Milnor K-theory via commuting automorphisms

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Contents

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Declaration

The work presented in this thesis is my own except where I have stated otherwise. This work has not been presented for a degree at any other university.

Abstract

In this thesis, we give a presentation for Milnor K-theory of a field F whose generators are tuples of commuting automorphisms. This is similar to a presentation for Milnor K-theory given by the cohomology groups of Grayson. The main difference is that, in our presentation, we do not use a homotopy invariance relation, which we should not expect to hold for non-regular rings R.

We go on to study this presentation for R a local ring. We conjecture that it agrees with the usual definition of Milnor K-theory for any local ring. We give some evidence towards this, including showing that the natural map $K_n(R) \to \widetilde{K}_n(R)$ is injective when $n = 0, 1, 2$ or when R is a regular, local ring containing an infinite field. We also show a reciprocity result for $\widetilde{K}_n^M(R)$ any ring R, which, when R is a field, allows us to deduce surjectivity of the map.

We prove a version of the additivity, resolution, devissage and cofinality theorems for the groups $\widetilde{K}_n^M(R)$. We also construct a comparison homomorphsim from $\widetilde{K}_n^M(R)$ to the presentation of Quillen K-theory given by Grayson.

Chapter 1

Introduction

Milnor K-theory $K_n^M(F)$ of a field F is a sequence of abelian groups with a certain presentation. It was originally defined, by Milnor, in [14] based on the presentation of $K_2(F)$ of a field given by Matsumoto [12]. In this paper, Milnor conjectures results connecting Milnor K-theory mod 2 to quadratic forms and Galois cohomology. More precisely, he constructs homomorphisms

$$h_F\colon K_n^M(F)/2K_n^M(F) \to H^n(G;\mathbb{Z}/2\mathbb{Z})$$
$$s_n\colon K_n^M(F)/2K_n^M(F) \to I^n(F)/I^{n+1}(F),$$

where G is the Galois group of the separable closure of F and $I(F)$ is the fundamental ideal of the Witt ring, and conjectures that these maps are isomorphisms. These conjectures became known as the Milnor conjectures and were proven by Voevodsky, Orlov and Vishik in [16], by using methods in motivic cohomology.

Milnor K-theory of a local ring was first studied in [15] and [7]. In [15], it is shown that the maps

$$H_n(\mathrm{GL}_n(R)) \to H_n(\mathrm{GL}_{n+1}(R)) \to H_n(\mathrm{GL}_{n+2}(R)) \to \cdots \to H_n(\mathrm{GL}(R)),$$

induced by the natural inclusion

$$\mathrm{GL}_i(R) \to \mathrm{GL}_{i+1}(R)$$
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

are all isomorphisms when R is a local ring with infinite residue field. It is also shown that Milnor K-theory occurs as the obstruction to further stability i.e. that the map

$$K_n^M(A) \xrightarrow{\cong} H_n(\mathrm{GL}_n(A))/H_n(\mathrm{GL}_{n-1}(A))$$

is an isomorphism.

Nowadays, Milnor K-theory is part of motivic cohomology and there are several theorems relating Milnor K-theory of a field to various cohomology theories. In [15] and [20], it is also shown that there is an isomorphism

$$K_n^M(F) \xrightarrow{\cong} CH^n(F, n)$$

where $CH^n(F, n)$ are Bloch's higher Chow groups. Another connection with motivic cohomology is with Voevodsky's motivic cohomology groups [13]; there is an isomorphism

$$K_n^M(F) \xrightarrow{\cong} H^{n,n}(\mathrm{Spec}(F), \mathbb{Z})$$

Many of the proofs of the theorems above rely on some of the nice properties of Milnor K-theory. Of particular importance, is the existence of transfer maps

$$N_{L/F}^M : K_n^M(L) \to K_n^M(F)$$

where L/F is a finite field extension. These maps are defined using Milnor's exact sequence from [14], which computes the Milnor K-theory of $K_n^M(F(t))$ in terms of the groups $K_{n-1}^M(F[t]/p(t))$ and $K_n^M(F)$ where $p(t)$ is monic, irreducible.

In [10], Kerz constructs an analogue of this exact sequence for semi-local rings with infinite residue fields and uses this to construct transfer maps

$$N_{B/A} : K_n^M(B) \to K_n^M(A),$$

where A is a semi-local ring with infinite residue fields and B/A is an etale extension of semi-local rings with infinite residue field. The existence of transfer maps is used to prove the Gersten conjecture for Milnor K-theory in the equi-charactistic case and this is used to show the Bloch formula

$$H^n_{\text{zar}}(X, K^M) = CH^n(X)$$

for X a regular, excellent scheme over an infinite field.

Therefore, if one wishes to generalise some of these results to the realm of local rings with finite residue fields it would seem that the existence of transfer maps is important. Unfortunately, the naive generalisation of Milnor K-theory to local rings with finite residue field does not have transfers in general. However, in [11] Kerz gives a definition, based on ideas of Gabber, of improved Milnor K-theory of a local ring with finite residue field and shows that this definition has transfers. Furthermore, Kerz shows that improved Milnor K-theory agrees with Milnor K-theory when the residue field is sufficiently large and that this extension is unique. However, this definition is not given by a presentation as Milnor K-theory usually is.

The purpose of this thesis is to give a possible presentation of Milnor K-theory of any local ring, motivated by the motivic cohomology groups of Grayson. The idea is to replace the generators in Milnor K-theory, which are n-tuples of units in R^*, with n-tuples of commuting automorphisms of finitely generated, projective modules. This allows transfers to be naturally defined for any finite, flat extensions of local rings. This is in contrast to Milnor K-theory where the existence of transfers is not obvious, with the construction of these maps dependent on the existence of a certain exact sequence. In fact, replacing tuples of units with tuples of automorphisms allows transfers to be defined for finite, flat extensions of any ring. This presentation is similar to the one given by the motivic cohomology of Grayson [4]. The main difference is that we do not use a homotopy invarience relation which we do not expect to hold when R is not regular.

In chapter 2, we review some of the results in Milnor K-theory and related areas of mathematics that we will need. In the first section of the chapter, we review the construction of transfer maps for Milnor K-theory. We do this both for fields and semi-local rings with infinite residue fields. Along the way, we present the residue homomorphisms and the exact sequence necessary to define these transfers. We end the section by stating some of the properties of these transfer maps. In the next section we give the definition of improved Milnor K-theory studied in [11] and state, without proof, some of its properties. In the third section we give the definition of the motivic cohomology groups of Grayson. These groups motivate our goal to give a presentation of Milnor K-theory which has generators n-tuples of commuting automorphisms. In the fourth section, we present the construction of higher algebraic K-theory of Grayson [5] which gives a presentation of the Quillen K-theory of an exact category in terms of binary complexes.

In chapter 3, we give our definition of $\widetilde{K}_n^M$. The purpose of this chapter is to show that $K_n^M(F) \cong \widetilde{K}_n^M(F)$ when F is a field. We begin by showing that the groups agree when $n = 0, 1$ when F is a local ring. We go on to define transfers for $\widetilde{K}_n^M$, and to prove some of the analogous identities that hold in Milnor K-theory. We then show that the natural map $K_n^M(F) \to \widetilde{K}_m^M(F)$ is surjective, by showing $\widetilde{K}_m^M(F)$ is generated by images of transfers and that the transfers for $K_n^M(F)$ and $\widetilde{K}_m^M(F)$ are compatible. In the last section, we prove that the map is injective. To do this we construct an inverse by first mapping into the cohomlogy groups of Grayson and then constructing an inverse map from these groups to Milnor K-theory.

In chapter 4, we study some properties of the groups $\widetilde{K}_n^M$. We begin by proving a reciprocity law for $\widetilde{K}_n^M(R)$. Two immediate corollaries are that the transfer maps for $K_n^M(R)$ and $\widetilde{K}_n^M(R)$ are compatible and the transfers for $\widetilde{K}_n^M(R)$ satisfy naturality when R is a semi-local ring with infinite residue fields. The naturality property is enough to show that if $K^M(R)$ agrees with $\widetilde{K}_n^M(R)$ when R is a local ring with infinite residue field then $\widetilde{K}_n^M(R)$ will agree with the improved Milnor K-groups $\hat{K}_n^M(R)$ of Gabber-Kerz when R is a local ring with finite reisdue field. The remainder of this chapter is dedicated to proving some analogues of fundamental theorems, for Quillen K-theory, in our setting. In particular we prove versions of the additivity, resolution and devissage theorems.

In chapter 5, we construct a comparison homomorphism $\widetilde{K}_n^M(R) \to K_n^Q(R)$, such that the standard comparison homomorphism from $K_n^M(R)$ factors through this map. This provides further evidence that our definition for $\widetilde{K}_n^M$ is the correct one. To do this we use the presentation of K_n^Q, due to Grayson [5], which we reviewed in chapter 1. In the first section we review the proof of the bilinearity relation which we take from the thesis of Harris [19]. In the next section we prove the Steinberg relation holds in $K_n^Q(R)$ for any ring R. Before we do this, we prove a version of the cofinality theorem which will allow us to reduce to proving the Steinberg relation for free modules. We then go on to prove the Steinberg relation using homotopy invariance and functorality of K_n^Q. Because the comparison homomorphism is an isomorphism when $n = 2$ this allows us to show that the map $K_2^M(R) \to \widetilde{K}_2^M(R)$ is injective. More generally, we can conclude that the kernel of the map $K_n^M(R) \to \widetilde{K}_n^M(R)$ is a torsion group annihilated by $(n-1)!$.

In Chapter 6, we look at some further questions that we were not able to answer. We show, using the resolution theorem, that the groups $\widetilde{K}_n^M(R)$ are generated by images of transfers when R is regular, local. We use this to show that $K_n^M(R) \cong \widetilde{K}_n^M(R)$, for R a DVR, if the transfers for K_n^M are compatible with those for $\widetilde{K}_n^M$.

In the last section we construct a map

$$\widetilde{K}_n^M(R) \to H_n(\mathrm{GL}(R))$$

which we conjecture to be the composition $\widetilde{K}_n^M(R) \to K_n^Q(R) \to H_n(\mathrm{GL}(R))$.

Chapter 2

K-theory, Motivic cohomology and Homology

2.1 Milnor K-theory

In this section, we review some facts about Milnor K-theory including the construction of the transfer maps and its properties. We begin by reviewing the definition of Milnor K-theory.

Definition 2.1.1. *Let A be a commutative ring. We define Milnor K-theory, denoted $K_*^M(A)$, of A to be the graded ring*

$$K_*^M(A) := \mathrm{Tens}_{\mathbb{Z}}(A^*)/I$$

$\mathrm{Tens}_{\mathbb{Z}}(A^*)$ *is the tensor algebra $\bigoplus_{n=0}^{\infty}(A^*)^{\otimes n}$ where I is the two-sided ideal generated by elements of the form $a \otimes (1-a)$, for $a,\ 1-a \in A^*$. We define the n'th Milnor K-group $K_n^M(A)$ to be the abelian subgroup generated by elements of degree n.*

We denote an element $a_1 \otimes \ldots \otimes a_n \in K_n^M(A)$ by $\{a_1, \ldots, a_n\}$. As noted earlier, this definition is not the correct one in general when A is a local ring with finite residue field.

A fundamental result in Milnor K-theory is the short exact sequence which calculates the Milnor K-theory of a rational function field. This short exact sequence is used to construct the transfer maps for Milnor K-theory. We now give the definition, which we take from [14, Lemma 2.1], of the residue maps in 2.1.2 and use these to give a presentation of the short exact sequence of Milnor [14, Theorem 2.1] in 2.1.3.

Proposition 2.1.2. *Let F be a field, v be a discrete valuation on F and $F(v)$ be the residue field of F. There exists a unique homomorphism*

$$\partial_v \colon K_n^M(F) \to K_{n-1}(F(v)),$$

such that

$$\partial_v\{\pi, u_2, \ldots, u_n\} = \{\overline{u}_2, \ldots, \overline{u}_n\}$$

where π is any uniformizing element and u_i satisfy $v(u_i) = 0$.

Of particular importance, is the case when F is a field of rational functions. In this case we get a valuation for each monic, irreducible polynomial $p(t)$. We denote the associated residue map by $\partial_{p(t)}$. We also have a valuation with uniformizer $\frac{1}{t}$. We denote the residue map for this valuation as ∂_∞.

Theorem 2.1.3. *Let F be field. The sequence*

$$0 \to K_n^M(F) \to K_n^M(F(t)) \xrightarrow{\oplus\partial_\pi} \bigoplus_{\substack{\pi \text{ irreducible,} \\ \text{monic}}} K_{n-1}^M(F[t]/\pi) \to 0$$

is split exact.

One can use this sequence to define transfer maps for Milnor K-theory of fields. These were originally defined in [1].

Definition 2.1.4. *Let F be a field and $L := F[t]/p(t)$ be a simple field extension. We define a map*

$$N_{L/F}^M \colon K_n^M(L) \to K_n^M(F)$$

to be the composition

$$K_n^M(L) \to \bigoplus_{\substack{\pi \text{ irreducible,} \\ \text{monic}}} K_n^M(F[t]/\pi) \xrightarrow{\Psi} K_{n+1}(F(t)) \xrightarrow{-\partial_\infty} K_n(F)$$

where the first map is inclusion into the appropriate direct summand and Ψ *is any splitting map for the exact sequence in 2.1.3.*

We can also define transfer maps for an arbitrary finite field extension L/F by writing L as a tower of finite simple extensions. It was shown in [9] that this is independent of the tower of extensions chosen.

2.1.1 Transfer maps for Milnor K-theory of semi-local rings with infinite residue fields

In this section, we give the definition of transfer maps defined by Kerz in [10] for finite, etale extensions of semi-local rings with infinite residue fields. To do this we first give the analogue of the exact sequence 2.1.3. To give this sequence we only need to define the middle term and the residue maps. We do this in the following definitions taken from [10, Definition 5.2]:

Definition 2.1.5. *Let* A *be a semi-local ring. An* n*-tuple of rational functions*

$$\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)$$

with $p_i, q_i \in A[t]$ *together with a factorization*

$$p_i = a_i p_1^i \ldots p_{n_i}^i$$
$$q_i = b_i q_1^i \ldots q_{m_i}^i$$

such that $a_i, b_i \in A^*$ *and each* p_i, q_i *is monic irreducible, is called feasible if the fraction* $\frac{p_i}{q_i}$ *is reduced, if every irreducible factor is either equal or coprime and* $\mathrm{Disc}(p_i), \mathrm{Disc}(q_i) \in A^*$*, where* $\mathrm{Disc}(p_i)$ *is the discriminant of the polynomial* p_i*.*

Definition 2.1.6. *Let A be a semi-local ring. We define*

$$\mathscr{T}^{et}(A) := \mathbb{Z}\{(p_1, \dots, p_n) | (p_1, \dots, p_n) \text{ feasible}, \\ p_i \in A[t] \text{ irreducible or unit}\} / Linear.$$

Where Linear denotes the subgroup generated by elements

$$(p_1, \dots, a_i p_i, \dots, p_n) - (p_1, \dots, p_i, \dots, p_n) - (p_1, \dots, a_i, \dots, p_n)$$

where $a_i \in A^*$.

If we have an n-tuple of rational functions together with a choice of factorization as in 2.1.5 then we can define an element in $\mathscr{T}_n^{et}(A)$ by using multilinear factorization. We will now define the group $K_n^{et}(A)$ which will replace $K_n^M(F(t))$ in the semi-local ring version of the sequence in 2.1.3.

Definition 2.1.7. *Let A be a semi-local ring. We define*

$$K_n^{et}(A) = \mathscr{T}_n^{et}(A)/St^{et},$$

where St^{et} is the group generated by the elements in $\mathscr{T}_n^{et}(A)$ which are associated to feasible n-tuples

$$\Big(p_1, \dots, \tfrac{p}{q}, \tfrac{p-q}{q}, \dots, p_n\Big) \\ \Big(p_1, \dots, \tfrac{p}{q}, -\tfrac{p}{q}, \dots, p_n\Big)$$

with $(p, q) = 1$ and $(q - p, q) = 1$.

We can now define the residue maps, taken from [10, Lemma 4.6], in the cases we need them.

Proposition 2.1.8. *Let A be a semi-local ring with infinite residue fields. For every monic, irreducible polynomial $\pi \in A[t]$ there exists a homomorphism*

$$\partial_\pi : K_n^{et}(A) \to K_{n-1}^M(A[t]/\pi)$$

such that

$$\partial_\pi(\pi, u_2, \ldots, u_n) = \{\overline{u}_2, \ldots, \overline{u}_n\}.$$

where u_i are rational functions as in 2.1.5 such that each irreducible factor is invertible in $K^M_{n-1}(A[t]/\pi)$. There also exists a homomorphism

$$\partial_\infty : K^{et}_n(A) \to K^M_{n-1}(A)$$

such that

$$\partial_\infty(\frac{1}{t}, p_2(t^{-1}), \ldots, p_n(t^{-1})) = (p_2(0), \ldots, p_n(0))$$

where $p_i \in A[t]$ are such that $p_i(0) \in A^*$.

We can now state the version of the exact sequence 2.1.3 taken from [10, Theorem 4.4].

Theorem 2.1.9. *Let A be a semi-local ring with infinite residue fields. The sequence*

$$0 \to K^M_n(A) \to K^{et}_n(A) \to \oplus_\pi K^M_{n-1}(A[t]/\pi) \to 0$$

is split exact, where the sum is taken over all monic, irreducible $\pi \in A[t]$ such that $\mathrm{Disc}(\pi) \in A^*$.

We are now ready to define the transfer maps for finite etale extensions of semi-local rings with infinite residue fields. To do this we use the following proposition taken from [6] Proposition 18.4.5.

Proposition 2.1.10. *Let A be a local ring, k its residue field and B be a finite A-algebra. Suppose, moreover, that k is infinite, B is infinite, or that B is a local ring. Let n be the rank of $L := B \otimes_A k$ over k. Then B is etale if and only if there exists a monic polynomial $f \in A[t]$ with* $\mathrm{Disc}(f) \in A^*$ *such that*

$$B \cong A[t]/f.$$

Moreover, we have that $\deg(f) = n$.

We now give Kerz's definition of transfer maps for Milnor K-theory [10, Definition 5.5]

Definition 2.1.11. *Let A be a semi-local ring with infinite residue fields. Let*

$$B = A[t]/\pi(t)$$

where π is an irreducible monic polynomial with $Disc(\pi) \in A^*$. We define the transfer maps to be the composition

$$K_n^M(B) \to \bigoplus K_n^M(A[t]/\pi) \xrightarrow{\Psi} K_{n+1}^{et}(A) \xrightarrow{-\partial_\infty} K_n(A)$$

where Ψ is any section of the split exact sequence in 2.1.9 and the sum is taken over all π which are irreducible, monic and $\mathrm{Disc}(\pi) \in A^*$.

Kerz also proves the following compatibility result which we will need.

Proposition 2.1.12. *Let $i : A \to A'$ be a homomorphism of semi-local rings. Let B be as in the previous definition and let $i(\pi) = \prod_j \pi_j$ be a factorization into irreducible polynomials. Let $B'_j = A'[t]/\pi_j$. Then the following diagram commutes*

$$\begin{array}{ccc} K_n^M(B) & \longrightarrow & \bigoplus_j K_n^M(B'_j) \\ \Big\downarrow{\scriptstyle N_{B/A}} & & \Big\downarrow{\scriptstyle \oplus_j N_{B'_j/A'}} \\ K_n^M(A) & \longrightarrow & K_n^M(A') \end{array}$$

The transfer maps for Milnor K-theory satisfy the following properties

1. The map $N^M_{B|A} : K_0^M(B) \to K_0^M(A)$ is just multiplication by $[B : A]$.

2. The map $N^M_{K|k} : K_1^M(B) \to K_1^M(A)$ is gievn by

$$\{b\} \mapsto \{\det T_b\},$$

   where T_b is the A-linear map

$$T_b : B \to B$$
$$x \mapsto bx$$

3. (projection formula) Let $B|A$ be a finite, etale extension $\alpha \in K_n^M(A)$ and $\beta \in K_m^M(B)$ we have that

$$N_{B|A}^M(\{\alpha_B, \beta\}) = \{\alpha, N_{B|A}^M(\beta)\}$$

4. (Composition) Given a tower of etale extensions $C|B|A$, we have that

$$N_{C|A} = N_{B|A} \circ N_{C|B}$$

5. Let $B|A$ be a finite, etale extension and $i_* : K_n^M(A) \to K_n^M(B)$ be the map induced by the inclusion $A \to B$. Then

$$N_{B|A}^M \circ i_*(\alpha) = [B : A]\alpha$$

.

2.2 Improved Milnor K-theory

In this section we present the generalisation of Milnor K-theory to local rings with finite residue field due to Gabber [2] and studied in [11]. We will present this generalisation, more generally, for certain types of abelian sheaves. Let $\mathscr{C}$ be the category of abelian sheaves on the big Zariski site of all schemes. We define $\mathscr{N}\mathscr{C}$ to be the full subcategory of abelian sheaves in $\mathscr{C}$ such that for every finite etale extension of local rings $i\colon A \subset B$ there are a system of transfers

$$[N_{B'/A'} : F(B') \to F(A')]_{A'}$$

for any A' which is local A-algebra such that $B' := B \otimes_A A'$ is also local. We require these transfers to be compatible in the sense that if $A' \to A''$ are both local A-algebras with $B' = B \otimes_A A'$ and $B'' = B \otimes_A A''$ also local then the diagram

$$\begin{array}{ccc} F(B') & \longrightarrow & F(B'') \\ \big\downarrow{\scriptstyle N_{B'|A'}} & & \big\downarrow{\scriptstyle N_{B''|A''}} \\ F(A') & \longrightarrow & F(A'') \end{array}$$

commutes. We also assume that

$$N_{B'|A'} \circ i'_* = [B : A] id_{F(A)},$$

where $i'_*\colon F(A') \to F(B')$ is the map induced by $i'\colon A' \to B'$.

We denote by $\mathscr{N}\mathscr{C}^\infty$ the full subcategory of sheaves which have a system of compatible transfers for all finite, etale extensions $A \subset B$ of local rings such that A has infinite residue field. Clearly every sheaf in $\mathscr{N}\mathscr{C}$ gives a sheaf in $\mathscr{N}\mathscr{C}^\infty$. The following theorem, proved in [11, Theorem 7], proves that every continuous sheaf in $\mathscr{N}\mathscr{C}^\infty$ can be extended uniquely to a sheaf in $\mathscr{N}\mathscr{C}$.

Theorem 2.2.1. *For every continuous $F \in \mathscr{N}\mathscr{C}^\infty$ there exists a continuous $\hat{F} \in \mathscr{N}\mathscr{C}$ together natural transformation $F \to \hat{F}$, such that for any continuous $G \in \mathscr{N}\mathscr{C}$ and natural transformation $F \to G$ there exists a unique natural transformation $\hat{F} \to G$ such that the following diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \hat{F} \arrow[d] \\ & G \end{tikzcd}$$

commutes. Moreover for a local ring A with infinite residue field we have $F(A) = \hat{F}(A)$

We therefore define the improved Milnor K-theory of a local ring A to be $\hat{K}^M_n(A)$. Below we give a more explicit, but equivalent, definition and then we summarize some of the results proved in [11, Proposition 10].

Theorem 2.2.2. *Let A be a commutative ring. We define the subset $S \in A[t_1, \ldots, t_n]$*

to be

$$S := \{\sum_{\underline{i}\in\mathbb{N}^n} a_{\underline{i}} t^{\underline{i}} \in A[t_1,\ldots,t_n] | \langle a_{\underline{i}} | \underline{i} \in \mathbb{N}^n\rangle = A\}$$

The set is multiplicatively closed so we can define the ring of rational functions to be

$$A(t_1,\ldots,t_n) = S^{-1}A[t_1,\ldots,t_n].$$

We have maps $f_1, f_2 : A(t) \to A(t_1,t_2)$, where the map f_i maps t to t_i. Then we have that

$$\hat{K}_n^M(A) = \ker(K_n^M(A(t)) \xrightarrow{K_n^M(f_1)-K_n^M(f_2)} K_n^M(A(t_1,t_2)))$$

Proposition 2.2.3. *Let (A,m) be a local ring. Then:*

1. $\hat{K}_1^M(A) = A^*$.
2. $\hat{K}_*^M(A)$ *has a natural graded commutative ring structure.*
3. *For every $n \geq 0$ there exists a universal natural number M_n such that if $|A/m| > M_n$ the natural homomorphism*

   $$K_n^M(A) \to \hat{K}_n^M(A)$$

   is an isomorphism.
4. *There exists a homomorphism*

   $$K_n^Q(A) \to \hat{K}_n^M(A)$$

   such that the composition

   $$\hat{K}_n^M(A) \to K_n^Q(A) \to \hat{K}_n^M(A)$$

is multiplication by $(n-1)!$ *and the composition*

$$K_n^Q(A) \to \hat{K}_n^M(A) \to K_n^Q(A)$$

is the chern class $c_{n,n}$.

5. *Let* A *be regular and equicharacteristic,* $F = Q(A)$ *and* $X = Spec(A)$. *The Gersten conjecture holds for Milnor K-theory, i.e. the Gersten complex*

$$0 \to \hat{K}_n^M(A) \to K_n^M(F) \to \oplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \to \dots$$

is exact.

