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Corrigendum to: Around ℓ -independence

Bruno Chiarellotto and Christopher Lazda

ABSTRACT

We correct the proof of the main ℓ -independence result of [CL18] by showing that for any smooth and proper variety over an equicharacteristic local field, there exists a globally defined such variety with the same (p -adic and ℓ -adic) cohomology.

1. Introduction

It was pointed out to us by W. Zheng that the proof of [CL18, Theorem 6.1] is invalid. The problem is in the final step of the proof on p.237, where we showed that there was an exact sequence

$$0 \rightarrow H_\ell^{i+n}(X) \rightarrow H_\ell^{i+n}(X_0) \rightarrow H_\ell^{i+n}(X_1) \rightarrow \dots$$

and claimed to deduce ℓ -independence of $H_\ell^i(X)$ from ℓ -independence of all the other terms $H_\ell^{i+n}(X_n)$. Of course, this deduction doesn't work, since there might be infinitely many such other terms.

In their preprint [LZ18] Q. Lu and W. Zheng provide (amongst other things) an alternative proof of this ℓ -independence result, at least for $\ell \neq p$, see Theorem 1.4(2). In this corrigendum we will explain how to fix the proof of [CL18, Theorem 6.1] by instead proving a stronger version of [CL18, Corollary 5.5] where the semistable hypothesis is removed. In particular, this includes the case $\ell = p$.

Notations and conventions

We will use notations from [CL18] freely.

2. Log structures

We begin with a general result on semistable reduction and log schemes. Let R be a complete DVR with perfect residue field k , π a uniformiser for R , and let $\mathcal{X} \rightarrow \operatorname{Spec}(R)$ be a strictly semistable scheme. That is, \mathcal{X} is Zariski locally étale over $R[x_1, \dots, x_n]/(x_1 \cdots x_r - \pi)$ for some n, r . There is a natural log structure $\mathcal{M}_{\mathcal{X}}$ on \mathcal{X} given by functions invertible outside the special fibre X , and we let \mathcal{M}_X denote the pull-back of this log structure to X . We will also write X_i for the reduction of \mathcal{X} modulo π^{i+1} , and k^\times for k equipped with the log structure pulled back from the canonical log structure R^\times on R .

2010 *Mathematics Subject Classification* 11G20 (primary), 14F99 (secondary).

Keywords: local function fields, cohomology, L -functions, motives, unipotent fundamental groups

B. Chiarellotto was supported by the grants MIUR-PRIN 2015 "Number Theory and Arithmetic Geometry" and Padova PRAT CDPA159224/15. C. Lazda was supported by the Netherlands Organisation for Scientific Research (NWO).

PROPOSITION 2.1 Illusie, Nakayama [Nak98] Appendix A.4. If $\mathcal{X}, \mathcal{X}'$ are strictly semistable schemes over R , and $g : X_1 \rightarrow X'_1$ is an isomorphism between their mod π^2 -reductions, then g induces a canonical isomorphism $g : (X, \mathcal{M}_X) \xrightarrow{\sim} (X', \mathcal{M}_{X'})$ of log schemes over k^\times .

Sketch of proof. Use g to identify X_1 and X'_1 , and thus X and X' . Let \mathcal{M}_X and \mathcal{M}'_X be the log structures on X coming from \mathcal{X} and \mathcal{X}' respectively.

Near a closed point of X let $X^{(1)}, \dots, X^{(r)}$ be the irreducible components of X , and pick $x_1, \dots, x_r \in \mathcal{O}_{\mathcal{X}}$ such that $X^{(i)} = V(x_i)$. Similarly pick $x'_1, \dots, x'_r \in \mathcal{O}_{\mathcal{X}'}$ such that $X^{(i)} = V(x'_i)$. Let $v \in \mathcal{O}_{\mathcal{X}}^*$ and $v' \in \mathcal{O}_{\mathcal{X}'}^*$ be such that $x_1 \dots x_r = v\pi$ and $x'_1 \dots x'_r = v'\pi$. Then in a neighbourhood of p the morphisms $(\mathcal{X}, \mathcal{M}_{\mathcal{X}}) \rightarrow \text{Spec}(R^\times)$ and $(\mathcal{X}', \mathcal{M}_{\mathcal{X}'}) \rightarrow \text{Spec}(R^\times)$ can be described by the following diagrams:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}^* \oplus \mathbb{N}^r & \xrightarrow{(u, a_1, \dots, a_r) \mapsto ux_1^{a_1} \dots x_r^{a_r}} & \mathcal{O}_{\mathcal{X}} \\ \uparrow (\lambda, a) \mapsto (\lambda v^{-a}, a, \dots, a) & & \uparrow \\ R^* \oplus \mathbb{N} & \xrightarrow{(\lambda, a) \mapsto \lambda \pi^a} & R. \end{array}$$

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{X}'} \cong \mathcal{O}_{\mathcal{X}'}^* \oplus \mathbb{N}^r & \xrightarrow{(u, a_1, \dots, a_r) \mapsto ux_1'^{a_1} \dots x_r'^{a_r}} & \mathcal{O}_{\mathcal{X}'} \\ \uparrow (\lambda, a) \mapsto (\lambda v'^{-a}, a, \dots, a) & & \uparrow \\ R^* \oplus \mathbb{N} & \xrightarrow{(\lambda, a) \mapsto \lambda \pi^a} & R. \end{array}$$

