Life-cycle Decision Making Problems under Uncertainty

by

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30 September 2018

Zhezhi Hu

Declarations

I declare that this thesis is my own work and has not been submitted for any degree at any another university.

Abstract

This thesis focuses on modelling and solving one of the fundamental financial problems for an individual investor — the life-cycle decision making problem. In particular, we study three kinds of life-cycle problems that focusing on various preferences of the investor such as bequest motive and ambiguity aversion by using different financial instruments. The investment, consumption and saving decisions are affected by various uncertainties.

We first investigate a life-cycle consumption and asset allocation problem introducing habit formation preferences and demand for term life insurance. We consider an investor who is ambiguity-averse about stock returns and model the problem in a robust optimization framework. Our main contribution is to develop a robust life-cycle consumption and asset allocation model and show its tractability when integrating investor's subjective preferences and uncertainties. The empirical study shows the important consequences of degree of ambiguity aversion on life-cycle decisions, especially on the stock allocation. We also investigate the relationship between term life insurance demand and investor's ambiguity aversion/habit formation preferences.

Then we study a life-cycle consumption and asset allocation problem incorporating labour income ambiguity, stock market predictability and analyse effects of the correlation between stock return and labour income on investor's life-cycle decisions. We model the ambiguity aversion about labour incomes using the robust optimization framework and show its importance in explaining the strong (retirement) saving motive observed in empirical data. The computational results (obtained under assumptions of stock return

to be predicable or not) illustrate possible effect of stock market predictability on not only the stock allocation, but also the consumption and saving decisions.

Finally, we focus on the investor's consumption and housing decisions over the life cycle. We assume that the investor needs to make consumption choices between non-durable/durable goods as well as housing decisions. We introduce the letting decisions (given the homeownership) in the life-cycle model, along with renting and housing decisions. The underlying life-cycle consumption and housing problem is formulated as a Markov decision process and solved by traditional dynamic programming method using a backward induction. We empirically show how letting choices are driven by investor's preferences (such as risk aversion, elasticity of intertemporal substitution, bequest motive and the housing weight in the consumption utility) and hence influence renting and owning decisions. The computational results illustrate that, by including letting decisions, our calibrated life-cycle model performs well in matching the empirical data in terms of life-cycle homeownership rate and investor's living space patterns.

Chapter 1

Introduction and Background

1.1 Introduction

One of the fundamental financial problems individuals face is the life-cycle decision making problem, where the investor needs to i) choose consumption decisions between non-durable and durable goods, ii) decide the amount of wealth to save for future use or leave to her heirs and also iii) allocate wealth among different kinds of instruments such as housing assets, stocks, bonds and financial derivatives such as life insurance and annuity products. The life-cycle decision making problem appears in the investor's daily life, thus is really important, especially in a world where most of the financial decision making nowadays is shifting towards individuals, e.g., retirement plans.

On the other hand, this problem is notoriously difficult to solve [e.g., Samuelson, 1969; Merton, 1969, 1971; Cocco et al., 2005; Yao and Zhang, 2005] since life-cycle decisions are intertwined, and a small change in one decision at any time has impact on other decisions for the rest of life, not only for the individual herself, but also for other members of the household. Most importantly, the life-cycle decisions need to be made under different uncertainties.

Traditional approaches such as dynamic programming and stochastic programming often suffer from the curse of dimensionality, which restrict the modeller in terms of the number of model ingredients (hence unable to investigate the interactions between them), decision variables (such as the number of stocks) and decision periods (e.g., only considering specific periods during the life). As a result, many life-cycle models find it difficult to match the model results with empirical data or conventional wisdom in different aspects such as stock allocation, consumption and wealth pattern and insurance products investments. Therefore, advanced mathematical optimization methods are needed for solving the life-cycle decision making problem.

In this thesis, we study three life-cycle problems focusing on various model ingredients, investor's subjective preferences and uncertainties using traditional dynamic programming algorithms which are popular in finance literature and also the robust optimization approach which has not been applied to life-cycle problems before. By doing that, we obtain quite a few important but new findings on different life-cycle decisions and are also able to match the empirical data to some extent.

In general, our research proposes a modelling and solution framework for the individual investors' life-cycle decision-making problems and contributes to finance, insurance and operational research communities. This research helps financial institutions, insurance companies as well as governments to understand the optimal life-cycle decisions from the investor's perspective. Moreover, individual investors would be better equipped with decision support tools that help them prepare for a long life without financial difficulties.

In the next section, we describe the main characteristics of the life-cycle problems and the literature review of specific life-cycle problems is presented at corresponding chapters.

1.2 Description of Life-cycle Problems

A typical life-cycle problem [Cocco et al., 2005] aims to find the optimal consumption of non-durable goods and how much wealth to allocate in stocks at each time period (e.g., one year) during the life to maximize the total life-cycle utilities of consumption and/or bequest. This problem has also been extended in the literature to incorporate other realistic ingredients. For example, in real life, the investor does not only need to decide the amount to consume non-durable goods but also the durable goods such as homes and cars, which also contribute to the utility of consumption. In terms of the portfolio choices, besides the stocks and bonds, the investor can also allocate the wealth in housing assets and other financial derivatives such as life insurance and annuity products. All these ingredients together form a rich body of the life-cycle problems, which have drawn substantial interest in both academic circles and the industry.

