A SUNSPOT-BASED THEORY OF UNCONVENTIONAL MONETARY POLICY*

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Abstract

This paper is about the effectiveness of qualitative easing, a form of unconventional monetary policy that changes the risk composition of the central bank balance sheet. We construct a general equilibrium model where agents have rational expectations and there is a complete set of financial securities, but where some agents are unable to participate in financial markets. We show that a change in the risk composition of the central bank's balance sheet affects equilibrium asset prices and economic activity. We prove that, in our model, a policy in which the central bank stabilizes non-fundamental fluctuations in the stock market is self-financing and leads to a Pareto efficient outcome.

Key Words

Qualitative Easing, Unconventional Monetary Policy, Sunspots

Central banks throughout the world have engaged in two kinds of unconventional monetary policies: *quantitative easing* (QE), which is "an increase in the size of the balance sheet of the central bank through an increase in its monetary liabilities", and *qualitative easing* (QualE) which is "a shift in the composition of the assets of the central bank towards less liquid and riskier assets, holding constant the size of the balance sheet."[1]

Because qualitative easing is conducted by the central bank, it is often classified as a monetary policy. But because it adds risk to the public balance sheet that is ultimately borne by the taxpayer, QualE is better thought of as a fiscal or quasi-fiscal policy (Buiter, 2010). This distinction is important because, in order to be effective, QualE necessarily redistributes resources from one group of agents to another.

In theoretical papers that study the effectiveness of QualE, researchers often assume that financial markets are complete and that everyone has access to them. When these two conditions hold, a change in the risk composition of the central bank's balance sheet has no effect on asset prices (Woodford, 2012): QualE is ineffective because market participants are able to undo the effects of a portfolio shift by the central bank through private trades in securities. We will demonstrate here that when we relax the assumption of complete participation, QualE starts being effective because it redistributes resources across states of nature for agents unable to insure themselves in financial markets.

We make the case for the effectiveness of qualitative easing by constructing an analytically tractable general equilibrium model where agents are rational and have rational expectations, and where the financial markets are complete. Our setup has two important features. First, some people in our model do not trade in the financial markets. Second, people in our model use money as a medium of exchange. These assumptions ensure that, in the absence of uncertainty, the model possesses multiple equilibria. It also implies that the underlying mechanism is different from that in Arajo et al. (2015), and can be seen as an application of the insights of Cass and Shell (1983) and Balasko and Shell (1993) to the study of QualE effectiveness in a monetary economy.

[1] The quotes are from Buiter (2008).

We focus on sunspot shocks because we believe they play a key role in periods of financial distress and are important drivers of financial volatility.[2] We show that when these non-fundamental disturbances play a dominant role, then a central bank that takes risk onto its balance sheet can replicate the efficient, full-participation allocation, and that the optimal intervention is self-financing. Importantly, in such circumstances, the policy can be implemented by stabilizing equity prices so that the return to the stock market is equal in every state to the return on a one-period real government bond. While these implementation recommendations are sensitive to the assumed mixture of shocks, the broader result obtains even when all shocks are fundamental: In the presence of incomplete asset market participation, unconventional monetary policy matters and can be used to engineer the efficient, full-participation allocation.

1. How our Model is Related to Previous Literature

Our model is related to the work of Cass and Shell (1983). These authors construct a two-period, purely real, pure-exchange general-equilibrium model. In the first period households trade financial assets. In the second period they trade goods. In the Cass-Shell example there are multiple equilibria in the second period. They show that, if some households are not present in period 1, purely non-fundamental uncertainty can influence the equilibrium allocation of goods across households.

We adapt the Cass-Shell example in two ways. First, we introduce money as a medium of exchange. Second, we build a model with production rather than pure exchange. Adding money allows us to explain the distinction between conventional monetary policy, which alters the size of the central bank's balance sheet, and unconventional

[2] Standard accounts of asset market dynamics struggle to account for the volume of trades that we observe in real world asset markets (Milgrom and Stokey, 1982), and for the observed volatility in asset prices, relative to dividend movements (Shiller, 1981). In our view, significant portions of asset price fluctuations in the real world are caused by self-fulfilling shifts from one equilibrium to another that are associated with inefficient fluctuations in wealth (see also Farmer (2014) and Farmer (2015)).

monetary policy, which alters its composition between safe and risky assets. Adding production allows us to explain how unconventional monetary policy can alter output.[3]

Although it is possible to construct fully dynamic examples of the argument we make in this paper, we have chosen instead to use a two-period model to keep the argument as simple as possible.[4] That presents the challenge of explaining why the agents would choose to hold an asset, fiat money, that will be worthless when the model ends. To meet that challenge, we adopt a device proposed by Starr (1974) and used by Balasko and Shell (1993). We assume that money is required to pay taxes at the end of the second period. Money has value in our model because the government decrees it to be so.

Related work of which we are aware includes papers by McMahon and Polemarchakis (2011) and McMahon et al. (2018). Although the environments they consider are similar to ours, these authors do not study the optimal monetary policy and they do not explicitly model an equilibrium selection rule as we do here. Hall and Reis (2016) have studied the implications of policies that pay interest on reserves for price level stabilization and Reis (2016) studies the role of unconventional monetary policies in response to a future fiscal crisis. Neither of these approaches considers the implications, for monetary policy, of incomplete asset market participation.

Two alternative theories to ours include the market segmentation approach of Vayanos and Vila (2009) and Greenwood et al. (2015). These authors posit that different asset purchasers inhabit different segments of the market. Alternatively, Gertler and Karadi (2011), Curdia and Woodford (2011) and He and Krishnamurthy (2013) present theories in which capital constraints may be alleviated by large scale central bank asset purchases that offset the restrictions imposed by borrowing constraints. Our own work is complementary to both the segmentation approach and the capital constraints theories,

[3] To model money, we include the real value of money balances as an argument of utility functions. This approach originated with Patinkin (1956) and we think of it as a short-cut that explains why people choose to hold an asset that is dominated in rate-of-return. For convenience, and to simplify algebra, we assume that money, measured relative to the money wage, yields utility.

[4] See Farmer (2018) for an example of a dynamic model that uses a similar argument.

though we highlight an alternative reason for incomplete participation: the typical half-life of a company is about a decade (Daepp et al., 2015) and so prospective entrepreneurs may not be able to insure against fluctuations in future profits, simply because the firms that will generate those profits don't exist today.[5]

2. A simple Two-Period Model

In this section we construct a simple stylized model that is, nevertheless, rich enough to capture the main points of our argument.

2.1. **Assumptions about Workers and Entrepreneurs.** There are two periods, two types of people and two public agents: a central bank and a treasury. We refer to type 1 people as workers and to type 2 people as entrepreneurs, with corresponding variables indexed using the subscript $i \in \{1, 2\}$. Workers are alive in both periods and they are each endowed, in period 2, with one unit of leisure. Entrepreneurs are alive only in period 2 and they are endowed with a technology for producing a unique consumption good in that period. While these specific assumptions are meant to capture the relatively short lifespan of private companies, in a more complicated model with multiple periods there could be all types of agents present in all periods, as long as there would exist markets, which some agents would be unable to trade in.

In period 1, workers trade in asset markets with the central bank and with the treasury. Production and consumption take place only in period 2. There is a paper asset called money, that is an argument of workers' utility functions. We show in Appendix A that workers face the following life-cycle budget constraint

$$pc_1 + w(1 - n_1) + rM_1 \leq \mathcal{W}, \tag{1}$$

where,

$$\mathcal{W} \equiv w + \frac{TR}{Q} - \mathcal{T}_1 \tag{2}$$

is the dollar value, at date 2, of a worker's wealth, p is the dollar price of goods, w is the money wage, Q is the price of a dollar valued pure-discount bond and $r \equiv 1/Q - 1$

[5] Clearly, the life spans of workers, while typically longer, tend to be finite as well.

is the money interest rate. The terms n_1, c_1, M_1, TR and $\mathcal{T}_1$ are, respectively, labour supplied, consumption and money demanded, money transfers received by workers from the treasury and nominal tax obligations of the workers.

2.2. **Assumptions about the Treasury and the Central Bank.** The treasury finances transfers to workers in period 1 by issuing dollar denominated discount bonds that are worth B dollars in period 2 and sell for price Q in period 1. An amount A_{CB} of these bonds are purchased by the central bank to back the monetary base, M. Because the bank does not pay interest on its liabilities, the creation of money generates equity, E_{CB}, for the central bank, equal to the present value of the bank's seigniorage revenues, $\mathcal{S}$, where $\mathcal{S}$ is defined as $\mathcal{S} \equiv (A_{CB} - M) = rM$.

