a_{2,1}x_1y_2\\
&\quad -a_{2,2}x_2y_2 - a_{2,3}x_3y_2].
\end{aligned}$$

These are, modulo x_0^2, the 2×2 minors of the matrix $\widetilde{A} := A + x_0A_1$, where A is the matrix of Theorem 3.3.5:

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 & Q & y_1 & z \end{pmatrix}$$

and A_1 is defined as follows:

$$A_1 := \begin{pmatrix} 0 & -a_{1,4} & 0 & 0 \\ 0 & \ell_1 & \ell_2 & \mathfrak{s} \end{pmatrix}, \quad \text{with} \quad \begin{aligned} \ell_1 &:= a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3,\\ \ell_2 &:= a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3,\\ \mathfrak{s} &:= a_{3,1}x_1^2 + a_{3,2}x_1x_2 + a_{3,3}x_1x_3 + a_{3,5}x_2x_3 + a_{3,6}x_3^2. \end{aligned}$$

The remaining 3 relations are, modulo x_0^2, the 3 distinct entries of the symmetric matrix $\widetilde{A}\widetilde{M}({}^T\widetilde{A})$, where $\widetilde{M} := M + x_0M_1$. Once again I kept the notation for M:

$$M = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ex_1x_2 + y_1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

while M_1 is:

$$M_1 := \begin{pmatrix} \ell_3 & 0 & \frac{1}{2}a_{7,5}x_2 & \frac{1}{2}a_{7,10} \\ 0 & \ell_4 & 0 & \frac{1}{2}a_{7,11} \\ \frac{1}{2}a_{7,5}x_2 & 0 & \ell_5 & \frac{1}{2}a_{7,13} \\ \frac{1}{2}a_{7,10} & \frac{1}{2}a_{7,11} & \frac{1}{2}a_{7,13} & 0 \end{pmatrix}, \quad \text{with} \quad \begin{aligned} \ell_3 &:= a_{7,1}x_1 + a_{7,2}x_2 + a_{7,3}x_3,\\ \ell_4 &:= a_{7,4}x_1 + a_{7,7}x_3,\\ \ell_5 &:= a_{7,6}x_1 + a_{7,8}x_2 + a_{7,9}x_3. \end{aligned}$$

The 16 syzygies holding between our 9 lifted relations can be obtained in the same way described previously for the original halfcanonical ring $R(C,D)$. Repeating the process to extend $R(2C, D^{(2)})$ to $R(3C, D^{(3)})$ and so on, is actually easier at each step, because allowing higher powers of x_0 requires to consider fewer arbitrary coefficients when one imposes the congruences for the corresponding syzygies to lift. The process is particularly easy to be done by hand in the case of the first 6 relations; they can be put in the same determinantal in every degree and the final output, after renaming the coefficients, is of the form:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 - \delta_1 x_0 & x_3 & y_2 \\ x_2^2 - \delta_2 x_0^2 & Q + \ell_1 x_0 + \delta_6 x_0^2 & y_1 + \ell_2 x_0 + \delta_{10} x_0^2 & z + \mathfrak{s} x_0 + \ell_3 x_0^2 + \delta_{19} x_0^3 \end{pmatrix}, \quad (5.2)$$

$$\text{where:} \quad \begin{array}{rcl} \ell_1 & := & \delta_3 x_1 + \delta_4 x_2 + \delta_5 x_3, \\ \ell_2 & := & \delta_7 x_1 + \delta_8 x_2 + \delta_9 x_3, \\ \ell_3 & := & \delta_{11} x_1 + \delta_{12} x_2 + \delta_{13} x_3, \\ \mathfrak{s} & := & \delta_{14} x_1^2 + \delta_{15} x_1 x_2 + \delta_{16} x_1 x_3 + \delta_{17} x_2 x_3 + \delta_{18} x_3^2. \end{array}$$

Provided $\delta_1 \delta_2 \neq 0$, the first minor, $x_1(Q + \ell_1 x_0 + \delta_6 x_0^2) - (x_2^2 - \delta_2 x_0^2)(x_2 - \delta_1 x_0)$, allows me to assume that there are no terms involving powers of x_0 greater than 2 in the relation involving y_2^2 (that is, the relation that reduces to r_7 mod x_0, or the relation defining the double cover). This makes possible to fit the remaining relations in the same format. Before presenting the final ring, it is convenient to make coordinate changes of the form: $x_2 - \delta_1 x_0 \mapsto x_2$, $y_1 + \cdots \mapsto y_1$ and $z + \cdots \mapsto z$. These of course have the effect of making most of the entries of the matrix given in 5.2 to be equal to the ones of the original matrix used to define the halfcanonical ring $R(C, D)$, the little price I got to pay, is that I am forced to re-allow the term in $x_0 x_2^3$ when I lift r_7.

All these calculations are summarized in the following result:

Theorem 5.1.1. *Consider the graded ring* $R := k[x_0, x_1, x_2, x_3, y_1, y_2, z]/I$, *where* $\deg x_i = 1$, $\deg y_j = 2$, $\deg z = 3$ *and* I *is the homogeneous ideal generated by 9 elements defined as follows:*

$$\textit{Let } \mathcal{A} := \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 + a_1 x_0 x_2 + a_2 x_0^2 & \mathcal{Q} & y_1 & z \end{pmatrix},$$

$$\textit{with} \quad \mathcal{Q} := x_3^2 + a_3 x_1 x_2 + a_4 x_1^2 + (a_5 x_1 + a_6 x_2 + a_7 x_3) x_0 + a_8 x_0^2$$

$$\textit{and let } \mathcal{M} := \begin{pmatrix} \mathcal{Q}_1 & \frac{1}{2} a_{19} x_0^2 & \frac{1}{2}(a_{20} x_0 x_2 + a_{21} x_0^2) & \frac{1}{2} a_{22} x_0 \\ \frac{1}{2} a_{19} x_0^2 & \mathcal{Q}_2 & \frac{1}{2} a_{27} x_0^2 & \frac{1}{2} a_{28} x_0 \\ \frac{1}{2}(a_{20} x_0 x_2 + a_{21} x_0^2) & \frac{1}{2} a_{27} x_0^2 & \mathcal{Q}_3 & \frac{1}{2} a_{35} x_0 \\ \frac{1}{2} a_{22} x_0 & \frac{1}{2} a_{28} x_0 & \frac{1}{2} a_{35} x_0 & -1 \end{pmatrix},$$

$$
\text{with} \quad \begin{cases} \mathcal{Q}_1 := & a_9 x_1^2 + a_{10}x_1x_2 + a_{11}x_1x_3 + a_{12}x_2^2 + a_{13}x_2x_3 + a_{14}x_3^2 \\ & +(a_{15}x_1 + a_{16}x_2 + a_{17}x_3)x_0 + a_{18}x_0^2, \\ \mathcal{Q}_2 := & (a_{23}x_1 + a_{24}x_2 + a_{25}x_3)x_0 + a_{26}x_0^2, \\ \mathcal{Q}_3 := & a_{29}x_1x_2 + a_{30}y_1 + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)x_0 + a_{34}x_0^2. \end{cases}
$$

The first 6 generators are the 2×2 *minors of* $\mathcal{A}$ *and the last 3 are the distinct entries of the symmetric* 2×2 *matrix* $\mathcal{AM}({}^T\mathcal{A})$. *Then, for a general choice of parameters* $a_i \in \mathbb{C}$, $1 \le i \le 35$, $R = R(S, K_S)$ *where* S *is a surface of general type with* $p_g = 4$ *and* $K^2 = 7$ *whose canonical map is 2-to-1 onto a cubic surface in* $\mathbb{P}^3$, *that is,* S *is a surface belonging to subfamily (I.2) of Theorem 1.4.1 and conversely, up to a change of coordinates, the canonical ring of a surface of subfamily (I.2) is of the aforementioned form.*

5.1.3 An explicit deformation family

The *superelliptic* rings of Theorem 5.1.1 are presented in the $AM({}^TA)$ format which is *flexible* (that is, it carries not only with the 9 relations defining the ideal but also implies all the 16 syzygies needed to generate the corresponding module, as we explained earlier on). This implies that I can do small deformations in the entries of the matrices involved to obtain flat families of graded rings. Formally, this is a consequence of the following well known technical result:

Theorem 5.1.2. (cf. [Decker-Lossen], Theorem 5.12). *Let* $\mathfrak{D}$ *be an Artinian local* k*-algebra with residue field* k *and maximal ideal* $\mathfrak{m}$. *Let* $I = (f_1, \dots, f_r)$ *be an ideal of* $\mathfrak{D}[x_1, \dots, x_n]$ *and let* $\overline{f_1}, \dots \overline{f_r} \in k[x_1, \dots, x_n]$ *be the reductions of* $f_1, \dots, f_r$ *modulo* $\mathfrak{m}$. *Then the following are equivalent:*

1. $\mathfrak{D}[x_1, \dots, x_n]/I$ *is a flat* $\mathfrak{D}$*-module.*
2. $Tor_1^{\mathfrak{D}}(k, \mathfrak{D}[x_1, \dots, x_n]/I) = 0$.
3. *The syzygies between* $\overline{f_1}, \dots, \overline{f_r}$ *are generated by the reductions modulo* $\mathfrak{m}$ *of the syzygies between* $f_1, \dots, f_r$.

Exploiting this and using some computer algebra, we can prove the next result:

Theorem 5.1.3. *The canonical surfaces of general type with* $p_g = 4$ *and* $K^2 = 7$ *such that* $|K|$ *has only one simple base point and defines a map of degree 2 onto a cubic surface in* $\mathbb{P}^3$ *are degenerations of surfaces with the same invariants but whose canonical map has one simple base point and defines a birational map onto a* *surface of degree six in $\mathbb{P}^3$. In terms of Bauer's classification, surfaces of type (I.2) are degenerations of surfaces of type (I.1).*

Proof. I start taking a convenient subfamily of rings from Theorem 5.1.1 (keeping the same notation). Let $t \in \mathbb{C}$ be a small affine parameter and consider

$$\mathcal{A}_t := \mathcal{A} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ ty_2 & tx_1x_3 & 0 & 0 \end{pmatrix}$$

along with the restriction $a_4 = 0$ on $\mathcal{Q} = x_3^2+a_3x_1x_2+a_4x_1^2+(a_5x_1+a_6x_2+a_7x_3)x_0+a_8x_0^2$ (this forces $a_3 \neq 0$, otherwise the surface does not have canonical volume 7. Thus in the sequel, I will assume $a_3 = 1$) and let

$$\mathcal{M}_t := \mathcal{M} + \begin{pmatrix} ty_2 & 0 & 0 & 0 \\ & tx_2^2 & 0 & 0 \\ & & 0 & 0 \\ \text{sym} & & & -1 \end{pmatrix},$$

with $\mathcal{M}$ restricted as follows:

$$\begin{array}{rl} \mathcal{Q}_1 := & x_1^2 + (a_{15}x_1 + a_{16}x_2 + a_{17}x_3)x_0 + a_{18}x_0^2, \\ \mathcal{Q}_3 := & y_1 + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)x_0 + a_{34}x_0^2. \end{array}, \quad \text{no restriction on } \mathcal{Q}_2.$$

Define I_t to be the homogeneous ideal generated by $\bigwedge^2 \mathcal{A}_t$ and ${}^tr_7, {}^tr_8, {}^tr_9$ where

$$\mathcal{A}_t\mathcal{M}_t({}^T\mathcal{A}_t) = \begin{pmatrix} {}^tr_7 & {}^tr_8 \\ {}^tr_8 & {}^tr_9 \end{pmatrix}.$$