2.3 Grayson's motivic cohomology

In this section we present certain non standard cohomology groups studied in [4]. These groups serve as the motivation for our new definition of Milnor K-theory. One of the motivations for the development of motivic cohomology is that these groups should appear as terms in a spectral sequence

$$E_2^{pq} = H^{p-q}(X, \mathbb{Z}(-q)) \implies K_{-p-q}(R)$$

Grayson's approach to this is to study a filtration of the space $K(R)$

$$K(R) = W^0 \leftarrow W^1 \leftarrow \dots$$

due to Goodwillie and Lichtenbaum. We can then define the groups

$$H_G^m(X, \mathbb{Z}(t)) := \pi_{2t-m}(W^t/W^{t+1}).$$

We will first review the construction of W^t.

Given two rings R and S we let $\mathscr{P}(R,S)$ denoted the exact category of R-S-bimodules which as R-modules are finitely generated and projective. We define the

K-theory space

$$K(R, S) \coloneqq K(\mathscr{P}(R, S)).$$

Let $\mathbb{G}_m \coloneqq \operatorname{Spec} \mathbb{Z}[U, U^{-1}]$. Note that the category

$$\mathscr{P}(R, \mathbb{G}_m) \coloneqq \mathscr{P}(R, \mathbb{Z}[U, U^{-1}])$$

is isomorphic to the category whose objects are of the form $[P, \theta]$ where P is a finitely generated, projective module and θ is an automorphism of P. Similarly we can define, for $t \geq 0$,

$$\mathscr{P}(R, \mathbb{G}_m^t) = \mathscr{P}(R, \mathbb{Z}[U_1, U_1^{-1}, \dots, U_t, U_t^{-1}]).$$

We define

$$K_0(R, \mathbb{G}_m^{\wedge t}) \coloneqq K_0(R, \mathbb{G}_m^t)/\langle [P, A_1, \dots, I_P, \dots, A_t] \rangle.$$

We define the R-algebra $R\mathbb{A}^d$ as the algebraic analogue of an n-simplex

$$R\mathbb{A}^d = R[T_0, \dots, T_d]/(T_0 + \cdots + T^d - 1)$$

We can now define the filtration of Goodwillie and Lichtenbaum. We define

$$V^t \coloneqq K(R\mathbb{A}, \mathbb{G}_m^{\wedge t}) = |d \mapsto K(R\mathbb{A}^d, \mathbb{G}_m^{\wedge t})|$$
$$W^t \coloneqq \Omega^{-t} V^t$$

Grayson shows that this filtration satisfies the required properties when R is a regular noetherian ring and

$$W^t / W^{t+1} = \Omega^{-t} |d \mapsto K_0^{\oplus}(R\mathbb{A}^d, \mathbb{G}_m^{\wedge t})|$$

One can show that

$$\pi_n(W^t / W^{t+1}) \cong H^{-n+t}(K_0^{\oplus}(R\mathbb{A}, \mathbb{G}_m^{\wedge t})).$$

In [18] it is shown that these groups are isomorphic to Voevodsky's groups when X is a smooth variety over a field. We also have that Milnor K-theory is isomorphic to certain motivic cohomology groups. So we should have that

$$K_n^M(F) = H_G^n(\mathrm{Spec}(F), \mathbb{Z}(n)) = H^0(K_0^{\oplus}(R\mathbb{A}, \mathbb{G}_m^{\wedge n})).$$

In chapter 3, we shall prove directly that

$$K_n^M(F) = H^0(K_0^{\oplus}(R\mathbb{A}, \mathbb{G}_m^{\wedge n})).$$

in order to prove the main result that $K_n^M(F) \cong \widetilde{K}_n^M(F)$.

2.4 Grayson's presentation for Quillen K-theory

In this section we present Grayson's presentation for Quillen K-theory given in [5], and studied in [8] and [19]. We use this presentation in chapter 4 to construct our version of the comparison homomorphism. We will first give the definition of the category of chain complexes and of binary chain complexes.

Let $\mathscr{N}$ be an exact category. We first look at chain complexes in this category.

Definition 2.4.1. *Let $\mathscr{N}$ be an exact category. A chain complex is a sequence*

$$\ldots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \ldots$$

where $i \in \mathbb{Z}, C_i \in Ob(\mathscr{N})$ and $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We denote a chain complex by $C.$. A map of chain complexes $f.: C. \to D.$ is a collection of morphisms $f_i : C_i \to D_i$ such that the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"] & \ldots \\
\ldots \arrow[r] & D_{i+1} \arrow[r, "d'_{i+1}"] & D_i \arrow[r, "d'_i"] & D_{i-1} \arrow[r, "d'_{i-1}"] & \ldots
\end{tikzcd}$$

commutes. We say that a chain complex $C.$ *is bounded if* $\exists N \in \mathbb{Z}$ *such that* $C_i = 0$ *for all* $i \geq N$ *and* $i \leq -N$*. We define the category* $C\mathscr{N}$*, to be the category whose objects are bounded chain complexes of* $\mathscr{N}$ *and whose morphisms are maps of chain complexes. We say that a sequence of morphisms of chain complexes*

$$C. \xrightarrow{f.} D. \xrightarrow{g.} E_.$$

is exact, if

$$C_i \xrightarrow{f_i} D_i \xrightarrow{g_i} E_i$$

is exact for all i*. This gives* $C\mathscr{N}$ *the structure of an exact category.*

Because $C\mathscr{N}$ is exact we can inductively define

$$C^n\mathscr{N} := C(C^{n-1}\mathscr{N}).$$

We now define what it means for a chain complex to be acyclic.

Definition 2.4.2. *Let* $\mathscr{N}$ *be an exact category and* $C.$ *be an acyclic chain complex. We say that* $C.$ *is acyclic if the sequence factors as*

$$\begin{tikzcd}
 & & Z_i \arrow[dr] & & & & Z_{i-2} \\
\cdots \arrow[r] & C_{i+1} \arrow[ur] \arrow[rr] & & C_i \arrow[rr] \arrow[dr] & & C_{i-1} \arrow[r] \arrow[ur] & \cdots \\
Z_{i+1} \arrow[ur] & & & & Z_{i-1} \arrow[ur] & &
\end{tikzcd}$$

where

$$Z_i \longrightarrow C_i \longrightarrow Z_{i-1}$$

are exact for each i*. We define* $C^q(\mathscr{N})$ *to be the category of bounded, acyclic complexes.*

We now define binary complexes of an exact category. They will be the generators of the presentation of K_n^Q.

Definition 2.4.3. *A binary chain complex, of objects in some exact category $\mathscr{N}$, is a triple $(C., d, d')$ where both $(C., d)$ and $(C., d')$ are chain complexes. We call d the top differential and d' the bottom differential. A morphism between two binary complexes $(C., d, d')$ and $(D., \partial, \partial')$ is a morphism of the chain complexes*

$$f\colon (C., d) \to (D., \partial)$$
$$f\colon (C., d') \to (D., \partial').$$

i.e. f must commute with both the top and bottom differential.

We define $B\mathscr{N}$ to be the category of bounded, binary chain complexes. A sequence of morphisms is exact if it is exact on the underlying $\mathbb{Z}$-graded objects.

As with chain complexes, because $B\mathscr{N}$ is an exact category we can inductively define $B^n\mathscr{N} := B(B^{n-1}\mathscr{N})$.

Given a chain complex, there is a natural way to get a binary chain complex by taking both the top and bottom differentials to be the differential of the chain complex. Conversely, given a binary chain complex we can define two chain complexes, one by using the top differential, the other by using the bottom differential. This gives us three functors

$$\begin{aligned} \Delta : C\mathscr{N} &\to B\mathscr{N} \qquad & (C., d) &\mapsto (C., d, d) \\ \top : B\mathscr{N} &\to C\mathscr{N} \qquad & (C., d, d') &\mapsto (C., d) \\ \bot : B\mathscr{N} &\to C\mathscr{N} \qquad & (C., d, d') &\mapsto (C., d') \end{aligned}$$

We call Δ the diagonal functor, $\top$ the top functor and $\bot$ the bottom functor. We say that a binary complex is acyclic if its image under both $\top$ and $\bot$ is acyclic. We define $B^q\mathscr{N}$ to be the category of bounded, acyclic binary complexes. One can show that $B^q\mathscr{N}$ is an exact category, so we can define $(B^q)^n(\mathscr{N}) := B^q((B^q)^{n-1}\mathscr{N})$.

We can describe objects of $(B^q)^n(\mathscr{N})$ as $\mathbb{Z}^n$-graded collections of objects, together with acyclic differentials $d_1, d'_1, \ldots, d_n, d'_n$

$$d_i, d'_i \colon C_{(x_1,\ldots,x_i,\ldots,x_n)} \to C_{(x_1,\ldots,x_i-1\ldots,x_n)}$$

such that differentials in opposite direction commute. We call an object of $(B^q)^n\mathscr{N}$ an n-dimensional bounded acyclic binary multicomplex.

We can extend the functors above to the setting of multicomplexes. If I have an n-dimensional bounded acyclic binary multicomplex I can get a complex of $(n-1)$-dimensional bounded acyclic binary multicomplexes by forgetting one of the differentials. There are $2n$ ways to do this which gives us functors

$$\top^i : (B^q)^n\mathscr{N} \to C^q(B^q)^{n-1}\mathscr{N}$$
$$\perp^i : (B^q)^n\mathscr{N} \to C^q(B^q)^{n-1}\mathscr{N}.$$

We also have a version of the diagonal functor. Given any chain complex of $(n-1)$-dimensional bounded acyclic binary multicomplexes we can get a n-dimensional bounded acyclic binary multicomplex by duplicating the differential in the i'th direction. This gives us functors

$$\Delta^i : C^q(B^q)^{n-1}\mathscr{N} \to (B^q)^n\mathscr{N}$$

If a binary multicomplex is in the image of Δ_i, for some i, then it is called diagonal. We are now ready to state the main result of [5].

Theorem 2.4.4. *Let $\mathscr{N}$ be an exact category. We have a natural isomorphism*

$$K_n^Q(\mathscr{N}) \cong K_0((B^q)^n\mathscr{N})/Diag$$

where Diag is the subgroup of $K_0((B^q)^n$ generated by the diagonal binary multicomplexes.

Chapter 3

A new presentation for Milnor K-theory of a field

In this chapter, we give a presentation for Milnor K-theory of fields in terms of commuting automorphisms. We begin by giving some motivation and proving some of the basic identities for Milnor K-theory in this new setting. We then go on to show that the groups are isomorphic for a field F.

In section 2.3 we said that Milnor K-theory is isomorphic to Grayon's motivic cohomology groups. This suggests that a presentation of Milnor K-theory for local rings could be

$$\widetilde{K}_n^M(R) = \mathbb{Z}\{[P, A_1, \ldots, A_n]\}/(\text{some relations}).$$

However, the presentation of Grayson's cohomology groups includes a homotopy invariance relation which we should not expect to hold when R is not regular. In 3.5 we prove explicitly that these cohomology groups are isomorphic to Milnor K-theory for F a field. In the proof, we need the natural homomorphism to be well-defined and we need an exact sequence relation to hold. For the map to be well-defined we need the multilinearity and Steinberg relations to hold for rank one elements.

We also need transfers to exist, so we need the relations to hold for any commuting automorphisms of projective modules. This motivates the following definition:

Definition 3.0.1. *Let R be a commutative ring, we define the groups $\widetilde{K}_n^M(R)$ to be*

$$\widetilde{K}_n^M(R) \coloneqq \mathbb{Z}\{[P, A_1, \dots, A_n]\}/(1) - (3)$$

where P is a finitely generated, projective R-module, A_i are automorphisms of P that commute pairwise and relations are (1)-(3) are as follows:

1. $[P_1, A_1, \dots, A_n] + [P_3, C_1, \dots, C_n] = [P_2, B_1, \dots, B_n]$, *if there exists an exact sequence*

$$0 \to P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3 \to 0$$

such that

$$f \circ A_i = B_i \circ f \quad \textit{and} \quad g \circ B_i = C_i \circ g$$

for every i.

2. $[P, A_1, \dots, A_i A_i', \dots, A_n] = [P, A_1, \dots, A_i, \dots, A_n] + [P, A_1, \dots, A_i', \dots, A_n]$.

3. $[P, A_1, \dots, A_n] = 0$, *if $A_i + A_{i+1} = \mathrm{Id}_P$ for some i.*

We refer to (1) as the exact sequence relation, (2) as the multilinear relation and (3) as the Steinberg relation. More generally we define $\widetilde{K}_n^M \mathscr{E}$ for an exact category $\mathscr{E}$:

Definition 3.0.2. *Let $\mathscr{E}$ be an exact category. We define $\mathrm{Aut}^n(\mathscr{E})$ to be the category whose objects are elements of the form $[M, \Theta_1, \dots, \Theta_n]$ such that $M \in \mathrm{ob}(\mathscr{E})$ and Θ_i are automorphisms of M such that $\Theta_i \Theta_j = \Theta_j \Theta_i$ for all i,j. The morphisms between two objects $[M_1, \Theta_1, \dots, \Theta_n]$ and $[M_2, \Phi_1, \dots, \Phi_n]$ are the set of morphisms $f : M_1 \to M_2$ in $\mathscr{E}$ such that $f \circ \Theta_i = \Phi_i \circ f$ for every i. We say that a sequence*

$$[M_1, \Theta_1, \dots, \Theta_n] \xrightarrow{f} [M_2, \Phi_1, \dots, \Phi_n] \xrightarrow{g} [M_3, \Psi_1, \dots, \Psi_n]$$

is exact in $\mathrm{Aut}^n(\mathscr{E})$ *if*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is exact in $\mathscr{E}$.

This makes $\mathrm{Aut}^n(\mathscr{E})$ into an exact category. We can now define the Milnor K-groups of an exact category.

Definition 3.0.3. *We define* $\widetilde{K}_0^M(\mathscr{E})$ *to be the usual Grothendieck group of an exact category i.e.*

$$\widetilde{K}_0^M(\mathscr{E}) \coloneqq \mathbb{Z}\{\mathrm{ob}(\mathscr{E})\}/\textit{short exact sequences.}$$

We then define $\widetilde{K}_i^M(\mathscr{E})$ *for* $i \geq 1$ *as follows:*

$$\widetilde{K}_1^M(\mathscr{E}) \coloneqq \widetilde{K}_0^M(Aut^1(\mathscr{E}))/\langle [M, \Theta_1\Theta_2] = [M, \Theta_1] + [M, \Theta_2] \rangle$$
$$\widetilde{K}_i^M(\mathscr{E}) \coloneqq \widetilde{K}_0^M(Aut^i(\mathscr{E}))/H$$

where H *is the subgroup generated by any element of the two following forms:*

$$[M, \Theta_0, \dots, \Theta_i\Theta_{i+1}, \dots, \Theta_n] - [M, \Theta_0, \dots, \Theta_i, \dots, \Theta_n] - [M, \Theta_0, \dots, \Theta_{i+1}, \dots, \Theta_n]$$
$$[M, \Theta_1, \dots, \Theta_n] \textit{ whenever } \Theta_i + \Theta_{i+1} = \mathrm{Id}_M \textit{ for some } i$$

To simplify notation we define

$$\widetilde{K}_n^M(R) \coloneqq \widetilde{K}_n^M(\mathrm{Proj}_R)$$
$$\widetilde{G}_n^M(R) \coloneqq \widetilde{K}_n^M(\mathrm{Mod}_R)$$

where Proj_R is the category of finitely generated left projective R-modules and Mod_R is the category of finitely generated left R-modules. The purpose of this chapter is to show that the natural map

$$K_n^M(F) \to \widetilde{K}_n^M(F) \tag{3.1}$$
$$\{a_1, \dots, a_n\} \mapsto [F, a_1, \dots, a_n] \tag{3.2}$$

is an isomorphism when F is a field.

3.1 The isomorphism for K_0^M and K_1^M

In this section, we show that these groups agree with Milnor K-theory when $n = 0, 1$. In fact, we show this for any local ring For $n = 0$, this map is defined as

$$K_0^M(R) \to \widetilde{K}_0^M(R)$$
$$m \mapsto [R^m].$$

To show the map is an isomorphism we can define an inverse by mapping a finitely generated free module to its rank. This exact sequence relation holds by the rank-nullity theorem. We now deal with the case $n = 1$.

Proposition 3.1.1. *Let R be any commutative ring such that every matrix over R can be reduced to a diagonal matrix by elementary row and column operations e.g. local rings. Then the map*

$$g_1 : R^* \to \widetilde{K}_1^M(R)$$
$$a \mapsto [R, a]$$

is an isomorphism.

Proof. To show the map is injective we construct an inverse map. Define

$$\phi^{-1} \colon \widetilde{K}_1^M(R) \to R^*$$
$$[R^m, A] \mapsto \det(A).$$

To show the map is well-defined we only need to show that the relations in $\widetilde{K}_1^M(R)$ are satisfied. The multilinearity relation follows from the identity

$$\det(AB) = \det(A)\det(B).$$

The exact sequence relation follows from the identity

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C)$$

To show the map is surjective, we first define $e_{(i,j)}(\lambda)$ to be the matrix

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & \lambda & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

where the λ is in the i'th row and j'th column. We claim this element is trivial in $\widetilde{K}_1^M(F)$. We prove this by induction on the size of the matrix. For a 1×1 matrix the result is trivial. Assume it is true for an $n \times n$ matrix. Take a matrix $e_{(i,j)}(\lambda)$ and any standard basis vector e_k with $k \neq j$. Then we have an exact sequence

$$0 \to [F.e_k, 1] \to [F^n, e_{(i,j)}(\lambda)] \to [F^{n-1}, A] \to 0$$

where A is a matrix of the form $e_{(m,l)}(\lambda')$, where $\lambda' = 0$ or $\lambda' = \lambda$. By linearity $[F.e_k, 1] = 0$ and by induction $[F^{n-1}, A] = 0$.

Therefore, using the linearity relation we have that

$$[R^m, A] = [R^m, Ae_{(i,j)}(\lambda)]$$

for any $A \in \mathrm{GL}_m(R)$ and $\lambda \in R$. So given an element $[R^m, A] \in \widetilde{K}_1^M(R)$ we can use the above relation to row reduce A to a diagonal matrix. From there we can use the exact sequence relation to write

$$[R^m, A] = \sum_{i=1}^{m} [R, a_i]$$

for some $a_i \in R^*$. $\square$

3.2 Transfer maps for $\widetilde{K}_n^M$

In this section, we define the transfer maps for $\widetilde{K}_n^M$. First, we define multiplication on the graded abelian group

$$\widetilde{K}_*^M(R) \coloneqq \bigoplus_{i=0}^{\infty} \widetilde{K}_i^M(R).$$

by the formula

$$[P_1, A_1, \ldots, A_n] \otimes [P_2, B_1, \ldots, B_m] \coloneqq$$
$$[P_1 \otimes P_2, A_1 \otimes \mathrm{Id}_{P_2}, \ldots, A_n \otimes \mathrm{Id}_{P_2}, \mathrm{Id}_{P_1} \otimes B_1, \ldots, \mathrm{Id}_{P_2} \otimes B_m]$$

Proposition 3.2.1. *Given a map $i\colon R \to S$ of commutative we have a well-defined map*

$$i_*\colon \widetilde{K}_n^M(R) \to \widetilde{K}_n^M(S)$$
$$[P, \Theta_1, \ldots, \Theta_n] \mapsto [P \oplus_R S, \Theta_1 \otimes \mathrm{Id}_S, \ldots, \Theta_n \otimes \mathrm{Id}_S].$$

Definition/Proposition 3.2.2. *Let $R \to S$ be a finite map of commutative rings such that S is projective as an R-module. We define the transfer maps to be*

$$\widetilde{N}_{S/R}^M\colon \widetilde{K}_n^M(S) \to \widetilde{K}_n^M(R)$$
$$[M, \theta_1, \ldots, \theta_n] \mapsto [M, \theta_1, \ldots, \theta_n].$$

These maps are well-defined and satisfy the following:

1. *If R and S are local rings, then the map $\widetilde{N}_{S/R}^M\colon \widetilde{K}_0^M(S) \to \widetilde{K}_0^M(R)$ is just multiplication by $[S : R]$.*

2. *If R and S are local rings, the map $\widetilde{N}_{S/R}^M\colon \widetilde{K}_1^M(S) \to \widetilde{K}_1^M(R)$, is given by*

$$[V, \theta] \mapsto [R, \det_R(\theta)]$$

where $\det_R(\theta)$ is the determinant of θ as an R-linear map.

3. *(Composition) Let $R \to S \to T$ be a composition of finite maps such that S is a projectve R-module and T is a projective S-module, then*

$$\widetilde{N}^M_{T/R} = \widetilde{N}^M_{S/R} \circ \widetilde{N}^M_{T/S}$$

4. *(Projection formula) Let $i_*\colon \widetilde{K}^M_n(R) \to \widetilde{K}^M_n(S)$ be the map induced by inclusion, $[V, \Theta_1, \dots, \Theta_n] \in \widetilde{K}^M_n(R)$ and $[W, \Theta_{n+1}, \dots, \Theta_{n+m}] \in \widetilde{K}^M_m(S)$. Then*

$$\widetilde{N}^M_{S/R}(i_*([V, \Theta_1, \dots, \Theta_n]) \otimes_S [W, \Theta_{n+1}, \dots, \Theta_{n+m}]) = [V, \Theta_1, \dots, \Theta_n] \otimes_R \widetilde{N}^M_{S/R}([W, \Theta_{n+1}, \dots, \Theta_{n+m}])$$

5. *Let $R \to S$ be a map of rings such that S is finite, free R-module. Let $i_*\colon \widetilde{K}^M_n(R) \to \widetilde{K}^M_n(S)$, be the map induced by inclusion $i\colon R \to S$. Then*

$$\widetilde{N}^M_{S/R} \circ i_* = [S : R] \times \mathrm{Id}$$

Proof. To prove (1) not that $\widetilde{K}^M_n(R) = \widetilde{K}^M_n(S) = \mathbb{Z}$ because both R and S are local rings. Any homomorphism from $\mathbb{Z}$ to itself must be multiplication by some constant. To find this constant we need only to find the image of $[S]$. Then $S \cong R^{[S:R]}$ as an R-module so the map is just multiplication by $[S : R]$.

The proof of (2) follows similarly to the proof of proposition 3.1.1

(3) is trivially true.

To prove (4) note that

$$\begin{aligned}
&\widetilde{N}^M_{S/R}(i_*([V, \Theta_1, \dots, \Theta_n]) \otimes_S [W, \Theta_{n+1}, \dots, \Theta_{n+m}]) = \\
&\widetilde{N}^M_{S/R}([V \otimes_R S, \Theta_1 \otimes \mathrm{Id}_S, \dots, \Theta_n \otimes \mathrm{Id}_S] \otimes_S [W, \Theta_{n+1}, \dots, \Theta_{n+m}]) = \\
&\widetilde{N}^M_{S/R}([V \otimes_R W, \Theta_1 \otimes \mathrm{Id}_W, \dots, \Theta_n \otimes \mathrm{Id}_W, \mathrm{Id}_V \otimes \Theta_{n+1}, \dots, \mathrm{Id}_V \otimes \Theta_{n+m}]) = \\
&[V, \Theta_1, \dots, \Theta_n] \otimes_R \widetilde{N}^M_{S/R}([W, \Theta_{n+1}, \dots, \Theta_{n+m}])
\end{aligned}$$

(5) is a special case of (4) with $m = 0$, using the fact that the transfer on $\widetilde{K}^M_0$ is just multiplication by the degree of the extension.

□

We also have transfers of the form

$$\widetilde{N}_{S/R}^M\colon \widetilde{G}_n^M(S) \to \widetilde{G}_n^M(R)$$

for finite, flat maps $R \to S$.

3.3 Relations in $\widetilde{K}_n^M(R)$

In this section, we prove some of the standard identities for Milnor K-theory for $\widetilde{K}_n^M$. Usually the proofs of these theorems only hold for rings with many units, however in these new groups we can get around this by using matrices. This is one of the benefits of having more general transfers for $\widetilde{K}_n^M(R)$. We now prove the following useful identity which is used to prove graded commutativity as well as the reciprocity law.

Proposition 3.3.1. *Let $\mathscr{E}$ be an exact category. Let M be an object of $\mathscr{E}$ and Θ_i be automorphisms of M. Then*

$$[M, \Theta_1, \dots, \Theta_n] = 0 \in \widetilde{K}_n^M(\mathscr{E})$$

if $\Theta_i + \Theta_{i+1} = 0$ for some i.

Proof. We begin by proving the theorem in the case when $1 - \Theta_i$ is invertible. For this we use the identity

$$-\Theta_i = \frac{1 - \Theta_i}{1 - \Theta_i^{-1}}.$$

Using this we can see that

$$\begin{aligned}
[\Theta_1, \dots, \Theta_i, -\Theta_i, \dots, \Theta_n] &= [\Theta_1, \dots, \Theta_i, \frac{1 - \Theta_i}{1 - \Theta_i^{-1}}, \dots, \Theta_n] \\
&= [\Theta_1, \dots, \Theta_i, 1 - \Theta_i, \dots, \Theta_n] \\
&- [\Theta_1, \dots, \Theta_i, 1 - \Theta_i^{-1}, \dots, \Theta_n] \\
&= 0
\end{aligned}$$

We now prove the identity when $1-\Theta_i$ is not invertible. To do this we prove that $3[M, \Theta_1, \ldots, \Theta_n] = 0$ and $4[M, \Theta_1, \ldots, \Theta_n] = 0$.

