D-AFFINITY AND RATIONAL VARIETIES

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ABSTRACT. We investigate geometry of D-affine varieties. Our main result is that a D-affine rational projective surface over an algebraically closed field is a generalised flag variety of a reductive group.

Let us consider a connected smooth projective algebraic variety X over an algebraically closed field $\mathbb{K}$ of characteristic zero. By G/P we denote the generalised flag variety of a reductive algebraic group G.

Beilinson-Bernstein Localisation Theorem ([2] for G/B, [9, Th. 3.7] for G/P): *If $X \cong G/P$, then X is D-affine.*

It is a long-standing problem whether the converse statement holds or there are other smooth projective D-affine varieties (weighted projective spaces are D-affine but singular [21]). The converse statement is known for toric varieties [20] and homogeneous varieties [6]. Our main result is the converse statement for the rational surfaces:

Main Theorem:[1] (Corollary 2.3.) *If X is a D-affine rational surface, then $X \cong G/P$.*

In fact, we aim to cover the most general D-affine varieties with various intermediate statements. In particular, many results work in the positive characteristic as well. The reader should be aware that it is not known which of the partial flag varieties are D-affine in positive characteristic. Some of them are known to be D-affine: projective spaces [5], G/B in types A_2 [5] and B_2 [1, 18], odd-dimensional quadrics [15]. On the other hand, the grassmannian $Gr(2, 5)$ is not D-affine [14].

[1]Since this paper appeared as a preprint, Langer has shown that in characteristic 0 or $p > 7$ any D-affine surface is G/P [16]. It is a serious improvement of our result.

There are further notions of D-affinity and derived D-affinity in positive characteristic when instead of Grothendieck differential operators, either small differential operators [8, 11, 12, 19] or crystalline differential operators [3, 4, 16] are studied. These are not covered by the present paper, although some of our methods may prove useful for these unusual differential operators.

Let us explain the context of the paper section-by-section. In Section 1 we define D-affine varieties and make general observations about them and quasicoherent sheaves on varieties.

In Section 2 we study divisors on D-affine varieties. The main results are Theorems 2.1 and 2.2. Both are positivity statements about divisors on a D-affine variety. We use them to study D-affine surfaces in this section, proving Corollary 2.3, the main result of the paper.

In Section 3 we formulate two questions that could pave a way for further research for solving the converse Beilinson-Bernstein problem in general.

1. Preliminaries

Pushing for greater generality of some of our results, we will work over two algebraically closed field: a field $\mathbb{K}$ of characteristic zero and a field $\mathbb{F}$ of arbitrary characteristic.

Let X be an algebraic variety over $\mathbb{F}$, $\mathcal{O}_X-$Qcoh its category of quasicoherent sheaves. Following Kashiwara, we consider a sheaf $\mathcal{A}_X$ of $\mathbb{F}$-algebras on X together with a morphism of sheaves of algebras $j : \mathcal{O}_X \to \mathcal{A}_X$ such that $\mathcal{A}_X$ is a quasicoherent $\mathcal{O}_X$-module under left multiplication. Notice that the image of j is not necessarily central, so the left and the right module structures on $\mathcal{A}_X$ disagree. The main example of $\mathcal{A}_X$ are the structure sheaf $\mathcal{O}_X$ itself and the sheaf of Grothendieck differential operators $\mathcal{D}_X$.