Then, by construction, $R = R_0 := k[x_0, x_1, x_2, x_3, y_1, y_2, z]/I_0$ is the canonical ring of a surface of type (I.2) (provided I can choose the remaining free coefficients so that Proj R_0 is not badly singular, this is a sanity check that is better done by computer algebra). I claim that, for a small nonzero t, R_t is the canonical ring of a surface of type (I.1). Indeed, the hyperplane section ring $\overline{R}_t := R_t/(x_0)$ has relations given by

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ x_2^2 + ty_2 & x_3^2 + x_1x_2 + tx_1x_3 & y_1 & z \end{pmatrix},$$

plus $x_1^2(x_1^2 + ty_2) + tx_2^4 + x_3^2y_1 - y_2^2$ and two rolling factors forms of degrees 5 and 6. Eliminating variables of degrees 2 and 3 it is found that the equation of the image curve in $\mathbb{P}^2$ is given by the following sextic:

$$\begin{gathered}(t^3-1)x_2^6-(t^2+2)x_1^2x_2^4+2tx_1^2x_2^3x_3+2x_1x_2^3x_3^2-x_1^4x_2^2-t^2x_1x_2^2x_3^3\\ +(t^3+2t)x_1^4x_2x_3+(t^2+2)x_1^3x_2x_3^2+t^3x_1x_2x_3^4+t^2x_2x_3^5\\ -t^2x_1^4x_3^2-2tx_1^3x_3^3-x_1^2x_3^4,\end{gathered}$$

the terms are ordered with respect to the powers of x_2 purposely; the last 3 monomials are the only ones not involving x_2 and we can see that the sextic has 2 nodes with a tangency at the line joining them. It follows from Theorem 3.2.4 that $\overline{R}_t$ is the halfcanonical ring of a genus 8 curve. By construction, R_t is a flat extension of the halfcanonical Gorenstein codimension 4 ring $\overline{R}$. It follows that Proj R_t is a canonical surface S with $p_g = 4$ and $K_S^2 = 7$. Since the canonical image $\varphi_{K_S}(S) \subset \mathbb{P}^3$ is a sextic, the classification of Theorem 1.4.1 implies that S belongs to the subfamily (I.1) as stated. It is clear that all the surfaces of Theorem 5.1.1 can be obtained as deformations of the surfaces of the form Proj R_0. Finally, the flatness of the family R_t follows from Theorem 5.1.2 and the flexibility of the $AM({}^TA)$ format. Q.E.D.

Remark The reader can use the relevant Magma code in the appendix to verify that the varieties constructed in previous theorem are indeed nonsingular. The notation is the same and one can play around with the long lists of free coefficients as long as the stated restrictions are respected. Magma online calculator takes only a few seconds to test the nonsingularity for the values as they are and $0 \le t < 1$ (the surface is still nonsingular for $t = 1$, but it is easy to see that the corresponding halfcanonical ring does not give a nonsingular genus 8 curve. Thus our construction is no longer valid as it is. Of course this is harmless for our purposes, since we are interested in small deformations of the special fibre given by $t = 0$). One can also do the sanity check for the hyperplane section rings by erasing the generator x_0 from the first line of the code, then declare $x_0 := 0$; and corroborate that the one dimensional scheme one gets is a nonsingular genus 8 curve.

5.2 The Bauer-Catanese-Pignatelli case

In section §3.2, we studied rings $\overline{R} = R(C, \frac{1}{2}K_C)$ where $C \in |K_S|$ and S is a surface of general type belonging to the subfamily (I.1) of Bauer's classification. That is, $|K_S|$ has only one simple base point and defines a birational map onto a sextic. In particular, it was shown (cf. Theorem 3.2.5) that $R(C, D) = k[x_1, x_2, x_3, y_1, y_2, z]/I$

for an ideal I generated by the 9 independent 4×4 Pfaffians of a matrix of the form:

$$\begin{pmatrix} 0 & y_2 & Q_1 & y_1 + y_2 & z \\ & x_3 & x_1 & x_2 + x_3 & y_1 + y_2 \\ & & z + F_3 & x_1 Q_2 & Q_1 Q_2 \\ & & & x_2 Q_3 + x_3 Q_4 & y_1 Q_3 + y_2 Q_4 \\ & & & & 0 \end{pmatrix},$$

where $Q_1 := x_2 x_3$, F_3 is a homogeneous form of degree 3 in x_2, x_3 and Q_i are quadratic homogeneous forms such that Q_3 depends only on y_1, y_2 and the rest of them do not depend on these two variables. This matrix has a shape that is somewhat friendly to the geometry of the halfcanonical curve C as it was discussed in §3.2. However, it can be modified in several ways to get many other results.

First observe that Pfaffian 23.45: $x_3(x_2Q_3 + x_3Q_4) - x_1^2 Q_2 + (x_2 + x_3)(z + F_3)$ can be rewritten as

$$x_3^2(Q_4 - Q_3) - x_1^2 Q_2 + (x_2 + x_3)(z + F_3 + x_3 Q_3). \tag{5.3}$$

This is the first rolling factor relation of the ring and we know that for the construction to work, it is necessary for Pfaffians 13.45 and 13.46 to be rollings of (5.3) with respect to the matrix formed by rows 1, 2 and columns 3 to 6. One sees at once that changing the entry $y_1 Q_3 + y_2 Q_4$ for $y_2(Q_4 - Q_3)$ will do the trick. Thus if $\overline{F}_3 := F_3 + x_3 Q_3$ and $\overline{Q}_3 := Q_4 - Q_3$, the 4×4 Pfaffians of the following skew matrix generate the same ideal I:

$$\begin{pmatrix} 0 & y_2 & Q_1 & y_1 + y_2 & z \\ & x_3 & x_1 & x_2 + x_3 & y_1 + y_2 \\ & & z + \overline{F}_3 & x_1 Q_2 & Q_1 Q_2 \\ & & & x_3 \overline{Q}_3 & y_2 \overline{Q}_3 \\ & & & & 0 \end{pmatrix}.$$

Finally, it is natural to change coordinates $y_1 + y_2 \mapsto y_1$ and $x_2 + x_3 \mapsto x_2$ so after modifying the forms accordingly and re-naming them, one gets a matrix of the form:

$$\begin{pmatrix} 0 & y_2 & Q_1 & y_1 & z \\ & x_3 & x_1 & x_2 & y_1 \\ & & z+C & x_1Q_2 & Q_1Q_2 \\ & & & x_3Q_3 & y_2Q_3 \\ & & & & 0 \end{pmatrix}, \tag{5.4}$$

where C is a form of degree 3 and the Q_i are quadratics. This is precisely a matrix of the form obtained by Bauer, Catanese and Pignatelli in their article, (cf. [Bauer et al], Theorem 3.7). Thus the format of the curve contains all the information one needs for the surface case (so one can pretend that the canonical model of a surface of type (I.1) is unknown and perform the extension algorithm to happily see how the liftings of the relations fit on the format at every stage of the process). Before showing this format is flexible, I will give one last version of the matrix that should be more mind refreshing to the reader familiar with Tom and Jerry unprojections (cf. [Brown et al]). After performing row-column elementary operations, matrix (5.4) transforms into:

$$\begin{pmatrix} 0 & -x_3 & x_2 & y_1 & x_1 \\ & -y_2 & y_1 & z & Q_1 \\ & & x_1Q_2 & Q_1Q_2 & z+C \\ & & & 0 & x_3Q_3 \\ & & & & y_2Q_3 \end{pmatrix},$$

but I can get rid of the minus signs by changing coordinates and renaming the quadratics. Thus my final version will be as stated in next theorem, whose proof follows immediately from our previous results and from Theorem 3.7 of [Bauer et al]:

Theorem 5.2.1. *Let S be a canonical surface of general type with $K_S^2 = 7$ and $p_g = 4$ belonging to subfamily (I.1) of Theorem 1.4.1 (that is, whose canonical system has exactly one simple base point and maps S birationally onto a sextic). Then both, the canonical ring $R(S, K_S)$ and $R(C, \frac{1}{2}K_C)$, where C is a nonsingular canonical curve of S, have a presentation of the form*

$$k[x_0, x_1, x_2, x_3, y_1, y_2, z]/I, \quad (k[x_0, x_1, x_2, x_3, y_1, y_2, z]/(I, x_0), \textit{ respectively})$$

with indeterminates x_i, y_j and z of respective degrees 1, 2, 3 and I generated by the

4×4 *Pfaffians of a skew matrix of the form:*

$$\begin{pmatrix} 0 & x_3 & x_2 & y_1 & x_1 \\ & y_2 & y_1 & z & Q_1 \\ & & x_1 Q_2 & Q_1 Q_2 & z + C \\ & & & 0 & x_3 Q_3 \\ & & & & y_2 Q_3 \end{pmatrix}, \tag{5.5}$$

where all the Q_i are quadratics and C is a cubic.

5.2.1 Deforming to the base point free case

The skew matrix given in (5.5) has entry $(1,2)$ of degree 0. Take a small affine parameter $t \in \mathbb{C}$ and consider

$$N_t := \begin{pmatrix} t & x_3 & x_2 & y_1 & x_1 \\ & y_2 & y_1 & z & Q_1 \\ & & x_1 Q_2 & Q_1 Q_2 & z + C \\ & & & tQ_2 Q_3 & x_3 Q_3 \\ & & & & y_2 Q_3 \end{pmatrix},$$

as stated before, among its fifteen 4×4 Pfaffians, this matrix has 9 independent expressions and 6 redundancies because of its extrasymmetry. The complete list is given below:

$$I$$

$$\begin{aligned} 12.34 &= t x_1 Q_2 - x_3 y_1 + x_2 y_2 \\ 12.46 &= t x_3 Q_3 - x_2 Q_1 + x_1 y_1 \\ 12.36 &= t(z + C) - x_3 Q_1 + x_1 y_2 \\ 12.56 &= t y_2 Q_3 - y_1 Q_1 + x_1 z \\ 12.45 &= t^2 Q_2 Q_3 - x_2 z + y_1^2 \\ 12.35 &= t Q_1 Q_2 - x_3 z + y_1 y_2 \end{aligned}$$

$$II$$

$$\begin{aligned} 13.46 &= x_3^2 Q_3 - x_2(z + C) + x_1^2 Q_2 \\ 13.56 &= x_3 y_2 Q_3 - y_1(z + C) + x_1 Q_1 Q_2 \\ 23.56 &= y_2^2 Q_3 - z(z + C) + Q_1^2 Q_2 \end{aligned}$$

$$III$$

$$\begin{aligned} 13.45 &= Q_2(12.46) \\ 14.56 &= Q_3(12.34) \\ 23.45 &= Q_2(12.56) \\ 23.46 &= 13.56 \\ 24.56 &= Q_3(12.35) \\ 34.56 &= Q_2 Q_3(12.36) \end{aligned}$$