Let $\Phi : M^3 \to M^3$ be the automorphism given by the matrix

$$\begin{pmatrix} 0 & 0 & \mathrm{Id}_M \\ \mathrm{Id}_M & 0 & -\mathrm{Id}_M - \Theta_i \\ 0 & \mathrm{Id}_M & \Theta_i \end{pmatrix}.$$

Consider the element

$$[M^3, \Theta_1^{\oplus 3} \times \mathrm{Id}_{M^3}, \ldots, \Theta_i^{\oplus 3} \times \Phi, -\Theta_i^{\oplus 3} \times \Phi, \ldots, \Theta_n^{\oplus 3} \times \mathrm{Id}_{M^3}].$$

We claim that this element is 0 in $\widetilde{K}_n^M(\mathscr{E})$. To show this we only need to show that $\mathrm{Id}_{M^3} - \Theta_i \times \Phi$ is invertible which is easy to show. So the element above is trivial and using multilinearity we obtain

$$\begin{aligned} 0 = {} & [M^3, \Theta_1 \times \mathrm{Id}_{M^3}, \ldots, \Theta_i \times \mathrm{Id}_{M^3}, -\Theta_i \times \mathrm{Id}_{M^3}, \ldots, \Theta_n \times \mathrm{Id}_{M^3}] \\ & + [M^3, \Theta_1 \times \mathrm{Id}_{M^3}, \ldots, \Theta_i \times \mathrm{Id}_{M^3}, \Phi, \ldots, \Theta_n \times \mathrm{Id}_{M^3}] \\ & + [M^3, \Theta_1 \times \mathrm{Id}_{M^3}, \ldots, \Phi, \Theta_i \times \mathrm{Id}_{M^3}, \ldots, \Theta_n \times \mathrm{Id}_{M^3}] \\ & + [M^3, \Theta_1 \times \mathrm{Id}_{M^3}, \ldots, \Phi, -\Phi, \ldots, \Theta_n \times \mathrm{Id}_{M^3}]. \end{aligned}$$

We claim that the final 3 elements in this sum are 0. The last element is 0 because $1-\Phi$ is invertible. The other two are 0 because we can use elementary row and column operations to reduce Φ to the identity matrix. So we have proved that

$$0 = [M^3, \Theta_1 \times \mathrm{Id}_{M^3}, \ldots, \Theta_i \times \mathrm{Id}_{M^3}, -\Theta_i \times \mathrm{Id}_{M^3}, \ldots, \Theta_n \times \mathrm{Id}_{M^3}],$$

and using the exact sequence relation we get

$$3[M, \Theta_1, \ldots, \Theta_i, -\Theta_i, \ldots, \Theta_n] = 0$$

as required.

The proof that $4[M, \Theta_1, \ldots, \Theta_i, -\Theta_i, \ldots, \Theta_n] = 0$ is similar taking

$$\Phi : M^4 \to M^4$$

to be the morphism given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -\mathrm{Id}_M \\ \mathrm{Id}_M & 0 & 0 & \mathrm{Id}_M + \Theta_i \\ 0 & \mathrm{Id}_M & 0 & -\Theta_i \\ 0 & 0 & \mathrm{Id}_M & 0 \end{pmatrix}.$$

□

We have a few corollaries of this result. It gives us graded-commutativity of the multiplication defined on $\widetilde{K}_*^M(R)$.

Corollary 3.3.2. *Let $\mathscr{E}$ be an exact category. Then the identity*

$$[M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n] = -[M, \Theta_1, \ldots, \Theta_{i+1}, \Theta_i, \ldots, \Theta_n]$$

holds in $\widetilde{K}_n^M(\mathscr{E})$, for any $[M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n] \in \widetilde{K}_n^M(\mathscr{E})$. In particular if R is a commutative ring we have that

$$[P_1, A_1, \ldots, A_n] \otimes [P_2, B_1, \ldots, B_m] = (-1)^{mn}[P_2, B_1, \ldots, B_m] \otimes [P_1, A_1, \ldots, A_n]$$

in $\widetilde{K}_*^M(R)$.

Proof. The proof is the same as the proof that is given usually for Milnor K-theory.

$$\begin{aligned} 0 &= [M, \Theta_1, \ldots, \Theta_i\Theta_{i+1}, -\Theta_i\Theta_{i+1}, \ldots, \Theta_n] \\ &= [M, \Theta_1, \ldots, \Theta_i, -\Theta_i, \ldots, \Theta_n] + [M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n] \\ &+ [M, \Theta_1, \ldots, \Theta_{i+1}, -\Theta_{i+1}, \ldots, \Theta_n] + [M, \Theta_1, \ldots, \Theta_{i+1}, \Theta_i, \ldots, \Theta_n] \\ &= [M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n] + [M, \Theta_1, \ldots, \Theta_{i+1}, -\Theta_{i+1}, \ldots, \Theta_n] \end{aligned}$$

as required. □

A corollary of 3.3.2 is that $[M, \Theta_1, \ldots, \Theta_n] = 0 \in \widetilde{K}_n^M(\mathscr{E})$ if $\Theta_i + \Theta_j = 1$ or $\Theta_i = -\Theta_j$ for any $i \neq j$. Before moving on we need one final identity:

Corollary 3.3.3. *Let $\mathscr{E}$ be an exact category, then the identity*

$$[M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n] = [M, \Theta_1, \ldots, -\frac{\Theta_i}{\Theta_{i+1}}, \Theta_i + \Theta_{i+1}, \ldots, \Theta_n]$$

holds in $\widetilde{K}_n^M(\mathscr{E})$, whenever $\Theta_i + \Theta_{i+1}$ is invertible.

Proof. Using multilinearity we have that

$$\begin{aligned}
&[M, \Theta_1, \ldots, -\frac{\Theta_i}{\Theta_{i+1}}, \Theta_i + \Theta_{i+1}, \ldots, \Theta_n] \\
&\quad = [M, \Theta_1, \ldots, -\frac{\Theta_i}{\Theta_{i+1}}, \frac{\Theta_i}{\Theta_{i+1}} + 1 \ldots, \Theta_n] + [M, \Theta_1, \ldots, -\frac{\Theta_i}{\Theta_{i+1}}, \Theta_{i+1}, \ldots, \Theta_n]
\end{aligned}$$

The first element in the sum is trivial by the Steinberg relation. Then using multilinearity on the second term we see that the sum is equal to

$$-[M, \Theta_1, \ldots, -\Theta_{i+1}, \Theta_{i+1}, \ldots, \Theta_n] + [M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n]$$

The first term is trivial by 3.3.1 and so the sum is equal to

$$[M, \Theta_1, \ldots, \Theta_i, \Theta_{i+1}, \ldots, \Theta_n]$$

as required. □

3.4 Surjectivity of the map

In this section, we will show that the map (3.1)

$$K_n^M(F) \to \widetilde{K}_n^M(F)$$

is surjective when F is a field. To do this we will first show that the groups $\widetilde{K}_n^M(F)$ are generated by images of 1-dimensional elements of transfer maps. To finish the proof we then show that the transfer maps for $K_n^M(F)$ are compatible with the maps for $\widetilde{K}_n^M(F)$. We do the first part, more generally, for the groups $K_0(F, \mathbb{G}_m^n)$ defined in chapter 2.3 because it will be more useful later to have this result.

For any element $[F^m, A_1, \ldots, A_n] \in K_0(F, \mathbb{G}_m^n)$ we define a $F[t_1^{\pm}, \ldots, t_n^{\pm}]$-module F^m where multiplication by t_i is just multiplication by A_i. Note that this is well-defined because the matrices commute and are invertible. We call an element $[F^m, A_1, \ldots, A_n]$ simple if its associated $F[t_1^{\pm}, \ldots, t_n^{\pm}]$-module is simple. We claim that the simple elements generate the group $K_0(F, \mathbb{G}_m^n)$.

Lemma 3.4.1. *Every element $[F^m, A_1, \ldots, A_n]$ can be written as a sum of simple elements in $K_0(F, \mathbb{G}_m^n)$.*

Proof. Assume not, then there exists an element $[F^m, A_1, \ldots, A_n]$, with m minimal, which cannot be written as a sum of simple elements. $[F^m, A_1, \ldots, A_n]$ cannot be simple itself so there must be a subspace $V \subset F^m$ such that A_i restricts to an isomorphism on V. Therefore we have an exact sequence

$$0 \to [V, A_1, \ldots, A_n] \to [F^m, A_1, \ldots, A_n] \to [F^m/V, A_1, \ldots, A_n] \to 0.$$

Using the exact sequence relation we can write $[F^m, A_1, \ldots, A_n]$ as a sum of two elements each of which have rank less than m. So then each of these elements must be a sum of simple elements, hence so is $[F^m, A_1, \ldots, A_n]$. □

We will now show that the simple elements are images of some rank 1 element under some transfer map. Take any simple element $[F^m, A_1, \ldots, A_n]$. Then, as explained above, F^m is naturally a simple $F[t_1^{\pm}, \ldots, t_n^{\pm}]$-module. The simple $F[t_1^{\pm}, \ldots, t_n^{\pm}]$-modules are those of the form $F[t_1^{\pm}, \ldots, t_n^{\pm}]/m$ where m is a maximal ideal. So there is an $F[t_1^{\pm}, \ldots, t_n^{\pm}]$-module isomorphism where multiplication on F^m by A_i corresponds to multiplication on $F[t_1^{\pm}, \ldots, t_n^{\pm}]/m$ by t_i. We therefore have the following:

Proposition 3.4.2. *Let* $[F^m, A_1, \ldots, A_n] \in K_0(F, \mathbb{G}_m^n)$ *be simple and let*

$$F[t_1^{\pm}, \ldots, t_n^{\pm}]/m$$

be a finite extension of F *such that*

$$F[t_1^{\pm}, \ldots, t_n^{\pm}]/m = F^m$$

as a $F[t_1^{\pm}, \ldots, t_n^{\pm}]$*-module. Then*

$$N_{F[t_1,\ldots,t_n]/m|F}([F[t_1^{\pm}, \ldots, t_n^{\pm}]/m, t_1, \ldots, t_n]) = [F^m, A_1, \ldots, A_n]$$

Hence $K_0(F, \mathbb{G}_m^n)$ *is generated by images of rank 1 elements under transfer maps.*

We now give another proof of the fact that $K_0(F, \mathbb{G}_m^n)$ is generated by the images of transfer maps in the hope that one of these methods may generalise to the case of local rings considered later.

Take an element $[V, A_1, \ldots, A_n] \in K_0(F, \mathbb{G}_m^n)$. Take a polynomial, of minimal degree, $p(t) \in F[t]$ such that the nullity of $p(A_1)$ is greater than 0. That is, there exists a non-zero vector v such that $p(A_1)v = 0$. We claim that such a polynomial $p(t)$ is irreducible. Assume not then let

$$p(t) = p_1(t)p_2(t).$$

Then we must have that both $p_1(A_1)$ and $p_2(A_1)$ have nullity 0 by minimality. But then $p_1(A_1)$ must annihilate $p_2(A_1)v$ which gives a contradiction. So $p(t)$ must be irreducible. We define an F-subspace $V_{p(t)}$ to be the set annihilated by $p(A_1)$ i.e.

$$V_{p(t)} = \{v \in V \mid p(A_1)v = 0\}.$$

We claim that A_i restrict to automorphisms on $V_{p(t)}$. To show this, we only need to show that the map

$$A_i \colon V_{p(t)} \to V_{p(t)}$$

is well-defined, i.e. that the image of the map is is contained in $V_{p(t)}$. This follows from the commutativity of A_1 and A_i. Hence we have an exact sequence

$$0 \to [V_{p(t)}, A_1|_{V_{p(t)}}, \dots, A_n|_{V_{p(t)}}] \to [V, A_1, \dots, A_n] \to [W, B_1, \dots, B_n] \to 0.$$

Using the exact sequence relation and continuing inductively on W gives that $K_0(F, \mathbb{G}_m)$ is generated by elements of the form $[V_{p(t)}, A_1, \dots, A_n]$ where every vector $v \in V_{p(t)}$ is annihilated by $p(A_1)$. We now use a change of basis to put A_1 into rational canonical form which converts A_1 into a block diagonal matrix of the form

$$\begin{pmatrix} C_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_l \end{pmatrix} \tag{3.3}$$

where C_i is of the form

$$\begin{pmatrix} 0 & \dots & 0 & a_0^i \\ 1 & \dots & 0 & a_1^i \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_{m_i-1}^i \end{pmatrix}. \tag{3.4}$$

If any of these square matrices are of size less than $\deg(p) \times \deg(p)$ then there would exist a vector annihilated by a polynomial of smaller degree than p. This is impossible by the construction. Alternatively, if any of these blocks are larger than $\deg(p) \times \deg(p)$, then $p(C_i)e_1 \neq 0$ where e_1 is the first standard basis vector. So each matrix is square of size $\deg(p) \times \deg(p)$. We know that the characteristic polynomial of each matrix must be $p(t)$, otherwise $C_{C_i}(A_1) - p(A_1)$ would annihilate a non-zero vector. It is known that the characteristic polynomial of matrices of the form 3.4 is

$$C_{C_i}(t) = t^{m_i} - a_{m^i-1}^i t^{m_i-1} - \dots - a_0^i.$$

This shows that $C_i = C_j$ for every i and j. Furthermore if

$$p(t) = t^m - b_{m-1}t^{m-1} - \dots - b_0$$

then

$$C_i = \begin{pmatrix} 0 & \dots & 0 & b_0 \\ 1 & \dots & 0 & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & b_{m-1} \end{pmatrix}.$$

We have changed A_1 into an element which is an image of a transfer. We now look at what this change of basis does to A_i. One useful property of matrices of the form 3.4 is the following:

Lemma 3.4.3. *Let R be a commutative ring. Let $A \in \mathrm{GL}_n(R)$ be a companion matrix of the form* 3.4 *above. If a matrix B commutes with A then $B = b_n A^n + \ldots + b_0$ for some $b_i \in R$.*

To prove this we use the following:

Lemma 3.4.4. *Let R be a commutative ring and $A \in \mathrm{GL}_n(R)$ be a matrix of the form* 3.4 *above. If A commutes with a matrix of the form*

$$B = \begin{pmatrix} 0 & x_{(1,2)} & \dots & x_{(1,n-1)} & x_{(1,n)} \\ 0 & x_{(2,2)} & \dots & x_{(2,n-1)} & x_{(2,n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{(n,2)} & \dots & x_{(n,n-1)} & x_{(n,n)} \end{pmatrix}, \tag{3.5}$$

then $x_{(i,j)} = 0$, for every i, j

Proof. Denote the i'th column of the matrix B above by $\underline{c}_i$. If we multiply B by A on both sides, then using commutativity we obtain

$$\begin{pmatrix} 0 & A\underline{c}_2 & \dots & A\underline{c}_n \end{pmatrix} = \begin{pmatrix} \underline{c}_2 & \dots & \underline{c}_n & a_1\underline{c}_2 + \cdots + a_{n-1}\underline{c}_n \end{pmatrix}.$$

In particular, we obtain

$$\underline{c}_2 = 0 \text{ and } A\underline{c}_i = \underline{c}_{i+1}.$$

So by induction, we obtain that $\underline{c}_i = 0$, for all i. $\square$

We can now prove lemma 3.4.3.

Proof. Take an arbitrary matrix

$$B = \begin{pmatrix} x_{(1,1)} & x_{(1,2)} & \dots & x_{(1,n-1)} & x_{(1,n)} \\ x_{(2,1)} & x_{(2,2)} & \dots & x_{(2,n-1)} & x_{(2,n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(n,1)} & x_{(n,2)} & \dots & x_{(n,n-1)} & x_{(n,n)} \end{pmatrix}$$

which commutes with A. Consider the matrix

$$B - x_{(1,1)}I_n - \cdots - x_{(i,1)}A^{i-1} - \cdots - x_{(n,1)}A^{n-1}.$$

We claim this matrix satisfies the conditions of 3.4.4. This is easy to see based on the fact that the first column of A^i is $\underline{e}_{i+1}$, where $\underline{e}_j$ is the j'th standard unit basis vector. Therefore, the sum above must be equal to 0 and so

$$B = x_{(1,1)}I_n + \cdots + x_{(i,1)}A^{i-1} + \cdots + x_{(n,1)}A^{n-1}$$

□

Using Lemma 3.4.3 and the discussion above we have the following:

Lemma 3.4.5. *$K_0(F, \mathbb{G}_m^n)$ is generated by elements of the form*

$$\left[\begin{pmatrix} A & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A \end{pmatrix}, \quad B_2(A) \quad \dots \quad , B_n(A) \right]$$

where A is a companion matrix with irreducible characteristic polynomial and $B_i(t)$ are matrices of the form

$$\begin{pmatrix} p^i_{1,1}(t) & \dots & p^i_{m,1}(t) \\ \vdots & \ddots & \vdots \\ p^i_{m,1}(t) & \dots & p^i_{m,m}(t) \end{pmatrix}$$

where $p^k_{i,j} \in F[x]$.

The symbol in the above lemma is the image under some transfer map. It is equal to

$$N_{F[t]/c_A(t)}\left[\begin{pmatrix} t & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t \end{pmatrix}, \quad B_2(t), \quad \dots \quad, B_n(t)\right]$$

where $c_A(t)$ is the characteristic polynomial of A. Repeating this process with $B_2(t)$ in place of A_1 and continuing similarly gives that $K_0(F, \mathbb{G}_m^n)$ is generated by images of rank one elements of some transfer.

3.4.1 Compatibility of the transfers

The aim of this section is to show that the transfer maps commute. The proof we give here is based on the methods in [1] which allows us to reduce to proving the proposition for field extensions K/k with $[K:k] = p$ for some prime p, where k is a field which has no field extensions with degree coprime to p. It is simple to prove the proposition in this case however reducing to this case is difficult. We give a different proof later which works for semi-local rings and is more elementary.

Proposition 3.4.6. *For any finite extension $K|k$, the diagram*

$$\begin{array}{ccc} K_n^M(K) & \longrightarrow & \widetilde{K}_n^M(K) \\ \big\downarrow{\scriptstyle N_{K|k}^M} & & \big\downarrow{\scriptstyle \widetilde{N}_{K|k}^M} \\ K_n^M(k) & \longrightarrow & \widetilde{K}_n^M(k) \end{array}$$

commutes.

We first prove this proposition for the field extensions we mentioned above. We need the following lemma to do this which we take from [3, Lemma 7.2.9]:

Lemma 3.4.7. *Let $K = k(a)$ be a field extension obtained by adjoining an element a of degree d to k. Then $K_*^M(K)$ is generated as a left $K_*^M(k)$-module by elements* *of the form*

$$\{\pi_1(a), \ldots, \pi_m(a)\}$$

where π_i *are monic irreducible polynomials in* $k[t]$ *satisfying deg*$(\pi_1) < \cdots < \deg(\pi_m) \leq d-1$

This allows us to prove Proposition 3.4.6 for these certain field extensions.

Lemma 3.4.8. *Proposition 3.4.6 is true if* $[K : k] = p$.

Proof. To prove this, we use the properties of the tranfer map and lemma 3.4.7. Take an arbitrary generator given in Lemma 3.4.7. There are no irreducible polynomials of degree less than p which have degree greater than 1. So we know that $K_n^M(K)$ is generated by elements of the form

$$\{t + a_1, a_2, \ldots, a_n\}.$$

We know the transfer maps commute when $n = 1$ because the transfer for Milnor K-theory is given explicitly as

$$\{a\} \mapsto \{\det(T_a)\}$$

where T_a is the k-linear map

$$T_a : K \to K$$
$$b \mapsto b \times a$$

[3, Proposition 7.2.5]. Then using the projection formula and the fact that the transfer maps commute when $n = 1$ we are done. □

The following proposition allows us, by induction, to remove the assumption that $[K : k] = p$ in lemma 3.4.8. A proof can be found in [3, Lemma 7.3.7].

Proposition 3.4.9. *Let k be a field such that every finite extension of k has degree p^n for some prime p and let K/k be a proper finite extension. Then there exists a subfield $k \subset K_1 \subset K$ such that K_1/k is a normal extension of degree p.*

Using the composition of transfer maps we can deduce that proposition 3.4.6 holds whenever k is a field such that every finite field extension of k has order p^n for some prime p.

We now begin by trying to reduce the general case to this case. We first need the following nice property of the transfer map for $\widetilde{K}^M_n$.

Proposition 3.4.10. *Let $F[t]/p(t)|F$ be a simple field extension and L/F a field extension. Let*

$$p(t) = p_1(t) \dots p_l(t)$$

be the irreducible factorization of $p(t)$ in $L[t]$. Then diagram

$$\begin{array}{ccc} \widetilde{K}^M_n(F[t]/p(t)) & \xrightarrow{i_{F[t]/p(t)|L[t]/p_i(t)}} & \bigoplus \widetilde{K}^M_n(L[t]/p_i(t)) \\ \Big\downarrow{\scriptstyle \widetilde{N}^M_{F[t]/p(t)|F}} & & \Big\downarrow{\scriptstyle \sum \widetilde{N}^M_{L[t]/p_i(t)|L}} \\ \widetilde{K}^M_n(F) & \xrightarrow{i_{F|L}} & \widetilde{K}^M_n(L) \end{array}$$

commutes.

Proof. We first compute $i_{F|L} \circ \widetilde{N}^M_{F[t]/p(t)|F}$.

$$i_{F|L} \circ \widetilde{N}^M_{F[t]/p(t)}([F[t]/p(t), f_1(t), \dots, f_n(t)]) = [L^m, f_1(A), \dots, f_n(A)]$$

where A is the companion matrix whose characteristic polynomial is $p(t)$. We claim that we can choose an invertible matrix P such that PAP^{-1} is a block upper triangular matrix, which has companion matrices on the diagonal. Furthermore, we can choose P such that the characteristic polynomials of the matrices on the diagonal are precisely the irreducible factors of $p(t)$ in $L[t]$. The proof of this was essentially done in the second proof of the fact that $K_0(F, \mathbb{G}_m^n)$ is generated by images of transfers. Now, using the exact sequence relation to get rid of the elements above the diagonal, we can see that the image of the first composition is

$$\sum_i [L^{\deg p_i}, f_1(A_i), \ldots, f_n(A_i)],$$

where A_i is the companion matrix whose characteristic polynomial is p_i.

Next we compute the other composition. This is a similar calculation and so we get that the image under the other composition is

$$\sum_i [L^{\deg(p_i)}, f_1(A_i), \ldots, f_n(A_i)]$$

as required. $\square$

An analogous result to the above holds for Milnor K-theory a proof of which can be found in [3].

Remark 3.4.11. *We only proved* 3.4.10 *for rank* 1 *elements. This is enough to prove that the map g_* is surjective, which will give that the diagram commutes for all elements in $\widetilde{K}_n^M(F)$.*

We now begin to show Prop 3.4.6 for the general case. We define Δ to be the subgroup

$$\Delta := \langle (g_F \circ N^M_{F[t]/p(t)|F} - \widetilde{N}^M_{F[t]/p(t)|F} \circ g_{F[t]/p(t)})(\{a_1, \ldots, a_n\}) : \\ \{a_1, \ldots, a_n\} \in K_n^M(F[t]/p(t)) \rangle$$

where g_F is the map

$$K_n^M(F) \to \widetilde{K}_n^M(F)$$

Our aim is to show this group is trivial. We first show that it is a torsion group.

Proposition 3.4.12. *Let F be a field and L be an algebraic extension of F. The kernel of the natural map*

$$\widetilde{K}_n^M(F) \to \widetilde{K}_n^M(L)$$

is a torsion group.

Proof. Take an arbitrary element $[F^m, A_1, \ldots, A_n]$ in the kernel. If L is finite, the result follows from the projection formula for the transfer map. If L is not finite, it is true that there exists a finite field extension $F'|F$ such that

$$[F^m, A_1, \ldots, A_n] = 0 \in \widetilde{K}_n^M(F').$$

This is true because only finitely many relations are needed to reduce $[F^m, A_1, \ldots, A_n]$ to 0 in $\widetilde{K}_n^M(L)$. □

So to show Δ is a torsion group it suffices to show that Δ is in the kernel of the map $\widetilde{K}_n^M(F) \to \widetilde{K}_n^M(\overline{F})$.