Pulling back to k , we see that the morphisms $(X, \mathcal{M}_X) \rightarrow \text{Spec}(k^\times)$ and $(X, \mathcal{M}'_X) \rightarrow \text{Spec}(k^\times)$ can be described by the diagrams

$$\begin{array}{ccc} \mathcal{M}_X \cong \mathcal{O}_X^* \oplus \mathbb{N}^r & \xrightarrow{(u, a_1, \dots, a_r) \mapsto ux_1^{a_1} \dots x_r^{a_r}} & \mathcal{O}_X \\ \uparrow (\lambda, a) \mapsto (\lambda v^{-a}, a, \dots, a) & & \uparrow \\ k^* \oplus \mathbb{N} & \xrightarrow{(\lambda, a) \mapsto \lambda 0^a} & k. \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}'_X \cong \mathcal{O}_X^* \oplus \mathbb{N}^r & \xrightarrow{(u, a_1, \dots, a_r) \mapsto ux_1'^{a_1} \dots x_r'^{a_r}} & \mathcal{O}_X \\ \uparrow (\lambda, a) \mapsto (\lambda v'^{-a}, a, \dots, a) & & \uparrow \\ k^* \oplus \mathbb{N} & \xrightarrow{(\lambda, a) \mapsto \lambda 0^a} & k \end{array}$$

respectively, again in a neighbourhood of p . Since $V(x_i) = V(x'_i)$ inside X_1 , we must have $x_i = u_i x'_i$ for some $u_i \in \mathcal{O}_{X_1}^*$, and so we can define an isomorphism

$$\mathcal{M}_X \xrightarrow{\sim} \mathcal{M}'_X$$

of log structures by mapping

$$(u, a_1, \dots, a_r) \mapsto (u u_1^{a_1} \dots u_r^{a_r}, a_1, \dots, a_r).$$

This is checked to be a morphism of log structures over k^\times by using the above local descriptions. Note that any other choice u'_i must satisfy $(u_i - u'_i)x'_i = 0$ in \mathcal{O}_{X_1} , and hence we must have $u_i - u'_i \in (\pi)$. In particular, the above isomorphism doesn't depend on the choice of u_i . By a similar argument, neither does it depend on the choice of x_i and x'_i , and so it glues to give a global isomorphism $(X, \mathcal{M}_X) \cong (X, \mathcal{M}'_X)$ of log schemes over k^\times . \square

We will need to extend this result to cover morphisms between strictly semistable schemes over different bases. So suppose that $R \rightarrow S$ is a finite morphism of complete DVR's, with induced residue field extension $k \rightarrow k_S$. Let π_S be a uniformiser for S , and let $e = v_{\pi_S}(\pi)$. We do not assume that the induced extension $Q(R) \rightarrow Q(S)$ of fraction fields is separable.

Suppose that we have strictly semistable schemes $\mathcal{X}, \mathcal{X}'$ over R and $\mathcal{Y}, \mathcal{Y}'$ over S , and a pair of commutative diagrams

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array} \quad \begin{array}{ccc} \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) . \end{array}$$

As before, let us write Y_j for the reduction of \mathcal{Y} modulo π_S^{j+1} . Suppose that we have isomorphisms

$$g_Y : Y_e \xrightarrow{\sim} Y'_e, \quad g_X : X_1 \xrightarrow{\sim} X'_1$$

of S - and R -schemes respectively such that the diagram

$$\begin{array}{ccc} Y_e & \xrightarrow{f} & X_1 \\ g_Y \downarrow & & \downarrow g_X \\ Y'_e & \xrightarrow{f'} & X'_1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

commutes. Then by Proposition 2.1 we obtain isomorphisms

$$g_Y : (Y, \mathcal{M}_Y) \xrightarrow{\sim} (Y', \mathcal{M}_{Y'})$$

of log schemes over k_S^\times , as well as

$$g_X : (X, \mathcal{M}_X) \xrightarrow{\sim} (X', \mathcal{M}_{X'})$$

of log schemes over k^\times . The above commutative diagrams of strictly semistable schemes induce commutative diagrams

$$\begin{array}{ccc} (Y, \mathcal{M}_Y) & \xrightarrow{f} & (X, \mathcal{M}_X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_S^\times) & \longrightarrow & \mathrm{Spec}(k^\times) \end{array} \quad \begin{array}{ccc} (Y', \mathcal{M}_{Y'}) & \xrightarrow{f'} & (X, \mathcal{M}_{X'}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_S^\times) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

of log schemes. Note that the morphism of punctured points along the bottom of each square is given by

$$\begin{aligned} k^* \oplus \mathbb{N} &\rightarrow k_S^* \oplus \mathbb{N} \\ (\lambda, a) &\mapsto (\lambda u^a, ea) \end{aligned}$$

where $u \in S^*$ is such that $\pi = u\pi_S^e$.

PROPOSITION 2.2. *The diagram*

$$\begin{array}{ccc} (Y, \mathcal{M}_Y) & \xrightarrow{f} & (X, \mathcal{M}_X) \\ \cong \downarrow g_Y & & \cong \downarrow g_X \\ (Y', \mathcal{M}_{Y'}) & \xrightarrow{f'} & (X', \mathcal{M}_{X'}) \end{array}$$

of log schemes commutes.

Proof. Let us use g to identify $Y_e = Y'_e$ and $Y = Y'$, and let \mathcal{M}_Y and \mathcal{M}'_Y be the log structures on Y coming from \mathcal{Y} and \mathcal{Y}' respectively. Similarly identify $X_1 = X'_1$ and $X = X'$, and let \mathcal{M}_X and \mathcal{M}'_X be the log structures on X coming from \mathcal{X} and \mathcal{X}' respectively.

Locally on X and Y , choose functions $y_1, \dots, y_s \in \mathcal{O}_Y$, $y'_1, \dots, y'_s \in \mathcal{O}_{Y'}$ cutting out the irreducible components of Y , and functions $x_1, \dots, x_r \in \mathcal{O}_X$ and $x'_1, \dots, x'_r \in \mathcal{O}_{X'}$ cutting out the irreducible components of X . Write

$$f^*(x_i) = \alpha_i y_1^{d_{i1}} \dots y_s^{d_{is}}, \quad f'^*(x'_i) = \alpha'_i y'_1^{d'_{i1}} \dots y'_s^{d'_{is}},$$

since both d_{ij} and d'_{ij} are given by the multiplicity of the j th irreducible component of Y in the scheme theoretic preimage of the i th irreducible component of X inside Y_e , we must have $d_{ij} = d'_{ij}$. Moreover, since $V(f^*(x_i)) \subset V(\pi_S^e) = V(y_1^e \dots y_s^e)$ we must have $d_{ij} \leq e$ for all i, j .