Assets	Liabilities
QA_{CB}	M
$Q\mathcal{S}$	
	E_{CB}

TABLE 1. The Central Bank Balance Sheet

Table 1 represents the bank's balance sheet in period 1. At date 2, the treasury repays its debt by raising taxes $\mathcal{T}$ or from seigniorage revenues, $\mathcal{S}$. The fact that the treasury must remain solvent leads to the government's intertemporal budget constraint,

$$Q(\mathcal{T} + \mathcal{S}) = TR. \tag{3}$$

This equation clarifies that the dollar value of the transfer to the workers in period 1 is equal to the present value of tax revenues plus the present value of seigniorage revenue. Importantly, this can be seen as a variant of passive fiscal policy (Leeper, 1991) since the government is assumed to balance its books for any underlying price level.[6]

2.3. **The Equal Treatment Assumption.** We assume that people alive in each period are treated equally, and thus workers receive the entire transfer and entrepreneurs and workers share the tax burden equally. It follows from the equal treatment assumption

[6] Because both liabilities and tax proceeds are assumed to be nominal, therefore the government cannot rely on debt deflation to equate their respective values.

that the per-person values of taxes and transfers in period 2, as functions of B and M, are

$$\mathcal{T}_i = \left(\frac{B - rM}{2}\right), \ i \in \{1, 2\} \qquad \text{and} \qquad \frac{TR}{Q} = B. \tag{4}$$

Crucially, fiscal policy reallocates from one group to another: because entrepreneurs are not present in period 1, they do not benefit from the initial transfer; they do, however, incur half of the cost of paying for the resulting fiscal obligations in period 2.

These assumptions allow us to express workers' wealth in period 2 as follows,

$$\mathcal{W} \equiv w + \left(\frac{B}{2} + \frac{rM}{2}\right). \tag{5}$$

This expression clarifies that the value of the transfer depends on both fiscal policy, represented by B, and monetary policy, represented by M.[7]

3. Equilibria Under the Perfect Foresight Assumption

In this section we derive the demand and supply functions of workers and entrepreneurs and we define the concept of a competitive perfect foresight equilibrium. Our main result is that, because different price levels correspond to different real values of the nominal transfer, therefore real equilibrium allocations are affected by the value of the numeraire.

3.1. The Behavior of Workers Under Certainty.

3.1. The Behavior of Workers Under Certainty. Workers have logarithmic preferences defined over consumption, leisure and the real value of money balances in period 2

[7] In words, Equation (5) says that the period 2 money value of the wealth of a worker is equal to the money value of his leisure endowment plus 1/2 of the period 2 value of his transfer plus 1/2 of the government's seigniorage revenue from money creation. The first 1/2 fraction appears because workers receive the entire government transfer but only have to repay half of it ($1/2 = 1 - 1/2$). The fraction of seigniorage revenue follows from the fact that, for a given transfer, additional seigniorage revenues reduce the tax burden on both types. More generally, the wealth effect of a transfer policy will depend on the population growth rate and the period length.

with weights λ on consumption, μ on leisure and γ, on real money balances,

$$\mathcal{U}_1 = \lambda \log c_1 + \mu \log(1 - n_1) + \gamma \log \left(\frac{M_1}{w} \right),$$

where $\lambda + \mu + \gamma = 1$. While the assumption of logarithmic utility is not important for the construction of a perfect foresight equilibrium, it implies that entrepreneurs and workers are risk averse, which will affect outcomes when we introduce uncertainty in Section 5. Crucially, because people are risk averse, equilibria where non-fundamental shocks influence the allocation of goods across states are Pareto inefficient.

Workers maximize utility subject to the lifecycle budget constraint. The solution to this problem, given the assumption of logarithmic preferences, is for the workers' expenditure shares on leisure, consumption and money to equal the respective coefficients in the utility function times wealth, $\mathcal{W}$, that is,

$$w(1 - n_1) = \mu \mathcal{W}, \qquad pc_1 = \lambda \mathcal{W}, \qquad rM_1 = \gamma \mathcal{W}. \tag{6}$$

Rearranging terms in Equation (6), we obtain the following expression for the labour supply function,

$$n_1 = 1 - \mu \frac{\mathcal{W}}{w}. \tag{7}$$

3.2. **The Behavior of Entrepreneurs Under Certainty.** Entrepreneurs do not participate in the asset markets since, by assumption, the companies they own start operating in period 2. Each entrepreneur owns a decreasing returns-to-scale technology,

$$y = n_2^\alpha$$

that transforms labour into output. Entrepreneurs receive real profits, $\Pi \equiv n_2^\alpha - (w/p)n_2$, and they choose labour demand, n_2, to solve the problem

$$\max_{\{n_2\}} \mathcal{U}_2 = \log \left(n_2^\alpha - \frac{w}{p} n_2 - \frac{\mathcal{T}_2}{p} \right),$$

where the argument of the logarithmic utility function represents the consumption of entrepreneurs, which is assumed equal to the entrepreneur's after tax profit. The solution to this problem is characterized by the labour demand and output supply functions,

$$n_2 = \left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{1}{\alpha-1}}, \qquad y = \left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}}, \tag{8}$$

and the entrepreneurs' consumption demand function,

$$c_2 = (1-\alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B - rM)}{2p}, \tag{9}$$

where we have made use of Equation (4) to write the dollar value of the entrepreneur's taxes, $\mathcal{T}_2$, as a function of fiscal policy, represented by B, and monetary policy, represented by M.

3.3. **The Definition of Perfect Foresight Equilibria.** In this section we write down three equations that characterize equilibria. These are the excess demand equations for labour, consumption and money, which we set equal to zero in a competitive equilibrium. These excess demand functions are given by the expressions,

$$\overbrace{\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{1}{\alpha-1}}}^{\text{Labour Demand}} - \overbrace{\left(1 - \mu\frac{\mathcal{W}}{w}\right)}^{\text{Labour Supply}} = 0, \tag{10}$$

$$\overbrace{(1-\alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B - rM)}{2p}}^{\text{Entrepreneur's Consumption}} + \overbrace{\frac{w}{p}\lambda\frac{\mathcal{W}}{w}}^{\text{Workers' Consumption}} - \overbrace{\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}}}^{\text{Output}} = 0, \tag{11}$$

$$\overbrace{\gamma\frac{\mathcal{W}}{r}}^{\text{Money Demand}} - \overbrace{M}^{\text{Money Supply}} = 0. \tag{12}$$

The three goods in our model are labour, consumption, and money. The three dollar denominated prices are the money price of goods, p, the money wage, w, and the money interest rate, r. We will characterize equilibria as feasible solutions to these equations.[8]

Definition 1. *A competitive equilibrium is a monetary policy and a fiscal policy* $\{M, B\}$*, an allocation* $\{\{c_i, n_i\}_{i=1,2}, M_1, y\}$ *and a set of prices* $\{p, w, r\}$ *that satisfies non-negativity,*

[8] It follows from Walras law that if the excess demands for money, goods and labour are equal to zero then the quantity of bonds demanded is also equal to the quantity supplied.

budget balance and optimality. An equilibrium price system is a non-negative triple $\{p, w, r\}$ *such that equations (10), (11) and (12) hold.*

Our definition of a competitive equilibrium is fairly standard. Proposition 1 establishes that there is a continuum of equilibria and it characterizes them in closed form.

Proposition 1. *Let* $\{M, B\} \geq 0$ *characterize monetary and fiscal policy, and call* w *feasible if it satisfies,*

$$w \geq \frac{(\mu + \alpha\lambda)B}{(1 + \lambda - \mu) - 2\lambda\alpha}.$$

For all feasible values of w*, there exists a competitive equilibrium indexed by* w*. The equilibrium level of nominal wealth, the interest rate and the real wage are given by,*

$$\mathcal{W} = \frac{2w + B}{1 + \lambda + \mu}, \qquad r = \frac{\gamma}{M}\mathcal{W}, \qquad \frac{w}{p} = \alpha\left(1 - \frac{\mu\mathcal{W}}{w}\right)^{\alpha - 1},$$

and the equilibrium values of $\{n_i, c_i\}_{i \in 1,2}$*,* y *and* M_1 *are determined by equations (6),* (7)*, (8)and (9).* $\square$

The role of the feasibility condition is to rule out wages that would result in negative prices or negative allocations in one or more states. See Appendix B for a proof of this proposition.

The intuition behind there being a continuum of equilibria in our model is simple. The government effectively reallocates resources from entrepreneurs to workers: entrepreneurs pay taxes to finance government debt, but do not benefit from the initial transfer.[9] Since the transfer is nominal, changes in the price level pin down its real value. Different real values of the transfer correspond to different real equilibrium allocations.

It may seem surprising, however, that the equilibrium nominal price (or wage) level is not unique. Isn't that somehow special to our two-period model? We believe not, and there is, arguably, a fairly direct mapping from our setup to infinite-horizon, general equilibrium models typically used for policy analysis. In particular, our example features passive monetary policy – M is fixed in advance – and passive fiscal policy – the Treasury sets taxes in nominal terms, which implies that it cannot count on fluctuations in the

[9] While workers also help pay off a share of the debt, they repay less than they received.

price level to help balance its books. It is well established (Leeper, 1991), that a passive-passive combination of monetary and fiscal policies results in an indeterminate price level – and this is precisely what occurs in our setup. As pointed out above, the existence of a nominal, "intra-generational" transfer means that this nominal indeterminacy ends up having real implications.

The fact that multiple equilibria arise in our model because monetary policy is assumed to be passive may seem unappealing. Arguably, the Taylor principle, and the monetary activism that it is associated with, are widely accepted as characterizing "good policy". Furthermore, Leeper (1991) shows that the more realistic, active - passive combination of monetary and fiscal policies would guarantee price level uniqueness, which would, in turn, eliminate real indeterminacy in our setup. In recent work (Farmer and Zabczyk, 2019), however, we have demonstrated that Leeper's (1991) findings do not generalize to overlapping generations (OLG) models, and that the price level can be indeterminate even under an active-passive policy combination (and indeed, as we show, even under an active-active one) in the neighbourhood of dynamically efficient steady states.[10] It follows that our argument does not necessarily require monetary policy to be "passive", with Farmer and Zabczyk (2019) providing an example of a fully dynamic environment where equilibrium prices and wages are potentially subject to pure sunspot shocks, and where the QualE policy similar to the one we describe could help replicate the efficient, "full-participation" equilibrium.