When $t = 0$, the first group is formed (modulo $\pm$ signs) by the 2×2 minors of the matrix $\begin{pmatrix} x_1 & x_2 & x_3 & y_1 \\ Q_1 & y_1 & y_2 & z \end{pmatrix}$, whereas the second contains the 3 rolling factors relations and the third is already contained in the ideal generated by previous two groups. To see that this format is flexible, one requires to verify that the syzygy module of the ideal generated by these 9 independent Pfaffians has rank 16 and observe that the matrix carries with 16 independent syzygies. The first assertion can be checked in a few seconds by a computer algebra program, whereas the second (as noticed by Reid in [Reid D-E], § 5.9) is a consequence of skew Cramer rule. If I take the Pfaffian adjugate of N_t, that is, the skew matrix defined by:

$$P_{N_t} := \begin{pmatrix} 34.56 & -24.56 & 23.56 & -23.46 & 23.45 \\ & 14.56 & -13.56 & 13.46 & -13.45 \\ & & 12.56 & -12.46 & 12.45 \\ & & & 12.36 & -12.35 \\ & & & & 12.34 \end{pmatrix},$$

then the off-diagonal entries of $N_t P_{N_t}$ all vanish, giving the syzygies required. As a consequence, if I_t is the ideal generated by the entries of P_{N_t}, the 1-parameter family of rings

$$R_t := k[x_0, x_1, x_2, x_3, y_1, y_2, z]/I_t$$

is flat. By construction, Proj R_0 is a surface of type (I.1), but what happens for a small $t \neq 0$ is very interesting. Suppose that $t \neq 0$. Pfaffian 12.36 allows us to write z as a cubic in the remaining variables:

$$z = \frac{1}{t}(x_3 Q_1 - x_1 y_2) - C$$

and we are left with 2 more degree 3 relations and 3 degree 4 relations from the first group of Pfaffians. It is easy to see that these 5 relations are, modulo some negligible minus signs, the maximal diagonal Pfaffians of the following skew 5×5 matrix:

$$\begin{pmatrix} z & tQ_3 & Q_1 & y_1 \\ & -y_1 & -y_2 & tQ_2 \\ & & x_1 & x_2 \\ & & & x_3 \end{pmatrix}. \qquad (5.6)$$

The 3 relations in group II have now became redundant:

$$\begin{aligned} t13.46 &= x_1 12.34 - x_2 12.36 + x_3 12.46, \\ t13.56 &= Q_1 12.34 - y_1 12.36 + y_2 12.46, \\ t23.56 &= Q_1 12.35 - z12.36 + y_2 12.56. \end{aligned}$$

Therefore for any small $t \neq 0$, $R_t \cong k[x_0, x_1, x_2, x_3, y_1, y_2]/J$, where J is a codimension 3 Gorenstein ideal (this follows also from the classic Buchsbaum-Eisenbud structure theorem). The reader, no doubt, has noticed that matrix (5.6) defining the generators of J is of the form of Mukai's first syzygies matrices for halfcanonical rings of genus 8 curves with base point free halfcanonical linear system (cf. Theorem 3.1.3). It is a well known result that this presentation extends to the surface case. Thus every surface of type (I.1) is a degeneration of a surface of type (0).

5.2.2 An interesting question

So far, we have the following situation in the moduli space of surfaces with $K^2 = 7$, $p_g = 4$:

$$(0) \longrightarrow (I.1) \longrightarrow (I.2),$$

By openness of versality it is possible for a surface in (0) to degenerate to one in (I.2) without degenerating first to a surface of type (I.1). Can we get this degeneration using graded ring calculations?. In these last lines of the chapter, I discuss the difficulties in answering this question using our methods.

Consider a general superelliptic ring $R = k[x_0, x_1, x_2, x_3, y_1, y_2, z]/I$. The first 6 relations generating I are minors of a matrix of the form

$$\begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ Q_1 & Q_2 & y_1 & z \end{pmatrix}, \tag{5.7}$$

where Q_1 and Q_2 are quadratic forms not involving y_1 nor y_2. It is easy, using the $AM({}^TA)$ format, to deform the ring by slightly perturbing the entries of M so that the remaining 3 generators of I are rolling factors relations of the form:

$$\begin{aligned} & y_2^2 - x_1^2 Q_3 + x_2^2 Q_4 + x_3^2 Q_5, \\ & y_2 z - x_1 Q_1 Q_3 + x_2 Q_2 Q_4 + x_3 y_1 Q_5, \\ & z^2 - Q_1^2 Q_3 + Q_2^2 Q_4 + y_1^2 Q_5. \end{aligned} \tag{5.8}$$

For some quadratic forms Q_i, $3 \leq i \leq 5$. Next consider the deformation obtained by replacing y_1 and y_2 by $y_1 + ty_2$ and $y_2 + ty_1$ respectively. It is obvious that for $t \notin 0, 1$ we just get an isomorphic ring, ($t = 0$ gives the original ring and $t = 1$ leads to a surface with $K^2 < 7$). I like to think on $(0) \longrightarrow (I.2)$ as a limit case when $t \to 1$ of this situation, because on the other hand, if I write $\bar{y} := y_1 + y_2$ and I take a second very small affine parameter s, the pathological limit case is the special fibre of the following flat family of surfaces whose general member is of type (0):

Let $R_s := k[x_1, x_2, x_3, y_1, y_2, z]/J_s$ with J_s generated by the 4×4 Pfaffians of the following skew matrix:

$$\begin{pmatrix} s & Q_2' & Q_1' & \bar{y} & z \\ & x_2 & x_1 & x_3 & \bar{y} \\ & & z + x_3 Q_5' & x_1 Q_3' & Q_1' Q_3' \\ & & & x_2 Q_4' & Q_2' Q_4' \\ & & & & s Q_3' Q_4' \end{pmatrix}, \tag{5.9}$$

where the Q_i' may depend also on s. The key point is that on the limit $t = 1$, the last 2×2 block from left to right of matrix (5.7) becomes symmetric and allows me to render $\bar{y}^2$ as $x_3 z$ and consequently, to do the trick of writing the relations as Pfaffians. One observes once again that Pfaffian 12.34 of matrix (5.9) allows to write z in terms of the other variables and the rest of them are redundancies modulo the 4×4 Pfaffians of the following 5×5 skew matrix:

$$\begin{pmatrix} z & -Q_1' & -sQ_4 & \bar{y} \\ & -Q_2' & -\bar{y} & sQ_3' \\ & & x_1 & x_2 \\ & & & x_3 \end{pmatrix},$$

which define relations cutting a surface with base point free canonical linear system.

Chapter 6

Surfaces of type (I.3)

In this chapter I show that the canonical ring of a surface of type (I.3) of Theorem 1.4.1 can be presented in any of the formats given in Theorem 4.2.4 for the corresponding halfcanonical ring of the curve case. Although this can be done using the extension algorithm, the calculations are far too long and insubstantial to be written here in full. Thus I simply show how to write a minimal generating set for the module of syzygies, setting the first step of the procedure in case the reader is interested in doing the calculation. As an alternative, we can use a theorem of Zucconi that characterises the minimal model of such surfaces to construct the canonical ring and prove our claim. At the end, our main result (Theorem 6.3.2) shows that there is a 1-parameter flat family of rings with special fibre isomorphic to one of these *trigonal septic* rings and general fibre a canonical ring of a surface of type (I.1).

6.1 The 64 syzygies

Let $\overline{R}$ be a ring as in Theorem 4.2.4, that is:

$$\overline{R} = k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2]/I,$$

where the x_i, y_j and z_ℓ are indeterminates of degrees 1, 2 and 3 respectively, $P = P(x_i, y_j)$ is a homogeneous form of degree 4 and I is the ideal generated by the 2×2 minors of the following matrices:

$$A := \begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \end{pmatrix},$$

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & z_1 \\ y_1 & y_2 & y_3 & z_1 & z_2 & P \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & P \end{pmatrix},$$

then we have:

Proposition 6.1.1. *The ideal I of relations of the ring $\overline{R}$ is minimally generated by 20 elements with 64 independent syzygies holding between them.*

Proof. (Sketch) I list and give names to 20 of the 35 minors of matrices A, M and N. First take the 15 minors of M:

$$\begin{array}{lll} r_1 := x_1y_2 - x_2y_1 & r_6 := x_2y_3 - x_3y_2 & r_{11} := x_3z_2 - y_2y_3 \\ r_2 := x_1y_3 - x_3y_1 & r_7 := x_2z_1 - y_1y_2 & r_{12} := x_3P - y_3z_1 \\ r_3 := x_1z_1 - y_1^2 & r_8 := x_2z_2 - y_2^2 & r_{13} := y_1z_2 - y_2z_1 \\ r_4 := x_1z_2 - y_1y_2 & r_9 := x_2P - y_2z_1 & r_{14} := y_1P - z_1^2 \\ r_5 := x_1P - y_1z_1 & r_{10} := x_2z_1 - y_1y_3 & r_{15} := y_2P - z_1z_2 \end{array}$$

I only take 4 minors from matrix A:

$$r_{16} := x_1x_3 - x_2^2 \quad r_{17} := x_2y_2 - x_3y_1 \quad r_{18} := y_1y_3 - y_2^2 \quad r_{19} := y_2z_2 - y_3z_1$$

and one from matrix N:

$$r_{20} := y_3P - z_2^2.$$

We can see that $r_1, \ldots, r_{20}$ generate the same ideal as all the 35 minors together and that a minimal basis of the corresponding syzygy module has 64 elements using `Magma` (execute the code in appendix A.10).

If interested, we can write explicitly 64 linearly independent syzygies (but we will not for reasons of space) following this procedure:

1. The first 40 syzygies are obtained taking the 20 2×3 submatrices of matrix M and cloning each of its rows to get a 3×3 matrix whose determinant vanishes by construction.

2. Repeat the above procedure with the three 2×3 submatrices of N that have $\begin{pmatrix} y_3 & z_2 \\ z_2 & P \end{pmatrix}$ as a submatrix. This gives 6 more syzygies involving $r_1, \ldots r_{20}$ provided we write some of the relations appearing as 2×2 minors of N as linear combinations of the r_is. For example $x_2z_2 - y_1y_3$ is $r_8 - r_{19}$, etc.

3. The remaining 18 syzygies come from the following 9 submatrices of matrix A:

$$\begin{pmatrix} x_1 & x_2 & y_1 \\ x_2 & x_3 & y_2 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & y_2 \\ x_2 & x_3 & y_3 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & z_1 \\ x_2 & x_3 & z_2 \end{pmatrix}$$

$$\begin{pmatrix} x_2 & y_1 & z_1 \\ x_3 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} x_1 & y_1 & y_2 \\ x_2 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} x_2 & y_1 & y_2 \\ x_3 & y_2 & y_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 & y_2 & z_1 \\ y_2 & y_3 & z_2 \end{pmatrix} \begin{pmatrix} x_1 & y_2 & z_1 \\ x_2 & y_3 & z_2 \end{pmatrix} \begin{pmatrix} x_2 & y_2 & z_1 \\ x_3 & y_3 & z_2 \end{pmatrix}$$

Q.E.D.

Remark Listing the 64 syzygies was the reason because of which I found the format that uses separately matrices M, A and N. A more beautiful way of presenting the ideal as 2×2 minors of a matrix though, is to glue together A and N to get this double symmetric 3×5 matrix:

$$\mathcal{S} := \begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & P \end{pmatrix}$$

obviously $\bigwedge^2 \mathcal{S}$ generates I. $\mathcal{S}$ has two concatenated 3×3 symmetric blocks. If we forget about our grading and set all variables to have degree 1, including P, then

$$\text{Proj } R, \quad \text{where } R = k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, P] / \left\langle \bigwedge^2 \mathcal{S} \right\rangle \tag{6.1}$$

is the third Veronese embedding of the blowup of $\mathbb{P}^2$ in one point. If I recover the weights then (6.1) is a del Pezzo surface with two cyclic quotient singularities, one of type $\frac{1}{2}(1,1)$ and the other $\frac{1}{4}(1,1)$. It is polarised by an ample divisor $\widetilde{D}$ with $\widetilde{D}^2 = \frac{7}{4}$ and anticanonical divisor $2\widetilde{D}$. It is known that this surface is smoothable to the ordinary del Pezzo surface of degree 7, but not while preserving the anti semicanonical condition. Although this suggest a strategy to deform our trigonal rings to the ones corresponding to family (I.1) of Theorem 1.4.1, the calculation presents several difficulties. Therefore we will use a different strategy later on.