We consider the category of $\mathcal{O}$-quasicoherent left $\mathcal{A}_X$-modules $\mathcal{A}_X-$Qcoh (i.e., sheaves of left $\mathcal{A}_X$-modules, quasicoherent as $\mathcal{O}_X$-modules). Notice that the cohomology of an $\mathcal{O}$-quasicoherent left $\mathcal{A}_X$-module $\mathcal{F}$ are independent of the category are independent of the category. Indeed,

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{A}_X}(\mathcal{A}_X, \mathcal{F}).$$

Furthermore, an $\mathcal{O}$-quasicoherent left $\mathcal{A}_X$-module admits a resolution $\mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \ldots$ by injective $\mathcal{A}_X$-modules that are flabby as sheaves [10, Prop 1.4.14] (notice that the reference states this for $\mathcal{D}_X$ but the proof works for $\mathcal{A}_X$). Hence,

$$H^k_{\mathcal{A}_X}(X, \mathcal{F}) \cong H^k(\Gamma(X, \mathcal{I}_\bullet)) \cong H^k(X, \mathcal{F}) \cong H^k_{\mathcal{O}_X}(X, \mathcal{F})$$

for all k. In particular, *the acyclicity* of $\mathcal{F}$ for the functor $\Gamma(X, -)$ (i.e., vanishing of $H^k(X, \mathcal{F})$ for $k > 0$) is independent of the category. On the other hand, the generation by global sections depends on the category. The space of global sections $\mathbb{A} = \Gamma(X, \mathcal{A}_X)$ is an algebra. We say that $\mathcal{F}$ is $\mathcal{A}$-*generated by global sections* if the following natural map is surjective:

$$\mathcal{A}_X \boxtimes_{\mathbb{A}} \Gamma(X, \mathcal{F}) \to \mathcal{F}.$$

For instance, $\mathcal{A}_X$ is always $\mathcal{A}$-generated by global sections but may or may not be $\mathcal{O}$-generated by global sections. The following lemma is immediate:

Lemma 1.1.   (1) *If $\mathcal{F}$ is $\mathcal{O}$-generated by global sections, then it is $\mathcal{A}$-generated by global sections.*
  (2) *The following two statements are equivalent:*
    (a) *$\mathcal{A}_X$ is $\mathcal{O}$-generated by global sections,*
    (b) *Any $\mathcal{O}$-quasicoherent left $\mathcal{A}_X$-module $\mathcal{F}$, $\mathcal{A}$-generated by global sections, is $\mathcal{O}$-generated by global sections*

The variety X is called *A-affine* if

- $\Gamma : \mathcal{A}_X-\mathrm{Qcoh} \to \mathbb{A}-\mathrm{Mod}$ is exact,
- if $\mathcal{F} \in \mathcal{A}_X-\mathrm{Qcoh}$ and $\Gamma(\mathcal{F}) \cong 0$, then $\mathcal{F} \cong 0$.

Lemma 1.2. [10, 13] *The following statements (where statement (5) requires X to be projective with an ample line bundle $\mathcal{L}$) about an algebraic variety X over $\mathbb{F}$ are equivalent:*

(1) *X is A-affine.*
(2) *$\Gamma : \mathcal{A}_X-\mathrm{Qcoh} \to \mathbb{A}-\mathrm{Mod}$ is an equivalence.*
(3) *$\mathcal{A}_X \boxtimes_{\mathbb{A}} - : \mathbb{A}-\mathrm{Mod} \to \mathcal{A}_X-\mathrm{Qcoh}$ is an equivalence.*
(4) *Each sheaf $\mathcal{F} \in \mathcal{A}_X-\mathrm{Qcoh}$ is acyclic for the functor $\Gamma(X, -)$ and $\mathcal{A}$-generated by global sections.*
(5) *There exists $N > 0$ such that the following two statements hold for all $n > N$:*
    (a) *$\mathcal{A}_X(-n) = \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-n\otimes}$ is $\mathcal{A}$-generated by global sections,*
    (b) *the map $\Gamma(\mathcal{A}_X(-n)) \otimes_{\mathbb{K}} \Gamma(\mathcal{L}^{n\otimes}) \to \Gamma(\mathcal{A}_X)$ is surjective.*

In the light of the condition (3) of Lemma 1.2 it is useful (as also introduced by Langer [15]) to call a variety X *A-quasiaffine* if each $\mathcal{O}$-quasicoherent left $\mathcal{A}_X$-module is $\mathcal{A}$-generated by global sections. Let us call a variety X *A-O-quasiaffine* if each $\mathcal{O}$-quasicoherent left $\mathcal{A}_X$-module is $\mathcal{O}$-generated by global sections.

