HODGE NUMBERS AND DEFORMATIONS OF FANO 3-FOLDS

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Abstract. We show that index 1 Fano 3-folds which lie in weighted Grassmannians in their total anticanonical embedding have no infinitesimal deformations, and we relate the deformation theory of any Fano 3-fold that has a K3 elephant to its Hodge theory. Combining these results with standard Gorenstein projection techniques calculates both the number of deformations and the Hodge numbers of most quasismooth Fano 3-folds in low codimension. This provides detailed new information for hundreds of deformation families of Fano 3-folds.

1. Introduction

1.1. **Aims and context.** The classification of nonsingular Fano 3-folds [32, 33, 41] is a landmark in modern birational geometry. The result is a (finite) list of deformation families, documented by Iskovskikh–Prokhorov [34, §12.2], with detailed information about each family, including equation formats and the Hodge numbers of individual members. The need for generalisation to the 'Mori category' of $\mathbb{Q}$-factorial terminal Fano 3-folds has been well understood since the 1980s. The Mori-theoretic classification remains incomplete, but a wealth of information is known. For example, Kawamata's finiteness result [35, 37] leads to a finite list of Hilbert series which includes all those of Fano 3-folds; this list is documented in the Graded Ring Database [8]. At this stage, we have some understanding of a few hundred families of Fano 3-folds that form a subset of the ultimate classification, which may comprise a few thousand or even tens of thousands of families. The search by graded ring methods works systematically in increasing codimension (in the full anticanonical embedding that is intrinsic to a Fano; see §2.1). All families are known up to codimension 3, and in codimension 4 a collection of results (such as those by Takagi and others [59, 10, 9, 54, 60], and Suzuki, Prokhorov–Reid and others [58, 46, 47, 26, 20] in higher Fano index) suggest we know most families there.

The main results of this paper (Theorems 1, 3 and 10) are on the infinitesimal rigidity of Fano 3-folds (that is, $H^0(X, T_X) = 0$), their deformations and Hodge theory, and a Lefschetz theorem on weighted Grassmannians (we refer to [19] for weighted Grassmannians). The latter seems surprisingly delicate: in the weighted context, low-degree linear systems are seldom free and so the state of the art ([49, Theorem 1] and [29, Corollary 2.8] for example) does not apply directly. Put together, these establish a connection between the deformation theory of a range of Fano 3-folds and their Hodge theory by proving a formula that calculates $h^1(X, T_X)$ in terms of $h^{2,1}(X)$. The systematic use of weighted cones to relate results on manifolds to their analogues on a large set of orbifolds (the proof of Lemma 9, for example) seems new to us in this context.

As an application, this paper also contributes detailed numerical information analogous to that of [34] for the families up to codimension 3, and some cases in codimension 4. The theorems provide tools that we apply to calculate the Hodge numbers $h^{p,q}(X)$ and the number of moduli $h^1(X, T_X)$ of all known index 1 Fano 3-folds X in codimensions 1, 2 and 3. These results are presented in Tables 1, 2 and 3 respectively; the Picard rank is 1 in every case. It seems to be the nature of the birational classification of Fano 3-folds, or perhaps the array of different gradings that arise, that some results boil down to hundreds of calculations that cannot always be systematised in one go (compare Corti–Pukhlikov–Reid [17] and Cheltsov–Przyjalkowski–Shramov [13],

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both of which summarise extensive calculations in Big Tables). Thus in §4 we explain general approaches in different situations, and illustrate them with particular calculations, including some in codimension 4.

It is worth emphasising that computing these numbers seems hard: we do not have flexible techniques to hand for working with the orbifold Chern classes of non complete intersections, and so resort to birational techniques of projection. Nevertheless, most cases follow the models in §4 and can be worked out by hand: we explain in Appendix A.3 and §2.6 how Tables 1–3 encode both the strategy and the proof of the calculations, working up from hypersurfaces through a 'staircase' of projections (§2.5). Techniques here are similar to the ones used in [18, 16].

In the few cases where we do not have geometric projections to work with, we can recover numerical information from certain graded pieces of the deformation theory using computer algebra to calculate in certain Jacobian rings; this is explained in §2.3. The key is Di Natale–Fatighenti–Fiorenza's [23] characterisation of deformation theory in terms of Hodge theory, and this also provides an additional computer check on all our manual results.

To illustrate the computer algebra tool further, we compute the Hodge numbers of some Fano 3-folds that lie in codimension 4. General varieties in codimension 4 are beyond our theoretical methods today, seemingly because they tend not to lie in concrete formats related to key varieties, such as a Grassmannians, whose own deformation theory could be exploited. This lack of format is a recurring theme in Gorenstein codimension 4, notwithstanding the ancient wisdom of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^2$ (see [9]); compare with Reid's structure theory [53] and the commentary therein. Our calculations in codimension 4 give a computer-assisted verification of a result of Takagi [59, Theorem 0.3], and begin to answer the main open question of Brown–Kerber–Reid [10, 3.4] on the Picard ranks of Fano 3-folds in codimension 4.

1.2. **Formal statement of results.** A *Fano 3-fold* is a normal 3-dimensional complex projective variety X with ample anticanonical class $-K_X$ and $\mathbb{Q}$-factorial terminal cyclic quotient singularities. (Of course more general notions of Fano 3-fold exist in the literature, but our methods work with orbifolds, and so this definition is appropriate here. In the Mori-theoretic context, it would be natural to broaden the definition slightly, by allowing X to have arbitrary terminal singularities. But note that, in that case, if X does not have a terminal singularity of the exceptional type $\frac{1}{4}(1,1,3,2;2)$ (see [40] [51, Theorem (6.1)(2)]), then Sano [54, Theorem 1.5] shows that there is a small deformation that has only quotient singularities – this is a so-called *Q-smoothing*. Thus our restriction to cyclic quotient singularities is not so severe.)

The *index* q_X of a Fano 3-fold X is the largest integer q for which there exists a $\mathbb{Q}$-Cartier Weil divisor A with $-K_X \overset{\text{lin}}{\sim} qA$.

A *K3 elephant* of a Fano 3-fold X is an irreducible surface $E \subset X$ with canonical singularities that is linearly equivalent to $-K_X$. In particular, E has $K_E = 0$, and so E is a K3 surface.

This paper has three main ingredients. The first is an unprojection calculus (see §2.5 or [11]). The second is a relation between the Hodge numbers of a Fano 3-fold and the number of its moduli, and the third is an infinitesimal rigidity result; we summarise these two as follows.

Theorem 1. *Let X be a Fano 3-fold with K3 elephant $E \subset X$.*

(i) *Setting $\alpha_E = h^{1,1}(E) - h^0(E, -K_{X|E})$, we have*

$$h^1(X, T_X) - h^0(X, T_X) = \alpha_E + h^{2,1}(X) - h^{2,2}(X). \tag{1}$$

(ii) *Suppose in addition that X has index $q_X = 1$ and that X is a complete intersection in weighted projective space or in a weighted Grassmannian $w\,\mathrm{Gr}(2,5)$. Then $h^0(X, T_X) = 0$. In particular, $\mathrm{Aut}(X)$ is a finite group in this case.*

The final part of this result compares with a sequence of recent papers that compute the automorphism group of smooth Fano varieties. Smooth weighted complete intersections of dimension at least 3 have finite automorphism groups by [48]; indeed [48, Corollary 4.5] proves $h^0(X, T_X) = 0$ in that case, similarly exploiting [27] as we do here, and thus concludes finiteness. In contrast, [39, 13] classify smooth Fano 3-folds with infinite automorphism group.

Part (i) is proved in §3.1. In the case where $q_X = 1$ we may express α_E purely in terms of the geometry of E and X as $\alpha_E = h^{1,1}(E) - g_X - 1$, where $g_X = h^0(X, -K_X) - 2$ is the *genus* of X. Part (ii) is proved in §3.2. The fact that $h^{1,1}(X) = 1$ for most cases we consider is Theorem 3. We work over $\mathbb{C}$ throughout.

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2. Hodge numbers of Fano 3-folds

2.1. **Fano 3-folds in their anticanonical embeddings.**

We study a Fano 3-fold X using its anticanonical graded ring

$$R(X, -K_X) = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(-mK_X)).$$

A minimal set of generators $x_0, \ldots, x_n$ for $R(X, -K_X)$, whose degrees are denoted $a_0, \ldots, a_n$, present X as a subvariety $X \subset \mathbb{P}(a_0, \ldots, a_n)$ defined by the relations holding in the ring. By definition, the *codimension* of a Fano 3-fold X is its codimension in this embedding: $\operatorname{codim}(X) = n - 3$. (These numerical properties are well defined for each given X: each graded piece of $R(X, -mK_X)$ is finite dimensional, so choosing generators x_i inductively modulo products from lower degrees determines n and the a_i, even though there is choice for the x_i. But it is important to note that the Hilbert series P_X of $R(X, -K_X)$ does *not* determine these numerical quantities.)

According to [35, 37], the classification of Fano 3-folds consists of finitely many deformation families. The Hilbert series of members of those families whose generic element lies in codimension at most 4 are known [1, 2] and available on the Graded Ring Database [8]. They fall into $95 + 85 + 70 + 145 = 395$ cases, according to the minimum realised codimension. There may be more than one irreducible family for any given Hilbert series, they may lie in different codimensions [9], and in codimension 4 there are usually two or more families in each case [11]; in all known cases, the different families are distinguished by the Euler characteristic of their general member.

The relationship between the orbifold nature of X and its equations when embedded in this way is standard, following [30, §§6,8], though not without subtlety. A variety $X \subset \mathbb{P}(a_0, \ldots, a_n) = \mathbb{A}^{n+1} /\!\!/ \mathbb{C}^*$ is said to be *quasismooth* if its affine cone $C_X \subset \mathbb{A}^{n+1}$ is smooth away from the origin. This condition may be tested by the usual Jacobian condition (on the rank of the Jacobian matrix at every point). If the equations at a point $P \in X$ satisfy the Jacobian condition, then an analytic neighbourhood of P inside X is the quotient of a complex 3-ball by a finite cyclic group. Thus a quasismooth $X \subset \mathbb{P}(a_0, \ldots, a_n)$ is a V-manifold.

2.2. **The Hodge numbers of Fano 3-folds.**

Because it is a V-manifold, the cohomology of a quasismooth variety $X \subset \mathbb{P}(a_0, \ldots, a_n)$ carries a pure Hodge structure by Steenbrink [57, Theorem 1.12], defined as follows. Consider the smooth locus $j : X_0 \hookrightarrow X$ and set $\widehat{\Omega}^p_X := j_*\Omega^p_{X_0}$. Then the Hodge decomposition then takes the form

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \widehat{\Omega}^p_X),$$

and one defines $H^{p,q}(X) := H^q(X, \widehat{\Omega}^p_X)$. Since there is no risk of confusion as we only ever work with $\widehat{\Omega}^p_X$, we abuse notation from here on and write Ω^p_X instead of $\widehat{\Omega}^p_X$. It follows at once from the Lefschetz hyperplane theorem [57, Theorem (1.13)] and Kawamata–Viehweg vanishing [38, Theorem 2.70] that the Hodge diamond of a Fano 3-fold X has the form

$$\begin{matrix} &&& h^{3,3} &&& \\ && h^{3,2} && h^{2,3} && \\ & h^{3,1} && h^{2,2} && h^{1,3} & \\ h^{3,0} && h^{2,1} && h^{1,2} && h^{0,3} \\ & h^{2,0} && h^{1,1} && h^{0,2} & \\ && h^{1,0} && h^{0,1} && \\ &&& h^{0,0} &&& \end{matrix} = \begin{matrix} &&& 1 &&& \\ && 0 && 0 && \\ & 0 && h^{2,2} && 0 & \\ 0 && h^{2,1} && h^{1,2} && 0 \\ & 0 && h^{1,1} && 0 & \\ && 0 && 0 && \\ &&& 1 &&& \end{matrix}.$$

Since such a Hodge structure is pure and an appropriate version of the Hard Lefschetz theorem holds in this context ([57, Theorem (1.13)]), $h^{2,2}(X) = h^{1,1}(X)$ and the Euler characteristic $e(X)$ of X satisfies

$$e(X) = 2 + 2h^{1,1}(X) - 2h^{2,1}(X).$$

We calculate these three integers for X in the known families of Fano 3-folds with small codimension. We explain the different strategies we employ in §2.6 below.

The answer is well known in codimension 1: the Hodge numbers of weighted hypersurfaces are computed by results of Griffiths, Dolgachev and Dimca. (Recall that primitive cohomology is the kernel of the hyperplane operator: if X has dimension m and hyperplane class L, then

$$H^k(X,\mathbb{C})_{\mathrm{prim}} = \ker\left\{\cap L^{m-k+1}\colon H^k(X,\mathbb{C}) \to H^{2m+2-k}(X,\mathbb{C})\right\},$$

and $H^{p,q}_{\mathrm{prim}}(X) = H^{p,q}(X) \cap H^{p+q}(X,\mathbb{C})_{\mathrm{prim}}$. When X is a Fano 3-fold, then $b_5(X) = 0$ and so $H^{2,1}_{\mathrm{prim}}(X) = H^{2,1}(X)$.)

Theorem ([25, 24, 30])**.** *Let $X_d\colon (f=0) \subset \mathbb{P}(a_0,\dots,a_n)$ be a quasismooth hypersurface defined by a homogeneous polynomial f of degree d in weighted homogeneous coordinates $x_0,\dots,x_n$ of degrees $\deg x_i = a_i$. Then the Milnor algebra $\mathcal{M} = \mathbb{C}[x_0,\dots,x_n]/J_f$ of X is finite dimensional, and there is an isomorphism*

$$H^{n-p,p-1}_{\mathrm{prim}}(X) \cong \mathcal{M}^{pd-\sum a_i}.$$

The Hilbert Series $P_{\mathcal{M}}$ of the Milnor algebra $\mathcal{M}$ is given, in the notation of the theorem, by

$$P_{\mathcal{M}} = \frac{(1-t^{b_0})\cdots(1-t^{b_n})}{(1-t^{a_0})\cdots(1-t^{a_n})}, \quad \text{where } b_i = d - a_i.$$

For example, $X_{66} \subset \mathbb{P}(1,5,6,22,33)$ has

$$\begin{aligned} P_{\mathcal{M}} &= \frac{\prod_{b\in\{65,61,60,44,33\}}(1-t^b)}{\prod_{a\in\{1,5,6,22,33\}}(1-t^a)} \\ &= 1+t+t^2+t^3+t^4+2t^5+\cdots+118t^{64}+120t^{65}+122t^{66}+\cdots+t^{196}. \end{aligned}$$

Thus we read off $h^{2,1}(X) = \dim \mathcal{M}^{2\cdot 66-67} = \dim \mathcal{M}^{65} = 120$. We list all 95 cases in Table 1.