4. Introducing Uncertainty to the Model

In this section we expand the model to allow for non-fundamental uncertainty. We assume that the workers anticipate, correctly, that there are two possible future realizations of the money wage. In one state of the world, state H, the nominal money wage is high and in the other, state L, it is low.

[10] Incidentally, OLG models explicitly account for limited participation on account of mortality, instead of the very convenient, but equally special, assumption of an infinitely-lived, representative agent underlying extant work on the Fiscal Theory of the Price Level.

4.1. **Budget Constraints Under Uncertainty.** We assume the existence of complete insurance markets, represented by a pair of Arrow securities (Arrow, 1964), one for each state. The H security pays one dollar if and only if state H occurs and the L security pays one dollar if and only if state L occurs. The H security costs $Q(H)$ dollars in period 1 and the L security costs $Q(L)$ dollars. We use the symbol $Q \equiv Q(H)+Q(L)$, to denote the price of a pure discount bond that pays one dollar for sure.

Define the *state prices* of leisure and consumption as,

$$\tilde{w}(\varepsilon) \equiv \frac{Q(\varepsilon)}{Q}\frac{w(\varepsilon)}{\pi_\varepsilon} \quad \text{and} \quad \tilde{p}(\varepsilon) \equiv \frac{Q(\varepsilon)}{Q}\frac{p(\varepsilon)}{\pi_\varepsilon}, \quad \text{for} \quad \varepsilon \in \{H, L\}, \tag{13}$$

and notice from this definition that

$$\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)} = \frac{w(\varepsilon)}{p(\varepsilon)}.$$

We show in Appendix C that the assumption of complete markets allows us to write a single lifecycle budget constraint of a worker as follows,

$$\mathbb{E}\left[\tilde{p}(\varepsilon)\, c_1(\varepsilon) + \tilde{w}(\varepsilon)\,(1 - n_1(\varepsilon))\right] + rM_1 \leq \mathcal{W}, \tag{14}$$

and that worker's wealth under uncertainty is defined as

$$\mathcal{W} \equiv \mathbb{E}[\tilde{w}(\varepsilon)] + \left(\frac{B}{2} + \frac{rM}{2}\right). \tag{15}$$

Here, $\mathbb{E}$ is the expectations operator, defined using the probability distribution $\{\pi_H, \pi_L\}$ and the term in the last bracket is the net transfer from the government.

4.2. **The Behaviour of Workers Under Uncertainty.** Workers have preferences defined over the probability weighted logarithm of consumption and leisure in each state and over the logarithm of real balances,

$$\mathcal{U}_1 = \mathbb{E}\left[\lambda \log(c_1(\varepsilon)) + \mu \log(1 - n_1(\varepsilon)) + \gamma \log\left(\frac{M_1}{w(\varepsilon)}\right)\right],$$

where, as before, $\lambda + \mu + \gamma = 1$. Workers maximize expected utility subject to their lifecycle budget constraint represented by Equation (14). The solution to this problem is for workers' expenditure shares on leisure and consumption to equal the respective coefficients in the utility function, weighted by probabilities, and for the expenditure share on money to equal the unweighted utility coefficient,

$$\tilde{w}(\varepsilon)(1-n_1(\varepsilon)) = \mu\mathcal{W}, \qquad \varepsilon \in \{H, L\},$$

$$\tilde{p}(\varepsilon)\, c_1(\varepsilon) = \lambda\mathcal{W}, \qquad \varepsilon \in \{H, L\}, \tag{16}$$

$$rM_1 = \gamma\mathcal{W}.$$

The probabilities π_H and π_L enter these equations through the definition of state prices, Equation (13). Rearranging terms in the expenditure share for leisure leads to the labour supply function in state ε,

$$n_1(\varepsilon) = 1 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \qquad \varepsilon \in \{H, L\}.$$

4.3. **The Behaviour of Entrepreneurs Under Uncertainty.** The companies that entrepreneurs eventually run are assumed not to exist in the initial period, which means that they cannot insure against fluctuations in profits by trading Arrow securities. This assumption implies that they solve two different problems, one for each realization of the state. In state ε, entrepreneurs receive real profits $\Pi(\varepsilon) \equiv n_2(\varepsilon)^{\alpha} - (\tilde{w}(\varepsilon)/\tilde{p}(\varepsilon))n_2(\varepsilon)$ and they choose labour demand $n_2(\varepsilon)$, to solve the problem,

$$\max_{n_2(\varepsilon)} \mathcal{U}_2(\varepsilon) = \log\left(n_2(\varepsilon)^{\alpha} - \frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)} n_2(\varepsilon) - \frac{\mathcal{T}_2}{p(\varepsilon)}\right).$$

The argument of the logarithmic utility function represents the consumption of the entrepreneurs and is also equal to their after-tax profits. The solution to this problem is characterized by the state-dependent labour demand and output supply functions,

$$n_2(\varepsilon) = \left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}}, \qquad y(\varepsilon) = \left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}}, \qquad \varepsilon \in \{H, L\}, \tag{17}$$

and the state-dependent consumption demand function,

$$c_2(\varepsilon) = (1-\alpha)\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}} - \frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\frac{(B-rM)}{2w(\varepsilon)}, \qquad \varepsilon \in \{H, L\}. \tag{18}$$

Recall that the state price, $\tilde{w}(\varepsilon)$, is defined as $\tilde{w}(\varepsilon) \equiv [Q(\varepsilon) w(\varepsilon)] / [Q\pi_\varepsilon]$ and notice that $w(\varepsilon)$ enters Equation (18) independently of $\tilde{w}(\varepsilon)$. This is important and, mechanically, it is the reason why different beliefs about the money wage have real effects: For every self-fulfilling belief about w, there is a different real value of taxes and transfers.

5. Incomplete Participation Equilibrium in the World of Uncertainty

In Section 5, we explore the properties of equilibria in a world of uncertainty when the asset markets are complete but entrepreneurs cannot participate in this market. We show, in this world, that non-fundamental uncertainty may have real effects on the output produced and on its allocation across people.

5.1. **Rational Expectations Equilibrium with Incomplete Participation.**

The model with two states has five goods: consumption in states H and L, leisure in states H and L and money. There are four state-contingent prices, $\tilde{p}(H)$, $\tilde{p}(L)$, $\tilde{w}(H)$, $\tilde{w}(L)$, and one non state-contingent interest rate, r. The following five excess demand functions characterize equilibrium in the model with complete markets but incomplete participation[11]

$$\overbrace{\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}}}^{\text{Labour Demand}} - \overbrace{\left(1-\mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}\right)}^{\text{Labour Supply}} = 0, \qquad \varepsilon \in \{H, L\} \tag{19}$$

$$\overbrace{\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} + \frac{(B-rM)}{2w(\varepsilon)}}^{\text{Entrepreneur's Net Supply of Output}} - \overbrace{\lambda\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}}^{\text{Workers' Consumption}} = 0, \qquad \varepsilon \in \{H, L\} \tag{20}$$

$$\overbrace{\gamma\frac{\mathcal{W}}{r}}^{\text{Money Demand}} - \overbrace{M}^{\text{Money Supply}} = 0. \tag{21}$$

In any model with multiple equilibria we must take a stand on what selects an equilibrium. Following Farmer (2012b), we select what will happen by modeling the way

[11] Equation (20) is derived from equating the sum of the consumptions demands of entrepreneurs and workers to the supply of output, multiplying the equation by $\tilde{w}(\varepsilon)/\tilde{p}(\varepsilon)$ and rearranging terms.

people form beliefs. Specifically, to complete our characterization of a rational expectations equilibrium, we define the following belief function,

$$w(\varepsilon) = \varphi(M, \varepsilon) \equiv M + \varepsilon. \tag{22}$$

In a rational expectations equilibrium, these beliefs are not only fundamental, they are also rational, i.e. fully consistent with actual equilibrium outcomes.

We have included the policy variable M in the belief function to capture the idea that beliefs depend on observable variables. We have included the random variable ε in the belief function to capture the idea that non-fundamental uncertainty may matter simply because people believe that it will.[12]

Using the definitions of the market clearing equations, we define an incomplete participation rational expectations equilibrium as follows.

Definition 2. *An incomplete participation rational expectations equilibrium is:*

- *A monetary policy and a fiscal policy* $\{M, B\}$,
- *A belief function* $\varphi(M, \varepsilon)$,
- *An allocation which consists of*
  - *A labour supply function* $n_1(\varepsilon)$ *and consumption demands* $c_i(\varepsilon)$ *for* $i = \{1, 2\}$,
  - *Functions for aggregate output* $y(\varepsilon)$ *and labour demand* $n_2(\varepsilon)$,
  - *Money demand* M_1,
- *An equilibrium price system consisting of*
  - *A price function* $p(\varepsilon)$ *and a wage function* $w(\varepsilon)$,
  - *A security pricing function* $Q(\varepsilon)$,
  - *A money interest rate* $r = \dfrac{1}{Q(H) + Q(L)} - 1$.

[12] Our example, where money is the only fundamental that affects beliefs, is very special. More generally, beliefs about the future wage might depend on current and past wages, or on current and past output or employment. For an example of a dynamic model closed with a belief function see Farmer (2012a).

