Coherent Risk Measures, Reserving, and Transaction Costs

by

Sebastian Peter Armstrong

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Statistics

May 2018

Contents

Acknowledgments **iii**

Declarations **iv**

Abstract **v**

Chapter 1 Introduction **1**

1.1 Risk . . . 1
1.2 Basic functional analytic results . . . 4
1.3 Monetary, convex and coherent measures of risk . . . 7
1.3.1 Notation . . . 7
1.3.2 Basic definitions . . . 8
1.3.3 Robust representation of convex and coherent risk measures . 9
1.3.4 Examples . . . 14
1.3.5 Dynamic convex and coherent risk measures . . . 19
1.4 Transaction costs . . . 24
1.4.1 The Fundamental Theorem of Asset Pricing under transaction costs in discrete time . . . 25

Chapter 2 Predictable representation **28**

2.1 Introduction . . . 28
2.2 Pricing measures . . . 32
2.2.1 Time-consistency . . . 34
2.2.2 Predictable representability . . . 37
2.2.3 Stability properties . . . 38
2.2.4 Main result . . . 41
2.3 Examples . . . 42
2.3.1 Modelling transaction costs . . . 42
2.3.2 A Haezendonck–Goovaerts risk measure . . . 44

2.3.3 Reserving for cash flows 45
2.4 Proof of main result 46
2.4.1 Proofs of Lemmas 49

Chapter 3 Multi-currency reserving **55**
3.1 Introduction 55
3.2 Preliminaries 57
3.3 Optional representation and multi-currency time consistency 58
3.3.1 Time-consistency 58
3.3.2 Representability of claims 59
3.3.3 Stability 60
3.3.4 An Equivalence Theorem 62
3.4 The Fundamental Theorem of Multi-currency Reserving 65
3.5 Completing the Proof of Theorem 3.3.13 68
3.6 Associating a pricing mechanism to a market with proportional transaction costs 69
3.7 Appendix 74
3.7.1 Proofs of subsidiary results 74

Chapter 4 Trading to acceptability **77**
4.1 Trading to acceptability 78
4.1.1 Gain–loss based convex risk limits in discrete-time trading . . 79
4.1.2 Our model 80
4.1.3 Equivalence in one period 83

Bibliography **93**

Acknowledgments

I would like to express my deepest gratitude to my supervisor Saul Jacka for his constant guidance and kind support.

I gratefully acknowledge funding received from the EPSRC Doctoral Training Partnerships grant EP/M508184/1.

Declarations

The work contained in this thesis is original, except as acknowledged, and has not been submitted previously for a degree at any University. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made.

Chapter 2 is based on a joint paper with Saul Jacka and Abdelkarem Berkaoui, to appear in the Journal of Convex Analysis.

Abstract

This thesis deals with reserving for risk in a dynamic multi-asset market. Chapter 1 contains an exposition of the basic concepts of reserving for risks under convex and coherent risk measures.

In Chapter 2, we provide a dual characterisation of the weak*-closure of a finite sum of cones in L^∞ adapted to a discrete time filtration $\mathcal{F}_t$: the t^{th} cone in the sum contains bounded random variables that are $\mathcal{F}_t$-measurable. Hence we obtain a generalisation of Delbaen's m-stability condition [Delbaen, 2006a] for the problem of reserving in a collection of numéraires $\mathbf{V}$, called $\mathbf{V}$-m-stability, provided these cones arise from acceptance sets of a dynamic coherent measure of risk [Artzner et al., 1997, Artzner et al., 1999]. We also prove that $\mathbf{V}$-m-stability is equivalent to time-consistency when reserving in portfolios of $\mathbf{V}$, which is of particular interest to insurers.

In Chapter 3, we examine the problem of dynamic reserving for risk in multiple currencies under a general coherent risk measure. The reserver requires to hedge risk in a time-consistent manner by trading in baskets of currencies. We show that reserving portfolios in multiple currencies $\mathbf{V}$ are time-consistent when (and only when) a generalisation of Delbaen's m-stability condition [Delbaen, 2006a], termed optional $\mathbf{V}$-m-stability, holds. We prove a version of the Fundamental Theorem of Asset Pricing in this context. We show that this problem is equivalent to dynamic trading across baskets of currencies (rather than just pairwise trades) in a market with proportional transaction costs and with a frictionless final period.

Chapter 4 deals with the related problem of trading to acceptability, where

a claim X is acceptable if and only if the expected gain under each measure in a collection exceeds an associated floor.

Chapter 1

Introduction

1.1 Risk

Risk has been a useful concept to mankind through the ages. The avoidance of risk has arguably contributed to society right from its inception, when neolithic hunter-gatherers turned to agriculture for a more stable food supply. More recently, on the back of mathematical and probabilistic advances in the 17^{th} century, John Graunt is credited with producing the first life table in 1661, estimating the chance of death based on a person's age. This allowed Halley [Halley, 1693] to develop the first life annuity, and determine the premium that should be paid. This, in turn, lead to James Dodson founding the Equitable Life Assurance Society [Hickman, 2004] in 1762. Advancements in calculation and statistics improved actuarial understanding of pricing such products over the next two and a half centuries.

At the turn of the twentieth century, Bachelier [Bachelier, 1900] pioneered the idea of modelling stocks with Brownian Motion. In the 1970s, Black, Scholes, and Merton invented the risk-neutral argument for pricing derivatives. They showed that, if a financial claim could be replicated by the sum of the gains from a sequence of self-financing trades (delta hedges) made dynamically in the underlying assets, then the initial wealth required is the arbitrage-free price of the claim. Of course, such a sequence of trades does not take into account any risk preference, but instead hedges perfectly the risk at each time. Harrison and Kreps [Harrison and Kreps, 1979] introduced the concept of the risk-neutral probability measure, also known as the equivalent martingale measure, under which the stock price is a martingale. Such a probability measure exists if and only if the market is arbitrage-free; this is known as the (first) Fundamental Theorem of Asset Pricing [Delbaen and Schachermayer, 1997].

A complete market is one where every claim is perfectly hedgable. The second Fundamenteal Theorem of Asset Pricing is that the equivalent martingale measure is unique. Then, any claim may be priced as the (discounted) expectation of its terminal value under this equivalent martingale measure. In a market that is not complete, we cannot hedge every claim perfectly, but instead look to superhedge every claim; in the absence of arbitrage, the initial wealth required for superhedging a claim is the supremum of the expected value of the claim under each equivalent martingale measure.

Another critical concept in the field of Financial Mathematics is that of Markowitz's mean-variance criterion [Markowitz, 1952], which was a crucial insight into how to maximise expected return on a portfolio of stocks whilst controlling the risk. Equating the variance with the risk of the portfolio penalises equally quick losses and quick gains.

Regulators and investors seek to limit exposure to losses (the "downside" risk) without penalising gains. The problem of quantifying this downside risk gave rise to the theory of coherent risk measures, described in the seminal paper [Artzner et al., 1999]. The key idea is to give the downside risk of a claim X in terms of an amount of cash $\rho(X)$ to be added to the position so that the aggregate position $X + \rho(X)$ is acceptable to the regulator. We say a position is acceptable if the risk is less than zero. We model ρ as a functional on some space of random variables to the extended real line; this is to be defined more precisely in the next section.

The first and most widely used example of a monetary risk measure is Value at Risk, under which a position is acceptable if the probability of a loss is beneath a certain level. Value at Risk is essentially a quantile function of the distribution of the claim. While it is easy to see how this probability may be empirically estimated from historical data, the potential magnitude of the loss is not taken into account, and more importantly, Value at Risk is not sub-additive, and hence discourages diversification.

The axiomatic study of risk measures allows for a choice of a few basic tenets that the risk measure should satisfy, and from which come a rich and interesting discussion. The two axioms on which everyone can agree are monotonicity and cash-additivity. Monotonicity is that if a financial position is better in all states of the world, then it should have lower risk. Cash-additivity is that if a cash amount is added to a position, then the risk of the position is reduced by exactly the amount of cash.

Convexity is motivated in part by how Value at Risk discourages diversification. Convexity is the requirement that the risk of a convex combination of two positions should be less than the convex combination of the risks of the two positions. Thus we see that, if we have any two acceptable claims, then diversifying between the two positions yields something that is still acceptable. Under these three basic assumptions, and assuming that ρ has the Fatou property, we may represent the functional ρ as

$$\rho(X) = \sup_{\mathbb{Q}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha(\mathbb{Q})\},$$

for some penalty function α on probability measures $\mathbb{Q}$. We see that the dual representation takes into account expected losses under various probability measures $\mathbb{Q}$, with each $\mathbb{Q}$ given more or less weight according to the penalty function α.

An important technical assumption on ρ is that it have the Fatou property: whenever (X_n) is a sequence of random variables in L^∞ tending to X in probability such that $\sup \|X_n\|_\infty < \infty$, then

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

If ρ satisfies the further assumption of positive homogeneity, then we say that ρ is coherent. Positive homogeneity is the condition that if the position is scaled up by a positive number, then the risk scales up by that number. It is easy to see that the penalty function in the dual representation α is then a convex-analytical indicator function, taking the value 0 on a subset $\mathcal{Q}$ of probability measures, and $+\infty$ outside this set. The dual representation then takes the form

$$\rho(X) = \sup_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[-X].$$

A great deal of research has been conducted into risk measures that are law invariant. These are those risk measures that depend on a claim X only through the distribution of X. In fact, law invariant measures automatically have the Fatou property, as shown in [Jouini et al., 2006]. Law invariant risk measures admit the Kusuoka representation [Kusuoka, 2001]

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left\{ \int_{(0,1]} \mathrm{AVaR}_\alpha(X)\mu(d\alpha) - \beta(\mu) \right\}$$

where $\mathcal{M}_1((0,1])$ is the space of all probability measures on the interval $(0,1]$ absolutely continuous w.r.t. the Lebesgue measure, and β is a suitable penalty function.

1.2 Basic functional analytic results

We cover some basic functional analytic notions here. First, let $\mathcal{X}$ be a real vector space. There are several closely-related theorems under the name of Hahn-Banach. We present one such theorem.

Theorem 1.2.1 (Hahn-Banach). *Let $\mathcal{X}$ be a real vector space, and let $p : \mathcal{X} \to \mathbb{R}$ be a sub-linear function:*

$$p(x+y) \leq p(x) + p(y) \qquad \textit{and} \qquad p(\lambda x) = \lambda p(x) \quad \textit{for} \quad \lambda > 0.$$

Let $\mathcal{Y}$ be any *subspace of $\mathcal{X}$, and suppose that $f : \mathcal{Y} \to \mathbb{R}$ is a linear functional such that $f(y) \leq p(y)$ for all $y \in \mathcal{Y}$. Then there exists a linear functional $\tilde{f} : \mathcal{X} \to \mathbb{R}$ such that*

$$\tilde{f}|_{\mathcal{Y}} = f \qquad \textit{and} \qquad \tilde{f}(x) \leq p(x) \quad \textit{for all} \quad x \in \mathcal{X}.$$

See page 57 of [Rudin, 1991].

Let $(\mathcal{X}, \tau)$ be a topological vector space. $\mathcal{X}$ is *locally convex* if there is a non-empty family $\{p_\alpha\}_\alpha$ of seminorms on $\mathcal{X}$ (see [Rudin, 1991], p. 25). For example, any normed space is locally convex, and hence is any Banach space (a complete normed space), and hence is L^p for $p \geq 1$. An important non-example is L^0, the space of all measurable functions.

Example 1.2.2. L^0 with the topology of convergence in measure is not locally convex. For a proof, refer to Theorem 13.41 of [Aliprantis and Border, 2006].

The dual space of $\mathcal{X}$, denoted $\mathcal{X}^*$, is the vector space of the continuous linear functionals on $\mathcal{X}$. The *initial topology* of $\mathcal{X}$ with respect to a set E of linear functionals on $\mathcal{X}$ is denoted $\sigma(\mathcal{X}, E)$, and this is the coarsest topology under which all functionals in E are continuous.

The *weak topology* is the coarsest topology such that all linear functionals in $\mathcal{X}^*$ are continuous. The weak topology is denoted $\sigma(\mathcal{X}, \mathcal{X}^*)$. Of course, since $\sigma(\mathcal{X}, \mathcal{X}^*)$ is the coarsest topology such that every element of $\mathcal{X}^*$ is continuous, it must be that $\sigma(\mathcal{X}, \mathcal{X}^*) \subset \tau$.

The *weak*-topology* on $\mathcal{X}^*$ is the coarsest topology such that each element of $\mathcal{X}$ when viewed as a functional on $\mathcal{X}^*$ is continuous. The weak*-topology is denoted $\sigma(\mathcal{X}^*, \mathcal{X})$.

Example 1.2.3. For subsequent chapters, a particularly useful example to introduce is $\mathcal{X} = L^1(\Omega, \mathcal{F}, \mathbb{P})$ for a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. The duality is established through linear functionals of the form

$$Y : \mathcal{X} \to \mathbb{R}, \qquad X \mapsto \langle X, Y\rangle = \mathbb{E}[XY]$$

for $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and so $\mathcal{X}^* = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ (see [Fremlin, 2001], 243F). The dual of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the space $\mathrm{ba}(\Omega, \mathcal{F}, \mathbb{P})$ of all finitely additive finite signed measures on $\mathcal{F}$, absolutely continuous w.r.t. $\mathbb{P}$ (see [Dunford and Schwartz, 1958] p296), topologised by the total variation norm. We choose to work with L^∞ together with the weak*-topology $\sigma(L^\infty, L^1)$, yielding the dual space L^1. The double dual of our original space L^1 is again L^1.

The *closure* of a subset $E \subset \mathcal{X}$, denoted $\overline{E}$, is the smallest closed set containing E. Equivalently, $\overline{E}$ is the union of E together with all its limit points. If the closure is taken in the weak*-topology, then the closure is denoted $\overline{E}^{w^*}$; however, the w^* is omitted when it is clear from the context which closure is taken.

The spaces that we work with are not first-countable, i.e., they do not have a countable neighbourhood basis. Thus sequential closure is not equivalent to topological closure ([Sieradski, 1992] p 120). We require the stronger notion of nets for considering all limit points. A *net* in $\mathcal{X}$ is a function from a directed set $(A, \succcurlyeq)$ to $\mathcal{X}$. We shall denote a net as $(x_\alpha)_{\alpha \in A}$.

A function f from $\mathcal{X}$ to the extended real line $\mathbb{R} \cup \{\pm\infty\}$ is *weak*-lower semicontinuous* (l.s.c.) at a point $x \in \mathcal{X}$ if

$$\liminf f(x_\alpha) = \lim_{\beta \in A} \inf_{\alpha \succcurlyeq \beta} f(x_\alpha) \geq f(x)$$

for any net $x_\alpha \xrightarrow{w^*} x$, where $x_\alpha \xrightarrow{w^*} x$ denotes that the net (x_α) converges to x in the weak*-topology. A function f is l.s.c. if f is l.s.c. at every point $x \in \mathcal{X}$. A level set of a function f is a set

$$\{X \in \mathcal{X} : f(x) \leq \alpha\} \quad \text{for some} \quad \alpha \in \mathbb{R} \cup \{+\infty\}$$

A function f is weak*-l.s.c. if and only if the level sets of f are weak*-closed. We shall omit the weak* qualifier and assume that it is clear from the context henceforth.

A function f is convex if $f(\theta X + (1-\theta)Y) \leq \theta f(X) + (1-\theta) f(Y)$ for $0 \leq \theta \leq 1$ and $X, Y \in \mathcal{X}$.

The *Fenchel conjugate* of a function $f : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ is a function $f^* : \mathcal{X}^* \to \mathbb{R}$ defined by

$$f^*(Y) = \sup_{X \in \mathcal{X}} \{\langle X, Y\rangle - f(X)\}.$$

We define the Fenchel biconjugate $f^{**} : \mathcal{X}^{**} \to \mathbb{R}$ to be $(f^*)^*$: for $X \in \mathcal{X}^{**}$,

$$f^{**}(X) = \sup_{Y \in \mathcal{X}^*} \{\langle X, Y\rangle - f^*(Y)\}.$$

We shall assume further that $\mathcal{X}$ is Hausdorff, which is equivalent to saying that any net will have an unique limit, if the limit exists. The following is from [Borwein and Lewis, 2010], p76:

Theorem 1.2.4 (Fenchel-Moreau Duality). *Let $\mathcal{X}$ be a locally convex and Hausdorff topological vector space, and take $f : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$. The following are equivalent:*

(i) $f^{**} = f$*;*

(ii) f *is a proper, convex, and weak***-l.s.c. function.*

See also [Lai and Lin, 1988].

A *cone* is a set $C \subset \mathcal{X}$ such that $tx \in C$ for any $x \in C$, and $t \geq 0$. We denote the smallest convex set containing a subset $E \subset \mathcal{X}$ by $\operatorname{conv} E$; its closure is $\overline{\operatorname{conv}} E$. The *polar cone* of a cone C is

$$C^* = \{Y \in \mathcal{X}^* : \langle X, Y\rangle \leq 0\}.$$

The slight abuse of the * notation is explained via the convex analysis indicator function of a set E:

$$\delta_E(x) = \begin{cases} 0 & \text{for } x \in E, \\ +\infty & \text{otherwise.} \end{cases}$$

We see that $\delta_C^* = \delta_{C^*}$ for a cone C.

Theorem 1.2.5 (Bipolar Theorem). *Let $\mathcal{X}$ be a locally convex Hausdorff topological vector space. For a cone $\mathcal{C} \subseteq \mathcal{X}$,*

$$\mathcal{C}^{**} = \overline{\operatorname{conv}\{\mathcal{C}\}}.$$

This follows from the Fenchel-Moreau duality Theorem; see p57 of [Borwein and Lewis, 2010].

We now show a lemma that is useful for Chapters 2 and 3.

Lemma 1.2.6. *Suppose for each $t \in \mathbb{T}$, $\mathcal{C}_t \subset E$ is a closed convex cone. Then*

$$(\cap_t \mathcal{C}_t)^* = \overline{\operatorname{conv}\{\cup_t \mathcal{C}_t^*\}} = \overline{\oplus_t \mathcal{C}_t^*}.$$

Proof. The second equality is clear. For the first, we first show $(\cap_t \mathcal{C}_t)^* \supseteq \overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}}$:

$$
\begin{aligned}
& \cap \mathcal{C}_t \subseteq \mathcal{C}_s \quad \forall s \in \mathbb{T} \\
\Longrightarrow \; & (\cap_t \mathcal{C}_t)^* \supseteq \mathcal{C}_s^* \quad \forall s \in \mathbb{T} \\
\Longrightarrow \; & (\cap_t \mathcal{C}_t)^* \supseteq \cup_s \mathcal{C}_s^* \\
\Longrightarrow \; & (\cap_t \mathcal{C}_t)^* \supseteq \overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}}.
\end{aligned}
$$

since $(\cap_t \mathcal{C}_t)^*$ is closed and convex. Conversely, for $(\cap_t \mathcal{C}_t)^* \subseteq \overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}}$:

$$
\begin{aligned}
& \forall s \in \mathbb{T}, \quad \mathcal{C}_s^* \subseteq \overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}} \\
\Longrightarrow \; & \forall s \in \mathbb{T}, \quad \mathcal{C}_s \supseteq \left(\overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}}\right)^* && \text{using } \mathcal{C}_s^{**} = \mathcal{C}_s \\
\Longrightarrow \; & \cap_s \mathcal{C}_s \supseteq \left(\overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}}\right)^* \\
\Longrightarrow \; & (\cap_t \mathcal{C}_t)^* \subseteq \overline{\text{conv}\{\cup_t \mathcal{C}_t^*\}}.
\end{aligned}
$$

□

1.3 Monetary, convex and coherent measures of risk

A good introduction to convex risk measures is given by Föllmer and Schied [Föllmer and Schied, 2004]. Throughout, we work on the measurable space $(\Omega, \mathcal{F})$, and we suppose claim X belongs to the collection $\mathcal{X}$ of bounded measurable functions on $(\Omega, \mathcal{F})$ containing constant functions.

1.3.1 Notation

We fix a terminal time $T \in \mathbb{N}$, a discrete time set $\mathbb{T} := \{0, 1, \dots, T\}$. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the *reference measure* or *objective measure*. The filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ describes the information available at each time point. The space of all $\mathcal{F}$-measurable random variables is denoted $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$; we denote $L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ by L_t^0. The space of all $\mathbb{P}$-integrable (respectively $\mathbb{P}$-essentially bounded) $\mathcal{F}_t$-measurable random variables is L_t^1 (resp. L_t^∞). The space of $\mathcal{F}_t$-measurable (respectively integrable; essentially bounded) $\mathbb{R}^{d+1}$-valued random variables is denoted $\mathcal{L}_t^0 = L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{d+1})$ (resp. $\mathcal{L}_t^1$; $\mathcal{L}_t^\infty$). A subscript '+' denotes the positive orthant of a space, and '++' denotes strict positivity; for example, the set of non-negative (respectively strictly positive) essentially bounded random variables is L_+^∞ (resp. L_{++}^∞). Similarly, a subscript '−' denotes the negative orthant of a space.

1.3.2 Basic definitions

Definition 1.3.1. A functional $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ with $\rho(0) = 0$ is called a *monetary risk measure* if the following two properties hold:

(M) *Monotonicity*: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

(TI) *Translation invariance*: for every constant function m we have $\rho(m + X) = \rho(X) - m$.

A monetary risk measure taking only finite values is Lipschitz continuous with respect to the supremum norm, with Lipschitz constant 1. Indeed,

$$X \leq Y + \|X - Y\| \quad \overset{(M)}{\Longrightarrow} \quad \rho(X) \geq \rho(Y + \|X - Y\|) \overset{(TI)}{=} \rho(Y) - \|X - Y\|,$$

so $\rho(Y) - \rho(X) \leq \|X - Y\|$. Performing the same steps with X and Y interchanged, we see that

$$|\rho(X) - \rho(Y)| \leq \|X - Y\|.$$

Definition 1.3.2. For a monetary risk measure ρ, the set

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \leq 0\}$$

is the *acceptance set* of ρ.

We may treat the acceptance set $\mathcal{A}$ as the primitive object, and recover the risk measure $\rho_{\mathcal{A}}$ via

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}.$$

Thus a monetary risk measure of a claim X may be seen as the amount of capital required to ensure that the position X is acceptable to the investor. For the reader's convenience, we list some properties of acceptance sets.

Proposition 1.3.3. *The acceptance set $\mathcal{A}$ of a monetary risk measure ρ is non-empty and has the following properties:*

1. $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$*;*

2. ($\mathcal{A}$ is solid) $X \in \mathcal{A}$*,* $Y \in \mathcal{X}$*,* $Y \geq X \quad \Longrightarrow \quad Y \in \mathcal{A}$*;*

3. (closure property) *for* $X \in \mathcal{A}$ *and* $Y \in \mathcal{X}$*,*

$$\{\lambda \in [0,1] : \lambda X + (1-\lambda)Y \in \mathcal{A}\} \quad \textit{is closed in} \quad [0,1].$$

Definition 1.3.4. A monetary risk measure ρ is called a *convex risk measure* if, in addition to axioms (M) and (TI), ρ satisfies

(C) *Convexity*: $\rho(\theta X + (1-\theta)Y) \leq \theta\rho(X) + (1-\theta)\rho(Y) \quad$ for $0 \leq \theta \leq 1$.

In fact, in conjunction with (M) and (TI), we could assume the seemingly weaker

(QC) *Quasi-convexity*: $\rho(\theta X + (1-\theta)Y) \leq \max\{\rho(X), \rho(Y)\} \quad$ for $0 < \theta < 1$.

See [Föllmer and Schied, 2004] for details.

Definition 1.3.5. A convex risk measure ρ is called a *coherent risk measure* if, in addition to axioms (M), (TI) and (C), ρ satisfies

(PH) *Positive homogeneity*: for $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda\rho(X)$.

Under (PH), (C) is equivalent to

(S) *Subadditivity*: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

Proposition 1.3.6. *Suppose ρ is a monetary risk measure with acceptance set $\mathcal{A}_\rho$ as defined above.*

- *ρ is a convex risk measure if and only if $\mathcal{A}_\rho$ is a convex set;*
- *ρ is positively homogeneous if and only if $\mathcal{A}_\rho$ is a cone.*

In particular, ρ is a coherent risk measure if and only if $\mathcal{A}_\rho$ is a convex cone.

1.3.3 Robust representation of convex and coherent risk measures

Let $\mathcal{M}_1$ be all the probability measures on the space $(\Omega, \mathcal{F})$, and let $\mathcal{M}_{1,f}$ be the set of all finitely additive set functions μ on $(\Omega, \mathcal{F})$, normalised to $\mu(\Omega) = 1$. We write $\mathbb{E}_\mu[X]$ for the integral of a bounded, μ-measurable X with respect to $\mu \in \mathcal{M}_{1,f}$; see Part I, Chapter III, section 2 of [Dunford and Schwartz, 1958], or Appendix A.6 of [Föllmer and Schied, 2004]. Observe that for $\mathbb{Q} \in \mathcal{M}_{1,f}$, the functional $X \mapsto \mathbb{E}_\mathbb{Q}[-X] - \alpha(\mathbb{Q})$ is convex, monotone, and translation invariant on $\mathcal{X}$. These three properties are preserved when optimising over $\mathbb{Q} \in \mathcal{M}_{1,f}$, so

$$\rho(X) := \sup_{\mathbb{Q} \in \mathcal{M}_{1,f}} \big(\mathbb{E}_\mathbb{Q}[-X] - \alpha_{\min}(\mathbb{Q})\big)$$

is a convex risk measure. Interestingly, *every* convex measure is representable in this form:

Theorem 1.3.7. *Any convex risk measure ρ on $\mathcal{X}$ is of the form*

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{M}_{1,f}} \big(\mathbb{E}_{\mathbb{Q}}[-X] - \alpha_{\min}(\mathbb{Q})\big), \quad \textit{for} \quad X \in L^\infty(\mathbb{P}), \tag{1.1}$$

where the penalty function $\alpha_{\min}$ is given by

$$\alpha_{\min}(\mathbb{Q}) := \sup_{X\in\mathcal{A}_\rho} \mathbb{E}_{\mathbb{Q}}[-X] \quad \textit{for} \quad \mathbb{Q} \in \mathcal{M}_{1,f}.$$

Moreover, $\alpha_{\min}$ is the minimal penalty function which represents ρ, i.e., any penalty function α for which eq. (1.1) *holds satisfies $\alpha(\mathbb{Q}) \geq \alpha_{\min}(\mathbb{Q})$ for any $\mathbb{Q} \in \mathcal{M}_{1,f}$.*

This is Theorem 4.15 of [Föllmer and Schied, 2004]. Note that the supremum is attained for some finitely additive measure with total mass 1, so we can replace the supremum with a maximum. It is natural to consider under what conditions α is supported on $\mathcal{M}_1$, the set of probability measures (that is, those measures in $\mathcal{M}_{1,f}$ that are also sigma-additive).

Definition 1.3.8. A convex risk measure ρ has a *robust representation* whenever

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}_\rho} \{\mathbb{E}[-X] - \alpha(\mathbb{Q})\}, \tag{1.2}$$

where $\alpha : \mathcal{M}_1 \to (-\infty, +\infty]$ is a given penalty function, and

$$\mathcal{Q}_\rho = \{\mathbb{Q} \in \mathcal{M}_1 : \alpha(\mathbb{Q}) < \infty, \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}[X] \text{ is well-defined for any } X \in \mathcal{X}\}.$$

We may think of $\mathcal{Q}_\rho$ as a collection of credible scenarios, whose credibility is expressed through the penalty function; a lower value of $\alpha(\mathbb{Q})$ would signify a greater belief in the probabilistic model $\mathbb{Q}$.

Suppose ρ admits a robust representation (1.2) with penalty function α. Then the representation also holds for penalty function

$$\alpha_{\min}(\mathbb{Q}) = \sup_{X\in\mathcal{X}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \rho(X)\} = \sup_{X\in\mathcal{A}_\rho} \mathbb{E}_{\mathbb{Q}}[-X].$$

Furthermore, the penalty $\alpha_{\min}$ is the minimal such function, in the sense that $\alpha(\mathbb{Q}) \geq \alpha_{\min}(\mathbb{Q})$ for any $\mathbb{Q} \in \mathcal{M}_1$. We now provide sufficient conditions for which ρ has a robust representation.

Theorem 1.3.9. *Let ρ be a convex risk measure. The following two statements are equivalent.*

(i) ρ *is* continuous from below, *i.e.*

$$X_n(\omega) \uparrow X(\omega) \quad \textit{for each} \quad \omega \in \Omega \quad \Longrightarrow \quad \rho(X_n) \downarrow \rho(X);$$

(ii) ρ *satisfies the* Lebesgue property, *i.e. for any bounded sequence X_n converging pointwise to X on Ω,*

$$\lim_{n\to\infty} \rho(X_n) = \rho(X).$$

If (i) and (ii) are satisfied, then ρ admits the robust representation (1.2), *and further*

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{M}_1} \{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha_\rho(\mathbb{Q})\},$$

i.e. the minimal penalty function $\alpha_{\min}$ is concentrated on $\mathcal{M}_1$, and the supremum is attained.

Fixing a probabilistic model

Fix $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Recall that $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) = L^\infty(\mathbb{P})$ with the supremum norm is a Banach space. The dual space of $L^\infty(\mathbb{P})$ is $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$, the space of all bounded, finitely additive measures μ on $(\Omega, \mathcal{F})$ such that $\mu(A) = 0$ whenever $\mathbb{P}(A) = 0$. If $\mu \in \mathbf{ba}(\mathbb{P})$ satisfies $\mu(\Omega) = 1$, then we call μ a finitely additive probability measure. Let $\mathcal{M}_1(\mathbb{P}) \subset \mathbf{ba}(\mathbb{P})$ denote the set of probability measures absolutely continuous with respect to $\mathbb{P}$. The following is due to Delbaen, [Delbaen, 2000]:

Theorem 1.3.10. *Suppose $\rho : L^\infty(\mathbb{P}) \to \mathbb{R}$ is a coherent risk measure. Then there is a convex $\sigma(\mathbf{ba}(\mathbb{P}), L^\infty(\mathbb{P}))$-closed set $\mathcal{P}_{\mathbf{ba}}$ of finitely additive probabilities, such that*

$$\rho(X) = \sup_{\mu\in\mathcal{P}_{\mathbf{ba}}} \mathbb{E}_\mu[-X]. \tag{1.3}$$

For general convex measures, we have the following characterisation.

Theorem 1.3.11. *Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure. Then the following are equivalent.*

(a) ρ can be represented by some penalty function supported on $\mathcal{M}_1(\mathbb{P})$.

(b) ρ can be represented by the restriction of the minimal penalty function $\alpha_{\min}$ to $\mathcal{M}_1(\mathbb{P})$:

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha_{\min}(\mathbb{Q})\}.$$

(c) ρ *is* continuous from above*, i.e.*

$$X_n \downarrow X \quad \mathbb{P}\text{-}a.s. \qquad \Longrightarrow \qquad \rho(X_n) \uparrow \rho(X).$$

(d) ρ *satisfies the* Fatou property*, i.e. for any bounded sequence* X_n *converging* $\mathbb{P}$*-a.s. to* X*,*

$$\liminf_{n\to\infty} \rho(X_n) \geq \rho(X).$$

(e) ρ *is lower semicontinuous for the weak* topology* $\sigma(L^\infty, L^1)$*.*

(f) The acceptance set $\mathcal{A}_\rho$ *of* ρ *is weak* closed in* L^∞*, i.e.* $\mathcal{A}_\rho$ *is closed with respect to the topology* $\sigma(L^\infty, L^1)$*.*

See [Delbaen, 2000] for a proof. Following on from Theorem 1.3.10, we have:

Corollary 1.3.12. *The minimal penalty function* $\alpha_{\min}$ *of a coherent risk measure* ρ *takes only the values 0 and* $+\infty$*. In particular,*

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{Q}_{\max}} \mathbb{E}_{\mathbb{Q}}[-X] \quad \textit{for} \quad X \in L^\infty(\mathbb{P}),$$

for the convex set

$$\mathcal{Q}_{\max} = \{\mathbb{Q} \in \mathbf{ba}(\mathbb{P}) : \alpha_{\min}(\mathbb{Q}) = 0\},$$

and $\mathcal{Q}_{\max}$ *is the largest set of measures for which the representation* (1.3) *holds.*

The interpretation of the measure $\mathbb{P}$ needs some thought. In practice, probabilities are dependent on modelling choices, and different investors might assign different values to events. However, everyone must be able to agree which events are impossible, and hence, the possible events with probability strictly positive, without agreeing on the numerical value of such a probability. Only knowledge of events of probability zero is important. In this way, $\mathbb{P}$ is thought of as a representative from a class of equivalent probability measures $\mathcal{P}^{\approx} = \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}\}$, that define what is and what is not possible.

$L^p(\Omega, \mathcal{F}, \mathbb{P})$

For certain applications, a larger class of random variables are needed for the set of claims $\mathcal{X}$, particularly for modelling unbounded claims. A possible space with elegant theory readily available is $L^p(\Omega, \mathcal{F}, \mathbb{P})$, the space of L^p-integrable random variables, with $1 \leq p < \infty$. The extra choice in claims does not come without cost;

the set of probability measures in the dual space is limited:

$$\mathcal{M}_q = \{\mathbb{Q} \in \mathcal{M}_1 : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q\},$$

where $q = p/p - 1$ is the Hölder conjugate of p.

One wrinkle when evaluating claims that are possibly unbounded, say in $L^p(\mathbb{P})$ for $p \in (1, \infty)$, is that in general $L^p(\mathbb{P}) \neq L^p(\mathbb{Q})$ for $\mathbb{Q} \in \mathcal{P}^{\approx}$. Thus any results depend on the choice of objective measure.

Orlicz hearts

Continuing to generalise, Orlicz spaces are a generalisation of the Lebesgue spaces discussed above. See [Cheridito and Li, 2009] for more detail. Let Φ be a Young function, that is, a left-continuous function that is convex, $\lim_{x\to 0+} \Phi(x) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$. Define the Luxemburg norm[1]

$$\|X\|_\Phi := \inf\{\alpha > 0 : \mathbb{E}_\mathbb{P}[\Phi(|X|/\alpha)] \leq \Phi(1)\}.$$

The Orlicz space is the subset of L^0 of all measurable functions $L^\Phi := \{X \in L^0 : \|X\|_\Phi < \infty\}$.

The Orlicz heart for function Φ is

$$\mathcal{H}^\Phi := \{X \in L^\Phi : \mathbb{E}[\Phi(c|X|)] < \infty \quad \forall c > 0\}.$$

If Φ jumps to $+\infty$, then $\mathcal{H}^\Phi = \{0\}$. So we assume that Φ is finite-valued. In this case, Φ is its own Fenchel biconjugate, $\Phi = \Phi^{**}$. A risk measure on an Orlicz heart has the dual representation

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{M}^{\Phi^*}} \{\mathbb{E}_\mathbb{Q}[-X] - \alpha(\mathbb{Q})\} \quad \text{for } X \in \mathcal{H}^\Phi$$

where $\mathcal{M}^{\Phi^*} = \{\mathbb{Q} \in \mathcal{M}_1 : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^{\Phi^*}\}$. See [Cheridito and Li, 2009].