2.3. **Calculating T^1 and $h^{2,1}(X)$.** We recall the context and results of [23]. A *subcanonical pair* $(X, \mathcal{O}_X(1))$ consists of a quasismooth projective variety X and an ample sheaf $\mathcal{O}_X(1)$ which satisfies $\omega_X \cong \mathcal{O}_X(k_X)$ for some $k_X \in \mathbb{Z}$. (The results of [23] are stated with X smooth. However, the proofs apply verbatim to give the same conclusions in the case X quasismooth, as noted in [23] at the beginning of §2.2, and we use that level of generality here.)

Let $(X, \mathcal{O}_X(1))$ be a subcanonical pair. We denote by A_X the affine cone over X and by $U_X = A_X \setminus \{v\}$, where v is the vertex of the cone. The results of [23] require that $\mathrm{depth}_v A_X \geq 3$, which holds in our context since $H^1(X, \mathcal{O}_X(j)) = 0$ for any $j \in \mathbb{Z}$.

For $i \geq 0$, one defines (following [55], since X is projectively normal)

$$T^i_{A_X} := \operatorname{Ext}^i_{\mathcal{O}_{A_X}}(\Omega^1_{A_X}, \mathcal{O}_{A_X});$$

this admits a $\mathbb{Z}$-grading given by the natural $\mathbb{C}^*$-action on A_X, and we denote the graded piece in degree d by $T^i_{A_X}(d)$.

The space $T^1_{A_X}$ parametrizes the set of isomorphism classes of first order infinitesimal deformations of A_X. By [55], the degree 0 component $T^1_{A_X}(0)$ of the deformations of the affine cone parametrizes the embedded deformations of X; that is, the deformations of the pair $(X, \mathcal{O}_X(1))$. Furthermore, the negative components are identified with the smoothings of the affine cone, while the positive components parametrize equisingular deformations. In the case of a smooth projective hypersurface of degree d,

$$T^1_{A_X}(-d) \cong \mathbb{C}[x_0, \dots, x_n]/J_f,$$

the Jacobian ring of X, as in §2.2.

Theorem 2 ([23] Theorem 1.1)**.** *Let $(X, \mathcal{O}_X(1))$ be a subcanonical pair with $\omega_X \cong \mathcal{O}_X(k_X)$. Set $n = \dim X$. Then there is an isomorphism*

$$T^1_{A_X}(k) \cong \ker\left(\lambda\colon H^1(X, \Omega^{n-1}(k - k_X)) \longrightarrow H^2(X, \omega_X(k - k_X))\right),$$

where $\lambda(\eta) = c_1(\mathcal{O}_X(1)) \wedge \eta$.

When $k = k_X$, the statement becomes $T^1_{A_X}(k_X) \cong H^{n-1,1}_{\text{prim}}(X)$, the primitive cohomology.

2.4. **Calculating the Picard number.** Every Fano 3-fold in codimension up to 3 arises in one of the two situations of the Theorem 3, which calculates $h^{1,1}(X)$. (Recall that if $V \subset w\mathbb{P}^N$, in coordinates $x_0, \dots, x_N$, is a variety in weighted projective space, then a weighted cone $\mathrm{C}\,V$ on V is defined by the equations of V in a larger space $w\mathbb{P}^{N+\ell}$ with coordinates $x_0, \dots, x_N$, $y_1, \dots, y_\ell$. See for example [19, (2.5)].)

Theorem 3 (c.f. [17] Lemma 3.5, [45])**.** *Suppose that X is a quasismooth Fano 3-fold that is either*

(i) *a complete intersection in weighted projective space, or*

(ii) *a complete intersection in a weighted cone over a weighted* $\mathrm{Gr}(2, 5)$ *with index $q_X = 1$.*

Then $h^{1,1}(X) = 1$.

Proof. Part (i) is Proposition 2.3 of [45]. For part (ii), we prove that $T^2_{A_X}(-1) = 0$, where A_X is the affine cone on X. This is enough since by [23, Theorem 2.6] we have $H^{1,1}_{\text{prim}}(X) \cong T^2_{A_X}(-1) = 0$, and so $h^{1,1}(X) = 1$. (Note that X satisfies the arithmetically Cohen–Macaulay conditions $H^1(X, mK_X) = 0$ for all $m \in \mathbb{Z}$ required for [23] by Kawamata–Viehweg vanishing and Serre duality [38, Theorem 2.70, Corollary 5.27].)

Let $\mathrm{C}\,\mathbb{P}$ denote the ambient projective space for the Grassmannian in its Plücker embedding with the addition of the cone variables. It follows from [23] that $T^2_{A_X}(-1) \cong H^1(X, N_{X/\mathrm{C}\,\mathbb{P}}(-1))$. Indeed this is a graded piece of equation (2.3) of [23], together with the isomorphism $T^2_{A_X} \cong H^2(U_X, \Theta_{U_X})$ that follows it, given that $H^2(X, \mathcal{O}_X(-1)) = 0$.

From [56, §D.1, (D.3)] the flag of schemes $X \subset \mathrm{C}\,\mathrm{Gr} \subset \mathrm{C}\,\mathbb{P}$ determines a sequence of sheaves on X:

$$0 \to N_{X/\mathrm{C}\,\mathrm{Gr}} \to N_{X/\mathrm{C}\,\mathbb{P}} \to N_{\mathrm{C}\,\mathrm{Gr}/\mathrm{C}\,\mathbb{P}} \otimes \mathcal{O}_X \to 0,$$

where the last map is exact, by the argument in [56, §D.1, Lemma D.3(ii)]: we have that $Ext^1_{\mathcal{O}_X}(N^*_{X/\mathrm{C}\,\mathrm{Gr}}, \mathcal{O}_X) = H^1(X, N_{X/\mathrm{C}\,\mathrm{Gr}})$, and that group is 0 since X is arithmetically Cohen–Macaulay. Twisting by $\mathcal{O}_X(-1)$ we get

$$H^1(N_{X/\mathrm{C}\,\mathbb{P}}(-1)) \cong H^1(N_{\mathrm{C}\,\mathrm{Gr}/\mathrm{C}\,\mathbb{P}}(-1)|_X) = 0.$$

The latter equality follows from the Koszul complex, together with the description of the normal bundle of Gr(2, 5) as $\bigwedge^2 Q$, see [19] in the weighted case. This proves part (ii). □

Our proof of (ii) above also gives an alternative proof of (i), at least in the index 1 case: [55, 1.3] provides the required vanishing of $T^2_{A_X}(-1)$.

We found (ii) stated several times in the literature, such as [36], but we could not find a proof to cite. In this situation, one would like appeal to folklore and simply apply a weighted Lefschetz hyperplane theorem for ample systems. But unfortunately the linear systems we cut by to make X are rarely base-point free when there are nontrivial weights, so the strong results in the literature such as [49, Theorem 1] and [29, Corollary 2.8] do not apply directly.

Thus the strategy for most cases in codimension 1, 2 and 3 is to compute the Euler characteristic by some means, deduce the remaining Hodge numbers by Theorem 3, and finally compute deformations by Theorem 1. In codimension 3 there are three cases which don't have a simple projection we can use to compute $e(X)$. In these three cases we use computer algebra to calculate $h^{2,1}(X)$ directly, and then proceed as before; these three cases are labelled by T^1 in Table 3; see §4 for a sample calculation.

In codimension 4 we calculate a few first cases using a hybrid approach (§4.3): the projection calculus computes $e(X)$, then Theorem 1 computes deformations, and finally we use a computer calculation, similar to that of the three codimension 3 cases, to pick out one of the remaining Hodge numbers to complete the calculation. In these cases, the Picard rank can exceed 1.

2.5. **Fano 3-folds and projection.** Consider the following arrangement of projective 3-folds:

$$\begin{array}{ccc} & \widetilde{Y} & \to & X \\ & \downarrow & & \\ Y & \rightsquigarrow & \overline{Y} & \end{array} \tag{2}$$

where X and Y are quasismooth, $Y \rightsquigarrow \overline{Y}$ is a degeneration to a singular orbifold whose only non-quasismooth points are ordinary nodes, $\overline{Y} \leftarrow \widetilde{Y}$ is a projective small resolution of the nodes, and $\widetilde{Y} \to X$ is the contraction of a divisor $\widetilde{D} \subset \widetilde{Y}$. The passage from Y to $\widetilde{Y}$, that shrinks a number of vanishing cycles to nodes and then resolves the nodes by exceptional $\mathbb{P}^1$s, is well known as a *conifold transition.*

In our context, the exceptional divisor $\widetilde{D} \cong \mathbb{P}(a, b, c)$ maps to a divisor $\mathbb{P}(a, b, c) \to D \subset \overline{Y}$, and the nodes of $\overline{Y}$ lie on D. The small resolution is the relatively $\widetilde{D}$-ample resolution, so is projective, and $\widetilde{D} \to D$ is birational—often an isomorphism, in fact. With this setup, we recall from [50, §5]) (which follows Clemens [15], detailed in the same context as diagram (2) above):

Theorem 4. *Let X and Y be Fano 3-folds related as in diagram (2). Then*

$$e(X) = e(Y) + 2n - 2, \tag{3}$$

where n is the number of nodes of $\overline{Y}$. In particular, if $h^{1,1}(X) = h^{1,1}(Y)$, then

$$h^{2,1}(X) = h^{2,1}(Y) - n + 1. \tag{4}$$

The relevance of this is as follows (see [17, 2.6.3], [11, 3.2]). If X is a Fano 3-fold in codimension k, then it often happens that the *Gorenstein projection* from a quotient singularity sits in diagram (2) as $X \dashrightarrow \overline{Y}$, and that $\overline{Y}$ lies in codimension $< k$. If this nodal Fano $\overline{Y}$ deforms to a quasismooth Y whose Hodge numbers are known, then we may recover the invariants of X.

2.6. **An overview of the calculations.** We adopt different tactics to compute the Hodge numbers of a Fano 3-fold X according to its graded ring.

2.6.1. When X is a hypersurface, this calculation is well known (see §2.2).

2.6.2. When X is a complete intersection in weighted projective space, we may calculate using orbifold Chern classes (see [4] or §A.2).

2.6.3. If X is a complete intersection in weighted projective space or inside a weighted Grassmannian, then $h^{1,1}(X) = 1$ (Theorem 3). If X arises by (possibly multiple) unprojection from a hypersurface, then we can compute $e(X)$ and hence the whole Hodge diamond. This applies to most X that lie in codimension 2 or 3; see §§4.1–4.2. Up to codimension 3, this calculation can be done by hand—the key point is to confirm the existence of a nodal degeneration.

2.6.4. Denoting the affine cone over X by A_X, [23, Theorem 2.4] gives

$$H^{2,1}(X) \cong T^1_{A_X}(-1).$$

Indeed we are interested in complete intersections in weighted projective spaces and weighted Grassmannian, where the index 1 case is equivalent to having the amplitude equal to 1 (see [30, 6.14] and [19]). If X is given by explicit equations, we may use standard algorithms and implementations in computer algebra to calculate $h^{2,1}(X)$; see §2.3 and §A.1.

In these cases we compute $h^{2,1}(X)$ for a single quasismooth member of each family, expressed in the format we expect. Since $h^{p,q}$ are deformation invariants for orbifolds (since Steenbrink [57, Theorem 2] applies in the context of V-manifolds), the numbers we obtain are also the Hodge numbers of any orbifold Fano 3-fold in the family.

2.6.5. By [23, Theorem 2.6],

$$H^{1,1}_{\text{prim}}(X)(X) \cong T^2_{A_X}(-1),$$

and so if X is given by explicit equations we may compute $h^{1,1}(X)$; see Section 4.3 for an example. This algorithm seems to be more complicated, and in practice choosing good equations is delicate.

3. Moduli of Fano 3-folds

We explain a relation between $H^{2,1}(X)$ of a Fano threefold X and the tangent space to its versal deformation space $H^1(X, T_X)$. Since deformations of quasismooth Fano 3-folds X are unobstructed (by [54, Theorem 1.7]), this is the number of moduli of X.

3.1. **Deforming a Fano with an elephant.** The idea comes from Calabi–Yau 3-folds. Given such a V, it follows by Serre duality (non-canonically, involving a choice of determinant) that $H^{2,1}(V) \cong H^1(V, T_V)$; or one may observe that both are isomorphic to the same graded piece $T^1_{A_V}(0) \subset T^1_{A_V}$.

If a Fano 3-fold X has a K3 elephant $E = (x = 0) \subset X$, we may regard the pair (X, E) as a log Calabi–Yau and hope to mimic this relationship. In the index 1 case, one has $H^{2,1}(X) \cong T^1_{A_X}(-1)$ and $H^1(X, T_X) \cong T^1_{A_X}(0)$, and the analogue to the Calabi–Yau isomorphism is the multiplication map $x\colon H^{2,1}(X) \to H^1(X, T_X)$. This map is not an isomorphism, in general, but Theorem 6 below explains the difference in terms of the geometry of E. To make this intuition precise, we start with a more general lemma about Fano 3-folds of arbitrary index $m > 0$. Note that by [21, Proposition A.4.1], the tangent sheaf T_X is isomorphic to $Hom_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \cong \Omega^2_X(m)$, bearing in mind our abuse of notation writing Ω^i_X in place of $\widehat{\Omega}^i_X$.

