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Weak Cartels and Collusion-Proof Auctions*

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Abstract

We study private value auctions in which bidders may collude without using side-payments. In our model, bidders coordinate their actions to achieve an outcome that interim-Pareto dominates the noncooperative outcome. We characterize auctions that are collusion-proof in the sense that no such coordination opportunities exist, and show that the efficient and revenue maximizing auctions are not collusion-proof unless all bidders exhibit a concave distribution of valuations. We then solve for revenue maximizing collusion-proof auctions. If distributions of valuations are symmetric and single-peaked, the optimal selling mechanism is a standard auction with a minimum bid, followed by sequential negotiation in case no bidder bids above the minimum bid.

Keywords: Weak cartels, weakly collusion-proof auctions, optimal auctions.
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1 Introduction

Collusion is a pervasive problem in auctions, especially in public procurement. A canonical example is the famous "Great Electrical Conspiracy" in the 1950s, in which more than 40

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manufacturers of electrical equipment colluded in sealed bid procurement auctions, using a bid rotation scheme also known as "phase of the moon" agreement (see Smith (1961)). More recently, in 2012, the largest six construction companies of Korea—so-called "Big 6" according to the competition authority—were involved in bid rigging in the Four River Restoration Project.[1] As a result, each of the Big 6 won 2 sections of the rivers while two other companies, also part of the collusive agreement, won 1 section each.

Many bid-rigging cases uncovered by competition authorities fall into the category of what McAfee and McMillan (1992) labeled "weak cartels," namely cartels that do not involve exchange of side payments among cartel members.[2] Weak cartels usually operate by designating a winning bidder and suppressing competition from other cartel members. The winning bidder is designated through "market sharing" agreements (e.g., the Korean construction case), through "bid rotation" whereby firms took turns in winning contracts (e.g., the U.S. case of electrical equipment conspiracy), or through more complicated schemes. The designated bidders place bids somewhere around the reserve price, and bids from other cartel members are either altogether suppressed (the practice of "bid suppression") or submitted at non-competitive levels (the practice of "cover bidding").

Cartels may avoid side payments for fear that they will leave a trail of evidence for antitrust authorities.[3] Compensating losing bidders in money may also lure "pretenders" who join a cartel solely to collect "the loser compensation" without ever intending to win. We show in Section II of the Supplementary Material that the ability to use side payments and reallocate the winning object (e.g., via a "knock-out" auction) adds no value to a cartel if entry by such pretenders cannot be controlled.[4] Hence, while transfers and knockout auctions

[1]This construction project is considered the biggest national infrastructural project in Korean history and has received a great deal of attention. We emphasize that many large national procurement auctions are "one-off" kind. These auctions are often so important for bidders that, even though they know they may face each other in future auctions, they naturally perceive the interaction as a static one.

[2]For example, among 16 bidding rigging cases in Korea that have been filed by the Korea Fair Trade Commission during the first half of year 2014, some evidence of side transfers was found only in 2 cases while there was no such evidence in 8 cases. It is also unclear whether transfers have been used in other cases. Another recent instance of a weak cartel involves producers of high voltage power cables, fined for about 0.3 billion euros by the European Commission. According to the press release, "the European and Asian producers would stay out of each other's home territories and most of the rest of the world would be divided amongst them. In implementing these agreements, the cartel participants allocated projects between themselves according to the geographic region or customer."

[3]In practice, cartels may hide side payments under different guises. For instance, Marshall et al. (1994) suggests that members bring bogus lawsuits against one another and exchange settlements. Such settlements must pass the scrutiny of a legal system, and must involve lawyers, so they entail transaction costs.

[4]If transfers cannot be used, the ability to reallocate the object (e.g., via a knockout auction) makes no

are important features of bidding rings (see Marshall and Marx (2012)), a theory of weak cartels is applicable far beyond environments where transfers are never used or impossible.

A key question is how weak cartels can profitability suppress competition in a way that benefits all its members. A strong cartel can achieve this goal by promising low-value bidders a compensation for staying out of the auction, preserving allocative efficiency. Given that such compensation is unavailable to weak cartels, the only scope for their profitable operation is to manipulate the allocation, sometimes allocating the good to bidders that do not have the highest value. But since the latter entails efficiency loss, it is not clear when and how such a distortion may benefit cartel members.

That weak cartels can indeed profitably operate was first demonstrated by McAfee and McMillan (1992, henceforth MM). Assuming that the distribution of bidders' valuations exhibits increasing hazard rate, they showed that in a standard auction symmetric bidders would benefit from agreeing, before knowing their private valuation, to randomly select a single bidder to bid the reserve price. It also follows that the best the seller can do to respond to such cartel behavior is to raise the reserve price. However, as we will show, this largely negative view rests on the assumption, made by MM, that the cartel is formed ex ante, i.e., before bidders learn their private valuations. If bidders are already privately informed of their values when deciding on a cartel agreement, a reasonable assumption in many settings, they may not be willing to participate in the cartel even if the ex-ante benefit from joining is positive.

In the current paper, we study weak cartels by explicitly considering the bidders' interim incentive to collude. By doing so, we offer a theory of weak cartels that differs from existing theories, not only in terms of what auctions are susceptible to collusion and under what conditions, but also of how a weak cartel would behave when it is active and of how the auctioneer should respond to the threat of collusion. Weak cartels trade off allocative efficiency for reduced competition, by asking their members to make bids/participation decisions that are, to some extent, insensitive to their private valuation. The key to our characterization is that the resulting efficiency loss is not borne uniformly across bidders with different valuations. Instead, high value-bidders suffer most acutely from collusion, and are most likely to object. Whether this loss triggers high-value bidders to reject a cartel depends on what they expect from competitive bidding if they reject the cartel. It turns out that the (information) rent lost by colluding will be higher when the distribution of values is convex (or its den-

difference. Further, since we assume risk neutrality for bidders, a fractional/probabilistic assignment entails no loss of generality per se. Hence, arrangements such as counter-purchase agreements which may be used to fine-tune market shares add no additional value to our weak cartel. In other words, our notion of a weak cartel already subsumes such an arrangement via random assignment.

sity is increasing). This observation leads us to identify intervals of of so-called "susceptible types"—namely those that would benefit from colluding—based on the curvature of bidder's value distribution.

We consider a large class of auctions, which we call "winner-payable," that include all standard auctions.[5] We then consider a model of collusion wherein bidders coordinate their bidding behavior to achieve an outcome that interim-Pareto dominates (from the bidders' perspective) the noncooperative outcome that would arise if there were no collusion. This is a workhorse model of collusion studied by a number of authors (Laffont and Martimort (1997, 2000), Che and Kim (2006, 2009), Pavlov (2008)). As shown by Laffont and Martimort (1997, 2000, henceforth LM), this model can be microfounded by an extensive form game in which an uninformed but benevolent cartel-mediator proposes to all bidders a collusion scheme—a side-contract specifying how they should bid in the auction—before they participate in the auction (but after they learn their values); all bidders simultaneously accept or reject the plan; and in the (on-path) event of all accepting they play the auction game according to the collusion scheme, and in the (off-path) event of a bidder unilaterally refusing the scheme, bidders play the auction game noncooperatively without updating their beliefs. Of particular interest is when the cartel mediator finds no collusion scheme that would interim-Pareto dominate the noncooperative outcome. When this happens, we call the original auction *weak collusion-proof* (WCP).

We first provide a complete characterization of (winner-payable) auctions that are weak collusion-proof. Our Theorem 1 states that a winner-payable auction is weak collusion-proof if and only if it satisfies (1) *non-wastefulness* and (2) *pooling on susceptible types*. Non-wastefulness requires that the good be fully assigned to some bidder whenever there is a bidder whose valuation exceeds a reserve price. Pooling on susceptible types requires that each bidder's interim winning probability not vary across valuations within the same connected interval of susceptible types. Roughly speaking, these two properties eliminate any scope for the cartel to coordinate bidders' behavior to benefit them regardless of their types. Conversely, if either condition fails, then we can construct a bid-coordination mechanism that interim-Pareto dominates the original auction.[6] An implication of this characterization

[5]The formal definition is presented in Section 3. Roughly speaking, the winner payable auctions require only winners to make positive payments and possess sufficient flexibility in payment rules for (colluding) bidders. As will be seen, the required condition is satisfied by all standard auctions, including first-price and second-price auctions, as well as by a wide variety of negotiation schemes.

[6]Note that the characterization of weak collusion-proof auctions (within the winner payable auctions) is solely in terms of interim allocation rules they exhibit—i.e., without any regard to any other features of the underlying auctions such as payment rules except for the reserve price of an auction. While this feature is reminiscent of the revenue equivalence theorem, it does not follow from the usual envelope theorem

is that efficient auctions as well as the revenue-maximizing auction (à la Myerson) are susceptible to weak cartels unless the value distributions of bidders are strictly concave (above reserve prices).

We next study the outcome of collusion when the auction *fails* to be weak collusion-proof. We demonstrate in Theorem 2 that a cartel operating according to our model implements full assignement whenever there is a bidder whose valuation is above the reserve price and implements a random allocation among susceptible bidder types. In other words, the two conditions of weak collusion-proofness we have derived also characterize the optimal cartel behavior, thus leading to a *positive theory* of collusion.[7] This confirms and extends the original insight of MM that random allocations are crucial tools for collusion. This result is of practical significance in light of the prevalence of cartel practices such as bid rotation and cover pricing, which can be seen as ways of implementing random allocations.

Compared with MM, however, our characterization of collusive behavior is richer and more nuanced. If the density of bidders' valuations is single-peaked, as one would expect in many cases, only low-valuation types are susceptible, meaning that a special arrangement is needed to coax high-valuation (unsusceptible) types into participating in a cartel. Specifically, a cartel would let high-valuation bidders bid competitively, while suppressing competition among low-valuation bidders. This kind of "semi-competitive" bidding behavior induces a bi-modal distribution of bids: bids are concentrated around the reserve price and around higher, more competitive, levels. We believe this observation is useful in guiding empirical efforts to identify the presence of a cartel in auctions or to estimate bidders' preferences in its presence. Moreover, the collusive behavior described above divides the spoils of collusion asymmetrically across bidder types. Asymmetric treatment of cartel members is not unusual in practice, even for firms that compete within the same geographical and product market. For instance, in their empirical analysis of a Canadian gasoline cartel, **?** document how firms were sharing collusive profits asymmetrically, without using explicit monetary transfer, but by allowing selected cartel members to undercut prices for specific periods of time.

argument. In fact, this feature of collusion-proofness characterization rests on the weak collusion-proofness (and the winner payability of the underlying auction), which in turn involves a careful construction of a bid-coordination scheme in case an allocation rule fails either (1) or (2). The tractactiblility it offers in checking an auction's susceptibility to collusion can be seen as an advantage of the weak collusion-proofness notion.

[7]Formally, this result amounts to the so-called *collusion-proofness principle*: it is without loss of generality to focus one's attention to auction rules that are weak collusion-proof. Plainly, the same behavioral outcome characterized by (1) and (2) emerges whether or not the auctioneer deters, or allows for, collusion. An analogous result is established by LM.

The complete characterization of collusion-proof auctions enables us to study the following normative question: *How should one design an auction in the presence of a weak cartel?* Restricting attention to winner-payable auctions, we identify the optimal collusion-proof auction for the seller up to the choice of the individual reserve prices (Theorem 3). The optimal mechanism allocates the good to maximize the virtual value functions that are suitably ironed out for the susceptible types. An interesting feature of the optimal mechanism is asymmetric treatment of bidders who are ex-ante identical. For instance, when the bidders' value distributions are convex, the optimal mechanism takes the form of a sequential negotiation: the seller engages in a take-it-or-leave-it negotiation with each of the bidders sequentially in a predetermined order.

In the case of a single-peaked density, the optimal (collusion-proof) mechanism consists of an auction with reserve price followed by a negotiation with individual bidders if the auction ends without sale. This is reminiscent of the way public-procurement auctions are conducted in Italy. In fact, public procurement laws in Italy allow buyers to start a direct negotiation with potential sellers if the initial competitive tendering fails to deliver any bid clearing the reserve price. The outcome of this negotiation can end up being a price higher than the originally set reserve price.[8]

The rest of the paper is organized as follows. In the next section, we illustrate our main results via two simple examples. Then, section 3 introduces the class of "winner payable" auction rules that we study and the model of collusion. Section 4 characterizes the susceptibility of auctions to weak cartels. Section 5 characterizes optimal collusion-proof auctions. Section 6 discusses related literature. Section 7 concludes. Appendices A-B and Supplementary Material contain all the proofs not presented in the main body of the paper.

2 Illustrative Examples

We first illustrate via simple examples how bidders' interim incentives to participate in weak cartels—as opposed to ex-ante incentives—affect the formation of cartels and their behavior. We present two examples here, and others will be interspersed throughout the analysis.

Example 1. Suppose there are two bidders vying for a single object in a second-price auction with zero reserve price. Each bidder has a valuation drawn from the interval $[0, 1]$ according to a distribution function $F(v) = 1 - (1 - v)^2$. Its hazard rate $\frac{f}{1-F}$ is increasing, and, according to MM, this implies that bidders would benefit ex ante from a weak cartel.

[8] See **?** for more details on the Italian public procurement regulation.

Specifically, if bidders were to bid non-cooperatively, both bidding their values, each bidder would earn an ex-ante payoff of $\frac{2}{15}$, but if they form a cartel and select one bidder at random to win the object at zero price, each would enjoy a strictly higher ex-ante payoff of $\frac{1}{6}$.

However, if bidders have private information at the cartel formation stage, then the fact that a cartel is beneficial ex-ante need not guarantee it will be beneficial to all types. To see this, suppose initially that both bidders participate in the cartel regardless of their valuations. And suppose the cartel has each bidder win with probability one half at zero price. Then, a bidder would enjoy the "interim" payoff of $\frac{v}{2}$ if his valuation is v.

Suppose the same bidder refuses to join the cartel. Then, the cartel collapses, and in the ensuing noncooperative play, each bidder employs a dominant strategy of bidding his valuation. The bidder would earn the "interim" payoff of

$$U^0(v) := \int_0^v (v-s)dF(s) = v^2 - \frac{v^3}{3}.$$

As depicted in Figure 1, $U^0(v) > v/2$ if $v > \frac{1}{2}\left(3-\sqrt{3}\right) =: \bar{v}_0$. That is, any bidder with valuation greater than $\bar{v}_0$ will be better off from refusing to join the cartel.

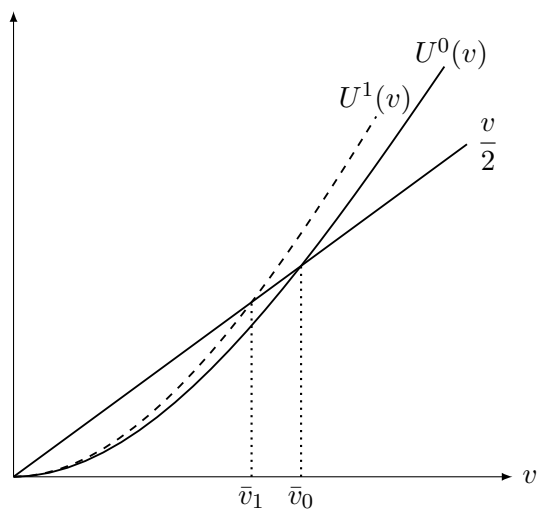


Figure 1: Cartel Unraveling

Given this, bidders *may* attempt to form a cartel that only operates when their valuations are both less than $\bar{v}_0$. Will such a "partial cartel" form? The answer is no. To see this, suppose to the contrary that a cartel forms if and only if the bidders' values are both less than $\bar{v}_0$. Also, suppose the cartel operates as before, randomly selecting a winner and having the loser bid zero. Given the agreement, the bidder will enjoy the interim payoff of $\frac{v}{2}$ as before, conditional on a cartel having been formed. But given the same event (i.e., his opponent

having $v < \bar{v}_0$), he would have earned

$$U^1(v) = \frac{\int_0^v (v-s)dF(s)}{F(\bar{v}_0)}$$

if he refused to join the cartel and bid his valuation in the noncooperative play. It turns out that $U^1(v) > v/2$ if and only if $v > \frac{1}{2}\left(3 - \sqrt{9 - 12\bar{v}_0 + 6\bar{v}_0^2}\right) =: \bar{v}_1$, which is strictly less than $\bar{v}_0$, as described in Figure 1. In other words, no bidder with valuation $v \in (\bar{v}_1, \bar{v}_0]$ will participate in the cartel.