While an A-affine variety is A-quasiaffine, a punctured affine space $X = \mathbb{K}^n \setminus \{0\}$, $n \geq 2$ gives an example of not-D-affine D-quasiaffine variety: the $\mathcal{D}_X$-module $\mathcal{O}_X$ has a non-vanishing higher cohomology.

It would be nice to have examples illustrating D-O-quasiaffinity (cf. Lemma 1.1).

Question 1.3. *Find an example of a variety X with a sheaf of algebras $\mathcal{A}_X$ such that X is A-affine but not A-O-quasiaffine variety.*

Question 1.4. *Is it true that a D-affine variety is necessarily D-O-quasiaffine?*

The following lemma is straightforward, so we skip a proof:

Lemma 1.5. *Let $\mathcal{F}, \mathcal{F}' \in \mathcal{A}_X-\mathrm{Qcoh}$ be $\mathcal{A}$-generated by global sections.*
   (1) *If $\mathcal{G} \in \mathcal{A}_X-\mathrm{Qcoh}$ is a quotient of $\mathcal{F}$, then $\mathcal{G}$ is $\mathcal{A}$-generated by global sections.*
   (2) *If $Y \subseteq X$ is a closed subscheme and $\mathcal{F}$ is $\mathcal{O}$-generated by global sections, then $\mathcal{F}|_Y$ is $\mathcal{O}_Y$-generated by global sections.*
   (3) *If $0 \to \mathcal{F}' \to \mathcal{G} \to \mathcal{F} \to 0$ is an exact sequence in $\mathcal{A}_X-\mathrm{Qcoh}$, then $\mathcal{G}$ is $\mathcal{A}$-generated by global sections.*

The third lemma is easy but contains not so well-known terminology, hence, we give a proof.

Lemma 1.6. *Let $\mathcal{F} \in \mathcal{A}_X-\mathrm{Qcoh}$ be normal in the sense of Barth as an $\mathcal{O}_X$-module. If for each $p \in X$ there exists an open neighbourhood $p \in U \subseteq X$ such that $X \setminus U$ is of codimension at least two and $\mathcal{F}|_U$ is $\mathcal{A}|_U$-generated by global sections, then $\mathcal{F}$ is $\mathcal{A}$-generated by global sections.*

Proof. Let $F = \mathcal{F}_X(X)$. Consider an open $U \subseteq X$ with $X \setminus U$ is of codimension at least two. Normality means that the restriction map $F = \mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism [7, p. 126].

Let Y be the support of the cokernel of the natural map $\gamma : \mathcal{A}_X \boxtimes F \to \mathcal{F}$. Generation of $\mathcal{F}|_U$ by global sections means that $U \cap Y = \emptyset$. Our condition means that no point p belongs to Y. Hence, $Y = \emptyset$ and γ is surjective. $\square$

The following well-known observation is sufficient for our ends. We believe that it is true for singular varieties as well: it should follow from the description of $\mathcal{D}_X$ as the dual $\mathcal{O}_X|\mathbb{F}$-algebra of the algebroid of functions on the formal neighbourhood of the diagonal $X \to X \times X$ [17, 2.4].