Lemma 5. *Let X a Fano threefold. If $E \subset X$ a K3 elephant $E \in |-K_X|$, then*

$$h^1(X, T_X) - h^0(X, T_X) = \alpha_E + h^{2,1}(X) - h^{2,2}(X),$$

where $\alpha_E = h^{1,1}(E) - g_X - 1$.

Proof. Suppose X is of index m with $-K_X \overset{\text{lin}}{\sim} mH$, for an ample $\mathbb{Q}$-Cartier divisor H. We write $\mathcal{F}(m)$ for $\mathcal{F} \otimes \mathcal{O}_X(mH)$.

Consider the standard exact sequence of $\mathcal{O}_X$-modules twisted by $\Omega^2(m)$,

$$0 \to \Omega^2_X \to \Omega^2_X(m) \to \Omega^2_X(m)|_E \to 0.$$

In cohomology this yields a long exact sequence

$$\begin{aligned} 0 &\to H^0(\Omega^2_X(m)) \to H^0(\Omega^2(m)_X|_E) \to H^1(\Omega^2_X) \\ &\to H^1(\Omega^2_X(m)) \to H^1(\Omega^2_X(m)|_E) \to H^2(\Omega^2_X) \to 0, \end{aligned} \tag{5}$$

where $H^0(\Omega^2_X) = 0$ (by Hodge theory) and $h^2(\Omega^2_X(m)) = 0$ by Akizuki–Kodaira–Nakano vanishing [3, Theorem 1].

On the other hand the relative exact tangent sequence

$$0 \to T_E \to T_X|_E \to \mathcal{O}_E(m) \to 0$$

yields a long exact sequence

$$0 \to H^0(E, T_X|_E) \to H^0(E, \mathcal{O}_E(m)) \to H^1(E, T_E) \to H^1(E, T_X|_E) \to 0, \tag{6}$$

where $H^1(E, \mathcal{O}_E(m)) = 0$ and $H^0(E, T_E) = H^0(E, \Omega^1_E) = 0$, since E is K3 surface. By (5) and (6) we get

$$\begin{aligned} h^0(X, \Omega^2_X|_E(m)) + h^1(X, \Omega^2_X(m)) + h^{2,2}(X) = \\ h^{2,1}(X) + h^1(X, \Omega^2_X(m)|_E) + h^0((X, \Omega^2_X(m)) \end{aligned} \tag{7}$$

and

$$h^1(T_X|_E) - h^0(T_X|_E) = h^1(T_E) - h^0(\mathcal{O}_E(m)).$$

We have (see [21, A.4]) $\Omega^2_X(m) \cong T_X$ from the pairing

$$\Omega^1_X \otimes \Omega^2_X \to \omega_X \cong \mathcal{O}_X(-m).$$

So with α_E defined as in the statement, we get

$$h^1(X, T_X) - h^0(X, T_X) = \alpha_E + h^{2,1}(X) - h^{2,2}(X)$$

as required. ☐

Theorem 6. *Let X be a Fano 3-fold with K3 elephant $E \subset X$ and α_E as defined in Lemma 5. If $h^0(X, T_X) = 0$, then*

$$h^1(X, T_X) - h^{2,1}(X) = \alpha_E - h^{2,2}(X).$$

This gives an estimate of the difference between the moduli and Hodge theory of X: when $b_2 = h^{2,2}(X)$ is small, we have a more moduli than $h^{2,1}$, while if $b_2 \gg 0$ the opposite holds.

Remark 1. The number $\alpha_E = h^{1,1}(E) - g_X - 1 = h^{1,1}(E) - h^0(E, \mathcal{O}_E(E))$ is a function of the polarised K3 surface $(E, (-K_X)_{|E})$. When E is smooth $h^{1,1}(E) = 20$, and so if X has Fano index 1 then $\alpha_E = 20 - h^0(E, \mathcal{O}_E(1))$. More generally, if E has canonical singularities with corresponding basket $\mathcal{B} = \left\{\frac{1}{r}(a, -a)\right\}$ (see [51, Theorem (9.1)(III)]), then

$$\alpha_E = 20 - \sum_{\mathcal{B}} (r-1) - h^0(E, \mathcal{O}_E(1)).$$

In every case that we know, when a general member X of a family of Fano 3-folds has a K3 elephant $E \subset X$, then both X and (the general) E are quasismooth; in particular, they both have only quotient singularities, and the basket of E is equal to the set of singularities of E.

3.2. **Automorphisms of Fano 3-folds in Grassmannians.**

Lemma 7. *Let X be a Fano 3-fold of index 1. If X is a weighted complete intersection (in its total anticanonical embedding), then $H^0(X, T_X) = 0$.*

Proof. Recall from Flenner [27, Satz 8.11] that if X is an n-dimensional weighted complete intersection, then $H^p(X, \Omega^q_X(t)) = 0$ whenever $p + q < \dim X$ and $t < q - p$.

The lemma follows by setting $q = 2$, $p = 0$, $t = 1$ together with Serre duality $T_X \cong \Omega^2_X(1)$. □

We prove an analogous result for complete intersection in weighted Grassmannians. Our main interest is in Fano 3-folds of index 1 in codimension 3, $X \subset \mathbb{P}(a_0, \dots, a_6)$, most of which arise in this way. We show in Theorem 10 below that $H^0(X, T_X) = 0$ in this case. We first show the vanishing result in standard (non-weighted) Grassmannians.

Lemma 8. *Let X a Fano 3-fold of index 1 that is a complete intersection of multidegree $(d_1, \dots, d_c)$, with every $d_i \geq 2$, in a cone $V = \mathrm{C}\,\mathrm{Gr}(2, n)$, on vertex a linear projective space that is disjoint from X, over a Grassmannian $\mathrm{Gr}(2, n)$ for some $n \geq 5$. Then $H^0(X, T_X) = 0$.*

Proof. We show that $H^0(X, \Omega^2_X(1)) = 0$, which suffices since $T_X \cong \Omega^2_X(1)$ for X a Fano 3-fold of index 1.

We consider the case $V = \mathrm{Gr}(2, n)$ first, with no cone structure. Suppose that $X = (f_1 = \cdots = f_c = 0) \subset G = \mathrm{Gr}(2, n)$, and denote $d_i = \deg f_i$. The Koszul complex of $\mathcal{O}_X$-modules for $\mathcal{O}_X$ twisted by $\Omega^2(1)|_G$ is

$$0 \to \Omega^2_G(1 - d_1 - \cdots - d_c) \to \cdots \to \bigoplus_{i,j,k} \Omega^2_G(1 - d_i - d_i - d_k) \to$$
$$\bigoplus_{i,j} \Omega^2_G(1 - d_i - d_j) \to \bigoplus_i \Omega^2_G(1 - d_i) \to \Omega^2_G(1) \to \Omega^2_G(1)|_X \to 0.$$

By [44, Lemma 0.1], $H^p(G, \Omega^2_G(t)) = 0$ for each of $p = 1, 2, 3$ and any $t \leq -1$, and also $H^0(G, \Omega^2_G(1)) = 0$. (But note that $H^2(G, \Omega^2_G) \neq 0$; this is why we exclude the case where some $d_i = 1$.) It follows, by splitting the Koszul sequence above into short exact sequences, that

$$H^0(X, \Omega^2_G(1)|_X) = H^1(X, \Omega^2_G(1)|_X) = H^1(X, \Omega^2_G(1 - d_i)|_X) = 0. \tag{8}$$

The conormal exact sequence of $X \subset G$ is

$$0 \to \bigoplus_{1 \leq i \leq c} \mathcal{O}_X(-d_i) \to \Omega^1_G|_X \to \Omega^1_X \to 0.$$

Taking its second exterior power and twisting by $\mathcal{O}_X(1)$ we get

$$0 \to \bigoplus_{1 \leq i,j \leq c} \mathcal{O}_X(1 - d_i d_j) \to \bigoplus_{1 \leq i \leq c} \Omega^2_G(1 - d_i)|_X \to \Omega^2_G(1)|_X \to \Omega^2_X(1) \to 0.$$

After splitting this into short exact sequences, the vanishing statements in (8) show at once that $H^0(X, \Omega^2_X(1)) = 0$, as required.

The proof for a cone is the same, replacing Ω^2_{Gr} by the extension of the pullback of Ω^2_{Gr} to the complement of the vertex, in which X is a complete intersection; this restricts to X as above, and the proof follows. □

The proof of Lemma 8 suggests that we need a Bott vanishing type of result to extend the vanishing statements to complete intersections in $w\,\mathrm{Gr}(2, 5)$. The following lemma gives the precise statement we need.

Lemma 9. *Let $wG = wGr(2, 5)$. Then $H^p(w\,\mathrm{Gr}, \Omega^2_{w\,\mathrm{Gr}}(-k)) = 0$ for $p = 1, 2, 3$ and any $k > 0$.*

Proof. If $A_G^\bullet$ denotes the punctured affine cone over the (weighted or not) Grassmannian, we have the following diagram

$$\begin{array}{ccc} & A_G^\bullet & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ Gr(2,5) & & wGr(2,5) \end{array}$$

where π_1 and π_2 denote the quotients by the standard and the weighted $\mathbb{C}^*$ actions respectively. We use the vanishing results from [44, Lemma 0.1] for the standard $\mathrm{Gr}(2,5)$ repeatedly.

The grading on the cohomology groups of $A^\bullet$ is interpreted in terms of local cohomology at the maximal ideal $\mathfrak{m}$ of the vertex of the affine cone A.

Consider the short exact sequence

$$0 \to \pi_1^*\Omega^1_G \to \Omega^1_{A^\bullet} \to \mathcal{O}_{A^\bullet} \to 0. \tag{9}$$

Since $H^i(G, \mathcal{O}_G(-k)) = 0$ for any $i < \dim(G)$, we have

$$H^1(A^\bullet, \Omega^1_{A^\bullet})(-k) = H^1(G, \Omega^1_G(-k)) = 0.$$

In the same way one also gets $H^0(A^\bullet, \Omega^1_{A^\bullet})(-k) = 0$.

Raising the short exact sequence (9) to the second exterior power we have

$$0 \to \pi_1^*\Omega^2_G \to \Omega^2_{A^\bullet} \to \pi_1^*\Omega^1_G \to 0;$$

by the vanishing statements above this reduces to

$$H^1(A^\bullet, \Omega^2_{A^\bullet})(-k) = H^1(G, \Omega^2_G(-k)) = 0.$$

Comsidering analogous exact sequences for the second projection π_2 gives

$$0 \to \pi_2^*\Omega^1_{wG} \to \Omega^1_{A^\bullet} \to \mathcal{O}_{A^\bullet} \to 0,$$

$$0 \to \pi_2^*\Omega^2_{wG} \to \Omega^2_{A^\bullet} \to \pi_2^*\Omega^1_{wG} \to 0.$$

Putting all these vanishing statements together with $H^0(\mathcal{O}_{wG}(-k)) = 0$ we get

$$H^1(wG, \Omega^2_{wG}(-k)) = H^1(A^\bullet, \Omega^2_{A^\bullet})(-k) = 0,$$

as required. The results for $i = 2, 3$ follow similarly. □

Theorem 10. *Let X a Fano 3-fold of index 1 that is a complete intersection of multidegree $(d_1, \ldots, d_c)$, with every $d_i \geq 2$, in a weighted cone $\mathrm{C}\,\mathrm{Gr}(2,5)$, with vertex a linearly-embedded weighted projective space that is disjoint from X. Then $H^0(X, T_X) = 0$.*

Corollary 11. *If $X \subset w\mathbb{P}^6$ is a quasismooth member of one of the 69 Pfaffian families of Fano 3-folds in codimension 3, then $H^0(X, T_X) = 0$.*

The point is that each of these is expressed as a complete intersection, as in Theorem 10, with no equations of degree 1. In practical terms, this is the observation that the number of Plücker variables of degree 1 (that is, above-diagonal entries of degree 1 in the skew-symmetric syzygy matrix) never exceeds the number of degree 1 variables of the ambient $w\mathbb{P}^6$.

Both the lemma and the theorem can be extended to weighted Grassmannians $w\,\mathrm{Gr}(2,n)$, for $n \geq 5$, using Bott-type vanishing theorems, but we only need the $\mathrm{Gr}(2,5)$ case here.

4. Explicit calculations

It takes a few hundred calculations to complete Tables 1–3 below. In this section, we give illustrative examples of each type.

4.1. **Codimension 2.** There are 85 deformation families of Fano 3-folds in codimension 2 ([30, 14]), each one a complete intersection with $h^{1,1}(X) = 1$. The case $X_{2,3} \subset \mathbb{P}^5$ is classical: $e(X) = c_3(T_X)$ can be calculated directly to give $e(X_{2,3}) = -36$ and so $h^{2,1}(X_{2,3}) = 20$.

More generally, Blache [4] describes a general theory of orbifold characteristic classes and their relations with the usual topological notions (see Appendix A.2). This gives an effective method for calculating the Euler characteristic of complete intersections. As a warmup for higher codimension, we recalculate the Euler characteristic by birational projection or by Gröbner basis: of the remaining 84 cases, 66 have a Type I projection (§4.1.1), a further 10 cases have a Type II_1 projection (§4.1.2), and 8 cases have no projection of either type (§4.1.3).

4.1.1. *66 cases with Type I projection.* Consider one of the families of Fano 3-folds of the form $X = X_{a_3+r,a_4+r} \subset \mathbb{P}(1, a, r - a, a_3, a_4, r)$ with $a < r$. The general member has a quotient singularity $\frac{1}{r}(1, a, r - a)$, and admits a Type I projection, as in diagram (2), to a hypersurface:

$$\begin{array}{rcl} X & \subset & \mathbb{P}(1, a, r - a, a_3, a_4, r) \\ \pi_r \downarrow & & \\ D \subset (x_3 A = x_4 B) = \overline{Y} & \subset & \mathbb{P}(1, a, r - a, a_3, a_4), \end{array}$$

where $D = (x_3 = x_4 = 0) = \mathbb{P}(1, a, r - a)$ and π_r is the projection from the final coordinate point of index r. In each one of these 66 cases, the general $\overline{Y}$ is quasismooth away from $n = \deg(A) \deg(B)/(a(r-a))$ nodes that lie on D (by Bertini's theorem), and it admits a Q-smoothing to a general $Y = Y_{a_3+a_4+r} \subset \mathbb{P}(1, a, r-a, a_3, a_4)$. Thus we calculate $e(X) = e(Y)+2n-2$ by (3).