Arguing recursively in this manner, one can see that no types of bidders are willing to participate in the cartel. Any cartel unravels. We shall later show that this is due to the density f being decreasing. Intuitively, declining density means that a higher valuation type forgoes relatively more from a non-cooperative play, in terms of the chance of winning the good. This creates the iterative process of high valuation types successively dropping out of collusion, leading to a full collapse, despite the fact that it is beneficial ex-ante.

Example 2. We next consider a situation in which a cartel is sustainable, but the way a cartel operates is crucially affected by the interim participation constraints. Suppose again two bidders participate in a second-price auction to obtain an object. Each bidder draws his valuation from a triangular distribution F with density $f(v) = 8v$ if $v \in [0, 1/4]$ and $f(v) = \frac{8}{3}(1-v)$ if $v \in [1/4, 1]$. The hazard rate is increasing everywhere, so bidders ex-ante payoff would be maximized by a random allocation, as shown by MM. However, since the density is decreasing in $[1/4, 1]$, a random allocation is not implementable by the cartel.[9]

Unlike the previous example, the density is not decreasing everywhere, and this feature will ensure profitability of a cartel, as our results in Section 4 will show. Such a cartel will, however, require a different arrangement than complete pooling (which violates interim participation constraint for high types, as showed by the dotted line crossing the non-collusive

[9]To see this, suppose that the bidders form a cartel and randomly allocate the object between them. A bidder will then earn the payoff of $v/2$ if his valuation is v. Suppose the same bidder refuses to join a cartel. From the ensuing non-cooperative bidding, the bidder will earn the payoff of

$$U(v) = \int_0^v (v-s)dF(s) = \int_0^v F(s)ds, \tag{1}$$

where

$$F(v) = \begin{cases} 4v^2 & \text{if } v \in [0, 1/4] \\ -\frac{1}{3} + \frac{8}{3}v - \frac{4}{3}v^2 & \text{if } v \in [1/4, 1]. \end{cases} \tag{2}$$

A simple calculation reveals that $U(v) > v/2$ for v sufficiently close to 1 (which is shown in Figure 2), meaning that a high valuation bidder will refuse to join such a cartel.

payoff in Figure 2). Suppose the cartel has each participating member send a cheap talk message, either H or L, depending on whether their values are above or below $\tilde{v} = 1/2$, respectively. Their bids are then coordinated as follow. Call a bidder who send message $j = H, L$ a j-bidder. Then, an H-bidder is instructed to bid his value. An L-bidder is instructed to bid $\frac{3}{4}v$, given his value v, if his opponent is an H-bidder (which prevents L from mimicking H). If both bidders are L-bidders, then one of them is chosen randomly to bid his value, and other bids zero.

This cartel arrangement implements pooling for bidders with $v < \tilde{v} = 1/2$, and competitive bidding for bidders with $v > \tilde{v} = 1/2$. Unlike complete pooling, this arrangement induces participation by all types. As can be seen in Figure 2, collusive payoff $\tilde{U}$ exceeds the non-collusive payoff U (specified in (1)) for all v.[10] Later we shall show (Theorem 2) that

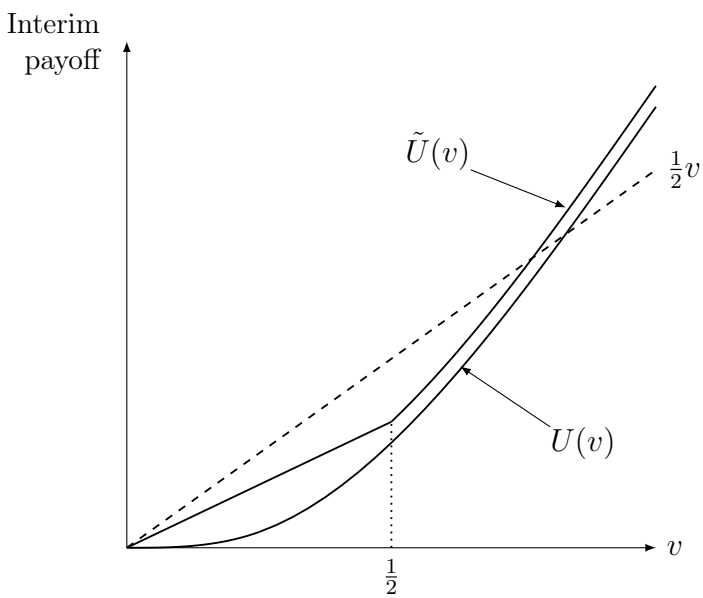


Figure 2: Profitability of Cartel Manipulation

the above cartel behavior is Pareto optimal among all sustainable cartel behaviors.

[10]Under the collusive arrangement, a type-v bidder obtains the object with probability

$$\tilde{Q}(v) := \begin{cases} \frac{F(\tilde{v})}{2} = \frac{1}{3} & \text{if } v \leq \tilde{v} \\ F(v) & \text{if } v > \tilde{v}, \end{cases}$$

and enjoys the payoff of

$$\tilde{U}(v) := \begin{cases} \frac{1}{3}v & \text{if } v \leq \tilde{v} \\ \frac{1}{3}\tilde{v} + \int_{\tilde{v}}^{v} F(s)ds & \text{if } v > \tilde{v}. \end{cases}$$

3 Model

3.1 Environment

A risk-neutral seller has a single object for sale. The seller's valuation of the object is normalized at zero. There are $n \geq 2$ risk neutral bidders, and $N := \{1, ..., n\}$ denotes the set of bidders. We assume that bidder i is privately informed of his valuation of the object, v_i, drawn from an interval $\mathcal{V}_i := [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$ according to a strictly increasing and continuous cumulative distribution function F_i (with density f_i).[11] We let $\mathcal{V} := \times_{i \in N} \mathcal{V}_i$ and assume that bidders' valuations are independently distributed. Each bidder's payoff from not obtaining the object and paying (or receiving) no money is normalized to zero.

The object is sold via an auction. An **auction** is defined by a triplet, $A := (\mathcal{B}, \xi, \tau)$, where $\mathcal{B} := \times_{i \in N} \mathcal{B}_i$ is a profile of message spaces (with $\mathcal{B}_i$ being i's message space), $\xi : \mathcal{B} \to \mathcal{Q}$ is a rule mapping a vector of messages ("bids") to a (possibly random) allocation of the object in $\mathcal{Q} := \{(x_1, \ldots, x_n) \in [0,1]^n | \sum_{i \in N} x_i \leq 1\}$, and $\tau : \mathcal{B} \to \mathbb{R}^n_+$ is a rule determining expected payments as a function of the messages. Let ξ_i and τ_i be i-th element of ξ and τ that corresponds to the allocation and payment rule for bidder i, respectively. We assume that the seller cannot force bidders to participate in the auction. Therefore, for each bidder, we require the message space $\mathcal{B}_i$ to include a non-participation option, b_i^0, which, when exercised, results in no winning and no payment for bidder i, $\xi_i(b_i^0, \cdot) := \tau_i(b_i^0, \cdot) = 0$.

It is useful to define the set $\mathcal{B}^i = \{b \in \mathcal{B} | \xi_i(b) > 0\}$ of bid profiles that lead bidder i to win with positive probability. (This set is assumed throughout to be nonempty.) Bidder i's **reserve price** under A is then defined as

$$r_i := \inf \left\{ \frac{\tau_i(b)}{\xi_i(b)} \geq 0 \middle| b \in \mathcal{B}^i \right\}, \tag{3}$$

the minimum per-unit price bidder i must pay to win with positive probability. Likewise, the maximum per-unit price bidder i could pay under auction A is given by

$$R_i := \sup \left\{ \frac{\tau_i(b)}{\xi_i(b)} \leq \overline{v}_i \middle| b \in \mathcal{B}^i \right\}.$$

Whether and how a cartel operates in an auction depends crucially on the details of its allocation and payment rule. Che and Kim (2009) show that if the seller faces no constraints in designing an auction, any outcome that involves no sale with sufficient probability can be

[11] Following Myerson (1981), we could add a common value component to the private valuations, by assuming that such component is common knowledge. In this case, our analysis remains unchanged. If bidders have private signals on the common value, however, collusion may facilitate information sharing, as pointed out by **?**. The analysis of this latter case is outside the scope of our paper.

implemented even in the presence of a cartel that can use side payment and even reallocate the object ex-post. The idea is to force the cartel to accept a fixed price (i.e., "selling the firm to the cartel"). This eliminates the scope for the cartel to manipulate the outcome. However, implementing this idea requires losing bidders to make payments as well—a feature seldom observed in practice.

In the current paper, we thus focus on more realistic auction formats in which losing bidders make no payments. Specifically, we restrict attention to a set $\mathcal{A}^*$ of auction rules that are winner-payable in the following sense.

Definition 1. *An auction A is* ***winner-payable*** *if, for all $i \in N$, there exist bid profiles $\underline{b}^i, \overline{b}^i \in \mathcal{B}^i$ such that $\xi_i(\underline{b}^i) = \xi_i(\overline{b}^i) = 1$, $\tau_i(\underline{b}^i) = r_i$, $\tau_i(\overline{b}^i) = R_i$, and $\tau_j(\underline{b}^i) = \tau_j(\overline{b}^i) = 0$, for $j \neq i$.*

In words, an auction is winner-payable if it is possible for bidders to coordinate their bids (possibly including non-participation) so that any given bidder can win the object for sure at the minimum per-unit price r_i (i.e., his reserve price) or at the maximum per-unit price R_i allowed by the bidding rule, and the other (losing) bidders pay nothing. Most of commonly observed auctions are winner-payable. Examples include first-price (or Dutch) auctions, second-price (or English) auctions, possibly with any reserve prices, and sequential take-it-or-leave-it offers.[12]

We note that our main result (Theorem 1) applies beyond winner-payable auctions as long as only the winner of the auction pays for the object and its equilibrium allocation is deterministic (i.e. for each profile of bids, the object is assigned with probability one to only one of the bidders, whenever it is assigned), or randomization is limited to tie-breaking and occurs with zero probability.

3.2 Characterization of Collusion-Free Outcomes

In the absence of collusion, an auction rule A in $\mathcal{A}^*$ induces a game of incomplete information where all bidders simultaneously submit messages (i.e. bids) to the seller. A pure strategy for player i is denoted $\beta_i : \mathcal{V}_i \to \mathcal{B}_i$, and $\beta = (\beta_1, \cdots, \beta_n)$ denotes a profile of strategies.

[12]Lotteries represent a notable exception. For instance, consider a mechanism where there is a fixed number $n \geq 2$ of lottery tickets, each bidder can buy a single ticket at a fixed price $p \in \mathbb{R}_+$, the auctioneer retains the unsold tickets, and the object is assigned to the holder of a randomly selected ticket. In this mechanism, $\mathcal{B}_i := \{0, 1\}$, $\xi_i(0, b_{-i}) = \tau_i(0, b_{-i}) = 0$, $\xi_i(1, b_{-i}) = 1/n$, and $\tau_i(1, b_{-i}) = p$. Winner-payability fails as there is no message profile that can guarantee the object to any of the players. On the other hand, fixed-prize raffles (see Morgan (2000)) are winner-payable.

Given a profile of equilibrium bidding strategies β^* of an auction A, its **outcome** corresponds to a **direct mechanism** $M_A \equiv (q,t) : \mathcal{V} \to \mathcal{Q} \times \mathbb{R}^n$, where for all $v \in \mathcal{V}$, $q(v) = \xi(\beta^*(v))$ is the allocation rule for the object and $t(v) = \tau(\beta^*(v))$ is the payment rule. Given M_A, we define the interim winning probability $Q_i(v_i) = \mathbb{E}_{v_{-i}}[q_i(v_i, v_{-i})]$ and interim payment $T_i(v_i) = \mathbb{E}_{v_{-i}}[t_i(v_i, v_{-i})]$ for bidder $i \in N$ with type $v_i \in \mathcal{V}_i$. We will refer to the mapping $Q = (Q_i)_{i \in N}$ and $T = (T_i)_{i \in N}$ as interim allocation and transfer rules, respectively. The equilibrium payoff of player i with value v_i is then expressed as

$$U_i^{M_A}(v_i) := Q_i(v_i)v_i - T_i(v_i).$$

Any collusion-free equilibrium outcome M_A must be **incentive compatible** (by definition of equilibrium) and **individually rational** (because bidders are offered the non-participation option). That is, for all $i \in N$ and $v_i \in \mathcal{V}_i$,

$$U_i^{M_A}(v_i) \geq v_i Q_i(\tilde{v}_i) - T_i(\tilde{v}_i), \text{ for all } \tilde{v}_i \in \mathcal{V}_i, \tag{IC}$$

$$U_i^{M_A}(v_i) \geq 0. \tag{IR}$$

As is well known, (IC) and (IR) are equivalent to the following conditions:

$$Q_i \text{ is nondecreasing}, \forall i \in N; \tag{M}$$

$$T_i(v_i) = v_i Q_i(v_i) - \int_{\underline{v}_i}^{v_i} Q_i(s)ds + T_i(\underline{v}_i) - \underline{v}_i Q_i(\underline{v}_i), \forall v_i \in \mathcal{V}_i, \forall i \in N; \tag{Env}$$

$$U_i^{M_A}(\underline{v}_i) = \underline{v}_i Q_i(\underline{v}_i) - T_i(\underline{v}_i) \geq 0, \forall i \in N. \tag{IR$'$}$$

In the later analysis, we often restrict attention to direct mechanisms (that may not be winner-payable). This restriction is without loss, however, as is shown next:

Lemma 1. *Given any direct mechanism $M = (q,t)$ that satisfies (IC) and (IR), there is a winner-payable auction rule $A \in \mathcal{A}^*$ whose equilibrium outcome yields the same interim outcome as M.*

Proof. See Section I of the Supplementary Material. ▌

3.3 Models of Collusion

Members of a weak cartel can only coordinate the bids submitted to the seller. Since non-participation is regarded as a possible bid in our model, this means that the bidders can also coordinate on their participation decisions. We abstract from the question of how a cartel

can enforce an agreement among its members, but rather focus on whether there will be an incentive compatible agreement that is beneficial for all bidders.[13]

Formally, a **cartel agreement** is a mapping $\alpha : \mathcal{V} \to \Delta(\mathcal{B})$ that specifies a lottery over possible bid profiles in auction A for each profile of valuations for the bidders. A cartel agreement leads bidders to play a game of incomplete information where each player's strategy is to report his type to the cartel and then outcomes are determined by the lottery α over bids and auction rule A. By the revelation principle, it is without loss to restrict attention to cartel agreements that make bidders report their true valuation to the mediator. Hence, for any cartel agreement α, one can equivalently consider the direct mechanism it induces.

Definition 2. *A direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ is a **cartel manipulation** of A if there exists a cartel agreement α such that*

$$\tilde{q}_i(v) = \mathbb{E}_{\alpha(v)}[\xi_i(b)] \quad \textit{and} \quad \tilde{t}_i(v) = \mathbb{E}_{\alpha(v)}[\tau_i(b)], \forall v \in \mathcal{V}, i \in N, \tag{4}$$

where $\mathbb{E}_{\alpha(v)}[\cdot]$ denotes the expectation taken using the probability distribution $\alpha(v) \in \Delta(\mathcal{B})$.

Since $\tilde{M}_A$ results from bidders' equilibrium play in the incomplete information game described above, it is incentive compatible, or satisfies (IC).[14]

Our goal is to investigate whether any auction $A \in \mathcal{A}^*$ is susceptible to some cartel manipulation $\tilde{M}_A$. To this end, we must first identify the set of cartel manipulations that would be agreed upon by the bidders in auction A. In a fully game-theoretic approach, this would require analyzing how the auction would proceed if some bidder refused to participate in a proposed manipulation. The latter in turn depends on the beliefs the refusing bidder and other bidders form against each other.

To address these issues, we follow the model of collusion proposed by Laffont and Martimort (1997) and studied further by Laffont and Martimort (2000), Che and Kim (2006, 2009), Pavlov (2008), and **?** in various contexts. This model postulates that all bidders play an auxiliary cartel game before the auction takes place. In the cartel game, an uninformed third party proposes a (possibly "null") cartel agreement and all bidders, after privately observing their valuations, simultaneously and independently decide whether to accept or reject the proposal. If all bidders accept the proposal, then the cartel agreement comes into force

[13]This is consistent with MM and LM and most of the literature analyzing static models of collusion in auctions.