Lemma 1.7.   (1) *If X and Y are schemes over $\mathbb{F}$, $(\mathcal{A}_X, j_X : \mathcal{O}_X \to \mathcal{A}_X)$ and $(\mathcal{A}_Y, j_Y : \mathcal{O}_Y \to \mathcal{A}_Y)$ are sheaves of algebras with morphisms of sheaves of algebras such that $\mathcal{A}_X$ and $\mathcal{A}_Y$ are quasicoherent $\mathcal{O}_X$-module and $\mathcal{O}_Y$-module under corresponding*

left multiplications, then $\mathcal{A}_X \boxtimes \mathcal{A}_Y$ is a sheaf of $\mathbb{F}$-algebras on $X \times Y$ and

$$j_{X \times Y} : \mathcal{O}_{X \times Y} \xrightarrow{\cong} \mathcal{O}_X \boxtimes \mathcal{O}_Y \xrightarrow{j_X \boxtimes j_Y} \mathcal{A}_X \boxtimes \mathcal{A}_Y$$

is a morphism of sheaves of algebras such that $\mathcal{A}_X \boxtimes \mathcal{A}_Y$ is a quasicoherent $\mathcal{O}_{X \times Y}$-module.

(2) *If X and Y are smooth varieties over $\mathbb{F}$, then the natural map $\varphi_{X,Y} : \mathcal{D}_X \boxtimes \mathcal{D}_Y \to \mathcal{D}_{X \times Y}$ is an isomorphism of sheaves of $\mathbb{F}$-algebras on $X \times Y$.*

Proof. Notice that

$$\mathcal{A}_X \boxtimes \mathcal{A}_Y = p_X^*(\mathcal{A}_X) \otimes_{\mathcal{O}_{X \times Y}} p_Y^*(\mathcal{A}_Y)$$

where $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ are the projections. It suffices to compute the sections of $\mathcal{A}_X \boxtimes \mathcal{A}_Y$ on an open affine subset $U \times V \subseteq X \times Y$. Let $R := \mathcal{O}_X(U)$, $A := \mathcal{A}_X(U)$, $S := \mathcal{O}_Y(V)$, $B := \mathcal{A}_Y(V)$. Then

$$p_X^*(\mathcal{A}_X)(U \times V) = A \otimes_{\mathbb{F}} S, \quad p_Y^*(\mathcal{A}_Y)(U \times V) = R \otimes_{\mathbb{F}} B$$

so that we have a natural isomorphism f of $R \otimes_{\mathbb{F}} S$-modules

$$\mathcal{A}_X \boxtimes \mathcal{A}_Y(U \times V) = (A \otimes_{\mathbb{F}} S) \otimes_{R \otimes_{\mathbb{F}} S} (R \otimes_{\mathbb{F}} B) \xrightarrow{f} A \otimes_{\mathbb{F}} B$$

given, together with its inverse, by

$$f\big((a \otimes s) \otimes (r \otimes b)\big) = j_X(r)a \otimes j_Y(s)b, \quad f^{-1}(a \otimes b) = (a \otimes 1) \otimes (1 \otimes b).$$

Thus, $\mathcal{A}_X \boxtimes \mathcal{A}_Y$ is a sheaf of algebras. The map $j_{X \times Y}$ is given locally by

$$j_{X \times Y}(r \otimes s) = j_X(r) \otimes j_Y(s) (\text{also equal to } (1 \otimes s) \otimes (r \otimes 1)).$$

The second statement also becomes clear on an open affine subset $U \times V \subseteq X \times Y$. In this case $\mathcal{D}_X(U)$ and $\mathcal{D}_Y(V)$ are subalgebras of $\mathcal{D}_{X \times Y}(U \times V)$ so that the natural map

$$\varphi_{X,Y} : \mathcal{D}_X \boxtimes \mathcal{D}_Y(U \times V) \cong \mathcal{D}_X(U) \otimes_{\mathbb{F}} \mathcal{D}_Y(V) \to \mathcal{D}_{X \times Y}(U \times V)$$

is given by multiplication $\varphi_{X,Y}(a \otimes b) = ab$. It is a homomorphism of algebras because $\mathcal{D}_X(U)$ and $\mathcal{D}_Y(V)$ commute inside $\mathcal{D}_{X \times Y}(U \times V)$.