[1] Some authors prefer the definition $\|X\|_\Phi := \inf\{\alpha > 0 : \mathbb{E}_\mathbb{P}[\Phi(|X|/\alpha)] \leq 1\}$; however, this leads to a messy constant $\Phi(1)$ appearing throughout. Under our definition, $\|1\|_\Phi = 1$.

1.3.4 Examples

Worst case risk measure

Consider the measure defined by

$$\rho_{\max}(X) = -\inf_{\omega\in\Omega} X(\omega) \qquad \forall X \in \mathcal{X}.$$

This is the least upper bound for the potential loss in any eventuality. It is the most conservative measure of risk, due to the inequality

$$\rho(X) \leq \rho(\inf_{\Omega} X(\omega)) = \rho_{\max}(X) \qquad \text{(assuming normalisation } \rho(0) = 0\text{)}.$$

The acceptance set is the positive orthant L^∞_+, the cone of all non-negative bounded random variables. Clearly $\rho_{\max}$ is coherent. Fixing $\mathbb{P}$, and taking $\mathcal{P}_{\mathbf{ba}}$ to be all probability measures in $\mathbf{ba}(\mathbb{P})$, we have the representation of Theorem 1.3.10.

Scenario measures and floors

Fix $\mathcal{Q}$ a set of probabilities, and consider "floors" $f : \mathcal{Q} \to \mathbb{R}$ with $\sup_{\mathcal{Q}} f(\mathbb{Q}) < \infty$. We define the acceptability set

$$\mathcal{A} = \{X \in L^\infty : \forall \mathbb{Q} \in \mathcal{Q}, \quad \mathbb{E}_{\mathbb{Q}}[X] \geq f(\mathbb{Q})\}.$$

We may represent this convex risk measure with the penalty function

$$\alpha(\mathbb{Q}) = \begin{cases} -f(\mathbb{Q}) & \text{for } \mathbb{Q} \in \mathcal{Q}, \\ +\infty & \text{otherwise} \end{cases}$$

We note that ρ is coherent whenever $f \equiv 0$ on $\mathcal{Q}$.

Risk measures from utility

Extending the previous example, fix a class of probability measures $\mathcal{Q}$, fix a utility function u on $\mathbb{R}$, and fix levels $c_{\mathbb{Q}} \in \mathbb{R}$ for each $\mathbb{Q} \in \mathcal{Q}$, such that $\sup_{\mathbb{Q}\in\mathcal{Q}} c_{\mathbb{Q}} < \infty$. Define the acceptability set

$$\mathcal{A} = \{X \in L^\infty : \forall \mathbb{Q} \in \mathcal{Q}, \quad \mathbb{E}_{\mathbb{Q}}[u(X)] \geq u(c_{\mathbb{Q}})\}.$$

The acceptability set is convex, inducing a convex risk measure.

Entropic risk measures

These measures are well-suited to Orlicz hearts. The entropic risk measure is

$$e_\gamma(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \left(\mathbb{E}_\mathbb{Q}[-X] - \gamma h(\mathbb{Q}|\mathbb{P})\right) = \gamma \ln \mathbb{E}_\mathbb{P}\left(\exp\left(-\frac{1}{\gamma}X\right)\right)$$

where the *relative entropy* h is defined as

$$h(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_\mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right) \qquad \text{whenever the integral is finite.}$$

This is studied in [Acciaio and Penner, 2011, Föllmer and Knispel, 2013, Barrieu and El Karoui, 2004].

Value at Risk

Fix a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$, and define the upper quantile function

$$q_X^+(\lambda) = \inf\{x : \mathbb{P}[X \le x] > t\}.$$

Then the *Value at Risk* at level λ is the monetary risk measure given by

$$\mathrm{VaR}_\lambda(X) = -q_X^+(\lambda) = \inf\{m \in \mathbb{R} : \mathbb{P}[m + X < 0] \le \lambda\}$$

It is clear that VaR_λ is positively homogeneous; the following example shows that VaR_λ is not convex, and hence VaR_λ is not a coherent risk measure.

Example 1.3.13. Consider an investment into two defaultable corporate bonds, each with return rate $r \in (0, 1)$ in a market with zero risk-free interest rate. Both bonds are independent and identically distributed, with payoff

$$X_i = \begin{cases} -1 & \text{with probability } p \in (0,1), \\ r & \text{with probability } 1-p. \end{cases} \qquad \text{for } i = 1, 2.$$

A smart investor wishing to invest capital 1 in these two bonds might diversify the risk posed by each individual bond by investing half of her funds in X_1 and the other half in X_2. Define $Y := (X_1 + X_2)/2$. The probability of a negative outcome when splitting funds between the bonds is the probability of at least one of the two bonds defaulting. Since $r < 1$, we see that this is larger than the probability of a negative outcome when investing in a single bond:

$$\mathbb{P}[Y < 0] = 1 - \mathbb{P}[\text{neither default}] = 1 - (1-p)^2 = p(2-p) > p.$$

Thus, taking $\lambda \in (p, p(2-p))$, we have

$$\mathrm{VaR}_\lambda(X_1) = -r < 0 \qquad \text{but} \qquad \mathrm{VaR}_\lambda(Y) = \frac{1-r}{2} > 0.$$

So VaR_λ discourages investing in Y compared to investing in X_1!

Average Value at Risk

An important example is the *Average Value at Risk*; this name is potentially the least misleading amongst other names in the literature, including *Tail Value at Risk*, *Expected Shortfall*, and *Conditional Value at Risk*. For a level $\lambda \in (0, 1]$, we define the Average Value at Risk to be

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\, d\alpha.$$

We may extend this definition to encompass $\lambda = 0$ by

$$\mathrm{AVaR}_0(X) =: \mathrm{VaR}_0(X) := \operatorname{ess\,sup}(-X).$$

This is consistent with another definition obtained via the optimized certainty equivalent/Fenchel-Moreau duality

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda}\inf_{z\in\mathbb{R}}\{\mathbb{E}[(z-X)^+] - \lambda z\} = \frac{1}{\lambda}\mathbb{E}[(q_X^+(\lambda) - X)^+] - q_X^+(\lambda).$$

Theorem 1.3.14. *For $\lambda \in (0, 1]$, AVaR_λ is a coherent risk measure which is continuous from below, with the robust representation*

$$\mathrm{AVaR}_\lambda(X) = \max_{\mathbb{Q}\in\mathcal{Q}_\lambda} \mathbb{E}[-X]$$

where $\mathcal{Q}_\lambda := \{\mathbb{Q} \ll \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\lambda}\}$. Moreover $\mathcal{Q}_\lambda$ is the maximal set for which a robust representation of the above form occurs.

We introduce a fascinating property of AVaR_λ with a definition:

Definition 1.3.15. A monetary risk measure ρ is *law-invariant* if $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution under $\mathbb{P}$.

All the previous examples, with the exception of the worst-case risk measure have been law-invariant; in particular, VaR_λ and AVaR_λ are law-invariant. The next result shows that AVaR_λ is, in some sense, a basic building block of *any* law-invariant convex risk measure.

Theorem 1.3.16. *A convex risk measure ρ is law-invariant and continuous from above if and only if*

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left\{ \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda) - \beta_{\min}(\mu) \right\},$$

where $\mathcal{M}_1((0,1])$ is the set of probability measures on $(0,1]$, and

$$\beta_{\min}(\mu) = \sup_{X \in \mathcal{A}_\rho} \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda).$$

For a proof, see Theorem 4.57 of [Föllmer and Schied, 2004]. When ρ is coherent, the positive scaling implies $\beta_{\min}(\mu) \in \{0, +\infty\}$ for any μ. We thus have the representation

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda). \tag{1.4}$$

Distortions and Choquet integrals

A distortion function is a non-decreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. Define the distorted probability measure $\mathbb{Q}$ by $\mathbb{Q}[A] = g(\mathbb{P}[A])$ for any $A \in \mathcal{F}$. Then we may define a monetary risk measure to be the negative expectation under the distorted measure $\mathbb{Q}$:

$$\rho_g(X) = \mathbb{E}_{\mathbb{Q}}[-X] = \int_{-\infty}^{0} (1 - g(\mathbb{P}[-X > x]))\,dx - \int_{0}^{\infty} g(\mathbb{P}[-X > x])\,dx. \tag{1.5}$$

The above may be written as a *Choquet integral*

$$\rho_g(X) = \int (-X)\,dc \qquad \text{where} \qquad c = g \circ \mathbb{P}.$$

Note that $\mathbb{Q} \sim \mathbb{P}$ if and only if the mapping g is continuous and one-to-one. By linearity of expectation, any ρ_g constructed in the above manner is positively homogeneous and translation invariant.

Theorem 1.3.17. *The distorted risk measure ρ_g is coherent if and only if g is concave.*

For a proof of this, see [Sereda et al., 2010]. A range of popular measures may be cast in this setting, including VaR_λ, which is obtained from distortion

$$g_{\mathrm{VaR}_\lambda}(x) = \mathbb{1}_{[\lambda,1]}(x),$$

and AVaR_λ, which is obtained from distortion

$$g_{\mathrm{AVaR}_\lambda}(x) = \frac{x}{\lambda}\mathbb{1}_{[0,\lambda]}(x) + \mathbb{1}_{[\lambda,1]}(x) = \left(\frac{x}{\lambda}\right) \wedge 1.$$

We note that $g_{\mathrm{AVaR}_\lambda}$ is indeed a concave distortion, whereas g_{VaR_λ} is not.

We have so far seen two representations of coherent risk measures, namely (1.4) and (1.5); it is natural to ask how they are related. To simplify matters, we fix a probability measure $\mu \in \mathcal{M}_1(0,1]$ and *assume* the supremum is attained in (1.4) for this μ:

$$\rho_\mu(X) := \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda).$$

We may find a concave distortion function g such that $\rho_\mu = \rho_g$ for ρ_g defined as in (1.5).

Theorem 1.3.18. *In the above notation, μ and g are related through*

$$\partial_+ g(\alpha) = \int_{(\alpha,1]} \lambda^{-1}\,\mu(d\lambda),$$

where ∂_+ denotes the right derivative.

As an illustration of why this might be true, we assume g is continuously differentiable, and so $\partial_+ g = g'$. By definition of AVaR_λ,

$$\begin{aligned}\rho_\mu(X) &= \int_{(0,1]} \mathrm{AVaR}_\lambda(X)\,\mu(d\lambda)\\ &= \int_{(0,1]} \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\,d\alpha\,\mu(d\lambda).\end{aligned}$$

By swapping the order of integration, and using the equality in the statement,

$$\begin{aligned}\rho_\mu(X) &= \int_0^1 d\alpha \left(\int_{(\alpha,1]} \lambda^{-1}\,\mu(d\lambda)\right) \mathrm{VaR}_\alpha(X)\\ &= -\int_0^1 g'(\alpha) q_X^+(\alpha)\,d\alpha,\end{aligned}$$

where q_X^+ is the upper quantile function defined previously. Now we employ the change of variables $t = g(\alpha)$:

$$\begin{aligned}\rho_\mu(X) &= -\int_0^1 q_X^+(g^{-1}(t))\,dt \\ &= -\mathbb{E}_{\mathbb{Q}}[X] \\ &= \rho_g(X).\end{aligned}$$

A rigorous discussion may be found in [Föllmer and Schied, 2004], Theorem 4.64. The interested reader may look to [Denneberg, 1990, Wang et al., 1997, Robert and Thérond, 2013, Föllmer and Knispel, 2013] for more.

1.3.5 Dynamic convex and coherent risk measures

In this subsection we briefly survey an area that has received much attention in the last decade. We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$, for $\mathbb{T} := [0, \infty)$. By suitable embeddings, we may reduce to the case of discrete time, finite time horizon, etc.. We assume $\mathcal{F}_0$ trivial, containing every $\mathbb{P}$-null set, and the filtration is right-continuous. Write $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}) = L_t^\infty$.

Definition 1.3.19. A map $\rho_t : L^\infty \to L_t^\infty$ is a *conditional convex risk measure* if it satisfies the following properties for all $X, Y \in L^\infty$:

(i) *Conditional translation invariance:* for all $m_t \in L_t^\infty$,

$$\rho_t(X + m_t) = \rho_t(X) - m_t.$$

(ii) *Monotonicity:* $X \leq Y \quad \Longrightarrow \quad \rho_t(X) \geq \rho_t(Y)$.

(iii) *Conditional convexity:* for any $\lambda \in L_t^\infty$ with $0 \leq \lambda \leq 1$,

$$\rho_t(\lambda X + (1 + \lambda)Y) \leq \lambda\rho_t(X) + (1 - \lambda)\rho_t(Y).$$

A conditional convex risk measure is a *conditional coherent risk measure* if, in addition to properties (i)-(iii), ρ_t satisfies

(iv) *Conditional positive homogeneity:* for any $\lambda \in L_t^\infty$ with $\lambda \geq 0$,

$$\rho_t(\lambda X) = \lambda\rho_t(X).$$

A sequence $(\rho_t)_{t\in\mathbb{T}}$ is a *dynamic convex risk measure* if ρ_t is a conditional convex risk measure for each $t \in \mathbb{T}$.

The conditional convex risk measure ρ_t induces an acceptance set

$$\mathcal{A}_t := \{X \in L^\infty : \rho_t(X) \leq 0\}.$$

As in the static case, we may recover the conditional risk measure ρ_t via

$$\rho_t(X) = \operatorname{ess\,inf}\{Y \in L^\infty_t : X + Y \in \mathcal{A}_t\} \quad \mathbb{P}\text{-a.s.}$$

so that $\rho_t(X)$ is the minimal conditional captial requirement to be added to claim X to achieve time-t acceptability.

Robust representation

We summarise robust representation results that carry over from the static case. As before, let $\mathcal{M}_1(\mathbb{P})$ denote the set of probability measures absolutely continuous with respect to $\mathbb{P}$.

Theorem 1.3.20 (From [Acciaio and Penner, 2011])**.** *For a conditional risk measure ρ_t the following are equivalent:*

1. *ρ_t has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_t}\{\mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_t] - \alpha_t(\mathbb{Q})\},$$

where

$$\mathcal{Q}_t := \{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) : \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}\}$$

and α_t maps $\mathcal{Q}_t$ to the set of $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R} \cup \{+\infty\}$, such that

$$\operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_t}\{-\alpha_t(\mathbb{Q})\} \in \mathbb{R}.$$

2. *ρ_t has the robust representation in terms of the minimal penalty function*

$$\alpha_t^{\min}(\mathbb{Q}) = \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} \mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_t].$$

3. *ρ_t has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_t^f}\{\mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_t] - \alpha_t^{\min}(\mathbb{Q})\} \qquad \mathbb{P}\text{-a.s.},$$

where

$$\mathcal{Q}_t^f := \{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) : \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}, \quad \mathbb{E}_\mathbb{Q}[\alpha_t^{\min}(\mathbb{Q})] < \infty\}.$$

4. *ρ_t has the Fatou property: for any bounded sequence $(X_n)_{n\in\mathbb{N}} \subseteq L^\infty$ converging $\mathbb{P}$-a.s. to some $X \in L^\infty$,*

$$\rho_t(X) \leq \liminf_{n\to\infty} \rho_t(X_n) \qquad \mathbb{P}\text{-a.s.}.$$

5. *ρ_t is continuous from above: for any bounded sequence $(X_n)_{n\in\mathbb{N}} \subseteq L^\infty$, and $X \in L^\infty$,*

$$X_n \downarrow X \quad \mathbb{P}\text{-a.s.} \qquad \Longrightarrow \qquad \rho_t(X_n) \uparrow \rho_t(X) \quad \mathbb{P}\text{-a.s.}.$$

6. *The acceptance set $\mathcal{A}_t \subseteq L^\infty$ of ρ_t is $\sigma(L^\infty, L^1)$-closed.*

Time-consistency

The fundamental idea behind time-consistency is the following: suppose at time t, an investor, when choosing between claims X and Y, is indifferent. Then at time $s \leq t$, she should be indifferent. There are many notions of time-consistency in the literature; we follow the approach of Delbaen [Delbaen, 2006b].

Fix $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F} = \mathcal{F}_\infty)$. Let $\mathcal{S}$ be a closed convex set of probability measures containing $\mathbb{P}$, where every element is absolutely continuous with respect to $\mathbb{P}$. We identify probability measures $\mathbb{Q}$ on $\mathcal{F}_\infty$ that are absolutely continuous w.r.t. $\mathbb{P}$, with their densities $\frac{d\mathbb{Q}}{d\mathbb{P}}$, so with a subset of functions in L^1. We write $\mathcal{S}^e$ for the set of measures in $\mathcal{S}$ that are also equivalent to $\mathbb{P}$.

For each stopping time τ and bounded r.v. X, we would like to define the coherent risk measure

$$\rho_\tau(X) \stackrel{?}{=} \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_\tau].$$

However, $\mathbb{Q}$ does not necessarily have to be in $\mathcal{S}^e$, the set of measures *equivalent* to $\mathbb{P}$. So the equality would not hold $\mathbb{P}$-a.s.. A more sensible definition, using the density of $\mathcal{S}^e$ in $\mathcal{S}$, is

$$\rho_\tau(X) := \operatorname{ess\,sup}\{\mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_\tau] : \mathbb{Q} \in \mathcal{S}, \mathbb{Q} \sim \mathbb{P}\},$$

for each stopping time τ and bounded r.v. X.

Definition 1.3.21. The set $\mathcal{S}$ is called *time consistent* if, for any pair of stopping times $\sigma \leq \tau$ and any pair of r.v.s $X, Y \in L^\infty$, we have that

$$\rho_\tau(X) \geq \rho_\tau(Y) \qquad \Longrightarrow \qquad \rho_\sigma(X) \geq \rho_\sigma(Y).$$

Multiplicative stability

Definition 1.3.22. Fix $\mathbb{Q}^0 \in \mathcal{S}$, $\mathbb{Q} \in \mathcal{S}^e$, and define their associated change of measure martingales $Z_t^0 = \mathbb{E}\left[\frac{d\mathbb{Q}^0}{d\mathbb{P}}|\mathcal{F}_t\right]$ and $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right]$. Fix a stopping time τ, and define L by

$$L_t := \begin{cases} Z_t^0 & \text{for } t \leq \tau, \\ Z_\tau^0 \frac{Z_t}{Z_\tau} & \text{for } t \geq \tau. \end{cases}$$

The set of probability measures $\mathcal{S} \subset L^1$, is *multiplicatively stable* (henceforth, *m-stable*) if any L constructed in the above way is a change of measure martingale defining a measure belonging to $\mathcal{S}$.

Theorem 1.3.23. *The following are equivalent:*

(1) The set $\mathcal{S}$ is m-stable.

(2) For every bounded random variable X, the family $\{\rho_T(X) : T$ is a stopping time$\}$ is recursive*:*

$$\textit{for any two stopping times } \sigma \leq \tau, \textit{ we have } \rho_\sigma(X) = \rho_\sigma(-\rho_\tau(X)).$$

(3) For every bounded r.v. X and for every stopping time σ, we have $\rho_0(X) \geq \rho_0(-\rho_\sigma(X))$.

(4) The set $\mathcal{S}$ is time consistent.

(5) The family $\{\rho_T(X) : T$ is a stopping time$\}$ satisfies the supermartingale property:

$$\forall \mathbb{Q} \in \mathcal{S} \textit{ and all pairs of stopping times } \sigma \leq \tau \textit{ we have } \rho_\sigma(X) \geq \mathbb{E}_\mathbb{Q}[\rho_\tau(X)|\mathcal{F}_\sigma].$$

Proof. The proof we give here is due to Delbaen, [Delbaen, 2006b].

(1) $\Longrightarrow$ (2): By the tower property of conditional expectation,

$$\rho_\sigma(X) = \operatorname*{ess\,sup}_{\mathbb{Q}} \mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_\sigma] = \operatorname*{ess\,sup}_{\mathbb{Q}} \mathbb{E}_\mathbb{Q}[\mathbb{E}_\mathbb{Q}[-X|\mathcal{F}_\tau]|\mathcal{F}_\sigma].$$

m-stability implies that the essential supremum is the same, whether using the same measure $\mathbb{Q} \in \mathcal{S}$ throughout, or "switching" to another measure $\widetilde{\mathbb{Q}} \in \mathcal{S}$ after stopping time σ, where $\widetilde{\mathbb{Q}}$ agrees with $\mathbb{Q}$ up to time σ. More formally stated, for stopping times $\nu \le \sigma \le \tau < \infty$, m-stability implies the equality of the sets

$$\left\{\left(\frac{Z_\tau}{Z_\sigma}, \frac{Z_\sigma}{Z_\nu}\right) : Z \text{ defines a measure in } \mathcal{S}^e\right\} = \left\{\left(\frac{Z'_\tau}{Z'_\sigma}, \frac{Z_\sigma}{Z_\nu}\right) : Z, Z' \text{ define measures in } \mathcal{S}^e\right\}.$$

Thus,

$$\operatorname{ess\,sup}_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[-X|\mathcal{F}_\tau]|\mathcal{F}_\sigma] = \operatorname{ess\,sup}_{\mathbb{Q}} \operatorname{ess\,sup}_{\widetilde{\mathbb{Q}}} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\widetilde{\mathbb{Q}}}[-X|\mathcal{F}_\tau]|\mathcal{F}_\sigma].$$

The second essential supremum optimises over measures $\widetilde{\mathbb{Q}}$ for events after σ, hence we may write

$$\begin{aligned}
\rho_\sigma(X) &= \operatorname{ess\,sup}_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\operatorname{ess\,sup}_{\widetilde{\mathbb{Q}}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[-X|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \\
&= \operatorname{ess\,sup}_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\rho_\tau(X)|\mathcal{F}_\sigma] \\
&= \rho_\sigma(\rho_\tau(X)).
\end{aligned}$$

(2) $\implies$ (3) is clear.

(3) $\implies$ (1): Suppose that Z^1 and Z^2 are change of measure martingales defining elements $\mathbb{Q}^1$ and $\mathbb{Q}^2$ in $\mathcal{S}$. Let σ be a stopping time, and suppose that $Z^1_\sigma \frac{Z^2_\infty}{Z^2_\sigma}$ is not in the set $\mathcal{S}$. The set $\mathcal{S}$ is closed and convex, so by the Hahn-Banach theorem, there is a random variable $X \in L^\infty$ such that

$$\mathbb{E}_{\mathbb{Q}^1}[\mathbb{E}_{\mathbb{Q}^2}[-X|\mathcal{F}_\sigma]] \equiv \mathbb{E}_{\mathbb{P}}\left[Z^1_\sigma \frac{Z^2_\infty}{Z^2_\sigma}(-X)\right] > \sup_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}[-X].$$

By definition of ρ_σ and ρ_0,

$$\mathbb{E}_{\mathbb{Q}^1}[\mathbb{E}_{\mathbb{Q}^2}[-X|\mathcal{F}_\sigma]] \le \mathbb{E}_{\mathbb{Q}^1}[\rho_\sigma(X)] \le \rho_0(\rho_\sigma(X)).$$

But by definition, $\rho_0(\rho_\sigma(X)) \equiv \sup_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}[-X]$, which shows the required contradiction.

(2) $\implies$ (4): Suppose that for bounded random variables X and Y and stopping times $\sigma \le \tau$, we have $\rho_\tau(X) \le \rho_\tau(Y)$. By hypothesis,

$$\rho_\sigma(X) = \rho_\sigma(-\rho_\tau(X)) \quad \text{and} \quad \rho_\sigma(Y) = \rho_\sigma(-\rho_\tau(Y)).$$

Since $-\rho_\tau(X) \ge -\rho_\tau(Y)$, we use monotonicity of ρ_σ to deduce $\rho_\sigma(X) \ge \rho_\sigma(Y)$.

(4) $\implies$ (2): Set $Y = -\rho_\tau(X)$, and observe that we have the equality $\rho_\tau(Y) =$

$\rho_\tau(-\rho_\tau(X)) = \rho_\tau(X)$, which we view as the system of inequalities

$$\rho_\tau(Y) \geq \rho_\tau(X) \qquad \text{and} \qquad \rho_\tau(X) \geq \rho_\tau(Y).$$

By hypothesis (4) applied to both of the above inequalities, we have $\rho_\sigma(X) = \rho_\sigma(Y) = \rho_\sigma(-\rho_\tau(X))$.

(1) $\iff$ (5): we provide this equivalent condition without proof as a sample of related results for convex risk functionals, which study the dynamics of the penalty function of time-consistent convex risk measures. The interested reader may refer to [Delbaen, 2006b], Theorem 12 for a proof of the statement as it appears here, and for a broader perspective, may refer to [Föllmer and Penner, 2006, Acciaio and Penner, 2011, Delbaen et al., 2010]. □

1.4 Transaction costs

In any market for an asset that may be both bought and sold, in general, there is no one true price of the asset; rather one for each of buying, and selling. The bid price ("bid") is the maximum that a market participant will offer to pay for the asset, and the ask price ("ask") is the minimum that a holder of the asset will accept to part with the asset.

In any sufficiently liquid market, for a particular asset, various types of orders come in and get matched. If the market is functioning correctly, and both prices exist, then the bid price will be below the ask. The more liquid the asset, the tighter the difference between bid and ask (the bid-ask spread). The less liquid an asset becomes, the wider the bid-ask spread. This widening represents an increase in risk associated to this asset: the market is, on the whole, less certain on the price, and less willing to buy and buy at bids close to asks and sell at asks close to bids. Market depth is another factor affecting the bid-ask spread, which is linked with liquidity.

Assuming that the market is sufficiently deep (relative to trade sizes) and liquid, a bid price will not move upon a small order to sell at that price, so we may focus modelling of such a market to just bid and ask prices, simplifying the information contained in the order book to just a pair of values at any particular time.

For a multi-asset market, where each asset satisfies the assumptions above, there is a pair of values for each asset. In such a multi-asset market, suppose that any asset in the market may be bought and sold in units of any other asset: for example a currency market where dollars may be exchanged for an amount of either pounds sterling, or euros[2]. In this case, the bids and asks at time t may be arranged in a matrix, denoted $\Pi_t = (\pi_t^{ij})_{i,j\in 1,2,\ldots,d}$, where π_t^{ij} is the amount of asset i that is required for 1 unit of asset j at time t. Of course, $1/\pi_t^{ji}$ is the amount that 1 of asset j is worth in units of i, so that the bid-ask spread of asset j in terms of i is

$$\left[\frac{1}{\pi_t^{ji}}, \pi_t^{ij}\right].$$

The exchanging of one unit of asset i to j via a third asset k should not result in more than exchanging from i to j directly: we may not create money from essentially no risk: we assume that there is no arbitrage. Mathematically,

$$\pi_t^{ik}\pi_t^{kj} \geq \pi_t^{ij}.$$

From this, and the fact that we assume the bid-ask prices in the matrix are positive real numbers, we have that $\pi_t^{ii} = 1$. From this, we have $\pi_t^{ij} \geq \frac{1}{\pi_t^{ji}}$.

Proportional transaction costs In many examples, where there is little uncertainty on the value of the asset being traded, the bid-ask spread will be of small order relative to either the bid or the ask, and the proportion

$$\frac{ask - bid}{bid} \approx \lambda$$

is approximately constant through time. This is a useful further simplification of the modelling of bid and ask prices, as we only need model a single price process, and then multiply by $(1 + \lambda)$.

1.4.1 The Fundamental Theorem of Asset Pricing under transaction costs in discrete time

An investor trades in the market under bid-ask spreads Π. Suppose that the investor starts with a zero initial endowment, denote the set of all claims attainable by terminal time T to be A_T.

Arbitrage considerations form a cornerstone of modern Financial Mathematics. A probabilistic consideration of arbitrage goes back to (at least) Ramsey's and de Finetti's Dutch Book Theorem in the 1930s.

An arbitrage is a way of making a riskless profit. Say that we are trading in an asset S, with no transaction costs. From a zero initial endowment, an arbitrage

[2] the case where, for example, dollars may be exchanged for a basket comprising a mixture of pounds and euros is naturally more general and shall be discussed later

is a self-financing investment strategy that replicates a claim that is positive, and not identically zero. On a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let the convex cone of all claims attainable from zero initial wealth be $\mathscr{A}$. Then $\mathscr{A}$ is *arbitrage-free* if

$$\mathscr{A} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\},$$

where $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ is the positive orthant of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. The Fundamental Theorem of Asset Pricing (FTAP) is that absence of arbitrage is equivalent to the existence of an equivalent martingale measure. An equivalent martingale measure is a measure under which the price process S is a martingale.

For more general probability spaces, we need stronger notions of no arbitrage. For example, the Kreps-Yan theorem (Theorem 5.2.2 of [Delbaen and Schachermayer, 2006]) shows that existence of an equivalent local martingale measure is equivalent to the *no free lunch* condition,

$$\overline{\mathscr{A}} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\},$$

where the closure is taken in the weak*-topology.

In the case where transaction costs are present, the FTAP fails. Assume that multiple assets are traded according to a bid-ask matrix Π, that satisfies, for each t,

- $\pi_t^{ij} > 0$ for all i, j;
- $\pi_t^{ii} = 1$; and
- $\pi_t^{ik}\pi_t^{kj} \geq \pi_t^{ij}$.

Indeed, in section 3 of [Schachermayer, 2004] there is an example of a set of claims $\mathscr{A} \subset L^0$ attainable from 0 endowment which satisfies no-arbitrage, but whose closure $\overline{\mathscr{A}}$ in L^0 has an arbitrage. In the same work, it is shown that if $\mathscr{A}$ satisfies a stronger assumption of *robust no-arbitrage*, then $\mathscr{A}$ is closed in L^0, and there exists a strictly consistent price process. The converse is also shown: that if there is a strictly consistent price process, then robust no-arbitrage holds.

A bid-ask process (Π_t) satisfies the robust no-arbitrage condition if there is a bid-ask process $(\widetilde{\Pi}_t)$ satisfying no-arbitrage, and such that each bid-ask interval $\left[\frac{1}{\widetilde{\pi}_t^{ji}}, \widetilde{\pi}_t^{ij}\right]$ is contained in the relative interior of $\left[\frac{1}{\pi_t^{ji}}, \pi_t^{ij}\right]$.

The solvency cone $K(\Pi_t)$ of a bid-ask matrix Π_t is the cone of all claims that may be liquidated to a non-negative value according to the prices Π_t. It is given by taking the cone of all positive units in each of the assets e_i, together with the exchanges $\pi_t^{ij}e_i - e_j$. A consistent price process (Z_t) is a process that, at each t, lives in the polar cone of the solvency cone $K(\Pi_t)^*$. A strictly consistent price process lies in the relative interior of $K(\Pi_t)^*$.

The theorem from [Schachermayer, 2004] is

Theorem 1.4.1 (Schachermayer)**.** *A bid-ask process satisfies the robust no-arbitrage condition if and only if it admits a strictly consistent pricing process.*

We may characterise the closure of the cone $\mathscr{A}$ in L^0 as follows, from [Jacka et al., 2008]:

Theorem 1.4.2 (Jacka, Berkaoui, Warren)**.** *There exists an adjusted bid-ask process $\widetilde{\Pi}$ such that the associated cone of claims $\widetilde{\mathscr{A}}$ satisfies $\mathscr{A} \subseteq \widetilde{\mathscr{A}} \subseteq \overline{\mathscr{A}}$. Moreover, either $\widetilde{\mathscr{A}}$ contains an arbitrage, or it is arbitrage-free and closed.*

Chapter 2

Predictable representation

2.1 Introduction

Insurers reserve for future financial risks by investing in a suitably prudent asset. Reserving is done in a particular unit of account, typically cash, or any other asset universally agreed to always hold positive value. We call such assets numéraires, examples of which include paper assets, such as currencies, or physical commodities. Reserving a sufficient amount ensures that the risk carried by the insurer is acceptable. In some circumstances, the choice of numéraire is clear; in others, it is not, for example insurers reserving for claims in multiple currencies. We model the sufficient amount to reserve by a coherent measure of risk.

Coherent risk measures were first introduced by Artzner, Delbaen, Eber and Heath [Artzner et al., 1997, Artzner et al., 1999], in order to give a broad axiomatic definition for monetary measures of risk. Financial positions are modelled as essentially bounded random variables on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A coherent risk measure is a real-valued functional on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ defined in definition 1.3.5. A coherent risk measure assigns a real value to every financial position: those with non-positive risk are deemed acceptable. We denote by $\mathcal{A}$ the set of acceptable claims. It is easily shown that $\mathcal{A}$ is a cone in L^∞.

A coherent risk measure is a reserving mechanism: we assume that an insurer is making a market in (or at least reserving for) risk according to a coherent risk measure ρ and they charge or reserve for a random claim X the price $\rho(X)$. Thus the aggregate position of holding the risky claim X and reserving adequately should always be acceptable to the insurer.

A risk measure ρ satisfies the Fatou property if, for any X^n converging to X

in probability,
$$\liminf_n \rho(X^n) \geq \rho(X).$$
A risk measure satisfies the Fatou property if and only if, for a set of probability measures $\mathcal{Q}$ absolutely continuous with respect to $\mathbb{P}$, we can represent ρ as

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X],$$

as shown in Theorem 1.3.11.

Fix a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that the dual of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all finitely additive measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$. The Fatou property allows us to restrict our search for dual optimisers to elements in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, identified with probability measures through their Radon-Nikodym derivative. We say that a probability measure $\mathbb{Q} \ll \mathbb{P}$ is *identified* to a random variable $Z \in L^1$ if $d\mathbb{Q}/d\mathbb{P} = Z$ holds $\mathbb{P}$-almost surely. We equip the space L^∞ with the weak* topology $\sigma(L^\infty, L^1)$, so the topological dual is L^1. The acceptance set $\mathcal{A}$ is weak*-closed.

We assume that the insurer can trade at finitely many times $\{0, 1, \ldots, T\}$. At each time t, the insurer can re-evaluate the risk, conditional on the information in the sigma algebra $\mathcal{F}_t$. A conditional coherent risk measure is the natural generalisation of a coherent risk measure; again, such a measure ρ_t satisfies the Fatou property if and only if, for a set $\mathcal{Q}_t$ of $\mathbb{P}$-absolutely continuous probability measures we may represent ρ_t by
$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} \mathbb{E}_Q[X|\mathcal{F}_t].$$
In what follows, we fix $\mathcal{Q}_t = \mathcal{Q}$ for all t, and define $\mathcal{A}_t$ as the set of all claims $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $\rho_t(X) \leq 0$. Of course, $\mathcal{A}_t$ is a cone.

The simplest act of reserving is to hold a set amount of cash $\rho(X)$ until the insurer must pay the claim X. More generally, starting with an amount $\rho(X)$ of cash, an insurer trades in any financial asset available, constructing a self-financing strategy with a terminal value equal to or exceeding the value of the claim X at maturity. If this strategy is built by trading in the set of assets $\mathbf{V} = (v^0, \ldots, v^d)$ as numéraires, then we shall say that the claim may be *represented* by the vector $\mathbf{V}$.