[14]The condition (IC) holds for all Bayes Nash equilibria. While dynamic nature of the game may impose additional restriction on cartel manipulation (in the form of sequential rationality), any such behavior must also satisfy (IC). In this sense, the current approach is permissive about a possible cartel manipulation.

and bidders are committed to play the agreement; if any bidder rejects the proposal then the original auction is played. One then focuses on an equilibrium in which the third-party mediator always propose a (possibily a null) cartel agreement that weakly interim-Pareto dominates the noncooperative outcome, bidders all accept the agreement, and in the off-path event of a bidder unilaterally rejecting the agreement, they play the noncooperative auction game without updating their prior beliefs about their opponents' types. That is, a *passive belief* is assumed off the path when collusion breaks down unexpectedly.[15]

Since noncooperative play under prior beliefs yields payoff $U_i^{M_A}(v_i)$ and cartel manipulation $\tilde{M}_A$ of auction A yields $U_i^{\tilde{M}_A}(v_i)$, the manipulation is **profitable** if it satisfies (IC) and

$$U_i^{\tilde{M}_A}(v_i) \geq U_i^{M_A}(v_i), \forall v_i, i, \text{ with strict inequality for some } v_i, i, \tag{C-IR}$$

where M_A is the collusion-free outcome at auction A. The susceptibility of auction to weak cartels is defined in the following way:

Definition 3. *An equilibrium outcome M_A of an auction A is* ***weakly collusion-proof (or WCP)*** *if it does not admit a profitable cartel manipulation.*

According to this definition, an auction is susceptible to bidder collusion if and only if there exists a cartel manipulation that interim Pareto dominates its collusion-free outcome.[16] This condition provides a reasonable test for the collusion-proofness of an auction rule. The presence of an interim Pareto dominating manipulation would make it a common knowledge that everyone will gain from collusion (as argued by Holmstrom and Myerson (1983)), making cartel-forming a clear cause for concern. By contrast, its absence would mean that no consensus exists among bidders to form a cartel.[17] We initially use WCP to study susceptibility (or lack thereof) of an auction to cartels. But later, we establish the

[15]This assumption has been adopted widely in the collusion literature, including the papers mentioned above. The assumption disciplines bidders' beliefs after a breakdown of collusion, by preventing them from being too pessimistic or optimistic about each other. The passive belief assumptions is also widely used in the contracting literature.

[16]Definition 3 implies that at least one type of one bidder must have a strict incentive to accept the cartel manipulation. If that was not the case, then the manipulation would yield exactly the same outcome as the original auction, including the same revenue for the seller. In this case, collusion would not be a concern.

[17] Not only is the assumed behavior a reasonable description of how cartels may operate, but the model also permits a now familiar auction-theoretic analytics pioneered by Meyerson: as will be seen, the constraint collusion imposes on a mechanism is characterized entirely via the interim allocation rule induced by the mechanism up to the choice of a reserve price, and thus can be tractably incorporated into a mechanism design framework. This enables us to explore an optimal auction in the presence of collusion, as will be done in Section 5.

collusion-proofness principle (Theorem 1 and Corollary 5), which will validate WCP as a notion of equilibrium outcome in the presence of collusion, whether collusion is deterred or not, that is, whether an auction rule is WCP or not.

Throughout, our analysis focuses on weak cartels. Weak cartels are characterized by two important restrictions on their behavior that differentiate them form strong cartels. First, they are unable to use side payments to compensate losers. Second, they cannot reallocate the object once it leaves the seller's hand. While realistic in many settings, these limitations are non-trivial. Therefore, one might expect that strong cartels will always serve collusive bidders better then weak cartels. For instance, transfers could be used to prevent the sort of cartel unraveling described in Section 2, by providing compensation for high-value bidders and allowing them to join the cartel. This is not necessarily the case. As we formally show in Section II of the Supplementary Material, a winner-payable auction that is resistant to weak cartels is also resistant to strong cartels, if bidders with values below the reserve price do not expect a positive payoff from joining the strong cartel, a condition labeled *entry exclusion constraint* (or EEC). Given this additional condition, all our results in sections 4 and 5 remain valid even for strong cartels.

The EEC condition is natural. As MM pointed out, a strong cartel would be reluctant to pay off non-serious bidders, especially when their entry in the cartel can not be controlled. If positive compensation is offered to low-value bidders that would never make a profit in the auction, then they will all wish to enter the market solely to receive the compensation. And if a large pool of low-value bidders exists, then this would dissipate collusive rents for serious bidders.

4 When Are Auctions Susceptible to Weak Cartels?

In this section, we first characterize outcomes of winner-payable auction that are susceptible to a weak cartel, and then show that the characterization also identifies optimal cartel behavior at such susceptible auctions.

4.1 Conditions for Weak Cartel Susceptibility

We begin by introducing one key definition.

Definition 4. *For each $i \in N$ and $r \in [\underline{v}_i, \overline{v}_i]$, the* ***concave closure*** *of F_i is: for each $v \in [r_i, \overline{v}_i]$,*

$$G_i(v; r_i) := \max\{sF_i(v') + (1-s)F_i(v'') | s \in [0,1], v', v'' \in [r_i, \overline{v}_i], \text{ and } sv' + (1-s)v'' = v\}.$$

In words, the concave closure $G_i(\cdot; r_i)$ is the lowest concave function above $F_i(\cdot)$ for all $[r_i, \overline{v}_i]$.[18] (To simplify notation, we will henceforth write $G_i(\cdot; r_i)$ as G_i.) Figure 3 depicts the concave closure G_i for a value distribution F_i that has a single-peaked density. Concave closure G_i is always linear on regions where F_i is linear or convex, but it may also be linear in areas where F_i is concave. Note that each concave function G_i admits density, denoted $g_i(v)$, for almost every $v \in [r_i, \overline{v}_i]$, whose derivative is well defined and satisfies $g_i'(v) \leq 0$ for almost every $v \in [r_i, \overline{v}_i]$. For each bidder i, we call **susceptible types** the set $\mathcal{V}_i^0(r_i) := \{v \geq r_i | g_i'(v) = 0\}$ —namely a subset of types above r_i where the concave closure is linear. In Figure 3, the susceptible types are an interval $[r_i, v^*(r_i)]$, while in general the set $\mathcal{V}_i^0(r_i)$ is a collection of disjoint intervals.

The intuition provided in the introduction suggests that susceptible types are prone to a cartel manipulation. This is formalized in the next theorem.

THEOREM 1. *Consider a winner-payable auction rule $A \in \mathcal{A}^*$.*

1. *(Necessity) Its equilibrium outcome M_A is weakly collusion-proof if M_A's interim allocation rule Q satisfies the following properties:*

   (i) Non-wastenefullness: $$\sum_{i \in N} q_i(v) = 1 \text{ if } v_i > r_i \text{ for some } i \in N; \tag{NW}$$

   (ii) Pooling on susceptible types: $$Q_i(v_i) = Q_i(v_i') \text{ if } [v_i, v_i'] \subset \mathcal{V}_i^0(r_i), \forall i \in N. \tag{PS}$$

2. *(Sufficiency) The converse also holds if, in addition to (PS) and (NW), A satisfies $r_i \geq \underline{v}_i$ for all $i \in N$.*

Proof. See Appendix A (page 31). ▌

The condition that $r_i \geq \underline{v}_i$ for all $i \in N$ is fairly natural. In fact, preventing the auctioneer from designing auctions that violate it is without loss of generality when the objective is revenue or welfare maximization.[19]

Weak collusion proofness requires that the object be fully allocated whenever there is a bidder whose valuation exceeds a reserve price (condition (NW)), and that a bidder's winning probability is constant for susceptible types (condition (PS)).

[18] We stress that G_i depends not only on type distribution F_i but also indirectly on the specific auction rule, which determines r_i.

[19] Indeed, one can show that for any auction rule $A \in \mathcal{A}^*$ whose outcome $M_A = (q, t)$ is WCP, there is a direct auction mechanism A' that satisfies $r_i \geq \underline{v}_i, \forall i \in N$, and whose outcome is WCP and achieves the same allocation and as much revenue for the seller as M_A does. The proof of this result is provided in Section III of the Supplementary Material.

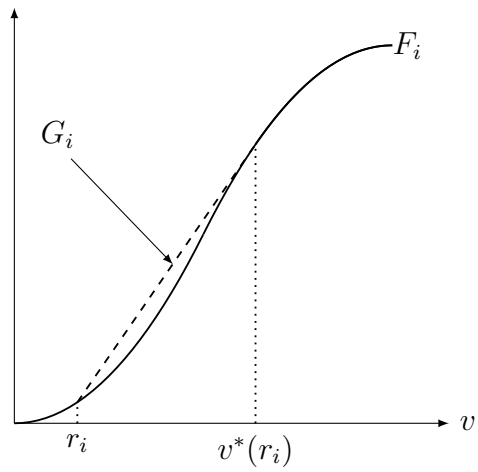


Figure 3: Type Distribution F and Its Concave Closure G

To understand the necessity of this latter condition, suppose that in some (collusion-free) equilibrium, bidder i's winning probability $Q_i(v_i)$ is strictly increasing within $[a, b]$.[20] Now, consider a cartel manipulation, labeled $\tilde{M}_A$, that: (i) leaves unchanged the interim winning probability and expected payments of all bidders other than bidder i and also of bidder i when his value is outside $[a, b]$ and (ii) gives the good to bidder i with a constant probability $\bar{p}$ if his value is inside $[a, b]$, where

$$\bar{p} = \frac{\int_a^b Q_i(s) f_i(s) ds}{F_i(b) - F_i(a)}, \tag{5}$$

that is, bidder i's average winning probability over the interval $[a, b]$ in M_A.

We investigate when this manipulation is uniformly profitable to all types—namely, when (C-IR) holds for all $v \in [a, b]$. This means that each type $v \in [a, b]$ must be getting at least its noncooperative payoff, which by (Env) equals

$$U^{M_A}(v) = U^{M_A}(a) + \int_a^v Q_i(s) ds. \tag{6}$$

Since Q is nondecreasing, this payoff is convex, as is described in Figures 4(a) and 4(b). For the cartel manipulation to be profitable, its payoff $U^{\tilde{M}_A}(v)$ must lie above the noncooperative payoff $U^{M_A}(v)$ for all $v \in [a, b]$, namely the shaded area in Figures 4(a) and 4(b). Since the manipulation randomizes assignment for types $[a, b]$, the cartel payoff $U^{\tilde{M}_A}(v)$ is linear for

[20] The explanation here also provides some intuition for the sufficiency of (CP) (together with the other conditions). The proof, however, requires a different argument since the auction must be resistant to all manipulations, not just the one considered here

the affected types, so the constraint is most binding for the highest affected type $v = b$, as can be seen clearly in the figures. By (Env), we can rewrite the cartel payoff for type v:

$$U^{\tilde{M}_A}(v) = U^{\tilde{M}_A}(a) + \int_a^v \tilde{Q}(s)ds = U^{M_A}(a) + \int_a^v \bar{p}ds$$
$$= U^{M_A}(a) + (v-a)\frac{\int_a^b Q_i(s)f_i(s)ds}{F_i(b) - F_i(a)}. \tag{7}$$

Combining (6) and (7), the crucial question is whether

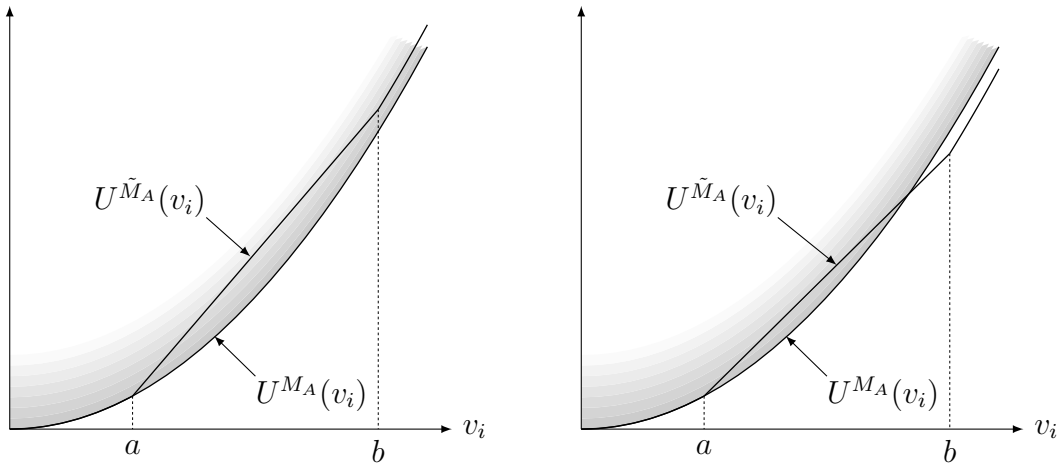
$$\int_a^b Q_i(s)ds \leq (b-a)\frac{\int_a^b Q_i(s)f_i(s)ds}{F_i(b) - F_i(a)},$$

namely, whether the conditional winning probability "contributed" by the affected types (the right hand side) is large enough to match the noncooperative rent (the left hand side).

This last question depends on the shape of the density function f. If f is (conditionally) uniform on $[a, b]$, then the two sides are equal, satisfying (C-IR) for all affected types. If f is increasing in $[a, b]$, this means that high types are relatively more abundant within $[a, b]$ than in the uniform case. Since higher types enjoy higher winning probability under M_A, this means that the winning probability contributed is larger. Hence, (C-IR) holds strictly for all types $[a, b]$, as depicted in Figure 4(a). Indeed, this is precisely when the concave closure G is linear, and the types $[a, b]$ become susceptible. In this case, a cartel succeeds. By the same token, if f is declining (i.e., the concave closure G is strictly concave), the exact opposite is true, and (C-IR) fails for types close to $v = b$, as depicted in Figure 4(b). In this case, a cartel fails to form.

Even when a manipulation $\tilde{M}_A$ is profitable (as in Figure 4(a)), implementing it can still be challenging for a weak cartel. In fact, pooling the types of bidder i in $[a, b]$ requires shifting the winning probability away from high value types toward low value types of bidder i, and it is not clear whether and how such a shifting of the winning probabilities can be made incentive compatible without using transfers. Without transfers, a cartel must coordinate its members' bids in the right way to replicate the exact interim transfers necessary to make $\tilde{M}_A$ incentive compatible. Winner-payability of an auction plays a role here: it allows the cartel to find, for each profile of reported values, a distribution of bids that implements the ex post allocation and transfers needed for the proposed manipulation. [21]

[21] To see this point observe that the cartel is able to assign an arbitrary winning probability to each bidder i by combining the non-participation option with bids $\underline{b}^i$ and $\overline{b}^i$. Moreover, whenever i is assigned the object, the cartel can induce any payment which is a convex combination of r_i (the minimum per-unit price that i could ever pay at the auction) and R_i (the maximum price that i could pay) by using $\underline{b}^i$ and $\overline{b}^i$. Winner-payability is thus sufficient for the cartel to attain any incentive compatible allocation for values



(a) G_i linear on $[a, b]$

(b) G_i strictly concave on $[a, b]$

Figure 4: Profitability of Weak-Cartel Manipulation

Theorem 1 suggests that a winner-payable auction which assigns the object with higher probability to bidders with higher values is vulnerable to weak cartels unless each bidder's value distribution is strictly concave for all types that obtain the good with positive probability. The next three corollaries state (under certain technical qualifications) that (i) standard auctions, (ii) revenue maximizing auctions (i.e. those which implement Myerson's optimal auction), and (iii) efficient auctions are all susceptible to weak cartels unless all distributions of values are strictly concave.

Corollary 1. *Letting $\overline{v} := \min_{i \in N} \overline{v}_i$ and $\underline{v} := \max_{i \in N} \underline{v}_i$, assume that $\overline{v} > \underline{v}$. Then, the collusion-free equilibrium outcomes (in weakly undominated strategies) of first-price, second-price, English, or Dutch auctions, with a reserve price $r < \overline{v}$, are not WCP if G_i is linear in some interval $(a, b) \subset \mathcal{V}_i$ with $b > r$ and $a \geq \underline{v}$ for some bidder i.*

Proof. See Appendix A (page 40). ▮

Corollary 2. *Suppose that the virtual valuation, $J_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, is strictly increasing in v_i for all $i \in N$. Suppose also that G_i is linear in some interval $(a, b) \subset (r_i, \overline{v}_i]$, $J_i(b) > 0$, and $\max_{j \neq i} J_j(\underline{v}_j) < J_i(b) < \max_{j \neq i} J_j(\overline{v}_j)$, for some bidder i. Then, all equilibrium outcomes of auctions in $\mathcal{A}^*$ that maximize the seller's revenue are not WCP.*

Proof. The hypotheses guarantee that there exists an interval $[b - \epsilon, b]$ with $\epsilon > 0$, where $Q_i(v_i)$ is strictly increasing in the optimal auction. The result follows from Theorem 1. ▮

above reserve prices. Therefore, enlarging the set of auctions beyond $\mathcal{A}^*$ would not make collusion any easier for the cartel.