6.2 Calculation of $R(S, K_S)$

The formats we obtained for the trigonal curve case suggest to construct surfaces of type (I.3) as regular pullbacks from a key variety $\mathcal{V}$ defined by the 20 two by two minors given in (6.1). Concretely, if $\mathcal{V} = \text{Spec } R_{\mathcal{V}}$ where

$$R_{\mathcal{V}} = k[\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{z}_1, \overline{z}_2, \overline{P}] / \left\langle \bigwedge^2 \mathcal{S} \right\rangle \tag{6.2}$$

and $\mathcal{S} = \begin{pmatrix} \overline{x}_1 & \overline{x}_2 & \overline{y}_1 & \overline{y}_2 & \overline{z}_1 \\ \overline{x}_2 & \overline{x}_3 & \overline{y}_2 & \overline{y}_3 & \overline{z}_2 \\ \overline{y}_1 & \overline{y}_2 & \overline{z}_1 & \overline{z}_2 & \overline{P} \end{pmatrix}$. We can consider the morphism

$$\mathrm{Spec}\ k[x_0,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2] \longrightarrow \mathrm{Spec}\ k[\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{y}_1,\overline{y}_2,\overline{y}_3,\overline{z}_1,\overline{z}_2,\overline{P}]$$

defined by $\overline{x}_i \mapsto x_i$, $\overline{y}_i \mapsto y_i$, $\overline{z}_j \mapsto z_j$ for $1 \le i \le 3$, $1 \le j \le 2$; $\overline{P} \mapsto P$ where P is $P(x_0,x_1,x_2,x_3,y_1,y_2,y_3)$, then take the pullback of the subscheme defined by the ideal generated by $\bigwedge^2 \mathcal{S}$ and take the quotent by the $k^\times$-action defining the grading to obtain a surface that must be of type (I.3). [1] As we will see, every surface of type (I.3) can be obtained this way. The proof uses a result of Francesco Zucconi that we state as the first part of the following theorem:

Theorem 6.2.1. *Consider the toric variety $\mathbb{T}$ defined as a $\mathbb{P}(1,1,1,2)-$bundle over $\mathbb{P}^1$ by the variables and weights of the following table:*

Variable	t_1	t_2	X_0	X_1	X_2	Y
Bi-degree	1	1	-1	-2	-4	-6
	0	0	1	1	1	2

Let A be the divisor class of a fibre of the natural projection $\pi : \mathbb{T} \longrightarrow \mathbb{P}^1$ and let T be a tautological divisor on $\mathbb{T}$. Then:

1. *S is a minimal surface of type (I.3) if and only if it is a complete intersection (F,G), where $F \in |-5A+2T|$, and $G \in |-8A+4T|$ are given by the vanishing of the following forms:*

$$F: \quad t_1Y - X_0X_2$$
$$G: \quad \alpha Y^2 + QY + c_1X_1^4 + X_2R$$

subject to the conditions:

   a) $c_1 \in k^\times$.

   b) $\alpha \in H^0(\mathbb{T},4A)$, $\alpha|_{(t_1=0)} \neq 0$.

   c) $Q = c_0X_0^2 + \alpha_1X_0X_1 + \alpha_2X_1^2 + \alpha_4X_1X_2 + \alpha_6X_2^2$, where $c_0 \in k^\times$ and $\alpha_i \in H^0(\mathbb{T},iA)$.

   d) $R = \beta_1X_1^3 + \beta_2X_1^2X_2 + \beta_3X_1X_2^2 + \beta_4X_2^3$, where $\beta_i \in H^0(\mathbb{T},2iA)$.

[1] Probably an analogous construction could also make sense to get Calabi-Yau 3-folds, Fano 4-folds, etc.

2. $R(S,K_S)$ is isomorphic to:

$$k[x_0,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2]/I,$$

where the x_i, y_j and z_ℓ have degrees 1, 2 and 3 respectively and there is a homogeneous form $P = P(x_i,y_j,z_\ell)$ of degree 4 such that I is generated by the 2×2 minors of the following matrix:

$$\begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & P \end{pmatrix}.$$

Proof. For part 1 see [Zucconi], Main Theorem.

It remains to compute $R(S,K_S)$. Since $K_{\mathbb{T}} = |11A - 5T|$ it follows that the canonical class of S is the restriction of $|-2A+T|$. Moreover for $n\in\mathbb{N}$:

$$H^0(S,nK_S) \cong H^0(\mathbb{T},-2nA+nT)|_S. \tag{6.3}$$

From here one sees that the canonical ring is minimally generated by the following ordered sets (I omit the symbols of restricting sections to S for simplicity):

$$\text{Degree 1:}\quad \{x_0,x_1,x_2,x_3\} := \{X_1, t_1^2X_2, t_1t_2X_2, t_2^2X_2\} \tag{6.4a}$$

$$\text{Degree 2:}\quad \{y_1,y_2,y_3\} := \{t_1^2Y, t_1t_2Y, t_2^2Y\} \tag{6.4b}$$

$$\text{Degree 3:}\quad \{z_1,z_2\} := \{t_1X_0Y, t_2X_0Y\} \tag{6.4c}$$

and it is clear that we have the following relations:

$$\bigwedge^2\begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \end{pmatrix},\quad \bigwedge^2\begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 \\ y_1 & y_2 & y_3 & z_1 & z_2 \end{pmatrix}.$$

The equation $F=0$ implies that $t_1^3X_0Y^2 = t_1^2X_0^2X_2Y$. Thus $y_1z_1 = x_1(X_0^2Y)|_S$. On the other hand, the equation $G=0$ subject to the condition c) of the first part of the theorem implies that

$$X_0^2Y = -\frac{1}{c_0}(c_1X_1^4 + X_2R + \alpha Y^2 + Y(\alpha_1X_0X_1 + \alpha_2X_1^2 + \alpha_4X_1X_2 + \alpha_6X_2^2)). \tag{6.5}$$

Therefore we have a relation of the form $y_1z_1 - x_1P$ where P is the right hand side of equation (6.5) rendered in terms of the x_i, y_j and z_j. The rest follows exactly as in the curve case from this point. Q.E.D.

Remark I will call the format given in part 2 of Theorem 6.2.1, $F(\frac{7}{4})$-format.

6.3 Relation with surfaces of type (I.1)

In this section I proof that the closures of the strata of canonical surfaces of type (I.1) and (I.3) of Theorem 1.4.1 meet at a stratum in the boundary of the moduli space. This stratum is formed by surfaces defined as Proj R where R is a ring of the following form:

$$R := k[x_0, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2]/I,$$

with x_i, y_j and z_ℓ of degrees 1, 2 and 3 respectively. I define $\overline{P}$ to be the degree 4 homogeneous form:

$$\overline{P} := a_1 x_0^3 x_2 + x_0^2 A_0 + x_1^2 A_1 + x_2^2 A_2 + x_3^2 A_3 + x_1 x_2 B_1 + x_2 x_3 B_2, \tag{6.6}$$

where

$$\begin{aligned}
A_0 &:= a_2 x_1^2 + a_3 x_1 x_2 + a_4 x_1 x_3 + a_5 x_2^2 + a_6 x_2 x_3 + a_7 x_3^2,\\
A_1 &:= a_8 x_0 x_1 + a_9 x_0 x_2 + a_{10} x_1^2 + a_{11} x_1 x_2 + a_{12} x_2^2 + a_{13} y_1 + a_{14} y_2 + a_{15} y_3,\\
A_2 &:= a_{16} x_0 x_1 + a_{17} x_0 x_2 + a_{18} x_0 x_3 + a_{19} x_1 x_2 + a_{20} x_1 x_3 + a_{21} x_2 x_3 + a_{22} x_3^2 + a_{23} y_3,\\
A_3 &:= a_{24} x_0 x_2 + a_{25} x_0 x_3 + a_{26} x_2 x_3 + a_{27} x_3^2 + a_{28} y_3\\
B_1 &:= a_{29} y_2 + a_{30} y_3,\\
B_2 &:= a_{31} y_3,
\end{aligned}$$

$a_1, \ldots, a_{31} \in \mathbb{C}$ and I is generated by the 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \\ y_1 & y_2 & z_1 & z_2 & \overline{P} \end{pmatrix}.$$

I will call the surfaces Proj R, with R a ring as defined above, *surfaces of type* $\widetilde{(I.3)}$.

Proposition 6.3.1. *Every surface of type* $\widetilde{(I.3)}$ *has a small deformation to a surface of type* $(I.3)$.

Proof. We are obtaining the rings corresponding to surfaces in the stratum $\widetilde{(I.3)}$ from those of $(I.3)$ simply by equating to zero some coefficients in the general degree

4 form $P = P(x_i, y_j, z_\ell)$ of second part of Theorem 6.2.1. Therefore the result follows immediately by the flexibility of the $F(\frac{7}{4})$-format. Q.E.D.

Remark 1. Surfaces of type $\widetilde{(I.3)}$ have singularities that are not rational double points and therefore they are not canonical surfaces with $K^2 = 7$ and $p_g = 4$.

2. The key feature of the degree 4 form $\overline{P}$ defined in (6.6) is that is the most general quartic in x_i, y_j, z_ℓ, up to the choice of different renderings obtained using the toric relations given by $\bigwedge^2 \begin{pmatrix} x_1 & x_2 & y_1 & y_2 & z_1 \\ x_2 & x_3 & y_2 & y_3 & z_2 \end{pmatrix}$, that can be rolled twice with respect to the following matrix:

$$\begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ y_1 & y_2 & y_3 & z_2 \end{pmatrix}.$$

This will play a crucial role in the next theorem.

Theorem 6.3.2. *Every ring defining a surface in stratum $\widetilde{(I.3)}$ is the central fibre of a flat family of rings over a small disc $\Delta_0 \subset \mathbb{C}$ whose general fibre is the canonical ring of a surface in the stratum $(I.1)$.*

Proof. Let $t \in \Delta_0$ and let $\overline{P}$ be as in (6.6). Consider the following family of rings:

$$R_t := k[x_0, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2]/I_t,$$

where I_t is the ideal generated by the following 3 sets of relations:

1. The 4×4 Pfaffians of the 6×6 skew matrix

$$G := \begin{pmatrix} t & z_1 & x_1 & x_2 & y_1 \\ & z_2 & x_2 & x_3 & y_2 \\ & & 0 & \overline{P} & 0 \\ & & & y_1 & 0 \\ & & & & -z_1 \end{pmatrix}.$$

2. The 4×4 Pfaffians of the 5×5 skew matrix

$$B := \begin{pmatrix} t & x_1 & x_2 & y_2 \\ & x_2 & x_3 & y_3 \\ & & y_1 & z_1 \\ & & & 0 \end{pmatrix}.$$

3. The 5 elements:

$$\begin{aligned} r_{16} &:= x_3 z_2 - y_2 y_3, \\ r_{17}(t) &:= y_2^2 - y_1 y_3 + t\overline{P}, \\ r_{18}(t) &:= y_2 z_2 - y_3 z_1 + t\overline{P}_1, \\ r_{19}(t) &:= y_2 - x_3\overline{P} + t\overline{P}_1, \\ r_{20}(t) &:= z_2^2 - y_3\overline{P} + t\overline{P}_2, \end{aligned}$$

where $\overline{P}_1$ is obtained from $\overline{P}$ by rolling factors with respect to the following matrix:

$$\begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ y_1 & y_2 & y_3 & z_2 \end{pmatrix} \tag{6.7}$$

and $\overline{P}_2$ is obtained from $\overline{P}_1$ in the same way. Explicitly:

$$\begin{aligned} \overline{P}_1 &:= a_1 x_0^3 y_2 + x_0^2 A_{0,1} + x_1 y_1 A_1 + x_2 y_2 A_2 + x_3 y_3 A_3 + y_1 x_2 B_1 + y_2 x_3 B_2, \\ \overline{P}_2 &:= a_1 x_0^3 z_2 + x_0^2 A_{0,2} + y_1^2 A_1 + y_2^2 A_2 + y_3^2 A_3 + y_1 y_2 B_1 + y_2 y_3 B_2, \end{aligned}$$

where

$$\begin{aligned} A_{0,1} &:= a_2 x_1 y_1 + a_3 y_1 x_2 + a_4 y_1 x_3 + a_5 x_2 y_2 + a_6 y_2 x_3 + a_7 x_3 y_3, \\ A_{0,2} &:= a_2 y_1^2 + a_3 y_1 y_2 + a_4 y_1 y_3 + a_5 y_2^2 + a_6 y_2 y_3 + a_7 y_3^2. \end{aligned}$$

By construction, the central fibre of this family is a ring whose Proj is a surface in stratum $\widetilde{(I.3)}$ and I claim that the general fibre is isomorphic to a codimension 4 canonical ring of a surface of type (I.1). To prove this I will discuss each of the 20 relations when $t \neq 0$, showing that only 11 of them are necessary to generate the ideal and that they fit in the desired format.