The components of $\mathbf{V}$ are $\mathcal{F}_T$-measurable. We need not liquidate the portfolio at any time but we shall adopt the view in the subsequent that we have identified a particular claim X for which we wish to reserve, in units of an identified numéraire, which we call the reference numéraire. For simplicity, we assume that this numéraire is the zero-th component v^0 of $\mathbf{V}$, and write $v^0_t \equiv 1$ for any time t. Thus, when we cash up at time T, the reserving portfolio $Y = (Y^0, \dots, Y^d)$ is liquidated frictionlessly, we obtain a value $Y \cdot \mathbf{V}$ in the reference numéraire, and used to cover the claim X. The net position will be $X - Y \cdot \mathbf{V}$ at time T.

If we allow ourselves a large enough collection of assets in the definition of $\mathbf{V}$, then representation is always possible: to hedge the bounded claim X we need only buy and hold a claim whose value is X. Interest, therefore, should be focused on choosing a parsimonious collection of representing numéraires $\mathbf{V}$, and in identifying when such a collection is representing.

Predictable representability For X to be *predictably representable*, we mean that X is attainable (representable) as a weak*-limit of nets $X = \lim_\alpha X^\alpha$ of sums of claims $X^\alpha = \sum_t C_t^\alpha$, where each C_t^α is realised over the time period $(t, t+1]$, pays out at time $t+1$, and is acceptable at time t.

A claim X is predictably representable in $\mathbf{V}$ if, starting from a reserve $\rho(X)$, we may transfer risk through each time period by trading in $\mathbf{V}$ in an acceptable manner, such that the terminal wealth equals the value of the claim: for portfolios $Y_t \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{d+1})$, we have

$$X = \rho_0(X) + \sum_{t=0}^{T-1} (Y_{t+1} - Y_t) \cdot \mathbf{V},$$

where each increment satisfies $\rho_t((Y_{t+1} - Y_t) \cdot \mathbf{V}) \leq 0$.

A reason for why we term this mechanism *predictable* is by analogy to the predictable representation result for martingales: a martingale M has the predictable representation property if, for any martingale X there exists a predictable process H such that X is the Ito integral of H with respect to M. A secondary reason for why we term this mechanism *predictable* is that when a claim is predictably representable, it must be the case that at time t, given a portfolio Y_t that reserves for a claim X, there will exist a portfolio Y_{t+1} at time $t+1$ such that the exchange of Y_t for Y_{t+1} is acceptable at time t. Thus the risk over $(t, t+1]$ is seen to be hedged before the time period, at time t.

We write $\mathcal{A}_t(\mathbf{V})$ for the set of all portfolios in $\mathbf{V}$ that are time-t acceptable. The acceptance set $\mathcal{A}_0$ is predictably $\mathbf{V}$-representable if it is the weak*-closure of the sum of the cones $K_t(\mathcal{A}, \mathbf{V}) := \mathcal{A}_t(\mathbf{V}) \cap L^\infty(\Omega, \mathcal{F}_{t+1}, \mathbb{P}; \mathbb{R}^{d+1})$,

$$\mathcal{A}_0(\mathbf{V}) = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V})}.$$

A key contribution of this chapter is to provide the dual characterisation of

$\mathbf{V}$-representability. Recall Delbaen's multiplicative stability (henceforth m-stability) condition, on the set of probability measures $\mathcal{Q}$. We identify probability measures in $\mathcal{Q}$ via Radon-Nikodym derivative with random variables in the dual cone

$$\mathcal{A}_0^* = \{Z \in L^1 : \mathbb{E}[ZX] \leq 0 \quad \forall X \in \mathcal{A}_0\}.$$

The dual cone $\mathcal{A}_0^*$ is m-stable if, for any stopping time τ and $Z_1, Z_2 \in \mathcal{A}_0^*$ such that $\mathbb{E}[Z_1 \mid \mathcal{F}_\tau] = \alpha\mathbb{E}[Z_2 \mid \mathcal{F}_\tau]$, then $\alpha Z_2 \in \mathcal{A}_0(\mathbf{V})^*$. See [Delbaen, 2006a]. Likewise, the dual cone $\mathcal{A}_0(\mathbf{V})^*$ is $\mathbf{V}$-m-stable if, for any stopping time τ and $Z_1, Z_2 \in \mathcal{A}_0(\mathbf{V})^*$ such that $\mathbb{E}[Z_1 \mid \mathcal{F}_\tau] = \alpha\mathbb{E}[Z_2 \mid \mathcal{F}_\tau]$, then $\alpha Z_2 \in \mathcal{A}_0(\mathbf{V})^*$. To show the equivalence of $\mathbf{V}$-m-stability and $\mathbf{V}$-representability, we present an elegant dual of each summand in the representation $K_t(\mathcal{A}, \mathbf{V})^* = \mathcal{M}_t(\mathcal{A}(\mathbf{V})^*)$, called the *predictable pre-image* of $\mathcal{A}_0(\mathbf{V})^*$ at time t. Aside from being useful in proving the equivalence of $\mathbf{V}$-predictable representability and predictable $\mathbf{V}$-m-stability, the predictable pre-image of a predictably m-stable convex cone $\mathcal{A}_0(\mathbf{V})^*$ at time t is a concrete description of the dual of the set of portfolios held at time t in order to maintain an acceptable position until time $t+1$.

A risk measure is time-consistent if $\rho_t = \rho_t \circ \rho_{t+1}$. That is, today's reserve for a claim X is precisely enough to reserve for tomorrow's reserve for X; see [Gianin, 2006, Delbaen, 2006a, Riedel, 2004, Roorda et al., 2005] for examples of such measures. The sequence (ρ_t) is not necessarily time-consistent; see for example [Boda and Filar, 2006, Cheridito and Stadje, 2009].

We prove that $\mathbf{V}$-representability is equivalent to time-consistency of the risk measure. A risk measure is time-consistent if $\rho_t = \rho_t \circ \rho_{t+1}$. That is, today's reserve for a claim X is precisely enough to reserve for tomorrow's reserve for X; see [Gianin, 2006, Delbaen, 2006a, Riedel, 2004, Roorda et al., 2005] for examples of such measures. The sequence (ρ_t) is not necessarily time-consistent; see for example [Boda and Filar, 2006, Cheridito and Stadje, 2009]. Considerations of time-consistency are important for banks modelling Risk-Weighted Assets (RWAs) under the Basel III accords. A recent consultative document [on Banking Supervision, 2013] highlights the change in methodology from using risk measures based on Value at Risk (VaR) to those based on Expected Shortfall (ES), also known as Average Value at Risk (AVaR, see [Embrechts et al., 2014]). As shown by Cheridito and Stadje [Cheridito and Stadje, 2009], AVaR is not time-consistent.

In section 2, we elaborate on our generalisations of the three properties: namely $\mathbf{V}$-time-consistency, $\mathbf{V}$-representability, and $\mathbf{V}$-m-stability. Throughout the section we illustrate our definitions with a toy example of Average Value at Risk.

The main result of this chapter is the equivalence of the three properties.

In section 3, we provide some examples. In section 4, we prove that the L^0 closure of the acceptance set is the sum of the L^0-closures of the cones $K_t(\mathcal{A}, \mathbf{V})$. In section 5, we prove the main result. A key step in the equivalence of $\mathbf{V}$-representability and $\mathbf{V}$-m-stability is the following result for $\mathcal{C}_t$ a sequence of closed convex cones:

$$(\cap_t \mathcal{C}_t)^* = \overline{\text{conv}} \{\cup_t \mathcal{C}_t^*\},$$

where $\overline{\text{conv}} A$ denotes the closure of the convex hull of a set A. We highlight the role that the filtration $(\mathcal{F}_t)_t$ plays.

2.2 Pricing measures

We recall some definitions and concepts. We fix a terminal time $T \in \mathbb{N}$, a discrete time set $\mathbb{T} := \{0, 1, \dots, T\}$. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the *reference measure* or *objective measure*. The filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ describes the information available at each time point. The space of all $\mathbb{P}$-essentially bounded $\mathcal{F}$-measurable random variables is $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$; we abbreviate $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ to L_t^∞. The space of essentially bounded $\mathbb{R}^d$-valued random variables is $\mathcal{L}^\infty(\mathbb{R}^d) = L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. We denote the cone of non-negative (respectively strictly positive) essentially bounded random variables by L_+^∞ (resp. L_{++}^∞). At each time $t \in \mathbb{T}$, we wish to price monetary risks using all information available at that time. Recall the following definition, adapted from [Detlefsen and Scandolo, 2005]:

Definition 2.2.1. A map $\rho_t : L^\infty \to L_t^\infty$ for $t \in \mathbb{T}$ is a *conditional convex risk measure* if, for all $X, Y \in L^\infty$, it has the following properties:

- Conditional cash invariance: for all $m \in L_t^\infty$,

$$\rho_t(X + m) = \rho_t(X) + m \qquad \mathbb{P}\text{-almost surely;}$$

- Monotonicity: if $X \leq Y$ $\mathbb{P}$-almost surely, then $\rho_t(X) \leq \rho_t(Y)$;
- Conditional convexity: for all $\lambda \in L_t^\infty$ with $0 \leq \lambda \leq 1$,

$$\rho_t(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_t(X) + (1 - \lambda)\rho_t(Y) \qquad \mathbb{P}\text{-almost surely;}$$

- Normalisation: $\rho_t(0) = 0$ $\mathbb{P}$-almost surely.

Furthermore, a conditional convex risk measure is called *coherent* if it also satisfies

- Conditional positive homogeneity: for all $\lambda \in L_t^\infty$ with $\lambda \geq 0$,

$$\rho_t(\lambda X) = \lambda \rho_t(X) \qquad \mathbb{P}\text{-almost surely.}$$

Our interest lies chiefly in reserving for and pricing liabilities. We see a positive random variable X as a gain, and a negative X as a loss, which explains the choice of sign in the cash invariance property, and the direction of monotonicity.

Definition 2.2.2. A convex risk measure satisfies the *Fatou property* if, for any bounded sequence $(X^n)_{n\geq 1} \subset L^\infty$ converging to $X \in L^\infty$ in probability, we have

$$\rho_t(X) \leq \liminf_{n\to\infty} \rho_t(X^n).$$

The Fatou property is equivalent to *continuity from above*: ρ_t is continuous from above if, whenever $(X^n)_{n\geq 1} \subset L^\infty$ is a non-increasing sequence such that $X_s^n \downarrow X_s$ $\mathbb{P}$-a.s. for all $s \in \mathbb{T}_t$, then

$$\rho_t(X^n) \downarrow \rho_t(X) \qquad \mathbb{P}\text{-a.s. as } n \to \infty$$

Definition 2.2.3. A *dynamic convex pricing measure* is a collection $\rho = (\rho_t)_{t=0,\dots,T}$, where each ρ_t is a conditional convex pricing measure satisfying the Fatou property with representing set of measures $\mathcal{Q}$:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_t].$$

The *acceptance set* of a conditional convex pricing measure $\rho_t : L^\infty \to L_t^\infty$ is

$$\mathcal{A}_t = \{X \in L^\infty : \rho_t(X) \leq 0\}.$$

For the following results, we refer the reader to Chapter 1. We equip the space L^∞ with the weak*-topology $\sigma(L^\infty, L^1)$, so that the topological dual will be L^1. Recall that a set $\mathcal{C}$ of claims is arbitrage-free whenever

$$\mathcal{C} \cap L_+^\infty = \{0\}.$$

Proposition 2.2.4. *Define $(\mathcal{A}_t)_t$ to be the acceptance sets of the dynamic conditional* coherent *pricing measure $\rho_t : L^\infty \to L_t^\infty$ satisfying the Fatou property. Then*

$\mathcal{A}_t$ is a weak-closed*[1] *convex cone that is stable under multiplication by bounded positive $\mathcal{F}_t$-measurable random variables, contains L^∞_-, and is arbitrage-free.*

Numéraires A *numéraire* is a random variable $v \in L^\infty_{++}$ such that $1/v \in L^\infty_{++}$. We shall from here on fix a finite collection of numéraires $\mathbf{V} = (v^0, \ldots, v^d)$, with $v^0 \equiv 1$.

2.2.1 Time-consistency

In this and the subsequent sections we identify the probability measures $\mathbb{Q}$ of the set $\mathcal{Q}$ with their Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$. We trust that which version is to be used will be clear from the context. The following definition is taken from Acciaio et al. [Acciaio et al., 2012].

Definition 2.2.5. A dynamic convex pricing measure for random variables $(\rho_t)_{t\in\mathbb{T}}$ is *(strongly) time-consistent* if for all $t \leq T-1$, and for all $X \in L^\infty$,

$$\rho_t(X) = \rho_t(\rho_{t+1}(X)).$$

We note that the reserve for X at time t is $\rho_t(X)$. The generalisation of strong time-consistency to $\mathbf{V}$-time-consistency is:

Definition 2.2.6. A dynamic convex risk measure $\rho = (\rho_t)_{t=0,\ldots,T}$ is *predictably* $\mathbf{V}$-*time-consistent* if, for any $X \in \mathcal{A}_0$, we may find a net X^α converging in the weak* sense to X, and a net $\pi^\alpha = (\pi^\alpha_t)_{t=0,\ldots,T-1}$ such that $\pi^\alpha_t \in L^\infty(\mathcal{F}_{t+1}; \mathbb{R}^{d+1})$, and

(i) for each t,
$$\rho_t(\pi^\alpha_t \cdot \mathbf{V}) \leq 0;$$

(ii) for each α,
$$\sum_{t=0}^{T-1} \pi^\alpha_t \cdot \mathbf{V} \geq X^\alpha \quad Q\text{-almost-surely for each } Q \in \mathcal{Q}.$$

When $\mathbf{V} \equiv 1$, strong time-consistency implies $\mathbf{V}$-time-consistency. Of course, for any $X \in L^\infty$, we have $X - \rho_0(X) \in \mathcal{A}_0$. Assuming strong time-consistency, take $\pi_t = \rho_{t+1}(X) - \rho_t(X)$, so that $\rho_t(\pi_t) = 0$ and

$$X - \rho_0(X) = \sum_{t=0}^{T-1} \pi_t.$$

[1] in L^∞, i.e., $\mathcal{A}_t$ is closed in the topology $\sigma(L^\infty_t, L^1_t)$

We illustrate predictable time-consistency in a finite sample space Ω with a sign-changed version of Average Value at Risk.

Example 2.2.7 (Average Value at Risk)**.** Consider the filtered probability space $\Omega = \{1, 2, 3, 4\}$ with $\mathcal{F}_0$ trivial, $\mathcal{F}_1 = \sigma(\{1, 2\}, \{3, 4\})$, $\mathcal{F}_2 = 2^\Omega = \mathcal{F}$ (describing a binary branching tree on two time steps). Define AVaR, the Average Value at Risk pricing measure, by

$$\mathrm{AVaR}(X) := \frac{1}{\lambda} \int_0^\lambda -\mathrm{VaR}_\alpha(X)\, d\alpha,$$

We may represent AVaR as

$$\mathrm{AVaR}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_\lambda} \mathbb{E}_\mathbb{Q}[X],$$

where

$$\mathcal{Q}_\lambda = \left\{ \text{probability measures } \mathbb{Q} \ll \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\lambda} \right\}$$

noting the sign change to make AVaR a pricing measure; see section 4.4 of [Föllmer and Schied, 2011]. We set $\lambda = \frac{1}{50}$, while the objective measure is given by

$$\mathbb{P}[\{1\}] = \frac{1}{100}, \qquad \mathbb{P}[\{2\}] = \mathbb{P}[\{3\}] = \frac{9}{100}, \qquad \text{and} \qquad \mathbb{P}[\{4\}] = \frac{81}{100}.$$

For notational convenience, we represent a probability measure $\mathbb{Q}$ by the a quartuple of its values on atoms, $\mathbb{Q}(\{i\}) =: q_i$, and similarly we write $X(i) = x_i$ for a random variable $X : \Omega \to \mathbb{R}$. It is easy to see that the representing set $\mathcal{Q}_\lambda$ is

$$\mathcal{Q}_\lambda = \{\mathbb{Q} = (q_1, q_2, q_3, q_4) : \sum_{i=1}^4 q_i = 1, \quad 0 \leq q_1 \leq \frac{1}{2}, q_i \in [0, 1] \text{ for } i = 2, 3, 4\}.$$

$\mathcal{Q}_\lambda$ is the convex hull of 6 points:

$$\begin{aligned} \mathcal{Q}_\lambda = \mathrm{conv}\{&(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0), \qquad (\tfrac{1}{2}, 0, \tfrac{1}{2}, 0), \qquad (\tfrac{1}{2}, 0, 0, \tfrac{1}{2}) \\ &(0, 1, 0, 0), \qquad (0, 0, 1, 0), \qquad (0, 0, 0, 1)\} \end{aligned}$$

The set of time-0 acceptable claims is

$$\mathcal{A}_0 = \{X = (x_1, x_2, x_3, x_4) : \sum_{i=1}^4 q_i x_i \leq 0 \quad \text{for } \mathbb{Q} \in \mathcal{Q}_\lambda\}.$$

Clearly, $X \in \mathcal{A}_0$ if and only if $\sum_{i=1}^4 q_i x_i \leq 0$ for each of the six extreme points $\mathbb{Q}$ of

$\mathcal{Q}_\lambda$. These six inequalities are neatly summarised as

$$\mathcal{A}_0 = \{X = (x_1, x_2, x_3, x_4) : x_i \leq 0 \text{ for } i = 1, 2, 3, 4; \\ \text{or} \quad x_1 \geq 0 \text{ and } x_i \leq -x_1 \text{ for } i = 2, 3, 4\}.$$

Define $X^0 := \mathbb{1}_{\{1\}} - \mathbb{1}_{\{2,3,4\}}$. Then it is clear that

$$\mathcal{A}_0 = \{\alpha X^0 - \beta : \alpha \geq 0,\ \beta \in L^\infty_+\}.$$

The time-1 acceptance set is

$$\mathcal{A}_1 = \{X = (x_1, x_2, x_3, x_4) : q_1 x_1 + q_2 x_2 \leq 0 \quad \text{and} \quad q_3 x_3 + q_4 x_4 \leq 0 \quad \text{for } \mathbb{Q} \in \mathcal{Q}_\lambda\} \\ = L^\infty_-.$$

Claim $(\mathrm{AVaR}_0, \mathrm{AVaR}_1)$ is not time-consistent.

Proof. It is easy to check that we have $\mathrm{AVaR}_0(X^0) = 0$, $\mathrm{AVaR}_1(X^0) = \mathbb{1}_{\{1,2\}} - \mathbb{1}_{\{3,4\}}$, and thus

$$\mathrm{AVaR}_0(\mathrm{AVaR}_1(X^0)) = \mathrm{AVaR}_0(\mathbb{1}_{\{1,2\}} - \mathbb{1}_{\{3,4\}}) = 1 > 0 = \mathrm{AVaR}_0(X^0).$$

□

Now we set $\mathbf{V} = (v^0, v^1)$, where $v^0 \equiv 1$ by convention, and $v^1 = X^0 + 2$, so that

$$v^1 = 3\mathbb{1}_{\{1\}} + \mathbb{1}_{\{2,3,4\}} > 0.$$

Claim $(\mathrm{AVaR}_0, \mathrm{AVaR}_1)$ is predictably $\mathbf{V}$-time-consistent.

Proof. For any acceptable risk $X \in \mathcal{A}_0$ we may set $X = \alpha X^0 - \beta$, where β is some non-negative random variable taking the value 0 on the event $\{1\}$. We reserve for X by holding α in v^1 and -2α in cash v^0, giving a mapping Y_0 from acceptable risks X to initial reserving portfolios in $\mathbf{V}$:

$$Y_0 = \begin{pmatrix} -2\alpha \\ \alpha \end{pmatrix}. \tag{2.1}$$

Clearly $Y_0 \cdot \mathbf{V} = \alpha X^0$.

Set

$$Y_1 = \begin{pmatrix} -(2\alpha + \frac{3}{2}\beta(2))\mathbb{1}_{\{1,2\}} - (\alpha + \beta(3) \wedge \beta(4))\mathbb{1}_{\{3,4\}} \\ (\alpha + \frac{1}{2}\beta(2))\mathbb{1}_{\{1,2\}} \end{pmatrix}, \tag{2.2}$$

so that

$$Y_1 \cdot \mathbf{V} = \alpha X^0 - \beta(2)\mathbb{1}_{\{2\}} - \beta(3) \wedge \beta(4)\mathbb{1}_{\{3,4\}}.$$

Now, let $\pi_0 = Y_0$ and $\pi_1 = Y_1 - Y_0$. We have $\rho_0(\pi_0 \cdot \mathbf{V}) \leq 0$, $\rho_1(\pi_1 \cdot \mathbf{V}) \leq 0$ and $\pi_0 \cdot \mathbf{V} + \pi_1 \cdot \mathbf{V} \geq \alpha X^0 - \beta$. Thus $(\mathrm{AVaR}_0, \mathrm{AVaR}_1)$ is predictably $\mathbf{V}$-time-consistent. □

2.2.2 Predictable representability

Given any cone $\mathcal{D}$ in L^∞ and our vector $\mathbf{V}$ of numéraires, we define the collection of portfolios attaining $\mathcal{D}$ to be

$$\mathcal{D}(\mathbf{V}) = \{Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{d+1}) : Y \cdot \mathbf{V} \in \mathcal{D}\}.$$

The set of time-t acceptable portfolios that are $\mathcal{F}_{t+1}$-measurable is $K_t(\mathcal{A}, \mathbf{V}) := \mathcal{A}_t(\mathbf{V}) \cap \mathcal{L}^\infty_{t+1}(\mathbb{R}^{d+1})$.

Definition 2.2.8. The cone $\mathcal{A}(\mathbf{V})$ is *predictably decomposable* if

$$\mathcal{A}(\mathbf{V}) = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V})},$$

where the closure is taken in the weak* topology. In this case, the cone $\mathcal{A}$ is *predictably represented* by $\mathbf{V}$.

Example 2.2.7 (Continued)**.** [Average Value at Risk] We return to the setting of Example 2.2.7.

Claim The acceptance set $\mathcal{A}_0$ is not predictably represented by 1.

Proof. We note that

$$\begin{aligned} K_0(\mathcal{A}_0, 1) &= \{X \in L^\infty(\mathcal{F}_1) : X \in \mathcal{A}_0\} = L^\infty_-(\mathcal{F}_1) \\ K_1(\mathcal{A}_0, 1) &= \mathcal{A}_1 = L^\infty_- \end{aligned}$$

If $\mathcal{A}_0$ is to be predictably represented by 1, we must have that $\mathcal{A}_0 = K_0(\mathcal{A}_0, 1) + K_1(\mathcal{A}_0, 1) = L^\infty_-$; however $\mathcal{A}_0$ contains X^0 which is not in L^∞_-. □

Now set $\mathbf{V} = (1, 3\mathbb{1}_{\{1\}} + \mathbb{1}_{\{2,3,4\}})$ as before.

Claim The set $\mathcal{A}_0$ is predictably represented by $\mathbf{V}$.

Proof. For any $X \in \mathcal{A}_0$ we may write $X = \alpha X^0 - \beta$, for $\alpha \geq 0$ and $\beta \in L^\infty_+$. Defining $\pi_0 = Y_0$ and $\pi_1 = Y_1 - Y_0$ for Y_0, Y_1 as in eqs. (2.1) and (2.2), we have that $X \leq \pi_0 \cdot \mathbf{V} + \pi_1 \cdot \mathbf{V} \in K_0(\mathcal{A}_0, \mathbf{V}) \oplus K_1(\mathcal{A}_0, \mathbf{V})$. Any non-positive random variable is in any of the $K_t(\mathcal{A}_0, \mathbf{V})$ for $t = 0, 1$, so X is in the sum, proving that $\mathcal{A}_0 \subseteq K_0(\mathcal{A}_0, \mathbf{V}) \oplus K_1(\mathcal{A}_0, \mathbf{V})$. The reverse inclusion is clear. $\square$

2.2.3 Stability properties

We recall Delbaen's m-stability condition, on a standard stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$:

Definition 2.2.9 (Delbaen [Delbaen, 2006a])**.** A set of probability measures $\mathcal{S} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ is *m-stable* if for elements $\mathbb{Q}^Z \in \mathcal{S}$ and $\mathbb{P} \sim \mathbb{Q}^W \in \mathcal{S}$, with associated density martingales $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}^Z}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$ and $W_t = \mathbb{E}\left[\frac{d\mathbb{Q}^W}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$, and for each stopping time τ, the martingale L defined as

$$L_t = \begin{cases} Z_t & \text{for } t \leq \tau \\ \frac{Z_\tau}{W_\tau} W_t & \text{for } t \geq \tau \end{cases}$$

defines an element in $\mathcal{S}$.

Note that a set $\mathcal{S}$ is m-stable if, whenever τ is a stopping time, and $Z, W \in \mathcal{S}$ are such that $Z_\tau = \alpha W_\tau$, then $\alpha W \in \mathcal{S}$. Just take $\alpha = \frac{Z_\tau}{W_\tau}$, and then $L = \alpha W$ in the above definition. We now define a vector-valued generalisation of m-stability, for a subset $\mathcal{D} \subset \mathcal{L}^1_+(\mathbb{R}^{d+1})$.

Definition 2.2.10. The subset $\mathcal{D} \subset \mathcal{L}^1_+(\mathbb{R}^{d+1})$ is *predictably m-stable* if, whenever $\tau \leq T$ is a stopping time, and whenever $Z, W \in \mathcal{D}$ with

$$\mathbb{E}\left[Z \,|\, \mathcal{F}_\tau\right] = \alpha \mathbb{E}\left[W \,|\, \mathcal{F}_\tau\right],$$

then αW is also in $\mathcal{D}$.

Note that α is $\mathcal{F}_\tau$-measurable.

Definition 2.2.11. The cone $\mathcal{D} \subset L^1_+$ is said to be *predictably* $\mathbf{V}$*-m-stable* if $\mathcal{D}\mathbf{V} = \{Y\mathbf{V} : Y \in \mathcal{D}\}$ is predictably m-stable.

Remark 2.2.12. In the case $d = 0$, we have $\mathbf{V} \equiv 1$ and so the requirement that a set of Radon-Nikodym derivatives $\mathcal{D} \subset L^1_+$ is 1-m-stable is precisely the requirement that $\mathcal{D}$ is m-stable.

Every random vector Z in $\mathcal{A}_0(\mathbf{V})^*$ can be written as a multiple of $\mathbf{V}$, that is, $Z = \widetilde{Z}\mathbf{V}$ with $\widetilde{Z} \in \mathcal{A}_0^*$. See Section 2.4.1 for a proof of the following

Lemma 2.2.13. *Suppose that* $\mathbf{V}$ *is a collection of* $d+1$ *numéraires, and* $\mathcal{D}$ *is a convex cone in* L^∞*. Then*

$$\mathcal{D}(\mathbf{V})^* = \mathcal{D}^*\mathbf{V}.$$

Remark 2.2.14. In light of Lemma 2.2.13, we may check that $\mathcal{A}_0(\mathbf{V})^* \equiv \mathcal{A}_0^*\mathbf{V}$ is predictably stable in the following way. We first associate to each $Z \in \mathcal{A}_0^*$ the probability measure $\mathbb{Q}^Z$, defined through its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^Z}{d\mathbb{P}} = \frac{Z}{\mathbb{E}[Z]}.$$

We note that if $Z, W \in \mathcal{A}_0(\mathbf{V})^*$, then we may find $\widetilde{Z}, \widetilde{W} \in \mathcal{A}_0^*$ such that $Z = \widetilde{Z}\mathbf{V}$ and $W = \widetilde{W}\mathbf{V}$. The assumption that $v^0 \equiv 1$ gives the equivalence of the condition $\mathbb{E}\left[Z \,|\, \mathcal{F}_\tau\right] = m\mathbb{E}\left[W \,|\, \mathcal{F}_\tau\right]$ with the condition

$$\mathbb{E}_{\mathbb{Q}^{\widetilde{Z}}}[\mathbf{V}|\mathcal{F}_\tau] = \mathbb{E}_{\mathbb{Q}^{\widetilde{W}}}[\mathbf{V}|\mathcal{F}_\tau]. \tag{2.3}$$

The set $\mathcal{A}_0(\mathbf{V})^*$ is predictably $\mathbf{V}$-m-stable if, for any stopping time $\tau \leq T$, whenever $\widetilde{Z}, \widetilde{W} \in \mathcal{A}_0^*$ are such that (2.3) holds, then

$$\frac{\mathbb{E}\left[\widetilde{Z} \,\middle|\, \mathcal{F}_\tau\right]}{\mathbb{E}\left[\widetilde{W} \,\middle|\, \mathcal{F}_\tau\right]} W \in \mathcal{A}_0^*(\mathbf{V}).$$

Example 2.2.7 (Continued)**.** We return to the setting of Example 2.2.7.

Claim $\mathcal{A}_0^*$ is not m-stable.

Proof. Define measures $\mathbb{Q}^1 = (\frac{1}{2}, \frac{1}{2}, 0, 0) \in \mathcal{Q}_\lambda$ and $\mathbb{Q}^2 = (\frac{1}{2}, 0, \frac{1}{2}, 0) \in \mathcal{Q}_\lambda$. We form the time-1 pasting of the measures $\mathbb{Q}^1$ and $\mathbb{Q}^2$ by setting

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = \frac{\mathbb{E}\left[\frac{d\mathbb{Q}^1}{d\mathbb{P}} \,\middle|\, \mathcal{F}_1\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}^2}{d\mathbb{P}} \,\middle|\, \mathcal{F}_1\right]} \frac{d\mathbb{Q}^2}{d\mathbb{P}}$$

so that $\widetilde{\mathbb{Q}} = (1, 0, 0, 0)$. Now $\widetilde{q}_1 = 1 > \frac{1}{2}$ which shows $\widetilde{\mathbb{Q}} \notin \mathcal{Q}_\lambda$, and so $\mathcal{Q}_\lambda$ is not m-stable. $\square$

Now set $\mathbf{V} = (1, 3\mathbb{1}_{\{1\}} + \mathbb{1}_{\{2,3,4\}})$ as before.

Claim $\mathcal{A}_0^*$ is $\mathbf{V}$-m-stable.

Proof. First, consider the pasting $\widetilde{\mathbb{Q}} = \mathbb{Q} \oplus_\tau \mathbb{Q}'$ of measures $\mathbb{Q}$ and $\mathbb{Q}'$ in $\mathcal{Q}_\lambda$ at the stopping time τ:

$$\begin{aligned}\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} &= \frac{\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_\tau\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}'}{d\mathbb{P}} \,\middle|\, \mathcal{F}_\tau\right]} \frac{d\mathbb{Q}'}{d\mathbb{P}} \\ &= \frac{d\mathbb{Q}'}{d\mathbb{P}} \mathbb{1}_{\{\tau=0\}} + \frac{\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_1\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}'}{d\mathbb{P}} \,\middle|\, \mathcal{F}_1\right]} \frac{d\mathbb{Q}'}{d\mathbb{P}} \mathbb{1}_{\{\tau=1\}} + \frac{d\mathbb{Q}}{d\mathbb{P}} \mathbb{1}_{\{\tau=2\}}.\end{aligned}$$

By Remark 2.2.14, we fix $\widetilde{Z}$ and $\widetilde{Z}'$ in $\mathcal{A}_0^*$ with associated probability measures $\mathbb{Q}$ and $\mathbb{Q}'$ that additionally satisfy

$$\mathbb{E}_{\mathbb{Q}}[v^1 | \mathcal{F}_\tau] = \mathbb{E}_{\mathbb{Q}'}[v^1 | \mathcal{F}_\tau],$$

and we aim to show that $\widetilde{\mathbb{Q}} \in \mathcal{Q}_\lambda$. On the event $\{\tau = 0\}$ (respectively $\{\tau = 2\}$), we have that $\widetilde{\mathbb{Q}} = \mathbb{Q}'$ (resp. $\widetilde{\mathbb{Q}} = \mathbb{Q}$) and the bound $\widetilde{\mathbb{Q}}(1) \leq \frac{1}{2}$ is trivially satisfied. The event $\{\tau = 1\}$ is one of $\varnothing$, $\{1,2\}$, $\{3,4\}$, Ω. Writing $\mathbb{Q} = (q_i)_{i=1}^4$, for $\omega \in \{1,2,3,4\}$,

$$\mathbb{E}_{\mathbb{Q}}[v^1 | \mathcal{F}_1](\omega) = \frac{3q_1 + q_2}{q_1 + q_2} \mathbb{1}_{\{q_1+q_2>0\}} \mathbb{1}_{\{1,2\}}(\omega) + \mathbb{1}_{\{q_3+q_4>0\}} \mathbb{1}_{\{3,4\}}(\omega)$$

We may paste measures $\mathbb{Q}$ and $\mathbb{Q}'$ that satisfy

$$\frac{3q_1 + q_2}{q_1 + q_2} \mathbb{1}_{\{q_1+q_2>0\}} \mathbb{1}_{\{1,2\}} + \mathbb{1}_{\{q_3+q_4>0\}} \mathbb{1}_{\{3,4\}} = \frac{3q_1' + q_2'}{q_1' + q_2'} \mathbb{1}_{\{q_1'+q_2'>0\}} \mathbb{1}_{\{1,2\}} + \mathbb{1}_{\{q_3'+q_4'>0\}} \mathbb{1}_{\{3,4\}}$$

on $\{\tau = 1\}$, which simplifies to the requirement that

$$\begin{aligned}&\frac{q_1}{q_2} \mathbb{1}_{\{q_1+q_2>0\}} \mathbb{1}_{\{1,2\}\cap\{\tau=1\}} + \mathbb{1}_{\{q_3+q_4>0\}} \mathbb{1}_{\{3,4\}\cap\{\tau=1\}} \\ &\qquad = \frac{q_1'}{q_2'} \mathbb{1}_{\{q_1'+q_2'>0\}} \mathbb{1}_{\{1,2\}\cap\{\tau=1\}} + \mathbb{1}_{\{q_3'+q_4'>0\}} \mathbb{1}_{\{3,4\}\cap\{\tau=1\}}.\end{aligned} \tag{2.4}$$

On $\{\tau = 1\} \supset \{1\}$, the pasting $\widetilde{\mathbb{Q}}$ weights $\{1\}$ as

$$\mathbb{Q} \oplus_\tau \mathbb{Q}'(\{1\}) = (q_1 + q_2) \frac{q_1'}{q_1' + q_2'} \mathbb{1}_{\{q_1'+q_2'>0\}} = (q_1 + q_2) \frac{\frac{q_1'}{q_2'}}{\frac{q_1'}{q_2'} + 1} \mathbb{1}_{\{q_1'+q_2'>0\}} \overset{(2.4)}{=} q_1 \mathbb{1}_{\{q_1+q_2>0\}}$$

The other cases are easy to check. Thus $\mathbb{Q} \oplus_\tau \mathbb{Q}' \in \mathcal{A}_0^*$, and $\mathcal{A}_0^*$ is $\mathbf{V}$-m-stable. $\square$

2.2.4 Main result

We fix numéraires $\mathbf{V}$, a coherent pricing measure $\rho = (\rho_t)_t$ with convex representing set of probability measures $\mathcal{Q}$, and take $\mathcal{A}_t$ to be the acceptance set of ρ_t for $t \in \mathbb{T}$. The main result is

Theorem 2.2.15. *The following are equivalent:*

(i) $(\rho_t)_{t\in\mathbb{T}}$ *is predictably* $\mathbf{V}$*-time-consistent;*

(ii) $\mathcal{A}_0$ *is predictably represented by* $\mathbf{V}$*;*

(iii) $\mathcal{A}_0(\mathbf{V})^*$ *is predictably m-stable.*

Proof. The proof may be found in Section 2.4. $\square$

We now highlight particularly interesting waypoints in the proof of the main Theorem.