Corollary 3. *Suppose that G_i is linear in some interval $(a, b) \in (r_i, \overline{v}_i]$ and $\max_{j \neq i} \underline{v}_j < b < \max_{j \neq i} \overline{v}_j$, for some bidder i. Then, all efficient equilibrium outcomes of auctions in $\mathcal{A}^*$ are not WCP.*

Proof. The hypotheses guarantee that there exists an interval $[b - \epsilon, b]$ with $\epsilon > 0$, where $Q_i(v_i)$ is strictly increasing in any efficient auction. The result follows from Theorem 1. ▮

On the flip side, we can identify conditions under which the auctions discussed in the previous corollaries are WCP.

Corollary 4. *If F_i is strictly concave for all $i \in N$, then the following are WCP: (i) the collusion-free equilibria of first-price, second-price, English, or Dutch auctions, with or without reserve price (ii) any equilibrium of any auction that results in an efficient allocation, and (iii) any equilibrium of any auction that maximizes the seller's revenue.*

Proof. The proof is immediate given Theorem 1 and the fact that there is no interval in $\mathcal{V}_i$ for any $i \in N$ where G_i is linear. ▮

4.2 Optimal Cartel Behavior

Understanding how a cartel operates is important for evaluating its outcome and for detecting its presence. As it turns out, the preceding characterization helps us to understand how a cartel operates, which in turn provides some clue on how one may empirically detect a cartel.

When a weak cartel is formed, it is natural to expect that it will seek to avoid outcomes that are suboptimal in a Pareto sense. Therefore, our positive analysis focuses on cartel behavior that is interim Pareto efficient, defined formally as follows.

Definition 5. *Suppose an equilibrium outcome of M_A auction A is not WCP. A profitable cartel manipulation $\tilde{M}_A$ of A is* ***cartel-optimal*** *if it is not interim Pareto dominated by another cartel manipulation: i.e., there does not exist another cartel manipulation $\tilde{M}'_A$ such that*

$$U_i^{\tilde{M}'_A}(v_i) \geq U_i^{\tilde{M}_A}(v_i), \forall v_i, i, \text{ with strict inequality for some } v_i, i. \tag{8}$$

The next result, which is analogous to the collusion-proofness principle of LM, follows from observing that if the cartel manipulation $\tilde{M}_A$ is cartel optimal, then any auction that induces $\tilde{M}_A$ as equilibrium behavior is also WCP according to Definition 3.

Theorem 2. *Suppose auction $A \in \mathcal{A}^*$ is not weakly collusion-proof. Then a profitable cartel manipulation $\tilde{M}_A$ of A is cartel-optimal if and only if it satisfies* (PS) *and* (NW).

Proof. To prove the sufficiency, suppose by way of contradiction that a profitable cartel manipulation $\tilde{M}_A$ satisfying (PS) and (NW) is not optimal. Then, there exists a profitable cartel manipulation $\tilde{M}'_A$ of A that interim Pareto dominates $\tilde{M}_A$ in the sense defined by (8). Since $\tilde{M}'_A$ is a cartel manipulation of A with respect to M_A, and since $\tilde{M}_A$ is also a cartel manipulation of A, it follows that $\tilde{M}'_A$ must be a cartel manipulation in turn of $\tilde{M}_A$ in the sense of satisfying (4) with respect to the bidding behavior specified by $\tilde{M}_A$. Since $\tilde{M}_A$ satisfies (PS) and (NW), Theorem 1 suggests that $\tilde{M}_A$ is weak collusion proof, or does not admit a further profitable manipulation, which implies that $\tilde{M}'_A$ cannot interim Pareto dominates M'_A in the sense of (8). The necessity follows from the same argument applied in the reverse order. If cartel manipulation $\tilde{M}_A$ does not satisfies (PS) and (NW), it in turn admits a profitable cartel manipulation $\tilde{M}'_A$. Since the latter is in turn a cartel manipulation of the original auction A and interim Pareto dominates $\tilde{M}_A$, the latter cannot be cartel-optimal. ▮

Combining Theorem 1 and Theorem 2 provides a powerful prediction of the allocation that would emerge in the presence of a cartel.

Corollary 5. *In the presence of collusion that implements a cartel-optimal manipulation, the equilibrium outcome of any auction $A \in \mathcal{A}^*$ must satisfy* (PS) *and* (NW).

The predicted collusive behavior is much richer and more nuanced than the simple random allocation at the reserve price predicted by MM. In our model, for any specific auction that is not collusion-proof, there is typically a range of optimal collusive behaviors that may differ in terms of revenue and efficiency.

To see this, consider an example where there are two bidders, each with value drawn from $[0, 1]$ according to the triangular density: $f(v) = 4v$ for $v \leq 1/2$ and $f(v) = 4(1-v)$ for $v > 1/2$. Suppose the seller naively employs a standard auction with a reserve price 0.4—an optimal level assuming no collusion. Given this auction, there exists a family of cartel-optimal manipulations, indexed by $\tilde{v} \in [0.541, 0.6]$, whereby types in $[r, \tilde{v}]$ pool (i.e., submit the same bid), and types $[\tilde{v}, \overline{v}]$ bid competitively. Interestingly, optimal cartel manipulations in this family can be unambiguously ranked in terms of efficiency and revenue.[22] In particular, the behavior with the most pooling, with $\tilde{v} = 0.6$, generates the lowest efficiency and revenue for the seller, but yields the largest ex ante rents for the cartel.

Figure 5 compares the distribution of winning bids under a collusion-free outcome and the cartel manipulation with most pooling. We have randomly drawn 10,000 pairs of valuations

[22] All cartel manipulations in the family are interim Pareto undominated, which is why they all constitute optimal cartel behaviors.

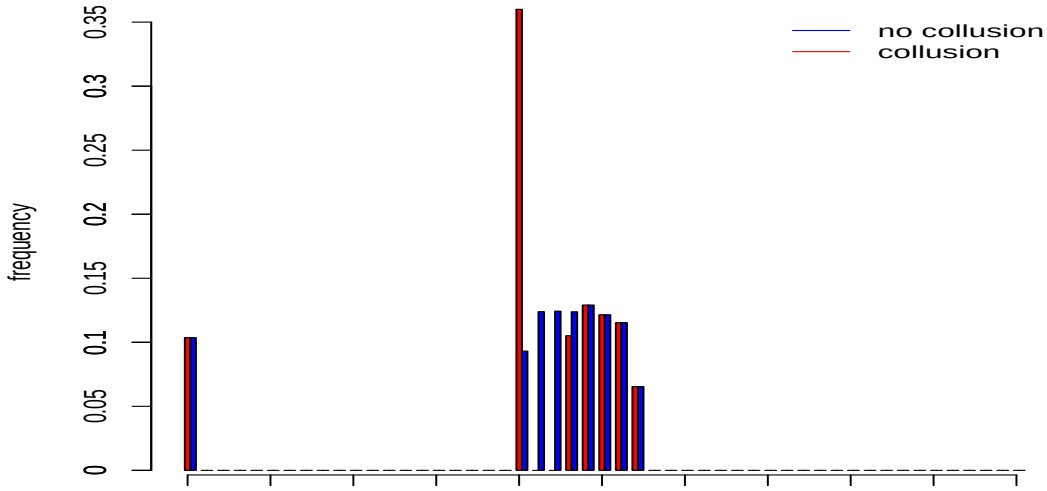


Figure 5: Distribution of winning bids under first-price auction with or without collusion

according to the above density function, and analyzed behavior under a first-price auction.[23] As seen in the figure, collusive bids are *double-peaked* despite the single-peaked density: bidder types pooling at the reserve price results in a spike of bids at the reserve price and absence of bids just above it; other types bid competitively and generate a higher peak of bids around 0.5. This example illustrates how one can potentially use our theory to empirically detect a cartel; but it also suggests how failing to control for possible collusion may lead to biased estimates of bidders' valuations.

If the seller herself implements an auction which induces minimal pooling — with $\tilde{v} = 0.541$, the resulting auction will satisfy (PS) and (NW) and it will not be any further susceptible to a cartel. The resulting outcome results in higher efficiency and revenue, due to reduced pooling. This point suggests a sense in which the seller could benefit strictly from intervention, beyond her choice of the reserve price. This motivates our next section.

5 Optimal Collusion-Proof Auctions

In this section, we look for an auction that maximizes the seller's revenue among all winner-payable WCP auctions. This exercise is not trivial; as Corollary 2 shows, in a wide range of circumstances the auction that maximizes the seller's revenue in the absence of collusion will not be collusion-proof.

To begin, observe first that Corollary 5 and Theorem 1 imply that there is no loss of

[23] The 0 bid in x-axis of Figure 5 represents non-participation.

generality in restricting attention to WCP auctions where bidders do not wish to manipulate the outcome. Next, write $J_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ for the virtual valuation of bidder i with value v_i, and henceforth assume, as standard, that it is strictly increasing. Theorem 1 allow us to represent a winner payable WCP auction by a direct mechanism that satisfies (PS) and (NW). Specifically, the seller's problem becomes

$$[P] \qquad \max_{(q_i,t_i)_{i\in N}} \sum_{i\in N} \int_{r_i}^{\overline{v}_i} J_i(v_i)Q_i(v_i)dF_i(v_i)$$

subject to (M), (Env), (PS), and (NW), given a reserve price $r_i = \inf\{\frac{t_i(v)}{q_i(v)} \mid q_i(v) > 0\}$ for each $i \in N$. The objective function represents the seller's expected revenue.[24]

Our main result identifies the optimal weakly collusion-proof auction up to the choice of the reserve prices $(r_1, \ldots, r_n)$. To state our result, we need to introduce some further notation. Recall first that $\mathcal{V}_i^0(r_i) \subset [r_i, \overline{v}_i]$ denotes the set of susceptible types. Note that $\mathcal{V}_i^0(r_i) \subset [r_i, \overline{v}_i]$ is a disjoint union of countably many intervals $I_i^k = [a_i^k, b_i^k], k \in K_i \subset \mathbb{N}$, for which G_i is linear. Then, for every bidder i, we define the "ironed" virtual valuation as follow:

$$\bar{J}_i(v_i) := \begin{cases} J_i(v_i) & \text{if } v_i \in \mathcal{V}_i \backslash \mathcal{V}_i^0(r_i) \\ \frac{\int_{a_i^k}^{b_i^k} J_i(s)dF_i(s)}{F_i(b_i^k)-F_i(a_i^k)} & \text{if } v_i \in I_i^k \text{ for some } k \in K_i. \end{cases} \tag{9}$$

The ironed virtual valuation is constant within each interval I_i^k for which (PS) requires bidder i to receive the object with a constant probability. For any value in I_i^k, it coincides with the conditional expected value of the virtual valuation in that interval.[25]

The following result shows that the optimal allocation rule is the one that, under optimal reserve prices, always assigns the object to the bidder with the highest ironed virtual value. As standard, the payoff equivalence allows us to focus on the allocation rule only.

THEOREM 3. *For any $v \in \mathcal{V}$, let $W(v) := \{j \in N \mid \bar{J}_j(v_j) = \max_{k\in N} \bar{J}_k(v_k)\}$ and let $\#W(v)$ denote the cardinality of this set. Then, there is an auction rule $(q_i^*, t_i^*)_{i\in N}$ that solves $[P]$*

[24] It is well known that the expression is obtained by substituting for the payments T_i into the original objective function using the condition (Env) and noting that (IR′) must be binding at the optimum for the lowest types, i.e., for all $i \in N$, $T_i(\underline{v}_i) = \underline{v}_i Q_i(\underline{v}_i)$. In the above expression we also used the facts that $Q_i(v_i) = 0$ for all $v_i < r_i$.

[25] The idea of ironing is in the spirit of Myerson (1981). In our case, ironing is needed to deal with the collusion-proofness constraint even though the virtual valuation is increasing; in Myerson (1981), ironing is required to satisfy the the monotonicity constraint that becomes binding in regions where the virtual value is decreasing.

such that $r_i \geq J_i^{-1}(0) := \inf\{v \geq \underline{v}_i : J_i(v) \geq 0\}, \forall i \in N$, *and where*

$$q_i^*(v) = \begin{cases} 0 & \text{if } v_i < r_i \text{ or } \bar{J}_i(v_i) < \max_{j \in N} \bar{J}_j(v_j) \\ \frac{1}{\#W(v)} & \text{if } v_i \geq r_i \text{ and } i \in W(v). \end{cases} \tag{10}$$

Proof. See Appendix B (page 40). ▮

The theorem characterizes the optimal WCP auction up to the choice of reserve prices. Therefore, once the allocation is chosen according to Theorem 3 for each $(r_1, \ldots, r_n)$, the revenue maximizing WCP auction is obtained by choosing $(r_1, \ldots, r_n)$ to maximize the resulting objective function in $[P]$. Note that $r_i \geq J_i^{-1}(0)$ for all $i \in N$ follows immediately as inspection of $[P]$ reveals that it is never optimal for the seller to sell to bidders with negative virtual valuations.

To understand this result, observe that the optimal auction under the threat of collusion is constructed exactly as the optimal auction in Myerson (1981), except that the seller is forced to pool together susceptible types so as to satisfy (PS). The pooling of susceptible types means that their average virtual value becomes the relevant criterion for these types in the revenue maximization.

We can obtain a more complete characterization of the optimal auction by focusing on two special cases (a) nondecreasing density (b) symmetric auction with single-peaked density.[26]

5.1 Monotone Nondecreasing Densities

If all bidders have nondecreasing densities, then for any $r_i \in [\underline{v}_i, \overline{v}_i]$, the function G_i will be linear in $[r_i, \overline{v}_i]$. Hence, bidder i must expect a constant probability of obtaining the object for all his values above r_i. This implies that the seller's problem takes on a much simpler form.

Corollary 6. *Suppose that all f_i's are nondecreasing. Then, the program $[P]$ simplifies to*

$$\max_{(r_i)_{i \in N}} \Big[\sum_{i \in N} \Big(\prod_{j : \pi(j) < \pi(i)} F_j(r_j) \Big) (1 - F_i(r_i))\, r_i \Big], \tag{11}$$

where $\pi : N \to N$ is any permutation function that satisfies $\pi(j) < \pi(i)$ if $r_j > r_i$.

Proof. See Appendix B (page 43). ▮

[26] We have already argued that if all bidders have monotone *decreasing* density, then the Myerson's revenue maximizing auction is WCP (see Corollary 4).

Interestingly, Corollary 6 suggests that the optimal WCP auction can be implemented via a sequential negotiation process. Bidders are ordered from first to last and the seller approaches them in sequence and makes them take-it-or-leave-it offers. If bidder i refuses the offer, the seller proceeds to make an offer to the next bidder; and the process continues until either some bidder j accepts an offer and pays r_j to the seller, or the object remains unsold. Not surprisingly, the seller's optimal offer falls with each rejected offer and the last bidder in the sequence, say i, must receive an offer at price $J_i^{-1}(0)$.[27]

This result suggests that the seller can compute optimal reserve prices, recursively, for all possible orders of bidders and then select the order that is optimal. The order becomes irrelevant when bidders are ex-ante symmetric, and the problem is further simplified in this case as illustrated by the next corollary, which is stated without proof.

Corollary 7. *Suppose that $f_i = f$ for all $i \in N$ and f is nondecreasing. Then, it is an optimal WCP auction to approach all bidders in sequence (i.e., in any arbitrary sequence) and offer to the k-th buyer a price r_k that maximizes*

$$r_k(1 - F(r_k)) + F(r_k)V_{n-k},$$

where V_{n-k} is the revenue the seller gets from an subproblem dealing only with $n-k$ bidders (and $V_0 = 0$).

One insight that emerges from this corollary is that, contrary to Myerson (1981), reserve prices in an optimal WCP auction may be different even when bidders are ex-ante symmetric. To see this point, consider the recursive nature of the problem. It is straightforward to see that $0 = V_0 < V_1 < \cdots < V_{n-1}$, which implies that $r_n = J_n^{-1}(0) < r_{n-1} < \cdots < r_1$. Therefore, the optimal reserve price charged to bidder i will be different from the one charged to bidder j, for any $i, j \in N$. For instance, suppose there are three bidders, 1,2 and 3, with valuations drawn uniformly from $[0, 1]$. Then, the optimal policy for the seller is to make take-it-or-leave-it offers of $r_1 = 89/128, r_2 = 5/8, r_3 = 1/2$, sequentially to the three bidders. This yields revenue of 0.4835, 10.5% higher than the revenue 0.4375 that the seller obtains if she selects a revenue maximizing auction and the cartel optimally responds by pooling all bidder types above the reserve price.[28]

The optimality of treating ex ante identical bidders asymmetrically extends beyond this case. Because virtual valuations are strictly increasing, optimal price discrimination calls for

[27] This is the optimal take-it-or-leave-it offer for a single bidder, which also corresponds to i's reserve price in the Myerson's optimal auction without collusion.