Example 12. Working from the bottom up in diagram (2), let $Y_4 \subset \mathbb{P}^4$ be a smooth quartic. We know $e(Y_4) = -56$ and $h^{2,1}(Y_4) = 30$. Imposing a linear plane $D = \mathbb{P}^2$ on Y_4 gives, in coordinates x, y, z, t, u of $\mathbb{P}^4$,

$$\mathbb{P}^2 = D = (x = y = 0) \subset \overline{Y}_4 = (Ax = By) \subset P^4,$$

where A, B are general cubic forms. Such $\overline{Y}$ has 9 nodes at $(A = B = 0) \subset D$. The unprojection of $D \subset Y$ is a quasismooth variety $X_{3,3} \subset \mathbb{P}(1^5, 2)$, which has Fano Hilbert series No. 20522. By (3) we have $e(X_{3,3}) = e(Y_4) + 18 - 2 = -40$, and so $h^{2,1}(X_{3,3}) = 30$.

This calculation is recorded in Table 2, together with the numerical data described here.

4.1.2. *10 cases with Type II_1 projection.* Again we work from bottom up in diagram (2). Thus, for example, to study X whose Hilbert series P_X is no. 6858 in the GRDB [8], we observe from that database (or by hand with the methods of [2]) that the numerics suggest a Type II_1 projection to $\overline{Y}$ with Hilbert series $P_{\overline{Y}}$ no. 5837, whose general member we know to be of the form $Y_{10} \subset \mathbb{P}(1, 1, 2, 2, 2, 3)$. The task in this case is to impose a divisor D onto a special (nodal) member of this family, where the divisor D may be singular, but its normalisation is $\widetilde{D} \cong \mathbb{P}^2$.

Example 13. Consider $X = X_{4,6} \subset \mathbb{P}(1, 1, 2, 2, 2, 3)$, which has Fano Hilbert series no. 6858 in [8]. As in Example 12 we work bottom up, first constructing $D \subset \overline{Y}_{10} \subset \mathbb{P}(1, 1, 2, 2, 5)$ and then unprojecting. We follow Reid [52, §9] and Papadakis [43] for Type II_1 unprojections.

In coordinates x, y, z, t, u on $\mathbb{P}(1, 1, 2, 2, 5)$, the finite morphism

$$\begin{array}{rcl} \mathbb{P}^2 \cong \widetilde{D} & \longrightarrow & D \subset \mathbb{P}(1, 1, 2, 2, 5) \\ (a, b, c) & \mapsto & (a, b, c^2, (a - b)c, abc^3 + c^5) \end{array}$$

has image D defined by the 2×2 minors of

$$M = \begin{pmatrix} t & u & (x - y)z & (xy + z)z^2 \\ x - y & (xy + z)z & t & u \end{pmatrix}.$$

The surface D has two singular points, each of which has a length 2 preimage in $\widetilde{D}$: the point $(1:1:0:0:0)$ is the pinched image of $(1:1:0)\in\widetilde{D}$, and $(1:1:-1:0:0)$ is the image of two points $(1:1:\pm i)$.

A general $\overline{Y}_{10}$ containing this D has 34 nodes, all of which lie on D. (Two lie at the singularities of D, so the preimage in $\widetilde{D}$ of the singular subscheme of $\overline{Y}$ has length 36 on $\widetilde{D}$.)

The unprojection of $D\subset\overline{Y}$ is given by the maximal Pfaffians of the skew 5×5 matrix

$$\begin{pmatrix} x-y & (xy+z)z & t & u \\ & s_0 & 1 & s_1+A_3 \\ & & s_1 & B_6 \\ & & & zs_0+C_4 \end{pmatrix} \quad \text{with entries of degrees} \quad \begin{pmatrix} 1 & 4 & 2 & 5 \\ & 2 & 0 & 3 \\ & & 3 & 6 \\ & & & 4 \end{pmatrix}$$

in $\mathbb{P}(1,1,2,2,5,2,3)$ with coordinates x,y,z,t,u,s_0,s_1, where A,B,C may be determined by the unprojection calculus if we wish to know them explicitly. Eliminating u using the linear equation gives $X_{4,6}\subset\mathbb{P}(1,1,2,2,2,3)$, as required. We know $e(Y)=-124$, so conclude that $e(X)=-124+2\cdot 34-2=-58$ and $h^{2,1}(X)=31$.

4.1.3. *8 cases with no projection.* Our projection techniques do not work in these cases, but Theorem 2 can be realised in computer algebra instead.

Example 14. Consider a quasismooth Fano 3-fold $X_{6,6}\colon (f=g=0)\subset\mathbb{P}(1,2^3,3^2)$ with Fano Hilbert series number 3508, defined by

$$f=x^6+y^3+z^3+t^3+u^2+v^2 \quad \text{and} \quad g=y^2z+z^2t+t^2y+uv.$$

Ilten's Macaulay2 package [31] works as follows (compressing blank lines in the output):

```
Macaulay2, version 1.5
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : loadPackage "VersalDeformations"
o1 = VersalDeformations
o1 : Package
i2 : R = QQ[x,y,z,t,u,v,Degrees=>{1,2,2,2,3,3}];
i3 : I = ideal ( x^6 + y^3 + z^3 + t^3 + u^2 + v^2,
        y^2*z + z^2*t + t^2*y + u*v );
o3 : Ideal of R
i4 : CT^1(-1,I)
             2       24
o4 : Matrix R  <--- R
```

The answer is that $h^{2,1}(X)=\dim T^1_{A_X}(-1)=24$.

Since X has a K3 elephant $E=(x=0)\subset X$ with basket $9\times\frac{1}{2}(1,1)$ quotient singularities, and $h^0(X,T_X)=0$ by Theorem 1(ii), we know at this stage from the moduli formula Theorem 1(i) that $h^1(X,T_X)=34$. This can also be calculated directly by Macaulay2 as follows:

```
i5 : CT^1(0,I)
             2       34
o5 : Matrix R  <--- R
```

Again, the answer is that $h^1(X,T_X)=\dim T^1_{A_X}(0)=34$.

A similar calculation works with $X_{12,14}\colon (f=g=0)\subset\mathbb{P}(2,3,4,5,6,7)$, with Hilbert series number 37, with, for example,

$$f=x^6+y^4+z^3-u^2+tv \quad \text{and} \quad g=x^7+z^2u+xu^2+zt^2+v^2.$$

In this case there is no elephant $E \subset X$, so the moduli formula (1) does not apply as stated. However, the Macaulay2 results are that $h^{2,1}(X) = 18$ and $h^1(X, T_X) = 23$, and so the formula holds with "$\alpha_E = 6$", which is the correct number calculated on X from its basket indices and $h^0(X, \mathcal{O}(1)) = 0$.

4.2. **Codimension 3.** There are 70 known deformation families of Fano 3-folds in codimension 3. The complete intersection $X = X_{2,2,2} \subset \mathbb{P}^5$ is classical: the Chern class calculation and Lefschetz gives $e(X) = -24$, $\rho_X = 1$ and $h^{2,1}(X) = 14$. The remaining 69 cases are all complete intersections in weighted Grassmannians $w\,\mathrm{Gr}(2,5)$, and so $h^{1,1}(X) = 1$ in every case.

4.2.1. *64 cases Type I.* We say that a Fano 3-fold X has a *Type I staircase* if it admits a sequence of alternate Type I projections and Q-smoothings to a hypersurface. Concretely, if $X \subset w\mathbb{P}^6$ lies in codimension 3, then the staircase is

$$\begin{array}{ccccccc} & & & & \widetilde{Y} & \to & X \\ & & & & \downarrow & & \\ & & \widetilde{Y} & \to\ Y \rightsquigarrow & \overline{Y} & & \\ & & \downarrow & & & & \\ Z & \rightsquigarrow & \overline{Z} & & & & \end{array} \tag{10}$$

where $X \dashrightarrow \overline{Y} \subset w\mathbb{P}^5$ eliminates a single variable, $Y \subset w\mathbb{P}^5$ is a general Q-smoothing of $\overline{Y}$, and $Y \dashrightarrow \overline{Z}$ is a projection to a nodal hypersurface $\overline{Z} \subset w\mathbb{P}^4$ as in §4.1. Counting nodes on $\overline{Y}$ and $\overline{Z}$ and using the formula of Theorem (4) completes the calculation of $e(X)$ and $h^{2,1}(X)$.

Of the 64 Fano 3-folds in codimension 3 with a Type I projection, 57 have a Type I staircase to a hypersurface.

Example 15. Consider the family with Hilbert series no. 20523 in [8]. A typical member $X \subset \mathbb{P}(1,1,1,1,1,2,3)$, in coordinates $x_{1\dots5}, y, z$, is given by the five maximal Pfaffians of a skew 5×5 matrix of forms

$$\begin{pmatrix} x_1 & x_2 & A & D \\ & x_3 & B & E \\ & & C & F \\ & & & z \end{pmatrix} \quad \text{where the entries have degrees} \quad \begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}.$$

It has a quotient singularity $\frac{1}{3}(1,1,2)$ at the z-coordinate point $P_z \in X$.

Projection from that point is calculated by eliminating z from these equations. Doing that leaves the two Pfaffians of degree 3, which define

$$\overline{Y}_{3,3} \colon \left\{ \begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix} \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} = \underline{0} \right\} \subset \mathbb{P}(1,1,1,1,1,2).$$

For general degree 2 forms $A, \dots, F$, the image $\overline{Y}$ has 6 nodes (by Hilbert–Burch) and a Q-smoothing $Y_{3,3}$ which was computed in Example 12 above. Making the projection from $Y_{3,3}$ as in Example 12 completes the staircase. In any case, using the result of Example 12 gives $e(X) = e(Y) + 2 \cdot 6 - 2 = -40 + 12 - 2 = -30$, and so $h^{2,1}(X) = 17$.

Of the remaining 7 cases, 4 have a Type I projection to a family that arises by Type II_1 unprojection from a hypersurface, so again have a staircase, but with a more complicated second step. A fifth case has a Type I projection to the classical family $Y_{2,3} \subset \mathbb{P}^5$, so also works.

But in two remaining cases, the image of the Type I projection lies in a family whose Hodge numbers were computed using the algorithms for $\dim T^1$; in this paper, these cases remain dependent on computational algebra.

4.2.2. *2 cases Type II_1.* Of the cases without a Type I projection, two have a Type II_1 projection: $X_{7,8,8,9,10} \subset \mathbb{P}(1,2,3,3,4,4,5)$ has a Type II_1 projection from $\frac{1}{4}(1,1,3)$ and $X_{10\dots14} \subset \mathbb{P}(1,3,4,5,5,6,7)$ has a Type II_1 projection from $\frac{1}{5}(1,2,3)$. We consider the latter in detail, following Reid [52, 9.5] and Papadakis [43, 4.4].

Consider $D \subset \mathbb{P}(1,3,4,5,6)$ defined by the maximal minors of

$$M_D = \begin{pmatrix} t & v & yz & z^2 \\ y & z & t & v \end{pmatrix}.$$

This D is the image of $\mathbb{P}(1,2,3) \to \mathbb{P}(1,3,4,5,6)$ given by $(a,b,c) \mapsto (a,c,b^2,bc,b^3)$; the normalising variable b is recovered as the ratio of the rows of M_D.

The general hypersurface $\overline{Y}_{18}$ containing D has the form

$$\overline{Y}_{18} = (A_{12}m_{12} + B_{11}m_{13} + 2B_{12}m_{23} + B_{22}m_{24} = 0) \subset \mathbb{P}(1,3,4,5,6),$$

where m_{ij} denotes the minor of M_D involving columns i and j.

The unprojection of $D \subset \overline{Y}_{18}$ is a codimension 3 variety $X \subset \mathbb{P}(1,3,4,5,5,6,7)$, in coordinates x,y,z,t,u,v,w, defined by the maximal Pfaffians of the skew 5×5 matrix

$$\begin{pmatrix} y & z & t & v \\ & -u & -B_{22} & w + B_{12} \\ & & -w + B_{12} & -B_{11} \\ & & & -uz - A_{12} \end{pmatrix}.$$

For example, setting

$$A_{12} = yv + y^3 + x^9, \quad B_{11} = yt + x^8, \quad B_{12} = 0 \quad \text{and} \quad B_{22} = v$$

results in a quasismooth X, and $\overline{Y}_{18}$ whose non-quasismooth locus is defined by the equations

$$zt - yv, \quad y^2z - t^2, \quad yz^2 - tv, \quad x^9y + y^4 + y^2v + 2v^2, \quad x^9z - 2x^8t - yt^2 + yzv,$$
$$z^3 - v^2, \quad x^9t + y^3t + 2z^2v + ytv, \quad x^8y^2 + y^3t + z^2v, \quad 2x^8yz - x^9v + y^3v - yv^2$$

and consists of 22 nodes, all of which lie on $D \subset \overline{Y}_{18}$.

The general $Y_{18} \subset \mathbb{P}(1,3,4,5,6)$ has $e(Y_{18}) = -80$, so $e(X) = -38$ and $h^{2,1}(X) = 21$.

4.2.3. *No Type I or II_1 projection.* The three remaining cases are $X_{12\dots16} \subset \mathbb{P}(1,4,5,5,6,7,8)$, $X_{16\dots20} \subset \mathbb{P}(1,5,6,7,8,9,10)$ and $X_{14\dots18} \subset \mathbb{P}(1,5,5,6,7,8,9)$. The first has only a type IV projection, while the other two do not have any Gorenstein projections at all. We compute T^1 in these cases: we work out the first in detail here; the other two are similar.

Example 16. A particular $X_{12\dots16} \subset \mathbb{P}(1,4,5,5,6,7,8)$, in coordinates x,y,z,t,u,v,w, is given by the maximal Pfaffians of the skew 5×5 matrix

$$\begin{pmatrix} y & z & u & v \\ & u & v & y^2 + w \\ & & -y^2 + w & x^9 + yz \\ & & & zt + t^2 \end{pmatrix}$$

in the usual antisymmetric notation. One checks that the scheme defined by those equations is quasismooth. We compute $h^{2,1}(X) = 20$ and $h^1(X, T_X) = 23$ by Macaulay2 as before.