Thinking of the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_{t+1}]$ as a projection from $\mathcal{L}^1(\mathbb{R}^{d+1})$ to $\mathcal{L}^1_{t+1}(\mathbb{R}^{d+1})$, we define the *predictable pre-image* of $\mathcal{D}$ at time t by first projecting $\mathcal{D}$ to $\mathcal{L}^1_{t+1}(\mathbb{R}^{d+1})$, then taking the $\mathcal{F}_t$-cone, and finally taking the pre-image under the projection $\mathbb{E}[\cdot|\mathcal{F}_{t+1}]$. The $\mathcal{F}_t$-cone of a set E is

$$\operatorname{cone}_{\mathcal{F}_t} E = \{\alpha w_1 + \beta w_2 : \alpha, \beta \in L^\infty_+(\mathcal{F}_t),\, w_1, w_2 \in E\}.$$

More concisely:

Definition 2.2.16. For $\mathcal{D} \subset \mathcal{L}^1_+$, we define for each time t the *predictable pre-image* of $\mathcal{D}$ by

$$\begin{aligned}\mathcal{M}_t(\mathcal{D}) := \{Z \in \mathcal{L}^1(\mathbb{R}^{d+1}) :&\exists \alpha_t \in L^0_{t,+}, \exists Z' \in \mathcal{D} \\ &\text{such that } \alpha_t Z' \in \mathcal{L}^1(\mathbb{R}^{d+1}) \text{ and } \mathbb{E}\left[Z \,|\, \mathcal{F}_{t+1}\right] = \alpha_t \mathbb{E}\left[Z' \,\middle|\, \mathcal{F}_{t+1}\right]\}.\end{aligned} \tag{2.5}$$

The predictable pre-image of a set $\mathcal{D} \subset \mathcal{L}^1_+$ is key to understanding predictably stable convex cones, as shown in the following two lemmas:

Lemma 2.2.17. *Suppose* $\mathcal{D} \subset \mathcal{L}^1_+$*. If* $\mathcal{D}$ *is a predictably stable convex cone, then*

$$\mathcal{D} = \bigcap_{t=0}^{T-1} \mathcal{M}_t(\mathcal{D}).$$

Lemma 2.2.18. *Suppose $\mathcal{D} \subset \mathcal{L}^1_+$, and define*

$$[\mathcal{D}] := \bigcap_{t=0}^{T-1} \left(\overline{\mathrm{conv}} \mathcal{M}_t(\mathcal{D})\right),$$

where $\mathcal{M}_t(\mathcal{D})$ is as defined in (2.5), the symbol $\overline{\mathrm{conv}}$ denotes the closure in $\mathcal{L}^1$ of the convex hull.

(a) $[\mathcal{D}]$ is the smallest stable closed convex cone in $\mathcal{L}^1$ containing $\mathcal{D}$;

(b) $\mathcal{D} = [\mathcal{D}]$ if and only if $\mathcal{D}$ is a stable closed convex cone in $\mathcal{L}^1$.

We prove both these lemmas in Section 2.4.1.

The proof of equivalence of statements (ii) and (iii) of Theorem 2.2.15 is underpinned by the following

Theorem 2.2.19. *For any $t \in \{0, 1, \ldots, T-1\}$, for $\mathcal{D} \subset \mathcal{L}^1_+$,*

$$K_t(\mathcal{A}, \mathbf{V}) = (\mathcal{M}_t(\mathcal{A}(\mathbf{V})^*))^*. \tag{2.6}$$

Thus we have characterised each "summand" in the representation (cf. definition 2.2.8) as a dual set of the predictable pre-image of the dual of the set of acceptable portfolios in $\mathbf{V}$.

2.3 Examples

In this section we present a brief exposition of the versatility of the framework.

2.3.1 Modelling transaction costs

We now present an example motivated by buying and selling a stock in a market with transaction costs across two time periods ($T = 2$). Let N_1 and N_2 be two independent and identically distributed standard Gaussian random variables under objective measure $\mathbb{P}$. Fix $M > 0$ and define the truncated random variables $\widetilde{N}_i := N_i \wedge M$, for $i = 1, 2$. Define the constant a_M such that $\mathbb{E}_{\mathbb{P}}[\exp(\widetilde{N}_i - a_M)] = 1$:

$$a_M := \log \mathbb{E}_{\mathbb{P}}[\exp(\widetilde{N}_1)] = \log\left(e^{\frac{1}{2}}\Phi(M-1) + e^M(1 - \Phi(M))\right).$$

Define the filtration by $\mathcal{F}_0$ trivial, $\mathcal{F}_1 = \sigma(\widetilde{N}_1)$, and $\mathcal{F}_2 = \sigma(\widetilde{N}_1, \widetilde{N}_2)$.

The market consists of a "cash account" $v_0 \equiv 1$ and a "stock" with time-2 price

$$v_1 = \exp\left(\widetilde{N}_1 + \widetilde{N}_2 - 2a_M\right).$$

Set $\mathbf{V} = (v_0, v_1)$. At time 0, to buy 1 unit of v_1 a purchaser must pay $1+\lambda$ cash, and to sell 1 unit of v_1 a vendor receives $1-\lambda$. At time 1, knowing the value of $\widetilde{N}_1$, buying 1 unit of v_1 costs $(1+\lambda)e^{\widetilde{N}_1 - a_M}$, and selling 1 unit of v_1 makes $(1-\lambda)e^{\widetilde{N}_1 - a_M}$. Define the $\mathcal{F}_1$-cone of a set E by $\operatorname{cone}_{\mathcal{F}_1} E = \{\alpha w : \alpha \in L^\infty_+(\mathcal{F}_1),\ w \in \operatorname{conv} E\}$. If we also allow wealth to be consumed, we arrive at the following set of claims to which we may trade from zero initial wealth:

$$\begin{aligned}\mathcal{A} = &\operatorname{cone}\{(-(1+\lambda),1),(1-\lambda,-1)\}\cdot\mathbf{V}\\ &\oplus \operatorname{cone}_{\mathcal{F}_1}\left\{(-(1+\lambda)e^{\widetilde{N}_1-a_M},1),((1-\lambda)e^{\widetilde{N}_1-a_M},-1)\right\}\cdot\mathbf{V}\\ &\oplus(-L^\infty_+).\end{aligned}$$

In the sum, the first term describes those claims that can be realised at time 0, the second term describes those claims that can be realised at time 1, and the last describes consumption of wealth at any time. It is easy to show that the dual of $\mathcal{A}$ is

$$\mathcal{Q} := \left\{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}_{\mathbb{Q}}[v_1] \in [1-\lambda, 1+\lambda] \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}[\exp(\widetilde{N}_2 - a_M)|\mathcal{F}_1] \in [1-\lambda, 1+\lambda]\right\}.$$

Note that $\mathcal{Q}$ is a convex set of probability measures that is not m-stable. Define a coherent pricing measure by $\rho_t(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ for $t = 0, 1$. We have $\rho_0(v_1) = 1+\lambda$, but

$$\rho_1(v_1) = e^{\widetilde{N}_1 - a_M} \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[e^{\widetilde{N}_2 - a_M}|\mathcal{F}_1] = (1+\lambda)e^{\widetilde{N}_1 - a_M},$$

and so

$$\rho_0(\rho_1(v_1)) = (1+\lambda)\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[e^{\widetilde{N}_1 - a_M}] = \frac{(1+\lambda)^2}{1-\lambda} > \rho_0(v_1).$$

The last line follows from the inequalities for any $\mathbb{Q}\in\mathcal{Q}$:

$$1+\lambda \ge \mathbb{E}_{\mathbb{Q}}[v_1] = \mathbb{E}_{\mathbb{Q}}[e^{\widetilde{N}_1 - a_M}\mathbb{E}_{\mathbb{Q}}[e^{\widetilde{N}_2 - a_M}|\mathcal{F}_1]] \ge (1-\lambda)\mathbb{E}_{\mathbb{Q}}[e^{\widetilde{N}_1 - a_M}].$$

Now, we may show that $\mathcal{Q}$ must be $\mathbf{V}$-m-stable: we take two measures $\mathbb{Q}^\Lambda$ and $\mathbb{Q}^M$ with Radon-Nikodym derivatives Λ and M, form the pasting at a stopping time $\tau \in \{0,1,2\}$, and check that the pasted measure $\widetilde{\mathbb{Q}}$, defined by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = \frac{M}{\mathbb{E}[M|\mathcal{F}_\tau]}\mathbb{E}[\Lambda|\mathcal{F}_\tau]$$

is also in $\mathcal{Q}$. Noting that $\mathbb{1}_{\{\tau=2\}} = 1 - \mathbb{1}_{\{\tau\leq 1\}} \in L_1^\infty$, we calculate

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{Q}}}[\exp(\widetilde{N}_2 - a_M)|\mathcal{F}_1] &= \mathbb{E}\left[\left(\frac{M}{\mathbb{E}[M|\mathcal{F}_1]}\mathbb{1}_{\{\tau\leq 1\}} + \frac{\Lambda}{\mathbb{E}[\Lambda|\mathcal{F}_1]}\mathbb{1}_{\{\tau=2\}}\right)\exp(\widetilde{N}_2 - a_M)\middle|\mathcal{F}_1\right] \\ &= \mathbb{E}_{\mathbb{Q}^M}[\exp(\widetilde{N}_2 - a_M)|\mathcal{F}_1]\mathbb{1}_{\{\tau\leq 1\}} + \mathbb{E}_{\mathbb{Q}^\Lambda}[\exp(\widetilde{N}_2 - a_M)|\mathcal{F}_1]\mathbb{1}_{\{\tau=2\}},\end{aligned}$$

so we see that the condition $\mathbb{E}_{\widetilde{\mathbb{Q}}}[\exp(\widetilde{N}_2 - a_M)|\mathcal{F}_1] \in [1-\lambda, 1+\lambda]$ is satisfied. To satisfy the definition of **V**-m-stability, we need only check that $\widetilde{\mathbb{Q}} \in \mathcal{Q}$ for those $\mathbb{Q}^\Lambda$ and $\mathbb{Q}^M$ that satisfy

$$\mathbb{E}_{\mathbb{Q}^\Lambda}[v_1|\mathcal{F}_\tau] = \mathbb{E}_{\mathbb{Q}^M}[v_1|\mathcal{F}_\tau].$$

Hence, we now calculate

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{Q}}}[v_1] &= \mathbb{E}\left[\frac{\mathbb{E}[\Lambda|\mathcal{F}_\tau]}{\mathbb{E}[M|\mathcal{F}_\tau]}\mathbb{E}[Mv_1|\mathcal{F}_\tau]\right] \\ &= \mathbb{E}\left[\mathbb{E}[\Lambda|\mathcal{F}_\tau]\mathbb{E}_{\mathbb{Q}^M}[v_1|\mathcal{F}_\tau]\right] \\ &= \mathbb{E}\left[\mathbb{E}[\Lambda|\mathcal{F}_\tau]\mathbb{E}_{\mathbb{Q}^\Lambda}[v_1|\mathcal{F}_\tau]\right] \\ &= \mathbb{E}_{\mathbb{Q}^\Lambda}[v_1].\end{aligned}$$

Thus $\mathcal{Q}$ is **V**-m-stable.

2.3.2 A Haezendonck–Goovaerts risk measure

The following is an example employing the so-called Haezendonck–Goovaerts risk measures; we refer the reader to the work of Bellini and Rosazza Gianin [Bellini and Gianin, 2008]. Consider a two-period binary branching tree, with $\mathbb{P}\{\omega\} = \frac{1}{4}$ for all four elements $\omega \in \Omega$. We choose a (normalised) Young function $\Phi(x) = x^2$, and define the Orlicz premium principle to be the unique solution $H_\alpha(X)$ of the equation

$$\mathbb{E}\left[\Phi\left(\frac{X}{H_\alpha(X)}\right)\right] = 1 - \alpha \qquad \text{for } X \neq 0; \qquad H_\alpha(0) := 0.$$

Fix $\alpha = \frac{1}{2}$, and rearrange the above to see $H_{\frac{1}{2}}(X) = \sqrt{2}\|X\|_2 = \sqrt{2}\left(\mathbb{E}\left[X^2\right]\right)^{\frac{1}{2}}$. We now define the *Haezendonck measure* to be

$$\rho_0(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_\mathbb{Q}[X] \qquad \text{where} \qquad \mathcal{Q} := \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}_\mathbb{Q}[Y] \leq H_{\frac{1}{2}}(Y) \quad \forall Y \in L_+^\infty\}.$$

We write $\mathbb{Q}(\{i\}) =: q_i$ for a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, and $X(i) = x_i$ for a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$. First, we characterise $\mathcal{Q}$. Note that the constraint in the definition of $\mathcal{Q}$ implies

$$\sup_{0 \neq Y \in L^\infty_+} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \frac{Y}{\|Y\|_2}\right] \leq \sqrt{2}.$$

The supremum is attained upon choosing $Y = \frac{d\mathbb{Q}}{d\mathbb{P}}$, so the above inequality implies $\left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_2^2 \leq 2$, thus

$$\mathcal{Q} = \left\{\mathbb{Q} = (q_1, \ldots, q_4) : q_i \geq 0, \quad \sum_{i=1}^4 q_i = 1, \quad \sum_{i=1}^4 q_i^2 \leq \frac{1}{2}\right\}.$$

$\mathcal{Q}$ is not m-stable Define measures $\mathbb{Q}^\Lambda$ and $\mathbb{Q}^M$ from $\Lambda_2 = 2 \times \mathbb{1}_{\{1,2\}}$ and $M_2 = 2 \times \mathbb{1}_{\{1,3\}}$ respectively. We see that both are elements of $\mathcal{Q}$, and their restrictions to $(\Omega, \mathcal{F}_1)$ are described by $\Lambda_1 = \mathbb{E}[\Lambda_2|\mathcal{F}_1] = 2 \times \mathbb{1}_{\{1,2\}}$, and $M_1 = \mathbb{E}[M_2|\mathcal{F}_1] = 1$. We form the time-1 pasting of the measures $\mathbb{Q}^\Lambda$ and $\mathbb{Q}^M$ by setting

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = \frac{\Lambda_1}{M_1} M_2 = 4 \times \mathbb{1}_{\{1\}}.$$

Here, $\sum_{i=1}^4 \widetilde{q}_i^2 = 1 > \frac{1}{2}$, so $\widetilde{\mathbb{Q}} \notin \mathcal{Q}$, and the set $\mathcal{Q}$ is not m-stable.

Now, set

$$\mathbf{V} = \left(1, \sqrt{2}\mathbb{1}_{\{1\}} + 1, \sqrt{2}\mathbb{1}_{\{3\}} + 1\right).$$

$\mathcal{Q}$ is V-m-stable We calculate $\mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_1]$ as in Example 2.2.7, to see that, for $\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}$, our additional condition is

$$\frac{q_1}{q_2} \mathbb{1}_{\{q_1+q_2>0\}} = \frac{q_1'}{q_2'} \mathbb{1}_{\{q_1'+q_2'>0\}} \quad \text{and} \quad \frac{q_3}{q_4} \mathbb{1}_{\{q_3+q_4>0\}} = \frac{q_3'}{q_4'} \mathbb{1}_{\{q_3'+q_4'>0\}}.$$

Thus we see that any pasting $\mathbb{Q} \oplus_{t=1} \mathbb{Q}'$ that satisfies this condition is in fact equal to $\mathbb{Q}$, which is trivially in $\mathcal{Q}$.

2.3.3 Reserving for cash flows

We describe a probabilistic approach to wealth processes using the notation of Acciaio, Föllmer, and Penner [Acciaio et al., 2012]. As before, we fix a terminal time $T < \infty$, a discrete time set $\mathbb{T} := \{0, 1, \ldots, T\}$, and a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$. On the product space $\underline{\Omega} := \Omega \times \mathbb{T}$, define the optional σ-algebra up to time $t \in \mathbb{T}$ as

$$\underline{\mathcal{F}}_t := \sigma\left(A_s \times \{s\}, A_t \times \mathbb{T}_t : s \le t, A_s \in \mathcal{F}_s\right), \qquad \text{where } \mathbb{T}_t := \{t, t+1, \dots, T\},$$
$$\underline{\mathcal{F}} := \underline{\mathcal{F}}_T.$$

Define the reference probability measure $\underline{P} := \mathbb{P} \otimes \mu$ on $(\underline{\Omega}, \underline{\mathcal{F}})$ via the expectation

$$\mathbb{E}_{\underline{P}}[X] = \mathbb{E}\left[\sum_{s=0}^{T} X_s \mu_s\right]$$

where $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ and μ is an *optional random probability measure* on $\mathbb{T}$, i.e., an $\mathcal{F}_t$-adapted process such that $\mu_t > 0$ for all $t \in \mathbb{T}$ and $\sum_{t \in \mathbb{T}} \mu_t = 1$.

We use the underline to denote multiperiod variants of standard notation; for example $\underline{L}^\infty := L^\infty(\underline{\Omega}, \underline{\mathcal{F}}, \underline{P})$ is the space of all bounded random variables on the extended probability space $(\underline{\Omega}, \underline{\mathcal{F}}, \underline{P})$, elements of which may alternatively be viewed as processes $X = (X_t)_{t \in \mathbb{T}}$. We write $\underline{\mathcal{L}}^1(\mathbb{R}^{d+1}) := L^1(\underline{\Omega}, \underline{\mathcal{F}}, \underline{P}; \mathbb{R}^{d+1})$ (respectively $\underline{\mathcal{L}}^\infty(\mathbb{R}^{d+1})$) for $\underline{P}$-integrable (resp. bounded) random variables X such that each X_t is $\mathbb{R}^{d+1}$-valued, for $t \in \mathbb{T}$. Non-negative elements of $\underline{L}^\infty$ are denoted by $\underline{L}^\infty_+$, and $\underline{\mathcal{F}}_t$-measurable elements of $\underline{L}^\infty$ are denoted by $\underline{L}^\infty_t$.

For $0 \le t \le s \le T$, define the projection $\pi_{s,t} : \underline{L}^\infty \to \underline{L}^\infty$

$$\pi_{s,t}(X)_r = \mathbb{1}_{\{s \le r\}} X_{r \wedge t}, \qquad \text{for } r \in \mathbb{T}.$$

Define $\mathcal{R}^\infty$ to be those *adapted* processes $X \in \underline{L}^\infty$, and set $\mathcal{R}^\infty_{t,s} = \pi_{s,t}(\mathcal{R}^\infty)$ and $\mathcal{R}^\infty_t = \pi_{t,T}(\mathcal{R}^\infty)$. We use the notation $X|_t$ for the conditional expectation $\mathbb{E}_{\underline{P}}[X|\underline{\mathcal{F}}_t] \equiv \mathbb{E}[X|\underline{\mathcal{F}}_t]$, which may be viewed as a process, constant after time t; we write X_t to denote the time-t realisation of the process X.

We remark that there is a one-to-one correspondence between pricing measures for processes $\rho_t : \mathcal{R}^\infty_t \to L^\infty_t$ and pricing measures $\underline{\rho}_t : \mathcal{R}^\infty \to \underline{L}^\infty_t$ for random variables on $\underline{\Omega}$ equipped with the optional σ-algebra, via

$$\underline{\rho}_t(X) = \sum_{s=0}^{t-1} X_s \mathbb{1}_{\{s\}} + \rho_t(\pi_{t,T}(X)) \mathbb{1}_{\mathbb{T}_t}. \tag{2.7}$$

2.4 Proof of main result

First, we show equivalence of predictable time-consistency and predictable representability.

Proof of Theorem 2.2.15, equivalence of (i) and (ii). (i) ⇒ (ii): Since the sets $K_t(\mathcal{A}, \mathbf{V}) := \mathcal{A}_t(\mathbf{V}) \cap \mathcal{L}^\infty_{t+1}(\mathbb{R}^{d+1})$ are all subsets of $\mathcal{A}_0$, we only need demonstrate that $\mathcal{A}_0 \subseteq \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V})}$.

Take $X \in \mathcal{A}_0$, and suppose that (i) holds. We proceed by backwards induction on t. For each t, there exist nets $\pi_t^\alpha \in L^\infty(\mathcal{F}_{t+1}; \mathbb{R}^{d+1})$ such that

$$\lim_\alpha \rho_t(\pi_t^\alpha \cdot \mathbf{V}) \leq 0 \quad \text{and} \quad X = \lim_\alpha \sum_{s=0}^{T-1} \pi_s^\alpha \cdot \mathbf{V}.$$

Set

$$\varepsilon_t^\beta = \sup_{\alpha \geq \beta} \{\rho_t(\pi_t^\alpha \cdot \mathbf{V})\},$$

and note that for $\alpha \geq \beta$, for any t,

$$\pi_t^\alpha - \varepsilon_t^\beta e^0 \in K_t(\mathcal{A}, \mathbf{V}),$$

where $e^0 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^{d+1}$. Summing over times t, we obtain

$$\sum_{s=0}^{T-1} (\pi_s^\alpha - \varepsilon_s^\beta e^0) \cdot \mathbf{V} \in \oplus_{s=0}^{T-1} K_s(\mathcal{A}, \mathbf{V}) \cdot \mathbf{V}.$$

Limiting over α, we have for any β, the weak* limit $X - \sum_{s=0}^{T-1} \varepsilon_s^\beta$ is an element of $\overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V}) \cdot \mathbf{V}}$.

For each s, we note that

$$\varepsilon_s := \lim_\beta \varepsilon_s^\beta = \limsup_\alpha \rho_s(\pi_s^\alpha \cdot \mathbf{V}) = \lim_\alpha \rho_s(\pi_s^\alpha \cdot \mathbf{V}) \leq 0.$$

Limiting over β, we have

$$X - \sum_{s=0}^{T-1} \varepsilon_s \in \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V}) \cdot \mathbf{V}}.$$

The set $\overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V}) \cdot \mathbf{V}}$ is a cone containing the negative orthant, thus it contains $\sum_{s=0}^{T-1} \varepsilon_s$ as well. Thus $X = \left(X - \sum_{s=0}^{T-1} \varepsilon_s\right) + \sum_{s=0}^{T-1} \varepsilon_s \in \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V}) \cdot \mathbf{V}}$, and the desired inclusion is proved.

(ii) ⇒ (i): Suppose that $\mathcal{A}_0(\mathbf{V}) = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, \mathbf{V})}$. Then, for any $X \in \mathcal{A}_0$, for each t there exists a sequence $\pi_t^n \in K_t(\mathcal{A}, \mathbf{V}) \subseteq L^\infty(\mathcal{F}_{t+1}; \mathbb{R}^{d+1})$, so that $X = \lim_{n\to\infty} \sum_{s=0}^{T-1} \pi_s^n \cdot \mathbf{V}$, and $\rho_t(\pi_t^n \cdot \mathbf{V}) \leq 0$ for each n, so in particular the limit as $n \to \infty$ is also non-positive. □

Proof of Theorem 2.2.15, equivalence of (ii) and (iii). To simplify the notation, denote $\mathcal{A}(\mathbf{V})$ by $\mathcal{B}$. Assume now that $\mathcal{B}$ is a weak*-closed convex cone in $\mathcal{L}^\infty(\mathbb{R}^{d+1})$ which is arbitrage-free, so that $\mathcal{B}^{**} = \mathcal{B}$. Also, define

$$K_t(\mathcal{B}) := \{X \in \mathcal{L}^\infty(\mathcal{F}_{t+1}, \mathbb{R}^{d+1}) : \alpha X \in \mathcal{B} \text{ for any } \alpha \in L^\infty_+(\mathcal{F}_t)\}.$$

Recall that $\mathcal{B}$ is predictably representable if

$$\mathcal{B} = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{B})}.$$

We must show the equivalence of the two conditions

(ii') $\mathcal{B}$ is predictably representable; and

(iii') $\mathcal{B}^*$ is predictably stable.

(ii') ⇒ (iii'): Assuming $\mathcal{B}$ is predictably representable, it follows from Theorem 2.2.19 that

$$\mathcal{B} = \overline{\oplus_t K_t(\mathcal{B})}^{w^*} = \overline{\oplus_t \mathcal{M}_t(\mathcal{B}^*)^*}^{w^*}.$$

Taking the dual, we find that

$$\mathcal{B}^* = \cap_t \mathcal{M}_t(\mathcal{B}^*)^{**} = \cap_t \overline{\mathrm{conv}} \mathcal{M}_t(\mathcal{B}^*)$$

where the last equality follows from the Bipolar Theorem. Hence, $\mathcal{B}^* = [\mathcal{B}^*]$, and by Lemma 2.2.18, $\mathcal{B}^*$ is predictably stable.

(iii') ⇒ (ii'): Assuming $\mathcal{B}$ is a weak*-closed convex cone, note that $\mathcal{B}^*$ is a convex cone closed in $(\mathcal{L}^1, \sigma(\mathcal{L}^1, \mathcal{L}^\infty))$. Assuming further that $\mathcal{B}^*$ is *stable*,

$$\begin{aligned} \mathcal{B}^* &= \cap_t \mathcal{M}_t(\mathcal{B}^*) && \text{by Lemma 2.2.17} \\ &= \cap_t K_t(\mathcal{B})^* && \text{by eq. (2.6).} \end{aligned}$$

Now we may apply Lemma 1.2.6 to deduce

$$\mathcal{B} \equiv \mathcal{B}^{**} = \overline{\oplus_t K_t(\mathcal{B})}^{w^*}$$

and $\mathcal{B}$ is predictably representable, as required. □

Proof of Theorem 2.2.19. We set $\mathcal{B} = \mathcal{A}(\mathbf{V})$, as above.

First we prove that $\mathcal{M}_t(\mathcal{B}^*) \subset K_t(\mathcal{B})^*$. For arbitrary $Z \in \mathcal{M}_t(\mathcal{B}^*)$, there exist $Z' \in \mathcal{B}^*$ and $\alpha \in L^0_+(\mathcal{F}_t)$ with $\alpha Z' \in \mathcal{L}^1$ and $Z|_{t+1} = \alpha Z'|_{t+1}$.

Note that, for any $X \in K_t(\mathcal{B})$,

$$\mathbb{E}[Z \cdot X] = \mathbb{E}[Z|_{t+1} \cdot X] = \mathbb{E}[\alpha Z'|_{t+1} \cdot X] = \lim_{n\to\infty} \mathbb{E}[(\alpha \mathbb{1}_{\{\alpha \le n\}} X) \cdot Z'|_{t+1}] \le 0,$$

since $\alpha \mathbb{1}_{\{\alpha\le n\}} X \in \mathcal{B}$ and $Z' \in \mathcal{B}^*$. Hence $Z \in K_t(\mathcal{B})$, and since Z is arbitrary, we have shown that $\mathcal{M}_t(\mathcal{B}^*) \subset K_t(\mathcal{B})^*$.

For the reverse inclusion, $\mathcal{M}_t(\mathcal{B}^*)^* \subset K_t(\mathcal{B})$, note that $\mathcal{B}^* \subset \mathcal{M}_t(\mathcal{B}^*)$ implies $\mathcal{M}_t(\mathcal{B}^*)^* \subset \mathcal{B}$, and

$$\begin{aligned} \mathcal{L}^\infty_+(\mathcal{F}_t)\mathcal{M}_s(\mathcal{D}) = \mathcal{M}_s(\mathcal{D}) \quad &\Longrightarrow \quad \text{for } X \in \mathcal{M}_t(\mathcal{B}^*)^*, \quad g \in \mathcal{L}^\infty_+(\mathcal{F}_t), \quad \mathbb{E}[X \cdot gZ] \le 0 \\ &\Longrightarrow \quad \mathcal{L}^\infty_+(\mathcal{F}_t)\mathcal{M}_t(\mathcal{B}^*)^* = \mathcal{M}_t(\mathcal{B}^*)^*. \end{aligned}$$

Define

$$\mathcal{B}_t := \{X \in \mathcal{L}^\infty(\mathcal{F}_T, \mathbb{R}^{d+1}) : gX \in \mathcal{B} \text{ for any } g \in L^\infty_+(\mathcal{F}_t)\}.$$

Thus $\mathcal{M}_t(\mathcal{B}^*)^* \subseteq \mathcal{B}_t$. To finish the proof, we need only show that $X \in \mathcal{M}_t(\mathcal{B}^*)^*$ is $\mathcal{F}_{t+1}$-measurable, since $\mathcal{B}_t \cap \mathcal{L}^\infty(\mathcal{F}_{t+1}, \mathbb{R}^{d+1}) = K_t(\mathcal{B})$.

To this end, note that for any $Z \in \mathcal{L}^1(\mathbb{R}^{d+1})$, it is true that $Z - Z|_{t+1} \in \mathcal{M}_t(\mathcal{B}^*)$, whence $\mathbb{E}[(Z - Z|_{t+1}) \cdot X] \le 0$. We deduce that

$$\mathbb{E}[(Z - Z|_{t+1}) \cdot X] = \mathbb{E}[(X - X|_{t+1}) \cdot Z] \le 0 \qquad \forall Z \in \mathcal{L}^1(\mathbb{R}^{d+1}),$$

and $X = X|_{t+1}$ $\mathbb{P}$-a.s.. □

2.4.1 Proofs of Lemmas

Lemma 2.4.1. *Let $\mathcal{D} \subset \mathcal{L}^1_+(\mathbb{R}^{d+1})$. The following are equivalent:*

(i) for each $t \in \{0, 1, \dots, T\}$, whenever $Y, W \in \mathcal{D}$ are such that there exists $Z \in \mathcal{D}$, a set $F \in \mathcal{F}_t$, positive processes $\alpha, \beta \in \mathcal{L}^0(\mathcal{F}_t)$ with $\alpha Y, \beta W \in \mathcal{L}^1(\mathbb{R}^{d+1})$ and

$$X := \mathbb{1}_F \alpha Y + \mathbb{1}_{F^c} \beta W \quad \text{satisfies} \quad \mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[Z \mid \mathcal{F}_t],$$

then X is also a member of $\mathcal{D}$;

(ii) $\mathcal{D}$ is predictably stable, that is, for each stopping time $\tau \le T$, whenever $Z, W \in \mathcal{D}$ are such that

$$\mathbb{E}[Z \mid \mathcal{F}_\tau] = m\mathbb{E}[W \mid \mathcal{F}_\tau],$$

then mW is also a member of $\mathcal{D}$.

Proof of Lemma 2.4.1. *(ii) $\Longrightarrow$ (i):* We suppose that (ii) holds, and fix $t \in \mathbb{T}$.

We aim for a triple of random variables Y, W, Z in $\mathcal{D}$, together with an $F \in \mathcal{F}_t$, and α, β as required in condition (i), such that we can apply (ii) twice to show that the resulting X defined in condition (i) is a member of $\mathcal{D}$.

First, let $\tau = T\mathbb{1}_F + t\mathbb{1}_{F^c}$ and suppose $Z, W \in \mathcal{D}$ satisfy $\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_\tau\right] = m\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_\tau\right]$ for all i. By (ii), we have $\widetilde{X} := mW \in \mathcal{D}$. Writing

$$\beta := \frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_t\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_t\right]} \mathbb{1}_{\{\mathbb{E}[W^i \mid \mathcal{F}_t] > 0\}},$$

we may express $\widetilde{X} = Z\mathbb{1}_F + \beta W \mathbb{1}_{F^c}$.

Second, let $\widetilde{\tau} = t\mathbb{1}_F + T\mathbb{1}_{F^c}$ and suppose $Y \in \mathcal{D}$ satisfies $\mathbb{E}\left[\widetilde{X}^i \,\middle|\, \mathcal{F}_\tau\right] = \widetilde{m}\mathbb{E}\left[Y^i \,\middle|\, \mathcal{F}_\tau\right]$ for all i. By (ii), we have $X := \widetilde{m}Y \in \mathcal{D}$. Writing

$$\alpha := \frac{\mathbb{E}\left[\widetilde{X}^i \,\middle|\, \mathcal{F}_t\right]}{\mathbb{E}\left[Y^i \,\middle|\, \mathcal{F}_t\right]} \mathbb{1}_{\{\mathbb{E}[Y^i \mid \mathcal{F}_t] > 0\}},$$

we may express $X = \alpha Y \mathbb{1}_F + \beta W \mathbb{1}_{F^c}$.

Now, we have a t fixed, $Y, W, Z \in \mathcal{D}$, a set $F \in \mathcal{F}_t$, and positive r.v.s $\alpha, \beta \in \mathcal{L}^0(\mathcal{F}_t)$. We have already that $X \in \mathcal{D}$, thus it remains to check[2] that X and Z as defined above satisfy $\mathbb{E}\left[X \mid \mathcal{F}_t\right] = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$.

$$\begin{aligned}
\mathbb{E}\left[X \mid \mathcal{F}_t\right] &= \mathbb{1}_F \mathbb{E}\left[\alpha Y \mid \mathcal{F}_t\right] + \mathbb{1}_{F^c}\mathbb{E}\left[\beta W \mid \mathcal{F}_t\right] \\
&= \mathbb{1}_F \mathbb{E}\left[\frac{\mathbb{E}\left[\widetilde{X}^i \,\middle|\, \mathcal{F}_t\right]}{\mathbb{E}\left[Y^i \,\middle|\, \mathcal{F}_t\right]} \mathbb{1}_{\{\mathbb{E}[Y^i \mid \mathcal{F}_t] > 0\}} Y \,\middle|\, \mathcal{F}_t\right] \\
&\quad + \mathbb{1}_{F^c}\mathbb{E}\left[\frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_t\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_t\right]} \mathbb{1}_{\{\mathbb{E}[W^i \mid \mathcal{F}_t] > 0\}} W \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{1}_F \mathbb{E}\left[\widetilde{X} \,\middle|\, \mathcal{F}_t\right] + \mathbb{1}_{F^c}\mathbb{E}\left[Z \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[Z \mid \mathcal{F}_t\right],
\end{aligned}$$

which establishes statement (i).

(i) $\implies$ (ii): Say (i) holds; then (ii) holds for when $\tau = T$ trivially. Now suppose that (ii) holds for any stopping time $\tau \geq k+1$ a.s., and proceed by backward induction on the lower bound of the stopping times. Fix an arbitrary stopping time $\widetilde{\tau} \geq k$ a.s., and define $F = \{\widetilde{\tau} \geq k+1\}$ and the stopping time $\tau^* := \widetilde{\tau}\mathbb{1}_F + T\mathbb{1}_{F^c}$. Note that $\tau^* \geq k+1$, since $F^c = \{\widetilde{\tau} = k\}$.

We shall now take $Z, W \in \mathcal{D}$ that satisfy $\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right] = m\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]$ for all

[2] the integrability conditions $\alpha Y, \beta W \in \mathcal{L}^1(\mathbb{R}^{d+1})$ are easily verified.

i, and aim to show that mW is indeed an element of $\mathcal{D}$, with the help of condition (i).