Now choose $u_i \in \mathcal{O}_{X_1}^*$ such that $x_i = u_i x'_i$, and $v_j \in \mathcal{O}_{Y_e}^*$ such that $y_j = v_j y'_j$. Then the isomorphisms of log structures induced by g_Y and g_X are given by

$$\begin{aligned} \mathcal{M}_Y &= \mathcal{O}_Y^* \oplus \mathbb{N}^s \rightarrow \mathcal{M}'_Y = \mathcal{O}_{Y'}^* \oplus \mathbb{N}^s \\ (v, b_1, \dots, b_s) &\mapsto (v v_1^{b_1} \dots v_s^{b_s}, b_1, \dots, b_s) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_X &= \mathcal{O}_X^* \oplus \mathbb{N}^r \rightarrow \mathcal{M}'_X = \mathcal{O}_{X'}^* \oplus \mathbb{N}^r \\ (u, a_1, \dots, a_r) &\mapsto (u u_1^{a_1} \dots u_r^{a_r}, a_1, \dots, a_r) \end{aligned}$$

respectively, and the morphisms $\mathcal{M}_X \rightarrow \mathcal{M}_Y$ and $\mathcal{M}'_X \rightarrow \mathcal{M}'_Y$ are defined by

$$(u, a_1, \dots, a_r) \mapsto (f^*(u) \alpha_1^{a_1} \dots \alpha_r^{a_r}, \sum_i d_{i1} a_i, \dots, \sum_i d_{is} a_i)$$

and

$$(u, a_1, \dots, a_r) \mapsto (f^*(u) \alpha'_1^{a_1} \dots \alpha'_r^{a_r}, \sum_i d_{i1} a_i, \dots, \sum_i d_{is} a_i)$$

respectively. Hence in the diagram

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{g_X} & \mathcal{M}'_X \\ f \downarrow & & \downarrow f \\ \mathcal{M}_Y & \xrightarrow{g_Y} & \mathcal{M}'_Y \end{array}$$

the composite $f \circ g_X$ is given by

$$(u, a_1, \dots, a_r) \mapsto (f^*(u) (\alpha'_1 f^*(u_1))^{a_1} \dots (\alpha'_r f^*(u_r))^{a_r}, \sum_i d_{i1} a_i, \dots, \sum_i d_{is} a_i)$$

and the composite $g_Y \circ f$ is given by

$$(u, a_1, \dots, a_r) \mapsto (f^*(u) (\alpha_1 v_1^{d_{11}} \dots v_s^{d_{1s}})^{a_1} \dots (\alpha_r v_1^{d_{r1}} \dots v_s^{d_{rs}})^{a_r}, \sum_i d_{i1} a_i, \dots, \sum_i d_{is} a_i).$$

We thus need to show that $\alpha'_i f^*(u_i) = \alpha_i v_1^{d_{i1}} \dots v_s^{d_{is}}$ in \mathcal{O}_Y^* for all i . But now we write

$$\alpha_i y_1^{d_{i1}} \dots y_s^{d_{is}} = f^*(x_i) = f^*(u_i x'_i) = f^*(u_i) \alpha'_i y_1^{d_{i1}} \dots y_s^{d_{is}}$$

in \mathcal{O}_{Y_e} and so deduce that

$$\alpha_i v_1^{d_{i1}} \dots v_s^{d_{is}} y_1^{d_{i1}} \dots y_s^{d_{is}} = f^*(u_i) \alpha'_i y_1^{d_{i1}} \dots y_s^{d_{is}}.$$

We deduce that the difference $\beta_i = \alpha'_i f^*(u_i) - \alpha_i v_1^{d_{i1}} \dots v_s^{d_{is}}$ annihilates $y_1^{d_{i1}} \dots y_s^{d_{is}}$ inside \mathcal{O}_{Y_e} , and since each $d_{ij} \leq e$ we deduce that in fact β_i annihilates π_S^e , and therefore must lie in (π_S) . Hence $\beta_i = 0$ in \mathcal{O}_Y and the proof is complete. \square

3. Functoriality of comparison isomorphisms

We will also need to know that the comparison isomorphisms [CL18, Propositions 5.3, 5.4] are compatible with morphisms of semistable schemes over different bases. So let us suppose that we are again in the above set-up, where we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

of strictly semistable schemes \mathcal{Y} and \mathcal{X} over S and R respectively, with S the integral closure of R in some finite extension of its fraction field. Let us assume that R , and hence S , is of equicharacteristic $p > 0$, with fraction fields F and F_S respectively, whose absolute Galois groups we will denote by G_F and G_{F_S} . Fix an embedding $F^{\mathrm{sep}} \hookrightarrow F_S^{\mathrm{sep}}$ of separable closures, note that this sends F^{tame} into F_S^{tame} and induces an injective homomorphism $G_{F_S} \rightarrow G_F$ with finite cokernel.

Let \mathcal{X}^\times and \mathcal{Y}^\times denote these semistable schemes endowed with their canonical log structures, and X^\times and Y^\times the corresponding log special fibres. We therefore have a commutative diagram

$$\begin{array}{ccc} Y^\times & \longrightarrow & X^\times \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_S^\times) & \longrightarrow & \mathrm{Spec}(k^\times) \end{array}$$

of log schemes. For every finite subextension $F \subset L \subset F^{\mathrm{tame}}$, let X_L^\times denote the corresponding base change of X^\times , and $X^{\times, \mathrm{tame}}$ the inverse limit of the étale topoi of all such X_L^\times ; we have $Y^{\times, \mathrm{tame}}$ defined entirely similarly. Via the embedding $F^{\mathrm{tame}} \hookrightarrow F_S^{\mathrm{tame}}$ this induces a G_{F_S} -equivariant morphism of topoi

$$Y^{\times, \mathrm{tame}} \rightarrow X^{\times, \mathrm{tame}}$$

and hence a G_{F_S} -equivariant morphism

$$H_{\mathrm{ét}}^i(X^{\times, \mathrm{tame}}, \mathbb{Q}_\ell) \rightarrow H_{\mathrm{ét}}^i(Y^{\times, \mathrm{tame}}, \mathbb{Q}_\ell)$$

in cohomology, for any $\ell \neq p$. On the other hand we have a natural G_{F_S} -equivariant map

$$H_{\mathrm{ét}}^i(\mathcal{X} \times_R F^{\mathrm{sep}}, \mathbb{Q}_\ell) \rightarrow H_{\mathrm{ét}}^i(\mathcal{Y} \times_S F_S^{\mathrm{sep}}, \mathbb{Q}_\ell),$$

and by [Nak98, Proposition 4.2] equivariant isomorphisms

$$\begin{aligned} H_{\text{ét}}^i(X^{\times, \text{tame}}, \mathbb{Q}_\ell) &\xrightarrow{\sim} H_{\text{ét}}^i(\mathcal{X} \times_R F^{\text{sep}}, \mathbb{Q}_\ell) \\ H_{\text{ét}}^i(Y^{\times, \text{tame}}, \mathbb{Q}_\ell) &\xrightarrow{\sim} H_{\text{ét}}^i(\mathcal{Y} \times_S F_S^{\text{sep}}, \mathbb{Q}_\ell). \end{aligned}$$