We verify the moduli formula (i) of Theorem 1. The basket of X is

$$\mathcal{B}_X = \left\{ \frac{1}{2}(1,1,1), \frac{1}{4}(1,1,3), 2 \times \frac{1}{5}(1,1,4), \frac{1}{5}(1,2,3) \right\}.$$

The K3 elephant $E = (x=0) \subset X$ is the unique member of $|-K_X|$. It has $h^0(\mathcal{O}_E(1)) = 0$ and $h^{1,1}(E) = 20 - \sum r_i - 1$, where the r_i are the indices of singularities of $\mathcal{B}_X$. Thus

$$h^1(T_X) - h^{2,1}(X) = \alpha_E - h^{2,2}(X) = (20 - 1 - 3 - 3 \cdot 4) - 1 = 3,$$

which agrees with $23 - 20$.

The other two cases work similarly; in each case $h^{2,1}(X) = 20$.

4.3. **Codimension 4.** All the calculations in codimensions 4 in this section depend on computer algebra: we use Magma [6] to compute examples of the codimension 4 equations by unprojection, and Macaulay2 [28, 31] for the Hodge numbers.

When a Hilbert series is realised by a Fano 3-fold in codimension 4, it frequently happens that there is more than one deformation family of such Fano 3-folds. For 116 of Hilbert series listed in [8] in codimension 4, [11] computes the different families, and observes that they are distinguished by the Euler characteristic of a quasismooth member. However it does not compute the Picard rank of these Fano 3-folds, in part because there is no known format in which they lie as complete intersections, and so we have no Lefschetz theorem to apply directly (although see [9] for some special cases where $\rho_X = 2$, including the case of Hilbert series 24078 in Example 18 below). But the computational methods of this paper still apply, in conjunction with the unprojection construction of [11, 42]. We compute a few examples here as first calculations.

Example 17. Fano Hilbert series 24097. By [11] there are 3 families of Fano 3-folds $Y \subset \mathbb{P}(1^6, 2^2)$ with (typically) two $\frac{1}{2}(1,1,1)$ quotient singularities, each with the Hilbert series No.24097 in [8]. They arise by unprojection of

$$\mathbb{P}^2 = D \subset \overline{Y} \subset \mathbb{P}(1^6, 2),$$

where $D \subset \mathbb{P}(1^6, 2)$ is a linearly embedded plane, and $\overline{Y}$ is defined by the vanishing of Pfaffians of a skew 5×5 matrix of forms of weights

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{pmatrix}. \tag{11}$$

The three families arise as so-called "Tom" and "Jerry" unprojections (see [11, §2.3] for details), and the three different results are listed in the Big Table [12]: Tom_1, Jer_{12} and Jer_{15}. Takagi's analysis [59, Theorem 0.3] of prime Fano 3-folds with index 2 terminal singularities shows that the first and third of these families have $h^{1,1}(X) = 1$. Using the Macaulay2 computation, and Theorem 1(i) (which holds since each unprojection does indeed carry a quasismooth elephant E with $\alpha_E = 19 - 1 - 5 = 13$), we complete the table below.

unproj type	# nodes	e_X	$h^{1,1}(X)$	$h^{2,1}(X)$	$h^1(X, T_X)$	$h^0(X, T_X)$
Tom_1	6	-14	1	9	21	0
Jer_{12}	8	-10	3	9	19	0
Jer_{15}	9	-12	1	8	20	0

For example, the Jer_{12} case above uses $\overline{Y}$ defined by Pfaffian matrix

$$\begin{pmatrix} t & u & v & w \\ & v & t+u & ux \\ & & x & y^2 - z^2 \\ & & & yz + t^2 + u^2 \end{pmatrix}$$

in the coordinates x, y, z, t, u, v and w of $\mathbb{P}(1^6, 2)$. Such $\overline{Y}$ contains the plane $D = (t = u = v = w = 0)$. Unprojecting $D \subset \overline{Y}$ gives $X \subset \mathbb{P}(1^6, 2^2)$, defined by

$$xt - tu - u^2 + v^2, \qquad y^2t - z^2t - xu^2 + vw, \qquad yzt + t^3 + tu^2 - xuv + tw + uw,$$
$$yzu + t^2u + u^3 - y^2v + z^2v + xw, \qquad x^2u - y^2u + z^2u - xu^2 + yzv + t^2v + u^2v + vw,$$
$$x^2v - xw + ts, \qquad -xyz - xt^2 - xu^2 - xw - us, \qquad -x^3 + xy^2 - xz^2 + x^2u + vs,$$
$$x^2y^2 - y^4 - x^2z^2 + 3y^2z^2 - z^4 + yzt^2 - xy^2u + xz^2u + yzu^2 +$$
$$+ y^2uv - z^2uv + xtuv + yzw - xuw - tuw + u^2w - ws$$

in coordinates x, y, z, t, u, v, w and unprojection variable s.

Example 18. Fano Hilbert series 24078. By [11] there are 3 families of Fano 3-folds $X \subset \mathbb{P}(1^6, 2, 3)$ with (typically) two $\frac{1}{3}(1, 1, 2)$ quotient singularities, each with the Hilbert series No.24078 in [8]. They arise by unprojection of

$$\mathbb{P}^2 = D \subset \overline{Y} \subset \mathbb{P}(1^6, 2),$$

where $D \subset \mathbb{P}(1^6, 2)$ is a linearly embedded $\mathbb{P}(1, 1, 2)$, and $\overline{Y}$ is defined by the vanishing of Pfaffians of a skew 5×5 matrix of forms of the same weights as (11) above.

The three different results [12] are: Tom_1, Tom_5 and Jer_{12}. In this case the elephant $E \subset X$ has $\alpha_E = 13$, and the table below summarises the results.

unproj type	# nodes	e_X	$h^{1,1}(X)$	$h^{2,1}(X)$	$h^1(X, T_X)$	$h^0(X, T_X)$
Tom_1	5	-16	1	10	22	0
Tom_5	4	-18	2	12	23	0
Jer_{12}	6	-14	1	9	21	0

These calculations seem to be on the limit of what we can do, as they terminate only when the equations are relatively small. For example, the Tom_5 case above uses $\overline{Y}$ defined by Pfaffian matrix

$$\begin{pmatrix} z & t & v+u & w \\ & u & t & xv + zu \\ & & z & w - y^2 \\ & & & x^2 - v^2 \end{pmatrix}$$

in the coordinates x, y, z, t, u, v and w of $\mathbb{P}(1^6, 2)$.

Of the 145 Hilbert series of Fano 3-folds listed in [8] as presented naturally in codimension 4, 116 have the numerical properties consistent with having a Type I unprojection. The unprojection analysis of these is the subject of [11], with the results presented in [12], and in principle they could all be computed as above. A further 16 Hilbert series have the numerical properties of a Type II_1 projection, and a computational approach following Papadakis [43] is conceivable; the constructions are part of Taylor's thesis [60].

Some of the remaining 13 cases have more complicated projections that we do not know how to work with systematically yet, but four cases have no Gorenstein projections at all, and some other approach is required (even to write down examples by equations). These cases are:

No. 25	$X \subset \mathbb{P}(2, 5, 6, 7, 8, 9, 10, 11)$	No. 282	$X \subset \mathbb{P}(1, 6, 6, 7, 8, 9, 10, 11)$
No. 166	$X \subset \mathbb{P}(2, 2, 3, 3, 4, 4, 5, 5)$	No. 308	$X \subset \mathbb{P}(1, 5, 6, 6, 7, 8, 9, 10)$.

4.4. **A quasismooth unprojection from codimension 4.** As a final, related curiosity, we construct a codimension 4, quasismooth Fano 3-fold $X \subset \mathbb{P}(1^6, 2^2)$ with Hilbert series number 24097 which contains a quasismooth divisor $E \subset X$ that is itself a complete intersection; this contrasts with the more typical nodal cases above, and is a novelty to us. We adapt Example 17 so that the codimension 3 projection $Y \subset \mathbb{P}(1^6, 2)$ contains two divisors: the coordinate planes

$D = \mathbb{P}^2$ and $E = \mathbb{P}(1,1,2)$ meeting along the coordinate line $\mathbb{P}^1$. Indeed define Y by the maximal Pfaffians of

$$\begin{pmatrix} t & u & v & w \\ & v & u & -zv-u^2 \\ & & z-t & yz-x^2 \\ & & & y^2-t^2 \end{pmatrix}$$

in the coordinates x, y, z, t, u, v and w of $\mathbb{P}(1^6, 2)$. Then $D = (t = u = v = w = 0) = \mathbb{P}^2$ lies inside Y in Jer_{12} format while $E = (z = t = u = v = 0) = \mathbb{P}(1,1,2)$ lies inside Y in Tom_5 format.

Altogether Y has 8 nodes; these all lie on D (in accordance with Jer_{12} unprojection of D to construct Hilbert series 24097), and 4 of them lie on the intersection $D \cap E$ (in accordance with the Tom_5 unprojection or E to construct Hilbert series 24078).

We may unproject either divisor, and we choose to unproject $D \subset Y$ to give $X \subset \mathbb{P}(1^6, 2^2)$. All the 8 nodes are resolved by this, and X is quasismooth. The Fano 3-fold X has Picard rank $\rho_X = 3$ (as in Example 17 above).

Furthermore, $E \subset Y$ has birational image in X, which we also denote $E \subset X$ defined by equations

$$E = (z = t = u = v = 0) \cap X \subset \mathbb{P}(1^6, 2^2),$$

in coordinates x, y, z, t, u, v, w, s. Computing the unprojection shows that $E \cong (x^4 - y^4 - w^2 + ws = 0) \subset \mathbb{P}(1^2, 2^2)$ in coordinates x, y, w, s, which is $\mathbb{P}(1,1,2)$ blown up in 4 points on the coordinate line $L = \mathbb{P}(1,1)$ followed by the contraction of the resulting -2-curve $\widetilde{L}$, the birational transform of L. Thus it is a index 2 Fano surface with two $\frac{1}{2}(1,1)$ quotient singularities, Picard rank 4 and $K_E^2 = 4$. It can be unprojected to an ordinary, isolated cDV singular point (the cone on E, in new local coordinates) on an otherwise smooth complete intersection $Z_{2,2,2} \subset \mathbb{P}^6$.

Appendix A. Hodge numbers of Fano 3-folds

Tables 1–3 in A.3 below list the invariants for all known families of Fano 3-folds in codimension at most 3. The majority of the calculations can be carried out by hand. We use computer algebra in the three cases where not (denoted by T^1 in Table 3), and also use it as a check on all results.

In codimensions 1 and 2 respectively the Fano 3-folds come from Iano-Fletcher ([30] Tables 5 and 6 respectively; in codimensions 3 and 4 they are from Altınok ([1]). The graded ring database identifier (denoted 'Grdb' in the tables) is that of [8].

A.1. **Our use of computer algebra.** The explicit calculations we need are standard, although sometimes rather involved. There are three places computer algebra may assist.

(i) Checking that a variety is quasismooth can usually be done with Bertini's theorem. In codimension 3 and 4, this can be carried out as in [7, §3–4], for example, when Type I projections (and staircases) are available. In other cases, we check the Jacobian condition by machine. This, or some equivalent (such as [61, Theorem 5.5] or [5]), can be checked by computer algebra given explicit equations.

(ii) Checking that a variety has only ordinary nodes as singularities, and counting those nodes, can again usually be done by Bertini's theorem together with a Chern class calculation when we have Type I projections; see for example [7, §4] for the nodes and [11, §7] for the count. In other cases, we use computer algebra following [11, §6].

(iii) Computing the dimensions of graded pieces of spaces $T^1_{A_X}$ seems too hard by hand in most cases, but there are algorithms to do this based on Gröbner basis; see [31].

We are indebted to the developers of the computer algebra systems Macaulay2 [28], Magma [6] and Singular [22] that we used for these calculations, and to Ilten [31] for the Versal Deformation package for Macaulay2. (The latter conveniently handles the gradings on variables automatically when computing graded pieces of $T^1_{A_X}$; on other systems we had to pick out the graded piece given generators for the whole module "by hand".)

In practice, most computations here work when the equations of the Fano 3-fold are fairly sparse, and as the codimension increases it becomes harder to find such sparse representatives.

A.2. **Blache's orbifold formula.** Let V be a projective orbifold of dimension n, embedded as a quasismooth subvariety of weighted projective space $V \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_N)$. We suppose, in addition, that V is a manifold away from a finite set of strictly orbifold points $Q_1, \dots, Q_s \in V$.

We define the orbifold total Chern class $c_{\mathrm{orb}}(T_{\mathbb{P}}) = 1 + c_{1,\mathrm{orb}}(T_P) + \dots + c_{\mathrm{orb},n}(T_{\mathbb{P}})$ of $\mathbb{P}$ via

$$0 \to \mathcal{O}_{\mathbb{P}} \to \oplus_{i=0}^N \mathcal{O}_{\mathbb{P}}(a_i) \to T_{\mathbb{P}} \to 0.$$

Taking the restriction of this to V, we derive the top Chern class $c_{\mathrm{orb}}(V)$ of V from the tangent exact sequence

$$0 \to T_V \to T_{\mathbb{P}|V} \to N_{V|\mathbb{P}} \to 0$$

exactly as in the smooth case: that is, we make the formal computation

$$1 + c_{\mathrm{orb},1}(T_{\mathbb{P}}) + \dots + c_{\mathrm{orb},N}(T_{\mathbb{P}}) = c_{\mathrm{orb}}(T_{\mathbb{P}}) := \prod (1 + a_i h),$$

where $H^2(\mathbb{P}, \mathbb{Q}) = h\mathbb{Q}$ and $c_{\mathrm{orb},j} \in H^{2j}(\mathbb{P}, \mathbb{Q})$, and then

$$\big(1 + c_{\mathrm{orb},1}(T_V) + \dots + c_{\mathrm{orb},n}(T_V)\big)\, c(N_{V|\mathbb{P}}) = c_{\mathrm{orb}}(T_{\mathbb{P}}).$$

Then we define the orbifold euler class $e_{\mathrm{orb}}(V)$ by

$$e_{\mathrm{orb}}(V) := \int_V c_{\mathrm{orb},n}(V) \in \mathbb{Q}.$$

This is a formal computation that ignores orbifold behaviour. However, it is related to the topological euler characteristic $e(V)$ by the following theorem of Blache [4].