To this end, define

$$Y := W\frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_{\tau^*}\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_{\tau^*}\right]}\mathbb{1}_{\{\mathbb{E}[W^i\,|\mathcal{F}_{\tau^*}]>0\}} = \mathbb{1}_F W\frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]}\mathbb{1}_{\{\mathbb{E}[W^i\,|\mathcal{F}_{\widetilde{\tau}}]>0\}} + Z\mathbb{1}_{F^c}.$$

By the inductive hypothesis, Y is in $\mathcal{D}$, thanks to the bound $\tau^* \geq k+1$, .

Now, we have $t = k$ fixed, $Y, W, Z \in \mathcal{D}$, a set $F \in \mathcal{F}_t$, and positive random variables $\alpha \equiv 1$, $\beta := \mathbb{1}_{F^c}\frac{\mathbb{E}\left[Z^i\,|\mathcal{F}_k\right]}{\mathbb{E}[W^i\,|\mathcal{F}_k]}$. Define

$$\begin{aligned}
X &:= \mathbb{1}_F\alpha Y + \mathbb{1}_{F^c}\beta W \\
&= W\mathbb{1}_F\frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]}\mathbb{1}_{\{\mathbb{E}[W^i\,|\mathcal{F}_{\widetilde{\tau}}]>0\}} + W\mathbb{1}_{F^c}\frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_k\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_k\right]}\mathbb{1}_{\{\mathbb{E}[W^i\,|\mathcal{F}_k]>0\}} \\
&= W\frac{\mathbb{E}\left[Z^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]}{\mathbb{E}\left[W^i \,\middle|\, \mathcal{F}_{\widetilde{\tau}}\right]}\mathbb{1}_{\{\mathbb{E}[W^i\,|\mathcal{F}_{\widetilde{\tau}}]>0\}}.
\end{aligned}$$

It is elementary to check that X and Z as defined above satisfy $\mathbb{E}\left[X\,|\,\mathcal{F}_k\right] = \mathbb{E}\left[Z\,|\,\mathcal{F}_k\right]$. Thus by (i), X is an element of $\mathcal{D}$, which completes the inductive step. $\square$

Proof of Lemma 2.2.13. First take $Z \in \mathcal{D}^*$. For any $X \in \mathcal{D}(\mathbf{V})$ we have $\mathbb{E}[ZV \cdot X] \leq 0$ and so $ZV \in \mathcal{D}(\mathbf{V})^*$, thus $\mathcal{D}(\mathbf{V})^* \supseteq \mathcal{D}^*\mathbf{V}$.

For the reverse inclusion, denote by e_i the ith canonical basis vector in $\mathbb{R}^{d+1}$. First, since $\mathbf{V} \cdot \alpha(v^i e_j - v^j e_i) = 0$, we have

$$\alpha(v^i e_j - v^j e_i) \in \mathcal{D}(\mathbf{V}) \qquad \forall \alpha \in L^\infty.$$

Take $Z \in \mathcal{D}(\mathbf{V})^*$. Now, for any $i, j \in \{1, \ldots, d\}$, $\alpha \in L^\infty$, we have

$$\mathbb{E}[Z \cdot \alpha(v^i e_j - v^j e_i)] \leq 0.$$

Reversing i and j in the above, we may write $\mathbb{E}[Z \cdot \alpha(v^i e_j - v^j e_i)] = 0$, and allowing first $\alpha = \mathbb{1}_{\{Z\cdot(v^i e_j - v^j e_i)>0\}}$ then $\alpha = \mathbb{1}_{\{Z\cdot(v^i e_j - v^j e_i)<0\}}$, we see that in fact,

$$Z \cdot (v^i e_j - v^j e_i) = 0 \qquad \text{a.s. for any } i, j,$$

and so, taking $i = 0$ we have $Z^j = Z^0 v^j$ a.s. for each j, thus any $Z \in \mathcal{D}(\mathbf{V})^*$ must be of the form $Z^0\mathbf{V}$ for some $Z^0 \in L^1$. Now, given $C \in \mathcal{D}$, take X such that $X \cdot \mathbf{V} = C$

(which implies that $X \in \mathcal{D}(\mathbf{V})$), then

$$0 \geq \mathbb{E}[W\mathbf{V} \cdot X] = \mathbb{E}[WC],$$

and since C is arbitrary, it follows that $W \in \mathcal{D}^*$. Hence $\mathcal{D}(\mathbf{V})^* \subseteq \mathcal{D}^*\mathbf{V}$. $\square$

Proof of Lemma 2.2.17. The inclusion $\mathcal{D} \subset \cap_{t=0}^{T-1}\mathcal{M}_t(\mathcal{D})$ is trivial. In the following, we write $Z|_t$ for $\mathbb{E}\left[Z \,|\, \mathcal{F}_t\right]$.

Now $Z \in \cap_{t=0}^{T-1}\mathcal{M}_t(\mathcal{D})$, and we aim to show that $Z \in \mathcal{D}$. So, for all $t \in \{0, 1, \dots, T-1\}$, there exist $\beta_t \in L^0_{t,+}$ and $Z^t \in \mathcal{D}$ such that $\beta_t Z \in \mathcal{L}^1$ and $Z|_{t+1} = \beta_t Z^t|_{t+1}$.

Define

$$\begin{aligned} \xi^{T-1} &= Z^{T-1} \\ \xi^t &= \mathbb{1}_{F_t}\kappa_t\xi^{t+1} + \mathbb{1}_{F_t^c}Z^t \qquad\qquad \text{for } t \in \{0, 1, \dots, T-2\}, \end{aligned}$$

where $F_t = \{\beta_t > 0\}$ and $\kappa_t = \beta_{t+1}/\beta_t$.

Note $Z = Z|_T = \beta_{T-1}Z^{T-1}|_T = \beta_{T-1}\xi^{T-1}$ and

$$Z = \beta_0\kappa_0\kappa_1 \cdots \kappa_{T-2}\xi^{T-1} = \beta_0\xi^0.$$

Thus we only need to show ξ^0 is in the cone $\mathcal{D}$ to deduce that $Z = \beta_0\xi^0$ is in $\mathcal{D}$.

Claim For all $t \in \{0, 1, \dots, T-1\}$, we have $\xi^t|_{t+1} = Z^t|_{t+1}$ and $\xi^t \in \mathcal{D}$.

We shall proceed by backwards induction, starting from the observation $\xi^{T-1} = Z^{T-1} \in \mathcal{D}$. Suppose that for $s \geq t+1$, we have $\xi^s|_{s+1} = Z^s|_{s+1}$ and $\xi^s \in \mathcal{D}$.

$$\begin{aligned} \xi^t|_{t+1} &= \mathbb{E}\left[\mathbb{1}_{F_t}\kappa_t\xi^{t+1} + \mathbb{1}_{F_t^c}Z^t \,\middle|\, \mathcal{F}_{t+1}\right] \\ &= \mathbb{E}\left[\mathbb{1}_{F_t}\kappa_t Z^{t+1} + \mathbb{1}_{F_t^c}Z^t \,\middle|\, \mathcal{F}_{t+1}\right] \end{aligned}$$

Now, whilst $\beta_t > 0$, i.e. on the event F_t,

$$\mathbb{E}\left[\kappa_t Z^{t+1} \,\middle|\, \mathcal{F}_{t+1}\right] = \frac{1}{\beta_t}\mathbb{E}\left[\beta_{t+1}Z^{t+1} \,\middle|\, \mathcal{F}_{t+1}\right] = \frac{1}{\beta_t}\mathbb{E}\left[Z|_{t+2} \,|\, \mathcal{F}_{t+1}\right] = \frac{Z|_{t+1}}{\beta_t} = Z^t|_{t+1}$$

allowing us to conclude

$$\xi^t|_{t+1} = \mathbb{E}\left[\mathbb{1}_{F_t}Z^t|_{t+1} + \mathbb{1}_{F_t^c}Z^t \,\middle|\, \mathcal{F}_{t+1}\right] = Z^t|_{t+1}.$$

By hypothesis $\mathcal{D}$ is stable, so by the alternative definition of stability (Lemma 2.4.1), we see that $\xi^t \in \mathcal{D}$. $\square$

Proof of Lemma 2.2.18. It is clear that $[\mathcal{D}]$ is a closed convex cone in $\mathcal{L}^1$. To see that $[\mathcal{D}]$ is stable, we use the definition of stability according to Lemma 2.4.1. Fix $t \in \{0, 1, \dots, T\}$, and suppose $Y, W \in [\mathcal{D}]$ are such that there exists $Z \in [\mathcal{D}]$, a set $F \in \mathcal{F}_t$, positive processes $\alpha, \beta \in \mathcal{L}^0(\mathcal{F}_t)$ with $\alpha Y, \beta W \in L^1(\mathbb{R}^{d+1})$ and

$$X := \alpha Y \mathbb{1}_F + \beta W \mathbb{1}_{F^c}$$

satisfies $\mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[Z \mid \mathcal{F}_t]$. We aim to show X is also a member of $[\mathcal{D}]$, that is,

$$X \in \overline{\text{conv}} \mathcal{M}_s(\mathcal{D}) \qquad \forall 0 \leq s \leq T-1.$$

First consider $s \in \{0, 1, \dots, t-1\}$. From the definition of $\mathcal{M}_s(\mathcal{D})$,

$$Z \in \operatorname{conv} \mathcal{M}_s(\mathcal{D}) \quad \text{and} \quad \mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[Z \mid \mathcal{F}_t] \quad \Longrightarrow \quad X \in \operatorname{conv} \mathcal{M}_s(\mathcal{D}),$$

since the membership of an integrable Z in $\mathcal{M}_s(\mathcal{D})$ only depends on its conditional expectation $\mathbb{E}[Z \mid \mathcal{F}_{s+1}]$. More generally, we show

$$Z \in \overline{\text{conv}} \mathcal{M}_s(\mathcal{D}) \quad \text{and} \quad \mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[Z \mid \mathcal{F}_t] \quad \Longrightarrow \quad X \in \overline{\text{conv}} \mathcal{M}_s(\mathcal{D}).$$

Take a sequence $(Z^n) \subset \operatorname{conv} \mathcal{M}_s(\mathcal{D})$ such that $Z^n \to Z$ in $\mathcal{L}^1$. Define the sequence

$$X^n := \mathbb{E}[Z^n \mid \mathcal{F}_t] + X - \mathbb{E}[X \mid \mathcal{F}_t].$$

Note that $X^n \to X$ as $n \to \infty$ and for each n, $\mathbb{E}[X^n \mid \mathcal{F}_t] = \mathbb{E}[Z^n \mid \mathcal{F}_t]$. So $X^n \in \operatorname{conv} \mathcal{M}_s(\mathcal{D})$, thus $X \in \overline{\text{conv}} \mathcal{M}_s(\mathcal{D})$.

Now consider $s \in \{t, t+1, \dots, T-1\}$. We begin by choosing sequences $(Y^n), (W^n) \subset \operatorname{conv} \mathcal{M}_s(\mathcal{D})$ such that $Y^n \to Y$ and $W^n \to W$ in $\mathcal{L}^1$. Define, for $n, K \in \mathbb{N}$,

$$X^{n,K} := \mathbb{1}_{\{\alpha \leq K\}} \alpha Y^n \mathbb{1}_F + \mathbb{1}_{\{\beta \leq K\}} \beta W^n \mathbb{1}_{F^c}.$$

The fact that $X^{n,K} \in \operatorname{conv} \mathcal{M}_s(\mathcal{D})$ follows from the following two elementary properties:

1. if $Z \in \operatorname{conv} \mathcal{M}_s(\mathcal{D})$ and $g \in \mathcal{L}^\infty_+(\mathcal{F}_t)$, then $gZ \in \operatorname{conv} \mathcal{M}_s(\mathcal{D})$;[3] and

[3]Let $Z \in \mathcal{M}_s(\mathcal{D})$. Then

$$\begin{aligned} &\exists \alpha_t \in L^0_{t,+}, \exists Z' \in \mathcal{D} \text{ such that } \alpha_t Z \in \mathcal{L}^1 \text{ and } Z|_{t+1} = \alpha_t Z'|_{t+1} \\ \Longrightarrow\ &\exists \alpha_t g \in L^0_{t,+}, \exists Z' \in \mathcal{D} \text{ such that } \alpha_t g Z \in \mathcal{L}^1 \text{ and } gZ|_{t+1} = \alpha_t g Z'|_{t+1} \end{aligned}$$

2. if $Z^i \in \operatorname{conv} \mathcal{M}_s(\mathcal{D})$ for $i = 1, 2$, then $Z^1 + Z^2 \in \operatorname{conv} \mathcal{M}_s(\mathcal{D})$.

Now, for any K fixed, $\mathbb{1}_{\{\alpha \leq K\}} \alpha Y^n \to \mathbb{1}_{\{\alpha \leq K\}} \alpha Y$ as $n \to \infty$, and similarly $\mathbb{1}_{\{\beta \leq K\}} \beta W^n \to \mathbb{1}_{\{\beta \leq K\}} \beta W$. Since αY and βW are integrable, we now send $K \to \infty$ to see that

$$X = \lim_{K \to \infty} \lim_{n \to \infty} X^{n,K} \in \overline{\operatorname{conv}} \mathcal{M}_s(\mathcal{D})$$

which completes the proof that X is indeed a member of $[\mathcal{D}]$.

To show minimality of $[\mathcal{D}]$ in the class of stable closed convex cones containing $\mathcal{D}$, we note that if $\mathcal{D} \subset \mathcal{D}'$ then $[\mathcal{D}] \subset [\mathcal{D}']$. Taking $\mathcal{D}'$ to be another stable closed convex cone containing $\mathcal{D}$, we have $\mathcal{D}' = [\mathcal{D}']$ by Lemma 2.2.17, and so $\mathcal{D}'$ contains $[\mathcal{D}]$. To show the equivalence in statement (b), the forward implication is due to the stability of $[\mathcal{D}]$, and the reverse is Lemma 2.2.17. □

---

and then take convex hulls.

Chapter 3

Multi-currency reserving

3.1 Introduction

Coherent risk measures (CRMs) were introduced in [Artzner et al., 1999]. A key example was based on the Chicago Mercantile Exchange's margin requirements. The Basel III accords mandate the use of Average Value at Risk (a coherent risk measure unlike the widely-used Value at Risk (VaR)measure, which is not coherent) for reserving risk-capital for certain derivatives-based liabilities [on Banking Supervision, 2013]. Many financial institutions have regulatory or other reasons for testing their reserves and a dynamic version of coherent risk measures is a model for this process.

In the previous chapter we outlined an approach to reserving for risk based on CRM's. The potential drawback of reserving with CRM's, as has been pointed out repeatedly, is the problem of time-consistency (see, for example [Bielecki et al., 2017] and references therein): one can view the time-t reserve for a liability payable at a later time T as itself a liability, payable at time t. A serial version of this shows that (for example) a regulator who imposes the reserving requirements implicit in the CRM is actually requiring a sequence of reserves $\rho_t(X)$ – one at each time-point where reserves are audited– for a liability X, and consequently it can be argued that one actually needs an initial reserve of $\rho_0 \circ \cdots \circ \rho_{T-1}(X)$. Delbaen [Delbaen, 2006a] gave a necessary and sufficient condition, termed multiplicative stability (henceforth m-stability), for this latter quantity to equal $\rho_0(X)$, which does not hold in general, although the inequality

$$\rho_0 \circ \cdots \circ \rho_{T-1}(X) \geq \rho_0(X)$$

does. In particular, Average- or Tail-Value at Risk (also known as Expected Shortfall) is not, in general, time-consistent.

It is normally assumed, in the context of CRM's, that assets and liabilities are discounted to time-0 values. Since CRM's are measures of *monetary* risk for amounts payable at time t, we think it is clearer to take the *prospective view* that liabilities are expressed in terms of time-T units and so at time 0, the risk or reserve is expressed in terms of units of a zero-coupon bond (or currency) payable at T. Of course, as soon as one adopts this approach it is clear that our assets need not just correspond to the unit of account and we should consider the possibility of holding multiple currencies or assets to perform the reserving function. In [Jacka et al., 2019] we showed how multiple currencies allowed the possibility of an extended version of time-consistency: predictable $\mathbf{V}$-time consistency. We envisaged a set of assets numbered $0, 1, \ldots, d$ with random terminal values $\mathbf{V} = (v^0, v^1, \ldots, v^d)$ (given in the distinguished unit of account) and gave a necessary and sufficient condition (Theorem 2.15 of [Jacka et al., 2019]) for time-consistent, multi-asset reserving to work for any specific CRM.

Examples of CRMs include superhedging prices in incomplete frictionless markets and (as we shall see) minimal hedging endowments in markets with proportional transaction costs.

In this paper we consider a stronger version of multi-asset time-consistency which corresponds to explicitly adjusting portfolios (and which therefore seems appropriate to situations where trading of the assets held as reserves is possible) which includes both these situations. We term this version *optional time-consistency*, and see it as the appropriate setting for many situations, including those mentioned above.

We shall give necessary and sufficient conditions for $\mathcal{A}$, the cone of acceptable claims corresponding to a CRM, ρ, to be expressible as the (closure in the appropriate topology of the) sum, over times t, of trades in the underlying assets which are acceptable at time t (Theorem 3.3.13). We will then show that under this condition we obtain a version of the Fundamental Theorem of Asset Pricing for CRMs (Theorem 3.4.5). Finally, in Theorem 3.6.1 we shall show the equivalence between optionally time-consistent CRMs and a generalisation (corresponding to permitting trades in baskets of assets) of the models for trading with proportional transaction costs introduced by Jouini and Kallal [Jouini and Kallal, 1995], developed by Cvitanic and Karatzas [Cvitanić and Karatzas, 1996], Kabanov [Kabanov, 1999], Kabanov and Stricker (see [Kabanov and Stricker, 2001]) and further studied by Schachermayer [Schachermayer, 2004] and Jacka, Berkaoui and Warren [Jacka et al., 2008], amongst others. For more recent developments see Bielecki, Cialenco and Rodriguez [Bielecki et al., 2015] and their survey paper [Bielecki et al., 2017].

3.2 Preliminaries

Insurers reserve for future financial risks by investing in suitably prudent and sufficiently liquid assets, typically bonds, or any other asset universally agreed always to hold positive value. We call such assets numéraires, examples of which include paper assets, such as currencies, and physical commodities. Reserving a sufficient amount ensures that the risk carried by the insurer is acceptable to the insurer and (possibly) to regulatory authorities, customers and their agents. In some circumstances, the choice of numéraire is clear; in others, it is not, for example when insurers reserve for claims in multiple currencies. It is common to calculate reserves by a "prudent" calculation of expected value in a pessimistic or "worst realistic case" scenario. We assume that the minimal amount sufficient to form the reserve is modelled by a coherent risk measure (CRM); see [Föllmer and Schied, 2011] for an introduction to CRM's.

We assume the availability of a finite collection of numéraires numbered $(0, \ldots, d)$. We examine the problem of reserving for a risk at a terminal time T, through adjusting the reserving portfolio held in the numéraire "currencies" at discrete times $t = 0, 1, \ldots, T$. The terminal value of the numéraires (in units of account) is denoted $\mathbf{V} = (v^0, v^1, \ldots, v^d)$, and we assume that each v^i is a strictly positive, $\mathcal{F}_T$-measurable, bounded random variable, with Euclidean norm bounded away from 0. Thus, to value any portfolio Y of holdings in the elements in $\mathbf{V}$, we take the inner product $Y \cdot \mathbf{V}$ and, conversely, any bounded X may be written in the form $Y \cdot \mathbf{V}$ with Y bounded . We regard the portfolio Y as corresponding to a *liability* of $Y \cdot \mathbf{V}$ at terminal time T.

A coherent risk measure is a reserving mechanism: we assume that an insurer is reserving for risk according to a conditional coherent risk measure ρ_t, at each time t. They reserve the amount $\rho_t(X)$ for a random claim X. Thus the aggregate position of holding the risky claim X and reserving adequately should always be acceptable to the insurer. The set $\mathcal{A}_t$ of acceptable claims at time t consists of those $\mathcal{F}_T$-measurable bounded random variables with non-positive ρ_t. We shall say that the portfolio Y_t reserves at time t for a claim X if

$$X - Y_t \cdot \mathbf{V} \in \mathcal{A}_t.$$

3.3 Optional representation and multi-currency time consistency

3.3.1 Time-consistency

An insurer who has insured the claim X needs to hold a sequence of portfolios $Y_0, Y_1, \ldots, Y_T$ (one for each time point at which a reserve calculation is to be made) so that the risk is adequately reserved for, and so that no unacceptable risk is assumed in any one exchange of portfolios. That is to say, in the optional case, from time $t-1$ until just before time t, the insurer holds a portfolio $Y_{t-1} \in \mathcal{L}^\infty(\mathcal{F}_{t-1})$ of the numéraires as an acceptable reserve for X, and will wish to exchange to a new reserving portfolio Y_t. The insurer may only exchange to the new portfolio Y_t if the risk of the adjustment is acceptable, i.e. $\rho_t(Y_t \cdot \mathbf{V} - Y_{t-1} \cdot \mathbf{V}) \leq 0$. Thus all the transfer of risk occurs instantaneously at time t (we shall see in section 3.6 that the analogy with a trading set-up is no coincidence). This is in contrast to the predictable case developed in [Jacka et al., 2019], where the idea is that the time-$(t-1)$ reserve is an adequate reserve for the hedging portfolio needed at time t. In the predictable case, the acceptable risk is carried between the time points $t-1$ and t, whereas in the optional case an explicit exchange of known amounts of the numéraires needs to take place at time t to update the reserve portfolio.

We shall say that the dynamic risk measure is (optionally) $\mathbf{V}$-time-consistent if this property holds (at least in a limiting sense) for each claim X, starting from an initial reserve $\rho_0(X)$.

Definition 3.3.1. A dynamic convex risk measure $\rho = (\rho_t)_{t=0,\ldots,T}$ is *optionally* $\mathbf{V}$-*time-consistent* if, for any $X \in \mathcal{A}$, we may find a sequence X^n in $\mathcal{A}$ and a sequence $\pi^n = (\pi^n_t)_{t=0,\ldots,T-1}$ such that $\pi^n_t \in \mathcal{L}^\infty(\mathcal{F}_t)$ for each t, and

(i)
$$X^n \to X \text{ almost surely} \tag{3.1}$$

(ii) for each t,
$$\rho_t(\pi^n_t \cdot \mathbf{V}) \leq 0 \ \mathbb{P} \text{ a.s.};$$

(iii) for each $n \in \mathbb{N}$,
$$\sum_{t=0}^{T-1} \pi^n_t \cdot \mathbf{V} = X^n \quad \mathbb{P}\text{-almost-surely.}$$

Remark 3.3.2. By the subsequence property, we can replace the almost sure convergence in (3.1) by convergence in L^0 without affecting the definition.

3.3.2 Representability of claims

We may view optional $\mathbf{V}$-time-consistency as a condition on the sequences of portfolios that can superhedge a claim X. Given a $\mathbf{V}$-time-consistent dynamic CRM (ρ_t), we may (at least in a limiting sense) express X as the sum of the initial reserve $\rho_0(X)$ and the $(T+1)$ adjustments at times $0, 1, \dots T$ (where we set Y_{-1} to be any vector in $\mathcal{L}_0$ with $\rho_0(Y \cdot \mathbf{V}) = \rho_0(X)$, for example $\frac{\rho_0(X)}{\rho_0(\mathbf{V}\cdot 1)}\mathbf{1}$):

$$X = \rho_0(X) + \sum_{t=0}^{T}(Y_t - Y_{t-1}) \cdot \mathbf{V},$$

where each adjustment satisfies $\rho_t((Y_t - Y_{t-1}) \cdot \mathbf{V}) \leq 0$. Each adjustment $Y_t - Y_{t-1}$ is an $\mathcal{F}_t$-measurable portfolio with t-acceptable valuation; we call the set of such portfolios $K_t(\mathcal{A}, \mathbf{V})$. We seek to answer the question "Is it possible to represent every claim in $\mathcal{A}$ by a series of such adjustments?"

Given any cone $\mathcal{D}$ in L^∞ and our vector $\mathbf{V}$ of numéraires, we define the collection of portfolios attaining $\mathcal{D}$ to be

$$\mathcal{D}(\mathbf{V}) = \{Y \in \mathcal{L}^\infty : Y \cdot \mathbf{V} \in \mathcal{D}\}.$$

The set of time-t acceptable portfolios that are $\mathcal{F}_t$-measurable is denoted

$$K_t(\mathcal{A}, \mathbf{V}) := \mathcal{A}_t(\mathbf{V}) \cap \mathcal{L}^\infty(\mathcal{F}_t). \tag{3.2}$$

Definition 3.3.3. The cone $\mathcal{A}$ in L^∞ is said to be *optionally* $\mathbf{V}$*-representable* if

$$\mathcal{A}(\mathbf{V}) = \overline{\oplus_{t=0}^{T} K_t(\mathcal{A}, \mathbf{V})}, \tag{3.3}$$

where the closure is taken in the weak* topology. If this is the case, we also say that $\mathcal{A}$ is *optionally represented* by $\mathbf{V}$. When $\mathbf{V}$ is fixed, we also say that $\mathcal{A}(\mathbf{V})$ is optionally represented if (3.3) holds.

Remark 3.3.4. It is an easy exercise to show that

$$K_t(\mathcal{A}, \mathbf{V}) = \{X \in \mathcal{L}^\infty(\mathcal{F}_t, \mathbb{R}^{d+1}) : \alpha X \in \mathcal{A} \text{ for any } \alpha \in L^\infty_+(\mathcal{F}_t)\}. \tag{3.4}$$

This characterisation is used repeatedly in what follows and in the proof of Theorem 3.3.18.

From now on, where there is no ambiguity, we shall write K_t for $K_t(\mathcal{A}, \mathbf{V})$.

3.3.3 Stability

We recall Delbaen's m-stability condition, on a standard stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$:

Definition 3.3.5 (Delbaen [Delbaen, 2006a])**.** A set of probability measures $\mathcal{Q} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ is *m-stable* if for elements $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$, with associated density martingales $\Lambda_t^{\mathbb{Q}^1} = \mathbb{E}\left[\frac{d\mathbb{Q}^1}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$ and $\Lambda_t^{\mathbb{Q}^2} = \mathbb{E}\left[\frac{d\mathbb{Q}^2}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$, and for each stopping time τ, the martingale L defined as

$$L_t = \begin{cases} \Lambda_t^{\mathbb{Q}^1} & \text{for } t \leq \tau \\ \frac{\Lambda_\tau^{\mathbb{Q}^1}}{\Lambda_\tau^{\mathbb{Q}^2}} \Lambda_t^{\mathbb{Q}^2} & \text{for } t \geq \tau \end{cases}$$

defines an element, $\mathbb{Q}$, in $\mathcal{Q}$. The probability measure $\mathbb{Q}$ is also defined by the properties that

$$\mathbb{Q}\big|_{\mathcal{F}_\tau} = \mathbb{Q}^1\big|_{\mathcal{F}_\tau} \text{ and } \mathbb{Q}(\cdot|\mathcal{F}_\tau) = \mathbb{Q}^2(\cdot|\mathcal{F}_\tau) \text{ } \mathbb{P}\text{a.s.},$$

so $\mathbb{Q}$ pastes together the laws $\mathbb{Q}^1$ and $\mathbb{Q}^2$ at time τ.

We generalise m-stability by allowing extra freedom over one time period when pasting two measures together *and* by only pasting measures satisfying a consistency condition relating to $\mathbf{V}$:

Definition 3.3.6. Let τ be a stopping time, and $\mathbb{Q}^1, \mathbb{Q}^2$ be two probability measures absolutely continuous with respect to $\mathbb{P}$. The set $\mathbb{Q}^1 \oplus_\tau^{\text{opt}} \mathbb{Q}^2$ of *optional pastings of* $\mathbb{Q}^1$ *and* $\mathbb{Q}^2$ consists of all $\widetilde{\mathbb{Q}}$ such that

(i) $\tilde{\mathbb{Q}}\big|_{\mathcal{F}_\tau} = \mathbb{Q}^1\big|_{\mathcal{F}_\tau}$,

and

(ii) for any $A \in \mathcal{F}_T$, $\widetilde{\mathbb{Q}}(A|\mathcal{F}_{(\tau+1)\wedge T}) = \mathbb{Q}^2(A|\mathcal{F}_{(\tau+1)\wedge T})$.

We make explicit the freedom over the time period $(\tau, \tau+1]$ by writing any optional pasting in terms of the two measures being pasted, and a "one-step density":

Lemma 3.3.7. *For τ a stopping time, and $\mathbb{Q}^1$, $\mathbb{Q}^2$ two probability measures,*

$$\mathbb{Q}^1 \oplus_\tau^{\text{opt}} \mathbb{Q}^2 = \left\{ \widetilde{\mathbb{Q}} \ll \mathbb{P} : \Lambda^{\widetilde{\mathbb{Q}}} = \Lambda_\tau^{\mathbb{Q}^1} R \frac{\Lambda^{\mathbb{Q}^2}}{\Lambda_{(\tau+1)\wedge T}^{\mathbb{Q}^2}}, \quad \textit{for some } R \in L^1_+(\mathcal{F}_{(\tau+1)\wedge T}) \textit{ s.t. } \mathbb{E}\left[R \,|\, \mathcal{F}_\tau\right] = 1 \right\}$$

Definition 3.3.8. The set of probability measures $\mathcal{Q}$ is *optionally* $\mathbf{V}$*-m-stable* if, whenever τ is a stopping time, $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$, and $\widetilde{\mathbb{Q}} \in \mathbb{Q}^1 \oplus_\tau^{\mathrm{opt}} \mathbb{Q}^2$ has the (additional) property that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathbf{V}|\mathcal{F}_\tau] = \mathbb{E}_{\mathbb{Q}^1}[\mathbf{V}|\mathcal{F}_\tau], \tag{3.5}$$

then $\widetilde{\mathbb{Q}}$ is also in $\mathcal{Q}$.

Example 3.3.9. It is easy to check that given an $(\mathcal{F}_t)_{0\le t\le T}$-adapted and bounded process X, the collection, $\mathcal{Q}_X$, of Equivalent Martingale Measures for X is optionally X_T-m-stable.

Note that a set that is optionally 1-m-stable is automatically m-stable but the converse is false.

The following proposition gives an equivalent definition of optional $\mathbf{V}$-m-stability in terms of the dual cone $\mathcal{A}(\mathbf{V})^*$.

Proposition 3.3.10. *Suppose, without loss of generality, that the set of pricing measures $\mathcal{Q}$ is convex and closed (so the set of densities is closed in the topology of L^1), and let $\mathcal{D} = \mathcal{A}(\mathbf{V})^*$. The following are equivalent:*

(i) $\mathcal{Q}$ is optionally $\mathbf{V}$-m-stable

(ii) for each $t \in \{0, 1, \ldots, T\}$, whenever $Y, W \in \mathcal{D}$ are such that there exists $Z \in \mathcal{D}$, an event $F \in \mathcal{F}_t$, positive random variables $\alpha, \beta \in \mathcal{L}^0_+(\mathcal{F}_{t+1})$ with $\alpha Y, \beta W \in \mathcal{L}^1$ and $X := \mathbb{1}_F \alpha Y + \mathbb{1}_{F^c} \beta W$ satisfies

$$\mathbb{E}\left[X \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[Z \,|\, \mathcal{F}_t\right], \tag{3.6}$$

then X is a member of $\mathcal{D}$.

The proof can be found in Section 3.7.1.

Definition 3.3.11. We shall say that an arbitrary cone $\tilde{\mathcal{D}} \subseteq \mathcal{L}^1_T$ satisfying condition (ii) of Proposition 3.3.10 is *optionally m-stable.*

Lemma 3.3.12. *Suppose that* $\mathbf{V}$ *is a collection of $d+1$ numéraires, and $\mathcal{D}$ is a convex cone in L^∞. Then*

$$\mathcal{D}(\mathbf{V})^* = \mathcal{D}^* \mathbf{V}.$$

The proof can be found in Section 3.7.1.

3.3.4 An Equivalence Theorem

Our first result is a set of conditions equivalent to optional **V**-time-consistency, including a precise statement of **V** representability, and a dual characterisation which pertains to the convex set of probability measures $\mathcal{Q}$ that define the risk measure.

This result resembles that obtained in [Jacka et al., 2019] for *predictable* versions of these concepts.

To show the equivalence of **V**-m-stability and **V**-representability, we find the dual of each K_t, which we call the *optional pre-image* of $\mathcal{A}(\mathbf{V})^*$ at time t. Aside from its utility in proving the equivalence of **V**-optional representability and optional **V**-m-stability, the optional pre-image of an optionally m-stable convex cone $\mathcal{A}(\mathbf{V})^*$ at time t is a concrete description of the dual of the set of portfolios held at time t in order to maintain an acceptable position at time t.

We fix the vector of numéraires **V**, a coherent pricing measure $\rho = (\rho_t)_t$ with a closed, convex representing set of probability measures $\mathcal{Q}$, and take $\mathcal{A}_t$ to be the acceptance set of ρ_t for $t \in \mathbb{T}$. The main result is

Theorem 3.3.13. *The following are equivalent:*

(i) $(\rho_t)_{t\in\mathbb{T}}$ *is optionally* **V***-time-consistent;*

(ii) $\mathcal{A}$ *is optionally represented by* **V***;*

(iii) $\mathcal{Q}$ *is optionally* **V***-m-stable.*

Example 3.3.14. [A generic example] Given a positive $X \in \mathcal{L}^1_T$, and a sequence of random, closed, convex sets $\mathcal{I} := (I_t)_{t=0,\dots,T}$ in $\mathbb{R}^{d+1}$, each measurable with respect to $\mathcal{E}(\mathbb{R}^{d+1}, \mathcal{F}_t)$, the relevant Effros σ-algebra (see Remark 4.2 of [Jacka et al., 2008]), let

$$\mathcal{Q}^X_{\mathcal{I}} := \{\mathbb{Q} \sim \mathbb{P} : \ \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \in I_t \text{ for each } t\},$$

then $\mathcal{Q}^X_{\mathcal{I}}$ is optionally X-m-stable. Note that X need not be in $\mathcal{L}^\infty$.

To recover the case of EMMs, simply take X to be M_T, the terminal value of a positive $\mathbb{P}$-martingale and I_t to be the singleton $\{M_t\}$. Of course, X is not necessarily in $\mathcal{L}^\infty$, but we may rectify this by taking $\mathbf{V} = (v_0, \dots, v_d)$, where $v_i = \frac{M^i_T}{\sum_j M^j_T}$, and letting $\mathcal{Q}$ be defined as the set $\mathcal{Q} := \{\mathbb{Q} : \ \Lambda^{\mathbb{Q}} = \frac{\sum_j M^j_T}{\sum_j \mathbb{E}_{\tilde{\mathbb{Q}}} M^j_T} \Lambda^{\tilde{\mathbb{Q}}} \text{ for some } \tilde{\mathbb{Q}} \in \mathcal{Q}^X_{\mathcal{I}}\}$.

We give the proof of Theorem 3.3.13 in two steps. First, we will show equivalence of (ii) optional **V**- representability and (iii) optional **V**-m-stability. The proof of the equivalence of (i) optional **V**-time-consistency and optional representability is given after we have proved Theorem 3.4.5 – a version of the Fundamental Theorem of Asset Pricing.