The allocation is such that equations (19), (20) and Equation (21) hold when the money wage in each state is given by Equation (22) and the state prices are defined from the equilibrium price system by Equation (13)*.*

We next define a set of feasibility conditions on the properties of a belief function.

Definition 3. *A belief function $\varphi(M,\varepsilon)$ is feasible under monetary policy M and fiscal policy B if $w(\varepsilon)=\varphi(M,\varepsilon)$ satisfies,*

$$w\left(\varepsilon\right)\geq\frac{\left(\mu+\alpha\lambda\right)B}{\left(1+\lambda-\mu\right)-2\lambda\alpha},\quad\varepsilon\in\left\{H,L\right\}.\tag{23}$$

We now turn to a proposition that characterizes the properties of an incomplete participation rational expectations equilibrium. To complete the statement of this proposition we will need the following lemma.

Lemma 1. *Let $\varphi(M,\varepsilon)$ be a feasible belief function and define the following numbers θ, X_L, Y_L, X_H, Y_H, θ_1 and θ_2,*

$$\theta\equiv\frac{\lambda+\mu}{\gamma},$$

$$X_L\equiv\left[2w\left(L\right)+B\right],\qquad X_H\equiv\left[2w\left(H\right)+B\right],\qquad Y_L\equiv2\pi_L\theta,\qquad Y_H\equiv2\pi_H\theta,$$

$$\theta_1\equiv\frac{\left[X_H\left(1+Y_L\right)-X_L\left(1+Y_H\right)\right]}{X_LY_H},\qquad\theta_2\equiv\frac{X_HY_L}{X_LY_H}.$$

The quadratic equation

$$q^2-\theta_1q-\theta_2=0.$$

has a unique real positive solution.

Lemma 1 is proved in Appendix D.

Proposition 2. *Let $\{M,B\}\geq0$ characterize public sector policy. For every feasible belief function, $\varphi(M,\varepsilon)$, let $\theta,X_L,Y_L,X_H,Y_H,\theta_1,\theta_2$ be the numbers defined in Lemma 1.*

There exists a unique incomplete participation rational expectations equilibrium, characterized by the following conditions:

- *The equilibrium ratio of Arrow security prices $q \equiv Q(L)/Q(H)$, is the unique positive solution to the quadratic equation*

$$q^2 - \theta_1 q - \theta_2 = 0. \tag{24}$$

- *The equilibrium Arrow security prices satisfy*

$$Q(H) = \frac{(q + Y_L[q+1])\, M}{X_L q (1+q) + (1+q)(q + Y_L[q+1])\, M}, \tag{25}$$

and

$$Q(L) = qQ(H). \tag{26}$$

- *The equilibrium state wages $\tilde{w}(\varepsilon)$ are equal to*

$$\tilde{\omega}(L) = \frac{w(L)}{(1+q^{-1})\,\pi_L} \qquad \text{and} \qquad \tilde{\omega}(H) = \frac{w(H)}{(1+q)\,\pi_H}. \tag{27}$$

- *The equilibrium state prices are equal to*

$$\tilde{p}(\varepsilon) = \frac{\tilde{w}(\varepsilon)}{\alpha}\left(1 - \frac{\mu \mathcal{W}}{\tilde{w}(\varepsilon)}\right)^{1-\alpha}, \quad \text{for} \quad \varepsilon \in \{H, L\}. \tag{28}$$

- *The price of a safe bond, Q, the money interest rate r and the date 2 value of the wealth of workers are given by the expressions,*

$$Q = Q(L) + Q(H), \qquad r \equiv \frac{1-Q}{Q}, \qquad \text{and} \qquad \mathcal{W} = rM. \tag{29}$$

Conditional on the $\tilde{w}(\varepsilon), \tilde{p}(\varepsilon)$ and $Q(\varepsilon)$ characterized above, the equilibrium quantities $c_1(\varepsilon), n_1(\varepsilon)$ and M_1 are given by Equations (16), while $n_2(\varepsilon), y(\varepsilon)$ and $c_2(\varepsilon)$ are given by Equations (17) *and* (18).

Proposition 2, proved in Appendix E, establishes a mapping from beliefs to equilibrium prices and allocations. The following corollary confirms that these beliefs don't only affect nominal prices but also the corresponding real allocations.

Corollary 1. *Whenever $w(L) \neq w(H)$,*

$$n_i(L) \neq n_i(H), \quad i \in \{1,2\} \qquad \text{and} \qquad c_i(L) \neq c_i(H), \quad i \in \{1,2\}. \tag{30}$$

Corollary 1 is proved in Appendix F. This is an example, for a monetary economy, of Cass and Shell's (1983) result that, when there is incomplete asset market participation, "sunspots matter".

6. Complete Participation Equilibrium in the World of Uncertainty

In this section we consider a counter-factual economy in which entrepreneurs are present in the asset markets that open before their companies start operating and we derive their decision rules in these markets. Although there are still multiple equilibria in the complete participation case, sunspots cease to have real effects.

6.1. **Entrepreneur's Choice under Complete Participation.** We continue to assume that entrepreneurs only care about consumption and receive no part of the government transfer. We alter the assumptions of the previous section by allowing entrepreneurs to trade assets in period 1, subject to the first period constraint,

$$\sum_{\varepsilon \in \{L,H\}} Q(\varepsilon) A_2(\varepsilon) = 0. \tag{31}$$

When the entrepreneur chooses labour optimally, her pre-tax profit in each state is given by,

$$\Pi(\varepsilon) = (1-\alpha)\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}},$$

and her consumption in each state is constrained by the single budget constraint,

$$p(\varepsilon) c_2(\varepsilon) \leq p(\varepsilon) \Pi(\varepsilon) - \mathcal{T}_2 + A_2(\varepsilon), \tag{32}$$

where $\mathcal{T}_2 \equiv (B - rM)/2$ is her nominal tax liability and the first term on the right side of this inequality is the money value of profits. Substituting Inequality (32) into (31) and using the no arbitrage condition and the definition of state prices leads to the entrepreneur's lifecycle constraint,

$$\mathbb{E}[\tilde{p}(\varepsilon) c_2(\varepsilon)] \leq \mathbb{E}[\tilde{p}(\varepsilon)\Pi(\varepsilon)] - \left(\frac{B - rM}{2}\right).$$

When the entrepreneur allocates consumption across states to maximize expected utility she will choose the following consumption demands,

$$c_2(\varepsilon) = \frac{\mathbb{E}[\tilde{p}(\varepsilon)\Pi(\varepsilon)]}{\tilde{p}(\varepsilon)} - \frac{1}{\tilde{p}(\varepsilon)}\left(\frac{B - rM}{2}\right). \tag{33}$$

Notice, and this is important, that dollar prices $p(\varepsilon)$ or $w(\varepsilon)$ no longer separately appear in the entrepreneur's budget constraint, which can be expressed entirely using state prices. This corresponds to the fact that, instead of consuming the after tax profit in each state, access to an insurance market allows the entrepreneur to smooth her consumption.

6.2. **Rational Expectations Equilibrium with Complete Participation.** In this sub-section we characterize the equations that define equilibrium in the complete participation economy.

Recall that $\mathcal{W}$, the wealth of a worker, is defined as

$$\mathcal{W} = \mathbb{E}[\tilde{w}(\varepsilon)] + \left(\frac{B}{2} + \frac{rM}{2}\right). \tag{34}$$

Using this definition, the labour market equilibrium condition is given by Equation (35),

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = 1 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \tag{35}$$

and equating the consumption demands of workers and entrepreneurs to the supply of output, the goods market equilibrium condition is,

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}} = \frac{1}{\tilde{p}(\varepsilon)}\left\{\mathbb{E}[\tilde{p}(\varepsilon)\Pi(\varepsilon)] - \left(\frac{B - rM}{2}\right)\right\} + \lambda\frac{\mathcal{W}}{\tilde{p}(\varepsilon)}. \tag{36}$$

Finally, equality of the demand and supply of money is represented by Equation (37),

$$\gamma\mathcal{W} = rM. \tag{37}$$

The important difference of the equations that characterize the complete participation economy from the incomplete participation economy, is to be found in Equation (33), which no longer contains terms in $w(L)$ or $w(H)$. With the entrepreneur and worker both able to fully insure, the actual realization of the nominal wage does not affect their choices and, consequently, the labour and goods market equilibrium allocations.

Definition 4. *A complete participation rational expectations equilibrium comprises the same elements as an incomplete participation rational expectations equilibrium (Definition 2). The allocation is such that Equations* (35), (36) *and Equation* (37) *hold when the money wage in each state is given by Equation (22) and the state prices are defined from the equilibrium price system by Equation* (13).

Next, we turn to a proposition that characterizes the properties of employment, output and the distribution of output in the complete participation rational expectations equilibrium.