Theorem 19 ([4] (2.11–14))**.** *Let V be a projective orbifold with finite orbifold locus as above. Then $e_{\mathrm{orb}}(X) \in \mathbb{Q}$ satisfies*

$$e(X) = e_{\mathrm{orb}}(X) + \sum_{Q \in \mathcal{B}} \frac{r-1}{r},$$

where $r = r(Q)$ is the local index of the orbifold point Q.

For a hypersurface $X_d \subset \mathbb{P}(a_0, \dots, a_{n+1})$ we have

$$e_{\mathrm{orb}}(X) = \text{the coefficient of } h^n \text{ in series expansion of } \frac{\prod(1 + a_i h)}{1 + dh} \deg(X).$$

For example, Fano number 337 is $X_{28} \subset \mathbb{P}(1, 4, 6, 7, 11)$ and has basket

$$\mathcal{B} = \left\{ 2 \times \frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5), \frac{1}{11}(1,4,7) \right\}.$$

Calculating as above gives

$$\begin{aligned}
e(X) &= e_{\mathrm{orb}}(X) + 2 \times \frac{1}{2} + \frac{5}{6} + \frac{10}{11} \\
&= \mathrm{coeff}_{h^3} \left((1 + 29h + 309h^2)(1 - 28h + 784h^2 - 21952h^3) \right) \frac{28}{4 \cdot 6 \cdot 7 \cdot 11} \\
&\qquad + 2 \times \frac{1}{2} + \frac{5}{6} + \frac{10}{11} \\
&= \mathrm{coeff}_{h^3} (1 + h + 281h^2 - 6385h^3) \frac{1}{66} + 2 \times \frac{1}{2} + \frac{5}{6} + \frac{10}{11} \\
&= \frac{-6385}{66} + 1 + 5/6 + 10/11 \\
&= -94.
\end{aligned}$$

This agrees with our calculation $h^{2,1}(X) = 49$ and $e(X) = 4 - 2 \times 49$.

A.3. **Tables of results.** Tables 1–3 list the Hodge number $h^{2,1}(X)$, the topological euler characteristic $e(X)$ and the number of moduli $h^1(T_X) = \dim H^1(X, T_X)$ for quasismooth members X of the families of Fano 3-folds in codimensions 1–3 respectively.

In codimension 1, we apply the Griffith's Residue Theorem in §2.2 together with the formulas of Theorem 1. In codimension 2, Table 2 documents the method we use to compute the invariants. This could be the conventional Chern class calculation, indicated by $c_3(T_X)$, a projection calculation, indicated by I or II_1 depending on the type of the projection, or a computer calculation of $T^1_{A_X}$, indicated by T^1 (which we also use as a check on all the calculations). Where we use a projection, we also list the centre $\frac{1}{r}$ of projection (leaving the polarising weights of $\frac{1}{r}(1, a, -a)$ implicit), the number of nodes on the image of projection, and the number of that image in the GRDB. Where there is more than one possible centre of projection, we list them all. Combining this data with the results of Table 1 and Theorems 1 and 4, one can quickly check the calculations by hand. For example, number 25022, $X_{3,3} \subset \mathbb{P}(1^5, 2)$ (the second line in Table 2) projects to number 20521 with 9 nodes; the Euler charactistic of the smoothed image is listed in Table 1 as -56, and so the for $X_{3,3}$ it is $-56 + 2 \times 9 - 2 = -40$, as displayed.

In codimension 3, Table 3 documents the method we use in the 70 cases as follows:

(i) 57 cases have at least one 'staircase' of two Type I projections to a hypersurface. This is indicated by I–I.
(ii) 4 cases have a Type I projection to a codimension 2 family that has as a Type II_1 projection to a hypersurface (indicated by I–II_1).
(iii) 2 cases have a Type II_1 projection directly to a hypersurface (II_1).
(iv) 2 cases have a Type I projection to a codimension 2 family with no projection (I–T^1).
(v) 1 case has a Type I projection to a known smooth Fano (I–smooth).
(vi) 1 case is a known smooth Fano complete intersection ($c_3(T_X)$).
(vii) 3 cases have no Type I or II_1 projections at all (T^1).

Again, where there is a projection from X we list the centre $\frac{1}{r}$, the number of nodes and the GRDB identifier for each possibility, and applying Theorems 1 and 4 together with data from previous tables calculates the invariants.

Table 1: Codimension 1: $h^{1,1}(X) = 1$ and $h^0(X, T_X) = 0$ in all cases.

GRDB	variety	$h^{2,1}$	$e(X)$	$h^1(T_X)$
20521	$X_4 \subset \mathbb{P}^4$	30	-56	43
16203	$X_5 \subset \mathbb{P}(1,1,1,1,2)$	38	-72	51
16202	$X_6 \subset \mathbb{P}(1,1,1,1,3)$	52	-100	66
11101	$X_6 \subset \mathbb{P}(1,1,1,2,2)$	41	-78	55
10981	$X_7 \subset \mathbb{P}(1,1,1,2,3)$	51	-98	63
10980	$X_8 \subset \mathbb{P}(1,1,1,2,4)$	64	-124	78
10960	$X_9 \subset \mathbb{P}(1,1,1,3,4)$	71	-138	83
10959	$X_{10} \subset \mathbb{P}(1,1,1,3,5)$	85	-166	98
10958	$X_{12} \subset \mathbb{P}(1,1,1,4,6)$	111	-218	125
5838	$X_8 \subset \mathbb{P}(1,1,2,2,3)$	45	-86	58
5837	$X_{10} \subset \mathbb{P}(1,1,2,2,5)$	64	-124	79
5257	$X_9 \subset \mathbb{P}(1,1,2,3,3)$	49	-94	62
5157	$X_{10} \subset \mathbb{P}(1,1,2,3,4)$	56	-108	66
5153	$X_{11} \subset \mathbb{P}(1,1,2,3,5)$	65	-126	74
5152	$X_{12} \subset \mathbb{P}(1,1,2,3,6)$	75	-146	88

Continued on next page

Table 1 continued from previous page

5137	$X_{12} \subset \mathbb{P}(1,1,2,4,5)$	70	−136	81
5136	$X_{14} \subset \mathbb{P}(1,1,2,4,7)$	90	−176	102
5134	$X_{15} \subset \mathbb{P}(1,1,2,5,7)$	97	−190	106
5133	$X_{16} \subset \mathbb{P}(1,1,2,5,8)$	108	−212	119
5132	$X_{18} \subset \mathbb{P}(1,1,2,6,9)$	128	−252	141
4984	$X_{12} \subset \mathbb{P}(1,1,3,4,4)$	60	−116	73
4909	$X_{13} \subset \mathbb{P}(1,1,3,4,5)$	66	−128	73
4907	$X_{15} \subset \mathbb{P}(1,1,3,4,7)$	82	−160	89
4906	$X_{16} \subset \mathbb{P}(1,1,3,4,8)$	91	−178	102
4893	$X_{15} \subset \mathbb{P}(1,1,3,5,6)$	78	−152	87
4892	$X_{18} \subset \mathbb{P}(1,1,3,5,9)$	104	−204	114
4891	$X_{21} \subset \mathbb{P}(1,1,3,7,10)$	126	−248	133
4890	$X_{22} \subset \mathbb{P}(1,1,3,7,11)$	136	−268	144
4889	$X_{24} \subset \mathbb{P}(1,1,3,8,12)$	154	−304	165
4835	$X_{16} \subset \mathbb{P}(1,1,4,5,6)$	77	−150	83
4834	$X_{20} \subset \mathbb{P}(1,1,4,5,10)$	108	−212	119
4822	$X_{18} \subset \mathbb{P}(1,1,4,6,7)$	88	−172	94
4821	$X_{22} \subset \mathbb{P}(1,1,4,6,11)$	120	−236	127
4820	$X_{28} \subset \mathbb{P}(1,1,4,9,14)$	165	−326	172
4819	$X_{30} \subset \mathbb{P}(1,1,4,10,15)$	182	−360	190
4807	$X_{21} \subset \mathbb{P}(1,1,5,7,8)$	99	−194	104
4806	$X_{26} \subset \mathbb{P}(1,1,5,7,13)$	137	−270	143
4805	$X_{36} \subset \mathbb{P}(1,1,5,12,18)$	211	−418	218
4794	$X_{24} \subset \mathbb{P}(1,1,6,8,9)$	110	−216	115
4793	$X_{30} \subset \mathbb{P}(1,1,6,8,15)$	154	−304	160
4792	$X_{42} \subset \mathbb{P}(1,1,6,14,21)$	240	−476	247
2402	$X_{12} \subset \mathbb{P}(1,2,2,3,5)$	47	−90	59
2401	$X_{14} \subset \mathbb{P}(1,2,2,3,7)$	60	−116	74
1389	$X_{12} \subset \mathbb{P}(1,2,3,3,4)$	40	−76	54
1162	$X_{14} \subset \mathbb{P}(1,2,3,4,5)$	45	−86	52
1160	$X_{16} \subset \mathbb{P}(1,2,3,4,7)$	54	−104	62
1159	$X_{18} \subset \mathbb{P}(1,2,3,4,9)$	65	−126	76
1155	$X_{15} \subset \mathbb{P}(1,2,3,5,5)$	48	−92	60
1149	$X_{17} \subset \mathbb{P}(1,2,3,5,7)$	56	−108	60
1147	$X_{18} \subset \mathbb{P}(1,2,3,5,8)$	61	−118	66
1146	$X_{20} \subset \mathbb{P}(1,2,3,5,10)$	72	−140	82
1144	$X_{21} \subset \mathbb{P}(1,2,3,7,9)$	72	−140	78
1143	$X_{24} \subset \mathbb{P}(1,2,3,7,12)$	89	−174	97
1142	$X_{24} \subset \mathbb{P}(1,2,3,8,11)$	87	−170	93
1141	$X_{26} \subset \mathbb{P}(1,2,3,8,13)$	99	−194	106
1140	$X_{30} \subset \mathbb{P}(1,2,3,10,15)$	121	−238	131
1113	$X_{20} \subset \mathbb{P}(1,2,4,5,9)$	62	−120	70
1112	$X_{22} \subset \mathbb{P}(1,2,4,5,11)$	72	−140	81
1079	$X_{20} \subset \mathbb{P}(1,2,5,6,7)$	55	−106	60
1078	$X_{26} \subset \mathbb{P}(1,2,5,6,13)$	80	−156	87
1076	$X_{27} \subset \mathbb{P}(1,2,5,9,11)$	77	−150	79
1075	$X_{32} \subset \mathbb{P}(1,2,5,9,16)$	100	−196	104

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Table 1 continued from previous page

1074	$X_{42} \subset \mathbb{P}(1,2,5,14,21)$	144	−284	150
1067	$X_{30} \subset \mathbb{P}(1,2,6,7,15)$	88	−172	96
866	$X_{15} \subset \mathbb{P}(1,3,3,4,5)$	40	−76	52
545	$X_{18} \subset \mathbb{P}(1,3,4,5,6)$	42	−80	49
539	$X_{19} \subset \mathbb{P}(1,3,4,5,7)$	45	−86	47
537	$X_{20} \subset \mathbb{P}(1,3,4,5,8)$	48	−92	53
536	$X_{24} \subset \mathbb{P}(1,3,4,5,12)$	63	−122	71
534	$X_{24} \subset \mathbb{P}(1,3,4,7,10)$	57	−110	58
533	$X_{28} \subset \mathbb{P}(1,3,4,7,14)$	72	−140	80
532	$X_{30} \subset \mathbb{P}(1,3,4,10,13)$	74	−144	75
531	$X_{34} \subset \mathbb{P}(1,3,4,10,17)$	90	−176	92
530	$X_{36} \subset \mathbb{P}(1,3,4,11,18)$	97	−190	101
529	$X_{42} \subset \mathbb{P}(1,3,4,14,21)$	120	−236	125
508	$X_{21} \subset \mathbb{P}(1,3,5,6,7)$	45	−86	51
507	$X_{33} \subset \mathbb{P}(1,3,5,11,14)$	74	−144	74
506	$X_{38} \subset \mathbb{P}(1,3,5,11,19)$	92	−180	93
505	$X_{48} \subset \mathbb{P}(1,3,5,16,24)$	126	−248	130
500	$X_{24} \subset \mathbb{P}(1,3,6,7,8)$	48	−92	56
356	$X_{24} \subset \mathbb{P}(1,4,5,6,9)$	45	−86	47
355	$X_{30} \subset \mathbb{P}(1,4,5,6,15)$	62	−120	69
353	$X_{25} \subset \mathbb{P}(1,4,5,7,9)$	46	−88	46
352	$X_{32} \subset \mathbb{P}(1,4,5,7,16)$	65	−126	69
351	$X_{44} \subset \mathbb{P}(1,4,5,13,22)$	91	−178	91
350	$X_{54} \subset \mathbb{P}(1,4,5,18,27)$	120	−236	121
337	$X_{28} \subset \mathbb{P}(1,4,6,7,11)$	49	−94	50
336	$X_{34} \subset \mathbb{P}(1,4,6,7,17)$	65	−126	67
296	$X_{27} \subset \mathbb{P}(1,5,6,7,9)$	42	−80	42
295	$X_{30} \subset \mathbb{P}(1,5,6,8,11)$	46	−88	45
294	$X_{38} \subset \mathbb{P}(1,5,6,8,19)$	64	−124	64
293	$X_{66} \subset \mathbb{P}(1,5,6,22,33)$	120	−236	120
289	$X_{40} \subset \mathbb{P}(1,5,7,8,20)$	64	−124	68
271	$X_{36} \subset \mathbb{P}(1,7,8,9,12)$	42	−80	41
270	$X_{50} \subset \mathbb{P}(1,7,8,10,25)$	63	−122	62