Proposition 3.4.13. *For any field F with algebraic closure $\overline{F}$ we have that $i_{\overline{F}|F}(\Delta) = 0$.*

Proof. Take an arbitrary element

$$(g_F \circ N^M_{F[t]/p(t)|F} - \widetilde{N}^M_{F[t]/p(t)|F} \circ g_{F[t]/p(t)})(\{f_1(t), \ldots, f_n(t)\})$$

in Δ. It is simple to see that

$$i_{F|\overline{F}} \circ g_F \circ N^M_{F[t]/p(t)|F}\{f_1(t), \ldots, f_n(t)\} = g_{\overline{F}} \circ i_{F|\overline{F}} \circ N^M_{F[t]/p(t)|F}\{f_1(t), \ldots, f_n(t)\}.$$

By [3, Corollary 7.3.11], we have a commutative diagram

$$\begin{tikzcd}
K_n^M(F[t]/p(t)) \arrow[r, "\oplus i_{F[t]/p(t)|\overline{F}[t]/t-a_i}"] \arrow[d, "N^M_{F[t]/p(t)|F}"] & \bigoplus_{i=1}^k K_n^M(\overline{F}[t]/(t-a_i)) \arrow[d, "\sum N^M_{\overline{F}[t]/t-a_i|\overline{F}}"] \\
K_n^M(F) \arrow[r, "i_{F|\overline{F}}"] & K_n^M(\overline{F})
\end{tikzcd}$$

where a_i are the roots of p.The transfer $N^M_{\overline{F}[t]/t-a_i|\overline{F}}$, is just evaluation at a_i so we have that

$$\begin{aligned} g_{\overline{F}} \circ i^M_{F|\overline{F}} \circ N^M_{F[t]/p(t)|F}\{f_1(t), \dots, f_n(t)\} &= g_{\overline{F}} \circ \sum N^M_{F[t]/p(t)|\overline{F}[t]/t-a_i}\{f_1(t), \dots, f_n(t)\} \\ &= \sum_i [\overline{F}, f_1(a_i), \dots, f_n(a_i)]. \end{aligned}$$

Where the sum ranges over the roots of p in $\overline{F}$. A similar calculation shows that $i_{\overline{F}|F} \circ \widetilde{N}^M_{F[t]/p(t)|F} \circ g_{F[t]/p(t)} = \sum_i [\overline{F}, f_1(a_i), \dots, f_n(a_i)]$. □

So we have shown that Δ is a torsion group. To continue we need the following proposition:

Proposition 3.4.14. *Let F be a field, p be a prime and let G_n be $\widetilde{K}^M_n$ or K^M_n. Then there exists an algebraic extension L of F such that every finite extension of L has order a power of p and such that the map $G_n(F)_{(p)} \to G_n(L)$ is injective.*

Proof. First we set some notation. We define an ordinal to be an equivalence class of totally ordered set. For any ordinal α and any $x \in \alpha$ we define $x + 1$ to be the smallest element in the set

$$\{y \in \alpha : y > x\}.$$

Let Ω be the set of fields contained in $\overline{F}$ which contain F. The cardinality of Ω is less than the cardinality of $\overline{F}$ so it is a set. We put a partial order on Ω by saying $L \leq K$ if L is a subfield of K. We define a tower of field extensions to be a function from an ordinal to Ω which strictly preserves the ordering and preserves all limits when they exist. We define a p-tower to be a tower $f\colon \alpha \to \Omega$, such that, for every $x \in \alpha$, $f(x+1)/f(x)$ is a finite extension with degree prime to p. We define the set $\mathscr{T}_p$ to be the set of all p-towers. We put a partial order on p-towers by saying that $f \leq g$, where

$$f\colon \alpha \to \Omega \quad g\colon \beta \to \Omega,$$

if there is an injective map of sets

$$i\colon \alpha \to \beta,$$

such that $i(0) = 0$, $i(x+1) = i(x)+1$ and i preserves limits, such that $f(x) = g(i(x))$. We now use Zorn's lemma. Take any non-empty chain

$$\mathscr{C} = \{C_j\colon \alpha_j \to \Omega\colon j \in J\} \subset \mathscr{T}_p.$$

We can take an upper bound by taking the disjoint union of α_j and identifying two points if one is the image of the other under the inclusion map.

So by Zorn's Lemma there exists a maximal element $f\colon \alpha \to \Omega$. We define L to be

$$L \coloneqq \bigcup_{x\in\alpha} f(x).$$

because f is maximal it must be true that $L = f(y)$ for some $y \in \alpha$. We also have that L must have no non-trivial, finite field extensions of degree prime to p, else f would not be maximal.

To complete the proof we only need to show that the map

$$G_n(F)_{(p)} \to G_n(L)_{(p)}$$

is injective. Assume not, then let z be the minimal element such that the map

$$G_n(F)_{(p)} \to G_n(f(z))_{(p)}$$

is not injective. We consider two cases.

Assume first that there exists $z' \in \alpha$ such that $z' + 1 = z$. By minimality of z the map

$$G_n(F)_{(p)} \to G_n(f(z))_{(p)}$$

is injective. Using the projection formula we know that the composition

$$G_n(f(z))_{(p)} \to G_n(f(z'))_{(p)} \to G_n(f(z))_{(p)}$$

is multiplication by $|f(z') : f(z)|$. Because $|f(z') : f(z)|$ is coprime to p we deduce that the composition is an isomorphism and hence the first map is injective. By commutativity of the diagram

$$\begin{tikzcd} G_n(F)_{(p)} \arrow[rr] \arrow[dr] & & G_n(f(z'))_{(p)} \arrow[dl] \\ & G_n(f(z))_{(p)} & \end{tikzcd}$$

we can see that the map $G_n(F)_{(p)} \to G_n(f(z))_{(p)}$ is injective, giving a contradiction.

Lastly assume that z' does not exist. In this case we have that

$$z = \lim\{x \in \alpha \colon x < z\},$$

and because f preserves limits we have that

$$f(z) = \bigcup_{x \in \alpha} f(x).$$

Because the map $G_n(F)_{(p)} \to G_n(f(z))$ is not injective, there exists a non-zero element $s \in G_n(F)_{(p)}$ that maps to 0. Hence, there exists $a_1, \ldots, a_m \in f(z)$ such that $s = 0 \in G_n(F(a_1, \ldots, a_m))$. Hence, because $f(z)$ is a union of all the elements less than it, there exists $z'' < z$ such that $F(a_1, \ldots, a_n) \subset f(z'')$. Hence, $s = 0 \in G_n(F(z'')$ contradicting the minimality of z. □

The following Lemma finally completes the proof of Proposition 3.4.6.

Lemma 3.4.15. *The p-primary component* Δ_p *is trivial for every prime p. Hence* $\Delta = 0$.

Proof. We want to first show that the following diagram commutes

$$\begin{array}{ccc}
K_n^M(F[t]/p(t)) & \longrightarrow & \widetilde{K}_n^M(F[t]/p(t)) \\
\Big\downarrow {\scriptstyle i_{L|F}\circ \widetilde{N}^M_{F[t]/p(t)|F}} & & \Big\downarrow {\scriptstyle i_{L|F}\circ N^G_{F[t]/p(t)|F}} \\
K_n^M(L) & \longrightarrow & \widetilde{K}_n^M(L)
\end{array}$$

By 3.4.10 this is equivalent to showing that

$$\begin{array}{ccc}
K_n^M(F[t]/p(t)) & \longrightarrow & \widetilde{K}_n^M(F[t]/p(t)) \\
\Big\downarrow & & \Big\downarrow \\
\bigoplus K_n^M(L[t]/p_i(t)) & \longrightarrow & \bigoplus \widetilde{K}_n^M(L[t]/p_i(t)) \\
\Big\downarrow & & \Big\downarrow \\
K_n^M(L) & \longrightarrow & \widetilde{K}_n^M(L)
\end{array}$$

commutes. The top square obviously commutes because the vertical maps are just the maps induced by inclusion. The bottom square commutes because we have already shown proposition 3.4.6 for field extensions of this form. This gives that $i_{L|F}(\Delta) = 0$ and $i_{L|F}$ is injective on Δ_p so $\Delta_p = 0$. □

Finally, we can prove the map g is surjective:

Proposition 3.4.16. *The map*

$$K_n^M(F) \to \widetilde{K}_n^M(F)$$

is surjective.

Proof. We have shown that $\widetilde{K}_n^M(F)$ is generated by elements of the form

$$[F[t_1^{\pm}, \ldots, t_n^{\pm}]/m, t_1, \ldots, t_n].$$

Hence it suffices to show that elements of this form are in the image. We have also shown that the diagram

$$\begin{array}{ccc}
K_n^M(F[t_1^{\pm}, \ldots, t_n^{\pm}]/m) & \longrightarrow & \widetilde{K}_n^M(F[t_1^{\pm}, \ldots, t_n^{\pm}]/m) \\
\Big\downarrow {\scriptstyle N^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/m|F}} & & \Big\downarrow {\scriptstyle \widetilde{N}^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/m|F}} \\
K_n^M(F) & \longrightarrow & \widetilde{K}_n^M(F)
\end{array}$$

is commutative. Hence we have that

$$\begin{aligned}[F[t_1^{\pm},\ldots,t_n^{\pm}]/m, t_1,\ldots,t_n] &= \widetilde{N}^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/m|F} \circ g(\{t_1,\ldots,t_n\}) \\ &= g \circ N^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/m|F}(\{t_1,\ldots,t_n\}),\end{aligned}$$

as required. □

3.5 Injectivity and homotopy invariance

In this section we will complete the proof that the map

$$K_n^M(F) \to \widetilde{K}_n^M(F)$$

is an isomorphism. To do this, we will construct an inverse map by first mapping into $H^n_G(\mathrm{Spec}(F), \mathbb{Z}(n))$ and then mapping to $K_n^M(F)$.

3.5.1 Relations in motivic cohomology

In this section we construct a map

$$\widetilde{K}_n^M(F) \to H^n_G(\mathrm{Spec}(F), \mathbb{Z}(n)).$$

We denote the group $H^n_G(\mathrm{Spec}(F), \mathbb{Z}(n))$ by $K_n^G(F)$. One can show that these groups are given by the following presentation, which we take as our definition of $K_n^G(F)$ throughout this chapter.

Definition 3.5.1. *Let F be a field. We define the groups $K_n^G(F)$, for each $n \in \mathbb{N}$, to be*

$$\begin{aligned}K_n^G(F) \coloneqq \mathbb{Z}[\{[F^m, A_1,\ldots,A_n] :\ & m \in \mathbb{N},\ A_i \in \mathrm{GL}_m(F) \\ & \textit{and } A_iA_j = A_jA_i \textit{ for every } 1 \le i,j \le n\}]/(1)-(4)\end{aligned}$$

1. $[F^{m_1+m_2}, A_1 \oplus B_1, \ldots, A_n \oplus B_n] = [F^{m_1}, A_1, \ldots, A_n] + [F^{m_2}, B_1, \ldots, B_n]$

2. $[F^m, A_1, \ldots, A_n] = [F^m, PA_1P^{-1}, \ldots, PA_nP^{-1}]$ *for any* $P \in \mathrm{GL}_m(F)$.

3. $[F^m, A_1, \ldots, A_n] = 0$ *if* $A_i = I_m$ *for some* i.

4. $[F^m, A_1(1), \ldots, A_n(1)] = [F^m, A_1(0), \ldots, A_n(0)]$ *where* $A_i(t) \in \mathrm{GL}_m(F[t])$ *and* $A_i(t)A_j(t) = A_j(t)A_i(t)$ *for every* $1 \leq i, j \leq n$.

We refer to relation 4 as polynomial homotopy. A simple consequence of this is the following relation:

$$\left[F^{m_1+m_2}, \begin{pmatrix} A_1 & B_1 \\ 0 & C_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & B_n \\ 0 & C_n \end{pmatrix}\right] = \left[F^{m_1+m_2}, \begin{pmatrix} A_1 & 0 \\ 0 & C_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & C_n \end{pmatrix}\right]$$

which is derived from relation 4 by using the homotopy.

$$\left[F^{m_1+m_2}, \begin{pmatrix} A_1 & B_1 t \\ 0 & C_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & B_n t \\ 0 & C_n \end{pmatrix}\right]$$

This relation is just the exact sequence relation. The above groups fit together to form a graded ring where multiplication is given by

$$\begin{aligned}[F^{m_1}, A_1, \ldots, A_{n_1}] &\times [F^{m_2}, B_1, \ldots, B_{n_2}] \\ &= [F^{m_1} \otimes F^{m_2}, A_1 \otimes I_{m_2}, \ldots, A_{n_1} \otimes I_{m_2}, I_{m_1} \otimes B_1, \ldots, I_{m_1} \otimes B_{n_2}]\end{aligned}$$

We denote this ring by $K_*^G(F)$. Note that when $m_1 = m_2 = 1$ the above multiplication is just concatenation of the symbols as it is for Milnor K-theory. We will now prove some useful relations.

Proposition 3.5.2. *Let F be a field. The relation*

$$[F^m, AB, C_2, \ldots, C_n] = [F^m, A, C_2, \ldots, C_n] + [F^m, B, C_2, \ldots, C_n]$$

holds in $K_n^G(F)$.

Proof. We first show

$$[F^m, A_1, A_2, \ldots, A_n] + [F^m, A_1^{-1}, A_2, \ldots, A_n] = 0$$

Using the direct sum relation this is equivalent to showing

$$\left[F^{2m}, \begin{pmatrix} A_1 & 0 \\ 0 & A_1^{-1} \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & A_n \end{pmatrix}\right] \tag{3.6}$$

We use Whitehead's lemma to give a homotopy

$$A_1(t) := \begin{pmatrix} 1 & 0 \\ A_1^{-1}t & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-A_1)t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-A_1^{-1})t \\ 0 & 1 \end{pmatrix}$$

Then using the homotopy

$$\left[F[t]^{2m},\ A_1(t), \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & A_n \end{pmatrix}\right]$$

gives a homotopy between (3.6) and

$$\left[F^{2m}, \mathrm{Id}_{F^{2m}}, \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & A_n \end{pmatrix}\right]$$

So to show the identity

$$\Big[F^m, AB, C_2, \ldots, C_n\Big] - \Big[F^m, A, C_2, \ldots, C_n\Big] - \Big[F^m, B, C_2, \ldots, C_n\Big] = 0$$

it suffices to show

$$\Big[F^m, AB, C_2, \ldots, C_n\Big] + \Big[F^m, A^{-1}, C_2, \ldots, C_n\Big] + \Big[F^m, B^{-1}, C_2, \ldots, C_n\Big] = 0.$$

Using additivity this is equivalent to the relation

$$\left[F^{3m}, \begin{pmatrix} AB & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & B^{-1} \end{pmatrix}, \begin{pmatrix} C_2 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_2 \end{pmatrix}, \ldots, \begin{pmatrix} C_n & 0 & 0 \\ 0 & C_n & 0 \\ 0 & 0 & C_n \end{pmatrix}\right] = 0$$

Note that the first matrix can be factored as

$$\begin{pmatrix} AB & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B^{-1} \end{pmatrix}$$

Using Whitehead's lemma, as in the first part of the proof, we can see that each of these factors are homotopic to the identity, hence so is their product. □

Next we show anti-commutativity still holds as in Milnor K-theory.

Proposition 3.5.3. *The relation*

$$[F^m, A, B, C_3, \ldots, C_n] = -[F^m, B, A, C_3, \ldots, C_n]$$

holds in $K_n^G(F)$.

Proof. We first show that

$$[F^m, AB, AB, C_3, \ldots, C_n] = [F^m, A, A, C_3, \ldots, C_n] + [F^m, B, B, C_3, \ldots, C_n].$$

To do this we show that

$$\left[F^{3m}, \begin{pmatrix} AB & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & B^{-1} \end{pmatrix}, \begin{pmatrix} AB & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix}, \ldots, \begin{pmatrix} C_n & 0 & 0 \\ 0 & C_n & 0 \\ 0 & 0 & C_n \end{pmatrix} \right]$$

We use the homotopy defined in the previous proof on the first matrix in this tuple. The homotopy commutes with the second matrix in the tuple because A and B commute.

Using the 3.5.2 we can also show that

$$\begin{aligned}[F^m, AB, AB, C_3, \ldots, C_n] &= [F^m, A, A, C_3, \ldots, C_n] + [F^m, A, B, C_3, \ldots, C_n] \\ &\quad + [F^m, B, A, C_3, \ldots, C_n] + [F^m, B, B, C_3, \ldots, C_n],\end{aligned}$$

which gives the result. □

The Steinberg relation

It follows from Proposition 3.5.2 that the obvious map $\widetilde{K}_n^M(F) \to K_n^G(F)$ is well-defined when $n = 1, 0$. In this section, we prove that the map is well-defined for $n \geq 2$ by proving the Steinberg relation. We use a similar technique to the proof of the Steinberg relation in motivic cohomology [13, Proposition 5.9]

Lemma 3.5.4. *Let F be a field.*

1. *If $\omega \in F$, is such that $\omega^3 = 1$ and $\omega \neq 1$, then $2[F, a^3, 1-a^3] = 0 \in K_2^G(F)$ for every $a \in F^*$, such that $1 - a^3 \in F^*$.*

2. *If F has no such element ω, then $4[F, a^3, 1-a^3] = 0 \in K_2^G(F)$ for every $a \in F^*$, such that $1 - a^3 \in F^*$.*

Proof. Assume first $\omega \in F$. Consider the homotopy given by

$$\Big[F[t]^3, \quad A(t), \quad 1 - A(t)\Big]$$

where

$$A(t) = \begin{pmatrix} 0 & 0 & a^3 \\ 1 & 0 & -t(a^3+1) \\ 0 & 1 & t(a^3+1) \end{pmatrix}$$

Using this we have that

$$\left[F^3, \begin{pmatrix} 0 & 0 & a^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -a^3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\right] =$$

$$\left[F^3, \begin{pmatrix} 0 & 0 & a^3 \\ 1 & 0 & -(a^3+1) \\ 0 & 1 & a^3+1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -a^3 \\ -1 & 1 & a^3+1 \\ 0 & -1 & 1-(a^3+1) \end{pmatrix}\right]$$

Assuming that $a^3 \neq \omega$ and $a^3 \neq \omega^2$ we can diagonalize these matrices to give

$$\left[F^3, \begin{pmatrix} a & 0 & 0 \\ 0 & a\omega & 0 \\ 0 & 0 & a\omega^2 \end{pmatrix}, \begin{pmatrix} 1-a & 0 & 0 \\ 0 & 1-a\omega & 0 \\ 0 & 0 & 1-a\omega^2 \end{pmatrix}\right] = \left[F^3, \begin{pmatrix} a^3 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & -\omega^2 \end{pmatrix}, \begin{pmatrix} 1-a^3 & 0 & 0 \\ 0 & 1+\omega & 0 \\ 0 & 0 & 1+\omega^2 \end{pmatrix}\right]$$

If $a = \omega$ or $a = \omega^2$ we instead but these matrices in Jordan canonical form, in either case the same argument works. Using the exact sequence relation we have that

$$\left[F, \ a, \ 1-a\right] + \left[F, \ a\omega, \ 1-a\omega\right] + \left[F, \ a\omega^2, \ 1-a\omega^2\right] = \left[F, \ a^3, \ 1-a^3\right] + \left[F, \ -\omega, \ 1+\omega\right] + \left[F, \ -\omega^2, \ 1+\omega^2\right]$$

expanding the second and third term in the sum gives

$$\left[F, \ a, \ 1-a\right] + \left[F, \ a, \ 1-a\omega\right] + \left[F, \ \omega, \ 1-a\omega\right] + \left[F, \ a, \ 1-a\omega^2\right] + \left[F, \ \omega^2, \ 1-a\omega^2\right] = \left[F, \ a^3, \ 1-a^3\right] + \left[F, \ -\omega, \ 1+\omega\right] + \left[F, \ -\omega^2, \ 1+\omega^2\right]$$

Then recombining terms using the multilinear relation gives

$$[F, \ a, \ 1-a^3] + [F, \ \omega, \ (1-\omega a)(1-\omega^2 a)^2] = [F, \ a^3, \ 1-a^3] + [F, \ -\omega, \ 1+\omega] + [F, \ -\omega^2, \ 1+\omega^2].$$

Multiplying both sides by 3 eliminates all terms involving ω because

$$3[F, \omega, b] = 0 \ \text{ and } \ [F, \ -1, \ 1+\omega] + [F, \ -1, \ 1+\omega^2] = 0$$

as $(1+\omega)(1+\omega^2) = 1$ So we have shown that $2[F, a^3, 1-a^3] = 0$ when $\omega \in F^*$.

For the case $\omega \notin F$, we consider the field $E := F(\omega)$. Let $i_*\colon K_2^G(F) \to K_2^G(E)$ be the map induced by the inclusion $i\colon F \to E$. Then the element

$$i_*[F, a, 1-a] = [E, a, 1-a], \text{ satisfies}$$
$$2i_*[F, a, 1-a] = 0.$$

If we apply the transfer to both sides of this equation and use the projection formula then we obtain

$$4[F, a, 1-a] = 0,$$

as required. □

Corollary 3.5.5. *For any field F, we have that $12[F,\ a,\ 1-a] = 0 \in K_2^G(F)$ for every $a \in F \backslash \{0, 1\}$.*

Proof. Using lemma 3.5.4 we know that $4[a^3, 1-a^3] = 0$ for any field. If $\sqrt[3]{a} \in F$ then we clearly have $4[a, 1-a] = 0$. Otherwise, we have that $4i_*[F, a, 1-a] = 0$ over $K_2^G F(\sqrt[3]{a})$, where i_* is the map induced by the inclusion $i\colon F \to F(\sqrt[3]{a})$. Applying the transfer map and using the projection formula gives $12[a, 1-a] = 0$ as required. □

Lemma 3.5.6. *Let F be a field and $n \in \mathbb{N}$. If $n[K, a, 1-a] = 0 \in K_2^G(K)$ for every finite field extension K/F and every $a \in K$, then $[F, a, 1-a] = 0 \in K_2^G(F)$ for every $a \in F \backslash \{0, 1\}$.*

Proof. Take any $a \in F \backslash \{0, 1\}$. Let

$$c_i(t) := b_0^i + \ldots + b_{l_i-1}^i t^{l_i-1} + t^{l_i}$$

be the irreducible factors of the polynomial $t^n - a$ over $F[t]$. By assumption, we have that $n[F[t]/c_i(t),\ t,\ 1-t] = 0$, so using proposition 3.5.2 we have that

$$[F[t]/c_i(t),\ a,\ 1-t] = 0.$$

Applying the transfer and using the projection formula gives

$$[F^{l_i},\ a\,\mathrm{Id}_{F^{l_i}},\ 1 - A_i] = 0,$$

where A_i is the matrix

$$\begin{pmatrix} 0 & \dots & 0 & -b_0 \\ 1 & \dots & 0 & -b_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & -b_{l_i-1} \end{pmatrix}.$$

The determinant of $1 - A_i$ is $c_i(1)$ and so

$$[F, a, c_i(1)] = 0.$$

Because $t^n - a = c_1(t) \dots c_n(t)$ we have that $[F, a, 1 - a] = 0$. □

We can now show that the Steinberg relation holds for matrices.

Corollary 3.5.7. *Let F be a field and $[F^m, A_1, \dots, A_n] \in K_n^G(F)$. If $A_i + A_j = 1$ for some i, j then we have that*

$$[F^m, A_1, \dots, A_n] = 0 \in K_n^G(F)$$

Proof. Multiplication of rank 1 elements in $K_*^G(F)$ is concatenation of symbols so we have that $[F^m, A_1, \dots, A_n] = 0$ when $m = 1$. We have also shown, in section 3.4, that $K_0(F, \mathbb{G}_m^n)$ is generated by images of rank 1 elements under some transfer map. Hence, we can write $[F^m, A_1, \dots, A_n]$ as a sum of images of transfers each of which will be 0. □

As a result of proposition 3.5.7 we have that the map from $\widetilde{K}^M$ to K^G is well-defined.

Corollary 3.5.8. *For any field F the map*

$$g_*\colon \widetilde{K}_*^M(F) \to K_*^G(F)$$
$$[F, a_1, \dots, a_n] \mapsto [F, a_1, \dots, a_n]$$

is a well-defined homomorphism of graded rings.

3.5.2 The map $K_n^G(F) \to K_n^M(F)$

In this section, we prove that the map $K_n^M \to \widetilde{K}_n^M$ is injective by constructing an inverse map Θ. Our strategy is to define a map

$$K_n^G(F) \to K_n^M(F)$$

and then compose with the map $\widetilde{K}_n^M(F) \to K_n^G(F)$.

Take an element $[F^m, A_1, \dots, A_n] \in K_n^G(F)$. As noted in the previous section we can associate to this element a $S = F[t_1^{\pm}, \dots, t_n^{\pm}]$-module M. We then define

$$\Theta([F^m, A_1, \dots, A_n]) \coloneqq \sum_{\substack{m \subset S,\\ \text{m maximal}}} l_{S_m}(M_m) N^M_{S/m|F}(t_1, \dots, t_n) \in K_n^M(F) \tag{3.7}$$

We actually show this homomorphism is well-defined on a slightly different group which we define in the following.

Definition 3.5.9. *Let R be a commutative ring. We define the group*

$$K_0(\mathscr{M}(R, \mathbb{G}_m^{\wedge n})) \coloneqq K_0(\mathscr{M}(R, \mathbb{G}_m^n))/I$$

where $\mathscr{M}(R, \mathbb{G}_m^n)$ is the category whose objects are of the form

$$[M, \phi_1, \dots, \phi_n]$$

where M is a finitely generated R-module, ϕ_i are commuting automorphisms and I is the subgroup generated by elements of the form $[M, \phi_1, \dots, \phi_n]$, with $\phi_i = \mathrm{Id}_M$ for some i.