Definition 3.3.15. For $\mathcal{D} \subset \mathcal{L}^1_+$, we define, for each time t, the *optional pre-image* of $\mathcal{D}$ by

$$\begin{aligned} \mathcal{M}_t(\mathcal{D}) := \{Z \in \mathcal{L}^1 : &\exists \alpha_t \in L^0_{t,+}, \exists Z' \in \mathcal{D} \\ &\text{such that } \alpha_t Z' \in \mathcal{L}^1(\mathbb{R}^{d+1}) \text{ and } \mathbb{E}\left[Z \,|\, \mathcal{F}_t\right] = \alpha_t \mathbb{E}\left[Z' \,\middle|\, \mathcal{F}_t\right]\}. \end{aligned} \tag{3.7}$$

The optional pre-image of a set $\mathcal{D} \subset \mathcal{L}^1_+$ is key in understanding optionally stable convex cones, as shown in the following two lemmas:

Lemma 3.3.16. *Suppose $\mathcal{D} \subset \mathcal{L}^1_+$. If $\mathcal{D}$ is an optionally stable convex cone, then*

$$\mathcal{D} = \bigcap_{t=0}^{T} \mathcal{M}_t(\mathcal{D}).$$

If $S \subset \mathcal{L}^1$, we denote by the $\overline{\text{conv}}(S)$ the closure in $\mathcal{L}^1$ of the convex hull of S.

Lemma 3.3.17. *Suppose $\mathcal{D} \subset \mathcal{L}^1_+$, and define*

$$[\mathcal{D}] := \bigcap_{t=0}^{T} \left(\overline{\text{conv}} \mathcal{M}_t(\mathcal{D})\right),$$

(where $\mathcal{M}_t(\mathcal{D})$ is as defined in (3.7)). Then

(a) $[\mathcal{D}]$ is the smallest stable closed convex cone in $\mathcal{L}^1$ containing $\mathcal{D}$;

(b) $\mathcal{D} = [\mathcal{D}]$ if and only if $\mathcal{D}$ is a stable closed convex cone in $\mathcal{L}^1$.

We prove both these lemmas in Section 3.7.1. The proof of equivalence of statements (ii) and (iii) of Theorem 3.3.13 is underpinned by the following

Theorem 3.3.18. *For any $t \in \{0, 1, \ldots, T-1\}$,*

$$K_t = (\mathcal{M}_t(\mathcal{A}(\mathbf{V})^*))^*. \tag{3.8}$$

The proof is given in Section 3.7.1.

Thus we characterise each "summand" in the representation (cf. Definition 3.3.3) as the dual of the optional pre-image of the dual of the set of acceptable portfolios in $\mathbf{V}$.

Proof of Theorem 3.3.13, equivalence of (ii) and (iii). By assumption, $\mathcal{A}(\mathbf{V})$ is a weak*-closed convex cone in $\mathcal{L}^\infty(\mathbb{R}^{d+1})$ which is arbitrage-free, so that $\mathcal{A}(\mathbf{V})^{**} = \mathcal{A}(\mathbf{V})$. Recall that $\mathcal{A}(\mathbf{V})$ is optionally representable if

$$\mathcal{A}(\mathbf{V}) = \overline{\oplus_{t=0}^{T} K_t}.$$

Thanks to Proposition 3.3.10, we must show the equivalence of the two conditions

(ii') $\mathcal{A}(\mathbf{V})$ is optionally representable; and

(iii') $\mathcal{A}(\mathbf{V})^*$ is optionally $\mathbf{V} - m$-stable.

(ii') ⇒ (iii'): Assuming $\mathcal{A}(\mathbf{V})$ is optionally representable, it follows from Theorem 3.3.18 that

$$\mathcal{A}(\mathbf{V}) = \overline{\oplus_t K_t} = \overline{\oplus_t \mathcal{M}_t(\mathcal{A}(\mathbf{V})^*)^*}.$$

Taking the dual, we find that

$$\mathcal{A}(\mathbf{V})^* = \cap_t \mathcal{M}_t(\mathcal{A}(\mathbf{V})^*)^{**} = \cap_t \overline{\mathrm{conv}} \mathcal{M}_t(\mathcal{A}(\mathbf{V})^*)$$

where the second equality follows from the Bipolar Theorem. Hence, $\mathcal{A}(\mathbf{V})^* = [\mathcal{A}(\mathbf{V})^*]$, and so by Lemma 3.3.17, $\mathcal{A}(\mathbf{V})^*$ is optionally stable.

(iii') ⇒ (ii'): Assuming $\mathcal{A}(\mathbf{V})$ is a weak*-closed convex cone, note that $\mathcal{A}(\mathbf{V})^*$ is a convex cone closed in $(\mathcal{L}^1, \sigma(\mathcal{L}^1, \mathcal{L}^\infty))$. Assuming further that $\mathcal{A}(\mathbf{V})^*$ is *stable*,

$$\begin{aligned} \mathcal{A}(\mathbf{V})^* &= \cap_t \mathcal{M}_t(\mathcal{A}(\mathbf{V})^*) && \text{by Lemma 3.3.16} \\ &= \cap_t K_t^* && \text{by eq. (3.8).} \end{aligned}$$

Now we may apply the Bipolar Theorem to deduce

$$\mathcal{A}(\mathbf{V}) \equiv \mathcal{A}(\mathbf{V})^{**} = \overline{\oplus_t K_t}$$

and $\mathcal{A}(\mathbf{V})$ is optionally representable, as required.

□

3.4 The Fundamental Theorem of Multi-currency Reserving

As announced in the introduction, we now discuss closure properties in $\mathcal{L}^0$ of the decomposition of a $\mathbf{V}$-optionally representable acceptance set $\mathcal{A}$.

By analogy to the definition in [Schachermayer, 2004], we define a trading cone as follows:

Definition 3.4.1. $C \subset \mathcal{L}^0(\mathbb{R}^{d+1}, \mathcal{F}_t)$ is said to be a *(time-t) trading cone* if C is closed in $\mathcal{L}^0$ and is closed under multiplication by non-negative, bounded, $\mathcal{F}_t$-measurable random variables.

We recall Lemma 4.6 of [Jacka et al., 2008] which we quote here (suitably rephrased) for ease of reference:

Theorem 3.4.2. *Let C be a closed convex cone in $\mathcal{L}^0(\mathcal{F})$, then*

$$C \text{ is stable under multiplication by (scalar) elements of } L^\infty_+(\mathcal{F}) \tag{3.9}$$

if and only if there is a random closed cone M^C such that

$$C = \{X \in \mathcal{L}^0(\mathcal{F}) : \ X \in M^C \ a.s.\}. \tag{3.10}$$

We shall demonstrate that if $\mathcal{A}$ is $\mathbf{V}$-representable then $(K^0_t)_{0 \le t \le T}$, the L^0-closures of the cones K_t, are trading cones, whose sum is closed, and equal to the L^0-closure of $\mathcal{A}(\mathbf{V})$, which is is arbitrage-free.

This is a version of the (First) Fundamental Theorem of Asset Pricing (FTAP).

For the rest of this section closures in L^0 or $\mathcal{L}^0$ will be denoted by a simple overline, whereas weak* closure of a set S will be denoted $\overline{S}^w$ We set $\mathcal{A}^0(\mathbf{V}) := \overline{\mathcal{A}(\mathbf{V})}$, the closure of $\mathcal{A}(\mathbf{V})$ in $\mathcal{L}^0$. Recall that $\mathcal{A}^0(\mathbf{V})$ is arbitrage-free whenever

$$\mathcal{A}^0(\mathbf{V}) \cap \mathcal{L}^0_+ = \{0\},$$

and define the trading cone

$$C_t = \{X \in \mathcal{L}^0_t : cX \in \mathcal{A}^0(\mathbf{V}) \text{ for all } c \in L^\infty_+(\mathcal{F}_t)\}.$$

Note that closure in $\mathcal{L}^0$ of C_t follows immediately from the closure of $\mathcal{A}^0(\mathbf{V})$.

Lemma 3.4.3. *For each t, K^0_t is a trading cone and if $X \in \mathcal{L}^0_t$ then $X \in K^0_t$ iff $X.\mathbb{E}_{\mathbb{Q}}[V|\mathcal{F}_t] \le 0$ for all $\mathbb{Q} \in \mathcal{Q}$.*

Proof. Now if $X \in \mathcal{L}_t^\infty$ then $X \in K_t$ if and only if $\mathbb{E}_{\mathbb{Q}}[X.V|\mathcal{F}_t] = X.\mathbb{E}_{\mathbb{Q}}[V|\mathcal{F}_t] \leq 0$ for all $\mathbb{Q} \in \mathcal{Q}$. It follows that $K_t^0 = \{X \in \mathcal{L}_t^0 : \ X.\mathbb{E}_{\mathbb{Q}}[V|\mathcal{F}_t] \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\}$ and this is obviously a trading cone. □

It follows from Theorem 3.4.2 and Lemma 3.4.3 that

Lemma 3.4.4. *There are random closed cones M_t^C and M_t^K such that*

$$C_t = \{Y \in \mathcal{L}_t^0 : Y \in M_t^C \ a.s.\}$$

$$K_t^0 = \{Y \in \mathcal{L}_t^0 : Y \in M_t^K \ a.s.\}$$

and the polar (in $\mathbb{R}^{d+1}$) of M_t^K is $\overline{cone}(\{\mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_t] : \ \mathbb{Q} \in \mathcal{Q}\})$, the random closed cone generated by $\{\mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_t] : \ \mathbb{Q} \in \mathcal{Q}\}$.

We now give the main theorem of this section:

Theorem 3.4.5. *The set $\mathcal{G} := \oplus_t C_t$ is closed in L^0, arbitrage-free and equals $\mathcal{H} := \oplus_t K_t^0(\mathcal{A}, \mathbf{V})$. Moreover, if $\mathcal{A}$ is $\mathbf{V}$-representable, then their common value is $\mathcal{A}^0(\mathbf{V})$ and then $\mathcal{A}^0$, the closure in L^0 of $\mathcal{A}$ is given by*

$$\mathcal{A} = \mathcal{A}^0(\mathbf{V}).\mathbf{V} = \oplus_t K_t^0(\mathcal{A}, \mathbf{V}).\mathbf{V}. \tag{3.11}$$

Proof. The proof is in three steps. We will show that:

1. $\mathcal{G}$ is closed in $\mathcal{L}^0$.
2. $C_t = K_t^0(\mathcal{A}, \mathbf{V})$ (and $\mathcal{G}$ is arbitrage-free) establishing equality of $\mathcal{G}$ and $\mathcal{H}$.
3. $\mathcal{A}^0(\mathbf{V}) = \mathcal{H}$ if $\mathcal{A}$ is $\mathbf{V}$-representable and $\mathcal{A}^0 = \mathcal{A}^0(\mathbf{V}).\mathbf{V}$.

Proof of 1. We recall Definition 2.6 and Lemma 2.7 (suitably rephrased) from [Jacka et al., 2008]

Definition 3.4.6. Suppose $\mathcal{J}$ is a sum of convex cones in $\mathcal{L}^0$:

$$\mathcal{J} = M_0 + \ldots + M_T.$$

We call elements of $M_0 \times \ldots \times M_T$ whose components almost surely sum to 0, null-strategies (with respect to the decomposition $M_0 + \ldots + M_T$) and denote the set of them by $\mathcal{N}(M_0 \times \ldots \times M_T)$.

For convenience we denote $C_0 \times \ldots \times C_T$ by $\mathscr{C}$.

Lemma 3.4.7. (Lemma 2 in [Kabanov et al., 2003]) *Suppose that*

$$\mathcal{J} = M_0 + \ldots + M_T$$

is a decomposition of $\mathcal{J}$ into trading cones; then $\mathcal{J}$ is closed if $\mathcal{N}(M_0 \times \ldots \times M_T)$ is a vector space and each M_t is closed in $\mathcal{L}^0$.

Since we have already established that each C_t is a trading cone, applying Lemma 3.4.7 to the decomposition of $\mathcal{G}$, we only need to prove that the null strategies $\mathcal{N}(\mathscr{C})$ form a vector space. The argument is standard: since $\mathcal{G}$ is a cone, we need only show that $\xi = (\xi_0, \ldots, \xi_T) \in \mathcal{N}(\mathscr{C})$ implies that $-\xi \in \mathcal{N}(\mathscr{C})$. To do this, given $\xi \in \mathcal{N}(\mathscr{C})$, fix a t and a bounded non-negative $c \in L^0_t$ with a.s. bound b. Then, since ξ is null,

$$-c\xi_t = b\xi_0 + \ldots b\xi_{t-1} + (b-c)\xi_t + b\xi_{t+1} + \ldots + b\xi_T,$$

and each term in the sum is clearly in the relevant C_s and hence in $\mathcal{A}^0$. Since c and t are arbitrary, $-\xi_t \in C_t$ for each t and so $-\xi \in \mathcal{N}(\mathscr{C})$.

It is clear from (3.4) that $K_t \subseteq C_t$ and hence, by closure of C_t that $K^0_t \subseteq C_t$. Thus $\mathcal{H} \subseteq \mathcal{G}$.

Proof of 2. Recall from [Jacka et al., 2008] that consistent price processes for $\mathcal{H}$ are those martingales valued in $(M^K_t)^*$ at each time step. Since $\Lambda^{\mathbb{Q}}_t \mathbf{V}^{\mathbb{Q}}_t$ is such a martingale (for any $\mathbb{Q} \in \mathcal{Q}$), the collection of consistent price processes for the sequence of trading cones $K^0_t(\mathcal{A}, \mathbf{V})$ is non-empty and so, by Theorem 4.11 of [Jacka et al., 2008], $\overline{\mathcal{H}}$ is arbitrage-free.

The consistent price processes for $\oplus_t C_t$ are those martingales valued in $(M^C_t)^*$ at each time step. We now claim that, for each t,

$$(M^C_t)^* = (M^K_t)^*.$$

Once we establish this, equality follows on taking the random polar cones in $\mathbb{R}^{d+1}$.

First, observe that $C_t \supseteq K^0_t(\mathcal{A}, \mathbf{V})$ implies $(M^C_t)^* \subseteq (M^K_t)^*$ almost surely. So, assume that $(M^C_t)^*$ is a strict subset of $(M^K_t)^*$. Then there exists $\mathbb{Q} \in \mathcal{Q}$ such that

$$\mathbb{P}(\mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_t] \notin M^C_t) > 0.$$

For this $\mathbb{Q}$, we form the consistent price process $Z_t = \Lambda^{\mathbb{Q}}_t \mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_t] \in (M^K_t)^*$. Form the frictionless trading cones

$$C_t(Z) := \{X \in L^0_t : X \cdot Z_t \leq 0\}$$

and we have an arbitrage-free and closed cone $\widetilde{\mathcal{A}} = \oplus_t C_t(Z)$ from the FTAP. Clearly $\widetilde{\mathcal{A}}$ contains $\mathcal{A}^0(\mathbf{V})$, and so $C_t(Z)$ is contained in C_t, whence $Z_t \in M_t^C$ a.s., contradicting the assumption of strict inclusion.

Proof of 3. If $\mathcal{A}$ is $\mathbf{V}$-representable then

$$\mathcal{A}^0(\mathbf{V}) = \overline{\overline{(\oplus_t K_t)}^w} = \overline{\oplus_t K_t^0} = \overline{\mathcal{H}}$$

but, as we have already established, $\mathcal{H}$ is closed. Finally, since $\mathcal{A} \supset K_t.\mathbf{V}$ it is clear that $\mathcal{A}^0 \supseteq \mathcal{H}.\mathbf{V}$. Conversely, since $\mathcal{A}^0 = \overline{\mathcal{A}(\mathbf{V}).\mathbf{V}} = \overline{\overline{\oplus K_t}^w.\mathbf{V}} = \overline{\oplus K_t.\mathbf{V}}$, it follows that $\mathcal{A}^0 \subseteq \mathcal{H}.\mathbf{V}$ □

3.5 Completing the Proof of Theorem 3.3.13

Proof of Theorem 3.3.13: the equivalence of (i) and (ii). We shall use the result from Theorem 3.4.5 that if (iii) (and hence (ii)) holds then

$$\mathcal{A}^0(\mathbf{V}) = \oplus_t K_t^0(\mathcal{A}, \mathbf{V})$$

and

$$\mathcal{A}^0 = \oplus_t K_t^0(\mathcal{A}, \mathbf{V}).\mathbf{V}, \tag{3.12}$$

where the superscript 0 represents closure in L^0 or $\mathcal{L}^0$. Now define

condition (iv):

$$\mathcal{A} \subseteq \oplus_t K_t^0(\mathcal{A}, \mathbf{V}). \tag{3.13}$$

Clearly (3.12)⇒(3.13) and hence that (ii)⇒(iv). It is also clear that (i)⇔(iv)

So it just remains to prove that (iv)⇒(ii). Suppose (iv) holds. We shall show that $\mathcal{A} \subseteq \overline{\oplus_t K_t.\mathbf{V}}$ or, equivalently (since $\mathcal{A}$ is a closed convex cone), that

$$(\oplus_t K_t.\mathbf{V})^* \subseteq \mathcal{A}^*. \tag{3.14}$$

Define

$$G_t = (\oplus_{s=t}^T K_s.\mathbf{V})^* \text{ and } B_t := L^\infty \cap \oplus_{s=t}^T K_s^0.\mathbf{V}$$

Now, since $(Ł^\infty \cap \oplus_t K_t^0.\mathbf{V})^* \subseteq \mathcal{A}^*$ we may show (3.14) by proving, by induction that

$$G_t \subseteq B_t^*. \tag{3.15}$$

Clearly (3.15) holds for $t = T$ since $K_T = \{X \in \mathcal{L}^\infty : \ X.\mathbf{V} \leq 0 \text{ a.s. }\}$ and $K_T^0 = \{X \in \mathcal{L}^0 : \ X.\mathbf{V} \leq 0 \text{ a.s.}\}$ so $B_T = L_-^\infty = K_T.\mathbf{V}$.

Now suppose that (3.15) holds for $t = u + 1$. Take arbitrary $Z \in G_u$ and $X \in B_u$. Then $X = \alpha_u.\mathbf{V} + Y$, for some $Y \in \oplus_{s=u+1}^{T} K_s^0.\mathbf{V}$ and $\alpha_u \in K_u^0$. For integer $n > 0$ set $F_n = \{||\alpha_u|| \leq n$, then $\alpha_u \mathbf{1}_{F_n} \in K_u$ and $Y\mathbf{1}_{F_n} = X\mathbf{1}_{F_n} - \alpha_u \mathbf{1}_{F_n} \in L^\infty$ (since $X \in L^\infty$). Since each K_t^0 is closed under multiplication by $\mathbf{1}_{F_n}$ for $t \geq u$, it follows that $Y\mathbf{1}_{F_n} \in B_{u+1}$ with $X\mathbf{1}_{F_n} = \alpha_u.\mathbf{V}\mathbf{1}_{F_n} + Y\mathbf{1}_{F_n}$ and hence $X\mathbf{1}_{F_n} \in B_u$. Since $Z \in G_{u+1}$ it follows from the induction hypothesis that $\mathbb{E}ZX\mathbf{1}_{F_n} = \mathbb{E}Z(Y\mathbf{1}_{F_n} + \alpha_u.\mathbf{V}\mathbf{1}_{F_n}) \leq \mathbb{E}Z(\alpha_u.\mathbf{V}\mathbf{1}_{F_n})$. Now $\alpha_u.\mathbf{V}\mathbf{1}_{F_n} \in K_u.\mathbf{V}$ and $Z \in (K_u.\mathbf{V})^*$ so $\mathbb{E}[ZX\mathbf{1}_{F_n}] \leq 0$. Thus, by dominated convergence, $\mathbb{E}[ZX] \leq 0$ and since X is an arbitrary element of B_u it follows that $Z \in B_u^*$ and so $G_u \subseteq B_u^*$, establishing the inductive step. □

3.6 Associating a pricing mechanism to a market with proportional transaction costs

Having made the connection in Section 3.4 between optionally-representable CRM's and trading cones, in this section, we directly associate the reserving mechanism to a hedging strategy in a market with transaction costs. This is achieved by adding an extra time period $(T, T+1]$ to the market with transaction costs, in which all positions are cashed out into a base numéraire v^0. We do this by imposing numéraire risks that are so disadvantageous as to force a risk-averse agent to sell-up at time T, rather than in the additional period.

Let $e_0 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ denote the canonical basis of $\mathbb{R}^{d+1}$. Recall that in a market with transaction costs the basic set-up has a collection of assets (labelled $0, \ldots, d$) and random bid-ask prices $\pi_t^{i,j}$ at each trading time $t \in \{0, \ldots, T\}$. Thus $\pi_t^{i,j}$ is the number of units of asset i that can be exchanged for one unit of asset j at time t. The corresponding trading cone, which we denote by $\tilde{K}_t^0(\pi_t)$ is generated by these trades together the possibility of consumption so that $\tilde{K}_t(\pi_t)$ is the (closed) cone generated by non-negative $\mathcal{F}_t$-measurable multiples of the vectors $-e_i$ and $e_j - \pi_t^{ij} e_i$, for $i, j \in \{0, 1, \ldots, d\}$. The set of claims available from zero endowment is then

$$\mathcal{B}_T(\pi) = \bigoplus_{t=0}^{T} \tilde{K}_t^0(\pi_t).$$

We (initially) assume that the closure of $\mathcal{B}_T(\pi)$ in $\mathcal{L}^0$ is arbitrage-free. Note that thanks to Theorem 1.2 of [Jacka et al., 2008] we may (and shall) then assume that, by amending the bid-ask prices if necessary, $\mathcal{B}_t(\pi)$ is closed. The proof of this theorem also establishes that the null strategies for the resulting trading cones form a vector space.

We denote the L^0-closure of the set of acceptable claims under a risk measure generated by a collection of absolutely continuous probability measures, $\mathcal{Q}$, by $\mathcal{A}^0_{\mathcal{Q}}$. We will show that each market corresponds to a CRM

Theorem 3.6.1. *For the sequence of transaction cost matrices $(\pi^{ij}_t)_{t=0,1,\ldots,T}$, , there is a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$, a vector of numéraires $\mathbf{V} \in L^\infty(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}; \mathbb{R}^{d+1})$, and a set of optionally $\mathbf{V}$-m-stable probability measures $\mathcal{Q}$ such that the closure (in $\mathcal{L}^0$) of the corresponding set of $\mathcal{F}_T$-measurable attainable claims is the collection of claims attainable by trading in the underlying assets:*

$$\mathcal{B}_T(\pi) = \mathcal{A}^0_{\mathcal{Q}}(\mathbf{V}) \cap L^0(\mathcal{F}_T).$$

The key element in the proof is to add an extra trading period $(T, T+1]$ at the end in which all positions are cashed out into asset 0. However, we impose numéraire risks that are so disadvantageous as to force the agent to sell up in the preceding time period, rather than in the additional period. To generate the final, frictionless prices, we add on a simple "coin spin" for each other asset. We encode the d binary choices (either buy or sell each of the other d numéraires) as $\{0,1\}^d$, and define μ to be the uniform measure on $(\{0,1\}^d, 2^{\{0,1\}^d})$. Thus, we define the augmented sample space $\widetilde{\Omega} := \Omega \times \{0,1\}^d$, and define the product sigma-algebra and measure:

$$\begin{aligned}
\widetilde{\mathcal{F}} &:= \mathcal{F} \otimes 2^{\{0,1\}^d}), \\
\widetilde{\mathbb{P}} &:= \mathbb{P} \otimes \mu.
\end{aligned}$$

We augment the filtration trivially, by setting $\widetilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \{\varnothing, \{0,1\}^d\}$. We employ the obvious embedding of $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ in $L^\infty(\widetilde{\Omega}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}})$; it should be clear from context to which version of L^0 we are referring.

Fix $0 < \varepsilon < 1$ small. Define the $\mathbb{R}^{d+1}$-valued random variable $\tilde{\mathbf{V}} = (\tilde{v}^0, \tilde{v}^1, \ldots, \tilde{v}^d)$, for $\omega \in \Omega$, $\omega' \in \{0,1\}^d$, by

$$\tilde{v}^0(\omega, \omega') = 1, \tag{3.16}$$

$$\tilde{v}^i(\omega, \omega') = (1 - \omega'_i)(1 - \varepsilon)\frac{1}{\pi^{i,0}_T(\omega)} + \omega'_i(1 + \varepsilon)\pi^{0,i}_T(\omega). \tag{3.17}$$

The interpretation of $\tilde{\mathbf{V}}$ is this: arriving at time T at a bid-ask spread

$\left[\frac{1}{\pi_T^{i,0}(\omega)}, \pi_T^{0,i}(\omega)\right]$ for numéraire i in state ω, we spin a coin. If the coin shows heads ($\omega_i' = 1$), the $(T+1)$-price of asset i is slightly higher than the T-ask price, and any negative holding of i to time $T+1$ makes a loss compared to cashing out at time T. If the coin shows tails ($\omega_i' = 0$), the $(T+1)$-price of asset i is slightly lower than the T-bid price, and any positive holding of i makes a loss. Any risk-averse agent will seek to avoid these losses by cashing out into asset 0 at time T.

Now we define the frictionless bid-ask matrix at time $T+1$ by

$$\pi_{T+1}^{ij} := \frac{\tilde{v}^j}{\tilde{v}^i}.$$

The trading cone $\tilde{K}_{T+1}^0(\pi_{T+1})$ is generated by positive $\mathcal{F}_{T+1}$-measurable multiples of the vectors $-e_i$ and $e_j - \pi_{T+1}^{ij} e_i$, for $i, j \in \{0, 1, \ldots, d\}$. Define the cone $\mathcal{B}_{T+1}(\pi) = \mathcal{B}_T(\pi) + \tilde{K}_{T+1}^0(\pi_{T+1})$.

The collection of consistent price processes for the original set of claims $\mathcal{B}_T(\pi)$ is

$$\mathcal{B}_T^\circ(\pi) = \{Z \in L_T^1(\mathbb{R}^{d+1}) : \mathbb{E}[Z|\mathcal{F}_t] \in \tilde{K}_t^0(\pi_t)^* \text{ a.s. for } t = 0, 1, \ldots, T\}.$$

By Theorem 4.11 of [Jacka et al., 2008], since $\mathcal{B}_T(\pi)$ is closed and has no arbitrage, there exists at least one consistent price process Z for $\mathcal{B}_T(\pi)$. The following proposition shows that the cone $\mathcal{B}_{T+1}(\pi)$ is arbitrage-free.

Proposition 3.6.2. *There is a consistent price process for $\mathcal{B}_{T+1}(\pi)$.*

Proof. We extend any consistent price process for $\mathcal{B}_T(\pi)$ to a consistent price process for $\mathcal{B}_{T+1}(\pi)$ by multiplying by the Radon-Nikodym derivative for the martingale measure for each coin spin. For any $Z \in \mathcal{B}_T^\circ$, define $\lambda^Z > 0$ such that the one-period process $\left(Z^i/Z^0,\, \lambda^Z \tilde{v}^i\right)$ is a $\widetilde{\mathbb{P}}$-martingale for each i. Then

$$Z_{T+1} = Z^0 \lambda^Z \tilde{\mathbf{V}} \tag{3.18}$$

defines a consistent price process for the cone $\mathcal{B}_{T+1}(\pi)$.

We first show that such a λ^Z always exists. Note that $Z \in \tilde{K}_T^0(\pi_T)^*$ gives that, ω-a.e.,

$$\frac{Z_T^j(\omega)}{Z_T^i(\omega)} \leq \pi_T^{ij}(\omega) \leq \pi_T^{i,0}(\omega)\pi_T^{0,j}(\omega) < \frac{1+\varepsilon}{1-\varepsilon}\pi_T^{i,0}(\omega)\pi_T^{0,j}(\omega) = \frac{\tilde{v}^j(\omega, 1)}{\tilde{v}^i(\omega, 0)},$$

with $\tilde{v}^j(\omega, 1)$ understood to be $\tilde{v}^j(\omega, \omega')\big|_{\omega_j'=1}$ etc. Fixing $\omega \in \Omega$ and $i \neq 0$, we see that

$$\overline{Z}^i_T(\omega) := \frac{Z^i_T(\omega)}{Z^0_T(\omega)} \in (\tilde{v}^i(\omega,0), \tilde{v}^i(\omega,1)).$$

The martingale measure for such an one-period binary tree model is determined by the probability of "heads"

$$\theta(\omega,i) = \frac{\overline{Z}^i_T - \tilde{v}^i(\omega,0)}{\tilde{v}^i(\omega,1) - v^i(\omega,0)}.$$

Now set

$$\lambda^Z(\omega,\omega') = 2^d \prod_{i=1}^{d} \theta(\omega,i)^{\omega'_i}(1-\theta(\omega,i))^{1-\omega'_i}$$

Clearly, λ^Z is a.s. positive and bounded, $\tilde{\mathbb{E}}[\lambda^Z|\mathcal{F}_T] = 1$ and $Z^i_{T+1} \in \mathcal{L}^1$ since $\tilde{\mathbb{E}}[Z^i_{T+1}] = \tilde{\mathbb{E}}[\lambda^Z \tilde{\mathbf{V}}_i Z^0_T] = \mathbb{E}[Z^i_T]$ by the the positivity of Z^i_{T+1}, Fubini's Theorem and the definition of μ and λ^Z

Similarly, for any $X_T \in L^\infty_T(\mathbb{R}^{d+1})$,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}[X_T \cdot Z_{T+1}] = \mathbb{E}_{\mathbb{P}}[X_T \cdot Z^0_T \mathbb{E}_{\widetilde{\mathbb{P}}}[(\lambda^Z \tilde{\mathbf{V}})|\mathcal{F}_T]] = \mathbb{E}_{\mathbb{P}}[X_T \cdot Z_T]. \tag{3.19}$$

Setting $X_T = \mathbb{1}_A e_i$ for $A \in \mathcal{F}_T$, for any i, we see that $Z_T = \mathbb{E}_{\widetilde{\mathbb{P}}}[Z_{T+1}|\mathcal{F}_T]$, and Z_{T+1} is thus a consistent price process as required. □

Proposition 3.6.3. *The cone* $\mathcal{B}_{T+1}(\pi) := \oplus_{t=0}^{T+1} \tilde{K}^0(\pi_t)$ *is closed in* L^0*, and is arbitrage-free.*

Proof. From Theorem 4.11 of [Jacka et al., 2008], we have that the closure of $\mathcal{B}_{T+1}(\pi)$ in L^0 is arbitrage-free. We will show that the set of null strategies

$$\mathcal{N}(\tilde{K}^0_0(\pi_0), \ldots, \tilde{K}^0_T(\pi_T), \tilde{K}^0_{T+1}(\pi_{T+1}))$$

is a vector space, and conclude from Lemma 3.4.6 that the cone $\mathcal{B}_{T+1}(\pi)$ is closed in L^0, and we are done.

Take

$$(x_0, \ldots, x_{T+1}) \in \mathcal{N}(\tilde{K}^0_0(\pi_0), \ldots, \tilde{K}^0_T(\pi_T), \tilde{K}^0_{T+1}(\pi_{T+1})).$$

Let

$$x = x_0 + \cdots + x_T, \text{ so that } x + x_{T+1} = 0. \tag{3.20}$$

We see that x_{T+1} is an $\mathcal{F}_T$-measurable element of $\tilde{K}^0_{T+1}(\pi_{T+1})$. We claim that $x_{T+1} \in \mathcal{B}_T(\pi)$: since $x_{T+1} \in \mathcal{B}_{T+1}$, for any consistent price process Z for the cone

$\mathcal{B}_{T+1}(\pi)$, and any $n \in \mathbb{N}$,

$$0 \geq \mathbb{E}[Z_{T+1} x_{T+1} \mathbb{1}_{\{||x_{T+1}||<n\}}] = \mathbb{E}[Z_T x_{T+1} \mathbb{1}_{\{||x_{T+1}||<n\}}],$$

so $x_{T+1} \mathbb{1}_{\{||x_{T+1}||<n\}} \in \mathcal{B}_T(\pi)$ for all n, and so $x_{T+1} \in \mathcal{B}_T(\pi)$ by closure of $\mathcal{B}_T(\pi)$.

Since $x_{T+1} \in \mathcal{B}_T(\pi)$, there exist $y_0 \in \tilde{K}_0^0(\pi_0), \ldots, y_T \in \tilde{K}_T^0(\pi_T)$ with $x_{T+1} = y_0 + \cdots + y_T$. Then, rewriting (3.20), we see that

$$(x_0 + y_0) + \ldots + (x_T + y_T) = 0.$$

Since each term in this sum is in the relevant trading cone, we see that $(x_0 + y_0), \ldots, (x_T + y_T))$ is in $\mathcal{N}(\tilde{K}_0^0, \ldots, \tilde{K}^0)_T$. Now, by assumption, this is a vector space so that each $-(x_t + y_t) \in \tilde{K}_t^0(\pi_t)$, and so, since $\tilde{K}_t^0(\pi_t)$ is a cone containing y_t, each $-x_t$ is in $\tilde{K}_t^0(\pi_t)$.

The bid-ask prices are frictionless at time $T+1$, so $x_{T+1} \in \tilde{K}_{T+1}^0(\pi_{T+1})$ may be written as $u_1 - u_2$, where $u_1 \in lin(\tilde{K}_{T+1}^0(\pi_{T+1}))$, and $u_2 \geq 0$. Note that

$$0 \leq u_2 = u_1 - x_{T+1} = u_1 + x \in \mathcal{B}_T(\pi),$$

but since $\mathcal{B}_T(\pi)$ is arbitrage-free, $u_2 = 0$, and so $-x_{T+1} \in \tilde{K}_{T+1}^0(\pi_{T+1})$ and thus the set of null strategies is a vector space. $\square$

The final prices $\tilde{\mathbf{V}}$ above are, in general unbounded, so we transform these by normalising, setting

$$\mathbf{V} = (v_0, \ldots, v_d) \text{ where } v_i := \frac{\tilde{v}_i}{\sum_j \tilde{v}_j}.$$

Finally, we define the set of measures

$$\mathcal{Q} = \{\mathbb{Q}^Z : \ Z \text{ is a consistent price process for } \mathcal{B}_T(\pi)\}, \text{ where } \frac{d\mathbb{Q}^Z}{d\tilde{\mathbb{P}}} := \frac{Z_T^0 \lambda^{Z_T}(\tilde{v}_0)}{\sum_j Z_0^j}.$$

It is easy to check that these are probability measures from the fact that the Z's are consistent price processes and hence strictly positive, vector-valued martingales.