Table 2: Codimension 2: $h^{1,1}(X) = 1$ and $h^0(X, T_X) = 0$ in all cases.

grdb	variety	method	$\frac{1}{r}$, #nodes, target id	$h^{2,1}$	$e(X)$	$h^1(T_X)$
24076	$X_{2,3} \subset \mathbb{P}^5$	$c_3(T_X)$		20	−36	34
20522	$X_{3,3} \subset \mathbb{P}(1,1,1,1,1,2)$	I	$\frac{1}{2}, 9, 20521$	22	−40	36
16225	$X_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$	I	$\frac{1}{2}, 12, 16203$	27	−50	41
16204	$X_{4,4} \subset \mathbb{P}(1,1,1,1,2,3)$	I	$\frac{1}{3}, 8, 16203$	31	−58	45
11435	$X_{4,4} \subset \mathbb{P}(1,1,1,2,2,2)$	I	$\frac{1}{2}, 16, 11101$	26	−48	39
11102	$X_{4,5} \subset \mathbb{P}(1,1,1,2,2,3)$	I	$\frac{1}{2}, 20, 10981$; $\frac{1}{3}, 10, 11101$	32	−60	45
11002	$X_{4,6} \subset \mathbb{P}(1,1,1,2,3,3)$	I	$\frac{1}{3}, 12, 10981$	40	−76	53

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Table 2 continued from previous page

10983	$X_{5,6} \subset \mathbb{P}(1,1,1,2,3,4)$	I	$\frac{1}{2}, 30, 10960;\ \frac{1}{4}, 10, 10981$	42	-80	55
10982	$X_{6,6} \subset \mathbb{P}(1,1,1,2,3,5)$	I	$\frac{1}{5}, 6, 10981$	46	-88	59
10961	$X_{6,8} \subset \mathbb{P}(1,1,1,3,4,5)$	I	$\frac{1}{5}, 12, 10960$	60	-116	73
6858	$X_{4,6} \subset \mathbb{P}(1,1,2,2,2,3)$	II_1	$\frac{1}{2}, 34, 5837$	31	-58	43
5857	$X_{5,6} \subset \mathbb{P}(1,1,2,2,3,3)$	I	$\frac{1}{3}, 15, 5838$	31	-58	42
5843	$X_{6,6} \subset \mathbb{P}(1,1,2,2,3,4)$	I	$\frac{1}{4}, 12, 5838$	34	-64	45
5839	$X_{6,7} \subset \mathbb{P}(1,1,2,2,3,5)$	I	$\frac{1}{5}, 7, 5838$	39	-74	50
5514	$X_{6,6} \subset \mathbb{P}(1,1,2,3,3,3)$	I	$\frac{1}{3}, 18, 5257$	32	-60	42
5261	$X_{6,7} \subset \mathbb{P}(1,1,2,3,3,4)$	I	$\frac{1}{3}, 21, 5157;\ \frac{1}{4}, 14, 5257$	36	-68	46
5258	$X_{6,8} \subset \mathbb{P}(1,1,2,3,3,5)$	I	$\frac{1}{3}, 24, 5153;\ \frac{1}{5}, 8, 5257$	42	-80	52
5200	$X_{6,8} \subset \mathbb{P}(1,1,2,3,4,4)$	I	$\frac{1}{4}, 16, 5157$	41	-78	51
5161	$X_{7,8} \subset \mathbb{P}(1,1,2,3,4,5)$	I	$\frac{1}{3}, 28, 5137;\ \frac{1}{5}, 14, 5157$	43	-82	53
5159	$X_{6,9} \subset \mathbb{P}(1,1,2,3,4,5)$	I	$\frac{1}{4}, 18, 5153;\ \frac{1}{5}, 9, 5157$	48	-92	58
5158	$X_{8,9} \subset \mathbb{P}(1,1,2,3,4,7)$	I	$\frac{1}{7}, 6, 5157$	51	-98	61
5156	$X_{6,10} \subset \mathbb{P}(1,1,2,3,5,5)$	I	$\frac{1}{5}, 10, 5153$	56	-108	66
5155	$X_{8,10} \subset \mathbb{P}(1,1,2,3,5,7)$	I	$\frac{1}{3}, 40, 5134;\ \frac{1}{7}, 8, 5153$	58	-112	68
5154	$X_{9,10} \subset \mathbb{P}(1,1,2,3,5,8)$	I	$\frac{1}{8}, 6, 5153$	60	-116	70
5138	$X_{8,10} \subset \mathbb{P}(1,1,2,4,5,6)$	I	$\frac{1}{6}, 16, 5137$	55	-106	65
5135	$X_{10,14} \subset \mathbb{P}(1,1,2,5,7,9)$	I	$\frac{1}{9}, 10, 5134$	88	-172	98
4985	$X_{8,9} \subset \mathbb{P}(1,1,3,4,4,5)$	I	$\frac{1}{4}, 24, 4909;\ \frac{1}{5}, 18, 4984$	43	-82	51
4936	$X_{8,10} \subset \mathbb{P}(1,1,3,4,5,5)$	I	$\frac{1}{5}, 20, 4909$	47	-90	55
4912	$X_{9,10} \subset \mathbb{P}(1,1,3,4,5,6)$	I	$\frac{1}{4}, 30, 4893;\ \frac{1}{6}, 18, 4909$	49	-94	57
4911	$X_{8,12} \subset \mathbb{P}(1,1,3,4,5,7)$	I	$\frac{1}{5}, 24, 4907;\ \frac{1}{7}, 8, 4909$	59	-114	67
4910	$X_{10,12} \subset \mathbb{P}(1,1,3,4,5,9)$	I	$\frac{1}{9}, 6, 4909$	61	-118	69
4908	$X_{12,14} \subset \mathbb{P}(1,1,3,4,7,11)$	I	$\frac{1}{11}, 6, 4907$	77	-150	85
4894	$X_{10,12} \subset \mathbb{P}(1,1,3,5,6,7)$	I	$\frac{1}{7}, 20, 4893$	59	-114	67
4848	$X_{10,12} \subset \mathbb{P}(1,1,4,5,6,6)$	I	$\frac{1}{6}, 24, 4835$	54	-104	61
4837	$X_{11,12} \subset \mathbb{P}(1,1,4,5,6,7)$	I	$\frac{1}{5}, 33, 4822;\ \frac{1}{7}, 22, 4835$	56	-108	63
4836	$X_{12,15} \subset \mathbb{P}(1,1,4,5,6,11)$	I	$\frac{1}{11}, 6, 4835$	72	-140	79
4823	$X_{12,14} \subset \mathbb{P}(1,1,4,6,7,8)$	I	$\frac{1}{8}, 24, 4822$	65	-126	72
4808	$X_{14,16} \subset \mathbb{P}(1,1,5,7,8,9)$	I	$\frac{1}{9}, 28, 4807$	72	-140	78
4795	$X_{16,18} \subset \mathbb{P}(1,1,6,8,9,10)$	I	$\frac{1}{10}, 32, 4794$	79	-154	85
3508	$X_{6,6} \subset \mathbb{P}(1,2,2,2,3,3)$	T^1		24	-44	34
2419	$X_{6,8} \subset \mathbb{P}(1,2,2,3,3,4)$	II_1	$\frac{1}{3}, 33, 2401$	28	-52	37
2409	$X_{6,10} \subset \mathbb{P}(1,2,2,3,4,5)$	II_1	$\frac{1}{4}, 25, 2401$	36	-68	45
2403	$X_{9,10} \subset \mathbb{P}(1,2,2,3,5,7)$	I	$\frac{1}{7}, 9, 2402$	39	-74	47
1390	$X_{8,9} \subset \mathbb{P}(1,2,3,3,4,5)$	I	$\frac{1}{5}, 12, 1389$	29	-54	36
1249	$X_{8,10} \subset \mathbb{P}(1,2,3,4,4,5)$	II_1	$\frac{1}{4}, 36, 1159$	30	-56	37
1179	$X_{9,10} \subset \mathbb{P}(1,2,3,4,5,5)$	I	$\frac{1}{5}, 15, 1162$	31	-58	37
1171	$X_{8,12} \subset \mathbb{P}(1,2,3,4,5,6)$	II_1	$\frac{1}{5}, 30, 1159$	36	-68	43
1165	$X_{10,11} \subset \mathbb{P}(1,2,3,4,5,7)$	I	$\frac{1}{7}, 11, 1162$	35	-66	41
1164	$X_{9,12} \subset \mathbb{P}(1,2,3,4,5,7)$	I	$\frac{1}{5}, 18, 1160;\ \frac{1}{7}, 9, 1162$	37	-70	43
1163	$X_{10,12} \subset \mathbb{P}(1,2,3,4,5,8)$	I	$\frac{1}{8}, 8, 1162$	38	-72	44
1161	$X_{12,14} \subset \mathbb{P}(1,2,3,4,7,10)$	I	$\frac{1}{10}, 8, 1160$	47	-90	53
1156	$X_{10,12} \subset \mathbb{P}(1,2,3,5,5,7)$	I	$\frac{1}{5}, 20, 1149;\ \frac{1}{7}, 12, 1155$	37	-70	42
1154	$X_{10,14} \subset \mathbb{P}(1,2,3,5,7,7)$	I	$\frac{1}{7}, 14, 1149$	43	-82	48
1152	$X_{10,15} \subset \mathbb{P}(1,2,3,5,7,8)$	I	$\frac{1}{7}, 15, 1147;\ \frac{1}{8}, 10, 1149$	47	-90	52

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Table 2 continued from previous page

1151	$X_{12,14} \subset \mathbb{P}(1,2,3,5,7,9)$	I	$\frac{1}{5}, 28, 1144;\ \frac{1}{9}, 12, 1149$	45	-86	50
1150	$X_{14,15} \subset \mathbb{P}(1,2,3,5,7,12)$	I	$\frac{1}{12}, 6, 1149$	51	-98	56
1148	$X_{15,16} \subset \mathbb{P}(1,2,3,5,8,13)$	I	$\frac{1}{13}, 6, 1147$	56	-108	61
1145	$X_{14,18} \subset \mathbb{P}(1,2,3,7,9,11)$	I	$\frac{1}{11}, 14, 1144$	59	-114	64
1121	$X_{10,12} \subset \mathbb{P}(1,2,4,5,5,6)$	II_1	$\frac{1}{5}, 40, 1112$	33	-62	39
1114	$X_{10,14} \subset \mathbb{P}(1,2,4,5,6,7)$	II_1	$\frac{1}{6}, 35, 1112$	38	-72	44
1083	$X_{12,16} \subset \mathbb{P}(1,2,5,6,7,8)$	II_1	$\frac{1}{5}, 48, 1067;\ \frac{1}{7}, 40, 1078$	41	-78	46
1080	$X_{14,15} \subset \mathbb{P}(1,2,5,6,7,9)$	I	$\frac{1}{9}, 15, 1079$	41	-78	45
1077	$X_{18,22} \subset \mathbb{P}(1,2,5,9,11,13)$	I	$\frac{1}{13}, 18, 1076$	60	-116	63
1068	$X_{14,18} \subset \mathbb{P}(1,2,6,7,8,9)$	II_1	$\frac{1}{8}, 45, 1067$	44	-84	49
867	$X_{10,12} \subset \mathbb{P}(1,3,3,4,5,7)$	I	$\frac{1}{7}, 10, 866$	31	-58	36
640	$X_{10,12} \subset \mathbb{P}(1,3,4,4,5,6)$	T^1		28	-52	33
547	$X_{12,13} \subset \mathbb{P}(1,3,4,5,6,7)$	I	$\frac{1}{7}, 13, 545$	30	-56	34
546	$X_{12,15} \subset \mathbb{P}(1,3,4,5,6,9)$	I	$\frac{1}{9}, 9, 545$	34	-64	38
544	$X_{12,14} \subset \mathbb{P}(1,3,4,5,7,7)$	I	$\frac{1}{7}, 14, 539$	32	-60	35
542	$X_{12,15} \subset \mathbb{P}(1,3,4,5,7,8)$	I	$\frac{1}{7}, 15, 537;\ \frac{1}{8}, 12, 539$	34	-64	37
541	$X_{14,15} \subset \mathbb{P}(1,3,4,5,7,10)$	I	$\frac{1}{10}, 10, 539$	36	-68	39
540	$X_{14,16} \subset \mathbb{P}(1,3,4,5,7,11)$	I	$\frac{1}{11}, 8, 539$	38	-72	41
538	$X_{15,16} \subset \mathbb{P}(1,3,4,5,8,11)$	I	$\frac{1}{11}, 10, 537$	39	-74	42
535	$X_{20,21} \subset \mathbb{P}(1,3,4,7,10,17)$	I	$\frac{1}{17}, 6, 534$	52	-100	54
509	$X_{14,15} \subset \mathbb{P}(1,3,5,6,7,8)$	I	$\frac{1}{8}, 14, 508$	32	-60	35
453	$X_{12,14} \subset \mathbb{P}(1,4,4,5,6,7)$	T^1		28	-52	32
359	$X_{14,16} \subset \mathbb{P}(1,4,5,6,7,8)$	T^1		29	-54	32
358	$X_{12,20} \subset \mathbb{P}(1,4,5,6,7,10)$	II_1	$\frac{1}{7}, 27, 355$	36	-68	39
357	$X_{18,20} \subset \mathbb{P}(1,4,5,6,9,14)$	I	$\frac{1}{14}, 8, 356$	38	-72	40
354	$X_{18,20} \subset \mathbb{P}(1,4,5,7,9,13)$	I	$\frac{1}{13}, 10, 353$	37	-70	38
338	$X_{16,18} \subset \mathbb{P}(1,4,6,7,8,9)$	T^1		30	-56	33
297	$X_{18,20} \subset \mathbb{P}(1,5,6,7,9,11)$	I	$\frac{1}{11}, 12, 296$	31	-58	32
279	$X_{18,30} \subset \mathbb{P}(1,6,8,9,10,15)$	T^1		36	-68	38
265	$X_{24,30} \subset \mathbb{P}(1,8,9,10,12,15)$	T^1		30	-56	31
37	$X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$	T^1		18	-32	23