We now wish to define a map

$$e_s\colon K_0(\mathscr{M}(R,\mathbb{G}_m^{\wedge n})) \to K_0(\mathscr{M}(R/s,\mathbb{G}_m^{\wedge n}))$$

for any $s \in R$. This will give us our homotopy relation. One might guess that the map e_s might take the form

$$[M,\Theta_1,\ldots,\Theta_n] \mapsto [M \otimes_R R/s, \Theta_1 \otimes \mathrm{Id}_{R/s},\ldots,\Theta_n \otimes \mathrm{Id}_{R/s}].$$

However, R/s is not necessarily a flat R-module so this map will not be well-defined because it will not preserve the exact sequence relation. However, given a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

the corresponding exact sequence

$$M_1 \otimes_R R/s \to M_2 \otimes_R R/s \to M_3 \otimes_R R/s \to 0,$$

can be extended to a long exact sequence involving the Tor functor. More precisely we have a long exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{Tor}_2^R(M_2,R/s) \arrow[r] & \mathrm{Tor}_2^R(M_3,R/s) \arrow[dll] \\
\mathrm{Tor}_1^R(M_1,R/s) \arrow[r] & \mathrm{Tor}_1^R(M_2,R/s) \arrow[r] & \mathrm{Tor}_1^R(M_3,R/s) \arrow[dll] \\
M_1 \otimes_R R/s \arrow[r] & M_2 \otimes_R R/s \arrow[r] & M_3 \otimes_R R/s \arrow[r] & 0
\end{tikzcd}$$

If s is a non-zero divisor then R/s has a free resolution of length 1 and so we also have that $\mathrm{Tor}_i^R(M,R/s) = 0$ for $i \geq 2$. In this case we have that $\mathrm{Tor}_1^R(M,R/s) = \mathrm{ann}_M(S)$ and $\mathrm{Tor}_0^R(M,R/s) = M \otimes_R R/s$. This motivates the following proposition which holds even when s s a zero-divisor.

Proposition 3.5.10. *Let R be a commutative ring and $s \in R$. The map*

$$\begin{aligned} e_s \colon K_0(\mathscr{M}(R, \mathbb{G}_m^{\wedge n})) &\to K_0(\mathscr{M}(R/s, \mathbb{G}_m^{\wedge n})) \\ [M, \Theta_1, \ldots, \Theta_n] \mapsto &[M \otimes_R R/s, \Theta_1 \otimes \mathrm{Id}_{R/s}, \ldots, \Theta_n \otimes \mathrm{Id}_{R/s}] \\ &- [\mathrm{ann}_M(s), \Theta_1, \ldots, \Theta_n], \end{aligned}$$

where

$$\mathrm{ann}_M(s) = \{x \in M \colon sx = 0\},$$

is well-defined.

Proof. We show first that Θ_i restrict to well-defined automorphisms on $\mathrm{ann}_M(s)$. We clearly have that $\Theta_i(\mathrm{ann}_M(s)) \subset \mathrm{ann}_M(s)$ because if $x \in \mathrm{ann}_M(s)$ then

$$s\Theta_i(x) = \Theta_i(sx) = \Theta_i(0) = 0.$$

The map Θ_i will obviously still be injective so we only need to show that it is surjective. Take any $y \in \mathrm{ann}_M(s)$. Θ_i is surjective as a map from M to M, so there exists $x \in M$ such that $\Theta_i(x) = y$. Then

$$\Theta_i(sx) = s\Theta_i(x) = sy = 0,$$

and so $sx = 0$ because Θ_i is injective. To complete the proof we only need to show the necessary relations hold. If any of the Θ_i are the identity then $[M, \Theta_1, \ldots, \Theta_n]$ will clearly map to 0 because the image of Θ_i will still be the identity.

To prove the exact sequence relation holds take any exact sequence

$$0 \longrightarrow [M_1, \Theta_1, \ldots, \Theta_n] \xrightarrow{g} [M_2, \Phi_1, \ldots, \Phi_n] \xrightarrow{h} [M_3, \Psi_1, \ldots, \Psi_n] \longrightarrow 0$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [M_1, \Theta_1, \ldots, \Theta_n] & \xrightarrow{g} & [M_2, \Phi_1, \ldots, \Phi_n] & \xrightarrow{h} & [M_3, \Psi_1, \ldots, \Psi_n] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times s} & & \downarrow{\scriptstyle \times s} & & \downarrow{\scriptstyle \times s} & & \\ 0 & \longrightarrow & [M_1, \Theta_1, \ldots, \Theta_n] & \xrightarrow{g} & [M_2, \Phi_1, \ldots, \Phi_n] & \xrightarrow{h} & [M_3, \Psi_1, \ldots, \Psi_n] & \longrightarrow & 0 \end{array}$$

The kernel of the vertical maps are precisely $\mathrm{ann}_{M_i}(s)$ and the cokernel of these maps are $M_i \otimes_R R/s$. Then, by the snake lemma, we have a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ann}_{M_1}(s) \arrow[r] & \mathrm{ann}_{M_2}(s) \arrow[r] & \mathrm{ann}_{M_3}(s) \arrow[dll] & \\ & M_1 \otimes_R R/s \arrow[r] & M_2 \otimes_R R/s \arrow[r] & M_3 \otimes_R R/s \arrow[r] & 0 \end{tikzcd}$$

These maps are also morphisms in $\mathscr{M}(R, \mathbb{G}_m^{\wedge n})$ so we have that the alternating sum of elements in the sequence are equal to 0. But this sum is exactly the image of the exact sequence relation and so we are done. □

We are now ready to define groups $H_n(F)$, which will be the domain of our inverse map. We define $H_n(F)$ to be $K_0(\mathscr{M}(F, \mathbb{G}_m^{\wedge n}))$ with the extra relation that an element is 0 if it is in the image of $e_t - e_{t-1}$. That is

$$H_n(F) \coloneqq \mathrm{coker}(K_0(\mathscr{M}(F[t], \mathbb{G}_m^{\wedge n})) \xrightarrow{e_{t-1}-e_t} K_0(\mathscr{M}(F, \mathbb{G}_m^{\wedge n}))).$$

We will now begin to show that the inverse map, given above, is well-defined on $H_n(F)$. We first show it is well-defined on $K_0(\mathscr{M}(F, \mathbb{G}_m^{\wedge n})))$.

To check that this gives a homomorphism we must check that the sum on the right hand side is finite and all the relations are satisfied. To check that the sum is finite, observe that the maximal ideals for which $l_{R_m}(M_m) \neq 0$ are the maximal ideals which contain $\mathrm{Ann}(M)$. To see this simply note that M_m has length 0, if and only if $M_m = 0$, if and only if there exists $r \notin m$ such that $rm = 0$. Because M is a finitely generated R-module this is true if and only if there exists an $r \notin m$ such that $rM = 0$. We claim that there are only finitely many maximal ideals which contain $\mathrm{Ann}(M)$. To show this we only need to show that $F[t_1^{\pm}, \ldots, t_n^{\pm}]/\mathrm{Ann}(M)$ is a finitely generated F-module. This is true because for each i, there exists a monic polynomial $p_i(t_i) \in \mathrm{ann}(M)$, which has invertible constant term. One such polynomial is the characteristic polynomial $C_{A_i}(t_i)$. This polynomial is clearly monic and has invertible constant term equal to the determinant of A_i. Then $F[t_1^{\pm}, \ldots, t_n^{\pm}]/\operatorname{Ann}(M)$ has only finitely many maximal ideals because it is artinian.

We now begin to show the necessary relations hold for the map to be well-defined on $K_0(\mathscr{M}(F, \mathbb{G}_m^{\wedge n}))$. We first show the exact sequence relation holds. Take any exact sequence

$$[M^1, \phi_1, \ldots, \phi_n] \hookrightarrow [M^2, \psi_1, \ldots, \psi_n] \twoheadrightarrow [M^3, \theta_1, \ldots, \theta_n].$$

This gives us an exact sequence of $F[t_1^{\pm}, \ldots, t_n^{\pm}]$-modules

$$M^1 \hookrightarrow M^2 \twoheadrightarrow M^3.$$

Then given any maximal ideal m, we get an exact sequence of $F[t_1^{\pm}, \ldots, t_n^{\pm}]_m$-modules

$$M^1_m \hookrightarrow M^2_m \twoheadrightarrow M^3_m.$$

because localisation is an exact functor. Then using the exact sequence and the properties of length we get that

$$l_{F[t_1^{\pm},\ldots,t_n^{\pm}]_m}(M^2_m) = l_{F[t_1^{\pm},\ldots,t_n^{\pm}]_m}(M^1_m) + l_{F[t_1^{\pm},\ldots,t_n^{\pm}]_m}(M^3_m)$$

which gives the exact sequence relation.

We now need to show that an element $[M, \phi_1, \ldots, \phi_n]$, maps to 0 if ϕ_i is the identity for some i. If m is a maximal ideal such that $t_i - 1 \in m$, then $t_i = 1 \in F[t_1^{\pm}, \ldots, t_n^{\pm}]/m$ and so $N^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/m|F}\{t_1, \ldots, t_n\} = 0$. If $t_i - 1 \notin m$ we claim that $l_{F[t_1^{\pm},\ldots,t_n^{\pm}]_m}(M_m) = 0$. This happens if and only if $M_m = 0$. As mentioned above, this can only happen if $\operatorname{Ann}(M) \nsubseteq m$ which holds in this case because $t_i - 1 \in \operatorname{Ann}(M)$ and $t_i - 1 \notin m$. Therefore we have a well-defined map

$$K_0(\mathscr{M}(F, \mathbb{G}_m^{\wedge n})) \to K_n^M(F).$$

To complete the proof that the inverse is well-defined, we only need to show that the composition

$$K_0(\mathscr{M}(F[t], \mathbb{G}_m^{\wedge n})) \to K_0(\mathscr{M}(F, \mathbb{G}_m^{\wedge n})) \to K_n^M(F). \tag{3.8}$$

is 0. To do this, we first describe a certain set of generators for $K_0(\mathscr{M}(F[t], \mathbb{G}_m^{\wedge n}))$. Given any element

$$[M, \phi_1, \ldots, \phi_n] \in K_0(\mathscr{M}(F[t], \mathbb{G}_m^{\wedge n})),$$

consider the induced $F[t, t_1^{\pm}, \ldots, t_n^{\pm}]$-module M. Now M is finitely generated as an $F[t, t_1^{\pm}, \ldots, t_n^{\pm}]$-module, so is noetherian. So there exists a series of $F[t, t_1^{\pm}, \ldots, t_n^{\pm}]$-modules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M,$$

such that each quotient M_{i+1}/M_i is isomorphic as a $F[t, t_1^{\pm}, \ldots, t_n^{\pm}]$-module to

$$F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p$$

for some prime ideal p. Then using the exact sequence relation we can deduce that every element in $K_0(\mathscr{M}(F[t], \mathbb{G}_m^{\wedge n}))$ can be written as a sum of elements of the form

$$[F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p\ , t_1, \ldots, t_n]$$

for some prime ideal p. So we only need to show that these elements map to 0 under the composition above. To do this we use a corollary to Weil reciprocity for Milnor K-theory, which we state and use without proof. For a proof of the Weil reciprocity see [3, Corollary 7.2.4], for a proof of the following corollary see [13, corollary 5.5.].

Theorem 3.5.11. *Suppose L is an algebraic function field over k. For each discrete valuation w on L there is a map*

$$\delta_w : K_{n+1}^M(L) \to K_n^M(k(w))$$

and for every $x \in K^M_{n+1}(L)$:

$$\sum_w N_{k(w)/k}\delta_w(x) = 0$$

Corollary 3.5.12. *Let $p : Z \to \mathbb{A}^1_F$ be a finite surjective morphism and suppose that Z is integral. Let $f_1, \ldots, f_n \in \mathcal{O}^*(Z)$ and:*

$$p^{-1}(\{0\}) = \amalg n_i^0 z_i^0 \quad p^{-1}(\{1\}) = \amalg n_i^1 z_i^1$$

where $n_i^\epsilon$are the multiplicities of the points $z_i^\epsilon = Spec(E_i^\epsilon)$ $(\epsilon = 0, 1)$. Define

$$\phi_0 = \sum n_i^0 N^M_{E_i^0/F}(\{f_1, \ldots, f_n\}_{E_i^0}), \quad \phi_1 = \sum n_i^1 N^M_{E_i^1/F}(\{f_1, \ldots, f_n\}_{E_i^1})$$

Then we have

$$\phi_0 = \phi_1 \in K^M_n(F)$$

We need to show that $[F[t, t_1^\pm, \ldots, t_n^\pm]/p, t_1, \ldots, t_n]$ maps to 0 under the composition for any prime p such that $F[t, t_1^\pm, \ldots, t_n^\pm]/p$ is a finitely generated $F[t]$-module. To do this we consider cases.

For the first case assume that $p \cap F[t] \neq 0$. So $p \cap F[t] = (f(t))$ for some irreducible polynomial $f(t)$. We claim in this case that

$$(e_t - e_{t-1})[F[t, t_1^\pm, \ldots, t_n^\pm]/p, t_1, \ldots, t_n] = 0$$

so clearly the composition is 0.

To prove this, first assume that $f(t) \neq t$ and $f(t) \neq t - 1$. In this case both t and $t - 1$ are invertible in $F[t, t_1^\pm, \ldots, t_n^\pm]/p$. So

$$F[t, t_1^\pm, \ldots, t_n^\pm]/p \otimes_{F[t]} F[t]/t = 0 = F[t, t_1^\pm, \ldots, t_n^\pm]/p \otimes_{F[t]} F[t]/t - 1$$
$$\mathrm{ann}_{F[t,t_1^\pm,\ldots,t_n^\pm]/p}(t) = 0 = \mathrm{ann}_{F[t,t_1^\pm,\ldots,t_n^\pm]/p}(t - 1)$$

hence $e_t - e_{t-1} = 0$. If $f(t) = t$ the same logic as above gives us that $e_{t-1} = 0$. To see that $e_t = 0$ note that

$$F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p \otimes_{F[t]} F[t]/t = F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p = \mathrm{ann}_{F[t,t_1^{\pm},\ldots,t_n^{\pm}]/p}(t)$$

so $e_t = 0$. Similar logic allows us to conclude that $e_t - e_{t-1} = 0$ when $f(t) = t - 1$.

Hence we can assume that p is such that $p \cap F[t] = 0$. In this case the map $F[t] \to F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p$ is injective. By the going-up theorem, we can conclude that the map

$$\mathrm{Spec}(F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p) \to \mathrm{Spec}(F[t])$$

is surjective. Therefore we can apply corollary 3.5.12 with

$$Z = \mathrm{Spec}(F[t, t_1^{\pm}, \ldots, t_n^{\pm}]/p)$$

to get the following identity in Milnor K-theory:

$$\sum_{\substack{q \subset F[t_1^{\pm},\ldots,t_n^{\pm}], \\ \text{q minimal} \\ p(1) \subset q}} l_{R/p(1)_q}(F[t_1^{\pm}, \ldots, t_n^{\pm}]/p(1)_q) N^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/q|F}(t_1, \ldots, t_n) = \phi_1$$

$$= \phi_0 = \sum_{\substack{q \subset F[t_1^{\pm},\ldots,t_n^{\pm}], \\ \text{q minimal} \\ p(0) \subset q}} l_{R/p(0)_q}(F[t_1^{\pm}, \ldots, t_n^{\pm}]/p(0)_q) N^M_{F[t_1^{\pm},\ldots,t_n^{\pm}]/q|F}(t_1, \ldots, t_n)$$

where $p(0)$, $p(1)$ are the ideals p evaluated at 0, 1 respectively.

Next we calculate the image of one of these generators under the composition. The image under the map (3.8) is

$$[F[t_1^{\pm}, \ldots, t_n^{\pm}]/p(0), t_1, \ldots, t_n] - [F[t_1^{\pm}, \ldots, t_n^{\pm}]/p(1), t_1, \ldots, t_n]$$

Then the image of this element under the inverse map (3.7) is

$$\sum_{\substack{m\subset F[t_1^\pm,\ldots,t_n^\pm],\\ \text{m maximal}\\ p(0)\subset m}} l_{R_m}(F[t_1^\pm,\ldots,t_n^\pm]/p(0)_m)N^M_{F[t_1^\pm,\ldots,t_n^\pm]/m|F}(t_1,\cdots,t_n)$$
$$-\sum_{\substack{m\subset F[t_1^\pm,\ldots,t_n^\pm],\\ \text{m maximal}\\ p(1)\subset m}} l_{R_m}(F[t_1^\pm,\ldots,t_n^\pm]/p(1)_m)N^M_{F[t_1^\pm,\ldots,t_n^\pm]/m|F}(t_1,\cdots,t_n)$$

because if a maximal ideal does not contain p_i the localisation will be 0. The minimal primes containing $p(1), p(0)$ will be maximal because $F[t_1^\pm,\ldots,t_n^\pm]/p(i)$ is a finitely generated F-module. So to complete the proof we need only to show that

$$l_{F[t_1^\pm,\ldots,t_n^\pm]/p(0)_p}(F[t_1^\pm,\ldots,t_n^\pm]/p(0)_p) = l_{F[t_1^\pm,\ldots,t_n^\pm]_p}(F[t_1^\pm,\ldots,t_n^\pm]/p(0)_p)$$

which is easy to see. So we have constructed the inverse map on the groups $H_n(F)$.

Lemma 3.5.13. *The map*

$$H_n(F)\to K_n^M(F)$$

defined in (3.7) is well-defined.

We have a natural homomorphism $K_n^G(F)\to H_n(F)$ so we define the inverse map to be the composition of this map with the map (3.7). Hence we have shown the following

Theorem 3.5.14. *Let F be a field. The map*

$$K_n^M(F)\to K_n^G(F)$$

is an isomorphism.

Proof. We showed in section 3.4 that the map is surjective. It only remains to show that the map

$$\Theta([F,a_1,\ldots,a_n]) = \sum_{\substack{m\subset S,\\ \text{m maximal}}} l_{S_m}(M_m)N^M_{S/m|F}(t_1,\cdots,t_n) = \{a_1,\ldots,a_n\}\in K_n^M(F).$$

If $m \subset S$ is such that $t_i - a_i \notin m$ then $M_m = 0$ because $(t_i - a_i)M = 0$. So we must have $t_i - a_i \in m$ for all i. Hence $m = (t_1 - a_1, \ldots, t_n - a_n)$ and we are done. $\square$

In 3.3 we showed that the natural map

$$\widetilde{K}_n^M(F) \to K_n^G(F)$$

is well-defined. We have shown that the composition

$$K_n^M(F) \to \widetilde{K}_n^M(F) \to K_n^G(F) \tag{3.9}$$

is an isomorphism, hence the first map is injective. We have also shown that the first map is surjective. Hence we have shown

Theorem 3.5.15. *Let F be a field. The map*

$$K_n^M(F) \to \widetilde{K}_n^M(F)$$

is an isomorphism.

As a result, we have that the second map in 3.9 is an isomorphism. Hence we have the following homotopy invariance relation:

Theorem 3.5.16 (Weak homotopy invariance)**.** *Let F be a field. The map*

$$\widetilde{K}_n^M(F[t]) \xrightarrow{ev_{t=1} - ev_{t=0}} \widetilde{K}_n^M(F)$$

is the zero map.

Chapter 4

Fundamental theorems for Milnor K-theory

In this chapter, we prove analogues of the additivity, resolution and devisage theorems from [17] for the groups $\widetilde{K}_n^M$. We also prove a reciprocity result for $\widetilde{K}_n^M(R)$ which we use to show compatability of the transfers for semi-local rings.

4.1 Compatibility of the transfers for local rings

In this section we prove that the transfer maps for K_n^M and $\widetilde{K}_n^M$ commute. That is we aim to prove the following:

Theorem 4.1.1. *Let A be a semi-local ring with infinite residue fields and $\pi \in A[t]$ be a monic irreducible polynomial such that* $\mathrm{Disc}(\pi) \in A^*$. *Then the diagram*

$$\begin{array}{ccc} K_n^M(A[t]/\pi) & \longrightarrow & \widetilde{K}_n^M(A[t]/\pi) \\ \Big\downarrow{\scriptstyle N^M_{A[t]/\pi|A}} & & \Big\downarrow{\scriptstyle \widetilde{N}^M_{A[t]/\pi|A}} \\ K_n^M(A) & \longrightarrow & \widetilde{K}_n^M(A) \end{array}$$

commutes.

To prove that the diagram commutes it is enough to show that they commute on generators. We will use the following result which gives us generators for $K_n^M(A[t]/\pi)$ which we take from [10, Appendix Theorem 8.1].

Proposition 4.1.2. *The group $K_n^M(A[t]/\pi)$ is generated by elements of the form $\{p_1(t),\ldots,p_n(t)\}$, where $p_i(t)$ are all irreducible in $A[t]$, each $p_i(t)$ is monic or constant and*

$$(p_i(t),p_j(t)) = A[t]$$

for $i \neq j$. Furthermore, we can choose the p_i such that $\mathrm{Disc}(p_i) \in A^*$ and $\deg(p_i) < \deg(\pi)$.

If any of these $p_i(t)$ are in A^* then we can show that the diagram above commutes for this element using the projection formula and induction. We therefore only need to show that the diagram commutes for elements with $p_i(t)$ non-constant. Recall from chapter 2 that we have a split exact sequence

$$0 \to K_n^M(A) \to K_n^{et}(A) \to \oplus K_{n-1}^M(A[t]/\pi) \to 0$$

Consider the splitting map

$$\phi_\pi : K_{n-1}^M(A[t]/\pi) \to K_n^{et}(A). \tag{4.1}$$

We claim that

$$\begin{aligned} \phi_\pi\{p_1,\ldots,p_n\} &= (\pi,p_1,\ldots,p_n) && (4.2)\\ &+\sum_{i=1}^{n}(-1)^{i+1}\phi_{p_i}\{\pi,p_1,\ldots,p_{i-1},\hat{p}_i,p_{i+1},\ldots,p_n\} \in K_n^{et}(A) && (4.3)\end{aligned}$$

To see this, observe that $\phi_f(\{p_1,\ldots,p_n\})$ is the unique element such that

$$\partial_g(\phi_f(\{p_1,\ldots,p_n\})) = \begin{cases} 0 \text{ if } g \neq f \\ \{p_1,\ldots,p_n\} \text{ if } g = f. \end{cases}.$$

and

$$s_\infty(\phi_f(\{p_1, \ldots, p_n\})) = 0$$

where s_∞ is the retraction map which sends an element to its leading coefficient. Then we only need to show that the RHS of (4.2) satisfies these which is a simple calculation.

Now composing 4.2 with $-\partial_\infty$ we can see that

$$N^M_{A[t]/\pi|A}\{p_1, \ldots, p_n\} = -\partial_\infty(\pi, p_1, \ldots, p_n) \tag{4.4}$$

$$+ \sum_{i=1}^{n} (-1)^{i+1} N^M_{A[t]/p_i|A}\{\pi, p_1, \ldots, p_{i-1}, \hat{p}_i, p_{i+1}, \ldots, p_n\} \tag{4.5}$$

We use this identity to prove that the transfer maps commute. We assume, inductively, that the transfer maps commute for $A[t]/f$ where $\deg(f) < \deg(\pi)$. Then to complete the proof we only need to show the analogous version of (4.4) for $\widetilde{K}^M_n$. To do this, we first need to compute $\partial_\infty(\pi, p_1, \ldots, p_n)$.

Proposition 4.1.3. *Let $p_1, \ldots, p_n$ be monic, pairwise coprime, irreducible polynomials. Then*

$$\partial_\infty((p_1, \ldots, p_n)) = \prod_{i=1}^{n} \deg(p_i)\{-1, \ldots, -1\} \in K^M_{n-1}(A)$$

Proof. Let

$$p_i(t) := t^{d_i} + a_{d_i-1,i}t^{d_i-1} + \cdots + a_{0,i}.$$

We can factorise $p_i(t)$ as

$$p_i(t) = q_i(t)r_i(t)$$

where

$$q_i(t) := (t^{-1})^{-d_i}$$
$$r_i(t) := 1 + a_{d_i-1,i}t^{-1} + \cdots + a_{0,i}(t^{-1})^{d_i}.$$

Using this factorisation we can expand $(p_1, \dots, p_n)$ using multilinearity. The term $(r_1(t), \dots, r_n(t))$ maps to 0 because r_i is a polynomial in t^{-1} with $r_i(0) \in A^*$. Any term in the expansion which has both a polynomial q_* and r_* in the symbol, also maps to 0. Using anti-commutativity and the identity $(t^{-1}, t^{-1}) = (t^{-1}, -1)$ we can write these symbols in the form

$$m(t^{-1}, -1, \dots, -1, r_*(t), \dots, r_{**}(t))$$

for $m \in \mathbb{Z}$. This element maps to

$$m\{-1, \dots, -1, 1, \dots, 1\} = 0 \in K_n^M(A).$$

The only element left to consider is

$$\begin{aligned} ((t^{-1})^{-d_1}, \dots, (t^{-1})^{-d_n}) &= \pm(\prod_{i=1}^{n} d_i)(t^{-1}, \dots, t^{-1}) \\ &= \pm(\prod_{i=1}^{n} d_i)(-t^{-1}, -1, \dots, -1). \end{aligned}$$

This element maps to

$$(\prod_{i=1}^{n} d_i)\{-1, \dots, -1\} \in K_n^M(A)$$

□

Lemma 4.1.4. *Let A be a semi-local ring and $\pi \in A[t]$ be an irreducible, monic polynomial. Then*

$$\widetilde{N}^M_{A[t]/\pi|A}(\{p_1(t), \dots, p_n(t)\}) = \widetilde{N}^M_{A[t,x_1,\dots,x_n]/(\pi, p_1(x_1), \dots, p_n(x_n))|A}(\{t - x_1, \dots, t - x_n\})$$

where p_i are all monic polynomials.