The Pfaffians of G are:

$12.45 = ty_1 - x_1 x_3 + x_2^2$	$24.56 = -x_2 z_1 + y_1 y_2$	$23.45 = y_1 z_2 - x_2\overline{P}$
$12.56 = -tz_1 - x_2 y_2 + x_3 y_1$	$12.36 = y_1 z_2 - y_2 z_1$	$23.56 = -z_1 z_2 + y_2\overline{P}$
$12.35 = t\overline{P} - x_3 z_1 + x_2 z_2$	$13.45 = y_1 z_1 - x_1\overline{P}$	$13.46 = 0$
$12.46 = x_2 y_1 - x_1 y_2$	$13.56 = -z_1^2 + y_1\overline{P}$	$23.46 = 0$
$12.34 = x_1 z_2 - x_2 z_1$	$14.56 = -x_1 z_1 + y_1^2$	$34.56 = 0$

using these, is easy to see that B gives us only 3 Pfaffians not included in previous list that I call as follows:

$$r_{13}(t) := tz_1 - x_1 y_3 + x_2 y_2 \quad r_{14} := x_3 y_2 - x_2 y_3 \quad r_{15} := y_1 y_3 - x_3 z_1.$$

It is convenient start discussing these 3 relations: r_{14} is $-\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}$. Adding $r_{13}(t)$ and 12.56 gives $-\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}$. Using 12.56 we have $tx_3z_1 = x_3^2y_1 - x_2x_3y_2$, but by the previous 2 determinantal relations we have $x_3^2y_1 = x_1x_3y_3$ and $x_2x_3y_2 = x_2^2y_3$. Thus $tx_3z_1 = x_1x_3y_3 - x_2^2y_3$ and using 12.45 we get $tx_3z_1 = ty_1y_3$. Therefore r_{15} is redundant, but allows me to write $r_{17}(t)$ as $\overline{r}_{17}(t) := y_2^2 - x_3z_1 + t\overline{P}$.

Next, I continue with the list of Pfaffians of G: 12.45 and 12.56 allow us to write y_1 and z_1 in terms of the remaining variables. This will decrease the codimension of the deformed ring by 2 whereas 12.35 is the degree 4 relation that we will roll twice with respect to matrix (6.7). Moreover, substracting 12.35 to relation $\overline{r}_{17}(t)$ gives $y_2^2 - x_2z_2$, which is $-\begin{vmatrix} x_2 & y_2 \\ y_2 & z_2 \end{vmatrix}$.
Pfaffian 12.46 is $-\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$ whereas substracting 24.56 from 12.34 gives $\begin{vmatrix} x_1 & z_2 \\ y_1 & y_2 \end{vmatrix}$. This along with previous deduced determinantal relations and r_{16} gives us already the following 6 relations in the general fibre ring:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ y_1 & y_2 & y_3 & z_2 \end{pmatrix}.$$

Now using 12.56 we have $ty_2z_1 = x_3y_1y_2 - x_2y_2^2$. But by previous observation, $x_2y_2^2 = x_2^2z_2$ and by 24.56 and 12.34, $x_3y_1y_2 = x_1x_3z_2$. Thus $ty_2z_1 = x_1x_3z_2 - x_2^2z_2 = ty_1z_2$, showing that 12.36 is redundant. Similarly, one shows that the remaining Pfaffians of G are also redundant.

Finally, multiplying $r_{17}(t)$ by x_3 gives $tx_3\overline{P} = x_3y_1y_3 - x_3y_2^2$, but $x_3y_2^2 = x_2y_2y_3$, so using 12.56 we have $tx_3\overline{P} = ty_3z_1$. Thus $x_3\overline{P} = y_3z_1$, proving that $r_{18}(t)$ and $r_{19}(t)$ are equivalent. Clearly $r_{18}(t)$ is obtained from 12.35 by rolling factors with respect to matrix (6.7) and $r_{20}(t)$ is obtained from $r_{19}(t)$ the same way. Therefore, the general fibre ring R_t is generated by $ty_1 - x_1x_3 + x_2^2$, $-tz_1 - x_2y_2 + x_3y_1$ and the following 9 relations:

$$\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 & y_2 \\ y_1 & y_2 & y_3 & z_2 \end{pmatrix}, \qquad \begin{array}{lcl} x_2z_2 - x_3z_1 + t\overline{P} & & \\ y_2z_2 - y_3z_1 + t\overline{P}_1 & \equiv & y_2z_2 - x_3\overline{P} + t\overline{P}_1 \\ z_2^2 - y_3\overline{P} + t\overline{P}_2 & & \end{array}$$

which proves my claim.

It remains to check that the family is flat. This can be done asking `Magma` to verify that the Hilbert polynomial of R_t is the same as that of R_0. Since this calculation is essential for the proof, I include the codes here and not in the appendix. The code to compute the Hilbert numerator of the general fibre ring is:

```
RR<t,dd,a1,a2,a3,a4,a5,a6,
a7,a8,a9,a10,a11,a12,
a13,a14,a15,a16,a17,a18,
a19,a20,a21,a22,a23,a24,
a25,a26,a27,a28,a29,a30,
b1,b2,b3,
b4,b5,b6,
b7,b8,b9,
x0,x1,x2,x3,y1,y2,y3,z1,z2> :=
PolynomialRing(Rationals(),
[0,0,0,0,0,0,0,0,
0,0,0,0,0,0,
0,0,0,0,0,0,
0,0,0,0,0,0,
0,0,0,0,0,0,
0,0,0,
0,0,0,
0,0,0,
1,1,1,1,2,2,2,3,3]);
A0:=a1*x1^2+a2*x1*x2+a3*x1*x3
+a4*x2^2+a5*x2*x3+a6*x3^2;
A1:=a7*x0*x1+a8*x0*x2+a9*x0*x3
+a10*x1^2+a11*x1*x2+a12*x1*x3
+a13*x2^2+a14*x2*x3
+a15*x3^2;
A2:=a16*x0*x1+a17*x0*x2+a18*x0*x3
+a19*x1*x2+a20*x1*x3
+a21*x2^2+a22*x2*x3
+a23*x3^2;
A3:=a24*x0*x1+a25*x0*x2+a26*x0*x3
+a27*x1*x2+a28*x1*x3
+a29*x2*x3
+a30*x3^2;
```

```
B1:=b1*y1+b2*y2+b3*y3;
B2:=b4*y1+b5*y2+b6*y3;
B3:=b7*y1+b8*y2+b9*y3;
A01:=a1*x1*y1+a2*y1*x2+a3*y1*x3+a4*x2*y2+a5*y2*x3+a6*x3*y3;
A02:=a1*y1^2+a2*y1*y2+a3*y1*y3+a4*y2^2+a5*y2*y3+a6*y3^2;
P:=x0^3*x2+x0^2*A0+dd*x0*x1*x2*x3
+x1^2*A1+x2^2*A2+x3^2*A3
+x1*x2*B1+x1*x3*B2+x2*x3*B3;
P1:=x0^3*y2+x0^2*A01+dd*x0*y1*x2*x3
+x1*y1*A1+x2*y2*A2+x3*y3*A3
+y1*x2*B1+y1*x3*B2+y2*x3*B3;
P2:=x0^3*z2+x0^2*A02+dd*x0*y1*y2*x3
+y1^2*A1+y2^2*A2+y3^2*A3
+y1*y2*B1+y1*y3*B2+y2*y3*B3;
G:=AntisymmetricMatrix
([t,z1,z2,x1,x2,0,x2,x3,P,y1,y1,y2,0,0,-z1]);
B:=AntisymmetricMatrix([t,x1,x2,x2,x3,y1,y2,y3,z1,0]);
Pf1:=Pfaffians(G,4);
Pf2:=Pfaffians(B,4);
U:=[x3*z2-y2*y3, t*P-y1*y3+y2^2,
y2*z2-y3*z1+t*P1, y2*z2-x3*P+t*P1,
z2^2-y3*P+t*P2];
I1:=Ideal(Pf1);
I2:=Ideal(Pf2);
I3:=Ideal(U);
I0:=I1+I2+I3;
I:=MinimalBasis(I0);
I;
#I;
HilbertNumerator(I0);
```

To compute the Hilbert numerator of the special fibre one can use the same code, erasing the degree zero generator and declaring `t:=0;`. In both cases one gets:

$$t^{17} - t^{15} - 4t^{14} - 3t^{13} + 7t^{12} + 10t^{11} + 3t^{10} - 13t^9 - 13t^8 + 3t^7 + 10t^6 + 7t^5 - 3t^4 - 4t^3 - t^2 + 1.$$

Q.E.D.

Chapter 7

Surfaces of type (III): conjectures and future work

In this last chapter I discuss the main unsolved problems on surfaces with $K^2 = 7$ and $p_g = 4$. I expect these problems to be solved in the forthcoming months and I state conjectures supported by particular examples and calculations.

I also mention a problem regarding a different class of surfaces (namely, those with $K^2 = 6$ and $p_g = 4$) that should be solvable using our methods in the near future.