Proposition 3. *Let* $\{M, B\} \geq 0$ *characterize public sector policy. For every feasible belief function,* $\varphi(M, \varepsilon)$*, there exists a unique complete participation rational expectations equilibrium, characterized by the following conditions:*

- *The equilibrium Arrow security prices satisfy*

$$Q(H) = \frac{M(2-\gamma)\pi_H}{(1+q)(M(2-\gamma)+B\gamma)\pi_H + 2\gamma w(H)} \qquad \textit{and} \qquad Q(L) = qQ(H) \quad (38)$$

where $q \equiv Q(L)/Q(H)$ *is equal to*

$$q = \frac{w(H)}{w(L)} \frac{\pi_L}{\pi_H}. \quad (39)$$

- *The equilibrium state wages* $\tilde{w}(\varepsilon)$ *are equal to*

$$\tilde{\omega}(L) = \frac{w(L)}{(1+q^{-1})\pi_L} \qquad \textit{and} \qquad \tilde{\omega}(H) = \frac{w(H)}{(1+q)\pi_H}. \quad (40)$$

- *The equilibrium state prices are equal to*

$$\tilde{p}(\varepsilon) = \frac{\tilde{w}(\varepsilon)}{\alpha}\left(1 - \frac{\mu\mathcal{W}}{\tilde{w}(\varepsilon)}\right)^{1-\alpha} \quad \textit{for} \quad \varepsilon \in \{H, L\}. \quad (41)$$

- *The price of a safe bond,* Q*, the money interest rate* r *and the date 2 value of the wealth of workers are given by the expressions,*

$$Q = Q(L) + Q(H), \qquad r \equiv \frac{1-Q}{Q}, \qquad \textit{and} \qquad \mathcal{W} = rM. \quad (42)$$

Conditional on the state prices, $\tilde{w}(\varepsilon)$, $\tilde{p}(\varepsilon)$ and $Q(\varepsilon)$, the equilibrium quantities $c_1(\varepsilon)$, $n_1(\varepsilon)$ and M_1 can be found from Equations (16) *while $n_2(\varepsilon), y(\varepsilon)$ and $c_2(\varepsilon)$ are characterized in Equations (17) and (33).* □

See Appendix G for a proof of this proposition. We also have the following corollary,

Corollary 2. *Under full participation, the equilibrium associated with **any** belief function has the property that*

$$n_i(L) = n_i(H), \quad i \in \{1,2\} \qquad \textit{and} \qquad c_i(L) = c_i(H), \qquad i \in \{1,2\}.$$

Proof. The proof follows directly from the proposition. The formula for q implies that state-wages $\tilde{w}(\varepsilon)$ are the same in both states

$$q \equiv \frac{Q(L)}{Q(H)} = \frac{w(H)}{w(L)}\frac{\pi_L}{\pi_H} \quad \Leftrightarrow \quad \frac{Q(L)w(L)}{\pi_L Q} = \frac{Q(H)w(H)}{\pi_H Q} \quad \Leftrightarrow \quad \tilde{\omega}(L) = \tilde{\omega}(H).$$

Accordingly, straight from the definition, so are state-prices $\tilde{p}(\varepsilon)$. The solutions to the workers' optimization problems, then imply that the corresponding real allocations are state-invariant. This establishes that, in a complete participation rational expectations equilibrium, there is complete insurance. ∎

To clarify what is happening in the model, we now characterize the entrepreneur's asset portfolio.

Proposition 4. *In the full participation model, the entrepreneur's asset position is given by*

$$A_2(H) = \pi_L\left(\frac{B - rM}{2}\right)\left(1 - \frac{w(H)}{w(L)}\right), \qquad A_2(L) = \pi_H\left(\frac{B - rM}{2}\right)\left(1 - \frac{w(L)}{w(H)}\right).$$

Proposition 4 is proved in Appendix H.

An immediate implication of this proposition and the fact that $w(H) > w(L)$ is that $A_2(H)$ is negative, while $A_2(L)$ is positive. The entrepreneur uses the asset market to buy insurance from the workers against the $w(L)$ outcome and to sell insurance to the workers against the $w(H)$ outcome.

When the entrepreneur is excluded from trade in Arrow securities, her utility is higher in the high wage state for two reasons. First, the entrepreneur pays for part of the nominal transfer which workers receive from the government. Higher nominal wages mean that the *real* value of the transfer is lower which makes her better off when $\varepsilon = H$. Second, the fact that workers are poorer in state $\varepsilon = H$ means that they consume less leisure and that equilibrium employment, output and the real value of profits are higher. In contrast, workers are worse off in state $\varepsilon = H$. Both groups of agents will trade Arrow securities up to the point at which their *real* consumption and leisure are constant across states.

6.3. **Nominal Bond and Equity Portfolios.** In this sub-section we translate the abstract notion of Arrow securities into the more familiar case in which agents cross-insure using debt and equity. We assume that a nominal bond pays a dollar in both states, while equities entitle their owners to a share of the entrepreneur's nominal profit stream, which we denote with the symbol $\tilde{\Pi}(\varepsilon)$ to distinguish it from the real profit stream, $\Pi(\varepsilon)$,

$$\tilde{\Pi}(\varepsilon) \equiv p(\varepsilon)\, y(\varepsilon) - w(\varepsilon) n_2(\varepsilon).$$

Using these definitions we prove the following proposition.

Proposition 5. *In the full participation model, the entrepreneur purchases nominal bonds with a face value of*

$$B_2 \equiv \frac{B - rM}{2},$$

where

$$r = \frac{1-Q}{Q}.$$

The purchase of bonds by entrepreneurs is financed by selling a share ψ of the entrepreneur's profit stream where

$$\psi = \frac{QB}{Q(L)\,\tilde{\Pi}(L) + Q(H)\,\tilde{\Pi}(H)}.$$

Proof. See Appendix I. ∎

If workers and firms were to trade two assets, debt and equity, the entrepreneur would use nominal bonds to insure herself against volatility in real tax expenditures. In equilibrium, fluctuations in the nominal price level cause fluctuations in the real value of tax liabilities that are perfectly insured by the nominal bonds she purchases from workers. Workers provide this insurance by purchasing a share in the firm. This share is risk free because fluctuations in the nominal profit stream are offset, in equilibrium, by fluctuations in the price level.

The equilibrium with complete participation Pareto dominates the equilibrium in the absence of complete participation because it provides an additional opportunity for trade. In Section 7 we show how the government can restore Pareto efficiency, even if entrepreneurs are not present in period 1, by trading on their behalf.

7. The Role of Qualitative Easing in a World of Incomplete Participation

We are now ready to discuss the role of qualitative easing; a policy in which the Central Bank, or the Treasury, makes trades of debt for equity in the asset markets. In a complete markets environment, with complete participation, a policy of this kind will have no real effects. We show that, in an environment with incomplete participation, central bank trades in the asset markets can help restore that Pareto efficient allocation.

We return to the case where entrepreneurs are excluded from participating in asset trades, and we assume that the treasury makes dollar denominated lump-sum transfers worth QB to workers, paid for by issuing nominal debt with a face value of B. We retain the assumption that workers trade two Arrow securities and we additionally account for the possibility of trading debt and equity. In this environment, there are redundant assets since the returns to debt and equity can be replicated by trades in Arrow securities.

A bond is a claim to B dollars in state ε that can be replicated by the purchase of B Arrow securities of type L and B securities of type H. Equity, is a claim to $\psi\tilde{\Pi}(\varepsilon)$ dollars in state ε that can be replicated by the purchase of a portfolio of $\psi\tilde{\Pi}(L)$ securities of type L and $\psi\tilde{\Pi}(H)$ securities of type H where ψ denotes the share of the firm bought.

We assume that workers purchase Arrow securities and they do not buy or sell bonds or equities.[13] We continue to assume that the central bank purchases debt A_{CB} where,

$$M = QA_{CB}.$$

In addition, we allow the bank to make supplementary trades in debt and equity, subject to the constraint that these supplementary security purchases are self-financing,

$$M = QA_{CB} + Q\tilde{A}_{CB} + P_E\psi_{CB}.$$

The self-financing condition implies that

$$Q\tilde{A}_{CB} + P_E\psi_{CB} = 0.$$

Here, $\tilde{A}_{CB}$ are additional purchases of debt by the central bank that may be positive or negative, ψ_{CB} is the number of shares to the nominal profit stream that is bought or sold by the central bank and

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H),$$

is the price of the whole firm. We allow short sales so that ψ_{CB} may be negative.

Let $\mathcal{S}$ denote seigniorage revenues associated with money issuance and define

$$\tilde{\mathcal{S}}(\varepsilon) = \mathcal{S} + \left[\tilde{\Pi}(\varepsilon) - \tilde{A}_{CB}\right],$$

where the term in the square brackets is the additional profit or loss associated with the risky component of the bank's balance sheet. Using these definitions, we arrive at the balance sheet of the central bank presented in Table 2. As in our previous model, the seigniorage revenues from money creation are repaid to the treasury. However, there is now risk associated with the central bank's "unconventional" asset holdings. The following modified definition of an equilibrium accounts for the fact that the central bank trades in the asset markets.

[13] This assumption is made for convenience. Because bonds and equities are redundant securities, the allocation of worker's assets across the two Arrow securities plus debt and equity is indeterminate.

Assets	Liabilities
QA_{CB}	M
$Q\mathcal{S}$	
$Q\tilde{A}_{CB}$	$P_E\psi_{CB}$
	E_{CB}

Table 2. The Central Bank Balance Sheet

Definition 5. *A rational expectations equilibrium with a self-financing stabilization policy is a monetary and a fiscal policy* $\{M, B, A_{CB}, \psi_{CB}\}$*, a belief function* $\varphi(M, \varepsilon)$*, an allocation* $\{c_i(\varepsilon), n_i(\varepsilon)\}_{i=1,2}, y(\varepsilon), M_1\}_{\varepsilon\in\{H,L\}}$ *and a set of state-dependent prices* $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ *that satisfies budget balance, optimality and the self-financing condition,*

$$Q\tilde{A}_{CB} + \psi_{CB}P_E = 0,$$

where

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H).$$

An equilibrium price system, $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$*, is a non-negative 7-tuple such that equations (35) and (36) hold in each state, Equation (37) holds, and the money wage in each state is given by the belief function, Equation (22).*

Proposition 6, establishes that there exists a set of central bank trades that leads to the same real allocations as those in the complete participation case of Proposition 3.