Proof. We show that

$$\widetilde{N}^M_{A[t]/\pi|A}(\{p_1(t), \dots, p_n(t)\}) = \widetilde{N}^M_{A[t,x_1]/(\pi, p_1)|A}(\{t - x_1, p_2(t), \dots, p_n(t)\})$$

and continue the process inductively to obtain the result. To show this, it suffices to show that

$$\widetilde{N}^M_{A[t,x_1]/(\pi(t),p_1(x_1))|A[t]/\pi}(\{x_1 - t, p_2(t), \dots, p_n(t)\}) = \{p_1(t), \dots, p_n(t)\} \tag{4.6}$$

in $\widetilde{K}^M_n(A[t]/\pi(t))$ because

$$\widetilde{N}^M_{A[t]/\pi|A} \circ \widetilde{N}^M_{A[t,x_1]/(\pi,p_1)|A[t]/\pi} = \widetilde{N}^M_{A[t,x_1]/(\pi,p_1)|A}.$$

To compute (4.6) we can use the projection formula to get that

$$\widetilde{N}^M_{A[t,x_1]/(\pi,p_1)|A[t]/\pi}(\{t - x_1, p_2(t), \dots, p_n(t)\}) = \{d, p_2(t) \dots, p_n(t)\}$$

where d is the determinant of the $A[t]/\pi(t)$-linear map

$$\times(t - x_1) : A[t, x_1]/(\pi, p_1) \to A[t, x_1]/(\pi, p_1)$$

We claim that the determinant of this map is $p_1(t)$. Let

$$p_1(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$$

The matrix corresponding to the map above is

$$\begin{pmatrix} t & 0 & \dots & 0 & a_0 \\ -1 & t & \dots & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t & a_{n-2} \\ 0 & 0 & \dots & -1 & t + a_{n-1} \end{pmatrix} \tag{4.7}$$

To calculate the determinant of this matrix we use induction. For a 1×1 matrix of the form above, the result is trivial. To calculate the determinant of the $n \times n$ case we expand the top row. Doing this we get that the determinant is equal to

$$t \times \det \begin{pmatrix} t & 0 & \dots & 0 & a_1 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t & a_{n-2} \\ 0 & 0 & \dots & -1 & t + a_{n-1} \end{pmatrix} + -(-1)^{n+1} a_0 \det \begin{pmatrix} -1 & t & \dots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & t \\ 0 & 0 & \dots & -1 \end{pmatrix}$$

We can calculate the determinant of the first matrix using induction. So we get the determinant of (4.7) is

$$t \times (a_1 + \cdots + a_{n-1} t^{n-2} + t^{n-1}) + (-1)^{n+1} \times (-1)^{n-1} \times a_0 = p(t)$$

as required. □

So if we want to prove the identity (4.4), by (4.1.4), it is enough to show that

$$\begin{aligned} &\widetilde{N}^M_{A[t,x_1,\dots,x_n]/(\pi(t),p_1(x_1),\dots,p_n(x_n)|A}(\{t - x_1, \dots, t - x_n\}) \\ &\qquad = \deg(\pi)\deg(p_1)\dots\deg(p_n)\{-1, \dots, -1\} \\ &+ \sum_{i=1}^{n} (-1)^{i+1} \widetilde{N}^M_{A[t,x_1,\dots,x_n]/(\pi(t),p_1(x_1),\dots,p_n(x_n))|A}(\{x_i - t, x_i - x_1 \dots, \widehat{x_i - x_i}, \dots, x_i - x_n\}) \end{aligned} \tag{4.8}$$

To prove this we use the following identity:

Lemma 4.1.5. *Let R be a commutative ring and $x_0, \dots, x_n \in R$ be such that $x_i - x_j \in R^*$, for all i, j. Then*

$$\sum_{i=0}^{n} (-1)^i [x_i - x_0, \dots, x_i - x_{i-1}, x_i - x_{i+1}, \dots, x_i - x_n] = [-1, \dots, -1]$$

in $\widetilde{K}^M_n(R)$.

Proof. We prove the result by induction on n. Let $n = 1$, then

$$[x_0 - x_1] - [x_1 - x_0] = [\frac{x_0 - x_1}{x_1 - x_0}] = [-1]$$

Now assume the identity holds when $n = k$. Then we have the identity

$$\sum_{i=0}^{k}(-1)^i[x_i - x_0, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_k] = [-1, \ldots, -1].$$

In an attempt to introduce x_{k+1} into the equation we multiply both sides by $[x_0 - x_{k+1}]$ to give

$$\sum_{i=0}^{k}(-1)^i[x_i - x_0, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_k, x_0 - x_{k+1}]$$
$$= [-1, \ldots, -1, x_0 - x_{k+1}].$$

Applying the identity $[c, d] = [-\frac{c}{d}, c + d]$ from 3.3.3 to the first and last coordinates of the elements in sum gives

$$\sum_{i=1}^{k}(-1)^i[-\frac{x_i - x_0}{x_0 - x_{k+1}}, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_k, x_i - x_{k+1}]$$
$$+ [x_0 - x_1, \ldots, x_0 - x_{k+1}] = [-1, \ldots, -1, x_0 - x_{k+1}].$$

Expanding the first term in the sum gives

$$\sum_{i=1}^{k}(-1)^{i+1}[-x_0 + x_{k+1}, x_i - x_1, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_k, x_i - x_{k+1}]$$
$$+ \sum_{i=0}^{k}(-1)^i[x_i - x_0, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_k, x_i - x_{k+1}]$$
$$= [-1, \ldots, -1, x_0 - x_{k+1}].$$

The second term is almost the sum we require, so by adding

$$(-1)^{k+1}[x_{k+1} - x_0, \ldots, x_{k+1} - x_k]$$

to both sides of the equation we reduce the proof to proving that

$$\sum_{i=1}^{k}(-1)^{i+1}[-x_0 + x_{k+1}, x_i - x_1, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_k, x_i - x_{k+1}]$$
$$= [-1, \ldots, -1, x_0 - x_{k+1}] + (-1)^{k+1}[x_{k+1} - x_0, \ldots, x_{k+1} - x_k] + [-1, \ldots, -1].$$

By rearranging this equation we reduce to showing

$$\sum_{i=1}^{k+1}(-1)^i[x_{k+1}-x_0, x_i-x_1, \ldots, x_i-x_{i-1}, x_i-x_{i+1}, \ldots, x_i-x_k, x_i-x_{k+1}]$$
$$= [-1, \ldots, -1, x_{k+1}-x_0] \quad (4.9)$$

The term on the right hand side has order 2 and so by graded commutativity is equal to $[x_{k+1}-x_0, -1, \ldots, -1]$. So we can see that the identity 4.9 holds by taking the reciprocity formula for $x_1, \ldots x_{k+1}$ and multiplying on the left by $x_{k+1}-x_0$, and so by induction we are done. $\square$

So we can use this identity, in the ring $A[t, x_1, \ldots, x_n]/(\pi(t), p_1(x_1), \ldots, p_n(x_n))$, to prove 4.8 using the fact that

$$\widetilde{N}^M_{A[t,x_1,\ldots,x_n]/(\pi(t),p_1(x_1),\ldots,p_n(x_n))|A}([-1, \ldots, -1])$$
$$= \deg(\pi)\deg(p_1)\ldots\deg(p_n)[-1, \ldots, -1]$$

This completes the proof of the following reciprocity result

Theorem 4.1.6 (reciprocity)**.** *Let A be a ring and $p_0, p_1, \ldots, p_n \in A[t]$ be monic, pairwise coprime polynomials. Then*

$$\sum_{i=0}^{n}(-1)^i\widetilde{N}^M_{A[t]/p_i}([p_0, \ldots, \hat{p}_i, \ldots, p_n]) = \prod_{i=0}^{n}\deg(p_i)[-1, \ldots, -1] \in \widetilde{K}^M_n(A)$$

4.2 Consequences of reciprocity

In this section we look at some consequences of reciprocity. In particular, we will show that if K^M_n is isomorphic to $\widetilde{K}^M_n(R)$ when R is a local ring with infinite residue field then $\widetilde{K}^M_n(R)$ agrees with the improved Milnor K-groups when R has finite residue field.

To do this we only need to show that our system of transfers satisfies the properties stated in 2.2. This is shown in the following proposition.

Proposition 4.2.1. *Let A be a local ring with infinite residue field and let $A \subset B$ be a finite, etale extension of local rings. Let $A' \to A''$ be a morphism of local A-algebras. Assume further that both*

$$B' := B \otimes_A A' \quad B'' := B \otimes_A A''$$

are local. Then we have that

1. *The composition*

$$\widetilde{K}_n^M(A') \xrightarrow{i} \widetilde{K}_n^M(B') \xrightarrow{\widetilde{N}_{B'/A'}} \widetilde{K}_n^M(A')$$

   is just multiplication by $[B : A]$.

2. *The diagram*

$$\begin{array}{ccc} \widetilde{K}_n^M(B') & \longrightarrow & \widetilde{K}_n^M(B'') \\ \downarrow & & \downarrow \\ \widetilde{K}_n^M(A') & \longrightarrow & \widetilde{K}_n^M(A'') \end{array}$$

   commutes on rank one elements in $\widetilde{K}_n^M(B')$.

Proof. Etale morphisms are preserved under base change so we have that the map $A' \to B'$ is an etale morphism. By 2.1.10 we can choose a monic $\pi \in A[t]$ with $\mathrm{Disc}(\pi) \in A^*$ such that

$$B = A[t]/\pi(t).$$

Furthermore, denoting the image of π in $A'[t]$ by π', we have

$$B' = A'[t]/\pi'(t).$$

To prove the first result, note that the projection formula gives that the composition is equal to multiplication by $[B' : A']$. The result follows from the fact that

$$[B : A] = \deg(\pi) = \deg(\pi') = [B' : A']$$

To prove the second result we need to show that the diagram

$$\begin{array}{ccc} \widetilde{K}_n^M(A'[t]/\pi'(t)) & \longrightarrow & \widetilde{K}_n^M(A''[t]/\pi''(t)) \\ \downarrow & & \downarrow \\ \widetilde{K}_n^M(A') & \longrightarrow & \widetilde{K}_n^M(A'') \end{array}$$

commutes on rank 1 elements. Take a generator for $\widetilde{K}_n^M(A'[t]/\pi'(t))$ of the form $[A'[t]/\pi', p_1'(t), \ldots, p_n'(t)]$ with the $p_i(t)$ monic, irreducible and pairwise coprime with $\operatorname{Disc} p_i' \in A'^*$. Using reciprocity we can write the composition $i_{A'|A''} \circ \widetilde{N}_{A'[t]/\pi'|A'}$ as

$$\begin{aligned} i_{A'|A''} \circ \widetilde{N}_{A'[t]/\pi'|A'}[p_1', \ldots, p_n'] = \sum_{i=1}^{n} i_{A''|A'} \circ \widetilde{N}_{A'[t]/p_i'|A'}(-1)^{i+1}[\pi', p_1', \ldots, \hat{p}_i', \ldots, p_n'] \\ + \deg(\pi')\deg(p_1')\ldots\deg(p_n')[A'', -1, \ldots, -1]. \end{aligned}$$

Using induction we can swap the order of composition in the summation to obtain

$$\begin{aligned} & i_{A'|A''} \circ \widetilde{N}_{A'[t]/\pi'|A'}[p_1', \ldots, p_n'] \\ &\qquad = \sum_{i=1}^{n} \widetilde{N}_{A''[t]/p_i''|A''}(-1)^{i+1}[\pi'', p_1'', \ldots, \hat{p}_i'', \ldots, p_n''] \\ &\qquad\qquad + \deg(\pi')\deg(p_1')\ldots\deg(p_n')[A'' - 1, \ldots, -1]. \end{aligned}$$

The right hand side of which is $\widetilde{N}_{A''[t]/\pi''|A''} \circ i_{A''[t]/\pi''|A'[t]/\pi'}$ □

We have shown the following:

Corollary 4.2.2. *Assume that $\widetilde{K}_n^M \in \mathscr{NC}$. If the map*

$$K_n^M(R) \to \widetilde{K}_n^M(R)$$

is an isomorphism when R is a local ring with infinite residue field then there is a unique isomorphism

$$\hat{K}_n^M(R) \to \widetilde{K}_n^M(R)$$

for R any local ring, such that the diagram

$$\begin{tikzcd} & \hat{K}_n^M(R) \arrow[dr] & \\ K_n^M(R) \arrow[ur] \arrow[rr] & & \widetilde{K}_n^M(R) \end{tikzcd}$$

commutes.

Proof. Let $\mathscr{N}\mathscr{C}, \mathscr{N}\mathscr{C}^\infty$ be the categories defined in section 2.2. By assumption we have that $\widetilde{K}_n^M \cong K_n^M \in \mathscr{N}\mathscr{C}^\infty$. Hence we have that $\hat{K}_n^M$ is naturally isomorphic to $\hat{\widetilde{K}}_n^M$. However if $\widetilde{K}_n^M \in \mathscr{N}\mathscr{C}$, we must have that

$$\widetilde{K}_n^M \cong \hat{\widetilde{K}}_n^M$$

$\square$

Remark 4.2.3. *Using the explicit description of $\hat{K}_n^M(R)$ as*

$$\hat{K}_n^M(R) = \ker(K_n^M(R(t)) \xrightarrow{K_n^M(f_1)-K_n^M(f_2)} K_n^M(R(t_1,t_2))),$$

we can see that there is always a map, regardless of whether the map

$$K_n^M(R) \to \widetilde{K}_n^M(R)$$

is an isomorphism for R with infinite residue field. To show this we simply need to show that

$$\widetilde{K}_n^M(R) = \ker(\widetilde{K}_n^M(R(t)) \xrightarrow{\widetilde{K}_n^M(f_1)-\widetilde{K}_n^M(f_2)} \widetilde{K}_n^M(R(t_1,t_2))),$$

The proof of this is identical to the proof of the analogous identity for Milnor K-theory [11].

4.3 The additivity theorem

The aim of this section is to prove a version of the additivity theorem for $\widetilde{K}_n^M$. The proof is similar to the proof for K_0 we only need to check that the relations are satisfied.

Definition 4.3.1. *Let $\mathscr{A}, \mathscr{C}$ be exact subcategories of an exact category $\mathscr{B}$. We define a category $\mathscr{E}(\mathscr{A}, \mathscr{B}, \mathscr{C})$, which we call the extension category, whose objects are short exact sequences*

$$0 \to A \to B \to C \to 0$$

with $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $C \in \mathscr{C}$ and whose morphisms are commuting diagrams.

Theorem 4.3.2. *With notation as in 4.3.1 we have an isomorphism*

$$\widetilde{K}_n^M(\mathscr{E}(\mathscr{A}, \mathscr{B}, \mathscr{C})) \cong \widetilde{K}_n^M(\mathscr{A}) \times \widetilde{K}_n^M(\mathscr{C})$$

Proof. We first define maps

$$\phi : \widetilde{K}_0(Aut^n(\mathscr{E}(\mathscr{A}, \mathscr{B}, \mathscr{C}))) \to \widetilde{K}_0(Aut^n(\mathscr{A})) \times \widetilde{K}_0(Aut^n(\mathscr{C}))$$
$$\psi : \widetilde{K}_0(Aut^n(\mathscr{A})) \times \widetilde{K}_0(Aut^n(\mathscr{C})) \to \widetilde{K}_0(Aut^n(\mathscr{E}(\mathscr{A}, \mathscr{B}, \mathscr{C})))$$

and then show these maps satisfy the necessary relations.

Take an element $[E, \theta_1, \ldots, \theta_n] \in \widetilde{K}_0(Aut^n(\mathscr{E}(\mathscr{A}, \mathscr{B}, \mathscr{C})))$ where

$$E = 0 \to A \to B \to C \to 0 \text{ and } \theta_i \text{ is}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{i,A}} & & \downarrow{\scriptstyle \theta_{i,B}} & & \downarrow{\scriptstyle \theta_{i,C}} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

We define $\phi([E,\theta_1,\ldots,\theta_n]) = ([A,\theta_{1,A},\ldots,\theta_{n,A}],[C,\theta_{1,C},\ldots,\theta_{n,C}])$. Given an element $([A,\theta_{1,A},\ldots,\theta_{n,A}],[C,\theta_{1,C},\ldots,\theta_{n,C}])$ we define the map ψ to be

$$\psi([A,\theta_{1,A},\ldots,\theta_{n,A}],[C,\theta_{1,C},\ldots,\theta_{n,C}]) = [E,\theta_{1,A\oplus C},\ldots,\theta_{n,A\oplus C}] \text{ where}$$

$$E = 0 \to A \to A\oplus C \to C \to 0 \text{ and } \theta_{i,A\oplus C}$$

$$E = 0 \to A \to B \to C \to 0 \text{ and } \theta_i \text{ is}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & A\oplus C & \longrightarrow & C & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \theta_{i,A}} & & \big\downarrow{\scriptstyle \theta_{i,A}\oplus\theta_{i,C}} & & \big\downarrow{\scriptstyle \theta_{i,C}} & & \\
0 & \longrightarrow & A & \longrightarrow & A\oplus C & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

It is a simple calculation to show that the exact sequence relation is satisfied so these maps are well-defined. To show that the composition is the identity is suffices to show that

$$\begin{aligned}
&[0\to A\to B\to C\to 0,\theta_1,\ldots,\theta_n] \\
&\qquad\qquad = [0\to A\to A\oplus C\to C\to 0,\theta_{1,A}\oplus\theta_{1,C},\ldots,\theta_{n,A}\theta_{n,C}]
\end{aligned}$$

in $K_0(Aut^n(\mathscr{E}(\mathscr{A},\mathscr{B},\mathscr{C})))$. This follows by using the exact sequence relation on the following exact sequence in $\mathscr{E}(\mathscr{A},\mathscr{B},\mathscr{C})$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow[Id_A]{} & A & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle Id_A} & & \big\downarrow{\scriptstyle f} & & \big\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle Id_C} & & \\
0 & \longrightarrow & 0 & \longrightarrow & C & \xrightarrow[Id_C]{} & C & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Each column is exact so this is an exact sequence of elements in $\mathscr{E}(\mathscr{A},\mathscr{B},\mathscr{C})$. We have to show that this gives an exact sequence in $Aut^n(\mathscr{E}(\mathscr{A},\mathscr{B},\mathscr{C}))$. We show that the morphism between $0 \to A \to A \to 0 \to 0$ and $0 \to A \to B \to C \to 0$ gives a morphism between

$$[0 \to A \to A \to 0 \to 0, \theta_{1,A}, \ldots, \theta_{n,A}] \text{ and } [0 \to A \to B \to C \to 0, \theta_1, \ldots, \theta_n]$$

To show this we observe that the diagram

$$\begin{tikzcd}
& A \arrow[rr, "Id_A"] \arrow[dl, "Id_A"'] \arrow[dd, "\theta_{i,A}"] & & A \arrow[rr] \arrow[dl, "f"'] \arrow[dd, "\theta_{i,A}"] & & 0 \arrow[dl] \arrow[dd] \\
A \arrow[rr, "f"] \arrow[dd, "\theta_{i,A}"] & & B \arrow[rr, "g"] \arrow[dd, "\theta_{i,B}"] & & C \arrow[dd] & \\
& A \arrow[rr, "Id_A"] \arrow[dl, "Id_A"'] & & A \arrow[rr] \arrow[dl, "f"'] & & 0 \arrow[dl] \\
A \arrow[rr, "f"] & & B \arrow[rr, "g"] & & C &
\end{tikzcd}$$

commutes. Hence the maps ϕ and ψ are inverse to each other. It can also be shown that the necessary relations are satisfied, this implies that ϕ and ψ induce maps on $\widetilde{K}_n^M$ which are mutually inverse. $\square$

4.4 The resolution theorem

The main result of this section is the following:

Theorem 4.4.1. *Let R be a regular local ring. Then the natural map*

$$\widetilde{K}_i^M(\mathscr{P}) \to \widetilde{K}_i^M(\mathscr{M})$$
$$[P, \Theta_1, \ldots, \Theta_i] \mapsto [P, \Theta_1, \ldots, \Theta_i]$$

is an isomorphism.

To prove this we will construct an inverse map

$$\widetilde{K}_i^M(\mathscr{M}) \to \widetilde{K}_i^M(\mathscr{P})$$

We will first show that there is an map

$$K_0(Aut^i(\mathscr{P})) \to K_0(Aut^i(\mathscr{M}))$$

and then that this map preserves the necessary relations.

The first thing to show is that every element in $Aut^i(\mathscr{M})$ has a resolution with elements in $Aut^i(\mathscr{P})$. This is known for $i = 0$ because R is regular. For the general case we need the following lemma.

Lemma 4.4.2. *Let R be any commutative ring and M a finitely generated R-module. Let $\Theta : M \to M$ be an automorphism of M as an R-module. Then there exists a polynomial $r(t) \in R[t]$ such that r is monic, $r(0) = 1$ and $r(\Theta) = 0$.*

Proof. M is finitely generated as an R-module so there exists a surjective R-module homomorphism

$$f : R^n \twoheadrightarrow M$$
$$(r_1, \ldots, r_n) \mapsto \sum_{i=1}^{n} r_i m_i$$

because Θ is invertible we can lift the maps Θ and Θ^{-1} to maps on R^n so that we have commutative diagrams.

$$\begin{array}{ccc} R^n & \xrightarrow{A} & R^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\Theta} & M \end{array} \qquad \begin{array}{ccc} R^n & \xrightarrow{B} & R^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\Theta^{-1}} & M \end{array}$$

So we must have monic polynomials p and q of degree n, such that $p(\Theta) = q(\Theta^{-1}) = 0$ (take, for example, the characteristic polynomials of A and B). Then define $r(t)$ to be

$$r(t) := t^n(p(t) + q(t^{-1})).$$

One easily checks that $r(t)$ satisfies the required properties. $\square$

We use this to construct a resolution of $[M, \Theta_1, \ldots, \Theta_i]$ with elements in $Aut^i(\mathscr{P})$.

Proposition 4.4.3. *Let $[M, \Theta_1, \ldots, \Theta_i] \in Aut^i(\mathscr{M})$. Then there exists a long exact sequence*

$$0 \to [P_n, A_{(n,1)}, \ldots, A_{(n,i)}] \to \cdots \to [P_0, A_{(0,1)}, \ldots, A_{(0,i)}] \to [M, \Theta_1, \ldots, \Theta_i] \to 0$$

such that $P_i \in \mathscr{P}$ for every i.

Proof. We show that there is a surjective map

$$[P_0, A_1, \ldots, A_i] \to [M, \Theta_1, \ldots, \Theta_i]$$

with P_0 projective. We then proceed by induction.

M is finitely generated so we have a homomorphism $f : R^n \to M$ defined by $f(r_1, \ldots, r_n) = \sum_{j=1}^n r_j m_j$. By 4.4.2 there are monic polynomials $r_j(t)$ of degree 2n with $r_j(0) = 1$ and $r_j(\Theta_j) = 0$. We define P_0 to be the $R[T_1, \ldots, T_i]$-module

$$P_0 := (R[T_1, \ldots, T_i]/\langle r_1(T_1), \ldots, r_i(T_i)\rangle)^n$$

The $r_j(T_j)$ are monic so P_0 is a free R-module. We define an $R[T_1, \ldots, T_n]$-module homomorphism

$$\tilde{f} : P_0 \to M$$
$$\tilde{f}(q_1(T_1, \ldots, T_i), \ldots, q_i(T_1, \ldots, T_i)) = \sum_{j=1}^n q_j(\Theta_1, \ldots, \Theta_i) m_j$$

This map is surjective because f is surjective and is well-defined because $r_j(\Theta_j) = 0$. We define the maps A_j to be multiplication by T_j. These maps clearly commute and are invertible because $r_j(T_j)$ has constant term 1 so we can find an inverse of T_j in $R[T_1, \ldots, T_i]/\langle r_1(T_1), \ldots r_i(T_i)\rangle$. To complete the proof of the claim we only need to show that $\tilde{f}$ gives a homomorphism from $[P_0, \times T_1, \dots, \times T_n]$ to $[M, \Theta_1, \dots, \Theta_n]$, i.e. that the following square commutes

$$\begin{array}{ccc} P_0 & \xrightarrow{\times T^i} & P_0 \\ \Big\downarrow f & & \Big\downarrow f \\ M & \xrightarrow{\Theta_i} & M \end{array}$$

which is simple to show. Hence we have an exact sequence

$$0 \to [ker(\tilde{f}), A_1, \dots, A_i] \to [P_0, A_1, \dots, A_i] \to [M, \Theta_1, \dots, \Theta_i] \to 0$$

Continuing this process with $[ker(\tilde{f}), A_1, \dots, A_i]$ replacing $[M, \Theta_1, \dots, \Theta_i]$ gives a projective resolution for $[M, \Theta_1, \dots, \Theta_i]$. The process must terminate because R is regular. □

Note that the above proposition gives us that the map in Theorem 4.4.1 is surjective for any regular ring because the resolution allows us to write each element as an alternating sum of the elements in its projective resolution. To show it is injective, we shall define an inverse map to be the alternating sum of the elements in its resolution and show that this is independent of the choice of resolution. We know that the map is well defined because $Aut^n(\mathscr{P})$ and $Aut^n(\mathscr{M})$ satisfy the conditions for the resolution theorem for K_0. Therefore, we have a map from $K_0^M(Aut^i(\mathscr{M})) \to K_i^M(\mathscr{P})$ which takes an element to the alternating sum of the elements in its projective resolution. We need to show it satisfies linearity and the Steinberg relation.