PROPOSITION 3.1. *The diagram*

$$\begin{array}{ccc} H_{\text{ét}}^i(X^{\times, \text{tame}}, \mathbb{Q}_\ell) & \longrightarrow & H_{\text{ét}}^i(\mathcal{X} \times_R F^{\text{sep}}, \mathbb{Q}_\ell) \\ \downarrow & & \downarrow \\ H_{\text{ét}}^i(Y^{\times, \text{tame}}, \mathbb{Q}_\ell) & \longrightarrow & H_{\text{ét}}^i(\mathcal{Y} \times_S F_S^{\text{sep}}, \mathbb{Q}_\ell) \end{array}$$

commutes.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} Y^{\times, \text{tame}} & \xrightarrow{i_Y} & \mathcal{Y}^{\times, \text{tame}} & \xleftarrow{j_Y} & \mathcal{Y}_{F_S^{\text{sep}}} \\ \downarrow f & & \downarrow f & & \downarrow f \\ X^{\times, \text{tame}} & \xrightarrow{i_X} & \mathcal{X}^{\times, \text{tame}} & \xleftarrow{j_X} & \mathcal{X}_{F_S^{\text{sep}}} \end{array}$$

of topoi as in [Nak98, §3], where $\mathcal{Y}^{\times, \text{tame}}$ and $\mathcal{X}^{\times, \text{tame}}$ are defined by ‘base change’ along $F_S \rightarrow F_S^{\text{tame}}$ and $F \rightarrow F^{\text{tame}}$ respectively. Then the isomorphism

$$H_{\text{ét}}^i(Y^{\times, \text{tame}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^i(\mathcal{Y} \times_S F_S^{\text{sep}}, \mathbb{Q}_\ell)$$

is given as the composite

$$H_{\text{ét}}^i(Y^{\times, \text{tame}}, \mathbb{Q}_\ell) \xleftarrow{\sim} H_{\text{ét}}^i(\mathcal{Y}^{\times, \text{tame}}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^i(\mathcal{Y} \times_S F_S^{\text{sep}}, \mathbb{Q}_\ell)$$

using the proper base change theorem in log-étale cohomology [Nak97, Theorem 5.1], and there is a similar statement for \mathcal{X} . The claim then follows simply from commutativity of the above diagram of log schemes. \square

We will also need a version of this result for p -adic cohomology. Write $W = W(k)$, $W_S = W(k_S)$, let $K = W[1/p]$, $K_S = W_S[1/p]$, and let $\mathcal{R}_K \supset \mathcal{E}_K^\dagger \subset \mathcal{E}_K$, and $\mathcal{R}_{K_S} \supset \mathcal{E}_{K_S}^\dagger \subset \mathcal{E}_{K_S}$ denote copies of the Robba ring, the bounded Robba ring and the Amice ring over K and K_S respectively. Lift the extension $F \rightarrow F_S$ to a finite flat morphism $\mathcal{E}_K^\dagger \rightarrow \mathcal{E}_{K_S}^\dagger$ which extends to finite flat morphisms $\mathcal{R}_K \rightarrow \mathcal{R}_{K_S}$ and $\mathcal{E}_K \rightarrow \mathcal{E}_{K_S}$. Then as above, the morphism of log schemes $Y^\times \rightarrow X^\times$ induces a morphism

$$H_{\text{log-cris}}^i(X^\times/K^\times) \rightarrow H_{\text{log-cris}}^i(Y^\times/K_S^\times)$$

in log crystalline cohomology, and the morphism $\mathcal{Y}_{F_S} \rightarrow \mathcal{X}_F$ induces a morphism

$$H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K) \rightarrow H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})$$

in Robba-ring valued rigid cohomology. Then following [CL18, Proposition 5.4] we can construct isomorphisms

$$\begin{aligned} H_{\text{log-cris}}^i(X^\times/K^\times) \otimes_K \mathcal{R}_K &\xrightarrow{\sim} H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K) \\ H_{\text{log-cris}}^i(Y^\times/K_S^\times) \otimes_{K_S} \mathcal{R}_{K_S} &\xrightarrow{\sim} H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S}) \end{aligned}$$

as follows. Let t denote a co-ordinate on \mathcal{E}_K^\dagger and t_S a co-ordinate on $\mathcal{E}_{K_S}^\dagger$ such that $t \in W_S[[t_S]]$. Write $S_K = K \otimes W[[t]]$ and $S_{K_S} = K_S \otimes W_S[[t_S]]$. Equip $W[[t]]$ (resp. $W_S[[t_S]]$) with the log

structure defined by the ideal $(t) \subset W[[t]]$ (resp. $(t_S) \subset W[[t_S]]$) and define the log-crystalline cohomology groups

$$\begin{aligned} H_{\log\text{-cris}}^i(\mathcal{X}^\times/S_K) &:= H_{\log\text{-cris}}^i(\mathcal{X}^\times/W[[t]]) \otimes_{\mathbb{Z}} \mathbb{Q} \\ H_{\log\text{-cris}}^i(\mathcal{Y}^\times/S_{K_S}) &:= H_{\log\text{-cris}}^i(\mathcal{Y}^\times/W_S[[t_S]]) \otimes_{\mathbb{Z}} \mathbb{Q}, \end{aligned}$$

these are naturally endowed with the extra structure of \log -(φ, ∇)-modules over S_K and S_{K_S} respectively. Moreover, we have isomorphisms of φ -modules

$$\begin{aligned} H_{\log\text{-cris}}^i(\mathcal{X}^\times/S_K) \otimes_{S_K, t \mapsto 0} K &\xrightarrow{\sim} H_{\log\text{-cris}}^i(Y^\times/K_S^\times) \\ H_{\log\text{-cris}}^i(\mathcal{Y}^\times/S_{K_S}) \otimes_{S_{K_S}, t_S \mapsto 0} K_S &\xrightarrow{\sim} H_{\log\text{-cris}}^i(Y^\times/K_S^\times), \end{aligned}$$