If X and Y are smooth, the tangent sheaves $\mathcal{T}_X$ and $\mathcal{T}_Y$ are locally free with local frames $\partial/\partial x_n$ and $\partial/\partial y_m$. Then $\mathcal{D}_X$ and $\mathcal{D}_Y$ have local bases $\partial^{|\alpha|}/\partial x^\alpha$ and $\partial^{|\beta|}/\partial y^\beta$, given by multi-indices α and β. An easy calculation in this basis shows that $\varphi_{X,Y}$ is an isomorphism. $\qquad\square$

Lemma 1.8. *If X and Y are D-affine D-O-quasiaffine varieties over $\mathbb{F}$ such the map $\varphi_{X,Y}$ from Lemma 1.7 is an isomorphism, then any open subset $j : U \hookrightarrow X \times Y$ is a D-O-quasiaffine variety.*

Proof. Decompose the global sections as a composition of functors

$$\Gamma : \mathcal{D}_{X \times Y} - \mathrm{Qcoh} \xrightarrow{(X \times Y \to X)_*} (D(X) \boxtimes \mathcal{D}_Y) - \mathrm{Qcoh} \xrightarrow{\Gamma} D(X \times Y) - \mathrm{Mod}.$$

The assumed tensor product decomposition together with D-affinity of X and Y imply that both functors are equivalences. Hence, $X \times Y$ is a D-affine variety. It is also D-O-quasiaffine by a combination of Lemma 1.1 and Lemma 1.7

If $\mathcal{F} \in \mathcal{D}_U - \mathrm{Qcoh}$, its direct images $j_* \mathcal{F}$ in the categories of D-modules, quasicoherent shaves and topological sheaves coincide. By D-affinity of $X \times Y$, $j_* \mathcal{F}$ is $\mathcal{O}$-generated by global sections. Hence, $\mathcal{F}$ is $\mathcal{O}$-generated by global sections. $\square$

2. DIVISORS

We go straight to the main result of this section.

Theorem 2.1. *Let X be an irreducible D-O-quasiaffine algebraic variety over $\mathbb{F}$, $Y \subset X$ an effective Cartier divisor. Then there exists n such that for all m, divisible by n the tensor power of the normal sheaf $\mathcal{N}_Y^{\otimes m}$ is $\mathcal{O}_Y$-generated by global sections.*

Proof. Let $U := X \setminus Y$ be the open complement and $j : U \hookrightarrow X$ its embedding. Observe that $j_*(\mathcal{O}_U) = \mathcal{O}_X(*Y)$ is a $\mathcal{D}_X$-submodule of the sheaf $\mathcal{M}_X$ of rational functions and a union of invertible sheaves $\mathcal{O}_X(nY)$. On an open subset $V \subseteq X$ (from some cover of X) the divisor Y is defined by a single function h so that

$$\mathcal{O}_X(nY)(V) = \{\frac{f}{h^n}\}, \quad \mathcal{O}_X(*Y)(V) = \{\frac{f}{h^k}\}, \quad f \in \mathcal{O}_X(V), k \in \mathbb{Z}.$$

Since $\mathcal{O}_X(*Y)$ is a D-module, it is $\mathcal{O}$-generated by global sections $s_1, s_2 \ldots$ Each s_k is a global section of a line bundle $\mathcal{O}_X(n_k Y)$. Out of the cover $X = \cup_k \{s_k \neq 0\}$ choose a finite subcover. Let n be the least common multiple of all n_k from the finite subcover. At every point of X one of the sections from the finite subcover does not vanish. It follows that for all m with $n|m$ the invertible sheaf $\mathcal{O}_X(mY)$ is $\mathcal{O}$-generated by global sections because one of s_k^{m/n_k} does not vanish at every point.