[28] The optimal revenue is also 2.3 % higher than the revenue of 0.4725 that would have obtained if the seller charged the optimal posted price $r = 1/\sqrt[3]{4}$ which is what MM prescribed for the seller facing a cartel that forms at the ex-ante stage.

assigning the object to bidders with the highest values. However, the collusion-proof constraint makes this allocation infeasible. The asymmetric allocation, implicit in the sequential negotiation, accomplishes partial price discrimination without violating the collusion-proof constraint.

5.2 Single-Peaked Density and Symmetric Auctions

Suppose now that bidders are ex-ante symmetric and that the (common) virtual valuation J is strictly increasing. In addition, assume that the (common) density f is (weakly) increasing in $[\underline{v}, \hat{v}]$ and strictly decreasing $[\hat{v}, \overline{v}]$ around a peak $\hat{v} \in [\underline{v}, \overline{v}]$. Let $\hat{v} > r^M := J^{-1}(0)$ to avoid the trivial result in which the Myerson's optimal auction is WCP. Observe that for any $r_i \geq \hat{v}$, there exists $v^*(r_i) \leq \hat{v}$ such that $\mathcal{V}_i^0(r_i) = [r_i, v^*(r_i)]$ (recall Figure 3), and also that $v^*(r_i)$ is decreasing in r_i and satisfies $v^*(\hat{v}) = \hat{v}$.[29]

The single-peaked density case allows us to illustrate how the choice of the reserve prices interacts with the (endogenous) level of ironing at the optimal mechanism. Moreover, the case of single-peaked density covers a general class of many plausible and well-known distributions, including Uniform, Triangular, Cauchy, Exponential, Logistic, Normal, and Weibull.

While a general optimal mechanism (which by Theorem 3 must be asymmetric) is of interest, it is also useful to consider an optimal auction among collusion-proof winner-payable auctions that treat bidders in a non-discriminatory way.[30] Nondiscriminatory, or symmetric, auctions are of practical interest since sellers, particularly government agencies, are often compelled to treat bidders identically. Theorem 3 characterizes the optimal WCP auction even in this case. Formally, in addition to the assumptions that bidders are ex-ante symmetric and that the density is single-peaked, we impose a *symmetry* requirement that $Q_i = Q$ for all $i \in N$.[31]

Under the stated assumptions, for any reserve price $r \leq \hat{v}$ (which must be the same for all bidders), there is a value $v^*(r) \in [\hat{v}, \overline{v}]$ such that the concave closure of F on the interval $[r, \overline{v}]$ is linear in $[r, v^*(r)]$ and strictly concave in $[v^*(r), \overline{v}]$.

[29] The proof of these statements is straightforward and thus omitted.

[30] The solution to the asymmetric optimal WCP auction problem for the single-peaked density case is omitted to save space but is contained in a working-paper version which is available upon request.

[31] Deb and Pai (2013) study auctions where the allocation and payment rule cannot depend on the identity of bidders and show that almost any interim allocation can be implemented using anonymous auctions. In contrast, we require the expected final outcome to be nondiscriminatory for ex-ante identical bidders.

Corollary 8. *The allocation that solves* $[P]$ *under the additional symmetry restriction is:*

$$q_i^*(v) = \begin{cases} \frac{1}{\#\{j\in N \mid v_j = \max_{k\in N} v_k\}} & \text{if } v_i = \max_{j\in N} v_j > v^*(r) \\ \frac{1}{\#\{j\in N \mid v_j \in [r, v^*(r)]\}} & \text{if } v_i \in [r, v^*(r)] \text{ and } \max_{j\in N} v_j \leq v^*(r) \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where r *is a value in* $(r^M, \hat{v}]$.

Proof. See Appendix B (page 43). ▮

Again, the characterization here is up to the choice of the reserve price. The auction allocates the object efficiently when bidders have high valuations, but allocates it randomly at a fixed price when bidders have low values. The optimal reserve price exceeds r^M since the region of efficient allocation $[v^*(r), \overline{v}]$ expands as r rises (and the seller benefits from this expansion). Formally, suppose the reserve price is raised from $r = r^M$ to $r^M + \varepsilon$. This entails only a second-order loss from withheld sale to the types in $[r^M, r^M + \varepsilon]$ since in that region virtual values are close to zero, but it results in the object being allocated efficiently among types $[v^*(r^M + \varepsilon), v^*(r^M)]$, which generates a first-order gain.

The optimal symmetric WCP auction given by (12) can be implemented by the following simple mechanism: First, hold either a first-price or second-price auction with a minimum bid m that satisfies $[v^*(r) - r]\,\overline{Q} = [v^*(r) - m]\,F(v^*(r))^{n-1}$, where $\overline{Q} := \frac{F(v^*)^n - F(r)^n}{n(F(v^*) - F(r))}$ denotes the constant winning probability for type $v \in [r, v^*(r)]$.[32] If the object is not sold in the auction, then the object is offered for sale at price r, with ties broken by a fair lottery (in case there are multiple buyers at that price).[33] As noted in the Introduction, this mechanism resembles the Italian procurement system which allows a procurer to negotiate with suppliers after an initial auction fails to attract a successful bid.

6 Related Literature

Seminal contributions to the literature on collusion in auctions include Robinson (1985), Graham and Marshall (1987), von Ungern-Stenberg (1988), Mailath and Zemsky (1991), and MM. Like us, they analyze cartel profitability at a single-unit auction and abstract from

[32] One can think of m as a reserve price in the conventional sense; we do not use the term to avoid confusion with the way we defined the term.

[33] Observe that the type $v^*(r)$ is indifferent between obtaining the object in the auction at the minimum price m and obtaining it in the posted-price sale at price r. Our characterization implies that in the case of the previous subsection, where all densities are increasing, an optimal symmetric auction would consist of posting a single price.

the enforcement issue—how members of a cartel may sustain collusion without a legally binding contract.[34] Unlike us, most of these authors focus on strong cartels and/or specific auction formats.

MM does consider weak cartels and show that they involve random allocation of the object for sale, much consistent with often observed practice of bid rotation.[35] Our approach is differentiated by its explicit consideration of the bidders' interim incentive to participate in the cartel. Besides, our model is more general than MM in several respects. First, we consider a more general class of auctions called "winner-payable auctions." Considering such a general class of auctions helps to isolate the features of auctions that make them vulnerable to cartels. Second, we relax the monotone hazard rate and symmetry assumptions.

Several authors study tacit collusion through repeated interaction (see Aoyagi (2003), Athey et al. (2004), Blume and Heidhues (2004), and Skrzypacz and Hopenhayn (2004)) or via implicit collusive strategies (see Engelbrecht-Wiggans and Kahn (2005), Brusco and Lopomo (2002), Marshall and Marx (2007, 2009), Garratt et al. (2009)). If types are distributed independently over time, repeated interaction enables members of a weak cartel to use their future market shares in a way similar to monetary transfers. If the types are persistent over time, as we envision to be more realistic, however, tampering with future market shares involves severe efficiency loss (see Athey and Bagwell (2008)). Indeed, assuming that bidders can commit to intertemporal collusive scheme and rely on explicit communication, our analysis remains valid for a sequence of second-price (or any ex-post implementable) auctions if types are fully persistent and the target equilibrium is a repetition of the stage game equilibrium.[36]

The current paper is also related to the literature that studies collusion-proof mechanism

[34]The likely scenario of enforcement involves the threat of retaliation through future interaction, multi-market contact, or organized crime.

[35]See also Condorelli (2012). This paper analyzes the optimal allocation of a single object to a number of agents when payments made to the designer are socially wasteful and cannot be redistributed. The problem addressed is analogous to that of a cartel-mediator designing an ex-ante optimal weak cartel agreement at a standard auction with no reserve price.

[36]This result follows from a couple of observations. First, with persistent types, the bidders' payoffs from any intertemporal collusive scheme can also be implemented by repeating some static collusive scheme in every period. Second, a second-price auction guarantees that the stage outcome of non-cooperative, truthful, bidding is constant across periods, which means that the non-collusive payoff in the repeated auctions is equal to that in the static one-shot auction (up to appropriate discounting). As a result, the comparison between collusive and noncollusive payoffs in the repeated auctions is no different from that in the static one-shot auction. Refer to Section V of the Supplementary Material for details. We expect this result does not hold for the first-price auction in which, as the auction is repeated, bidders update their beliefs and adjust their bidding behavior accordingly, which means the second observation above would fail.

design. This literature, pioneered by Laffont and Martimort (1997, 2000) and further generalized by Che and Kim (2006, 2009), models cartel as designing an optimal mechanism for its members (given the underlying auction mechanism they face), assuming that the members have necessary wherewithal to enforce whatever agreement they make. Similar to Laffont and Martimort (1997, 2000) and Che and Kim (2006), we explicitly consider the bidders' incentives to participate in the cartel. Unlike the current paper, though, their models allow a cartel to be formed only after bidders decide, noncooperatively, to enter the auction. This modeling assumption, while realistic in some internal organization setting, is not applicable to auction environments where the collusion often centers around withdrawing participation.

Che and Kim (2009) and Pavlov (2008) do consider collusion on participation. And they show that the second-best outcome (i.e., the Myerson (1981) benchmark) can be achieved even in the presence of a strong cartel, as long as the second-best outcome involves a sufficient amount of exclusion of bidders. However, the auction that accomplishes this requires losing bidders to pay, violating the ex-post individual rationality. Such auctions, while theoretically interesting, are never observed in practice. By contrast, the current paper has considered a more realistic, still broad, class of auctions rules.

7 Conclusion

The current paper analyzes weak cartels in auctions. Unlike the seminal work by McAfee and McMillan (1992), we explicitly consider the interim incentives of bidders to participate in a cartel. This perspective leads to a different characterization of when auctions are susceptible to a weak cartel, how a cartel would operate if it is active, and how the auctioneer should respond to a weak cartel.

We show that a large class of auction rules, called winner-payable, are susceptible to a weak cartel if and only if they seek to implement non-constant allocation for "susceptible" types, roughly those for which the distribution is locally nonconcave. This characterization stands in sharp contrast to the existing theory of MM. While the latter suggests that a (first-price sealed-bid) auction is susceptible to a cartel whenever a bidder's value distribution satisfies a nondecreasing hazard rate, a condition satisfied by most standard distribution functions, the condition we identify for vulnerability to a cartel is much stronger, and not necessarily satisfied by all standard distribution functions. In particular, standard auctions as well as classical revenue-maximizing auctions are never susceptible to weak cartels if bidders' distribution functions are strictly concave. Furthermore, we use our characterization to identify optimal weakly collusion-proof auctions. In case the bidders' distribution is single-

peaked, the optimal mechanism can be implemented by a competitive auction at a high reserve price followed by a sequential negotiation in case of the auction producing no sale.

Our results rest on the model of collusion in which bidders coordinate their actions to interim Pareto dominate a noncooperative outcome. While this model is a reasonable approximation of cartel behavior in many settings, questions arise as to whether our results are robust to the specific ways in which a cartel is formed (e.g., whether a cartel is all-inclusive or not, who proposes an agreement) and how it operates (i.e., whether members can commit to punish a defector or what beliefs they form in case a cartel collapses). These questions will constitute a worthy subject of future research.

A Proof for Section 4

Proof of Theorem 1:

Proof of Necessity: We provide separate proofs for the necessity of (PS) and (NW) while proving the former first. The proof of both results employs the following result (see Mierendorff (2011) or Che et al. (2013)):

LEMMA 2. *For any interim rule $(Q_i)_{i\in N}$, there exists an ex-post allocation rule that has Q as an interim allocation rule if and only if*

$$g(v) := 1 - \prod_{i\in N} F_i(v_i) - \sum_{i\in N} \int_{v_i}^{\overline{v}_i} Q_i(s)dF_i(s) \geq 0, \forall v = (v_i)_{i\in N} \in \mathcal{V}. \tag{B}$$

Necessity of (PS): Fix an equilibrium outcome $M_A = (q,t)$ of an auction $A \in \mathcal{A}^*$ and let (Q,T) denote its interim outcome. Suppose for a contradiction that M_A is WCP but Q_k is not constant in some interval $(a',b') \subset (r_k, \overline{v}_k]$ for some $k \in N$, where G_k is linear. Let (a,b) be the maximal (connected) interval in $[r_k, \overline{v}_k]$ containing (a',b') on which G_k is linear. Note that $F_k(s) = G_k(s)$ at $s = a, b$.

Let us define $\tilde{Q} = (\tilde{Q}_1, \cdots, \tilde{Q}_n)$ as follows:

$$\tilde{Q}_i(v_i) = \begin{cases} \bar{p} & \text{if } i = k \text{ and } v_i \in (a,b) \\ Q_i(v_i) & \text{otherwise} \end{cases}, \tag{13}$$

where $\bar{p}$ is defined to satisfy

$$\bar{p}(F_k(b) - F_k(a)) = \int_a^b Q_k(s)dF_k(s). \tag{14}$$

Observe first that $\tilde{Q}$ satisfies (M). For this, we only need to check that $Q_k(a) \leq \bar{p} = \frac{\int_a^b Q_k(s)dF_k(s)}{F_k(b)-F_k(a)} \leq Q_k(b)$, which clearly holds since Q_k is nondecreasing.

CLAIM 1. *The interim allocation rule $\tilde{Q}$ satisfies (B) and thus admits an ex-post allocation rule.*

Proof. Since Q satisfies (B), it suffices to show that for all $v = (v_1, \cdots, v_n) \in \mathcal{V}$,

$$\sum_{i\in N} \int_{v_i}^{\overline{v}_i} \tilde{Q}_i(s)dF_i(s) \leq \sum_{i\in N} \int_{v_i}^{\overline{v}_i} Q_i(s)dF_i(s),$$

which, given (13), will hold if for all $v_k \in [\underline{v}_k, \overline{v}_k]$,

$$\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s)dF_k(s) \leq \int_{v_k}^{\overline{v}_k} Q_k(s)dF_k(s). \tag{15}$$

Note that (15) clearly holds for $v_k \geq b$ since $\tilde{Q}_k(s) = Q_k(s), \forall s \in [b, \overline{v}_k]$. Let us pick $v_k \in [a, b)$ and then we obtain as desired

$$\begin{aligned}
\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s) dF_k(s) &= \int_{v_k}^{b} \bar{p} dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s) dF_k(s) \\
&= \left[\frac{F_k(b) - F_k(v_k)}{F_k(b) - F_k(a)}\right] \int_{a}^{b} Q_k(s) dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s) dF_k(s) \\
&\leq \int_{v_k}^{b} Q_k(s) dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s) dF_k(s) = \int_{v_k}^{\overline{v}_k} Q_k(s) dF_k(s), \qquad (16)
\end{aligned}$$

where the second equality follows from the definition of $\bar{p}$, and the inequality from the fact that $Q_k(\cdot)$ is nondecreasing and thus

$$\int_{a}^{b} \frac{Q_k(s)}{F_k(b) - F_k(a)} dF_k(s) \leq \int_{v_k}^{b} \frac{Q_k(s)}{F_k(b) - F_k(v_k)} dF_k(s).$$

Also, for $v_k < a$, we have

$$\begin{aligned}
\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s) dF_k(s) &= \int_{v_k}^{a} Q_k(s) dF_k(s) + \int_{a}^{\overline{v}_k} \tilde{Q}_k(s) dF_k(s) \\
&\leq \int_{v_k}^{a} Q_k(s) dF_k(s) + \int_{a}^{\overline{v}_k} Q_k(s) dF_k(s) = \int_{v_k}^{\overline{v}_k} Q_k(s) dF_k(s),
\end{aligned}$$

where the inequality follows from (16). Thus, we can invoke Lemma 2 to conclude that $\tilde{Q}$ admits an ex-post allocation rule. ▌

Let $\tilde{q}$ denote an ex post allocation rule that has $\tilde{Q}$ as the interim allocation rule. Next, we use the interim allocation $\tilde{Q}$ and (Env) to construct an interim payment rule $\tilde{T}$ satisfying $\tilde{T}_i(r_i) = T_i(r_i), \forall i \in N$. Given this, we construct an (ex post) payment rule $\tilde{t}$ defined by

$$\tilde{t}_i(v) = \tilde{q}_i(v) \frac{\tilde{T}_i(v_i)}{\tilde{Q}_i(v_i)}. \qquad (17)$$

Clearly, $E_{v_{-i}}[\tilde{t}_i(v)] = \tilde{T}_i(v_i)$. The direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ thus has $(\tilde{Q}, \tilde{T})$ as the interim rule. By construction, $\tilde{M}_A$ satisfies (IC). We next show that it satisfies (C-IR).