Proposition 6. *Let* $\left\{M, B, \tilde{A}_{CB}, \psi_{CB}\right\} \geq 0$ *characterize public sector policy, and let* $\left\{w(L), w(H)\right\} > 0$ *be wages implied by a feasible belief function* $\varphi(M, \varepsilon)$ *such that*

$$w(\varepsilon) \geq \frac{(\mu + \alpha\lambda)B}{(1 + \lambda - \mu) - 2\lambda\alpha}$$

Let the central bank buy debt equal to $\tilde{A}_{CB}$*, financed by selling equities,* ψ_{CB}*, where,*

$$\tilde{A}_{CB} = B - rM, \quad \text{and} \quad \psi_{CB} = -Q\left(\frac{B - rM}{Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H)}\right). \tag{43}$$

The prices $Q(L)$*,* $Q(H)$ *and* $r = (1 - Q)/Q$ *and the money value of profits in each state* $\tilde{\Pi}(L)$ *and* $\tilde{\Pi}(H)$*, are those defined in Proposition 4.*

Under this policy, there exists a unique equilibrium in which allocations are the same as those implemented in the complete participation rational expectations equilibrium.

Proof. See Appendix J. ∎

The fact that the equilibrium allocations are identical to those under complete participation means that the central bank is able to restore efficiency. In the proof of the proposition we establish that the central bank's position in the asset markets is twice the position that would be taken by the entrepreneur in the counter-factual complete markets equilibrium. Hence the workers' portfolios of risky assets will be larger under complete participation than without. In both cases, the real value of the workers' and entrepreneurs' after tax incomes will be stabilized under the optimal policy.

Corollary 3. *In the stabilization equilibrium, the return on a real indexed bond is equal to the real return from holding equity.*

Proof. In the Pareto efficient equilibrium the real value of profit is the same in both states. It follows immediately that the return to an indexed bond is the same as the return to an equity share. ∎

This corollary implies that, *when all uncertainty is non-fundamental*, the central bank can implement the optimal policy by standing ready to trade indexed bonds at the same price as claims to the stock market. By doing so, it would end up holding the optimal asset portfolio $\left\{\tilde{A}_{CB}, \psi_{CB}\right\}$ described in Proposition 6.

In our model quantitative easing and qualitative easing play two different roles. Since we have no goods market trade in period 1, we cannot talk about inflation in our model. But we can talk about the nominal price level. And it follows from our assumption that the money supply enters the belief function, that the central bank can influence the level of nominal prices and wages targeting the money supply, M.

Qualitative easing has a different purpose. When all uncertainty is non-fundamental, the optimal financial policy is to intervene in the asset markets by offering to trade indexed bonds for equity and to set the return on these two assets equal to each other. A policy of this kind cannot affect the mean or the variance of the price level. But, by targeting the risk composition of the central bank balances sheet, QualE can eliminate the real effects of nominal price level fluctuations.

8. Summary

Buiter (2008) made the distinction between Quantitative Easing, defined as an increase in the size of the central bank balance sheet, and Qualitative Easing, defined as a change in its risk composition. In this paper we have outlined a theory that provides a channel by which Qualitative Easing influences asset prices. According to our narrative, some agents cannot insure themselves against uncertain outcomes, leading to inefficient real fluctuations even when all uncertainty is non-fundamental. By trading debt for equity in the asset markets, the central bank can provide a substitute for the missing insurance market and, in so doing, stabilize asset price movements and make everyone better off.

Our explanation builds on the idea that nominal transfers in the presence of an indeterminate price level can lead to different real allocations. We highlight sunspot fluctuations because standard accounts of asset market dynamics struggle to account for the volume of trades that we observe in real world asset markets (Milgrom and Stokey, 1982), and for the the observed volatility in asset prices, relative to dividend movements (Shiller, 1981). We explain these features of data by exploiting shifts across different equilibria in the presence of incomplete participation. In our view, significant portions of asset price fluctuations in the real world are caused by self-fulfilling shifts from one equilibrium to another that are associated with inefficient fluctuations in wealth.

Although we have explained our case in a simple two-period model, our argument is more general than the model used to illustrate it. In recent work (Farmer and Zabczyk, 2019) on the overlapping generations model, for example, we have demonstrated that the price level can be indeterminate in the neighbourhood of dynamically efficient steady states even when monetary policy is active. Our work thus provides an example of a fully dynamic environment where equilibrium prices and wages are potentially subject to pure sunspot shocks, and where the QualE policy similar to the one we describe could help replicate the efficient, "full-participation" equilibrium. The fact that asset price volatility is Pareto inefficient provides, we believe, a strong case to make qualitative easing a permanent component of future financial policy.

Should the central bank implement a policy of asset price stabilization? Although our model provides a justification for stabilization of asset price volatility, our argument has two caveats.

First, if some or all of real-world asset price fluctuations have fundamental causes, the case for *fully* stabilizing asset prices breaks down. There would still exist asset-market policies, implemented by the central bank, that could increase the welfare of those unable to trade in assets; but these policies would not take the form of intervening in the asset markets to equate the real returns to stocks and bonds.

Second, we have modeled those active in asset markets (workers) as one homogenous group. If that was not the case, however, the results of Goenka and Préchac (2006) suggest that stabilizing asset prices could reduce welfare for at least some of those initially active in asset markets; in other words, the intervention we propose here might not constitute a Pareto improvement.[14] In line with a large literature on the benefits of trade, we conjecture that the central bank intervention could be augmented by an appropriate transfer scheme ensuring that no group of agents loses out on the stabilization package. Exploring how such transfers could be designed, in a more quantitatively realistic model, would, we think, constitute an interesting extension of our work.

[14] Kajii (2007) generalizes the Goenke-Préchac result to a larger set of economies and Kang (2019) proposes a measure of aggregate welfare in incomplete market economies. In a related paper, Cozzi et al. (2017) show that non-fundamental uncertainty may lead to welfare gains for some consumers in economies with information frictions.

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Appendix A. The Life-cycle Budget Constraint Under Certainty

Workers face the following budget constraint in period 1,

$$M_1 + QA_1 - TR = 0, \tag{A1}$$

where, TR is a nominal transfer to workers by the treasury. This transfer can be held as money, M_1, or interest bearing bonds, A_1 with Q denoting their period 1 price.

In period 2, workers face the constraint,

$$pc_1 + w(1 - n_1) \leq w + M_1 + A_1 - \mathcal{T}_1. \tag{A1}$$

Here, w is the money wage, p is the price of commodities, n_1 is labour supply, c_1 is consumption and $\mathcal{T}_1$ is a lump-sum, nominal tax obligation. Putting together the budget constraints of workers for periods 1 and 2, and rearranging terms, leads to the life-cycle constraint,

$$pc_1 + w(1 - n_1) + rM \leq \mathcal{W}, \tag{A2}$$

where

$$\mathcal{W} \equiv w + \frac{TR}{Q} - \mathcal{T}_1 \tag{A3}$$

is the dollar value, at date 2, of a worker's wealth and $r \equiv (1 - Q)/Q$ is the money interest rate. These are Equations (1) and (2) in the text.

Appendix B. Proof of Proposition 1

Proof. Combining the definition of wealth from Equation (5) with the money market clearing condition, Equation (12), and using the fact that $\lambda + \mu + \gamma = 1$, we have the following expression for wealth in equilibrium

$$\mathcal{W} = \frac{2w + B}{1 + \lambda + \mu}. \tag{B1}$$

To derive the expression for the equilibrium value of r, we use the money market equilibrium condition, Equation (12), while the real wage expression follows from inverting the labour market clearing condition, Equation (10).