7.1 The moduli space $\mathfrak{M}_{K^2=7,\ p_g=4}$

Our results together with previous work of Bauer, Catanese and Pignatelli, are summarised in the following picture concerning the 10 strata of the moduli space of surfaces with $K^2 = 7$ and $p_g = 4$ and the stratum $\widetilde{(I.3)}$ of Theorem 6.3.2:

$$\begin{tikzcd}
 & & (I.2) & \widetilde{(I.3)} & (I.3) \arrow[l] \\
 & (III.\gamma) & & (I.1) \arrow[ul] \arrow[u] & \\
 & & (III.\alpha) \arrow[ul] \arrow[dl] & (0) \arrow[u] & (III.\beta) \arrow[d] \\
(F.2) & (F.1) \arrow[l] & & & (F')
\end{tikzcd}$$

The colours indicate the irreducible component of the moduli space that each family belongs to. Following the notation of Theorem 1.4.2, the colour correspondence is as follows:

$$\mathfrak{M}_{7,4} = \overline{\mathfrak{M}_{(III.\alpha)}} \cup \overline{\mathfrak{M}_{(0)}} \cup \overline{\mathfrak{M}_{(III.\beta)}}.$$

It is known (also from Theorem 1.4.2) that

$$\overline{\mathfrak{M}_{(III.\alpha)}} \cap \overline{\mathfrak{M}_{(III.\beta)}} = \varnothing$$

and Bauer claims that she proves

$$\overline{\mathfrak{M}_{(0)}} \cap \overline{\mathfrak{M}_{(III.\alpha)}} \neq \varnothing.$$

However the following 2 questions remain open:

1. Exactly how $\overline{\mathfrak{M}_{(0)}}$ and $\overline{\mathfrak{M}_{(III.\alpha)}}$ intersect?
2. Is $\overline{\mathfrak{M}_{(III.\beta)}}$ a connected component of the moduli space? In other words: Does $\overline{\mathfrak{M}_{(0)}}$ intersect $\overline{\mathfrak{M}_{(III.\beta)}}$?

7.2 Setting up a deformation of a hyperelliptic ring

A starting point to answer these questions is to consider the deformation families of the codimension 8 rings described in Theorem 4.3.3. These rings have extensions to surfaces belonging to (III.α), (III.β) and (III.γ):

Let C be a nonsingular genus 8 curve admitting a linear system $|D|$ with three distinct base points, P_1, P_2, P_3 and satisfying the following properties:

1. $|D| = P_1 + P_2 + P_3 + 2g_2^1$.
2. $2D = K_C$.

Then the halfcanonical ring $R \coloneqq R(C, D) = \bigoplus_{n\geq 0} H^0(C, nD)$ is isomorphic to:

$$k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, z_4]/I,$$

with generators of degrees $1, 1, 1, 2, 2, 2, 3, 3, 3, 3$ respectively and the ideal I is generated by the homogeneous forms obtained by taking the 2×2 minors of the 2×12

matrix A, where

$$A := \begin{pmatrix} x_1 & x_2 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & z_1 & z_2 & z_3 \\ x_2 & x_3 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & z_2 & z_3 & z_4 \end{pmatrix}, \tag{7.1}$$

with $w_1 := y_1, w_5 := y_2, w_8 := y_3$ and the w_i for $i \in \{2,3,4,6,7\}$ are defined recursively:

$$\begin{aligned} w_4 &:= w_5 + x_1x_3 = w_5 + x_2^2, \\ w_3 &:= w_4 + x_1x_2, \\ w_2 &= w_3 + x_1^2; \end{aligned} \qquad \begin{aligned} w_6 &:= w_5 - x_2x_3, \\ w_7 &:= w_6 - x_3^2. \end{aligned}$$

Plus 7 rolling factors relations of the form:

$$z_1^2 - y_1^2 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_2^2 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2a}$$

$$z_1z_2 - y_1w_2 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_2w_6 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2b}$$

$$z_1z_3 - y_1w_3 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_2w_7 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2c}$$

$$z_1z_4 - y_1w_4 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_2y_3 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2d}$$

$$z_2z_4 - y_1y_2 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - w_6y_3 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2e}$$

$$z_3z_4 - y_1w_6 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - w_7y_3 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2f}$$

$$z_4^2 - y_1w_7 \cdot \left(y_1 + \sum_{i=1}^{7} a_i w_{i+1}\right) - y_3^2 \cdot \left(\sum_{j=1}^{8} a_{j+7} w_j\right), \tag{7.2g}$$

or any other rendering where possible. Moreover, the constants $a_1, \dots a_{15} \in k$ are required to satisfy:

1. $a_{15} \neq 0$,
2. $\sum_{i=1}^{15} a_i \neq -1$,
3. The polynomial $1 + \sum_{i=1}^{15} a_i t^i$ has 15 distinct roots.

One sees that the ideal generated by the 2×2 minors of the following submatrix of

A from (7.1):

$$\overline{A} := \begin{pmatrix} x_1 & x_2 & w_1 & w_4 & w_7 & z_1 & z_2 & z_3 \\ x_2 & x_3 & w_2 & w_5 & w_8 & z_2 & z_3 & z_4 \end{pmatrix} \tag{7.3}$$

suffice to generate the ideal

$$\left(\bigwedge^2 A\right).$$

I know several ways for presenting the rolling factor relations (7.2) together with some of the relations coming from $\bigwedge^2 \overline{A}$. For example, we can define

$$\mathcal{Q} := w_1 + \sum_{i=1}^{7} a_i w_{i+1}, \tag{7.4a}$$

$$\mathcal{R} := \sum_{j=1}^{8} a_{j+7} w_j \tag{7.4b}$$

and take the 4×4 Pfaffians of the following skew matrices:

$$B_1 := \begin{pmatrix} 0 & z_1 & z_2 & z_3 & w_1 & w_5 & w_7 \\ & z_2 & z_3 & z_4 & w_2 & w_6 & w_8 \\ & & 0 & 0 & -w_5\mathcal{R} & w_1\mathcal{Q} & w_3\mathcal{Q} \\ & & & 0 & -w_6\mathcal{R} & w_2\mathcal{Q} & w_4\mathcal{Q} \\ & & & & -w_7\mathcal{R} & w_3\mathcal{Q} & w_5\mathcal{Q} \\ & & & & & z_1 & z_2 \\ & & & & & & 0 \end{pmatrix}, \tag{7.5a}$$

$$B_2 := \begin{pmatrix} 0 & z_1 & z_2 & z_3 & w_1 & w_5 & w_7 \\ & z_2 & z_3 & z_4 & w_2 & w_6 & w_8 \\ & & 0 & 0 & -w_6\mathcal{R} & w_2\mathcal{Q} & w_4\mathcal{Q} \\ & & & 0 & -w_7\mathcal{R} & w_3\mathcal{Q} & w_5\mathcal{Q} \\ & & & & -w_8\mathcal{R} & w_4\mathcal{Q} & w_6\mathcal{Q} \\ & & & & & z_2 & z_4 \\ & & & & & & 0 \end{pmatrix}, \tag{7.5b}$$

both of weights $\begin{pmatrix} 1 & 3 & 3 & 3 & 2 & 2 & 2 \\ & 3 & 3 & 3 & 2 & 2 & 2 \\ & & 5 & 5 & 4 & 4 & 4 \\ & & & 5 & 4 & 4 & 4 \\ & & & & 4 & 4 & 4 \\ & & & & & 3 & 3 \\ & & & & & & 3 \end{pmatrix}$. As it is, this way of presenting the relations is not useful, because the matrices do not have any zero of degree zero suggesting us how to deform the relations. This can be improved defining

$$\mathcal{E}_1 := (x_1 + x_2 + x_3) w_1 \mathcal{Q}, \tag{7.6a}$$

$$\mathcal{E}_2 := \mathcal{E}_1 - x_1 w_1 \mathcal{Q}, \tag{7.6b}$$

$$\mathcal{Q}_1 := -w_1 \mathcal{Q} - w_6 \mathcal{R}, \tag{7.6c}$$

$$\mathcal{Q}_2 := -w_2 \mathcal{Q} - w_7 \mathcal{R}, \tag{7.6d}$$

$$\mathcal{Q}_3 := -w_5 \mathcal{Q} - w_7 \mathcal{R}, \tag{7.6e}$$

$$\mathcal{Q}_4 := -w_2 \mathcal{Q} - w_8 \mathcal{R} \tag{7.6f}$$

and replacing some of the 4×4 Pfaffians of (7.5) by the corresponding 4×4 Pfaffians of one (or more) of the following skew matrices:

$$D_1 := \begin{pmatrix} 0 & z_2 & x_1 & x_2 & w_5 \\ & z_3 & x_2 & x_3 & w_6 \\ & & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{E}_1 \\ & & & 0 & z_1 \\ & & & & z_2 \end{pmatrix}, \tag{7.7a}$$

$$D_2 := \begin{pmatrix} 0 & z_2 & x_1 & x_2 & w_5 \\ & z_3 & x_2 & x_3 & w_6 \\ & & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{E}_2 \\ & & & 0 & z_2 \\ & & & & z_3 \end{pmatrix}, \tag{7.7b}$$

$$D_3 := \begin{pmatrix} 0 & z_1 & x_1 & x_2 & w_5 \\ & z_2 & x_2 & x_3 & w_6 \\ & & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{E}_2 \\ & & & 0 & z_3 \\ & & & & z_4 \end{pmatrix}, \tag{7.7c}$$

the three having weights $\begin{pmatrix} 0 & 3 & 1 & 1 & 2 \\ & 3 & 1 & 1 & 2 \\ & & 4 & 4 & 5 \\ & & & 2 & 3 \\ & & & & 3 \end{pmatrix}$. This presentation suggests that is possible to deform 5 of the relations so we can write one of the degree 2 generators and the 4 degree 3 generators in terms of the remaining variables. This would deform the codimension 8 ring to a codimension 3 one. If we can extend such a calculation to the surface case, it could be possible to construct flat families of rings whose special fibre is the canonical ring of a surface in $\overline{\mathfrak{M}_{(III.\alpha)}}$ or $\overline{\mathfrak{M}_{(III.\beta)}}$ and whose general fibre is the canonical ring of a surface of type (0).

I managed to do this calculation in the curve case. I deform 10 of the relations defining the ideal of the ring of Theorem 4.3.3 and group them in two sets as follows: Let $f_3 = f_3(x_1, x_2, x_3, y_1, y_3)$ be a general homogeneous form of degree 3. Let $t \in \Delta_0 \subset \mathbb{C}$ be an affine parameter in a small disc around 0. Consider the following relations:

$$x_2^2 - x_1x_3 - t^2y_2 \tag{7.8a}$$
$$x_1w_2 - x_2y_1 - tz_1 \tag{7.8b}$$
$$x_2w_2 - x_3y_1 - tz_2 \tag{7.8c}$$
$$x_2y_2 - x_3w_4 - tz_3 \tag{7.8d}$$
$$x_2y_3 - x_3w_7 - tz_4 \tag{7.8e}$$

$$x_1y_2 - x_2w_4 + t^2x_3y_1 \tag{7.9a}$$
$$x_1y_3 - x_2w_7 + t^2x_3w_4 \tag{7.9b}$$
$$y_1w_7 - w_4^2 + tx_1f_3 \tag{7.9c}$$
$$y_1y_3 - y_2w_4 + tx_2f_3 \tag{7.9d}$$
$$w_4y_3 - y_2w_7 + t^3x_3f_3 \tag{7.9e}$$

If $t = 0$, these 10 relations are combinations of the 2×2 minors of matrix A of (7.1). However if $t \neq 0$, the first group implies that the general fibre ring is generated by x_1, x_2, x_3, y_1, y_3. Thus it has codimension 3. The second group of relations generate exactly the same ideal as the 4×4 Pfaffians of the following 5×5 *Mukai-type* skew matrix:

$$\begin{pmatrix} t^2f_3 & ty_1 & w_4 & y_2 \\ & tw_4 & w_7 & y_3 \\ & & x_1 & x_2 \\ & & & tx_3 \end{pmatrix}. \tag{7.10}$$

One then deforms the remaining relations so they become redundant with respect to (7.8) and (7.9). Finally, it can be proved that both, the special and general fibre rings have the same Hilbert series:

$$\frac{\mathrm{t}^4 + \mathrm{t}^3 + 3\mathrm{t}^2 + \mathrm{t} + 1}{\mathrm{t}^2 - 2\mathrm{t} + 1}.$$

Whence we have a flat family of Gorenstein rings.