Claim 2. *$\tilde{M}_A$ satisfies* (C-IR).

Proof. First, it is clear that all bidders other than k will have their payoffs unaffected. Moreover, bidder k's payoff will only be affected when his value is above a. To show that $U_k^{\tilde{M}_A}(v_k) \geq U_k^{M_A}(v_k)$ for all $v_k \in [a, \overline{v}_k]$, with strict inequality for some v_k, it suffices to show that $U_k^{\tilde{M}_A}(b) \geq U_k^{M_A}(b)$, since $U_k^{\tilde{M}_A}$ is linear in $[a, b]$ while $U_k^{M_A}$ is convex but not linear, and since $\tilde{Q}_k(v_k) = Q_k(v_k)$ for all $v_k \in (b, \overline{v}_k]$ so $\tilde{U}_k^{M_A}$ and $U_k^{M_A}$ have the same slope beyond b.

To do so, we let $\hat{\mathcal{V}}_k \subset [r_k, \overline{v}_k]$ denote the (countable) set of points at which Q_k is discontinuous (i.e., jumps up). Given the nondecreasing Q_k and the concave closure G_k of F_k over the interval $[a, b] \in [r_k, \overline{v}_k]$, we obtain

$$\begin{aligned}
&\int_a^b Q_k(s)(f_k(s) - g_k(s))ds \\
=&Q_k(s)(F_k(s) - G_k(s))\Big|_{a+}^{b-} - \sum_{v \in \hat{\mathcal{V}}_k \cap (a,b)} Q_k(s)(F_k(s) - G_k(s))\Big|_{v-}^{v+} - \int_a^b Q_k'(s)(F_k(s) - G_k(s))ds \\
=& \sum_{v \in \hat{\mathcal{V}}_k \cap (a,b)} (Q_k(v+) - Q_k(v-))(G_k(v) - F_k(v)) + \int_a^b Q_k'(s)(G_k(s) - F_k(s))ds \geq 0,
\end{aligned}$$

where $v-$ and $v+$ denote the left and right limit, respectively. Here the second equality follows from the fact that $F_k(s) = G_k(s)$ at $s = a, b$ and F_k and G_k are continuous, while the inequality from the fact that Q_k is nondecreasing and $G_k(s) \geq F_k(s), \forall s$. By the above inequality and the fact that g_k is constant over the interval $[a, b]$, we obtain

$$\begin{aligned}
\int_a^b Q_k(s)f_k(s)ds &\geq \int_a^b Q_k(s)g_k(s)ds \\
&= \left(\int_a^b Q_k(s)ds\right)\left(\frac{G_k(b) - G_k(a)}{b - a}\right) = \left(\int_a^b Q_k(s)ds\right)\left(\frac{F_k(b) - F_k(a)}{b - a}\right),
\end{aligned}$$

which yields

$$\bar{p} = \frac{\int_a^b Q_k(s)f_k(s)ds}{F_k(b) - F_k(a)} \geq \frac{\int_a^b Q_k(s)ds}{b - a}.$$

Thus, we obtain

$$U_k^{\tilde{M}_A}(b) - U_k^{\tilde{M}_A}(a) = \bar{p}(b - a) \geq \int_a^b Q_k(s)ds = U_k^{M_A}(b) - U_k^{M_A}(a).$$

or $U_k^{\tilde{M}_A}(b) \geq U_k^{M_A}(b)$ since $U_k^{\tilde{M}_A}(a) = U_k^{M_A}(a)$. ▌

Given Claim 2, the desired contradiction will follow if we show that $\tilde{M}_A$ can be implemented via a weak cartel manipulation. To this end, let $\tilde{B}_i(v_i) := \frac{\tilde{T}_i(v_i)}{\tilde{Q}_i(v_i)}$ if $v_i \in [r_i, \overline{v}_i]$ and $\tilde{B}_i(v_i) := 0$ otherwise.[37] We then exploit the winner-payability property to establish the following result.

Claim 3. *Given the winner-payability of A, for any given $v_i \in [r_i, \overline{v}_i]$, there exists $z_i(v_i) \in [0, 1]$, such that*

$$z_i(v_i)\tau_i(\underline{b}^i) + (1 - z_i(v_i))\tau_i(\overline{b}^i) = \tilde{B}_i(v_i). \tag{18}$$

[37]Note that $r_i = \inf\{v_i \in \mathcal{V}_i \,|\, \tilde{Q}_i(v_i) > 0\} = \inf\{v_i \in \mathcal{V}_i \,|\, Q_i(v_i) > 0\}$.

Proof. First, we show that

$$B_i(r_i) \leq \tilde{B}_i(v_i) \leq B_i(\overline{v}_i), \forall v_i \in [r_i, \overline{v}_i], \forall i, \tag{19}$$

where $B_i(v_i) = \frac{T_i(v_i)}{Q_i(v_i)}$ for $v_i \in [r_i, \overline{v}_i]$. This is immediate if $i \neq k$ or if $i = k$ and $v_k \in [\underline{v}_k, a]$ since in those cases, $B_i(v_i) = \tilde{B}_i(v_i)$ and B_i is nondecreasing.

Consider now $i = k$ and any $v_k \in (a, \overline{v}_k]$. The first inequality of (19) holds trivially. To prove the latter inequality, it suffices to show that $\tilde{B}_i(\overline{v}_i) \leq B_i(\overline{v}_i)$, since $\tilde{B}_i(\cdot)$ is nondecreasing. This inequality holds trivially if $\overline{v}_k = b$ since $B_k(b) \geq B_k(a) = \tilde{B}_k(a) = \tilde{B}_k(b)$. If $\overline{v}_k > b$, then $Q_k(\overline{v}_k) = \tilde{Q}_k(\overline{v}_k)$ and also

$$T(\overline{v}_k) - \tilde{T}(\overline{v}_k) = \overline{v}_k Q_k(\overline{v}_k) - \overline{v}_k \tilde{Q}_k(\overline{v}_k) + U_k^{\tilde{M}_A}(\overline{v}_k) - U_k^{M_A}(\overline{v}_k) = U_k^{\tilde{M}_A}(\overline{v}_k) - U_k^{M_A}(\overline{v}_k) \geq 0.$$

This implies $B_i(\overline{v}_i) \geq \tilde{B}_i(\overline{v}_i)$.

Next, we observe that for any $v_i \in [r_i, \overline{v}_i]$,

$$\inf\Big\{\frac{\tau_i(b)}{\xi_i(b)} \,|\, \xi_i(b) > 0, b \in \mathcal{B}\Big\} \leq B_i(v_i) \leq \sup\Big\{\frac{\tau_i(b)}{\xi_i(b)} \,|\, \xi_i(b) > 0, b \in \mathcal{B} \text{ and } \frac{\tau_i(b)}{\xi_i(b)} \leq \overline{v}_i\Big\}.$$

By definition, $\tau_i(\underline{b}^i)$ and $\tau_i(\overline{b}^i)$ equal respectively the first and the last terms in the above inequalities. Combining this with (19) means that for each $v_i \in [r_i, \overline{v}_i]$, $\tilde{B}_i(v_i) \in [\tau_i(\underline{b}^i), \tau_i(\overline{b}^i)]$, which guarantees the existence of $z_i(v_i)$ as in (18). ▌

It remains to show that $\tilde{M}_A$ is a cartel manipulation. To this end, we construct a cartel agreement α that implements $\tilde{M}_A$ in the sense of (4). Recall that $\alpha(v)(b)$ corresponds to the probability that the cartel submits a bid profile b given the report of type profile v. For each $v \in \mathcal{V}$ and $z_i(v)$ satisfying (19), let

$$\alpha(v)(b) = \begin{cases} \tilde{q}_i(v) z_i(v_i) & \text{for } b = \underline{b}^i \\ \tilde{q}_i(v)(1 - z_i(v_i)) & \text{for } b = \overline{b}^i \\ 1 - \sum_{i \in N} \tilde{q}_i(v) & \text{for } b = b^0. \end{cases} \tag{20}$$

Under this cartel agreement, given profile $v \in \mathcal{V}$ of (reported) values, bidder i obtains the object with probability $\tilde{q}_i(v)$ and pays $\tilde{q}_i(v)\tilde{B}_i(v_i)$ in expectation. Hence, for each $v \in \mathcal{V}$,

$$\tilde{q}_i(v) = \mathbb{E}_{\alpha(v)}[\xi_i(b)] \text{ and } \tilde{t}_i(v) = \tilde{q}_i(v)\tilde{B}_i(v_i) = \mathbb{E}_{\alpha(v)}[\tau_i(b)],$$

as it remained to be shown.

Necessity of (NW): We begin by introducing a few notation. For any $v, v' \in \mathcal{V}$, we denote $v \geq v'$ if $v_i \geq v'_i, \forall i \in N$, and $v > v'$ if $v \geq v'$ and $v \neq v'$. We will also denote

$F_{-i}(v_{-i}) = \prod_{j \neq i} F_j(v_j)$ to simplify notation. For any $v \in \mathcal{V}$, let $\mathcal{V}^i(v_i) = [v_i, \overline{v}_i] \times \mathcal{V}_{-i}$ and $\mathcal{V}(v) = \cup_{i \in N} \mathcal{V}^i(v_i)$ (that is, $\mathcal{V}(v)$ is the set of value profiles $v' \in \mathcal{V}$ such that $v'_i \geq v_i$ for at least one $i \in N$).

We first prove the following lemma:

Lemma 3. *For any $v = (v_i)_{i \in N} \geq r$, $g(v) = 0$ if and only if $\sum_{i: v'_i \geq v_i} q_i(v') = 1$ for almost every $v' \in \mathcal{V}(v)$.*

Proof. Fix any $v \geq r$ and observe

$$\begin{aligned}
\sum_{i \in N} \int_{v_i}^{\overline{v}_i} Q_i(s) dF_i(s) =& \mathbb{E}\left[\sum_{i \in N} q_i(v') \cdot 1_{\{v' \in \mathcal{V}^i(v_i)\}}\right] \\
\leq& \mathbb{E}\left[\left(\sum_{i \in N} q_i(v')\right) \cdot 1_{\{v' \in \mathcal{V}(v) = \cup_{i \in N} \mathcal{V}^i(v_i)\}}\right] \qquad (21) \\
\leq& \mathbb{E}\left[1_{\{v' \in \mathcal{V}(v)\}}\right] = 1 - \prod_{i \in N} F_i(v_i). \qquad (22)
\end{aligned}$$

Note that the first inequality becomes strict if $\sum_{i \in N} q_i(v') > \sum_{i: v'_i \geq v_i} q_i(v)$ while the second inequality becomes strict if $\sum_{i \in N} q_i(v') < 1$. This gives the desired result. ∎

Suppose now for a contradiction that Q fails (NW), which implies by Lemma 2 and Lemma 3 that $g(r) > 0$. We first construct an interim allocation $\tilde{Q}$ for manipulation.

Lemma 4. *For any interim allocation rule $Q = (Q_i)_{i \in N}$ for which $Q_i(v_i) = 0, \forall v_i < r_i, \forall i \in N$ and $g(r) > 0$, we can construct an alternative allocation rule $\tilde{Q} = (\tilde{Q}_i)_{i \in N}$ satisfying the following properties: for each $i \in N$,*

(a) $\tilde{Q}_i(v_i) = 0, \forall v_i < r_i$;

(b) $\tilde{Q}_i(v_i) \geq Q_i(v_i), \forall v_i \in \mathcal{V}_i$, which holds strictly for some $i \in N$ and a positive measure of v_i's;

(c) $\tilde{Q}_i$ satisfies (M), that is, it is non-decreasing;

(d) $\tilde{Q}$ satisfies (B).

Proof. Let us now prove a preliminary result:

Claim 4. *Consider any interim allocation rule $(Q_i)_{i \in N}$ satisfying the assumptions in the statement of this lemma. Then,*

(i) $g(v) > 0$ for for any $v \ngeq r$;

(ii) If $g(v) = g(\tilde{v}) = 0$, then $g(v \wedge \tilde{v}) = 0$, where $v \wedge \tilde{v}$ is the component-wise minimum of the two vectors v and $\tilde{v}$.

Proof. The statement (i) is obvious if $v_i = \underline{v}_i$ for some $i \in N$. Assume thus that $v_i > r_i, \forall i \in N$. Observe first that since $Q_i(v_i) = 0, \forall v_i < r_i$, $g(v)$ is strictly decreasing in v_i at any v with $v_i < r_i$. Given any $v = (v_i)_{i \in N} \ngeq r$, consider $v' = (v'_i)_{i \in N}$ such that $v'_i = v_i$ if $v_i \geq r_i$ while $v_i = r_i$ if $v_i < r_i$. Given the above observation and the fact that $v' \in \times_{i \in N}[r_i, \overline{v}_i]$, we have $g(v) > g(v') \geq 0$.

To prove (ii), note that $\mathcal{V}(v \wedge \tilde{v}) = \mathcal{V}(v) \cup \mathcal{V}(\tilde{v})$. Thus, by Lemma 3 and the assumption $g(v) = g(\tilde{v}) = 0$, if $v \in \mathcal{V}(v \wedge \tilde{v})$, then we have either $\sum_{i: v'_i \geq v_i} q_i(v') = 1$ for almost every $v' \in \mathcal{V}(v)$ or $\sum_{i: v'_i \geq \tilde{v}_i} q_i(v') = 1$ for almost every $v' \in \mathcal{V}(\tilde{v})$, which means $\sum_{i: v'_i \geq \min\{v_i, \tilde{v}_i\}} q_i(v') = 1$ for almost every $v' \in \mathcal{V}(v \wedge \tilde{v})$. This implies $g(v \wedge \tilde{v}) = 0$ by (ii). ▮

Since $g(r) > 0 = g(\overline{v})$, one can find some $\hat{v} > r$ such that $g(\hat{v}) = 0$ and $g(v') > 0$ for any v' such that $r \leq v' < \hat{v}$. We first argue that for each $i \in N$, $g(v') > 0, \forall v' \in \mathcal{V} \backslash \mathcal{V}^i(\hat{v}_i) = [\underline{v}_i, \hat{v}_i) \times \mathcal{V}_{-i}$. Suppose to the contrary that for some $i \in N$, we have $g(v') = 0$ for some $v' \in \mathcal{V} \backslash \mathcal{V}^i(\hat{v}_i)$. By (i) of Claim 4, we must have $v' \geq r$, which implies that $r \leq (v' \wedge \hat{v}) < \hat{v}$ since $v'_i < \hat{v}_i$. However, $g(v' \wedge \hat{v}) = 0$ by Lemma 3, which contradicts the selection of $\hat{v}$.

We now construct an allocation rule for manipulation $\tilde{Q} = (\tilde{Q}_i)_{i \in N}$ satisfying the properties (a) to (d). Choose any $i \in N$ such that $\hat{v}_i > r_i$. We let $\tilde{Q}_j = Q_j, \forall j \neq i$ and $\tilde{Q}_i(v_i) = Q_i(v_i), \forall v_i \in \mathcal{V}_i \backslash (r_i, \hat{v}_i)$. Clearly, $\tilde{Q}$ satisfies the property (a). We now construct $\tilde{Q}_i$ on the interval $(r_i, \hat{v}_i)$ such that the other properties are satisfied. We consider two cases depending on whether or not Q_i is constant on $(r_i, \hat{v}_i)$.