by smooth and proper base change in log-crystalline cohomology, as well as isomorphisms of (φ, ∇) -modules

$$\begin{aligned} H_{\log\text{-cris}}^i(\mathcal{X}^\times/S_K) \otimes_{S_K} \mathcal{R}_K &\xrightarrow{\sim} H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K) \\ H_{\log\text{-cris}}^i(\mathcal{Y}^\times/S_{K_S}) \otimes_{S_{K_S}} \mathcal{R}_{K_S} &\xrightarrow{\sim} H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S}), \end{aligned}$$

by [LP16, Proposition 5.45]. It therefore follows from the logarithmic form of Dwork's trick [Ked10, Corollary 17.2.4] that the (φ, ∇) -modules $H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K)$ and $H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})$ are unipotent, that there are isomorphisms

$$\begin{aligned} (H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K)[\log t])^{\nabla=0} &\cong H_{\log\text{-cris}}^i(X^\times/K^\times) \\ (H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S})[\log t_S])^{\nabla=0} &\cong H_{\log\text{-cris}}^i(Y^\times/K_S^\times) \end{aligned}$$

and moreover the connection ∇ on the rigid cohomology groups appearing on the LHS can be completely recovered from the monodromy operator N on the RHS. This allows us to construct isomorphisms of (φ, ∇) -modules

$$\begin{aligned} H_{\log\text{-cris}}^i(X^\times/K^\times) \otimes_K \mathcal{R}_K &\xrightarrow{\sim} H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K) \\ H_{\log\text{-cris}}^i(Y^\times/K_S^\times) \otimes_{K_S} \mathcal{R}_{K_S} &\xrightarrow{\sim} H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S}) \end{aligned}$$

where the LHS is endowed a natural connection coming from N , for more details see for example [Mar08, §3.2].

PROPOSITION 3.2. *The diagram*

$$\begin{array}{ccc} H_{\log\text{-cris}}^i(X^\times/K^\times) \otimes_K \mathcal{R}_K & \longrightarrow & H_{\text{rig}}^i(\mathcal{X}_F/\mathcal{R}_K) \\ \downarrow & & \downarrow \\ H_{\log\text{-cris}}^i(Y^\times/K_S^\times) \otimes_{K_S} \mathcal{R}_{K_S} & \longrightarrow & H_{\text{rig}}^i(\mathcal{Y}_{F_S}/\mathcal{R}_{K_S}) \end{array}$$

commutes.

Proof. Given the construction of the horizontal isomorphisms outlined above, it suffices to show that the diagram

$$\begin{array}{ccc} H_{\log\text{-cris}}^i(\mathcal{X}^\times/S_K) & \longrightarrow & H_{\log\text{-cris}}^i(X^\times/K^\times) \\ \downarrow & & \downarrow \\ H_{\log\text{-cris}}^i(\mathcal{Y}^\times/S_{K_S}) & \longrightarrow & H_{\log\text{-cris}}^i(Y^\times/K_S^\times) \end{array}$$

of log-crystalline cohomology groups commutes, which as in Proposition 3.1 simply follows from functoriality of log-crystalline cohomology. \square

4. Cohomology and global approximations

Now suppose that k is a finite field, $F = k((t))$, and X/F is a smooth and proper variety.

DEFINITION 4.1. We say that X is globally defined if there exist a smooth curve C/k , a k -valued point $c \in C(k)$, a smooth and proper morphism $\mathbf{X} \rightarrow (C \setminus \{c\})$ and an isomorphism $F \cong \widehat{k(C)}_c$ such that $\mathbf{X}_F \cong X$.

We will prove the following strengthened version of [CL18, Corollary 5.5].

THEOREM 4.2. *For any smooth and proper variety X/F there exists a globally defined smooth and proper variety Z/F such that*

$$H_\ell^i(X) \cong H_\ell^i(Z)$$

for all ℓ (including $\ell = p$).

Once we have shown this, the proof of [CL18, Theorem 6.1] can then be completed using [CL18, Proposition 5.8], exactly as in the proof of [CL18, Theorem 5.1].

To prove Theorem 4.2, first of all choose a proper and flat model \mathcal{X} for X over the ring of integers \mathcal{O}_F . By [dJ96, Theorem 6.5] we may choose an alteration $\mathcal{X}_0 \rightarrow \mathcal{X}$ and a finite extension F_0/F such that \mathcal{X}_0 is strictly semistable over \mathcal{O}_{F_0} .

Next, we take the fibre product $\mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$, and let \mathcal{X}'_1 denote the disjoint union of the reduced, irreducible components of $\mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$ which are flat over \mathcal{O}_{F_0} , or equivalently which map surjectively to $\text{Spec}(\mathcal{O}_{F_0})$. Once more applying [dJ96, Theorem 6.5] to each of the connected components of \mathcal{X}'_1 in turn enables us to produce

- a 2-truncated augmented simplicial scheme

$$\mathcal{X}_1 \rightrightarrows \mathcal{X}_0 \rightarrow \mathcal{X}$$

which is a proper hypercover after base changing to F ;

- a collection $F_{1,1}, \dots, F_{1,s}$ of finite field extensions of F_0 ;

such that \mathcal{X}_1 is a disjoint union of schemes $\mathcal{X}_{1,j}$, for $1 \leq j \leq s$, proper and strictly semistable over $\text{Spec}(\mathcal{O}_{F_{1,j}})$.