This calculation certainly extends straightforwardly to surfaces of type $(III.\gamma)$, since the liftings of the relations (7.8) and (7.9) have the same form because of the canonical image of the surface being the quadratic cone.

7.2.1 Stephen Coughlan's example

One problem with the above strategy is that for surfaces of types $(III.\alpha)$ and $(III.\beta)$, whose canonical image is $\mathbb{P}^1 \times \mathbb{P}^1$, the extensions of the halfcanonical rings are more subtle. For instance, if we want the halfcanonical ring to have the same form as in Theorem 4.3.3, we need to start writing the first 2×2 minor of matrix (7.1) in the form $\begin{vmatrix} sx_0 + x_1 & x_2 \\ sx_1 + x_2 & x_3 \end{vmatrix}$ for some constant s, or something similar. Despite these difficulties, I think that we can perform a completely analogous deformation calculation starting from a convenient presentation of a canonical ring of a surface of type $(III.\alpha)$ or $(III.\beta)$ not necessarily obtained as an extension of one of our halfcanonical curve rings.

The evidence that makes me think that such deformations might exist comes from some examples that I learnt from Stephen Coughlan. He constructs a surface with S of type $(III.\alpha)$ in an analogous way to Zucconi's construction of surfaces of type $(I.3)$ (cf. [Zucconi]):

Consider the toric variety $\mathbb{T}$ defined as a $\mathbb{P}(1,1,2,3)$−bundle over $\mathbb{P}^1$ by the variables and weights of the following table:

Variable	t_1	t_2	X_1	X_2	Y	Z
Bi-degree	1	1	0	0	−3	0
	0	0	1	1	2	3

Let A be the divisor class of a fibre of the natural projection $\pi : \mathbb{T} \longrightarrow \mathbb{P}^1$ and let T be a tautological divisor on $\mathbb{T}$. Consider $F \in H^0(\mathbb{T}, 2T)$ and $G \in H^0(\mathbb{T}, 6T)$ given by

$$F: \ t_1t_2(t_1+t_2)Y - X_1X_2 \tag{7.11a}$$

$$G: \ Z^2 - \beta_9 Y^3 - \beta_6\gamma_2 Y^2 - \beta_3\gamma_4 Y - \gamma_6. \tag{7.11b}$$

Then for a sufficiently general choice of $\beta_i \in S^i(t_1, t_2)$ and $\gamma_j \in S^j(X_1, X_2)$, the complete intersection $(F = 0) \cap (G = 0)$ is a surface of type $(III.\alpha)$.

For illustrative purposes, take

$$G := Z^2 - (t_1^9 + t_2^9)Y^3 - X_1^6 - X_2^6. \tag{7.12}$$

We have $K_{\mathbb{T}} = |A - 7T|$. Thus $K_S = |A + T|_S$. Then one sees that the canonical ring can be minimally generated by the following ordered sets (I omit the symbols of restricting sections to S for simplicity):

$$\text{Degree 1:} \quad \{x_0, x_1, x_2, x_3\} := \{t_1X_1, t_2X_1, t_1X_2, t_2X_2\} \tag{7.13a}$$

$$\text{Degree 2:} \quad \{y_1, y_2, y_3\} := \{t_1^5Y, t_1^3t_2^2Y, t_2^5Y\} \tag{7.13b}$$

$$\text{Degree 3:} \quad \{z_1, z_2, z_3, z_4\} := \{t_1^3Z, t_1^2t_2Z, t_1t_2^2Z, t_2^3Z\} \tag{7.13c}$$

It is also useful to name the restrictions of $S^5(t_1, t_2) \otimes Y$ to S as follows:

$$w_1 := t_1^5Y = y_1 \tag{7.14a}$$

$$w_2 := t_1^4t_2Y \tag{7.14b}$$

$$w_3 := t_1^3t_2^2Y = y_2 \tag{7.14c}$$

$$w_4 := t_1^2t_2^3Y \tag{7.14d}$$

$$w_5 := t_1t_2^4 \tag{7.14e}$$

$$w_6 := t_2^5Y = y_3 \tag{7.14f}$$

and using $F = 0$ so we get the following equations:

$$w_2 = x_0x_2 - y_2 \tag{7.15a}$$

$$w_4 = x_0x_3 - y_2 \tag{7.15b}$$

$$w_5 = x_3(x_1 - x_0) + y_2 \tag{7.15c}$$

Then by construction, we have the following relations in the canonical ring $R(S, K_S)$:

$$\bigwedge^2 \begin{pmatrix} x_0 & x_2 & w_1 & w_2 & w_3 & w_4 & w_5 & z_1 & z_2 & z_3 \\ x_1 & x_3 & w_2 & w_3 & w_4 & w_5 & w_6 & z_2 & z_3 & z_4 \end{pmatrix}. \tag{7.16}$$

These of course generate the same ideal as

$$\bigwedge^2 \begin{pmatrix} x_0 & x_2 & w_1 & w_2 & w_5 & z_1 & z_2 & z_3 \\ x_1 & x_3 & w_2 & w_3 & w_6 & z_2 & z_3 & z_4 \end{pmatrix}. \tag{7.17}$$

or

$$\bigwedge^2 \begin{pmatrix} x_0 & x_2 & w_1 & w_3 & w_5 & z_1 & z_2 & z_3 \\ x_1 & x_3 & w_2 & w_4 & w_6 & z_2 & z_3 & z_4 \end{pmatrix}. \tag{7.18}$$

Now we have a relation

$$z_1^2 - w_1^3 - w_3^2 w_6 - x_0^6 - x_2^6 \tag{7.19}$$

that can be deduced multiplying (7.12) by t_1^6. Finally, there are another 6 relations that can be obtained from (7.19) by rolling factors with respect to matrix (7.16), namely:

$$z_1 z_2 - w_1^2 w_2 - w_3 w_4 w_6 - x_0^5 x_1 - x_2^5 x_3 \tag{7.20a}$$
$$z_1 z_3 - w_1 w_2^2 - w_4^2 w_6 - x_0^4 x_1^2 - x_2^4 x_3^2 \tag{7.20b}$$
$$z_1 z_4 - w_2^3 - w_4 w_5 w_6 - x_0^3 x_1^3 - x_2^3 x_3^3 \tag{7.20c}$$
$$z_2 z_4 - w_2^2 w_3 - w_5^2 w_6 - x_0^2 x_1^4 - x_2^2 x_3^4 \tag{7.20d}$$
$$z_3 z_4 - w_2 w_3^2 - w_5 w_6^2 - x_0 x_1^5 - x_2 x_3^5 \tag{7.20e}$$
$$z_4^2 - w_3^3 - w_6^3 - x_1^6 - x_3^6. \tag{7.20f}$$

From this point, one checks that S =Proj R where

$$R := k[x_0, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, z_4]/I \tag{7.21}$$

and I is the ideal generated by the 2×2 minors of (7.17) and the 7 rolling factors relations (7.19) and (7.20) is an integral scheme of dimension 2. The invariants $K_S^2 = 7$ and $p_g = 4$ are given by construction.

The final step is to observe that the trick to deform the halfcanonical ring of section §7.2 can also be done in this example. Indeed, let $t \in \Delta_0 \subset \mathbb{C}$ be an affine parameter in a small neighbourhood of 0. Let $F_3 = F_3(x_0, x_1, x_2, x_3, y_1, y_3)$ be a homogeneous form of degree 3. Then the following set of relations:

$$x_0x_3 - x_1x_2 + t^2y_2 \tag{7.22a}$$
$$x_0w_2 - x_1y_1 + tz_1 \tag{7.22b}$$
$$x_0y_2 - x_1w_2 + tz_2 \tag{7.22c}$$
$$x_0y_3 - x_1w_5 + tz_3 \tag{7.22d}$$
$$x_2w_2 - x_3y_1 + tz_4 \tag{7.22e}$$

$$x_3w_2 - x_2y_2 + t^2x_1y_1 \tag{7.23a}$$
$$x_3w_5 - x_2y_3 + t^2x_1w_4 \tag{7.23b}$$
$$-w_2y_3 + y_2w_5 + t^3x_1F_3 \tag{7.23c}$$
$$-w_2w_4 + y_1w_5 + tx_2F_3 \tag{7.23d}$$
$$-y_2w_4 + y_1y_3 + tx_3F_3 \tag{7.23e}$$

restrict when $t = 0$ to relations identical to 10 relations in the ideal I, whereas for $t \neq 0$ (7.22) imply that the general fibre ring has codimension 3 and the five relations (7.23) generate the same ideal as the 4×4 Pfaffians of the 5×5 skew matrix

$$\begin{pmatrix} t^2F_3 & w_2 & y_2 & ty_1 \\ & w_5 & y_3 & tw_4 \\ & & tx_1 & x_2 \\ & & & x_3 \end{pmatrix}.$$

One sees that the Hilbert series remains invariant for all t: $\dfrac{-t^4 - t^3 - 3t^2 - t - 1}{t^3 - 3t^2 + 3t - 1}$. I do not see any serious reason for an analogous calculation not being possible if we consider some special surfaces of type $(III.\beta)$. Since $\overline{\mathfrak{M}_{(III.\beta)}}$ is an irreducible component of dimension 38 of the moduli space, we cannot deform a general member of the component to a surface in $\overline{\mathfrak{M}_{(0)}}$ which has dimension 36. Therefore one should start from a cleverly chosen particular subfamily for which our strategy can be followed. This leads to conjecture that $\overline{\mathfrak{M}_{(III.\beta)}}$ is not a connected component of the moduli space, but actually intersects $\overline{\mathfrak{M}_{(0)}}$. Although almost certainly there are even more possible degenerations/deformations to be found, I believe that the ones depicted below with purple arrows exist:

$$\begin{tikzcd}
 & & (I.2) & \widetilde{(I.3)} & (I.3) \arrow[l] \\
 & (III.\gamma) & & (I.1) \arrow[ul] \arrow[u] & \\
 & & (III.\alpha) \arrow[ul] \arrow[dl] & (0) \arrow[u] \arrow[l, dashed] \arrow[r, dashed] \arrow[ull, bend right] & (III.\beta) \arrow[d] \\
(F.2) & (F.1) \arrow[l] & & & (F')
\end{tikzcd}$$

Appendix
Magma codes

A.0 Sanity check for the baby example

```
RRgr<x1,x2,x3,y1,y2> :=
PolynomialRing(Rationals(),[1,1,1,2,2]);
A1:= AntisymmetricMatrix
(RRgr,[0,x1,x2,x2,x3,y2,y1,y2,-x2*x3^2,-x1^2*x2]);
A2:= AntisymmetricMatrix
(RRgr,[0,x1,x2,x2,x3,y1,y1,y2,-x2^2*x3,-x1^3]);
Pf1:=Pfaffians(A1,4);
Pf2:=Pfaffians(A2,4);
I1:=Ideal(Pf1);
I2:=Ideal(Pf2);
I:=I1+I2;
J:=MinimalBasis(I);
MinimalBasis(SyzygyModule(J));
```