Table 3: Codimension 3: $h^{1,1}(X) = 1$ and $h^0(X, T_X) = 0$ in all cases.

grdb	variety	method	$\frac{1}{r}$, #nodes, target id	$h^{2,1}$	$e(X)$	$h^1(T_X)$
26988	$X_{2,2\ldots} = X_{2,2,2} \subset \mathbb{P}^6$	$c_3(T_X)$		14	-24	27
24077	$X_{2,3\ldots} \subset \mathbb{P}(1,1,1,1,1,1,2)$	$\mathrm{I} - T^1$	$\frac{1}{2}, 7, 24076$	14	-24	27
20543	$X_{3,3\ldots} \subset \mathbb{P}(1,1,1,1,1,2,2)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{2}, 8, 20522$	15	-26	28
20523	$X_{3,3\ldots} \subset \mathbb{P}(1,1,1,1,1,2,3)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{3}, 6, 20522$	17	-30	30
16338	$X_{3,3\ldots} \subset \mathbb{P}(1,1,1,1,2,2,2)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{2}, 10, 16225$	18	-32	31
16226	$X_{3,4\ldots} \subset \mathbb{P}(1,1,1,1,2,2,3)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{2}, 11, 16204;\ \frac{1}{3}, 7, 16225$	21	-38	34
16205	$X_{4,4\ldots} \subset \mathbb{P}(1,1,1,1,2,3,4)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{4}, 7, 16204$	25	-46	38
12062	$X_{4,4\ldots} \subset \mathbb{P}(1,1,1,2,2,2,2)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{2}, 12, 11435$	15	-26	27
11436	$X_{4,4\ldots} \subset \mathbb{P}(1,1,1,2,2,2,3)$	$\mathrm{I} - \mathrm{I}$	$\frac{1}{2}, 14, 11102;\ \frac{1}{3}, 8, 11435$	19	-34	31

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Table 3 continued from previous page

11122	$X_{4,4\ldots} \subset \mathbb{P}(1,1,1,2,2,3,3)$	I − I	$\frac{1}{2}, 17, 11002;\ \frac{1}{3}, 9, 11102$	24	−44	36
11105	$X_{4,5\ldots} \subset \mathbb{P}(1,1,1,2,2,3,4)$	I − I	$\frac{1}{2}, 18, 10983;\ \frac{1}{4}, 8, 11102$	25	−46	37
11103	$X_{4,5\ldots} \subset \mathbb{P}(1,1,1,2,2,3,5)$	I − I	$\frac{1}{2}, 19, 10982;\ \frac{1}{5}, 5, 11102$	28	−52	40
11003	$X_{4,5\ldots} \subset \mathbb{P}(1,1,1,2,3,3,4)$	I − I	$\frac{1}{3}, 11, 10983;\ \frac{1}{4}, 9, 11002$	32	−60	44
10984	$X_{5,6\ldots} \subset \mathbb{P}(1,1,1,2,3,4,5)$	I − I	$\frac{1}{2}, 27, 10961;\ \frac{1}{5}, 9, 10983$	34	−64	46
10962	$X_{6,7\ldots} \subset \mathbb{P}(1,1,1,3,4,5,6)$	I − I	$\frac{1}{6}, 11, 10961$	50	−96	62
6859	$X_{4,5\ldots} \subset \mathbb{P}(1,1,2,2,2,3,3)$	I − II_1	$\frac{1}{3}, 11, 6858$	21	−38	32
5962	$X_{5,5\ldots} \subset \mathbb{P}(1,1,2,2,3,3,3)$	I − I	$\frac{1}{3}, 12, 5857$	20	−36	30
5865	$X_{5,6\ldots} \subset \mathbb{P}(1,1,2,2,3,3,4)$	I − I	$\frac{1}{3}, 13, 5843;\ \frac{1}{4}, 10, 5857$	22	−40	32
5858	$X_{5,6\ldots} \subset \mathbb{P}(1,1,2,2,3,3,5)$	I − I	$\frac{1}{3}, 14, 5839;\ \frac{1}{5}, 6, 5857$	26	−48	36
5844	$X_{6,6\ldots} \subset \mathbb{P}(1,1,2,2,3,4,5)$	I − I	$\frac{1}{5}, 10, 5843$	25	−46	35
5840	$X_{6,7\ldots} \subset \mathbb{P}(1,1,2,2,3,5,7)$	I − I	$\frac{1}{7}, 6, 5839$	34	−64	44
5515	$X_{6,6\ldots} \subset \mathbb{P}(1,1,2,3,3,3,4)$	I − I	$\frac{1}{3}, 15, 5261;\ \frac{1}{4}, 11, 5514$	22	−40	31
5302	$X_{6,6\ldots} \subset \mathbb{P}(1,1,2,3,3,4,4)$	I − I	$\frac{1}{3}, 17, 5200;\ \frac{1}{4}, 12, 5261$	25	−46	34
5267	$X_{6,7\ldots} \subset \mathbb{P}(1,1,2,3,3,4,5)$	I − I	$\frac{1}{3}, 18, 5161;\ \frac{1}{5}, 11, 5261$	26	−48	35
5264	$X_{6,6\ldots} \subset \mathbb{P}(1,1,2,3,3,4,5)$	I − I	$\frac{1}{3}, 19, 5159;\ \frac{1}{4}, 13, 5258;\ \frac{1}{5}, 7, 5261$	30	−56	39
5262	$X_{6,7\ldots} \subset \mathbb{P}(1,1,2,3,3,4,7)$	I − I	$\frac{1}{3}, 20, 5158;\ \frac{1}{7}, 5, 5261$	32	−60	41
5259	$X_{6,8\ldots} \subset \mathbb{P}(1,1,2,3,3,5,8)$	I − I	$\frac{1}{3}, 23, 5154;\ \frac{1}{8}, 5, 5258$	38	−72	47
5201	$X_{6,7\ldots} \subset \mathbb{P}(1,1,2,3,4,4,5)$	I − I	$\frac{1}{4}, 14, 5161;\ \frac{1}{5}, 12, 5200$	30	−56	39
5175	$X_{6,7\ldots} \subset \mathbb{P}(1,1,2,3,4,5,5)$	I − I	$\frac{1}{5}, 13, 5159;\ \frac{1}{5}, 8, 5161$	36	−68	45
5162	$X_{7,8\ldots} \subset \mathbb{P}(1,1,2,3,4,5,6)$	I − I	$\frac{1}{3}, 24, 5138;\ \frac{1}{6}, 12, 5161$	32	−60	41
5160	$X_{6,8\ldots} \subset \mathbb{P}(1,1,2,3,4,5,7)$	I − I	$\frac{1}{4}, 17, 5155;\ \frac{1}{7}, 7, 5159$	42	−80	51
5139	$X_{8,9\ldots} \subset \mathbb{P}(1,1,2,4,5,6,7)$	I − I	$\frac{1}{7}, 14, 5138$	42	−80	51
4999	$X_{8,8\ldots} \subset \mathbb{P}(1,1,3,4,4,5,5)$	I − I	$\frac{1}{4}, 19, 4936;\ \frac{1}{5}, 15, 4985$	29	−54	36
4988	$X_{8,9\ldots} \subset \mathbb{P}(1,1,3,4,4,5,6)$	I − I	$\frac{1}{4}, 20, 4912;\ \frac{1}{6}, 14, 4985$	30	−56	37
4986	$X_{8,9\ldots} \subset \mathbb{P}(1,1,3,4,4,5,9)$	I − I	$\frac{1}{4}, 23, 4910;\ \frac{1}{9}, 5, 4985$	39	−74	46
4937	$X_{8,9\ldots} \subset \mathbb{P}(1,1,3,4,5,5,6)$	I − I	$\frac{1}{5}, 17, 4912;\ \frac{1}{6}, 15, 4936$	33	−62	40
4914	$X_{9,10\ldots} \subset \mathbb{P}(1,1,3,4,5,6,7)$	I − I	$\frac{1}{4}, 25, 4894;\ \frac{1}{7}, 15, 4912$	35	−66	42
4913	$X_{8,9\ldots} \subset \mathbb{P}(1,1,3,4,5,6,7)$	I − I	$\frac{1}{6}, 17, 4911;\ \frac{1}{7}, 7, 4912$	43	−82	50
4895	$X_{10,11\ldots} \subset \mathbb{P}(1,1,3,5,6,7,8)$	I − I	$\frac{1}{8}, 17, 4894$	43	−82	50
4849	$X_{10,11\ldots} \subset \mathbb{P}(1,1,4,5,6,6,7)$	I − I	$\frac{1}{6}, 20, 4837;\ \frac{1}{7}, 18, 4848$	37	−70	43
4838	$X_{11,12\ldots} \subset \mathbb{P}(1,1,4,5,6,7,8)$	I − I	$\frac{1}{5}, 27, 4823;\ \frac{1}{8}, 18, 4837$	39	−74	45
4824	$X_{12,13\ldots} \subset \mathbb{P}(1,1,4,6,7,8,9)$	I − I	$\frac{1}{9}, 20, 4823$	46	−88	52
4809	$X_{14,15\ldots} \subset \mathbb{P}(1,1,5,7,8,9,10)$	I − I	$\frac{1}{10}, 23, 4808$	50	−96	55
4796	$X_{16,17\ldots} \subset \mathbb{P}(1,1,6,8,9,10,11)$	I − I	$\frac{1}{11}, 26, 4795$	54	−104	59
2420	$X_{6,7\ldots} \subset \mathbb{P}(1,2,2,3,3,4,5)$	I − II_1	$\frac{1}{5}, 8, 2419$	21	−38	29
2404	$X_{9,10\ldots} \subset \mathbb{P}(1,2,2,3,5,7,9)$	I − I	$\frac{1}{9}, 8, 2403$	32	−60	39
1409	$X_{7,8\ldots} \subset \mathbb{P}(1,2,3,3,4,4,5)$	II_1	$\frac{1}{4}, 21, 1389$	20	−36	27
1396	$X_{8,8\ldots} \subset \mathbb{P}(1,2,3,3,4,5,5)$	I − I	$\frac{1}{5}, 10, 1390$	20	−36	26
1394	$X_{8,9\ldots} \subset \mathbb{P}(1,2,3,3,4,5,7)$	I − I	$\frac{1}{7}, 8, 1390$	22	−40	28
1391	$X_{8,9\ldots} \subset \mathbb{P}(1,2,3,3,4,5,8)$	I − I	$\frac{1}{8}, 6, 1390$	24	−44	30
1252	$X_{8,9\ldots} \subset \mathbb{P}(1,2,3,4,4,5,5)$	I − II_1	$\frac{1}{5}, 11, 1249$	20	−36	26
1250	$X_{8,9\ldots} \subset \mathbb{P}(1,2,3,4,4,5,7)$	I − II_1	$\frac{1}{7}, 7, 1249$	24	−44	30
1184	$X_{8,9\ldots} \subset \mathbb{P}(1,2,3,4,5,5,6)$	I − II_1	$\frac{1}{5}, 13, 1171$	24	−44	30
1180	$X_{9,10\ldots} \subset \mathbb{P}(1,2,3,4,5,5,7)$	I − I	$\frac{1}{5}, 13, 1165;\ \frac{1}{7}, 9, 1179$	23	−42	28
1168	$X_{9,10\ldots} \subset \mathbb{P}(1,2,3,4,5,7,7)$	I − I	$\frac{1}{7}, 10, 1164;\ \frac{1}{7}, 8, 1165$	28	−52	33
1166	$X_{10,11\ldots} \subset \mathbb{P}(1,2,3,4,5,7,9)$	I − I	$\frac{1}{9}, 9, 1165$	27	−50	32

Continued on next page

Table 3 continued from previous page

1157	$X_{10,12\dots} \subset \mathbb{P}(1,2,3,5,5,7,12)$	I − I	$\frac{1}{5}, 19, 1150$; $\frac{1}{12}, 5, 1156$	33	−62	37
1153	$X_{10,12\dots} \subset \mathbb{P}(1,2,3,5,7,8,9)$	I − I	$\frac{1}{8}, 9, 1151$; $\frac{1}{9}, 11, 1152$	37	−70	41
1090	$X_{12,13\dots} \subset \mathbb{P}(1,2,5,6,7,7,8)$	I − II_1	$\frac{1}{7}, 15, 1083$	27	−50	31
1081	$X_{14,15\dots} \subset \mathbb{P}(1,2,5,6,7,9,11)$	I − I	$\frac{1}{11}, 12, 1080$	30	−56	33
868	$X_{10,12\dots} \subset \mathbb{P}(1,3,3,4,5,7,10)$	I − I	$\frac{1}{10}, 7, 867$	25	−46	29
641	$X_{10,11\dots} \subset \mathbb{P}(1,3,4,4,5,6,7)$	I − T^1	$\frac{1}{7}, 9, 640$	20	−36	24
568	$X_{10,11\dots} \subset \mathbb{P}(1,3,4,5,5,6,7)$	II_1	$\frac{1}{5}, 22, 545$	21	−38	25
548	$X_{12,13\dots} \subset \mathbb{P}(1,3,4,5,6,7,10)$	I − I	$\frac{1}{10}, 8, 547$	23	−42	26
543	$X_{12,14\dots} \subset \mathbb{P}(1,3,4,5,7,8,11)$	I − I	$\frac{1}{8}, 11, 540$; $\frac{1}{11}, 7, 542$	28	−52	30
510	$X_{14,15\dots} \subset \mathbb{P}(1,3,5,6,7,8,11)$	I − I	$\frac{1}{11}, 9, 509$	24	−44	26
454	$X_{12,13\dots} \subset \mathbb{P}(1,4,4,5,6,7,9)$	I − T^1	$\frac{1}{9}, 8, 453$	21	−38	24
392	$X_{12,13\dots} \subset \mathbb{P}(1,4,5,5,6,7,8)$	T^1		20	−36	23
326	$X_{14,15\dots} \subset \mathbb{P}(1,5,5,6,7,8,9)$	T^1		20	−36	22
298	$X_{16,17\dots} \subset \mathbb{P}(1,5,6,7,8,9,10)$	T^1		20	−36	22

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