Case of non-constant Q_i: We can find some $v'_i \in (r_i, \hat{v}_i)$ and $\varepsilon > 0$ such that $v'_i - \varepsilon \in (r_i, \hat{v}_i)$ and $Q_i(v'_i - \varepsilon) < Q_i(v_i)$ while Q_i is continuous at $v_i - \varepsilon$. Define $d_1 = \min_{v \in [r_i, v'_i] \times \mathcal{V}_{-i}} g(v)$. Since $g(v) > 0$ for all $v \in [r_i, v'_i] \times \mathcal{V}_{-i} \subset [\underline{v}_i, \hat{v}_i) \times \mathcal{V}_{-i}$, g is continuous, and the set $[r_i, v'_i] \times \mathcal{V}_{-i}$ is compact, we must have $d_1 > 0$. Now let

$$\tilde{Q}_i(v_i) = \begin{cases} \min\{Q_i(v'_i - \varepsilon) + d_1, Q_i(v'_i)\} & \text{for } v_i \in [v'_i - \varepsilon, v'_i] \\ Q_i(v_i) & \text{for } v_i \in (r_i, v'_i - \varepsilon) \cup (v'_i, \hat{v}_i). \end{cases}$$

Clearly, $\tilde{Q}_i$ satisfies (b) and (c). To check (d), let $\tilde{g}$ denote the same function as g except that Q_i is replaced by $\tilde{Q}_i$. For any $v \in \mathcal{V}$ with $v_i \geq v'_i$, we have $\tilde{g}(v) = g(v)$, so (d) is trivially satisfied. Thus, it suffices to check (d) for $v \in [\underline{v}_i, v'_i] \times \mathcal{V}_{-i}$. For any such v, we have

$$\tilde{g}(v) = 1 - \prod_{j \in N} F_j(v_j) - \sum_{j \in N} \int_{v_j}^{\overline{v}_j} Q_j(s) dF_j(s) - \int_{v'_i - \varepsilon}^{v'_i} \left[\tilde{Q}_i(s) - Q_i(s)\right] dF_i(s)$$

$$\geq g(v) - d_1 \left[F_i(v_i') - F_i(v_i' - \varepsilon)\right] \geq 0,$$

where the second inequality holds since $g(v) \geq d_1 = \min_{\tilde{v} \in [r_i, v_i'] \times \mathcal{V}_{-i}} g(\tilde{v})$.

Case of constant Q_i: Let $\overline{Q}_i = Q_i(v_i), \forall v_i \in (r_i, \hat{v}_i)$. We first argue that $\overline{Q}_i < F_{-i}(\hat{v}_{-i}) \leq Q_i(\hat{v}_i)$. To do so, note first that since $g(\hat{v}) = 0$, by Lemma 3, we have $\sum_{j: v_j \geq \hat{v}_j} q_j(\hat{v}_i, v_{-i}) = q_i(\hat{v}_i, v_{-i}) = 1$ for almost every $(\hat{v}_i, v_{-i})$ such that $v_{-i} \leq \hat{v}_{-i}$, which implies that $Q_i(\hat{v}_i) \geq F_{-i}(\hat{v}_{-i})$. Also, $\overline{Q}_i < F_{-i}(\hat{v}_{-i})$ since otherwise

$$g(\hat{v}_i, \hat{v}_{-i}) - g(r_i, \hat{v}_{-i}) = \int_{r_i}^{\hat{v}_i} \frac{\partial g}{\partial s}(s, \hat{v}_{-i}) ds = \int_{r_i}^{\hat{v}_i} \left[\overline{Q}_i - F_{-i}(\hat{v}_{-i})\right] dF_i(s) \geq 0,$$

which means $g(\hat{v}) \geq g(r_i, \hat{v}_{-i}) > 0$ (recall that $g(v') > 0$ for any $v' \in [\underline{v}_i, \hat{v}_i) \times \mathcal{V}_{-i}$), yielding a contradiction. Thus, $\overline{Q}_i < F_{-i}(\hat{v}_{-i}) \leq Q_i(\hat{v}_i)$ as desired. Also, $\min_{v_{-i} \in \mathcal{V}_{-i}} g(r_i, v_{-i}) > 0$ for the same reason as in the previous case. Let $d_2 = \min\{F_{-i}(\hat{v}_{-i}) - \overline{Q}_i, \min_{v_{-i} \in \mathcal{V}_{-i}} g(r_i, v_{-i})\}$ and

$$\tilde{Q}_i(v_i) = \overline{Q}_i + d_2, \forall v_i \in (r_i, \hat{v}_i).$$

Since $0 < d_2 \leq Q_i(\hat{v}_i) - \overline{Q}_i$, $\tilde{Q}_i$ satisfies the properties (b) and (c). To check (d), observe first that for any $v_i, v_i' \in [r_i, \hat{v}_i]$ and v_{-i},

$$\tilde{g}(v_i, v_{-i}) = g(v_i, v_{-i}) - d_2[F_i(\hat{v}_i) - F_i(v_i)] \tag{23}$$

$$g(v_i', v_{-i}) = g(v_i, v_{-i}) + \left[\overline{Q}_i - F_{-i}(v_{-i})\right] \left[F_i(v_i') - F_i(v_i)\right]. \tag{24}$$

Considering first the case of $v_{-i} \in \mathcal{V}_{-i}$ satisfying $F_{-i}(v_{-i}) \geq \overline{Q}_i + d_2$, we have

$$\begin{aligned}\tilde{g}(v_i, v_{-i}) &= g(v_i, v_{-i}) - d_2 \left[F_i(\hat{v}_i) - F_i(v_i)\right] \\ &= g(\hat{v}_i, v_{-i}) + \left[F_{-i}(v_{-i}) - \overline{Q}_i - d_2\right] \left[F_i(\hat{v}_i) - F_i(v_i)\right] \geq g(\hat{v}_i, v_{-i}) \geq 0,\end{aligned}$$

where the first two equalities follow from (23) and (24), respectively. For v_{-i} satisfying $F_{-i}(v_{-i}) < \overline{Q}_i + d_2$,

$$\begin{aligned}\tilde{g}(v_i, v_{-i}) &= g(v_i, v_{-i}) - d_2 \left[F_i(\hat{v}_i) - F_i(v_i)\right] \\ &= g(r_i, v_{-i}) + \left[\overline{Q}_i - F_{-i}(v_{-i})\right] \left[F_i(v_i) - F_i(r_i)\right] - d_2 \left[F_i(\hat{v}_i) - F_i(v_i)\right] \\ &= g(r_i, v_{-i}) - d_2 \left[F_i(\hat{v}_i) - F_i(r_i)\right] + \left[\overline{Q}_i + d_2 - F_{-i}(v_{-i})\right] \left[F_i(v_i) - F_i(r_i)\right] \\ &\geq g(r_i, v_{-i}) - d_2 \left[F_i(\hat{v}_i) - F_i(r_i)\right] \geq 0,\end{aligned}$$

where the first two equalities again follow from (23) and (24), while the second inequality from the fact that $g(r_i, v_{-i}) \geq \min_{v_{-i} \in \mathcal{V}_{-i}} g(r_i, v_{-i}) \geq d_2$.▮

Since $\tilde{Q}$ satisfies (B), there exists an ex-post allocation $(\tilde{q}_i)_{i\in N}$ whose interim allocation is $\tilde{Q}$. Next, we can use $\tilde{Q}$ and (Env) to construct an interim payment rule $\tilde{T} = (\tilde{T}_i)_{i\in N}$ satisfying $\tilde{T}_i(r_i) = T_i(r_i)$ for all $i \in N$. Given $\tilde{Q}$ and $\tilde{T}$, we can define an ex-post payment rule $(\tilde{t}_i)_{i\in N}$ as in (17). Then, a direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ has $(\tilde{Q}, \tilde{T})$ as the interim rule. We can then employ a collusive agreement α in (20) to generate $\tilde{M}_A$. Lastly, we draw a contradiction by showing that $\tilde{M}_A$ satisfies (IC) and (C-IR). First, (IC) is easily satisfied since $\tilde{Q}$ satisfies (M). To check (C-IR), note first that $U_i^{M_A}(v_i) = U_i^{\tilde{M}_A}(v_i) = 0, \forall v_i < r_i$ while $U_i^{M_A}(r_i) = Q_i(r_i)r_i - T_i(r_i) \leq \tilde{Q}_i(r_i)r_i - \tilde{T}_i(r_i) = U_i^{\tilde{M}_A}(r_i)$ since $T_i(r_i) = \tilde{T}_i(r_i)$ and $Q_i(r_i) \leq \tilde{Q}_i(r_i)$ by the property (b) of Lemma 4. For any $v_i > r_i$,

$$U_i^{M_A}(v_i) = U_i^{M_A}(r_i) + \int_{r_i}^{v_i} Q_i(s)ds \leq U_i^{\tilde{M}_A}(r_i) + \int_{r_i}^{v_i} \tilde{Q}_i(s)ds = U_i^{\tilde{M}_A}(v_i),$$

where the inequality holds due to (b) of Lemma 4. This inequality holds strictly for some bidder $i \in N$ and a positive measure of his types for whom the inequality in (b) of Lemma 4 holds strictly.

Proof of Sufficiency: Recall $\mathcal{V}_i^0(r_i) = \{v \in [r_i, \overline{v}_i] | g_i'(v) = 0\}$ is the set of susceptible types, and let $\mathcal{V}_i^- := [r_i, \overline{v}_i] \backslash \mathcal{V}_i^0(r_i)$ be the remaining set of types above r_i in which g_i is strictly decreasing. We let $\mathcal{V}_i^D \subset \mathcal{V}_i$ denote the set of types at which g_i drops discontinuously.

Fix any regular auction A which induces an equilibrium satisfying (PS) and (NW). We prove that A is unsusceptible to collusion.

Suppose for contradiction that there is a weak cartel manipulation $\tilde{M}_A = (\tilde{q}, \tilde{t})$ implementing an interim Pareto improvement. Since, by definition of r_i, we have $\tau_i(b) \geq \xi_i(b)\underline{v}_i, \forall i \in N, \forall b \in \mathcal{B}_i$ $\tilde{M}_A$ is a weak manipulation of A, for each $v_i \leq r_i$,

$$U_i^{\tilde{M}_A}(v) \leq \max_{b\in\mathcal{B}_i} \xi_i(b)v_i - \tau_i(b) \leq \max_{b\in\mathcal{B}_i} \xi_i(b)v_i - \xi_i(b)r_i \leq 0,$$

so (C-IR) implies that $U_i^{\tilde{M}_A}(v) = 0$, for $v_i \leq r_i$. That $\tilde{M}_A$ interim Pareto dominates M_A implies $U_i^{M_A}(v_i) = 0$, for $v_i \leq r_i$. Then, the interim Pareto domination implies that

$$X_i(v_i) := U_i^{\tilde{M}_A}(v_i) - U_i^{M_A}(v_i) = \int_{r_i}^{v_i} (\tilde{Q}_i(s) - Q_i(s))ds \geq 0, \forall i, v_i. \tag{25}$$

Next, by the condition (NW), we have

$$\sum_{i\in N} \int_{r_i}^{\overline{v}_i} Q_i(v_i)f_i(v_i)dv_i = 1 - \prod_{i\in N} F_i(r_i).$$

It follows from this equality and Lemma 2 that

$$\sum_{i\in N} \int_{r_i}^{\overline{v}_i} \tilde{Q}_i(v_i)f_i(v_i)dv_i \leq 1 - \prod_{i\in N} F_i(r_i) = \sum_{i\in N} \int_{r_i}^{\overline{v}_i} Q_i(v_i)f_i(v_i)dv_i,$$

or

$$\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))f_i(v_i)dv_i \leq 0. \tag{26}$$

Meanwhile,

$$\begin{aligned}
&\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))f_i(v_i)dv_i \\
=&\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))g_i(v_i)dv_i-\Big(\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))[g_i(v_i)-f_i(v_i)]dv_i\Big) \\
=&\sum_{i\in N}\left(X_i(\overline{v}_i)g_i(\overline{v}_i)-\sum_{v\in\mathcal{V}_i^D}[X_i(v)g_i(v)]_{v^-}^{v^+}-\int_{r_i}^{\overline{v}_i}X_i(v_i){g_i}'(v_i)dv_i\right) \\
&\quad+\sum_{i\in N}\sum_{v\in\mathcal{V}_i^{D'}}\Big[(G_i(v)-F_i(v))(\tilde{Q}_i(v)-Q_i(v))\Big]_{v^-}^{v^+}+\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(G_i(v_i)-F_i(v_i))[\tilde{Q}_i'(v_i)-Q_i'(v_i)]dv_i \\
=&\sum_{i\in N}\left(X_i(\overline{v}_i)g_i(\overline{v}_i)-\sum_{v\in\mathcal{V}_i^D}X_i(v)(g_i(v^+)-g_i(v^-))-\int_{r_i}^{\overline{v}_i}X_i(v_i){g_i}'(v_i)dv_i\right) \\
&\quad+\sum_{i\in N}\sum_{v\in\mathcal{V}_i^{D'}}(G_i(v)-F_i(v))\Big[\tilde{Q}_i(v^+)-\tilde{Q}_i(v^+)-(Q_i(v^+)-Q_i(v^-))\Big] \\
&\qquad+\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(G_i(v_i)-F_i(v_i))[\tilde{Q}_i'(v_i)-Q_i'(v_i)]dv_i \\
\geq&0,
\end{aligned} \tag{27}$$

where $\mathcal{V}_i^{D'}$ is the set of values at which either $\tilde{Q}_i$ or Q_i jumps up. The first equality follows from the integration by parts. The second equality holds since X_i, G_i and F_i are continuous. The inequality holds since, for each $i \in N$, $X_i(v) \geq 0$, $g_i'(v) \leq 0$ whenever it is well defined, and $g_i(v^+) - g_i(v^-) < 0$ for each $v \in \mathcal{V}_i^D$, and, whenever $G_i(v_i) > F_i(v_i)$, $Q_i(v^+) = Q_i(v^-)$ (since Q satisfies (PS)), $\tilde{Q}_i(v^+) \geq \tilde{Q}_i(v^-)$ and $Q_i'(v_i) = 0 \leq \tilde{Q}_i'(v_i)$ (by the monotonicity of $\tilde{Q}_i$).

The last inequality combined with (26) means that the inequality must hold as equality, which in turn implies that $X_i(\overline{v}_i) = 0$, and $X_i(v) = 0$ for a.e. $v \in \mathcal{V}_i^-$ for each $i \in N$.[38]

We now prove that $U_i^{\tilde{M}_A}(v) - U_i^{M_A}(v) = \int_{r_i}^{v}(\tilde{Q}_i(s) - Q_i(s))ds = X_i(v) \leq 0$ for all $v \geq r_i$, for all $i \in N$, which will have established the desired contradiction. Suppose to the contrary that there exists v' such that $X_i(v') > 0$. Recall that $X_i(r_i) = X_i(\overline{v}_i) = 0$ and that X_i is

[38] If $X_i(v) > 0$ for some $v \in \mathcal{V}_i^D$, or for a positive measure set of v's in $\mathcal{V}_i^-$, then the inequality in (27) becomes strict, a contradiction to (26).

continuous, and differentiable on $(r_i, \overline{v}_i)$. By the mean value theorem, there exists $v^1 \in (r_i, v')$ such that $X_i(v^1) > 0$ and that $X_i'(v^1) = \tilde{Q}_i(v^1) - Q_i(v^1) > 0$ and $v^2 \in (v', \overline{v}_i)$ such that $X_i(v^2) > 0$ and that $X_i'(v^2) = \tilde{Q}_i(v^2) - Q_i(v^2) < 0$. It follows that there exist $v'' \in (v^1, v^2)$ such that $X_i(v'') > 0$, and $X_i'(v) = \tilde{Q}_i(v) - Q_i(v)$ falls in v at $v = v''$, meaning either $\tilde{Q}_i(v) - Q_i(v)$ jumps down at $v = v''$ or $\tilde{Q}_i'(v'') - Q_i'(v'') < 0$. In either case, since $\tilde{Q}_i$ is nondecreasing, $Q_i(v)$ must increase in v at $v = v''$. This means that $v'' \in \mathcal{V}_i^-$ by the construction of Q_i. But then the above observation implies that $X_i(v'') = 0$, a contradiction. We thus conclude that $U_i^{\tilde{M}_A}(v) - U_i^{M_A}(v) = X_i(v) \leq 0$ for all v, and i. ▮

Proof of Corollary 1: Fix a bidder k for whom G_k is linear on some interval (a, b) with $b > r$ and $a \geq \underline{v}$. We show that in any standard auction, the winning probability of bidder k is non-constant in the interval $(\max\{a, r\}, b)$, which will imply by Theorem 1 that the auction is not WCP. Consider first the second-price and English auctions where each bidder bids his value in the undominated strategy. The interim winning probability of bidder k with $v_k \in (\max\{a, r\}, b)$ is equal to $Q_k(v_k) = \prod_{i \neq k} F_i(v_k)$, which is strictly increasing in the interval $(\max\{a, r\}, b)$.