Proposition 4.4.4. *The map $K_0^M(Aut^i(\mathscr{M})) \to K_i^M(\mathscr{P})$ factors through a map $K_i^M(\mathscr{M}) \to K_i^M(\mathscr{P})$.*

Proof. We show the Steinberg relation first. We do it for the case $i = 2$ to simplify notation. Take $[M, \Theta, 1 - \Theta]$. By 4.4.2 there exists monic polynomials $r_\Theta, r_{\Theta-1}$ such that

$$r_\Theta(0) = 1,\ r_\Theta(\Theta) = 0$$
$$r_{\Theta-1}(0) = 1,\ r_{\Theta-1}(\Theta - 1) = 0$$

Define the polynomial r to be

$$r(t) \coloneqq t^2 \times r_{\Theta-1}(t-1) + (t-1) \times r_\Theta(t).$$

We can see that $r(0) = -1$, $r(1) = 1$ and $r(\Theta) = 0$ and r is monic. Then t and $1-t$ are invertible in $R[t]/r(t)$. M is a finitely generated R-module. Hence there is an exact sequence

$$0 \to [N, A, 1-A] \to [(R[t]/r(t))^n, \times t, \times(1-t)] \xrightarrow{f} [M, \Theta, 1-\Theta] \to 0$$

where

$$f(p_1(t), \ldots, p_n(t)) = \sum_{i=1}^{n} p_i(\Theta) m_i$$

where $\{m_i\}$ are the generators of the R-module M. Continuing similarly with $[N, A, 1-A]$ we get a long exact sequence

$$0 \to [(R[t]/r(t))^{n_k}, \times t, \times(1-t)] \xrightarrow{f_k} \ldots \xrightarrow{f_1}$$
$$[(R[t]/r(t))^{n_0}, \times t, \times(1-t)] \xrightarrow{f_0} [M, \Theta, 1-\Theta] \to 0$$

To prove the linear relation we make the following claim

Lemma 4.4.5. *Let R be a regular local ring and M a finitely generated R-module. Given two elements of $[M, \Theta_0, \Theta_2, \ldots, \Theta_i]$ and $[M, \Theta_1, \Theta_2, \ldots, \Theta_i]$ of $K_0^M(Aut^i(\mathscr{M}))$ there exists projective resolutions*

$$0 \to [P_n, A_{(n,0)}, A_{(n,2)}, \ldots, A_{(n,i)}] \xrightarrow{f_n} \ldots$$
$$\ldots \xrightarrow{f_1} [P_0, A_{(0,0)}, A_{(0,2)}, \ldots, A_{(0,i)}] \xrightarrow{f_0} [M, \Theta_0, \Theta_2, \ldots, \Theta_i] \to 0$$

$$0 \to [P_n, A_{(n,1)}, A_{(n,2)}, \ldots, A_{(n,i)}] \xrightarrow{f_n} \ldots$$
$$\ldots \xrightarrow{f_1} [P_0, A_{(0,1)}, A_{(0,2)}, \ldots, A_{(0,i)}] \xrightarrow{f_0} [M, \Theta_1, \Theta_2, \ldots, \Theta_i] \to 0$$

Proof. We construct the first term and then we can continue similarly.

Define the polynomials $r_j(t)$ as in 2.2. M is finitely generated so we have a homomorphism $f : R^n \to M$ where $f(r_1, \ldots, r_n) = \sum_{j=1}^{n} r_j m_j$. We choose this map f so that n is minimal. Using Nakayama's Lemma we can show that there exist automorphisms $A_0, A_1 : R^n \to R^n$ which make the following diagram commute.

$$\begin{array}{ccccccc} R^n & \xrightarrow{A_0} & R^n & \quad & R^n & \xrightarrow{A_1} & R^n \\ \Big\downarrow f & & \Big\downarrow f & & \Big\downarrow f & & \Big\downarrow f \\ M & \xrightarrow{\Theta_0} & M & & M & \xrightarrow{\Theta_1} & M \end{array} \tag{4.10}$$

We define

$$S := R[t_2^{\pm}, \ldots, t_i^{\pm}]/\langle r_2(t_2), \ldots, r_i(t_i)\rangle.$$

We tensor S with the diagrams (4.10) and compose with the maps

$$g \colon M \otimes_R S \to M$$
$$m \otimes q(t_2, \ldots, t_n) \mapsto q(\Theta_2, \ldots, \Theta_n) * m$$

to obtain the diagram

$$\begin{array}{ccccccc} R^n \otimes_R S & \xrightarrow{A_0 \otimes_R \mathrm{Id}_S} & R^n \otimes_R S & \quad & R^n \otimes_R S & \xrightarrow{A_1 \otimes_R \mathrm{Id}_S} & R^n \otimes_R S \\ \Big\downarrow f\otimes_R \mathrm{Id}_S & & \Big\downarrow f\otimes_R \mathrm{Id}_S & & \Big\downarrow f\otimes_R \mathrm{Id}_S & & \Big\downarrow f\otimes_R \mathrm{Id}_S \\ M \otimes_R S & \xrightarrow{\Theta_0 \otimes_R \mathrm{Id}_S} & M \otimes_R S & & M \otimes_R S & \xrightarrow{\Theta_1 \otimes_R \mathrm{Id}_S} & M \otimes_R S \\ \Big\downarrow g & & \Big\downarrow g & & \Big\downarrow g & & \Big\downarrow g \\ M & \xrightarrow{\Theta_0} & M & & M & \xrightarrow{\Theta_1} & M \end{array} \tag{4.11}$$

The diagrams (4.11) commute so we can define maps

$$[R^n \otimes_R S, A_0 \otimes_R \mathrm{Id}_S, \mathrm{Id}_{R^n} \otimes_R t_2, \ldots, \mathrm{Id}_{R^n} \otimes_R t_n] \xrightarrow{g(f \otimes \mathrm{Id}_S)} [M, \Theta_0, \Theta_2, \ldots, \Theta_n]$$
$$[R^n \otimes_R S, A_1 \otimes_R \mathrm{Id}_S, \mathrm{Id}_{R^n} \otimes_R t_2, \ldots, \mathrm{Id}_{R^n} \otimes_R t_n] \xrightarrow{g(f \otimes \mathrm{Id}_S)} [M, \Theta_1, \Theta_2, \ldots, \Theta_n]$$

S is a free R-module hence so is $R^n \otimes_R S$. The map is surjective because f is, and so we can take the kernel and cotinue inductively. $\square$

To finish the proof, we take resolutions for

$$[M, \Theta_0, \Theta_2, \ldots, \Theta_i] \text{ and } [M, \Theta_1, \Theta_2, \ldots, \Theta_i]$$

of the form in Lemma 2.5. Then the following is a resolution for $[M, \Theta_0\Theta_1, \Theta_2, \ldots, \Theta_i]$

$$0 \to [P_n, A_{n,0}A_{(n,1)}, A_{(n,2)}, \ldots, A_{(n,i)}] \xrightarrow{f_n} \ldots$$
$$\ldots \xrightarrow{f_1} [P_0, A_{(0,0)}A_{(0,1)}, A_{(0,2)}, \ldots, A_{(0,i)}] \xrightarrow{f_0} [M, \Theta_0\Theta_1, \Theta_2, \ldots, \Theta_i] \to 0$$

Using linearity in $\widetilde{K}_i^M(\mathscr{P})$ gives the result. $\square$

4.5 Devissage

In this section, we prove a Devissage theorem for $\widetilde{K}_n^M$. To do this we mimic the proof for K_0. To finish the proof we only need to show that the necessary relations are satisfied.

Theorem 4.5.1. *Let I be an ideal of a noetherian ring R. Let $\mathrm{Mod}_I(R)$ be the abelian subcategory of $\mathrm{Mod}(R)$ whose objects are finitely generated modules M, such that $I^n M = 0$ for some M. Then*

$$\widetilde{K}_n^M(\mathrm{Mod}_I(R)) \cong \widetilde{K}_n^M(\mathrm{Mod}(R/I))$$

Proof. Given an R/I-module M, we can, by restriction of scalars, obtain an R-module M such that $IM = 0$. We therefore have an inclusion of abelian categories

$$\mathrm{Mod}(R/I) \subset \mathrm{Mod}_I(R).$$

This gives us an inclusion of abelian categories

$$\mathrm{Aut}^n(\mathrm{Mod}(R/I)) \subset \mathrm{Aut}^n(\mathrm{Mod}_I(R)).$$

This induces a homomorphisms on K_0

$$f\colon K_0(\mathrm{Aut}^n(\mathrm{Mod}(R/I))) \to K_0(\mathrm{Aut}^n(\mathrm{Mod}_I(R))).$$

To show this map is an isomorphism we only need to show that each object of $\mathrm{Aut}^n(\mathrm{Mod}_I(R))$ has a filtration with quotients in $\mathrm{Aut}^n(\mathrm{Mod}(R/I))$. Take any object $[M, \Theta_1, \ldots, \Theta_n]$ in $\mathrm{Aut}^n(\mathrm{Mod}_I(R))$. Then $[M, \Theta_1, \ldots, \Theta_n]$ has a filtration

$$[M, \Theta_1, \ldots, \Theta_n] \supset [IM, \Theta_1, \ldots, \Theta_n] \supset \cdots \supset [I^{m-1}M, \Theta_1, \ldots, \Theta_n] \supset 0.$$

Therefore, we can apply Devissage for K_0 to conclude that the map f is an isomorphism with inverse

$$\begin{aligned} f^{-1}\colon K_0(\mathrm{Aut}^n(\mathrm{Mod}_I(R))) &\to K_0(\mathrm{Aut}^n(\mathrm{Mod}(R/I))) \\ [M, \Theta_1, \ldots, \Theta_n] &\mapsto \sum_{i=0}^{m-1} [I^i M / I^{i+1} M, \Theta_1, \ldots, \Theta_n]. \end{aligned}$$

To get two mutually inverse maps on $\tilde{K}_n^M$ it remains to show that the multilinearity and Steinberg relations are satisfied under the maps

$$\begin{aligned} K_0(\mathrm{Aut}^n(\mathrm{Mod}(R/I))) &\to \tilde{K}_n^M(\mathrm{Mod}_I(R)) \\ K_0(\mathrm{Aut}^n(\mathrm{Mod}_I(R))) &\to \tilde{K}_n^M(\mathrm{Mod}(R/I)). \end{aligned}$$

Both relations hold trivially and so we are done. □

We now give a few special cases of the above theorem.

Corollary 4.5.2. *Let I be a nilpotent ideal of a noetherian ring R. Then the inclusion* $\mathrm{Mod}(R/I) \subset \mathrm{Mod}(R)$ *induces an isomorphism*

$$\widetilde{G}_n^M(R/I) \cong \widetilde{G}_n^M(R)$$

Corollary 4.5.3. *Let R be an artinian local ring. Then*

$$\widetilde{G}_*^M(R) \cong \widetilde{K}_*^M(R/m).$$

Proof. content... □

Corollary 4.5.4. *Let R be a local noetherian ring and $\mathrm{Mod}_{fl}(R)$ be the category of modules of finite length. Then*

$$\widetilde{K}_n^M(\mathrm{Mod}_{fl}(R)) \cong \widetilde{K}_n^M(R/m)$$

Proof. This follows from the fact that a module M over a local noetherian ring has finite length iff it is annihilated by a power of m. □

Chapter 5

The homomorphism to Quillen K-theory

In this chapter we will construct a homomorphism to Grayson's definition of higher K-theory. One consequence of this is that the kernel of the map $K_n^M(R) \to \widetilde{K}_n^M(R)$ is annihilated by $(n-1)!$. In particular, this shows the map is injective when $n = 2$. More precisely, we will show that the map which sends $[P, \Theta_1, \ldots, \Theta_n]$ to the n-dimensional cube whose top differential $d_i := A_i$ and whose bottom is the identity, is well-defined.

5.1 Multilinearity

In this section, we will give a sketch of a proof of the multilinear relation which we take from [8]. The proof uses the identity in 5.1.2, which is an analogue of an identiy of Nenashev.

Definition 5.1.1. *A bounded binary double complex $N_{..}$ is a pair of bounded double complexes which have the same objects in each position.*

Proposition 5.1.2. . *Let $N_{..}$ be a bounded binary double complex of objects in $(B)^{q-1}\mathscr{N}$ that is supported on $[0,m]\times[0,n]$, and whose rows and columns are acyclic. Let $N_{.,j}$ be the j^{th} row and $N_{i,.}$ the i^{th} row considered as objects in $(B^q)^n\mathscr{N}$. Then the equation*

$$\sum_{j=0}^{n}(-1)^j[N_{.,j}] = \sum_{i=0}^{m}(-1)^i[N_{i,.}]$$

holds in $K_n^Q(\mathscr{N})$.

Proof. Let $[P,\Theta_1,\dots,\Theta_n]$ denote the n-dimensional cube whose top differential d_i is Θ_i and whose bottom is the identity. To prove the multilinearity we wish to prove

$$[P,\Theta_0\Theta_1,\dots,\Theta_n] = [P,\Theta_0,\dots,\Theta_n] + [P,\Theta_1,\dots,\Theta_n]$$

Let $Q = [P,\Theta_2,\dots,\Theta_n]$. Consider the binary double complex

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & Q \arrow[r, "\Theta_1"] \arrow[d, "1"] & Q \arrow[d, "\Theta_0"] \\ 0 \arrow[r] & Q \arrow[r, "\Theta_0\Theta_1"] & Q \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & Q \arrow[r, "1"] \arrow[d, "1"] & Q \arrow[d, "1"] \\ 0 \arrow[r] & Q \arrow[r, "1"] & Q \end{tikzcd}$$

Using the relation 5.1.2 we get that

$$-[\ Q \overset{1}{\underset{1}{\rightrightarrows}} Q\ ]+[\ Q \overset{\Theta_0}{\underset{1}{\rightrightarrows}} Q\ ]=[\ Q \overset{\Theta_0\Theta_1}{\underset{1}{\rightrightarrows}} Q\ ]-[\ Q \overset{\Theta_1}{\underset{1}{\rightrightarrows}} Q\ ]$$

The first term in the sum is diagonal so is trivial. So

$$[P,\Theta_0\Theta_1,\Theta_2,\dots,\Theta_n] = [P,\Theta_0,\Theta_2,\dots,\Theta_n] + [P,\Theta_1,\Theta_2,\dots,\Theta_n],$$

as required. $\square$

5.2 The cofinality theorem

In section 5.1 we proved that the multilinearity relation holds in Grayson's definition of higher K-theory. In the next section we will show that the Steinberg relation holds. The purpose of this section is to prove the following theorem, which will reduce proving the Steinberg relation for projective modules to proving it just for free modules.

Theorem 5.2.1. *Let R be a ring and $\mathscr{F}$ be the category of finitely-generated, free left R-modules. Then the map*

$$\widetilde{K}_n^M(\mathscr{F}) \to \widetilde{K}_n^M(\mathscr{P})$$

is an isomorphism when $n \geq 1$

It is easy to see the map is surjective; take an element $[P, \Theta_1, \dots, \Theta_n] \in \widetilde{K}_n^M(R)$. Now P is projective so there exists Q such that $P \oplus Q$ is free. Because $n \geq 1$ we have that

$$[Q, \mathrm{Id}_Q, \dots, \mathrm{Id}_Q]$$

is trivial, so

$$[P, \Theta_1, \dots, \Theta_n] = \Big[P \oplus Q, \quad \Theta_1 \oplus \mathrm{Id}_Q, \quad \dots, \quad \Theta_n \oplus \mathrm{Id}_Q\Big]$$

which is in the image.

To show the map is injective we construct an inverse map. We define the inverse map s to be

$$s\colon \widetilde{K}_n^M(\mathscr{P}) \to \widetilde{K}_n^M(\mathscr{F})$$
$$[P, \Theta_1, \dots, \Theta_n] \mapsto \Big[P \oplus Q, \quad \Theta_1 \oplus \mathrm{Id}_Q, \quad \dots, \quad \Theta_n \oplus \mathrm{Id}_Q\Big]$$

We first show that the choice of Q is irrelevant. Let Q_1 and Q_2 be two left R-modules such that $P \oplus Q_1$ and $P \oplus Q_2$ are free. Then

$$\Big[P \oplus Q_1,\ \ \Theta_1 \oplus \mathrm{Id}_{Q_1},\ \ \ldots,\ \ \Theta_n \oplus \mathrm{Id}_{Q_1}\Big] = \\ \Big[P \oplus Q_1 \oplus P \oplus Q_2,\ \ \Theta_1 \oplus \mathrm{Id}_{Q_1} \oplus \mathrm{Id}_P \oplus \mathrm{Id}_{Q_2},\ \ \ldots,\ \ \Theta_n \oplus \mathrm{Id}_{Q_1} \oplus \mathrm{Id}_P \oplus \mathrm{Id}_{Q_2}\Big]$$

$$\Big[P \oplus Q_2,\ \ \Theta_1 \oplus \mathrm{Id}_{Q_2},\ \ \ldots,\ \ \Theta_n \oplus \mathrm{Id}_{Q_2}\Big] = \\ \Big[P \oplus Q_2 \oplus P \oplus Q_1,\ \ \Theta_1 \oplus \mathrm{Id}_{Q_2} \oplus \mathrm{Id}_P \oplus \mathrm{Id}_{Q_1},\ \ \ldots,\ \ \Theta_n \oplus \mathrm{Id}_{Q_2} \oplus \mathrm{Id}_P \oplus \mathrm{Id}_{Q_1}\Big]$$

These two terms are obviously equal.

Next we show the exact sequence relation. Take any exact sequence

$$0 \to [P_1, \phi_1, \ldots, \phi_n] \xrightarrow{f} [P_2, \psi_1, \ldots, \psi_n] \xrightarrow{g} [P_3, \Theta_1, \ldots, \Theta_n] \to 0$$

Let Q_1 and Q_3 be finitely generated modules such that $P_1 \oplus Q_1$ and $P_3 \oplus Q_3$ are free. Then there is an exact sequence of free modules

$$0 \to [P_1 \oplus Q_1, \phi_1 \oplus \mathrm{Id}_{Q_1}, \ldots, \phi_n \oplus \mathrm{Id}_{Q_1}] \xrightarrow{f} \\ [P_2 \oplus Q_1 \oplus Q_3, \psi_1 \oplus \mathrm{Id}_{Q_1} \oplus \mathrm{Id}_{Q_3}, \ldots, \psi_n \oplus \mathrm{Id}_{Q_1} \oplus \mathrm{Id}_{Q_3}] \xrightarrow{g} \\ [P_3 \oplus Q_3, \Theta_1 \oplus \mathrm{Id}_{Q_3}, \ldots, \Theta_n \oplus \mathrm{Id}_{Q_3}] \to 0.$$

Where $P_2 \oplus Q_1 \oplus Q_3$ is free because $P_2 \cong P_1 \oplus P_3$.

The multilinearity is simple to show. Take an element

$$[P, \Theta_0\Theta_1, \Theta_2, \ldots, \Theta_n]$$

this elements maps to an element of the form

$$[P \oplus Q, \Theta_0\Theta_1 \oplus \mathrm{Id}_Q, \Theta_2 \oplus \mathrm{Id}_Q, \ldots, \Theta_n \oplus \mathrm{Id}_Q]$$

in $\widetilde{K}_n^M(\mathscr{F})$. Using the multilinearity relation in $\widetilde{K}_n^M(\mathscr{F})$ this is equal to

$$[P \oplus Q, \Theta_0 \oplus \mathrm{Id}_Q, \Theta_2 \oplus \mathrm{Id}_Q, \ldots, \Theta_n \oplus \mathrm{Id}_Q] + \\ [P \oplus Q, \Theta_1 \oplus \mathrm{Id}_Q, \Theta_2 \oplus \mathrm{Id}_Q, \ldots, \Theta_n \oplus \mathrm{Id}_Q]$$

The Steinberg relation is more difficult. We first show the image of a Steinberg symbol is independent of the automorphisms of the module.

Lemma 5.2.2. *Let P be a finitely-generated projective module for which there exists an automorphism Ψ of P such that $1-\Psi$ is invertible. Then there exists a finitely generated module Q such that there is an automorphism Θ of Q with $1-\Theta$ invertible and $P \oplus Q$ is free.*

Proof. Because P is projective there obviously exists a Q such that $P \oplus Q$ is free. If Q satisfies the necessary properties then we are done. Otherwise we replace Q with $P \oplus Q \oplus Q$ and let

$$\Theta := \begin{pmatrix} \Psi & 0 & 0 \\ 0 & 0 & \mathrm{Id}_Q \\ 0 & \mathrm{Id}_Q & \mathrm{Id}_Q \end{pmatrix}$$

□

Lemma 5.2.3. *Let P be a projective module and let Θ_1, Θ_1' be automorphisms of P such that $1-\Theta_1'$ and $1-\Theta_1$ are both invertible. Then*

$$s[P, \Theta_1, 1-\Theta_1, \Theta_3, \ldots, \Theta_n] = s[P, \Theta_1', 1-\Theta_1', \Theta_3', \ldots, \Theta_n']$$

Proof. We take Q to be a projective as in lemma 5.2.2. Then

$$s[P, \Theta_1, 1-\Theta_1, \Theta_3, \ldots, \Theta_n] = s[P, \Theta_1, 1-\Theta_1, \Theta_3, \ldots, \Theta_n] \\ + [P \oplus Q, \Theta_1' \oplus \Psi, (1-\Theta_1') \oplus (1-\Psi), \Theta_3' \oplus \mathrm{Id}, \ldots, \Theta_n' \oplus \mathrm{Id}]$$

Combining these two terms using the exact sequence relation gives

$$[P \oplus Q \oplus P \oplus Q, \Theta_1 \oplus \mathrm{Id} \oplus \Theta_1' \oplus \Psi, (1-\Theta_1) \oplus \mathrm{Id} \oplus (1-\Theta_1') \oplus (1-\Psi), \\ \Theta_2 \oplus \mathrm{Id} \oplus \Theta_2' \oplus \mathrm{Id}, \ldots, \Theta_n \oplus \mathrm{Id} \oplus \Theta_n' \oplus \mathrm{Id}] \quad (5.1)$$

One can get the result from this by taking an exact sequence whose middle term is 5.1 and whose first term is just the inclusion of the first and last coordinate. □

Lemma 5.2.3 actually completes the proof that the Steinberg relation holds when $n \geq 3$ because we can just choose $\Theta_3' = \mathrm{Id}$. The only case left is the case $n = 2$. In this case we have shown that

$$s[P, \Theta, 1-\Theta] = s[P, \Theta', 1-\Theta']$$

whenever this makes sense. We denote an element $s[P, \Theta, 1-\Theta]$ by $s(P)$

Note that for projective modules M, N we have that $s(M \oplus N) = s(M) \oplus s(N)$ providing both $s(M)$ and $s(N)$ exist.

Our aim now is to show that $s(P) = 0$ whenever it exists. We begin with the following lemma.

Lemma 5.2.4. *Let Q be a projective R-module. If there is an automorphism θ of Q such that $1-\theta^2$ are invertible then*

$$3s(Q) = 0 \in \widetilde{K}_n^M(\mathscr{F})$$

Proof. We have that

$$\begin{aligned} s(Q) &= [Q, \theta^2, 1-\theta^2] \\ &= [Q, \theta^2, (1-\theta)(1+\theta)] \\ &= [Q, (-\theta)^2, 1+\theta] + [Q, \theta^2, 1-\theta] \\ &= 2[Q, -\theta^2, 1+\theta] + 2[Q, \theta, 1-\theta] = 2s(Q) + 2s(Q) \end{aligned}$$

which gives the result. □

From this we can show that for any projective module P we have that $3s(P^2) = 0$ and $3s(P^3) = 0$. To prove the first of these identities take

$$\theta = \begin{pmatrix} 0 & I_P \\ I_P & I_P \end{pmatrix},\ 1 - \theta^2 = \begin{pmatrix} 0 & -I_P \\ -I_P & -I_P \end{pmatrix}$$

and for the second identity take

$$\theta = \begin{pmatrix} 0 & 0 & I_P \\ I_P & 0 & I_P \\ 0 & I_P & 0 \end{pmatrix},\ 1 - \theta^2 = \begin{pmatrix} I_P & -I_P & 0 \\ 0 & 0 & -I_P \\ -I_P & 0 & 0 \end{pmatrix}$$

Then the two identities above give us the following

Lemma 5.2.5. *Let P be any projective R-module. We have that*

$$3s(P) = 0 \in \widetilde{K}_n^M(\mathscr{F})$$

To finish the proof we will show that $4s(P) = 0$. We do this by picking an explicit representation of $s(P^4)$. We take this to be

$$s(P^4) = \left[\begin{pmatrix} 0 & 0 & 0 & -I_P \\ I_P & 0 & 0 & I_P \\ 0 & I_P & 0 & -I_P \\ 0 & 0 & I_P & I_P \end{pmatrix}, \begin{pmatrix} I_P & 0 & 0 & I_P \\ -I_P & I_P & 0 & -I_P \\ 0 & -I_P & I_P & I_P \\ 0 & 0 & -I_P & 0 \end{pmatrix} \right]$$

One can check that both these maps are invertible. Furthermore, it is true that

$$\begin{pmatrix} 0 & 0 & 0 & -I_P \\ I_P & 0 & 0 & I_P \\ 0 & I_P & 0 & -I_P \\ 0 & 0 & I_P & I_P \end{pmatrix}^{10} = \begin{pmatrix} I_P & 0 & 0 & 0 \\ 0 & I_P & 0 & 0 \\ 0 & 0 & I_P & 0 \\ 0 & 0 & 0 & I_P \end{pmatrix}$$

So we have that $10s(P^4) = 0$, but we also have that $3s(P^4) = 0$ by the previous lemma so $s(P^4) = 0$, hence $4s(P) = 0$.