Let k_0 denote the residue field of F_0 , $k_{1,j}$ the residue field of $F_{1,j}$, and consider the intermediate extensions

$$F \subset F_0^{\text{un}} \subset F_0^s \subset F_0 \subset F_{1,j}^{\text{un}} \subset F_{1,j}^s \subset F_{1,j}$$

where F_0^{un}/F and $F_{1,j}^{\text{un}}/F_0$ are separable and unramified, F_0^s/F_0^{un} and $F_{1,j}^s/F_{1,j}^{\text{un}}$ are separable and totally ramified, and F_0/F_0^s and $F_{1,j}/F_{1,j}^s$ are totally inseparable, of degree p^{d_0} and $p^{d_{1,j}}$ respectively. Let t denote a uniformiser for F , t_0 one for F_0^s , and let P_0 be the minimal polynomial of t_0 over F_0^{un} . Then $t'_0 := t_0^{1/p^{d_0}}$ is a uniformiser for \mathcal{O}_{F_0} . Similarly, let $t_{1,j}$ be a uniformiser for $F_{1,j}^s$, and $P_{1,j}$ the minimal polynomial of $t_{1,j}$ over $F_{1,j}^{\text{un}}$. Then $t'_{1,j} := t_{1,j}^{1/p^{d_{1,j}}}$ is a uniformiser for $\mathcal{O}_{F_{1,j}}$.

Now choose a finitely generated sub- k -algebra $R \subset \mathcal{O}_F$, containing t , such that there exists a proper, flat scheme $\mathcal{Y} \rightarrow \text{Spec}(R)$ whose base change to \mathcal{O}_F is exactly \mathcal{X} . By [Spi99, Theorem 10.1], we may at any point increase R to ensure that it is in fact smooth over k . Next, enlarge R so that $R_0^{\text{un}} := R \otimes_k k_0 \subset \mathcal{O}_{F_0}^{\text{un}}$ contains all the coefficients of the minimal (Eisenstein) polynomial P_0 of t_0 , and let R_0^s denote the corresponding finite flat extension $R_0^{\text{un}}[x]/(P_0)$ of R_0^{un} . We can

thus consider $R_0^s \subset \mathcal{O}_{F_0^s}$ as a subring containing t_0 , we set $R_0 = R_0^s[t_0']$. Hence we have $R_0 \subset \mathcal{O}_{F_0}$ such that

$$R_0 \otimes_R \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F_0}.$$

Note also that R_0 is finite and flat over R , after localising R within \mathcal{O}_F we may in fact assume that R_0 is finite free over R .

Next we enlarge R so that there exists a proper and flat morphism $\mathcal{Y}_0 \rightarrow \mathrm{Spec}(R_0)$ whose base change to \mathcal{O}_{F_0} is \mathcal{X}_0 . Again, by further enlarging R we may in addition assume that the map $\mathcal{X}_0 \rightarrow \mathcal{X}$ arises from a proper surjective map

$$\mathcal{Y}_0 \rightarrow \mathcal{Y}$$

of R -schemes, and moreover that there exists an open cover of \mathcal{Y}_0 by schemes which are étale over $R_0[x_1, \dots, x_n]/(x_1 \dots x_r - t_0')$ for some n, r . In other words, \mathcal{Y}_0 is “strictly t_0' -semistable”.

We now repeat this process to produce further finite free extensions $R_0 \rightarrow R_{1,j}^{\mathrm{un}} \rightarrow R_{1,j}^s \rightarrow R_{1,j}$ for all j , and an injection $R_{1,j} \subset \mathcal{O}_{F_{1,j}}$ containing the image of $t_{1,j}'$ such that

$$R_{1,j} \otimes_R \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F_{1,j}}.$$

We can also find proper, strictly $t_{1,j}'$ -semistable schemes $\mathcal{Y}_{1,j} \rightarrow \mathrm{Spec}(R_{1,j})$ whose base change to $\mathcal{O}_{F_{1,j}}$ is $\mathcal{X}_{1,j}$, so that setting $\mathcal{Y}_1 := \coprod_j \mathcal{Y}_{1,j}$ (and again, possibly increasing R), we obtain a 2-truncated augmented simplicial scheme

$$\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0 \rightarrow \mathcal{Y}$$

which becomes a proper hypercover over a dense open subscheme of $\mathrm{Spec}(R)$, and whose base change to \mathcal{O}_F is exactly our original 2-truncated augmented simplicial scheme

$$\mathcal{X}_1 \rightrightarrows \mathcal{X}_0 \rightarrow \mathcal{X}.$$

Let $\iota : R \hookrightarrow \mathcal{O}_F$ denote the canonical inclusion, and $\iota^* : \mathrm{Spec}(\mathcal{O}_F) \rightarrow \mathrm{Spec}(R)$ the induced morphism of schemes. Note that since ι^* maps the generic point of $\mathrm{Spec}(\mathcal{O}_F)$ to that of $\mathrm{Spec}(R)$, the map $\mathcal{Y} \rightarrow \mathrm{Spec}(R)$ is generically smooth. We may thus choose an open subset $U \subset \mathrm{Spec}(R)$ such that $\mathcal{Y}_U \rightarrow U$ is smooth, and such that the base change of $[\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0 \rightarrow \mathcal{Y}]$ to U is a proper hypercover.

LEMMA 4.3. *For any $n \geq 0$ there exists a smooth curve C/k , a rational point $c \in C(k)$, a uniformiser t_c at c , and a locally closed immersion $C \rightarrow \mathrm{Spec}(R)$ such that $C \setminus \{c\} \subset U$, and the induced map*

$$\mathrm{Spec}(\mathcal{O}_{C,c}/\mathfrak{m}_c^n) \rightarrow \mathrm{Spec}(R)$$

agrees with the modulo t^n -reduction of ι^ via the isomorphism*

$$\widehat{\mathcal{O}}_{C,c} \xrightarrow{\sim} \mathcal{O}_F$$

sending t_c to t .

Proof. Since R is smooth, we may choose étale co-ordinates around the image $\iota^*(s)$ of the closed point of $\mathrm{Spec}(\mathcal{O}_F)$ under ι^* . This induces an étale map $\mathrm{Spec}(R) \rightarrow \mathbb{A}_k^n$ for some n , and it is a simple exercise to prove the corresponding claim for \mathbb{A}_k^n . We then just take the pull-back to $\mathrm{Spec}(R)$. \square

The canonical inclusion ι induces similar inclusions

$$\iota_0^\# : R_0^\# \hookrightarrow R_0^\# \otimes_R \mathcal{O}_F = \mathcal{O}_{F_0^\#}$$

for $\# \in \{\text{un}, s, \emptyset\}$, as well as

$$\iota_{1,j}^\# : R_{1,j}^\# \hookrightarrow R_{1,j}^\# \otimes_R \mathcal{O}_F = \mathcal{O}_{F_{1,j}^\#}$$

for all j , and again for $\# \in \{\text{un}, s, \emptyset\}$. We will need the following form of Krasner's lemma [Stacks, Section 0BU9].