Consider next the first-price auction (or Dutch auction since the two auctions are strategically equivalent). Note first that in undominated strategy equilibrium, (i) no bidder bids more than his value and (ii) no bidder puts an atom at any bid B if B wins with positive probability. Letting β_i denote bidder i's equilibrium strategy, note also that β_i is nondecreasing. Given (i), we must have $Q_k(v_k) > 0$ for all $v_k \in (\max\{a, r\}, b)$ since he can always bid some amount $B \in (\max\{a, r\}, v_k)$ and enjoy a positive payoff. Next, by (ii), there must be some $v_k \in (\max\{a, r\}, b)$ such that $\beta_k(v_k) < \beta_k(b)$ since otherwise $\beta_k(b)$ would be an atom bid. For such v_k, we must have $Q_k(v_k) < Q_k(b)$ so Q_k is non-constant in $(\max\{a, r\}, b)$. To see why, suppose to the contrary that $Q_k(v_k) = Q_k(b)$, which implies that no one else is submitting any bid between $\beta_k(v_k)$ and $\beta_k(b)$. Then, bidder k with value b can profitably deviate to lower his bid below $\beta_k(b)$ but above $\beta_k(v_k)$, a contradiction. ▮

B Proofs for Section 5

Proof of Theorem 3: We first establish a two lemmas, Lemma 5 and 6. To do so, some notation is first required. Given any vector $r = (r_1, \ldots, r_n) \in \mathcal{V}$, we write the allocation rule in (10) as $q^*(\cdot; r)$ to make its dependence on r explicit. Next, we define

$$t_i^*(v; r) = q_i^*(v; r)v_i - \int_{\underline{v}_i}^{v_i} q_i^*(s_i, v_{-i}; r)ds_i. \tag{28}$$

Then, from now on, we let $M^*(r)$ denote a direct mechanism $(q^*(\cdot;r), t^*(\cdot;r))$. It is straightforward to see that $M^*(r)$ is dominant-strategy implementable. For any $r = (r_i)_{i\in N}$, let $[P;r]$ be the same optimization program as $[P]$, except that it *ignores* the constraint $r_i = \inf\{\frac{t_i^*(v;r)}{q_i^*(v;r)} \mid q_i^*(v;r) > 0\}$, which we will refer to as the constraint (R) henceforth.

LEMMA 5. *For any $r = (r_i)_{i\in N}$ with $r_i \geq J^{-1}(0), \forall i \in N$, the mechanism $M^*(r)$ solves $[P;r]$.*

Proof. We first prove that $q^*(\cdot;r)$ maximizes the objective function of $[P;r]$. To do so, rewrite the objective function by incorporating the collusion-proofness constraint into it: For each $i \in N$ and $k \in K_i$, define $Q_i^k = Q_i(v_i)$ and let $J_i^k := \frac{\int_{a_i^k}^{b_i^k} J_i(s)dF_i(s)}{F_i(b_i^k)-F_i(a_i^k)}$ if $v_i \in I_i^k$. Then, express the seller's expected revenue as

$$
\begin{aligned}
&\sum_{i\in N}\int_{r_i}^{\overline{v}_i} J_i(v_i)Q_i(v_i)dF_i(v_i)\\
=&\sum_{i\in N}\int_{v_i\in[r_i,\overline{v}_i]\setminus\mathcal{V}_i^0(r_i)} J_i(v_i)Q_i(v_i)dF_i(v_i) + \sum_{i\in N}\sum_{k\in K_i}\int_{a_i^k}^{b_i^k} J_i(v_i)Q_i(v_i)dF_i(v_i)\\
=&\sum_{i\in N}\int_{v_i\in[r_i,\overline{v}_i]\setminus\mathcal{V}_i^0(r_i)} J_i(v_i)Q_i(v_i)dF_i(v_i) + \sum_{i\in N}\sum_{k\in K_i} Q_i^k\int_{a_i^k}^{b_i^k} J_i(v_i)dF_i(v_i)\\
=&\sum_{i\in N}\int_{v_i\in[r_i,\overline{v}_i]\setminus\mathcal{V}_i^0(r_i)} J_i(v_i)Q_i(v_i)dF_i(v_i) + \sum_{i\in N}\sum_{k\in K_i} J_i^kQ_i^k(F_i(b_i^k)-F_i(b_i^k))\\
=&\sum_{i\in N}\int_{r_i}^{\overline{v}_i} \bar{J}_i(v_i)Q_i(v_i)dF_i(v_i)\\
=&\mathbb{E}[\sum_{i\in N}\bar{J}_i(v_i)1_{\{v_i\geq r_i\}}q_i(v_i,v_{-i})].
\end{aligned}
$$

The expression within the expectation operator above is maximized by the allocation rule $q_i^*(\cdot;r)$ for each realization $v = (v_i)_{i\in N}$.

It is clear that $q^*(\cdot;r)$ satisfies (NW). Since $\bar{J}_i$ is (weakly) increasing, the interim allocation rule resulting from $q^*(\cdot;r)$ satisfies (M). Also, (Env) is easily satisfied since $M^*(r)$ is dominant-strategy implementable. Lastly, the constraint (PS) is satisfied because the fact that $\bar{J}_i$ is constant over I_i^k implies all types in the interval I_i^k receive the object with a constant probability under $q_i^*(\cdot;r)$ for each $i \in N$. We thus conclude that $M^*(r)$ solves $[P;r]$. ∎

However, there is no guarantee that the mechanism $M^*(r)$ satisfies the constraint (R). The following result shows that starting from $M^*(r)$ it is always possible to satisfy (R) without reducing the seller's revenue.

Lemma 6. *For any $r = (r_i)_{i\in N} \in \mathcal{V}$, there exists $\hat{r} \geq r$ such that $M^*(\hat{r})$ satisfies (R) and yields a (weakly) higher revenue for the seller than $M^*(r)$.*

Proof. As a first step, we prove the following claim:

Claim 5. *For any $\tilde{r}_i \geq r_i$, $\mathcal{V}_i^0(\tilde{r}_i) \subset \mathcal{V}_i^0(r_i)$.*

Proof. Consider any interval $I = [a, b] \subset \mathcal{V}_i^0(\tilde{r}_i)$ on which $G_i(\cdot; \tilde{r}_i)$ is linear. Then, for each $v_i \in I$, there is some $s \in [0, 1]$ and $v_i', v_i'' \in [\tilde{r}_i, \overline{v}_i]$ such that $G_i(v_i; \tilde{r}_i) = sF_i(v_i') + (1-s)F_i(v_i'')$. Since $\tilde{r}_i \geq r_i$ and thus $v_i', v_i'' \in [r_i, \overline{v}_i]$, we have $G_i(v_i; r_i) \geq sF_i(v_i') + (1-s)F_i(v_i'') = G_i(v_i; \tilde{r}_i)$ by definition of $G_i(v_i; r_i)$. Thus, we have $G_i(v_i; r_i) \geq G_i(v_i; \tilde{r}_i)$ for all $v_i \in I$. This implies that $G_i(\cdot; r_i)$ is also linear over the interval I since, if not, it must be the case that over some subinterval of I, $G_i(\cdot, r_i)$ is strictly concave and $G_i(\cdot, r_i) = F_i(\cdot) > G_i(\cdot; \tilde{r}_i)$, which cannot happen due to the fact that $G_i(\cdot; \tilde{r}_i)$ is the concave envelope of F_i. Thus, we have shown that $I \subset \mathcal{V}_i^0(r_i)$ so $\mathcal{V}_i^0(\tilde{r}_i) \subset \mathcal{V}_i^0(r_i)$. ∎

For any $r = (r_i)_{i\in N} \in \mathcal{V}$, let $Q^*(\cdot; r)$ denote the interim allocation rule corresponding to $q^*(\cdot; r)$, and define $r_i^*(r) := \inf\{v_i \in \mathcal{V}_i \mid Q_i^*(v_i; r) > 0\}$. (If $Q_i^*(\cdot, r) \equiv 0$, then let $r_i^*(r) = \overline{v}_i$.) Note that by construction of $q^*(\cdot; r)$, we have $r_i^*(r) \geq r_i, \forall i \in N$. Let $\pi^*(r)$ denote the seller's revenue that is generated by the mechanism $M^*(r)$.

Now fix any $r = (r_i)_{i\in N}$ and denote $r^1 = r$. Define $r^2 \in \mathcal{V}$ such that $r_i^2 = r_i^*(r^1)$ for each $i \in N$. Then, we must have $\pi^*(r^2) \geq \pi^*(r^1)$. To see this, note that $q^*(\cdot; r^1)$ satisfies all the constraints of $[P; r^2]$, in particular (PS) since, for each $i \in N$, $Q_i^*(v_i; r^1) = 0, \forall v_i \leq r_i^2$ and also since $Q_i^*(\cdot; r^1)$ is constant in each interval belonging to $\mathcal{V}_i^0(r_i^2)$, which is because $\mathcal{V}_i^0(r_i^2) \subset \mathcal{V}_i^0(r_i^1)$ by Claim 5 and the fact that $r_i^2 = r_i^*(r^1) \geq r_i^1$. Thus, $M^*(r^1)$ cannot yield a higher seller's revenue than $M^*(r^2)$, which is a solution of $[P; r^2]$. Define recursively $r^n \in \mathcal{V}$ for all $n \geq 2$ such that $r_i^n = r_i^*(r^{n-1})$ for each $i \in N$. By following the same reasoning as above, we have $\pi^*(r^n) \geq \pi^*(r^{n-1})$ for all $n \geq 2$. Also, the sequence $(r^n)_{n\in\mathbb{N}}$ is (weakly) increasing in the set $\mathcal{V}$, and thus has a limit $\hat{r} \in \mathcal{V}$ such that $\hat{r}_i = r_i^*(\hat{r})$. Then, we have $\pi^*(r) = \pi^*(r^1) \leq \pi^*(r^2) \leq \cdots \leq \pi^*(\hat{r})$.

It remains to show that $M^*(\hat{r})$ satisfies (R). Note first that for each $v_i > \hat{r}_i$, we have some v_{-i} such that $q_i^*(v_i, v_{-i}; \hat{r}) > 0$, since $Q_i^*(v_i; \hat{r}) > 0$. For such profile $v = (v_i, v_{-i})$, we have $t_i^*(v; \hat{r}) \leq q_i^*(v; \hat{r})v_i$ or $\frac{t_i^*(v;\hat{r})}{q_i^*(v;\hat{r})} \leq v_i$. Since this is true for all $v_i > \hat{r}_i$, we have $\inf\{\frac{t_i^*(v;\hat{r})}{q_i^*(v;\hat{r})} \mid q_i^*(v; \hat{r}) > 0\} \leq \hat{r}_i$. The desired result will follow if it is shown that this inequality cannot be strict. To do so, note first that for any $v_i < \hat{r}_i$, we have $q_i^*(v_i, v_{-i}; \hat{r}) = 0, \forall v_{-i}$. Also, for any $v_i \geq \hat{r}_i$,

$$t_i^*(v_i, v_{-i}; \hat{r}) \geq v_i q_i^*(v_i, v_{-i}; \hat{r}) - (v_i - \hat{r}_i)q_i^*(v_i, v_{-i}; \hat{r}) = \hat{r}_i q_i^*(v_i, v_{-i}; \hat{r}), \forall v_{-i},$$

where the inequality holds since $q_i^*(\cdot, v_{-i}; \hat{r})$ is nondecreasing. ▮

We are now ready to prove Theorem 3. Consider any profile of reserve prices $\tilde{r} = (\tilde{r}_i)_{i \in N}$ that results from solving $[P]$. Then, the optimal revenue cannot be greater than that from $M^*(\tilde{r})$ since $M^*(\tilde{r})$ solves $[P; \tilde{r}]$ according to Lemma 5. Then, by Lemma 6, one can find a profile r such that $M^*(r)$ satisfies all the constraints of $[P]$ and yields no less revenue for the seller than $M^*(\tilde{r})$ does, which means that $M^*(r)$ is a solution of $[P]$. The proof that $r_i \geq J_i^{-1}(0), \forall i \in N$ at the optimum of $[P]$ is straightforward and hence omitted. ▮

Proof of Corollary 6: We first observe that

$$\int_{r_i}^{\overline{v}_i} J_i(v_i) dF_i(v_i) = (1 - F_i(r_i)) r_i,$$

which can be readily verified using the definition of J_i and integration-by-parts. Thus, for any $v_i \in \mathcal{V}_i^0(r_i) = [r_i, \overline{v}_i]$, we have $\bar{J}_i(v_i) = r_i$. Then, the allocation rule in (10) requires allocating the object to bidder i if $v_i \geq r_i = \bar{J}_i(v_i) > \max\{r_j \,|\, v_j \geq r_j = \bar{J}_j(v_j) \text{ and } j \neq i\}$. This means that bidder i must always be given the priority to receive the object over bidder j if $r_i > r_j$. In case $r_i = r_j$, the priority can be given to either of bidder i and j. (Note that in the statement of Theorem 3 bidders with equal virtual values obtain the object with the same probability; it is without loss to treat them asymmetrically as we do here). Let such priority rule be denoted by a permutation function $\pi : N \to N$ satisfying that $\pi(j) < \pi(i)$ if $r_i < r_j$. The interim allocation rule that results from this priority rule is then given as $\prod_{j:\pi(j)<\pi(i)} F_j(r_j)$ for each bidder i with $v_i \geq r_i$. Also, the seller's expected revenue under this interim allocation rule coincides with the expression within the square bracket of (11), which must then be maximized by choosing $r = (r_i)_{i \in N}$ optimally. ▮

Proof of Corollary 8: First, by the symmetry of auction rule, we must have $r_i = r$ for all i and some $r \geq r^M$. Let us consider the case where $r \leq \hat{v}$ (while we will see below that $r \leq \hat{v}$ is required at the optimum). There is a value $v^*(r) \geq \hat{v}$ such that G is linear in $[r, v^*(r)]$ while it is strictly concave elsewhere, which implies that $\bar{J}(v)$ defined in (9) is constant for $v \in [r, v^*(r)]$ and strictly increasing for $v > v^*(r)$. Using this, it is straightforward to see that the allocation q_i^* in (10) coincides with (12).

We now show that $r^M < r \leq \hat{v}$ at the optimum. We first argue that $r \leq \hat{v}$. If $r > \hat{v}$, then there is no range where G is linear, which means that the corresponding optimal rule given by (10) is the one which allocates the object efficiently among the bidders whose values are greater than r. Clearly, this mechanism is revenue-dominated by a mechanism where $r' = \hat{v}$ and the object is efficiently allocated to the bidders whose values are greater than r', since $\hat{v} > r^M$ so the extra revenue can be generated from selling to bidders with values in $[\hat{v}, r]$.

We next show that $r > r^M$. Since we already know that $r \geq r^M$ at the optimum, we need to argue that $r \neq r^M$ at the optimum. Note first that the interim allocation rule is given by

$$Q^*(v) = \begin{cases} F(v)^{n-1} & \text{if } v > v^* \\ \frac{F(v^*)^n - F(r)^n}{n(F(v^*)-F(r))} = \frac{\sum_{k=0}^{n-1} F(v^*)^{n-1-k} F(r)^k}{n} & \text{if } v \in [r, v^*] \\ 0 & \text{otherwise.} \end{cases}$$

The seller's revenue from each bidder can then be written as

$$n \int_r^{\overline{v}} J(v) Q^*(v) f(v) dv = \Big(\sum_{k=0}^{n-1} F(v^*)^{n-1-k} F(r)^k \Big) \int_r^{v^*} J(v) f(v) dv + n \int_{v^*}^{\overline{v}} F(v)^{n-1} J(v) f(v) dv.$$

Keeping in mind that v^* is a function of r, we differentiate the above expressions with r, set $r = r^M$, and use $J(r^M) = 0$ to obtain

$$\underbrace{\left(\sum_{k=0}^{n-2} (n-1-k) F(v^*)^{n-2-k} F(r^M)^k f(v^*) \left(\frac{dv^*}{dr} \right) + \sum_{k=1}^{n-1} F(v^*)^{n-1-k} F(r^M)^{k-1} f(r^M) \right)}_{= A} \int_{r^M}^{v^*} J(v) f(v) dv$$
$$+ J(v^*) \underbrace{\left(\sum_{k=0}^{n-1} F(v^*)^{n-1-k} F(r^M)^k - n F(v^*)^{n-1} \right) f(v^*) \left(\frac{dv^*}{dr} \right)}_{= B}.$$

It is straightforward to check that

$$A \geq \left(\sum_{k=0}^{n-2} (n-1-k) F(v^*)^{n-2-k} F(r^M)^k \right) f(v^*) \left(\frac{dv^*}{dr} \right) = \frac{-B}{F(v^*) - F(r)}.$$

Thus the above expressions is no less than

$$\left(J(v^*) - \frac{\int_{r^M}^{v^*} J(v) f(v) dv}{F(v^*) - F(r^M)} \right) B > 0,$$

where the strict inequality holds since $v^* > r^M$ and $\frac{dv^*}{dr} > 0$ imply $B > 0$. Thus, it is profitable for the seller to raise r above r^M. ▮

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