Observe that we have the standard sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \xrightarrow{f} \mathcal{N}_{Y|X} \to 0$$

without any further restrictions on X or Y, where $\mathcal{N}_{Y|X}$ is the normal sheaf of the embedding $Y \hookrightarrow X$, considered as a sheaf on X. This

yields another exact sequence

$$0 \to \ker f^{\otimes m} \to \mathcal{O}_X(mY) \xrightarrow{f^{\otimes m}} \mathcal{N}_{Y|X}^{\otimes m} \to 0$$

for all m. By Lemma 1.5, if $n|m$, then $\mathcal{N}_{Y|X}^{\otimes m}$ is $\mathcal{O}$-generated by global sections. Hence, so is its restriction $\mathcal{N}_Y^{\otimes m} = (\mathcal{N}_{Y|X}^{\otimes m})|_Y$. □

We can derive some geometric consequences of D-affinity as soon as we can exhibit some interesting $\mathcal{D}_X$-modules. For example, $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, thus, if X is complete, we know some of its Hodge numbers

$$h^{0,0}(X) = 1, \ h^{0,m}(X) = 0 \ \text{ for } \ m > 0.$$

For a smooth projective surface X this means that $p_a = p_g = 0$. If X is also D-O-affine, Theorem 2.1 implies that the surface is minimal in a strong sense: $Y^2 \geq 0$ for any curve $Y \subseteq X$. Moreover,

$$c_2(X) = 2 + h^{2,2}(X), \ c_1^2(X) = 10 - h^{2,2}(X), \ 1 \leq h^{2,2}(X) \leq 10.$$

In the light of the next theorem, it would be interesting to classify minimal models with such numerical invariants that do not have any negative curves.

Theorem 2.2. *A projective D-affine variety X over $\mathbb{F}$ of dimension at least 2 cannot have any contractible divisors.*

Proof. Let $C \subseteq X$ be a contractible divisor. Let $\widetilde{X}$ be the blow-down of X at C. The centre of the blow-up $X \to \widetilde{X}$ is a subscheme $Y \subseteq \widetilde{X}$ supported at a finite number of points. We can pick a divisor $D \subseteq \widetilde{X}$ such that $Y \not\subset D$ and $\widetilde{X} \setminus D$ is affine. Let $U = f^{-1}(\widetilde{X} \setminus (D \cup Y))$. Then U is quasiaffine but not affine, while the complement $X \setminus U$ is a divisor. This contradicts Thomsen's Theorem that states if the complement of a divisor on a D-affine variety is quasiaffine, then it is affine [20, Lemma 1]. Notice that while Thomsen also assumes smoothness, it is never used in the proof. □

We are ready to prove the main theorem:

Corollary 2.3. *A rational smooth connected projective D-affine surface X over $\mathbb{F}$ is isomorphic to either P^2 or $P^1 \times P^1$.*

Proof. By Theorem 2.2, X is minimal. A minimal smooth rational surface is either P^2 or the Hirzebruch surface H_n, $n \geq 0$. Since H_n contains an irreducible curve C with $C^2 = -n$, we conclude that $n = 0$. Finally, $H_0 \cong P^1 \times P^1$. □

Given an arbitrary closed subvariety $Z \subseteq X$, we can produce some $\mathcal{D}_X$-modules supported on Z, for instance, functions on the formal neighbourhood of Z or local cohomology sheaves $\mathcal{H}_Z^n(\mathcal{F})$ where $\mathcal{F}$ is an $\mathcal{D}_X$-module, e.g., $\mathcal{F} = \mathcal{O}_X$ or $\mathcal{F} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}$ for a line bundle $\mathcal{L}$. It would be interesting to analyse how affinity of these sheaves affects geometry of X.

It is also interesting to resolve D-affinity of some particular varieties:

Question 2.4. *Can a fake projective space be D-affine?*

Question 2.5. *Can a cubic hypersurface in P^n, $n > 3$ be D-affine?*

3. Two further questions

We would like to state two further questions that could be quite useful for further research, including future attempts to settle the inverse Beilinson-Bernstein problem.