A.1 Rolling factors presentation, sextic with two nodes

```
RRgr<x1,x2,x3,y1,y2,z> :=
PolynomialRing(Rationals(),[1,1,1,2,2,3]);
M:= Matrix(RRgr,2,4,[x1,x2,x3,y1+y2,x2*x3,y1,y2,z]);
M0:=Minors(M,2);
L:=y1;
C1:=x1^2;
C2:=x2^2;
C3:=x3^2;
I1:=Ideal([(y1+y2)^2+x2*x3*L+x1^2*C1+x2^2*C2+x3^2*C3,
(y1+y2)*z+y1*x3*L+x1*x2*x3*C1+x2*y1*C2+x3*y2*C3,
z^2+y1*y2*L+x2^2*x3^2*C1+y1^2*C2+y2^2*C3]);
I0 := Ideal(M0);
I:=MinimalBasis(I1+I0);
X:=Scheme(Proj(RRgr), I);
IsReduced(X);
IsIrreducible(X);
IsSingular(X);
Dimension(X);
C:=Curve(X);
Genus(C);
```

A.2 Extrasymmetric presentation, sextic with two nodes

```
RRgr<x1,x2,x3,y1,y2,z> :=
PolynomialRing(Rationals(),[1,1,1,2,2,3]);
Q1:=x2*x3;
Q2:=x1^2;
Q3:=y1;
Q4:=x3^2;
F3:=-x2^3;
P:= AntisymmetricMatrix([0,y2,x3,Q1,x1,z+F3,y1+y2,x2+x3,
x1*Q2,x2*Q3+x3*Q4,z,y1+y2,Q1*Q2,y1*Q3+y2*Q4,0]);
Pf := Pfaffians(P,4);
I0:=Ideal(Pf);
I:=MinimalBasis(Ideal(I0));
S0:=SyzygyModule(I);
S:=MinimalBasis(S0);
S;
X:=Scheme(Proj(RRgr), I);
Dimension(X);
IsReduced(X);
IsIrreducible(X);
IsSingular(X);
C:=Curve(X);
Genus(C);
```

A.3 $AM(^TA)$-presentation, bielliptic family

```
RRgr<x1,x2,x3,y1,y2,z> :=
PolynomialRing(Rationals(),[1,1,1,2,2,3]);
a:=1;
b:=1;
Q1:=x1^2;
Q2:=x1*x3;
Q3:=x2*x3;
A:= Matrix(RRgr,2,4,[x1,x2,x3,y2,x2^2,x3^2-a*x1*x2-b*x1^2,y1,z]);
M:= Matrix(RRgr,4,4,[Q1,0,0,0,0,Q2,0,0,0,0,Q3+y1,0,0,0,0,-1]);
R:=A*M*(Transpose(A));
A0:= Minors(A,2);
I0:=Ideal(A0);
I1:=Ideal([R[1,1],R[1,2],R[2,2]]);
I:=I1+I0;
X:=Scheme(Proj(RRgr), I);
Dimension(X);
IsReduced(X);
IsIrreducible(X);
IsSingular(X);
C:=Curve(X);
Genus(C);
```

A.4 Rolling factors for trigonal curves

```
RRgr<x1,x2,x3,y1,y2,y3,z1,z2> :=
PolynomialRing(Rationals(),[1,1,1,2,2,2,3,3]);
F:=y1^2+y3^2;
G:=x1^4-x3^4;
H:=x2*x3*y1-x2^2*y2;
P:=F+G+H;
A:=Matrix(RRgr,2,5,[x1,x2,y1,y2,z1,x2,x3,y2,y3,z2]);
M:=Matrix(RRgr,2,6,[x1,x2,x3,y1,y2,z1,y1,y2,y3,z1,z2,P]);
N:=Matrix(RRgr,2,5,[x2,x3,y2,y3,z2,y1,y2,z1,z2,P]);
I1:=Ideal(Minors(A,2));
I2:=Ideal(Minors(M,2));
I3:=Ideal(Minors(N,2));
I0:=I1+I2+I3;
I:=MinimalBasis(Ideal(I0));
I;
X:=Scheme(Proj(RRgr), I);
C:=Curve(X);
Genus(C);
```

A.5 First Pfaffian presentation of the trigonal family

```
RRgr<x1,x2,x3,y1,y2,y3,z1,z2> :=
PolynomialRing(Rationals(),[1,1,1,2,2,2,3,3]);
F:=y1^2+y3^2;
G:=x1^4-x3^4;
H:=x2*x3*y1-x2^2*y2;
P:=F+G+H;
O:= AntisymmetricMatrix([0,x1,x2,x2,x3,x2,y1,y2,y1,0,0,0,
y1,y2,z1,0,0,y2,y3,z2,0,0,0,z1,z2,P,0,0]);
Pf := Pfaffians(O,4);
I0:=Ideal(Pf);
I:=MinimalBasis(Ideal(I0));
I;
X:=Scheme(Proj(RRgr), I);
C:=Curve(X);
Genus(C);
```

A.6 Nonsingularity of the trigonal curves

```
RRgr<x1,x2,x3,y1,y2,y3> :=
PolynomialRing(Rationals(),[1,1,1,2,2,2]);
F:=y1^2+y3^2;
G:=x1^4-x3^4;
H:=x2*x3*y1-x2^2*y2;
P:=F+G+H;
A:=Matrix(RRgr,2,4,[x1,x2,y1,y2,x2,x3,y2,y3]);
I1:=Ideal(Minors(A,2));
I2:=Ideal([y1^3-x1^2*P,y1^2*y2-x1*x2*P,y2^2*y1-x2^2*P,
y2^3-x3*x2*P,y2^2*y3-x3^2*P]);
I0:=I1+I2;
I:=MinimalBasis(Ideal(I0));
X:=Scheme(Proj(RRgr), I);
C:=Curve(X);
Genus(C);
IsSingular(C);
```

A.7 Two towers presentation of the trigonal family

```
RRgr<x1,x2,x3,y1,y2,y3,z1,z2> :=
PolynomialRing(Rationals(),[1,1,1,2,2,2,3,3]);
F:=y1^2+y3^2;
G:=x1^4-x3^4;
H:=x2*x3*y1-x2^2*y2;
P:=F+G+H;
T1:= AntisymmetricMatrix([0,z1,z2,x1,x2,0,x2,x3,P,
y1,y1,y2,0,0,-z1,y2,y3,0,0,-z2,0]);
T2:= AntisymmetricMatrix([0,z1,z2,x2,x3,0,x1,x2,P,
y2,y2,y3,0,0,-z2,y1,y2,0,0,-z1,0]);
Pf1 := Pfaffians(T1,4);
Pf2 := Pfaffians(T2,4);
I1:=Ideal(Pf1);
I2:=Ideal(Pf2);
I0:=I1+I2;
I:=MinimalBasis(Ideal(I0));
I;
X:=Scheme(Proj(RRgr), I);
C:=Curve(X);
Genus(C);
```

A.8 Rolling factors presentation, hyperelliptic family

```
RRgr<x1,x2,x3,y1,y2,y3,z1,z2,z3,z4> :=
PolynomialRing(Rationals(),[1,1,1,2,2,2,3,3,3,3]);
w1:=y1;
w5:=y2;
w8:=y3;
w4:=w5+x1*x3;
w3:=w4+x1*x2;
w2:=w3+x1^2;
w6:=w5-x2*x3;
w7:=w6-x3^2;
A:=Matrix(RRgr,2,12,[x1,x2,w1,w2,w3,w4,w5,w6,w7,
z1,z2,z3,x2,x3,w2,w3,w4,w5,w6,w7,w8,z2,z3,z4]);
I1:=Ideal(Minors(A,2));
Rf:=[z1^2-y1^3-y2^2*y3,
z1*z2-y1^2*w2-y2*w6*y3,
z1*z3-y1^2*w3-y2*w7*y3,
z1*z4-y1^2*w4-y2*y3^2,
z2*z4-y1^2*w5-w6*y3^2,
z3*z4-y1^2*w6-w7*y3^2,
z4^2-y1^2*w7-y3^3];
I2:=Ideal(Rf);
I0:=I1+I2;
I:=MinimalBasis(I0);
I;
X:=Scheme(Proj(RRgr), I);
Dimension(X);
IsReduced(X);
IsIrreducible(X);
C:=Curve(X);
Genus(C);
```

A.9 Nonsingularity of the codimension 4 deformation family

```
RR<x0,x1,x2,x3,y1,y2,z> :=
PolynomialRing(Rationals(),[1,1,1,1,2,2,3]);
t:=1/2;
a1:=1;
a2:=0;
a3:=1;
a4:=0;
a5:=0;
a6:=3;
a7:=1;
a8:=2;
Q:=x3^2-a3*x1*x2-a4*x1^2+x0*(a5*x1+a6*x2+a7*x3)+a8*x0^2;
a9:=1;
a10:=0;
a11:=0;
a12:=0;
a13:=0;
a14:=0;
a15:=4;
a16:=0;
a17:=3;
a18:=1;
Q1:=a9*x1^2+a10*x1*x2+a11*x1*x3+a12*x2^2+a13*x2*x3+a14*x3^2
+(a15*x1+a16*x2+a17*x3)*x0+a18*x0^2;
a23:=-1;
a24:=-2;
a25:=0;
a26:=1;
Q2:=(a23*x1+a24*x2+a25*x3)*x0+a26*x0^2;
a29:=0;
a30:=1;
a31:=0;
a32:=0;
a33:=3;
```

```
a34:=3;
Q3:=a29*x1*x2+a30*y1+(a31*x1+a32*x2+a33*x3)*x0+a34*x0^2;
a19:=0;
a20:=0;
a21:=0;
a22:=0;
a27:=0;
a28:=0;
a35:=0;
A:= Matrix(RR,2,4,[x1,x2,x3,y2,
x2^2+a1*x0*x2+a2*x0^2+t*y2,Q+t*x1*x3,y1,z]);
A1:= Minors(A,2);
I1:=Ideal(A1);
M:=Matrix(RR,4,4,[Q1+t*y2,(1/2)*(a19*x0^2),
(1/2)*(a20*x2*x0+a21*x0^2),(1/2)*a22*x0,
(1/2)*(a19*x0^2),Q2+t*x2^2,(1/2)*a27*x0^2,(1/2)*a28*x0,
(1/2)*(a20*x2*x0+a21*x0^2),(1/2)*a27*x0^2,Q3,(1/2)*a35*x0,
(1/2)*a22*x0,(1/2)*a28*x0,(1/2)*a35*x0,-1]);
R:=A*M*(Transpose(A));
I2:=Ideal([R[1,1],R[1,2],R[2,2]]);
I:=Ideal(I1+I2);
MinimalBasis(I);
X:=Scheme(Proj(RR), I);
Dimension(X);
IsSingular(X);
J:=EliminationIdeal(I,x0,x1,x2,x3);
MinimalBasis(J);
```

A.10 Generators and syzygies in codimension 6

```
RR<x1,x2,x3,y1,y2,y3,z1,z2,P> :=
PolynomialRing(Rationals(),[1,1,1,2,2,2,3,3,4]);
A:=Matrix(RR,2,5,[x1,x2,y1,y2,z1,x2,x3,y2,y3,z2]);
M:=Matrix(RR,2,6,[x1,x2,x3,y1,y2,z1,y1,y2,y3,z1,z2,P]);
N:=Matrix(RR,2,5,[x2,x3,y2,y3,z2,y1,y2,z1,z2,P]);
IA:=Ideal(Minors(A,2));
IM:=Ideal(Minors(M,2));
IN:=Ideal(Minors(N,2));
Ia:=Ideal([x1*x3-x2^2, x2*y2-x3*y1, y1*y3-y2^2, y2*z2-y3*z1]);
In:=Ideal([z2^2-y3*P]);
I1:=IA+IM+IN;
I2:=Ia+IM+In;
#MinimalBasis(I1);
#MinimalBasis(I2);
#MinimalBasis(SyzygyModule(MinimalBasis(I1)));
#MinimalBasis(SyzygyModule(MinimalBasis(I2)));
```

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