Feasibility requires that

$$n = 1 - \mu \frac{\mathcal{W}}{w} > 0, \tag{B2}$$

for both types. Combining Equations (B1) and (B2) leads to

$$w \geq \frac{\mu B}{1 + \lambda - \mu}. \tag{B3}$$

Feasibility also requires non-negative consumption for entrepreneurs,

$$c_2 = (1 - \alpha) \left(\frac{1}{\alpha} \frac{w}{p} \right)^{\frac{\alpha}{\alpha - 1}} - \frac{(B - rM)}{2p} > 0. \tag{B4}$$

Using the definition of equilibrium prices and wealth from Equation (1), and market clearing, Equations (8)–(9), evaluated at equilibrium prices, we arrive at

$$w \geq \frac{(\mu + \alpha\lambda) B}{(1 + \lambda - \mu) - 2\lambda\alpha}. \tag{B5}$$

Since

$$\frac{(\mu + \alpha\lambda) B}{(1 + \lambda - \mu) - 2\lambda\alpha} \geq \frac{\mu B}{(1 + \lambda - \mu) - 2\lambda\alpha} \geq \frac{\mu B}{(1 + \lambda - \mu)}$$

therefore the second inequality implies the first, leading to the condition in the statement of Proposition 1. ∎

Appendix C. The Life-cycle Budget Constraint Under Uncertainty

In period 1, workers receive a transfer TR that may be used to acquire money M_1, and buy or sell Arrow securities $A_1(\varepsilon)$,

$$\sum_{\varepsilon \in \{H,L\}} Q(\varepsilon) A_1(\varepsilon) + M_1 \leq TR. \tag{C1}$$

In period 2 state ε, workers face the constraint,

$$p(\varepsilon) c_1(\varepsilon) + w(\varepsilon)(1 - n_1(\varepsilon)) \leq w(\varepsilon) + A(\varepsilon) + M_1 - \mathcal{T}_1. \tag{C2}$$

Substituting for $A_1(\varepsilon)$ from Inequality (C2) into Inequality (C1) gives the following lifecycle budget constraint,

$$\sum_{\varepsilon\in\{H,L\}} Q(\varepsilon)\left[p(\varepsilon)c_1(\varepsilon)+w(\varepsilon)(1-n_1(\varepsilon))-w(\varepsilon)-M_1+\mathcal{T}_1\right]+M_1\leq TR. \tag{C3}$$

From the no-arbitrage assumption we have the following connection between Q, $Q(H)$ and $Q(L)$

$$\sum_{\varepsilon\in\{H,L\}} Q(\varepsilon)=Q, \tag{C4}$$

Using the definitions of state prices we may write the lifecycle budget constraint of a worker as follows,

$$\mathbb{E}\left[\tilde{p}(\varepsilon)c_1(\varepsilon)+\tilde{w}(\varepsilon)(1-n_1(\varepsilon))\right]+rM_1\leq\mathcal{W}, \tag{C5}$$

where workers' wealth in the model with uncertainty, but complete markets, is defined as

$$\mathcal{W}\equiv\mathbb{E}[\tilde{w}(\varepsilon)]+\left(\frac{B}{2}+\frac{rM}{2}\right). \tag{C6}$$

Here $\mathbb{E}$ is the expectations operator, defined using the probability distribution $\{\pi_H,\pi_L\}$ and the term in the last bracket denotes the net transfer from the government. These are Equations (14) and (15) in the text.

Appendix D. Proof of Lemma 1

Proof. From the statement of Lemma 1 we have that

$$\theta_1\equiv\frac{\left[X_H(1+Y_L)-X_L(1+Y_H)\right]}{X_LY_H},\quad\text{and}\quad\theta_2\equiv\frac{X_HY_L}{X_LY_H}. \tag{D1}$$

Define the quadratic equation in q,

$$q^2-\theta_1q-\theta_2=0, \tag{D2}$$

and let r_1 and r_2 be the roots of this quadratic, given by the expression

$$r=\frac{1}{2}\left(\theta_1\pm\sqrt{\theta_1^2+4\theta_2}\right). \tag{D3}$$

It follows from the fact that θ_2 is positive that both roots are real and that there is a unique non-negative root. ∎

Appendix E. Proof of Proposition 2

Proof. Consider the following three facts that follow from the definitions of market clearing, Equations (19)–(21). First, using money market clearing, Equation (21),

$$\mathcal{W} = \frac{(1-Q)\,M}{Q}\frac{1}{\gamma} \equiv \frac{x}{\gamma}, \tag{E1}$$

define the variable x,

$$x \equiv \frac{(1-Q)\,M}{Q} \equiv rM. \tag{E2}$$

Second, putting together labour and goods market clearing, Equations (19) and (20), with Equations (E1) and (E2) we have that,

$$1 - \frac{\theta x}{\tilde{w}(\varepsilon)} = \frac{x}{2w(\varepsilon)} - \frac{B}{2w(\varepsilon)}, \quad \varepsilon \in \{L, H\}, \tag{E3}$$

where we define

$$\theta \equiv \frac{\lambda + \mu}{\gamma}. \tag{E4}$$

Third, we use the definition of $\tilde{w}(\varepsilon)$,

$$\tilde{w}(\varepsilon) \equiv \frac{Q(\varepsilon)\,w(\varepsilon)}{Q\pi_\varepsilon}, \quad \varepsilon \in \{L, H\}. \tag{E5}$$

Substituting (E5) into (E3) gives,

$$1 - \frac{Q\pi_\varepsilon \theta x}{Q(\varepsilon)\,w(\varepsilon)} = \frac{x}{2w(\varepsilon)} - \frac{B}{2w(\varepsilon)}, \quad \varepsilon \in \{L, H\}. \tag{E6}$$

Rearranging this equation and using the no arbitrage condition, $Q = Q(L) + Q(H)$ leads to the following expression for x

$$x = \frac{[2w(L) + B]\,Q(L)}{Q(L) + 2\pi_L\theta\,[Q(L) + Q(H)]} = \frac{[2w(H) + B]\,Q(H)}{Q(H) + 2\pi_H\theta\,[Q(L) + Q(H)]}. \tag{E7}$$

Next use the definition of $q \equiv Q(L)/Q(H)$, and divide the numerator and denominator of Equation (E7) by $Q(H)$ to give,

$$\frac{[2w(L)+B]q}{q+2\pi_L\theta[1+q]} = \frac{[2w(H)+B]}{1+2\pi_H\theta[1+q]}. \tag{E8}$$

Rearranging this equation and using the definitions or terms from 1 leads to the quadratic equation in q,

$$q^2 - \theta_1 q - \theta_2 = 0. \tag{E9}$$

which, as we showed in Lemma 1, has a unique non-negative real root. Next note that

$$x \equiv M\frac{(1-Q)}{Q} \equiv \left(\frac{1}{Q_H} - 1 - q\right)\frac{M}{1+q} \tag{E10}$$

and use Equation (E8) to write,

$$x = \frac{X_L q}{q + Y_L(1+q)}. \tag{E11}$$

Combining Equations (E10) and (E11) leads to the expression for $Q(H)$, Equation (25) in Proposition 2. Equation (26) follows from the definition of Q. To derive Equations (27) use the definition of $w(\varepsilon)$ from Equations (13).

Equation (28) follows from the labour market clearing equation and Equation (29) follows from Equation (E1) and the no arbitrage assumption. It remains to check that Inequality (23) is sufficient to guarantee that labour supply is feasible and that (23) guarantees that entrepreneurs' consumption is non-negative. That follows from the fact that these assumptions guarantee feasibility in each state individually and therefore feasibility in a convex combination of the states as well. ∎

Appendix F. Proof of Corollary 1

Proof. Labour supply is given by the expression

$$n_1(\varepsilon) = 1 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \tag{F1}$$

and consumption of workers equals

$$c_1(\varepsilon) = \lambda \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}. \tag{F2}$$

Hence to establish inequalities (30) we need only show that

$$\tilde{w}(L) \neq \tilde{w}(H). \tag{F3}$$

But from Equation (E3) we have that

$$\frac{\theta x}{\tilde{w}(\varepsilon)} = \frac{B}{2w(\varepsilon)} - \frac{x}{2w(\varepsilon)} + 1, \tag{F4}$$

from which it follows that $\tilde{w}(L) = \tilde{w}(H)$ if and only if, $w(L) = w(H)$. The inequality of the consumption of entrepreneurs across states follows from the fact that their income is a function of the real wage. ∎

Appendix G. Proof of Proposition 3

Proof. Combining labour market equilibrium, Equation (35) with goods market equilibrium (36) leads to

$$\frac{1}{\alpha}\left(1 - \mu \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}\right) = \frac{1}{\tilde{w}(\varepsilon)}\left\{\mathbb{E}\left[\tilde{p}(\varepsilon)\Pi(\varepsilon)\right] - \left(\frac{B - rM}{2}\right)\right\} + \lambda \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \quad \varepsilon \in \{L, H\}. \tag{G1}$$

Because these two state equations are identical (the term in the wiggly brackets is a constant) it immediately follows that

$$\tilde{\omega}(L) = \tilde{w}(H) \equiv \tilde{w}. \tag{G2}$$

Using this fact, and the definition of $\tilde{w}(\varepsilon)$ gives,

$$q \equiv \frac{Q(L)}{Q(H)} = \frac{w(H)\pi_L}{w(L)\pi_H}. \tag{G3}$$

This establishes the expression for q, Equation (39), in the statement of Proposition 3.