LEMMA 4.4. *Let K be a local field, with ring of integers \mathcal{O}_K , and let $P(x)$ be an Eisenstein polynomial over \mathcal{O}_K . Let L be the corresponding finite totally ramified extension, and let α be a root of P in L . Then for any $m \geq 1$ there exists an $n \geq 2$ such that any $Q(x) \in \mathcal{O}_K[x]$ congruent to P modulo \mathfrak{m}_K^n is Eisenstein, and L contains a root β of Q such that $L = K(\beta)$ and $\alpha \equiv \beta$ modulo \mathfrak{m}_L^m .*

We will use this as follows: given $n_1 \geq \max_j \{[F_{1,j} : F]\}$ Lemma 4.4 shows that there exists some $n_0 \geq \max \{2, [F_0 : F]\}$ such that any polynomial $Q_{1,j}$ with coefficients in $\mathcal{O}_{F_{1,j}^\text{un}}$ which agrees with the minimal polynomial $P_{1,j}$ of $t_{1,j}$ modulo $(t'_0)^{n_0}$ is Eisenstein, and has a root in $\mathcal{O}_{F_{1,j}^s}$ which agrees with $t_{1,j}$ modulo $t_{1,j}^{n_1}$. Applying the lemma again shows the existence of some $n \geq 2$ such that any polynomial Q_0 with coefficients in $\mathcal{O}_{F_0^\text{un}}$ which agrees with P_0 modulo t^n is Eisenstein, and has a root in $\mathcal{O}_{F_0^s}$ which agrees with t_0 modulo $t_0^{n_0}$. Now choose a k -algebra homomorphism $\lambda : R \rightarrow \mathcal{O}_F$ as provided by Lemma 4.3: that is, factoring through the local ring of some smooth point on a curve inside $\text{Spec}(R)$ and agreeing with ι modulo t^n .

Since λ is a k -algebra homomorphism, we have a canonical isomorphism $R_0^\text{un} \otimes_{R,\lambda} \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F_0^\text{un}}$, which therefore induces a homomorphism

$$\lambda_0^\text{un} : R_0^\text{un} \rightarrow \mathcal{O}_{F_0^\text{un}}$$

extending λ and which agrees with ι_0^un modulo t^n . Now let $Q_0 = \lambda_0^\text{un}(P_0)$ denote the image under λ_0^un of the minimal polynomial P_0 of t_0 , this is therefore a monic polynomial with coefficients in $\mathcal{O}_{F_0^\text{un}}$, which agrees with P_0 modulo t^n . Thus it is also Eisenstein, and by the choice of n we know that $\mathcal{O}_{F_0^s}$ contains a root of $\lambda_0^\text{un}(P_0)$ which is congruent to t_0 modulo $t_0^{n_0}$ and generates $\mathcal{O}_{F_0^s}$ as an $\mathcal{O}_{F_0^\text{un}}$ -algebra. This then allows us to extend λ_0^un to a homomorphism

$$\lambda_0^s : R_0^s \rightarrow \mathcal{O}_{F_0^s}$$

which agrees with ι_0^s modulo $t_0^{n_0}$, and since $\lambda_0^s(t_0)$ generates $\mathcal{O}_{F_0^s}$ as an $\mathcal{O}_{F_0^\text{un}}$ -algebra, we deduce that the diagram

$$\begin{array}{ccc} R_0^s & \xrightarrow{\lambda_0^s} & \mathcal{O}_{F_0^s} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\lambda} & \mathcal{O}_F \end{array}$$

is coCartesian. We can then extend this to a homomorphism

$$\lambda_0 : R_0 \rightarrow \mathcal{O}_{F_0}$$

agreeing with ι_0 modulo $(t'_0)^{n_0}$, and forming a similar coCartesian diagram to λ_0^s . We now play exactly the same game for all of the $R_{1,j}$, to produce $\lambda_{1,j} : R_{1,j} \rightarrow \mathcal{O}_{F_{1,j}}$ extending all other λ_0 and all previous $\lambda_{1,j}^\#$, which agree with $\iota_{1,j}$ modulo $(t'_{1,j})^{n_{1,j}}$, and which form coCartesian

diagrams

$$\begin{array}{ccc} R_{1,j} & \xrightarrow{\lambda_{1,j}} & \mathcal{O}_{F_{1,j}} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\lambda} & \mathcal{O}_F. \end{array}$$

Now let \mathcal{Z} be the base change of \mathcal{Y} to \mathcal{O}_F via λ , note that the generic fibre \mathcal{Z}_F is globally defined by construction. Similarly define \mathcal{Z}_0 to be the base change of \mathcal{Y}_0 to \mathcal{O}_{F_0} via λ_0 , $\mathcal{Z}_{1,j}$ the base change of $\mathcal{Y}_{1,j}$ to $\mathcal{O}_{F_{1,j}}$ via $\lambda_{1,j}$, and $\mathcal{Z}_1 := \coprod_j \mathcal{Z}_{1,j}$, so we have a 2-truncated augmented simplicial scheme

$$\mathcal{Z}_1 \rightrightarrows \mathcal{Z}_0 \rightarrow \mathcal{Z}$$

over \mathcal{O}_F , which gives a proper hypercover after base changing to F . For any $m \geq 2$ we can therefore take $n_1 \geq m \max_j \{[F_{1,j} : F]\}$ to ensure that:

- \mathcal{Z}_0 is a proper and strictly semistable scheme over \mathcal{O}_{F_0} , and each $\mathcal{Z}_{1,j}$ is a proper and strictly semistable scheme over $\mathcal{O}_{F_{1,j}}$
- there is an isomorphism

$$[\mathcal{X}_1 \rightrightarrows \mathcal{X}_0] \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m \xrightarrow{\sim} [\mathcal{Z}_1 \rightrightarrows \mathcal{Z}_0] \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m$$

of 2-truncated simplicial schemes, such that

$$\mathcal{X}_0 \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m \xrightarrow{\sim} \mathcal{Z}_0 \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m$$

is in fact an isomorphism of $\mathcal{O}_{F_0}/(t^m)$ -schemes, and

$$\mathcal{X}_1 \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m \xrightarrow{\sim} \mathcal{Z}_1 \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m$$

is obtained as a disjoint union of isomorphisms

$$\mathcal{X}_{1,j} \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m \xrightarrow{\sim} \mathcal{Z}_{1,j} \otimes_{\mathcal{O}_F} \mathcal{O}_F/t^m$$

of $\mathcal{O}_{F_{1,j}}/(t^m)$ -schemes

Thus if we let $\mathcal{X}_{0,s}^\times$ and $\mathcal{Z}_{0,s}^\times$ denote the log schemes over k_0^\times given by the special fibres of \mathcal{X}_0 and \mathcal{Z}_0 , and $\mathcal{X}_{1,s}^\times$ and $\mathcal{Z}_{1,s}^\times$ the log schemes over $\coprod_{j=1}^s \text{Spec}(k_{1,j}^\times)$ given by the special fibres of \mathcal{X}_1 and \mathcal{Z}_1 , then by Proposition 2.2 there is an isomorphism

$$[\mathcal{Z}_{1,s}^\times \rightrightarrows \mathcal{Z}_{0,s}^\times] \cong [\mathcal{X}_{1,s}^\times \rightrightarrows \mathcal{X}_{0,s}^\times]$$

of 2-truncated simplicial log schemes over k^\times . Now by [CL18, Propositions 5.3, 5.4] there are isomorphisms

$$\begin{aligned} H_\ell^i(\mathcal{X}_{0,F_0}) &\cong H_\ell^i(\mathcal{Z}_{0,F_0}) \\ H_\ell^i(\mathcal{X}_{1,F_{1,j}}) &\cong H_\ell^i(\mathcal{Z}_{1,j,F_{1,j}}) \end{aligned}$$

between the cohomology of the generic fibres of $\mathcal{X}_0, \mathcal{X}_{1,j}$ and $\mathcal{Z}_0, \mathcal{Z}_{1,j}$, as Weil–Deligne representations over F_0 and $F_{1,j}$ respectively. If we define the category

$$\text{Rep}_{\mathbb{Q}_\ell}(\text{WD}_{F_1}) := \prod_{j=1}^s \text{Rep}_{\mathbb{Q}_\ell}(\text{WD}_{F_{1,j}})$$

of Weil–Deligne representations over $F_1 := \prod_j F_{1,j}$ to be the product of the categories of Weil

Deligne representations over each $F_{1,j}$, then by Propositions 3.1 and 3.2, the diagram

$$\begin{array}{ccc} H_\ell^i(\mathcal{X}_{0,F_0}) & \longrightarrow & H_\ell^i(\mathcal{X}_{1,F_1}) \\ \cong \downarrow & & \downarrow \cong \\ H_\ell^i(\mathcal{Z}_{0,F_0}) & \longrightarrow & H_\ell^i(\mathcal{Z}_{1,F_1}) \end{array}$$

(with horizontal arrows given by the differences of the two pullback maps) commutes via the restriction functor from Weil–Deligne representations over F_0 to Weil–Deligne representations over F_1 .

Let $\mathrm{Ind}_{F_i}^F$ denote a right adjoint to the restriction functor from Weil–Deligne representations over F to those over F_i : on the separable part this is the normal induction of representations, on the inseparable part it is a quasi-inverse to Frobenius pull-back, and $\mathrm{Ind}_{F_1}^F = \bigoplus_j \mathrm{Ind}_{F_{1,j}}^F$. We therefore have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ind}_{F_0}^F H_\ell^i(\mathcal{X}_{0,F_0}) & \longrightarrow & \mathrm{Ind}_{F_1}^F H_\ell^i(\mathcal{X}_{1,F_1}) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Ind}_{F_0}^F H_\ell^i(\mathcal{Z}_{0,F_0}) & \longrightarrow & \mathrm{Ind}_{F_1}^F H_\ell^i(\mathcal{Z}_{1,F_1}), \end{array}$$

in particular, the kernels of both horizontal maps are isomorphic as Weil–Deligne representations over F . The proof of Theorem 4.2 now boils down to the following claim.

PROPOSITION 4.5. *Let $X_1 \rightrightarrows X_0 \rightarrow X$ be a 2-truncated semisimplicial proper hypercover of a smooth and proper F -variety X , such that there exist finite field extensions F_0/F and $F_{1,j}/F_0$ for $1 \leq j \leq s$, with X_0 smooth over F_0 , and $X_1 = \coprod_j X_{1,j}$ with $X_{1,j}$ smooth over $F_{1,j}$. If we set $F_1 = \prod_{j=1}^s F_{1,j}$, then*

$$H_\ell^i(X) \cong \ker(\mathrm{Ind}_{F_0}^F H_\ell^i(X_0) \rightarrow \mathrm{Ind}_{F_1}^F H_\ell^i(X_1))$$

for all primes ℓ .

Proof. By taking \tilde{F}_1/F a sufficiently large finite extension such that all of the $F_{1,j}$ embed into \tilde{F}_1 and applying [dJ96, Theorem 4.1], we can extend $X_1 \rightrightarrows X_0 \rightarrow X$ to a full proper hypercover $X_\bullet \rightarrow X$ such that for $n \geq 2$ there exists a finite extension F_n/\tilde{F}_1 with X_n smooth over F_n . Now applying [CL18, Lemma 6.4] we can see that the terms in i th column of the resulting spectral sequence have to be ‘quasi-pure’ of weight i . Therefore the spectral sequence degenerates exactly as in the proof of [CL18, Theorem 6.1], and the proposition follows. \square

We now deduce from the proposition that $H_\ell^i(X) \cong H_\ell^i(\mathcal{Z}_F)$ as Weil–Deligne representations for all i, ℓ , and by construction \mathcal{Z}_F is globally defined. This completes the proof of Theorem 4.2

Remark 4.6. Note the use of the finite field hypothesis (via a weight argument) in the proof of Proposition 4.5. It might be possible to relax the assumption to k perfect using a more sophisticated argument.

ACKNOWLEDGEMENTS

Both authors would like to thank W. Zheng for pointing out the error in [CL18], as well as L. Illusie for useful discussions concerning log structures, in particular the proof of Proposition 2.1.

We would also like to thank the anonymous referee for a careful reading of an earlier version of the manuscript, in particular for correcting a mistake in our use of alterations.

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