The proof of the main result is now clear:

Proof of Theorem 3.6.1. We observe that

$$\begin{aligned} K_t^0(\mathcal{A}_{\mathcal{Q}}(\mathbf{V})) &= \{X \in \mathcal{L}_t^0 : \ X.\mathbb{E}_{\mathbb{Q}}[\mathbf{V}|\mathcal{F}_t] \leq 0 \text{ a.s. for all } \mathbb{Q} \in \mathcal{Q}\} \\ &= \{X \in \mathcal{L}_t^0 : \ X.Z_t \leq 0 \text{ a.s. for all consistent } Z\} \end{aligned}$$

It follows that

$$\mathcal{K}_t^0(\pi_t) = \{Y \in \mathcal{L}_t^0 : \ Y.Z_t \leq 0 \text{ a.s. for all consistent } Z\} = \{Y \in \mathcal{L}_t^0 : \ Y \in K_t^0(\mathcal{Q})\}$$

and so

$$\mathcal{B}_T = \mathcal{A}_{\mathcal{Q}}^0$$

□

3.7 Appendix

3.7.1 Proofs of subsidiary results

Proof of Proposition 3.3.10. By Lemma 3.3.12, we may write $\mathcal{D} = \text{cone}\{\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{V} : \mathbb{Q} \in \mathcal{Q}\}$.

(i) $\Longrightarrow$ (ii): We suppose that (i) holds, and fix $t \in \{0, 1, \ldots, T\}$, $Y, W, Z \in \mathcal{D}$, $F \in \mathcal{F}_t$, $\alpha, \beta \in \mathcal{L}_+^0(\mathcal{F}_{t+1})$ and $X \in \mathcal{L}_+^1$ as in the hypothesis of (ii). We show that $X \in \mathcal{D}$ by applying (i) twice. First, take $\tau = t\mathbb{1}_F + T\mathbb{1}_{F^c}$, and define $\mathbb{Q}^Z$ and Λ_t^Z via

$$\frac{d\mathbb{Q}^Z}{d\mathbb{P}} = \frac{Z^0}{\mathbb{E}[Z^0]} \quad \text{and} \quad \Lambda_t^Z = \mathbb{E}\left[\frac{d\mathbb{Q}^Z}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$$

and analogously for $\mathbb{Q}^Y$, Λ^Y. Note that $Z = \mathbb{E}[Z^0]\Lambda^Z\mathbf{V}$. We now form an optional pasting of $\mathbb{Q}^Z$ and $\mathbb{Q}^Y$ at time τ, as $\widehat{\mathbb{Q}}$, via

$$\widehat{\Lambda} = \Lambda_t^Z \left(\frac{\alpha\mathbb{E}[Y^0]\Lambda_{t+1}^Y}{\mathbb{E}[Z^0]\Lambda_t^Z}\right) \frac{\Lambda^Y}{\Lambda_{t+1}^Y}\mathbb{1}_F + \Lambda^Z\mathbb{1}_{F^c}.$$

This is an optional pasting, thanks to eq. (3.6): on F, we have $\mathbb{E}\left[\alpha Y^0 \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[Z^0 \,\middle|\, \mathcal{F}_t\right]$, and so the factor in parentheses has conditional $\mathcal{F}_t$-expectation of 1 on F. We shall apply (i) to deduce that $\widehat{\mathbb{Q}} \in \mathcal{Q}$, and for this we must show that

$$\mathbb{E}_{\widehat{\mathbb{Q}}}[\mathbf{V}|\mathcal{F}_\tau] = \mathbb{E}_{\mathbb{Q}^Z}[\mathbf{V}|\mathcal{F}_\tau].$$

We compute the left hand side to be

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}[\mathbf{V}|\mathcal{F}_\tau] &= \frac{1}{\widehat{\Lambda}_t}\mathbb{E}\left[\widehat{\Lambda}\mathbf{V} \,\middle|\, \mathcal{F}_t\right]\mathbb{1}_F + \mathbf{V}\mathbb{1}_{F^c} \\
&= \mathbb{E}\left[\left(\frac{\alpha\mathbb{E}[Y^0]\Lambda^Y_{t+1}}{\mathbb{E}[Z^0]\Lambda^Z_t}\right)\frac{\Lambda^Y}{\Lambda^Y_{t+1}}\mathbf{V} \,\middle|\, \mathcal{F}_t\right]\mathbb{1}_F + \mathbf{V}\mathbb{1}_{F^c} \\
&= \frac{1}{\mathbb{E}\left[Z^0 \,|\, \mathcal{F}_t\right]}\mathbb{E}\left[\alpha Y \,|\, \mathcal{F}_t\right]\mathbb{1}_F + \mathbf{V}\mathbb{1}_{F^c}
\end{aligned}$$

Condition (3.6) shows that, on F, $\mathbb{E}\left[\alpha Y \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[Z \,|\, \mathcal{F}_t\right]$, so we conclude that $\widehat{\mathbb{Q}} \in \mathcal{Q}$. We repeat the above steps for stopping time $\sigma = T\mathbb{1}_F + t\mathbb{1}_{F^c}$, measures $\widehat{\mathbb{Q}}$ and $\mathbb{Q}^W$,

$$\widetilde{\Lambda} = \widehat{\Lambda}\mathbb{1}_F + \widehat{\Lambda}_t\left(\frac{\beta\mathbb{E}[W^0]\Lambda^W_{t+1}}{\mathbb{E}[Z^0]\Lambda^Z_t}\right)\frac{\Lambda^W}{\Lambda^W_{t+1}}\mathbb{1}_{F^c}.$$

Condition (3.6) gives that $\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathbf{V}|\mathcal{F}_\tau] = \mathbb{E}_{\widehat{\mathbb{Q}}}[\mathbf{V}|\mathcal{F}_\tau]$, and so $\widetilde{\mathbb{Q}} \in \mathcal{Q}$ by (i). It is simple to show that $X = \mathbb{E}[Z^0]\widetilde{\Lambda}\mathbf{V}$, and thus $X \in \mathcal{D}$ as required.

(ii) $\Longrightarrow$ (i): Say (ii) holds; then (i) holds for when $\tau = T$ trivially. Now suppose that (i) holds for any stopping time $\tau \geq k+1$ a.s., and proceed by backward induction on the lower bound of the stopping times. Fix an arbitrary stopping time $\widetilde{\tau} \geq k$ a.s., and define $F = \{\widetilde{\tau} \geq k+1\}$ and the stopping time $\tau^* := \widetilde{\tau}\mathbb{1}_F + T\mathbb{1}_{F^c}$. Note that $\tau^* \geq k+1$, since $F^c = \{\widetilde{\tau} = k\}$.

We shall now take $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ and $\widetilde{\mathbb{Q}} \in \mathbb{Q}^1 \oplus^{\text{opt}}_{\widetilde{\tau}} \mathbb{Q}^2$ that satisfy eq. (3.5), and aim to show that $\widetilde{\mathbb{Q}}$ is indeed an element of $\mathcal{Q}$, with the help of condition (ii). Define $\Lambda^i = d\mathbb{Q}^i/d\mathbb{P}$ for $i = 1, 2$. Take a pasting of $\mathbb{Q}^1$ and $\mathbb{Q}^2$ at time τ^*, $\mathbb{Q}^* \in \mathbb{Q}^1 \oplus^{\text{opt}}_{\tau^*} \mathbb{Q}^2$, with Radon-Nikodym derivative

$$\Lambda^* = \Lambda^1_{\tau^*}R^*\frac{\Lambda^2}{\Lambda^2_{(\tau^*+1)\wedge T}}$$

with $R^* \in \mathcal{L}^1_+(\mathcal{F}_{(\tau^*+1)\wedge T})$ and $\mathbb{E}\left[R^* \,|\, \mathcal{F}_{\tau^*}\right] = 1$. We note that $\widetilde{\Lambda} := d\widetilde{\mathbb{Q}}/d\mathbb{P}$ can be written as

$$\widetilde{\Lambda} = \Lambda^1_{\widetilde{\tau}}\widetilde{R}\frac{\Lambda^2}{\Lambda^2_{(\widetilde{\tau}+1)\wedge T}} = \Lambda^1_{\tau^*}\widetilde{R}\frac{\Lambda^2}{\Lambda^2_{(\tau^*+1)\wedge T}}\mathbb{1}_F + \Lambda^1\mathbb{1}_{F^c}.$$

Set $X = \widetilde{\Lambda}\mathbf{V}$, $W = Z = \Lambda^1\mathbf{V}$, $Y = \Lambda^2$, $\alpha = \widetilde{R}/R^*$, $\beta = 1$ to satisfy the hypothesis of (ii). Thus, $X \in \mathcal{D}$, whence $\widetilde{\mathbb{Q}} \in \mathcal{Q}$. This completes the inductive step.

□

Proof of Theorem 3.3.18. We set $\mathcal{B} = \mathcal{A}(\mathbf{V})$, as above.

First we prove that $\mathcal{M}_t(\mathcal{B}^*) \subset K_t(\mathcal{B})^*$. For arbitrary $Z \in \mathcal{M}_t(\mathcal{B}^*)$, there exist $Z' \in \mathcal{B}^*$ and $\alpha \in L^0_+(\mathcal{F}_t)$ with $\alpha Z' \in \mathcal{L}^1$ and $Z|_t = \alpha Z'|_t$.

Note that, for any $X \in K_t(\mathcal{B})$,

$$\mathbb{E}[Z \cdot X] = \mathbb{E}[Z|_t \cdot X] = \mathbb{E}[\alpha Z'|_t \cdot X] = \lim_{n\to\infty} \mathbb{E}[(\alpha \mathbb{1}_{\{\alpha \le n\}} X) \cdot Z'|_t] \le 0,$$

since $\alpha \mathbb{1}_{\{\alpha \le n\}} X \in \mathcal{B}$ and $Z' \in \mathcal{B}^*$. Hence $Z \in K_t(\mathcal{B})$, and since Z is arbitrary, we have shown that $\mathcal{M}_t(\mathcal{B}^*) \subset K_t(\mathcal{B})^*$.

For the reverse inclusion, $\mathcal{M}_t(\mathcal{B}^*)^* \subset K_t(\mathcal{B})$, note that $\mathcal{B}^* \subset \mathcal{M}_t(\mathcal{B}^*)$ implies $\mathcal{M}_t(\mathcal{B}^*)^* \subset \mathcal{B}$, and

$$\begin{aligned} \mathcal{L}^\infty_+(\mathcal{F}_t)\mathcal{M}_s(\mathcal{D}) = \mathcal{M}_s(\mathcal{D}) \quad &\Longrightarrow \quad \text{for } X \in \mathcal{M}_t(\mathcal{B}^*)^*, \quad g \in \mathcal{L}^\infty_+(\mathcal{F}_t), \quad \mathbb{E}[X \cdot gZ] \le 0 \\ &\Longrightarrow \quad \mathcal{L}^\infty_+(\mathcal{F}_t)\mathcal{M}_t(\mathcal{B}^*)^* = \mathcal{M}_t(\mathcal{B}^*)^*. \end{aligned}$$

Define

$$\mathcal{B}_t := \{X \in \mathcal{L}^\infty(\mathcal{F}_T, \mathbb{R}^{d+1}) : gX \in \mathcal{B} \text{ for any } g \in L^\infty_+(\mathcal{F}_t)\}.$$

Thus $\mathcal{M}_t(\mathcal{B}^*)^* \subseteq \mathcal{B}_t$. To finish the proof, we need only show that $X \in \mathcal{M}_t(\mathcal{B}^*)^*$ is $\mathcal{F}_t$-measurable, since $\mathcal{B}_t \cap \mathcal{L}^\infty(\mathcal{F}_t, \mathbb{R}^{d+1}) = K_t(\mathcal{B})$.

To this end, note that for any $Z \in \mathcal{L}^1(\mathbb{R}^{d+1})$, it is true that $Z - Z|_t \in \mathcal{M}_t(\mathcal{B}^*)$, whence $\mathbb{E}[(Z - Z|_t) \cdot X] \le 0$. We deduce that

$$\mathbb{E}[(Z - Z|_t) \cdot X] = \mathbb{E}[(X - X|_t) \cdot Z] \le 0 \qquad \forall Z \in \mathcal{L}^1(\mathbb{R}^{d+1}),$$

and $X = X|_t$ $\mathbb{P}$-a.s. $\square$

Chapter 4

Trading to acceptability

In a single time period, we aim to assign a value at time zero to the risk entailed by holding contingent claim X due at time 1. A coherent risk measure assigns to each claim X a (possibly infinite) number, and is monotone, translation invariant, convex and positively homogeneous.

It may be argued that the requirement of positive homogeneity is not natural: risk may grow in a non-linear fashion as the size of a claim increases; an investor might invest \$10 aggressively, and \$10 million in a more cautious manner. Removing the requirement of positive homogeneity, we have the notion of a *convex risk measure*, as studied by Föllmer and Schied, and independently by Frittelli and Rosazza Gianin. We represent a convex risk measure as

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{M}_{1,f}} \left(\mathbb{E}_{\mathbb{Q}}[-X] - \alpha_{\min}(\mathbb{Q}) \right)$$

for $\mathcal{M}_{1,f}$ the set of all finitely additive set functions $\mathbb{Q} : \mathcal{F} \to [0,1]$ which are normalised to $\mathbb{Q}[\Omega] = 1$, and $\alpha_{\min}$ a *penalty function*.

Such risk measures induce a set of claims X that are acceptable: the set of claims X for which no additional capital is required to take on claim X, i.e. $\rho(X) \leq 0$. Alternatively we may regard the set of acceptable claims as a fundamental object, and define a risk measure based on that set. Carr, Geman and Madan [Carr et al., 2001] proposed that a claim should be acceptable whenever every reasonable person would agree that the potential gains from claim X adequately compensate for the potential losses. Formally, we have a finite collection of probability measures, one for each reasonable person, and to each measure we associate a 'floor'. A claim X is acceptable if and only if the expected gain under each measure exceeds the associated floor. The authors also extend the notion of no-arbitrage pricing.

Larsen et al. [Larsen et al., 2005] consider a similar situation, fixing these scenario (probabillity) measures and floors, and looking at trading in continuous time against a semimartingale price process to a position of acceptable wealth at the terminal time. The authors characterise the set of time-t acceptable positions, showing a result in the same vein as the first fundamental theorem of asset pricing: if one cannot form a martingale measure as a convex combination of scenario measures (with some scaling), then we may trade to acceptability from *any* initial wealth.

In section 2 we describe an extension of the ideas in Larsen et al. [Larsen et al., 2005] and Pınar [Pınar, 2011], and show a solution to the problem in one time period.

4.1 Trading to acceptability

We summarise the key results of Larsen et al. [Larsen et al., 2005]. The market consists of one risk-free asset with zero interest rate, and one risky asset S modelled as an L^2-integrable special local semimartingale, defined on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$, with time horizon T. The investor chooses an adapted trading strategy π in the Hardy space $\mathcal{H}^2(S)$ such that the stochastic integral $(\pi \cdot S)_t$ is also L^2-integrable:

$$\pi \in \mathcal{H}^2(S) \quad \Longrightarrow \quad \|\pi\|^2_{\mathcal{H}^2(S)} := \|\pi \cdot S\|^2_{\mathcal{H}^2} := \mathbb{E}\left(\left[\int_s^{\cdot} \pi_u \, dS_u\right]_T\right) < \infty.$$

Acceptability. We have a finite collection $\mathbb{P}^i$, $i = 1, \ldots, d$ of scenario measures, each absolutely continuous w.r.t. $\mathbb{P}$, and each associated with a floor $f^i \in \mathbb{R}$. We assume that each change of measure martingale Z^i, defined by

$$Z^i_T = \frac{d\mathbb{P}^i}{d\mathbb{P}}, \quad Z^i_t = \mathbb{E}[Z^i_T|\mathcal{F}_t] \quad \text{for} \quad 0 \leq t \leq T$$

is in L^2. Define the convex hull of the scenario measures to be $\mathcal{Q} = \text{conv}\big\{\mathbb{P}^i : i = 1, \ldots, d\big\}$. Any time-$T$ wealth $X \in L^2(\mathcal{F}_T)$ is acceptable if it falls in the set

$$\mathcal{G}_T = \left\{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) : \forall \mathbb{Q} \in \mathcal{Q}, \quad \mathbb{E}_{\mathbb{Q}}X \geq f^{\mathbb{Q}}\right\}$$

where

$$f^{\mathbb{Q}} := \sup\left\{\sum_{i=1}^d \alpha^i f^i : \sum_{i=1}^d \alpha^i \mathbb{P}^i = \mathbb{Q}, \quad 0 \leq \alpha^i \leq 1, \quad \sum_{i=1}^d \alpha^i = 1\right\}.$$

Now, any time-t wealth $X \in L^2(\mathcal{F}_t)$ for $t \leq T$ is (time-t) acceptable whenever it falls in the set

$$\mathcal{G}_t = \left\{ X \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}) : \exists \pi \text{ s.t. } X + \int_t^T \pi_u \, dS_u \in \mathcal{G}_T \right\}.$$

Theorem 4.1.1 (Larsen et al. [Larsen et al., 2005])**.** *For every $t \in [0, T]$, we have $\mathcal{G}_t = \mathcal{X}_t$, where*

$$\mathcal{X}_t := \{X \in L^2(\mathcal{F}_t, \mathbb{P}) : \forall \mathbb{Q} \in C_t, \quad \mathbb{E}_{\mathbb{Q}} X \geq f^{\mathbb{Q}}\},$$

for C_t the set of probability measures

$$C_t := \{\mathbb{Q} \in \mathcal{Q} : \mathbb{Q} \ll \mathbb{P}, \quad \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathcal{F}_T), \quad S \textit{ is a loc. mart. under } \mathbb{Q} \textit{ on } [t, T]\}.$$

4.1.1 Gain–loss based convex risk limits in discrete-time trading

Pınar [Pınar, 2011] considered the following development of the above problem. Assume that prices are supported on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_i : i = 1, \ldots, N\}$. The market evolves as a scenario tree, with root node 0, and leaf nodes $n \in \mathcal{N}_T$ which correspond one-to-one with $\omega \in \Omega$.

At every time $t \in \{1, \ldots, T-1\}$, at node $n \in \mathcal{N}_t$ we choose a strategy π_t^n. X is $\mathcal{F}_s$-measurable. Define $\Delta_t = S_{t+1} - S_t$. Wealth evolves as

$$W_t^{s,X,\pi} = X + \sum_{s}^{T-1} \pi_t \Delta_t$$

For the element $\omega \in \Omega$ corresponding to leaf node $n \in \mathcal{N}_T$, we have the stress measures defined as $\mathbb{P}^i(\omega) = p_n^i$, and recursively

$$p_n^i = \sum_{k \in C(n)} p_k^i$$

We redefine the notion of time-s acceptability to the (stronger) condition that there is a strategy π such that

$$\mathbb{E}_{\mathbb{P}^i}\left[\left(X+\sum_{u=s}^{T-1}\pi_t(S_{t+1}-S_t)\right)^+ \right.$$
$$\left. -\lambda\left(X+\sum_{u=s}^{T-1}\pi_t(S_{t+1}-S_t)\right)^-\right]\geq f^i \qquad \forall i=1,\dots,d.$$

Theorem 4.1.2. *For every $t\in[0,T]$, we have $\mathcal{G}_t=\mathcal{X}_t$, where*

$$\mathcal{X}_t:=\{X\in\mathcal{F}_t:\forall(\mathbb{Q},\mathbb{Y})\in C_t,\quad \mathbb{E}_{\mathbb{Y}}X\geq f^{\mathbb{Q}}\},$$

for C_t the set of pairs of measures

$$C_t:=\{(\mathbb{Q},\mathbb{Y}):\mathbb{Q}\in\mathcal{Q},\quad \frac{d\mathbb{Y}}{d\mathbb{Q}}\in[1,\lambda],\quad (S_u)_{t\leq u\leq T} \text{ is a martingale under } \tfrac{1}{\mathbb{Y}(\Omega)}\mathbb{Y}\}.$$

We point out that Pınar provides a proof via the duality theory of linear programming. In the next section, we aim to extend this setting somewhat.

4.1.2 Our model

Fix a finite time horizon T. We consider a market consisting of a bond with zero risk-free interest rate, and one risky stock $S=(S_t)_{0\leq t\leq T}$. On filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},\mathbb{P})$ we model S as an $L^\infty(\Omega,\mathcal{F},\mathbb{P})$-bounded martingale[1]. As usual, $\mathcal{F}_0$ is trivial and contains all $\mathbb{P}$-null sets in $\mathcal{F}$, and the filtration is right-continuous; furthermore assume $\mathcal{F}_T=\mathcal{F}$. We assume S is adapted to filtration $(\mathcal{F}_t)$, and in particular S_0 is almost surely constant.

We assume that the investor may trade between the stock and bond frictionlessly, so that the wealth process for initial endowment $X\in L^\infty(\Omega,\mathcal{F}_s,\mathbb{P})$ and adapted, $\mathbb{P}$-essentially bounded trading strategy $\pi=(\pi_t)_{s\leq t\leq T}$ is

$$W_t^{s,X,\pi}=X+\int_s^t\pi_u\,dS_u.$$

To begin, we fix a measure space (M,Σ) and a collection of *scenario measures*

$$\mathcal{P}=\{\mathbb{P}^\mu\sim\mathbb{P}:\mu\in M\},$$

[1] for now, we exploit the fact that L^∞ is the dual of L^1, and any measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ has Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}\in L^1$. We may relax this assumption later, as is done in [Delbaen, 2000].

each absolutely continuous with respect to objective measure $\mathbb{P}$. For each $\mu \in M$, we may define the Radon-Nikodym derivative

$$Z^\mu := \frac{d\mathbb{P}^\mu}{d\mathbb{P}} \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Let $\mathcal{M}_1(M, \Sigma)$ be the set of all probability measures on (M, Σ). Define the collection of L^1-integrable variables

$$\mathcal{Z} := \left\{ \int_M Z^\mu \, \nu(d\mu) : \nu \in \mathcal{M}_1(M, \Sigma) \right\},$$

which we identify with the "convex hull" of $\mathcal{P}$,

$$\mathcal{Q} = \left\{ \mathbb{Q} \sim \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{Z} \right\}.$$

Assumption 4.1.3. $\mathcal{Z}$ is norm-closed and convex.

Distances between probability measures in $\mathcal{Q}$ are taken to be the L^1-distances between the respective Radon-Nikodym derivatives in $\mathcal{Z}$.

For each of the measures $\mathbb{P}^\mu \in \mathcal{P}$ there is a *floor* f^μ, $\mu \in M$.

Assumption 4.1.4. $\sup_{\mu \in M} f^\mu < \infty$.

For fixed λ, the investor must trade such that at the terminal time T,

$$\mathbb{E}_{\mathbb{P}^\mu}[(W_T^{s,X,\pi})^+ - \lambda (W_T^{s,X,\pi})^-] \geq f^\mu \qquad \text{for each } \mu \in M.$$

Pınar's problem is precisely when the state space Ω is assumed finite, time is discrete, $M = \{1, \ldots, d\}$, and $\Sigma = 2^M$. Each measure $\mathbb{Q} \in \mathcal{Q}$ has a floor, defined by

$$f^{\mathbb{Q}} := \sup \left\{ \int_M f^\mu \, \nu(d\mu) : \nu \in \mathcal{M}_1(M, \Sigma), \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \int_M Z^\mu \, \nu(d\mu) \right\}$$

To this end, we take the set of all initial endowments for which the above conditions are satisfied:

$$\mathcal{G}_s(\lambda) = \left\{ X \in L^\infty(\Omega, \mathcal{F}_s, \mathbb{P}) : \exists \pi \text{ s.t. } \forall \mathbb{Q} \in \mathcal{Q}, \quad \mathbb{E}_{\mathbb{Q}} \varphi(W_T^{s,X,\pi}) \geq f^{\mathbb{Q}} \right\}$$

where

$$\varphi(W) := W^+ - \lambda W^-.$$

Pınar characterised $\mathcal{G}_s(\lambda)$ in the discrete time and finite state space setting as

$$\mathcal{G}_s(\lambda) = \mathcal{X}_s(\lambda) := \left\{ X \in L^\infty(\Omega, \mathcal{F}_s, \mathbb{P}) : \forall \mathbb{Q} \in \mathcal{Q}, \quad \forall \mathbb{Y} \in \tilde{\mathscr{Y}}_{\mathbb{Q},s}, \quad \int X \, d\mathbb{Y} \geq f^{\mathbb{Q}} \right\}$$

where

$$\mathscr{Y}_{\mathbb{Q}} := \left\{ \mathbb{Y} : \mathbb{Y} \sim \mathbb{Q} \text{ and } \frac{d\mathbb{Y}}{d\mathbb{Q}} \in [1, \lambda] \right\},$$

and for $s \in [0, T]$,

$$\tilde{\mathscr{Y}}_{\mathbb{Q},s} := \left\{ \mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}} : \int S_v \mathbb{1}_G \, d\mathbb{Y} = S_u \mathbb{1}_G \quad \forall s \leq u \leq v \leq T \text{ and } G \in \mathcal{F}_u \right\}.$$

We write $\tilde{\mathscr{Y}}_{\mathbb{Q},0} =: \tilde{\mathscr{Y}}_{\mathbb{Q}}$. From these definitions, we can easily deduce the following two results:

Proposition 4.1.5. $\mathbb{E}_{\mathbb{Q}} \varphi(W) = \inf_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \int W \, d\mathbb{Y}$.

Proof. We first show that the left hand side is less than the right hand side in the equation above.

$$\begin{aligned}
\inf_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \int W \, d\mathbb{Y} &= \inf_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \mathbb{E}_{\mathbb{Q}} \left[W \frac{d\mathbb{Y}}{d\mathbb{Q}} \right] \\
&= \inf_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[W^+ \frac{d\mathbb{Y}}{d\mathbb{Q}} \right] - \mathbb{E}_{\mathbb{Q}} \left[W^- \frac{d\mathbb{Y}}{d\mathbb{Q}} \right] \right\} \\
&\geq \inf_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \mathbb{E}_{\mathbb{Q}} \left[W^+ \frac{d\mathbb{Y}}{d\mathbb{Q}} \right] - \sup_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \mathbb{E}_{\mathbb{Q}} \left[W^- \frac{d\mathbb{Y}}{d\mathbb{Q}} \right] \\
&\geq \mathbb{E}_{\mathbb{Q}} \varphi(W).
\end{aligned}$$

For the reverse inequality, note the measure $\mathbb{Y}_0$ defined by

$$\frac{d\mathbb{Y}_0}{d\mathbb{Q}} = \mathbb{1}_{\{\{W>0\}\}} + \lambda \mathbb{1}_{\{\{W \leq 0\}\}}$$

attains this minimum. □

Proposition 4.1.6. $\mathcal{G}_s(\lambda) \subseteq \mathcal{X}_s(\lambda)$.

Proof. Take $X \in \mathcal{G}_s(\lambda)$. Then, by the previous proposition,

$$\exists \pi \text{ s.t. } \mathbb{E}_{\mathbb{Q}} \varphi(W_T^{s,X,\pi}) = \inf_{\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}} \int W_T^{s,X,\pi} \, d\mathbb{Y} \geq f^{\mathbb{Q}} \qquad \forall \mathbb{Q} \in \mathcal{Q}.$$

Using the fact that $\tilde{\mathscr{Y}}_{\mathbb{Q},s} \subseteq \mathscr{Y}_{\mathbb{Q}}$ for any $s \in [0,T]$,

$$\implies \exists \pi \text{ s.t. } \inf_{\tilde{\mathbb{Y}} \in \tilde{\mathscr{Y}}_{\mathbb{Q},s}} \int W_T^{s,X,\pi}\, d\tilde{\mathbb{Y}} \geq f^{\mathbb{Q}} \qquad \forall \mathbb{Q} \in \mathcal{Q}.$$

Finally, noting that $\int W_T^{s,X,\pi}\, d\tilde{\mathbb{Y}} = \int X\, d\tilde{\mathbb{Y}}$ for any $\tilde{\mathbb{Y}} \in \tilde{\mathscr{Y}}_{\mathbb{Q},s}$, we have $X \in \mathcal{X}_s(\lambda)$. $\square$

In the next subsection, we shall work on proving the reverse inclusion, namely $\mathcal{G}_s(\lambda) \supseteq \mathcal{X}_s(\lambda)$, in one period of discrete time.

4.1.3 Equivalence in one period

We assume $S_0 \in \mathbb{R}$ is non-random. Write $\Delta(\omega) = S_1(\omega) - S_0$. The problem posed to the investor is to optimally choose the strategy $\pi \in \mathbb{R}$ at time 0 to attain wealth $W = X + \pi\Delta$ at terminal time 1 satisfying the convex constraints. Fix $\lambda > 1$. In one period, the above definitions of $\mathcal{G}$ and $\mathcal{X}$ can be simplified to

$$\mathcal{G} = \left\{ X \in \mathbb{R} : \exists \pi \in \mathbb{R} \text{ s.t. } \forall \mathbb{Q} \in \mathcal{Q}, \quad \mathbb{E}_{\mathbb{Q}}\varphi(X + \pi\Delta) \geq f^{\mathbb{Q}} \right\},$$

$$\mathcal{X} = \left\{ X \in \mathbb{R} : \forall \mathbb{Q} \in \mathcal{Q}, \quad \forall \mathbb{Y} \in \tilde{\mathscr{Y}}_{\mathbb{Q}}, \quad X\mathbb{Y}(\Omega) \geq f^{\mathbb{Q}} \right\}.$$

We aim to show the following:

Theorem 4.1.7. *$\mathcal{X} = \mathcal{G}$.*

By work in the previous subsection, we already have the inclusion $\mathcal{G} \subseteq \mathcal{X}$. Thus it remains to show the implication

$$X \notin \mathcal{G} \implies X \notin \mathcal{X},$$

which is written more explicitly in the following proposition.

Proposition 4.1.8. *If, for every $\pi \in \mathbb{R}$, there exists $\mathbb{Q} \in \mathcal{Q}$ so that $\mathbb{E}_{\mathbb{Q}}\varphi(X+\pi\Delta) < f^{\mathbb{Q}}$, then there is a $\mathbb{Q} \in \mathcal{Q}$, and there is a $\mathbb{Y} \in \tilde{\mathscr{Y}}_{\mathbb{Q}}$ such that $X\mathbb{Y}(\Omega) < f^{\mathbb{Q}}$.*

We now set up some appropriate notation to show this result. Define the set of measures

$$R := \{\mathbb{Q} \in \mathcal{Q} : \mathbb{E}_{\mathbb{Q}}\varphi(\Delta) \leq 0 \leq -\mathbb{E}_{\mathbb{Q}}\varphi(-\Delta)\},$$

and note that R inherits convexity from $\mathcal{Q}$. Also, for fixed $\pi \in \mathbb{R}$, define the set of measures

$$\mathcal{Q}_\pi := \{\mathbb{Q} \in \mathcal{Q} : \mathbb{E}_{\mathbb{Q}}\varphi(X + \pi\Delta) < f^{\mathbb{Q}}\},$$

and again note that $\mathcal{Q}_\pi$ inherits convexity from $\mathcal{Q}$. We immediately observe

Lemma 4.1.9. *$\mathbb{Q} \in R$ if and only if $\tilde{\mathscr{Y}}_{\mathbb{Q}} \neq \varnothing$.*

Proof. Fix $\mathbb{Q} \in \mathcal{Q}$. Set

$$\underline{\beta} = \mathbb{1}_{\{\Delta \geq 0\}} + \lambda \mathbb{1}_{\{\Delta < 0\}} \qquad \text{and} \qquad \overline{\beta} = \lambda \mathbb{1}_{\{\Delta \geq 0\}} + \mathbb{1}_{\{\Delta < 0\}}.$$

For any $\mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}}$, define the Radon-Nikodym derivative β by

$$\frac{d\mathbb{Y}}{d\mathbb{Q}} = \beta \in [1, \lambda].$$

Then we have the bounds

$$\int \Delta \, d\mathbb{Y} = \int \beta \Delta \, d\mathbb{Q} \geq \int \underline{\beta} \Delta \, d\mathbb{Q} = \mathbb{E}_{\mathbb{Q}} \varphi(\Delta),$$

$$\int \Delta \, d\mathbb{Y} = \int \beta \Delta \, d\mathbb{Q} \leq \int \overline{\beta} \Delta \, d\mathbb{Q} = -\mathbb{E}_{\mathbb{Q}} \varphi(-\Delta).$$

So we have the inclusion

$$\left\{ \int \Delta \, d\mathbb{Y} : \mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}} \right\} \subseteq [\mathbb{E}_{\mathbb{Q}} \varphi(\Delta), -\mathbb{E}_{\mathbb{Q}} \varphi(-\Delta)].$$

We now show the reverse inclusion. For any $\theta \in [0, 1]$, the $L^1(\mathbb{Q})$-integrable random variable $\beta^\theta := \theta \underline{\beta} + (1 - \theta) \overline{\beta} \in [1, \lambda]$ is the Radon-Nikodym derivative w.r.t. $\mathbb{Q}$ of a measure $\mathbb{Y}^\theta \in \mathscr{Y}_{\mathbb{Q}}$ with

$$\int \Delta \, d\mathbb{Y}^\theta = \theta \int \underline{\beta} \Delta \, d\mathbb{Q} + (1 - \theta) \int \overline{\beta} \Delta \, d\mathbb{Q} = \theta \mathbb{E}_{\mathbb{Q}} \varphi(\Delta) + (1 - \theta)(-\mathbb{E}_{\mathbb{Q}} \varphi(-\Delta)),$$

so that the endpoints of the above interval are attained for $\theta = 0$ and $\theta = 1$. So we have shown the equality

$$\left\{ \int \Delta \, d\mathbb{Y} : \mathbb{Y} \in \mathscr{Y}_{\mathbb{Q}} \right\} = [\mathbb{E}_{\mathbb{Q}} \varphi(\Delta), -\mathbb{E}_{\mathbb{Q}} \varphi(-\Delta)].$$

To complete the proof, by definition of R,

$$\mathbb{Q} \in R \iff 0 \in [\mathbb{E}_{\mathbb{Q}} \varphi(\Delta), -\mathbb{E}_{\mathbb{Q}} \varphi(-\Delta)].$$

The above condition is equivalent to the existence of a $\theta \in [0, 1]$ such that

$$\int \Delta \, d\mathbb{Y}^\theta = 0,$$

since the integral on the left is a convex combination of the left and right limits of the interval $[\mathbb{E}_{\mathbb{Q}}\varphi(\Delta), -\mathbb{E}_{\mathbb{Q}}\varphi(-\Delta)]$. But if there exists such a θ, then $\mathbb{Y}^{\theta}$ is a martingale measure, and so $\mathbb{Y}^{\theta} \in \tilde{\mathscr{Y}}_{\mathbb{Q}} \neq \varnothing$. $\square$

A key idea in understanding the connection between the sets $\mathcal{G}$ and $\mathcal{X}$ is the following Proposition.

Proposition 4.1.10. *Fix a probability measure $\mathbb{Q} \in R$, a martingale measure $\mathbb{Y} \in \tilde{\mathscr{Y}}_{\mathbb{Q}}$, and a $\pi \in \mathbb{R}$. Then*

$$\mathbb{E}_{\mathbb{Q}}\varphi(X + \pi\Delta) = X\mathbb{Y}(\Omega) \iff \frac{d\mathbb{Y}}{d\mathbb{Q}} = \beta^{(\pi,\gamma)} \qquad a.s.\ on \qquad \{X + \pi\Delta \neq 0\},$$

where

$$\beta^{(\pi,\gamma)} := \mathbb{1}_{\{X+\pi\Delta>0\}} + \lambda\mathbb{1}_{\{X+\pi\Delta<0\}} + \gamma\mathbb{1}_{\{X+\pi\Delta=0\}}.$$

Proof. The Proposition crystallises the observation that

$$X\mathbb{Y}(\Omega) = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{Y}}{d\mathbb{Q}}(X + \pi\Delta)\right] \geq \mathbb{E}_{\mathbb{Q}}\left[\varphi(X + \pi\Delta)\right],$$

with equality if and only if

(i) on the set $\{X+\pi\Delta > 0\}$, the Radon-Nikodym derivative $\frac{d\mathbb{Y}}{d\mathbb{Q}}$ takes the essential infimal value of 1, and

(ii) on the set $\{X + \pi\Delta < 0\}$, $\frac{d\mathbb{Y}}{d\mathbb{Q}}$ takes the essential supremal value of λ.