Next we seek expressions for $Q(L)$ and $Q(H)$ individually. Combining the definition of workers wealth, Equation (15), with the money market equilibrium condition, Equation

(37), and using Equation (G2) gives the following equation linking $\tilde{\omega}$ and r,

$$\mathcal{W} = \frac{rM}{\gamma} = \tilde{w} + \frac{B}{2} + \frac{rM}{2} \tag{G4}$$

Note also that

$$\tilde{w} = \frac{w\left(H\right) Q\left(H\right)}{Q \pi_H}. \tag{G5}$$

The no arbitrage condition, $Q = Q\left(L\right) + Q\left(H\right)$ implies that

$$\frac{Q\left(H\right)}{Q} = \frac{1}{1+q}. \tag{G6}$$

Using no arbitrage and the definition of r we also have that

$$r = \frac{1-Q}{Q} = \frac{1-Q_L-Q_H}{Q_L+Q_H} = \frac{\frac{1}{Q_H} - \left(1+q\right)}{1+q}, \tag{G7}$$

which simplifies to give,

$$Q_H = \frac{1}{\left(1+r\right)\left(1+q\right)}. \tag{G8}$$

Next, we rearrange Equation (G4)

$$rM\left(\frac{1-\gamma}{\gamma}\right) = 2\frac{w\left(H\right)}{\left(1+q\right)\pi_H} + B, \tag{G9}$$

to find the following expression for $(1+r)$

$$\left(1+r\right) = \left(M\left(\frac{2-\gamma}{\gamma}\right) + 2\frac{w\left(H\right)}{\left(1+q\right)\pi_H} + B\right)\left(M\left(\frac{2-\gamma}{\gamma}\right)\right)^{-1}. \tag{G10}$$

Finally, combining (G10) with (G8) and using the definition of γ gives Equation (38) in the statement of Proposition 3, which is the expression we seek

$$Q_H = \frac{M\left(2-\gamma\right)\pi_H}{\left(1+q\right)\left(M\left(2-\gamma\right)+B\gamma\right)\pi_H + 2\gamma w\left(H\right)}. \tag{G11}$$

Equations (40) follow immediately from the definitions of state wages and Equation (41) follows from labour market clearing. The standard feasibility condition guarantees that labour supply for each worker and the consumption of entrepreneurs are each non-negative in equilibrium. ∎

Appendix H. Proof of Proposition 4

Proof. From the entrepreneur's budget constraint, Equation (32),

$$p\left(\varepsilon\right)c_{2}\left(\varepsilon\right)\leq p\left(\varepsilon\right)\Pi\left(\varepsilon\right)-\left(\frac{B-rM}{2}\right)+A_{2}\left(\varepsilon\right). \tag{H1}$$

From the solution to the entrepreneur's problem, we have that

$$c_{2}\left(\varepsilon\right)=\frac{\mathbb{E}\left[\tilde{p}\left(\varepsilon\right)\Pi\left(\varepsilon\right)\right]}{\tilde{p}\left(\varepsilon\right)}-\frac{1}{\tilde{p}\left(\varepsilon\right)}\left(\frac{B-rM}{2}\right). \tag{H2}$$

But from Proposition 3, $\tilde{p}\left(\varepsilon\right)$ is the same in both states and thus,

$$c_{2}\left(\varepsilon\right)=\Pi\left(\varepsilon\right)-\frac{1}{\tilde{p}\left(\varepsilon\right)}\left(\frac{B-rM}{2}\right). \tag{H3}$$

Rearranging Equation (H1) and combining it with (H3) gives the following expression,

$$A_{2}\left(\varepsilon\right)=\left(\frac{B-rM}{2}\right)\left(1-\frac{p\left(\varepsilon\right)}{\tilde{p}\left(\varepsilon\right)}\right). \tag{H4}$$

Finally, from the definitions of $\tilde{p}\left(\varepsilon\right)$ and q we have that

$$\frac{p\left(H\right)}{\tilde{p}\left(H\right)}=\pi_{H}\left(1+q\right),\qquad\frac{p\left(L\right)}{\tilde{p}\left(L\right)}=\pi_{L}\left(1+q^{-1}\right). \tag{H5}$$

Combining equations (H4) and (H5)and using the fact that

$$q=\frac{w\left(H\right)}{w\left(L\right)}\frac{\pi_{L}}{\pi_{H}}, \tag{H6}$$

gives

$$A_{2}\left(H\right)=\pi_{L}\left(\frac{B-rM}{3}\right)\left(1-\frac{w\left(H\right)}{w\left(L\right)}\right), \tag{H7}$$

$$A_{2}\left(L\right)=\pi_{H}\left(\frac{B-rM}{3}\right)\left(1-\frac{w\left(L\right)}{w\left(H\right)}\right), \tag{H8}$$

which are the expressions we seek. ∎

Appendix I. Proof of Proposition 5

Proof. By purchasing bonds with face value

$$B_2 = \frac{B - rM}{2}, \tag{I1}$$

the entrepreneur consumes

$$c_2(\varepsilon) = \frac{\tilde{\Pi}(\varepsilon)}{p(\varepsilon)} - \frac{(B - rM)}{2p(\varepsilon)} + \left[\frac{B_2}{p(\varepsilon)} - \psi\frac{\tilde{\Pi}(\varepsilon)}{p(\varepsilon)}\right], \tag{I2}$$

where

$$A_2(\varepsilon) \equiv B_2 - \psi\tilde{\Pi}(\varepsilon), \tag{I3}$$

is the dollar value of Arrow securities in state ε. This is equal to the face value of debt, B_2, minus the share of profits, $\psi\Pi(\varepsilon)$ that was sold to finance the purchase of debt. We established in Proposition 3, that $\tilde{\Pi}(\varepsilon)/p(\varepsilon)$ is the same in both states. It follows that if

$$B_2 = \frac{B - rM}{2}, \tag{I4}$$

then the entrepreneurs' consumption is independent of the state. The share of profits that the entrepreneur sells to workers, ψ, is defined by the entrepreneur's budget constraint in period 1,

$$QB_2 - \psi P_E = 0, \tag{I5}$$

where

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H), \tag{I6}$$

is the price of a claim to the money value of profits. ∎

Appendix J. Proof of Proposition 6

To prove this proposition we show first that, if the security prices $Q(L)$ and $Q(H)$ and the nominal profit streams $\tilde{\Pi}(L)$ and $\tilde{\Pi}(H)$ are equal to the equilibrium values defined in Proposition 4, then the portfolio defined by Equation (43) stabilizes the real value of tax revenues.

In state ε, the money value of tax revenues levied by the treasury is given by the expression,

$$\mathcal{T}(\varepsilon) = [B - \mathcal{S}] - \left[\tilde{A}_{CB} + \psi_{CB}\tilde{\Pi}(\varepsilon)\right]. \tag{J1}$$

To stabilize the real value of tax revenues the central bank must take a position such that

$$\frac{[B - \mathcal{S}] - \left[\tilde{A}_{CB} + \psi_{CB}\tilde{\Pi}(L)\right]}{w(L)} = \frac{[B - \mathcal{S}] - \left[\tilde{A}_{CB} + \psi_{CB}\tilde{\Pi}(H)\right]}{w(H)}. \tag{J2}$$

By holding additional bonds equal to

$$\tilde{A}_{CB} = [B - S], \tag{J3}$$

Equation (J2) gives

$$\frac{\mathcal{T}(L)}{w(L)} \equiv \psi_{CB}\frac{\tilde{\Pi}(L)}{w(L)} = \psi_{CB}\frac{\tilde{\Pi}(H)}{w(H)} \equiv \frac{\mathcal{T}(H)}{w(H)}. \tag{J4}$$

But from the definition of the money value of profits,

$$\tilde{\Pi}(\varepsilon) = \Pi(\varepsilon)\, p(\varepsilon) = \Pi(\varepsilon)\, w(\varepsilon)\, \frac{\tilde{p}(\varepsilon)}{\tilde{w}(\varepsilon)}, \tag{J5}$$

with the last equality implied by $p(\varepsilon)/w(\varepsilon) = \tilde{p}(\varepsilon)/\tilde{w}(\varepsilon)$.

Combining these expressions gives

$$\frac{\mathcal{T}(L)}{w(L)} \equiv \psi_{CB}\Pi(L)\,\frac{\tilde{p}(L)}{\tilde{w}(L)} = \psi_{CB}\Pi(H)\,\frac{\tilde{p}(H)}{\tilde{w}(H)} \equiv \frac{\mathcal{T}(H)}{w(H)}. \tag{J6}$$

where, from Proposition 3, $\Pi(H) = \Pi(L)$. Hence the portfolio

$$\tilde{A}_{CB} = B - rM, \quad \text{and} \quad \psi_{CB} = -Q\left(\frac{B - rB}{Q(L)\,\tilde{\Pi}(L) + Q(H)\,\tilde{\Pi}(H)}\right), \tag{J7}$$

is self-financing and stabilizes the real value of tax revenues as claimed.

Next we establish that this tax policy generates the same after tax wealth positions for entrepreneurs and workers that they would choose if entrepreneurs could self insure. We showed in Proposition that entrepreneurs would choose to hold debt equal to

$$B_2 = \frac{B - rM}{2}. \tag{J8}$$

In the counter-factual complete participation equilibrium the entrepreneur's wealth equals

$$\Pi(\varepsilon) - \frac{1}{2}\left(\frac{B - rM}{w(\varepsilon)}\right) + \left(\frac{B_2 - \psi P_E}{w(\varepsilon)}\right), \tag{J9}$$

where $\frac{1}{2}(B-rM)/w(\varepsilon)$, is the real value of her tax obligation and $\{B_2, \psi P_E\}$, is the debt and equity portfolio that she takes to offset fluctuations in after-tax wealth.

In contrast, in the equilibrium with policy stabilization, the after tax wealth of the entrepreneur is

$$\Pi(\varepsilon) - \frac{1}{2}\left(\frac{B - rM}{w(\varepsilon)}\right) + \frac{1}{2}\left(\frac{\tilde{A}_{CB} - \psi_{CB} P_E}{w(\varepsilon)}\right). \tag{J10}$$

It follows immediately that if the central bank chooses a policy where

$$\tilde{A}_{CB} = 2B_2 = B - rM, \tag{J11}$$

then the after tax wealth of the entrepreneur is identical in the equilibrium with stabilization as in the counter-factual complete markets equilibrium. It follows from Walras law that stabilizing the entrepreneurs income at its complete participation value also stabilizes workers' wealth at its complete participation value.