Question 3.1. *Let X be an irreducible D-affine algebraic variety over $\mathbb{F}$, $Y \subset X$ an effective Cartier divisor. Is $\mathcal{N}_Y$ necessarily $\mathcal{O}_Y$-generated by global sections?*

Question 3.2. *Characterize the class of varieties X such that for each point $p \in X$ there exists an open set $U \subseteq X$ such that*

- *$p \in U$,*
- *the complement $X \setminus U$ has a codimension at least 2,*
- *$\delta : U \to U^2$ is a scheme-theoretic complete intersection.*

References

[1] H. Andersen, M. Kaneda, On the D-affinity of the flag variety in type B_2, *Manuscripta Math.* **103** (2000), 393–399.

[2] A. Beilinson, J. Bernstein, Localisation de g-modules, *C. R. Acad. Sci. Paris Ser. I Math.* **292** (1981), 15–18.

[3] R. Bezrukavnikov, I. Mirković, D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, *Ann. of Math. (2)* **167** (2008), 945–991.

[4] R. Bezrukavnikov, I. Mirković, D. Rumynin, Singular localization and intertwining functors for reductive Lie algebras in prime characteristic, *Nagoya Math. J.*, 184 (2006), 1–55.

[5] B. Haastert, Über Differentialoperatoren und D-Moduln in positiver Charakteristik, *Manuscripta Math.* **58** (1987), 385–415.

[6] K. El Haloui, D. Rumynin, D-modules and projective stacks, *Journal of Algebra* **502** (2018), 515–537.

[7] R. Hartshorne, Stable reflexive sheaves, *Math. Ann.* **254** (1980), 121–176.

[8] Y. Hashimoto, M. Kaneda, D. Rumynin, On localization of $\overline{D}$-modules, *Representations of algebraic groups, quantum groups, and Lie algebras*, 43–62, Contemp. Math., 413, Amer. Math. Soc., Providence, 2006.

[9] M. Holland, P. Polo, K-theory of twisted differential operators on flag varieties, *Invent. Math.* **123** (1996), 377–414.

[10] R. Hotta, K. Takeuchi, T. Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Birkhauser, Boston, 2008.

[11] M. Kaneda, J. Ye, Equivariant localization of $\overline{D}$-modules on the flag variety of the symplectic group of degree 4, *J. Algebra* **309** (2007), 236–281.

[12] M. Kaneda, On the Frobenius direct image of the structure sheaf of a homogeneous projective variety, *J. Algebra* **512** (2018), 160–188.

[13] M. Kashiwara, Representation theory and D-modules on flag varieties, *Orbites unipotentes et représentations, III*, 55–109, Astérisque **173–174** (1989).

[14] M. Kashiwara, N. Lauritzen, Local cohomology and D-affinity in positive characteristic, *C. R. Math. Acad. Sci. Paris* **335** (2002), 993–996.

[15] A. Langer, D-affinity and Frobenius morphism on quadrics, *Int. Math. Res. Not.* (2008), no. 1, Art. ID rnm 145.

[16] A. Langer, On smooth projective D-affine varieties, arXiv:1906.00227.

[17] D. Rumynin, Duality for Hopf algebroids, *J. Algebra* **223** (2000), 237–255.

[18] A. Samokhin, On the D-affinity of flag varieties in positive characteristic, *J. Algebra* **324** (2010), 1435–1446.

[19] A. Samokhin, A vanishing theorem for differential operators in positive characteristic, *Transform. Groups* **15** (2010), 227–242.

[20] J. Thomsen, D-affinity and toric varieties, *Bull. London Math. Soc.*, **29** (1997), 317–321.

[21] M. Van den Bergh, *Differential operators on semi-invariants for tori and weighted projective spaces*, Topics in invariant theory (Paris, 1989/1990), 255–272, Lecture Notes in Math., 1478, Springer, Berlin, 1991.

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