On the set $\{X+\pi\Delta = 0\}$, there is no contribution to either integral on both sides of the inequality, so the Radon-Nikodym derivative $\frac{d\mathbb{Y}}{d\mathbb{Q}}$ may assume any value $\gamma \in [1, \lambda]$ for the equality in the statement to hold. In the case that $\mathbb{Q}\{X + \pi\Delta \neq 0\} > 0$, we are forced to choose the $\gamma \in [1, \lambda]$ such that $\beta^{(\pi,\gamma)}$ defines a martingale measure $\mathbb{Y}$; otherwise, the choice of γ is unrestricted in $[1, \lambda]$. $\square$

For the rest of this section, we assume the hypothesis of Proposition 4.1.8, namely

$$\text{for every } \pi \in \mathbb{R}, \qquad \mathcal{Q}_{\pi} \neq \varnothing. \tag{4.1}$$

Following on from Proposition 4.1.10, we look simultaneously for a pair $(\pi^*, \gamma^*) \in \mathbb{R} \times [1, \lambda]$ and a probability measure $\mathbb{Q}^* \in \mathcal{Q}_{\pi^*}$ such that $\beta^{(\pi^*,\gamma^*)}$ defines a martingale measure $\mathbb{Y}^*$ via

$$\frac{d\mathbb{Y}^*}{d\mathbb{Q}^*} = \beta^{(\pi^*,\gamma^*)}.$$

To this end, we define

$$C := \bigcap_{\pi \in \mathbb{R}} \mathcal{Q}_\pi,$$

the set of measures $\mathbb{Q}$ under which the expectation of $\varphi(X + \pi\Delta)$ fails to weakly exceed the floor $f^{\mathbb{Q}}$, no matter what the value of π. To proceed, we must examine two cases: $R \cap C \neq \varnothing$ and $R \cap C = \varnothing$. In the former case $R \cap C \neq \varnothing$, we may select a $\mathbb{Q}^* \in R \cap C$. Such a measure $\mathbb{Q}^*$ belongs to $\mathcal{Q}_\pi$ for any value of π; by virtue of membership in R, we may choose values of $\pi = \pi^*$ and $\gamma = \gamma^*$ to produce a martingale measure $\mathbb{Y}^*$ via the formula $\frac{d\mathbb{Y}^*}{d\mathbb{Q}^*} = \beta^{(\pi^*,\gamma^*)}$. Then,

$$\begin{aligned}
\mathcal{G}^c &= \Big\{ X \in \mathbb{R} : \forall \pi \in \mathbb{R}, \quad \exists \mathbb{Q} \in \mathcal{Q} \text{ s.t. } \mathbb{E}_{\mathbb{Q}} \varphi(X + \pi\Delta) < f^{\mathbb{Q}} \Big\} \\
&= \left\{ X \in \mathbb{R} : \begin{array}{l} \exists (\pi^*, \gamma^*) \in \mathbb{R} \times [1, \lambda], \quad \exists \mathbb{Q}^* \in \mathcal{Q}_{\pi^*} \\ \text{s.t. } \dfrac{d\mathbb{Y}^*}{d\mathbb{Q}^*} = \beta^{(\pi^*,\gamma^*)}, \quad \mathbb{Y}^* \in \tilde{\mathscr{Y}}_{\mathbb{Q}}, \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}^*} \varphi(X + \pi\Delta) = X\mathbb{Y}^*(\Omega) < f^{\mathbb{Q}} \end{array} \right\} \\
&\subseteq \Big\{ X \in \mathbb{R} : \exists \mathbb{Q}^* \in \mathcal{Q}, \quad \exists \mathbb{Y}^* \in \tilde{\mathscr{Y}}_{\mathbb{Q}^*}, \quad X\mathbb{Y}^*(\Omega) < f^{\mathbb{Q}^*} \Big\} \\
&= \mathcal{X}^c.
\end{aligned}$$

In the latter case $R \cap C = \varnothing$, we shall derive the following contradiction to our assumption (4.1),

Proposition 4.1.11. $R \cap C = \varnothing \implies \exists \pi \in \mathbb{R} : \mathcal{Q}_\pi = \varnothing$.

Topological considerations. We equip $\mathcal{Z} \equiv \{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q}\}$ with the induced (subspace) topology in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. We may speak of a topology on $\mathcal{Q}$, referring to the topology induced from the natural identification of $\mathcal{Q}$ with $\mathcal{Z}$. For $\pi \in \mathbb{R}$ we may establish the natural correspondence between $\mathcal{Q}_\pi$ and $\mathcal{Z}_\pi := \Big\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \quad \mathbb{Q} \in \mathcal{Q}_\pi \Big\}$. Define the function

$$\Psi : \quad \mathcal{Q} \times \mathbb{R} \ni (\mathbb{Q}, \pi) \mapsto \Psi(\mathbb{Q}, \pi) := \mathbb{E}_{\mathbb{Q}} \varphi(X + \pi\Delta) - f^{\mathbb{Q}}.$$

The function Ψ is central to the problem, and so it is worthwhile to present its properties, to be used in the sequel.

Lemma 4.1.12. Ψ *enjoys the following properties:*

(i) The function $\pi \mapsto \Psi(\mathbb{Q}, \pi)$ is concave on $\mathbb{R}$.

(ii) The function $\mathbb{Q} \mapsto \Psi(\mathbb{Q}, \pi)$ is convex on $\mathcal{Q}$.

(iii) For fixed $\mathbb{Q}$, we have the following bounds on $\pi \mapsto \Psi(\mathbb{Q}, \pi)$, for all $\pi \in \mathbb{R}$:

$$\mathbb{E}_{\mathbb{Q}}\varphi(\Delta\,\mathrm{sgn}(\pi))|\pi| - (\lambda|X| + f^{\mathbb{Q}}) \le \Psi(\mathbb{Q}, \pi) \le \mathbb{E}_{\mathbb{Q}}\varphi(\Delta\,\mathrm{sgn}(\pi))|\pi| + (2\lambda|X| - f^{\mathbb{Q}})$$

where $\mathrm{sgn}(x) = \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$ *is the signum function. In particular,*

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\varphi(\Delta) < 0 \qquad\qquad\quad &\iff \Psi(\mathbb{Q}, \pi) \to -\infty \quad \text{as} \quad \pi \to +\infty, \\ 0 < -\mathbb{E}_{\mathbb{Q}}\varphi(-\Delta) &\iff \Psi(\mathbb{Q}, \pi) \to -\infty \quad \text{as} \quad \pi \to -\infty. \end{aligned}$$

Proof. These statements follow easily from the definitions. □

Property (iii) formalises the idea that "far out, $\pi \mapsto \Psi(\mathbb{Q}, \pi)$ is of linear growth". From property (ii), we have that the set $\mathcal{Q}_\pi = \{\mathbb{Q} \in \mathcal{Q} : \Psi(\mathbb{Q}, \pi) < 0\}$ is open and convex in $\mathcal{Q}$ for all π, and the set $\mathcal{Q}_\pi^c$ is closed in $\mathcal{Q}$ for all π.

The case $R \cap C = \varnothing$. We first reduce the case $R \cap C = \varnothing$ to the case $C = \varnothing$.

Proposition 4.1.13.

$$C \equiv \bigcap_{\pi \in \mathbb{R}} \mathcal{Q}_\pi \subset R.$$

Proof. Define

$$\mathcal{Q}_\infty := \liminf_{\pi \to \infty} \mathcal{Q}_\pi \qquad \text{and} \qquad \mathcal{Q}_{-\infty} := \liminf_{\pi \to -\infty} \mathcal{Q}_\pi.$$

We aim to show

$$\mathcal{Q}_\infty \cap \mathcal{Q}_{-\infty} \subset R.$$

Note the two equations

$$\mathcal{Q}_{\pm\infty} = \{\mathbb{Q} \in \mathcal{Q} : \lim_{\pi \to \pm\infty} \Psi(\mathbb{Q}, \pi) \in [-\infty, 0)\}.$$

Due to $\lambda > 1$, no measure $\mathbb{Q}$ will produce a constant $\pi \mapsto \Psi(\mathbb{Q}, \pi)$ except in trivial situations, which we may exclude. Take $\mathbb{Q} \in \mathcal{Q}_\infty \cap \mathcal{Q}_{-\infty}$. Then at least one of the two limits $\lim_{\pi \to \pm\infty} \Psi(\mathbb{Q}, \pi)$ are $-\infty$. If both limits are $-\infty$, then $\mathbb{Q} \in R$ by the equivalences given in property (iii) of Lemma 4.1.12. Now suppose $\lim_{\pi \to +\infty} \Psi(\mathbb{Q}, \pi) > -\infty$ and $\lim_{\pi \to -\infty} \Psi(\mathbb{Q}, \pi) = -\infty$. Then, from the bounds on Ψ it must be that $\mathbb{E}_{\mathbb{Q}}\varphi(\Delta) = 0$, and that $0 < -\mathbb{E}_{\mathbb{Q}}\varphi(-\Delta)$, so $\mathbb{Q} \in R$. The case where $\lim_{\pi \to +\infty} \Psi(\mathbb{Q}, \pi) = -\infty$ and $\lim_{\pi \to -\infty} \Psi(\mathbb{Q}, \pi) > -\infty$ is analogous.

The result is clear, as $\bigcap_{\pi \in \mathbb{R}} \mathcal{Q}_\pi \subset \mathcal{Q}_\infty \cap \mathcal{Q}_{-\infty}$.

□

Proof of Proposition 4.1.11. We assume that:

(i) for every $\pi \in \mathbb{R}$, we have $\mathcal{Q}_\pi \neq \varnothing$, and

(ii) $\bigcap_{\pi\in\mathbb{R}} \mathcal{Q}_\pi = \varnothing$.

We aim for a contradiction, showing these two conditions cannot happen in tandem. Define

$$\Pi(\mathbb{Q}) = \{\pi \in \mathbb{R} : \mathbb{Q} \notin \mathcal{Q}_\pi\} = \{\pi \in \mathbb{R} : \Psi(\mathbb{Q}, \pi) \geq 0\}$$

and note by assumption (ii), there is no measure $\mathbb{Q}$ such that for any π, $\mathbb{Q} \in \mathcal{Q}_\pi$. So for any measure $\mathbb{Q}$ there is a π such that $\mathbb{Q} \notin \mathcal{Q}_\pi$, so

$$\text{for every } \mathbb{Q} \in \mathcal{Q}, \qquad \Pi(\mathbb{Q}) \neq \varnothing.$$

Suppose that $\bigcap_{\mathbb{Q}\in\mathcal{Q}} \Pi(\mathbb{Q}) \neq \varnothing$. Then we may find some strategy $\pi^* \in \bigcap_{\mathbb{Q}\in\mathcal{Q}} \Pi(\mathbb{Q})$ such that the function $\Psi(\mathbb{Q}, \pi^*) \geq 0$ for every $\mathbb{Q} \in \mathcal{Q}$, which contradicts our assumption (i). Thus we can consider the dual set of conditions

(i') $\bigcap_{\mathbb{Q}\in\mathcal{Q}} \Pi(\mathbb{Q}) = \varnothing$, and

(ii') for every $\mathbb{Q} \in \mathcal{Q}$, we have $\Pi(\mathbb{Q}) \neq \varnothing$.

If we can show that the conditions (i') and (ii') together lead to a contradiction, we have achieved our aim. We assume (i') and (ii').

By property (i) of Lemma 4.1.12, the set $\Pi(\mathbb{Q})$ is a closed interval in the extended real line $[-\infty, +\infty]$ with the standard topology.

Define the endpoints of $\Pi(\mathbb{Q})$ to be

$$\ell_{\mathbb{Q}} = \inf \Pi(\mathbb{Q}) \in [-\infty, \infty] \qquad \text{and} \qquad r_{\mathbb{Q}} = \sup \Pi(\mathbb{Q}) \in [-\infty, \infty].$$

We re-write condition (i'):

$$\bigcap_{\mathbb{Q}\in\mathcal{Q}} \Pi(\mathbb{Q}) = \varnothing \qquad \Longleftrightarrow \qquad \bar{\ell} := \sup_{\mathbb{Q}\in\mathcal{Q}} \ell_{\mathbb{Q}} > \inf_{\mathbb{Q}\in\mathcal{Q}} r_{\mathbb{Q}} =: r.$$

The strict inequality in the above line shows that we implicitly rule out both the cases where $\bar{\ell} = -\infty$ and $r = +\infty$. For each $N \in \mathbb{N}$, we choose probability measure $\mathbb{Q}^{(0,N)}$ so that

$$\begin{cases} \ell_{\mathbb{Q}^{(0,N)}} > \bar{\ell} - \frac{1}{N} & \text{whenever } \bar{\ell} < \infty, \\ \ell_{\mathbb{Q}^{(0,N)}} > N & \text{whenever } \bar{\ell} = \infty. \end{cases}$$

We choose $\mathbb{Q}^{(1,N)}$ so that

$$\begin{cases} r_{\mathbb{Q}^{(1,N)}} < r + \frac{1}{N} & \text{whenever } r > -\infty, \\ r_{\mathbb{Q}^{(1,N)}} < -N & \text{whenever } r = -\infty. \end{cases}$$

Now choose $N \in \mathbb{N}$ such that $\ell_{\mathbb{Q}^{(0,N)}} > r_{\mathbb{Q}^{(1,N)}}$. This is accomplished in the case $\bar{\ell} < \infty$ and $r > -\infty$ by any integer

$$N > \frac{2}{\bar{\ell} - r}.$$

Drop the N from the notation, $\mathbb{Q}^{(0,N)} = \mathbb{Q}^{(0)}$, etc.. For all $\theta \in [0,1]$, define the convex combination $\mathbb{Q}^\theta = \theta\mathbb{Q}^{(1)} + (1-\theta)\mathbb{Q}^{(0)}$. For legibility, we shall write

$$\ell_{\mathbb{Q}^\theta} = \ell(\theta), \qquad \text{and} \qquad r_{\mathbb{Q}^\theta} = r(\theta).$$

In this notation, we have

$$r(1) < \ell(0).$$

Claim. The subsets of $[0,1]$

$$\mathscr{U} := \{\theta \in [0,1] : \mathbb{Q}^\theta \in \mathcal{Q}^c_{\ell(0)}\}$$
$$\mathscr{V} := \{\theta \in [0,1] : \mathbb{Q}^\theta \in \mathcal{Q}^c_{r(1)}\}$$

are intervals closed in $[0,1]$.

Proof of Claim. We prove the statement for $\mathscr{U}$; the result for $\mathscr{V}$ is analogous. Take $\theta_1, \theta_2 \in \mathscr{U}$, $\theta_1 < \theta_2$. Suppose $\theta \in (\theta_1, \theta_2)$, but $\theta \notin \mathscr{U}$, which means that $\mathbb{Q}^\theta \in \mathcal{Q}_{\ell(0)}$. We have that $\mathbb{Q}^1 \in \mathcal{Q}_\pi$ for any $\pi > r(1)$, and in particular $\mathbb{Q}^1 \in \mathcal{Q}_{\ell(0)}$. By convexity of $\mathcal{Q}_{\ell(0)}$, we have $\mathbb{Q}^{\theta_2} \in \mathcal{Q}_{\ell(0)}$, that is, $\theta_2 \notin \mathscr{U}$, a contradiction. So $\mathscr{U}$ is an interval. Closure follows from the closedness of $\mathcal{Q}^c_{\ell(0)}$. $\square$

Set

$$\widetilde{a} = \sup\{\theta \in [0,1] : \mathbb{Q}^\theta \in \mathcal{Q}^c_{\ell(0)}\},$$
$$\widetilde{b} = \inf\{\theta \in [0,1] : \mathbb{Q}^\theta \in \mathcal{Q}^c_{r(1)}\}.$$

Claim. $\widetilde{a} < \widetilde{b}$.

Proof of Claim. Otherwise, take any $\theta \in [\widetilde{b}, \widetilde{a}]$, and note

$$\theta \geq \widetilde{b} \quad \Longrightarrow \quad \mathbb{Q}^\theta \in \mathcal{Q}^c_{r(1)}; \qquad \theta \leq \widetilde{a} \quad \Longrightarrow \quad \mathbb{Q}^\theta \in \mathcal{Q}^c_{\ell(0)}.$$

Furthermore,

$$\forall \pi > r(1), \quad \mathbb{Q}^1 \in \mathcal{Q}_\pi \qquad \text{and} \qquad \forall \pi < \ell(0), \quad \mathbb{Q}^0 \in \mathcal{Q}_\pi.$$

By convexity of each $\mathcal{Q}_\pi$ for $\pi \in (r(1), \ell(0))$, we have

$$\forall \pi \in (r(1), \ell(0)), \quad \forall \theta \in [0,1], \qquad \mathbb{Q}^\theta \in \mathcal{Q}_\pi.$$

Summarising,

$$\begin{array}{ccc} \mathbb{Q}^\theta \in \mathcal{Q}^c_{r(1)}, & \mathbb{Q}^\theta \in \bigcap_{\pi \in (r(1),\ell(0))} \mathcal{Q}_\pi, & \text{and} \quad \mathbb{Q}^\theta \in \mathcal{Q}^c_{\ell(0)}. \\ \text{i.e. } \Psi(\mathbb{Q}^\theta, r(1)) \geq 0, & \text{i.e. } \sup_{\pi \in (r(1),\ell(0))} \Psi(\mathbb{Q}^\theta, \pi) < 0, & \text{i.e. } \Psi(\mathbb{Q}^\theta, \ell(0)) \geq 0. \end{array}$$

which contradicts concavity of $\pi \mapsto \Psi(\mathbb{Q}^\theta, \pi)$. □

Set

$$\widetilde{\theta} = \frac{\widetilde{a} + \widetilde{b}}{2}.$$

We assume $\Pi(\mathbb{Q}^{\widetilde{\theta}}) \neq \varnothing$. By construction, we know that

$$\begin{aligned} \widetilde{\theta} > \widetilde{a} \quad &\Longrightarrow \quad \mathbb{Q}^{\widetilde{\theta}} \in \mathcal{Q}_{\ell(0)} \quad \Longrightarrow \quad \forall \pi \leq \ell(0), \quad \mathbb{Q}^{\widetilde{\theta}} \in \mathcal{Q}_\pi. \\ \widetilde{\theta} < \widetilde{b} \quad &\Longrightarrow \quad \mathbb{Q}^{\widetilde{\theta}} \in \mathcal{Q}_{r(1)} \quad \Longrightarrow \quad \forall \pi \geq r(1), \quad \mathbb{Q}^{\widetilde{\theta}} \in \mathcal{Q}_\pi. \end{aligned}$$

Thus, the interval $\Pi(\mathbb{Q}^{\widetilde{\theta}})$ does not intersect $[r(1), \ell(0)]$, so $\Pi(\mathbb{Q}^{\widetilde{\theta}})$ lies either wholly to the left or wholly to the right of $[r(1), \ell(0)]$. Either $r(\widetilde{\theta}) < r(1)$ or $\ell(\widetilde{\theta}) > \ell(0)$.

$$\text{If } \ell(\widetilde{\theta}) > \ell(0), \text{ then define } \begin{cases} a = \widetilde{\theta}, \\ b = \widetilde{b}. \end{cases}$$

$$\text{If } r(\widetilde{\theta}) < r(1), \text{ then define } \begin{cases} a = \widetilde{a}, \\ b = \widetilde{\theta}. \end{cases}$$

We now view the above workings as one step of an iterative scheme. Set

$a(0) = 0$ and $b(0) = 1$, and for $k = 0, 1, 2, \ldots$ define recursively

$$\widetilde{a}(k) = \sup\{\theta \in [a(k), b(k)] : \mathbb{Q}^{\theta} \in \mathcal{Q}^{c}_{\ell(a(k))}\},$$
$$\widetilde{b}(k) = \inf\{\theta \in [a(k), b(k)] : \mathbb{Q}^{\theta} \in \mathcal{Q}^{c}_{r(b(k))}\}.$$

As before, $\widetilde{a}(k) < \widetilde{b}(k)$; define

$$\widetilde{\theta}(k) = \frac{\widetilde{a}(k) + \widetilde{b}(k)}{2}.$$

We assume that $\Pi(\mathbb{Q}^{\widetilde{\theta}(k)}) \neq \varnothing$. As before, we have the following dichotomy:

$$\text{either } \ell(\widetilde{\theta}(k)) > \ell(a(k)), \text{ in which case set } \begin{cases} a(k+1) = \widetilde{\theta}(k), \\ b(k+1) = \widetilde{b}(k); \end{cases}$$
$$\text{or } r(\widetilde{\theta}(k)) < r(b(k)), \text{ in which case set } \begin{cases} a(k+1) = \widetilde{a}(k), \\ b(k+1) = \widetilde{\theta}(k). \end{cases}$$

The intervals $[a(k), b(k)]$ are nested, that is, $[a(k), b(k)] \supset [a(k+1), b(k+1)]$. At each step, the length of each interval at least halves:

$$b(k+1) - a(k+1) \leq \frac{1}{2}(b(k) - a(k))$$

thus these two sequences converge to the same limit $\theta^* \in [0, 1]$,

$$a(k) \uparrow \theta^* \quad \text{and} \quad b(k) \downarrow \theta^*.$$

We assume that $\Pi(\mathbb{Q}^{\theta^*}) \neq \varnothing$, thus $\Pi(\mathbb{Q}^{\theta^*}) = [\ell^*, r^*]$ for some real numbers $\ell^* < r^*$. By definition,

$$r(b(k)) \geq r(b(k+1)) \geq r \quad \text{and} \quad \ell(a(k)) \leq \ell(a(k+1)) \leq \overline{\ell} \qquad \forall k \geq 0.$$

So we have constructed bounded monotone sequences of real numbers, with limits defined to be

$$r(b(k)) \downarrow r_\infty \quad \text{and} \quad \ell(a(k)) \uparrow \ell_\infty, \qquad \text{where clearly} \qquad r_\infty < \ell_\infty.$$

We immediately derive the contradiction:

Lemma 4.1.14. *In the above notation, $\ell_\infty = \ell^*$ and $r_\infty = r^*$.*

Proof. We show the former equality; the latter is analogous.

We begin by noting that, by construction, we have $\Pi(\mathbb{Q}^{\theta^*}) \cap (r_\infty, \ell_\infty) = \varnothing$, so that the closed interval $\Pi(\mathbb{Q}^{\theta^*})$ is either wholly to the left or wholly to the right of the open interval (r_∞, ℓ_∞).

The central idea of this proof is the following:

Claim. For each k, $\mathbb{Q}^{a(k)} \in \mathcal{Q}^c_{\ell_\infty}$.

Proof of Claim. We know that $\ell(a(k)) \leq \ell_\infty$ for any k, so we have

$$\forall k, \quad \mathbb{Q}^{a(k)} \in \mathcal{Q}^c_{\ell_\infty} \qquad \Longleftrightarrow \qquad \forall k, \quad r(a(k)) \geq \ell_\infty.$$

Suppose the condition on the right is not satisfied: there is a k_0 such that $r(a(k_0)) < \ell_\infty$. By definition of $r(a(k_0))$, we have

$$\mathbb{Q}^{a(k_0)} \in \mathcal{Q}_\pi \qquad \forall \pi \in (r(a(k_0)), +\infty).$$

Using the fact that $\mathbb{Q}^{(1)} \in \mathcal{Q}_\pi$ for any $\pi > r(1)$, we certainly have

$$\mathbb{Q}^{(1)} \in \mathcal{Q}_\pi \quad \text{for} \quad \pi \geq r(a(k_0)) \geq \ell(a(k_0)) \geq \ell(0) > r(1).$$

By convexity of $\mathcal{Q}_\pi$ for $\pi > r(a(k_0))$, we have

$$\forall \pi > r(a(k_0)), \quad \forall \theta \in [a(k_0), \theta^*], \qquad \mathbb{Q}^\theta \in \mathcal{Q}_\pi.$$

Whence for any $k > k_0$, we have $\ell(a(k)) \leq r(a(k_0)) < \ell_\infty$, which contradicts $\ell(a(k)) \uparrow \ell_\infty$. □

The sequence $\mathbb{Q}^{a(k)}$ is norm convergent to $\mathbb{Q}^{\theta^*}$ as $k \to \infty$ since $a(k) \uparrow \theta^*$, and so by closure of $\mathcal{Q}^c_{\ell_\infty}$, we conclude

$$\mathbb{Q}^{\theta^*} \in \mathcal{Q}^c_{\ell_\infty}, \qquad \text{in particular,} \qquad \ell_\infty \in \Pi(\mathbb{Q}^{\theta^*}).$$

This has two crucial ramifications:

1. the interval $\Pi(\mathbb{Q}^{\theta^*})$, containing the value ℓ_∞, is wholly to the right of the interval (r_∞, ℓ_∞), so that $\ell^* \geq \ell_\infty$;

2. $\ell_\infty \in [\ell^*, r^*]$, so that $\ell^* \leq \ell_\infty$.

Hence $\ell^* = \ell_\infty$. □

This completes the proof of Proposition 4.1.11. □

Bibliography

[Acciaio et al., 2012] Acciaio, B., Föllmer, H., and Penner, I. (2012). Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles. *Finance and Stochastics*, 16(4):669–709.

[Acciaio and Penner, 2011] Acciaio, B. and Penner, I. (2011). Dynamic risk measures. In *Advanced Mathematical Methods for Finance*, pages 1–34. Springer.

[Aliprantis and Border, 2006] Aliprantis, C. D. and Border, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Science & Business Media.

[Artzner et al., 1997] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1997). Thinking coherently. *Risk*, RISK 10:68–71.

[Artzner et al., 1999] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

[Bachelier, 1900] Bachelier, L. (1900). Theory of speculation. *The random character of stock market prices*, 1018:17–78.

[Barrieu and El Karoui, 2004] Barrieu, P. and El Karoui, N. (2004). Optimal derivatives design under dynamic risk measures. *Contemporary Mathematics*, 351:13–26.

[Bellini and Gianin, 2008] Bellini, F. and Gianin, E. R. (2008). On Haezendonck risk measures. *Journal of Banking & Finance*, 32(6):986–994.

[Bielecki et al., 2017] Bielecki, T. R., Cialenco, I., and Pitera, M. (2017). A survey of time consistency of dynamic risk measures and dynamic performance measures in discrete time: LM-measure perspective. *Probability, Uncertainty and Quantitative Risk*, forthcoming(4):669–709.

[Bielecki et al., 2015] Bielecki, T. R., Cialenco, I., and Rodriguez, R. (2015). No-arbitrage pricing for dividend-paying securities in discrete-time markets with transaction costs. *Mathematical Finance*, 25(4):673–701.

[Boda and Filar, 2006] Boda, K. and Filar, J. A. (2006). Time consistent dynamic risk measures. *Mathematical Methods of Operations Research*, 63(1):169–186.

[Borwein and Lewis, 2010] Borwein, J. and Lewis, A. S. (2010). *Convex analysis and nonlinear optimization: theory and examples.* Springer Science & Business Media.

[Carr et al., 2001] Carr, P., Geman, H., and Madan, D. B. (2001). Pricing and hedging in incomplete markets. *Journal of Financial Economics*, 62(1):131 – 167.

[Cheridito and Li, 2009] Cheridito, P. and Li, T. (2009). Risk measures on orlicz hearts. *Mathematical Finance*, 19(2):189–214.

[Cheridito and Stadje, 2009] Cheridito, P. and Stadje, M. (2009). Time-inconsistency of VaR and time-consistent alternatives. *Finance Research Letters*, 6(1):40–46.

[Cvitanić and Karatzas, 1996] Cvitanić, J. and Karatzas, I. (1996). Hedging and portfolio optimization under transaction costs: a martingale approach. *Mathematical finance*, 6(2):133–165.

[Delbaen, 2000] Delbaen, F. (2000). Coherent risk measures. *Blätter der DGVFM*, 24(4):733–739.

[Delbaen, 2006a] Delbaen, F. (2006a). The structure of m–stable sets and in particular of the set of risk neutral measures. In *In Memoriam Paul-André Meyer*, pages 215–258. Springer.

[Delbaen, 2006b] Delbaen, F. (2006b). The structure of m–stable sets and in particular of the set of risk neutral measures. *In Memoriam Paul-André Meyer*, pages 215–258.

[Delbaen et al., 2010] Delbaen, F., Peng, S., and Gianin, E. R. (2010). Representation of the penalty term of dynamic concave utilities. *Finance and Stochastics*, 14(3):449–472.

[Delbaen and Schachermayer, 1997] Delbaen, F. and Schachermayer, W. (1997). Non-arbitrage and the fundamental theorem of asset pricing: Summary of main results. In *Summary of Main Results,Proceedings of Symposia in Applied Mathematics of the AMS*. Citeseer.

[Delbaen and Schachermayer, 2006] Delbaen, F. and Schachermayer, W. (2006). *The mathematics of arbitrage*. Springer Science & Business Media.

[Denneberg, 1990] Denneberg, D. (1990). Distorted probabilities and insurance premiums. *Methods of Operations Research*, 63(3).

[Detlefsen and Scandolo, 2005] Detlefsen, K. and Scandolo, G. (2005). Conditional and dynamic convex risk measures. *Finance and Stochastics*, 9(4):539–561.

[Dunford and Schwartz, 1958] Dunford, N. and Schwartz, J. T. (1958). Linear operators, vol. i. *Interscience, New York*, 1963.

[Embrechts et al., 2014] Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R., and Beleraj, A. (2014). An academic response to Basel 3.5. *Risks*, 2(1):25–48.

[Föllmer and Knispel, 2013] Föllmer, H. and Knispel, T. (2013). Convex risk measures: Basic facts, law-invariance and beyond, asymptotics for large portfolios. *Handbook of the Fundamentals of Financial Decision Making, Eds. LC MacLean and WT Ziemba, World Scientific.*

[Föllmer and Penner, 2006] Föllmer, H. and Penner, I. (2006). Convex risk measures and the dynamics of their penalty functions.

[Föllmer and Schied, 2004] Föllmer, H. and Schied, A. (2004). *Stochastic Finance: An Introduction in Discrete Time.* De Gruyter studies in mathematics. Walter de Gruyter.

[Föllmer and Schied, 2011] Föllmer, H. and Schied, A. (2011). *Stochastic Finance: an introduction in discrete time.* Walter de Gruyter.

[Fremlin, 2001] Fremlin, D. H. (2001). Measure theory, volume 2: Broad foundations. *Torres Fremlin.*

[Gianin, 2006] Gianin, E. R. (2006). Risk measures via g-expectations. *Insurance: Mathematics and Economics*, 39(1):19–34.

[Halley, 1693] Halley, E. (1693). An estimate of the degrees of the mortality of mankind, drawn from curious tables of the births and funerals at the city of breslaw; with an attempt to ascertain the price of annuities upon lives. *Philosophical Transactions of the Royal Society of London*, 17:596–610.

[Harrison and Kreps, 1979] Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory*, 20(3):381–408.

[Hickman, 2004] Hickman, J. (2004). History of actuarial profession. *Encyclopedia of actuarial science.*

[Jacka et al., 2019] Jacka, S., Berkaoui, A., and Armstrong, S. (2019). On representing and hedging claims for coherent risk measures. *Journal of Convex Analysis*, forthcoming.

[Jacka et al., 2008] Jacka, S., Berkaoui, A., and Warren, J. (2008). No arbitrage and closure results for trading cones with transaction costs. *Finance and Stochastics*, 12(4):583–600.

[Jouini and Kallal, 1995] Jouini, E. and Kallal, H. (1995). Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66(1):178 –197.

[Jouini et al., 2006] Jouini, E., Schachermayer, W., and Touzi, N. (2006). Law invariant risk measures have the Fatou property. *Advances in mathematical economics*, pages 49–71.

[Kabanov et al., 2003] Kabanov, Y., Rásonyi, M., and Stricker, C. (2003). On the closedness of sums of convex cones in l^0 and the robust no-arbitrage property. *Finance and Stochastics*, 7(3):403–411.

[Kabanov, 1999] Kabanov, Y. M. (1999). Hedging and liquidation under transaction costs in currency markets. *Finance and stochastics*, 3(2):237–248.

[Kabanov and Stricker, 2001] Kabanov, Y. M. and Stricker, C. (2001). The Harrison–Pliska arbitrage pricing theorem under transaction costs. *Journal of Mathematical Economics*, 35(2):185–196.

[Kusuoka, 2001] Kusuoka, S. (2001). On law invariant coherent risk measures. In *Advances in mathematical economics*, pages 83–95. Springer.

[Lai and Lin, 1988] Lai, H.-C. and Lin, L.-J. (1988). The Fenchel-Moreau theorem for set functions. *Proceedings of the American Mathematical Society*, pages 85–90.

[Larsen et al., 2005] Larsen, K., Pirvu, T. A., Shreve, S. E., and Tütüncü, R. (2005). Satisfying convex risk limits by trading. *Finance and Stochastics*, 9(2):177–195.

[Markowitz, 1952] Markowitz, H. (1952). Portfolio selection. *The journal of finance*, 7(1):77–91.

[on Banking Supervision, 2013] on Banking Supervision, B. C. (2013). Fundamental review of the trading book: A revised market risk framework. Bank for International Settlements: Basel, Switzerland.

[Pınar, 2011] Pınar, M. Ç. (2011). Gain–loss based convex risk limits in discrete-time trading. *Computational Management Science*, 8(3):299–321.

[Riedel, 2004] Riedel, F. (2004). Dynamic coherent risk measures. *Stochastic processes and their applications*, 112(2):185–200.

[Robert and Thérond, 2013] Robert, C. and Thérond, P.-E. (2013). Distortion risk measures, ambiguity aversion and optimal effort. Technical report, HAL.

[Roorda et al., 2005] Roorda, B., Schumacher, J. M., and Engwerda, J. (2005). Coherent acceptability measures in multiperiod models. *Mathematical Finance*, 15(4):589–612.

[Rudin, 1991] Rudin, W. (1991). Functional analysis. international series in pure and applied mathematics.

[Schachermayer, 2004] Schachermayer, W. (2004). The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. *Mathematical Finance*, 14(1):19–48.

[Sereda et al., 2010] Sereda, E. N., Bronshtein, E. M., Rachev, S. T., Fabozzi, F. J., Sun, W., and Stoyanov, S. V. (2010). Distortion risk measures in portfolio optimization. In *Handbook of Portfolio Construction*, pages 649–673. Springer.

[Sieradski, 1992] Sieradski, A. J. (1992). *An introduction to topology and homotopy.* PWS-Kent Publishing Company Boston.

[Wang et al., 1997] Wang, S. S., Young, V. R., and Panjer, H. H. (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, 21(2):173–183.