5.3 The Steinberg relation for Quillen K-theory

In this section we prove the Steinberg relation for Grayson's definition of higher K-theory of a ring.

Lemma 5.3.1. *Let R be any ring. Denote elements of the form*

$$\begin{tikzcd} R \arrow[r, shift left, "x"] \arrow[r, shift right, "1"'] \arrow[d, shift right, "1"'] \arrow[d, shift left, "y"] & R \arrow[d, shift right, "1"'] \arrow[d, shift left, "y"] \\ R \arrow[r, shift left, "x"] \arrow[r, shift right, "1"'] & R \end{tikzcd}$$

in $K_2^Q(R)$ by $[x,y]$. Then we have that

$$4[a^3, 1-a^3] = 0 \in K_2^Q(R)$$

for all $a^3, 1-a^3 \in R^*$.

Proof. We show that this relation holds when R is the ring $\mathbb{Z}[t, t^{-1}, (1-t^3)^{-1}]$. R is a regular ring so we know that $K_2^Q(R)$ is homotopy invariant. Using this we may show that

$$\left[\begin{pmatrix} 0 & 0 & t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -t^3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\right] =$$

$$\left[\begin{pmatrix} 0 & 0 & t^3 \\ 1 & 0 & -(t^3+1) \\ 0 & 1 & t^3+1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -t^3 \\ -1 & 1 & t^3+1 \\ 0 & -1 & 1-(t^3+1) \end{pmatrix}\right]$$

Using the homotopy

$$\left[\begin{pmatrix} 0 & 0 & t^3 \\ 1 & 0 & -x(t^3+1) \\ 0 & 1 & x(t^3+1) \end{pmatrix}, \begin{pmatrix} 1 & 0 & -t^3 \\ -1 & 1 & x(t^3+1) \\ 0 & -1 & 1-x(t^3+1) \end{pmatrix}\right]$$

We reduce these matrices to 1×1 matrices in the ring $R[\omega]/(\omega^2 + \omega + 1)$. We first use following change of bases matrices on the matrices above

$$\begin{pmatrix} t^2 & 0 & 1 \\ t & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The first column of each is an eigenvector changing bases gives the following:

$$\left[\begin{pmatrix} t & 1 & 0 \\ 0 & -t & 1 \\ 0 & -t^2 & 0 \end{pmatrix}, \begin{pmatrix} 1-t & -1 & 0 \\ 0 & 1+t & -1 \\ 0 & t^2 & 1 \end{pmatrix} \right] =$$

$$\left[\begin{pmatrix} t^3 & 0 & t^3 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1-t^3 & 0 & -t^3 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right]$$

Using the exact sequence relation we get that

$$[t, 1-t] + \left[\begin{pmatrix} -t & 1 \\ -t^2 & 0 \end{pmatrix}, \begin{pmatrix} 1+t & -1 \\ t^2 & 1 \end{pmatrix} \right] =$$

$$[t^3, 1-t^3] + \left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \right]$$

Using the change of bases matrices

$$\begin{pmatrix} 1 & 0 \\ -\omega^2 t & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

we get that

$$[t, 1-t] + \left[\begin{pmatrix} \omega t & 1 \\ 0 & \omega^2 t \end{pmatrix}, \begin{pmatrix} 1-\omega t & -1 \\ 0 & 1-\omega^2 t \end{pmatrix} \right] =$$

$$[t^3, 1-t^3] + \left[\begin{pmatrix} -\omega & -1 \\ 0 & -\omega^2 \end{pmatrix}, \begin{pmatrix} 1+\omega & 1 \\ 0 & 1+\omega^2 \end{pmatrix} \right]$$

Then using the exact sequence relation we get that

$$[t, 1-t] + [\omega t, 1-\omega t] + [\omega^2 t, 1-\omega^2 t] = [t^3, 1-t^3] + [-\omega, 1+\omega] + [-\omega^2, 1+\omega^2]$$

Using linearity we can get that

$$[t, 1-t^3] + [\omega, (1-\omega t)(1-\omega^2 t)^2] = [t^3, 1-t^3] + [-\omega, 1+\omega] + [-\omega^2, 1+\omega^2]$$

Multiplying both sides by 3 eliminates all terms involving ω because $3[\omega, b] = 0$ and $[-1, 1+\omega] + [-1, 1+\omega^2] = 0$ so we have shown that $2[t^3, 1-t^3] = 0$. We use the transfer map to get that $4[t^3, 1-t^3] = 0 \in K_2^Q(\mathbb{Z}[t, t^{-1}, (1-t^3)^{-1}])$. To get the result for a general ring R we use the fact we can take a homomorphism $\mathbb{Z}[t, t^{-1}, (1-t^3)^{-1}] \to R$ with $t \mapsto a$. $\square$

Corollary 5.3.2. *Let R be any ring, and $a, 1-a \in R^*$. Then $12[a, 1-a] = 0 \in K_2^Q(R)$.*

Proof. We prove this for the ring $R = \mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$ and the element $[t, 1-t]$. Consider the ring $S = \mathbb{Z}[t, t^{-1}, (1-t)^{-1}][x]/(x^3 - t)$. Then we know, by 5.3.1, that

$$4[x^3, 1-x^3] = 0 \in K_2^Q(S).$$

Hence taking the image under the transfer map we have that $12[t, 1-t] = 0 \in K_2^Q(R)$. $\square$

We are finally able to prove the Steinberg relation

Proposition 5.3.3. *Let R be any ring and $a, 1-a \in R^*$. Then*

$$[R, a, 1-a] = 0 \in K_2^Q(R)$$

Proof. We show that $[t, 1-t] = 0 \in K_2^Q(\mathbb{Z}[t, t^{-1}, (1-t)^{-1}])$. Let $R = \mathbb{Z}[t, t^{-1}, (1-t)^{-1}][x]/(x^{12} - t)$. Then we know, by 5.3.2, that

$$[t, 1-x] = 12[x, 1-x] = 0 \in K_2^Q(R)$$

Applying the transfer map to this element gives us the following 12×12 matrices

$$\left[\begin{pmatrix} t & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t \end{pmatrix} \quad \begin{pmatrix} 1 & \dots & 0 & -t \\ -1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & 1 \end{pmatrix}\right] = 0 \in K_2^Q(\mathbb{Z}[t, t^{-1}, (1-t)^{-1}])$$

We can now use elementary row and column operations to reduce the matrix on the right to

$$\left[\begin{pmatrix} t & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t \end{pmatrix}, \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1-t \end{pmatrix}\right]$$

The result follows. □

To finish the proof that the map $\widetilde{K}_n^M(R) \to K_n^Q(R)$ is well-defined we need to show that the identity holds for free modules.

First note the Steinberg relation holds for free modules of rank 1 in $K_n^Q(R)$ because

$$[a_1, 1-a_1, \dots, a_n] = [a_1, 1-a_1] \otimes [a_3, \dots, a_n]$$

We need to show that

$$\Big[P, A_1, 1-A_1, A_3, \dots, A_n\Big] = 0 \in K_n^Q(R).$$

Let S be the commutative subring of $M_n(R)$ generated by $A_1, A_3, \dots, A_n, A_1^{-1}, (1-A_1)^{-1}, A_3^{-1}, \dots, A_n^{-1}$. We know that $[S, A_1, 1-A_1, A_3, \dots, A_n] = 0 \in K_n^Q(S)$. We define a functor

$$F\colon \operatorname{Proj}_S \to \operatorname{Proj}_R$$
$$Q \mapsto P \bigotimes_S Q$$

and given a morphism $f\colon Q \to Q'$ we define $F(f) = \mathrm{Id}_P \otimes f$. This functor induces a map on K-theory $K_n^Q(S) \to K_n^Q(R)$. One can show that the image of S under this functor is P and the image of the homomorphism A_i is the matrix A_i.

We have shown the following:

Theorem 5.3.4. *Let R be a ring. There exists a homomorphism*

$$\phi : \widetilde{K}_n^M(R) \to K_n^Q(R)$$

such that the comparison homomorphism from Milnor K-theory to Quillen K-theory is equal to the composition

$$K_n^M(R) \to \widetilde{K}_n^M(R) \to K_n^Q(R)$$

We know that for a local ring with infinite residue field $K_2^M(R) \cong K_2^Q(R)$. We conjecture the map defined above is an isomorphism more generally.

Conjecture 5.3.5. *Let R be any ring. The map*

$$\widetilde{K}_2^M(R) \to K_2^Q(R)$$

is an isomorphism.

We know that this map is an isomorphism for R a field. We also know, because, by [15], the composition

$$K_2^M(R) \to \widetilde{K}_2^M(R) \to K_2^Q(R)$$

is an isomorphism for R a local ring with infinite residue, that $\hat{K}_2^M(R) \to K_2^Q(R)$ is surjective.

Corollary 5.3.6. *Let R be a regular, local ring with infinite residue field, then the map*

$$\widetilde{K}_n^M(R[t_1, \ldots, t_n]) \to K_2^Q(R[t_1, \ldots, t_n])$$

is surjective.

Proof. We do this by induction on n. The case $n = 0$ is considered above. Assume that this holds for $n = k$. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{K}_n^M(R[t_1,\ldots,t_k]) & \longrightarrow & K_n^Q(R[t_1,\ldots,t_k]) \\ \downarrow & & \downarrow \\ \widetilde{K}_n^M(R[t_1,\ldots,t_k,t_{k+1}]) & \longrightarrow & K_n^Q(R[t_1,\ldots,t_k,t_{k+1}]) \end{array}$$

by induction the top map is surjective and by homotopy invariance the right map is surjective. Hence, the bottom map is surjective. □

We can also use the map to Quillen K-theory to prove the following:

Corollary 5.3.7. *Let R be a local ring with infinite residue field. Then the kernel of the map*

$$K_n^M(R) \to \widetilde{K}_n^M(R)$$

is annihilated by $(n-1)!$. In particular, when $n = 2$ the map is injective.

Proof. By [15]here is a map

$$K_n^Q(R) \to K_n^M(R)$$

such that the composition

$$K_n^M(R) \to \widetilde{K}_n^M(R) \to K_n^Q(R) \to K_n^M(R)$$

is multiplication by $(n-1)!$. □

Chapter 6

Further questions

6.1 Surjectivity for local rings

In section 4.1 we showed that the transfers for Milnor K-theory are compatible with the transfers for $\widetilde{K}_n^M$. In the case when R is a field we can use compatibility of the transfer or the reciprocity laws to prove that the map is surjective. To do this we need that every element in $\widetilde{K}_n^M(F)$ is the image of transfers of rank one elements. In this section, we show that when R is a regular local ring $\widetilde{K}_n^M(R)$ is images of rank one elements under a transfer map. Unfortunetly, we do not have a reciprocity law to manipulate these elements nor do we have the corresponding transfers we need for Milnor K-theory.

Let R be a regular, local ring. We have shown, in section 4.4.1, that

$$\widetilde{K}_n^M(R) \to \widetilde{G}_n^M(R)$$

is an isomorphism. Take an element

$$[M, \Theta_1, \ldots, \Theta_n] \in \widetilde{G}_n^M(R).$$

Like in the field case, we can consider M as a $R[t_1^{\pm}, \ldots, t_n^{\pm}]$-module. We can take a filtration of M where each quotient is of the form

$$R[t_1^{\pm}, \ldots, t_n^{\pm}]/p$$

where p is prime. Hence every element in $\widetilde{G}_n^M(R)$ can be written as a sum of elements of the form

$$\widetilde{N}_{R[t_1^{\pm},\ldots,t_n^{\pm}]/p|R}[R[t_1^{\pm}, \ldots, t_n^{\pm}]/p, t_1, \ldots, t_n]$$

To complete the proof as in the field case we either need a more general version of reciprocity or transfers for Milnor K-theory.

In the proof of surjectivity for fields we gave an alternative proof that $\widetilde{K}_n^M(F)$ is generated by the image of rank one transfers. This also carries over, in some way, to the realm of regular local rings.

Take an element

$$[R^m, \Theta_1, \ldots, \Theta_n]$$

Let $c_{\Theta_1}(t)$ be the characteristic polynomial of Θ_1. Let

$$c_{\Theta_1}(t) = p_1(t) \ldots p_l(t)$$

be the factorizations into irreducibles in the field of fractions. As in the field case we can define M to be the subspace annihilated by some monic polynomial.

$$0 \to [M, \Theta_1, \ldots, \Theta_n] \to [[R^m, \Theta_1, \ldots, \Theta_n]] \to [R^m/M, \Theta_1, \ldots, \Theta_n] \to 0$$

Now M is a $R[t]/p(t)$-module where $t \times M = \Theta_1 \times M$.

Hence, we have that $K_n^M(R)$ is generated by transfers of the form

$$[R[t]/p(t), t] \otimes_{R[t]/p(t)} [M, \Theta_2, \ldots, \Theta_n] \in \widetilde{G}_n^M(R[t]/p(t))$$

where $p(t)$ is an irreducible, monic polynomial.

6.2 The case for DVRs

In the previous section, we showed that $\widetilde{K}_n^M(R)$ is generated by images of rank 1 elements under some transfer. In this section, we show that if R is a discrete valuation ring we can define the necessary transfers for Milnor K-theory. However we do not know whether these transfers commute.

Let R be a discrete valuation ring. We know that the group $\widetilde{G}_n^M(R)$ is generated by elements of the form

$$[R[t_1^{\pm}, \ldots, t_n^{\pm}]/p, t_1^{\pm}, \ldots, t_n^{\pm}]$$

where p is prime. Consider the map

$$R \to R[t_1^{\pm}, \ldots, t_n^{\pm}]/p.$$

First assume that the map is not injective. Then the kernel is a non-trivial prime ideal, so must be π. Hence the map factors as

$$R \to R/\pi \to R[t_1^{\pm}, \ldots, t_n^{\pm}]/p.$$

Hence the element $[R[t_1^{\pm}, \ldots, t_n^{\pm}]/p, t_1, \ldots, t_n]$ is in the image of the transfer

$$\widetilde{N}_{R/m|R}^M \colon \widetilde{G}_n^M(R/\pi) \to \widetilde{G}_n^M(R)$$

R/π is a field so we know that $\widetilde{G}_n^M(R/\pi)$ is a generated by elements of the form

$$[R/\pi, a_1, \ldots, a_n].$$

We claim that these elements are equal to 0 in $\widetilde{G}_n^M(R)$. Let $\widehat{a}_i$ be any lifting of a_i in R. We have an exact sequence

$$0 \to [R, \widehat{a}_1, \ldots, \widehat{a}_n] \xrightarrow{\times\pi} [R, \widehat{a}_1, \ldots, \widehat{a}_n] \to [R/\pi, a_1, \ldots, a_n] \to 0$$

So using the exact sequence relation we have that

$$[R/\pi, a_1, \dots, a_n] = [R, \widehat{a}_1, \dots, \widehat{a}_n] - [R, \widehat{a}_1, \dots, \widehat{a}_n] = 0$$

So we are left with the case $R \to R[t_1^{\pm}, \dots, t_n^{\pm}]/p$ is injective.

$R[t_1^{\pm}, \dots, t_n^{\pm}]/p$ is finite over R and so is 1-dimensional. Let S denote the integral closure of R in the field of fractions of $R[t_1^{\pm}, \dots, t_n^{\pm}]/p$. We know that S is a finite R-module which contains $R[t_1^{\pm}, \dots, t_n^{\pm}]/p$. We also know that S is a Dedekind domain. Consider the exact sequence

$$0 \to [R[t_1^{\pm}, \dots, t_n^{\pm}]/p, t_1, \dots, t_n] \xrightarrow{\times\pi} [S, t_1, \dots, t_n] \to [M, t_1, \dots, t_n] \to 0$$

where M is a finitely generated R/π-module. Using a similar argument to above we can deduce that $[M, t_1, \dots, t_n] = 0$.

So we have shown that $\widetilde{G}_n^M(R)$ is generated by elements of the form $[S, a_1, \dots, a_n]$ where S is a Dedekind domain. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_n^M(S) & \longrightarrow & K_n^M(L) & \xrightarrow{\oplus\partial_{\pi_i}} & \oplus_{\pi_i} K_n^M(S/\pi_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle N^M_{L/F}} & & \downarrow{\scriptstyle \sum N^M_{S/\pi_i|R/\pi}} & & \\
0 & \longrightarrow & K_n^M(R) & \longrightarrow & K_n^M(F) & \xrightarrow{\partial_\pi} & K_n^M(R/\pi) & \longrightarrow & 0
\end{array}$$

The diagram commutes and each of the rows are exact when R contains an infinite field. So we have constructed transfers

$$K_n^M(S) \to K_n^M(R)$$

If these transfers are compatible with those for $\widetilde{K}_n^M$ then we are done.

6.3 The map to homology

In this section, we give a possible map from $\widetilde{K}_n^M(R)$ to $H_n(\mathrm{GL}(R))/H_n(\mathrm{GL}_{n-1}(R))$ which agrees with map from Milnor K-theory.

We begin by defining a map

$$\phi\colon \mathbb{Z}\{[R^m, A_1, \dots, A_n]\} \to H_n(\mathrm{GL}_n(R))$$

Given an element $[R^m, A_1, \dots, A_n]$ we define

$$\phi([R^m, A_1, \dots, A_n]) = \sum_{\sigma\in S_n} sgn(\sigma)[A_{\sigma(1)}, \dots, A_{\sigma(n)}]$$

We need to show this map is well-defined. We first show that

$$\partial_i(\sum_{\sigma\in S_n} sgn(\sigma)[A_{\sigma(1)}, \dots, A_{\sigma(n)}]) = 0$$

when $0 < i < n$. Note that $S_n = S_{n,+} \bigcup (i, i{+}1)S_{n,+}$, where $S_{n,+}$ is the permutations which positive sign.

$$\partial_i(\sum_{\sigma\in S_n} sgn(\sigma)[A_{\sigma(1)}, \dots, A_{\sigma(n)}]) =$$

$$\sum_{\sigma\in S_{n,+}} [A_{\sigma(1)}, \dots, A_{\sigma(i)}A_{\sigma(i+1)}, \dots, A_{\sigma(n)}] - [A_{\sigma(1)}, \dots, A_{\sigma(i+1)}A_{\sigma(i)}, \dots, A_{\sigma(n)}]$$

this is 0 because all matrices commute. We claim that

$$\partial_0 + (-1)^n\partial_n = 0.$$

To show this we need to show that

$$\sum_{\sigma\in S_n|\sigma(1)=i} sgn(\sigma)[A_{\sigma(2)}, \dots, A_{\sigma(n)}] + (-1)^n \sum_{\sigma'\in S_n|\sigma'(n)=i} sgn(\sigma')[A_{\sigma'(1)}, \dots, A_{\sigma'(n-1)}] = 0$$

right multiplication by $(1, \dots, n)$ sends elements of S_n which send 1 to i to elements which send n to i.

$$\sum_{\sigma\in S_n|\sigma(1)=i} sgn(\sigma)[A_{\sigma(2)}, \dots, A_{\sigma(n)}]+$$

$$(-1)^n \sum_{\sigma'\in S_n|\sigma'(1,\dots,n)(n)=i} sgn(\sigma'(1, \dots, n))[A_{\sigma'(1,\dots,n)(1)}, A_{\sigma'(1,\dots,n)(2)} \dots, A_{\sigma'(1,\dots,n)(n-1)}] = 0$$

So we have constructed a symbol in $H_n(\mathrm{GL}(R))$. We denote this symbol by $\mu(A_1, \ldots, A_n)$. We show that this symbol satisfies the multlilinear relation. This means we have to show that

$$\mu([A_1A_2, A_3, \ldots, A_n]) - \mu([A_1, A_3, \ldots, A_n]) - \mu([A_2, A_3, \ldots, A_n])$$

is the image of some boundary map. We claim this is

$$\partial(\sum_{\sigma \in S_n | \sigma^{-1}(1) < \sigma^{-1}(2)} sgn(\sigma)[A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}])$$

First take $1 \leq i \leq n-1$. Then

$$\partial_i(\sum_{\sigma \in S_n | \sigma^{-1}(1) < \sigma^{-1}(2)} sgn(\sigma)[A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}]) =$$
$$\sum_{\sigma \in S_n | \sigma^{-1}(1) < \sigma^{-1}(2)} sgn(\sigma)[A_{\sigma(1)}, \ldots, A_{\sigma(i)}A_{\sigma(i+1)}, \ldots, A_{\sigma(n)}]$$

Using a similar argument to above (apply $(i, i+1)$) we can see that every element in the sum cancels unless $\sigma(i) = 1$ or $\sigma(i) = 2$. Again using similar argument as above we see that the only elements in the image of ∂_0 that do not cancel with some element in ∂_n are elements such that $\sigma(1) = 1$. Conversely, the only elements of ∂_n that do not cancel with some element of ∂_0 are elements of the form $\sigma(n) = 2$.

So we have that μ is a multilinear symbol. However μ is likely not additive unless $n = 1$. For μ to be additive when $n = 2$ we would need that

$$\mu(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}) = \mu([A, C]) + \mu([B, D])$$

Using linearity we can see that this is equivalent to

$$\mu(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}) = 0$$

For any A, D. This is not true in general. However, given a multilinear symbol we can define a additive symbol. We first do this for the case $n = 2$. We define

$$c(A, B) \coloneqq \mu(A, B) - \mu(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix})$$

$c(A, B)$ is also bilinear so we only need to see the identity holds above.

$$c(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}) = \mu(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}) - \mu(\begin{pmatrix} A & 0 & 0 & 0) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & D \end{pmatrix})$$

Changing basis and using the fact we are working in $\mathrm{GL}(R)$ gives the result.

We can rewrite this formula as

$$c(A, B) = \mu(\begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B^{-1} \end{pmatrix})$$

We outline how to construct an additive symbol generally and then present this map in the case $n = 3$.

Let $A_1, \ldots, A_n$ be commuting automorphisms of $R^m = P$ and let $f \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a function. Define $f[A_1, \ldots, A_n] = [B_1, \ldots, B_n]$ where B_i is an automorphism of P^n of the form

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & A_i & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

where A_i is in the $(f(i), f(i))$ position. Using the same argument as above we can see that a symbol is additive providing that $c(f[A_1, \ldots, A_n]) = 0$ whenever f with exactly 2 elements in the image. We will construct a symbol such $c(f[A_1, \ldots, A_n]) = 0$ whenever f is not constant.

Take any multilinear symbol μ. Define

$$c_n(A_1, \ldots, A_n) = \mu([A_1, \ldots, A_n]) - \mu(f[A_1, \ldots, A_n])$$

where f is the identity on $\{1, \ldots, n\}$. It is easy to see that $c_n(f[A_1, \ldots, A_n]) = 0$ for any bijective f. Inductively, we define

$$c_{i-1}([A_1, \ldots, A_n]) = c_i([A_1, \ldots, A_n]) - \sum_{f\colon \{1,\ldots,n\}\to\{1,\ldots,n\} s.t. |im(f)=i-1|} c_i(f[A_1, \ldots, A_n])$$

The sum is over functions $f\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $f(1) = 1$, the image has precisely $i-1$ elements and $f(j) \leq \max\{f(1), \ldots, f(n)\} + 1$ for all j

Example 6.3.1. *We do the above computation when $n = 3$. First let*

$$c_3([A_1, \ldots, A_3]) \coloneqq \mu([A_1, A_2, A_3]) - \mu(\begin{pmatrix} A & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & A_2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & A_3 \end{pmatrix})$$

Next we define c_2 to be

$$c_2([A_1, \ldots, A_3]) \coloneqq c_3([A_1, A_2, A_3]) - c_3([\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_3 \end{pmatrix}])$$
$$-c_3([\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_3 \end{pmatrix}]) - c_3([\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} A_3 & 0 \\ 0 & 1 \end{pmatrix}])$$

Writing this in terms of μ gives

$$c([A_1,A_2,A_3]) \coloneqq \mu([A_1,A_2,A_3]) - \mu([\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_3 \end{pmatrix}])$$

$$-\mu([\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_3 \end{pmatrix}]) - \mu([\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} A_3 & 0 \\ 0 & 1 \end{pmatrix}])$$

$$+2\mu(\begin{pmatrix} A_1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & A_2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & A_3 \end{pmatrix})$$

This map gives an additive multilinear symbol. We conjecture that the steinberg relation and the exact sequence relation hold under this map. One may be able to prove that they do by using homotopy invariance as was done in Grayson's definition of higher K-theory. Therefore we conjecture that the map

$$\widetilde{K}_n(R) \to H_n(\mathrm{GL}(R))/H_n(\mathrm{GL}_{n-1}(R))$$
$$[R^n, A_1, \ldots, A_n] \mapsto c(A_1, \ldots, A_n)$$

is well-defined.

Recall that the map $K_n^M(R) \to H_n(\mathrm{GL}(R))/H_n(\mathrm{GL}_{n-1}(R))$ is an isomorphism. One can see that the composition

$$K_n^M(R) \to \widetilde{K}_n^M(R) \to H_n(\mathrm{GL}(R))/H_n(\mathrm{GL}_{n-1}(R))$$

is equal to a constant multiple of the above map.This constant should be $(n-1)!$ but we have no proof of this. It should also be true that the map we have defined above factors as

$$\widetilde{K}_n^M(R) \to K_n^Q(R) \to H_n(\mathrm{GL}_n(R)).$$

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