Since the life-cycle problems are mainly individual-oriented, those individual investor's subjective preferences are also important when studying the life-cycle problems. First of all, the modeller needs to decide how to express the investor's life-cycle utility (or felicity), which is mostly the objective function of the life-cycle problem to maximize. There are two common choices in the literature: the constant relative risk aversion

(CRRA) utility and Epstein-Zin-Weil utility [Epstein and Zin, 1989, 1991; Weil, 1989]. The main difference between these two types of utility functions is that the latter one allows the modeller to disentangle the relative risk aversion coefficient for static gambles and the intertemporal elasticity of substitution for deterministic variations. On the other hand, the former one is defined only by the relative risk aversion coefficient and assumes the intertemporal elasticity of substitution to be the multiplicative inverse of the risk aversion. Other subjective preferences commonly introduced in the model include: the bequest motive, which shows the investor's attitudes towards heirs and beneficiaries; habit formation in consumption, which defines how each unit of consumption contributes to the consumption utility; housing weight in the consumption utility, which determines the investor preference of housing consumption over non-durable consumption; ambiguity aversion (or uncertainty aversion), which represents the investor's preference towards uncertainties;[1] etc.

Life-cycle problems involve long-horizon decision making, which usually covers the entire life of the investor. Therefore, they inevitably contain different kinds of uncertainties unknown to the investor when making investment and/or life-cycle decisions. The two most common uncertainties appeared in the life-cycle problem are the labour income and stock returns. The standard specification of labour income in the literature [e.g., Cocco et al., 2005; Gomes and Michaelides, 2005] consists of three components: a deterministic function and two random variables with respect to permanent and temporary shock of the labour income. In terms of the stock return, there are also two frequently used models to describe the stock return pattern during the decision horizon: the stock return is independently and identically distributed which means the stock return is not predictable or the stock return is time-varying, to some extent predictable, and is modelled by certain processes such as the mean-reversion and auto-regression processes. According to different ingredients introduced in the life-cycle problem, there are also other uncertainties such as the house prices, interest rates, returns on other financial assets such as bonds, variable (e.g., equity-linked) life insurance and annuities.

In this thesis, we focus on three specific life-cycle problems introducing model ingredients (such as life insurance products, stock market predictability, housing assets) and investor's subjective preferences (such as bequest motive, habit formation in consumption, risk aversion and ambiguity aversion). We also consider different kinds of uncertainties including stock returns, labour income and house prices. In general, risk

[1] Risk aversion and ambiguity aversion have different meanings in finance. If the investor is risk-averse, she prefers a sure lottery rather than a risky lottery (with known probabilities) given the same expected value. If the investor is ambiguity-averse, she prefer the lottery with known probability rather than unknown probability although both lotteries are risky.

aversion and ambiguity aversion are two commonly used terms in financial and economic literature but they have different meanings. Risk aversion measures the investor's preference between a risky alternative and its expected value, where the probabilities are known. Ambiguity aversion is the preference between a risky alternative with known probability and the ambiguous counterpart with unknown probability, where the expected values are the same [Ellsberg, 1961]. Mathematically, in the life-cycle problem, risk aversion usually appears in the utility function which defines the curvature of the utility while ambiguity aversion is modelled as the worst case towards uncertainty. We will describe each life-cycle problem in the corresponding chapter.

1.3 Solution Methods for Life-cycle Problems

All those model ingredients, investor's subjective preferences and uncertainties make the life-cycle problems quite sophisticated and require advanced mathematical optimization methods to model and solve. Traditionally, the life-cycle problems are solved by dynamic programming algorithms that require specific modelling framework using states and actions that correspond to random paths and decisions, respectively [e.g., Merton, 1969, 1971; Richard, 1975; Viceira, 2001; Gomes and Michaelides, 2003; Cocco et al., 2005]. However, those stochastic dynamic programming models suffer from the curse of dimensionality in state and action spaces of the system, especially when applied to such sophisticated life-cycle problems. Moreover, explicit optimal solutions may not exist with the inclusion of certain constraints as well as the investor's preferences.

Alternatively, the stochastic programming approach has been applied for solving the life-cycle problems [e.g., Geyer et al., 2009; Konicz, Pisinger, Rasmussen and Steffensen, 2015]. The stochastic programming methods assume that random variables arising in the underlying real-life problem follow a known probability distribution. The uncertainty can be modelled by a finite number of realizations (i.e., scenarios) given the distribution. The optimal strategy is determined in view of these scenarios [Dantzig and Infanger, 1993]. However, the scenario-based stochastic programming also suffers from the curse of dimensionality. The problem of size grows exponentially with the increase in decision periods and number of scenarios, which affects the computational tractability. In particular, some previous studies have to reduce the decision periods in the model due to tractability issues [Geyer et al., 2009] or to generate an efficient and effective scenario tree that leads a good approximate solution [e.g., Pflug, 2001; Gülpınar et al., 2004].

Another approach, which has not been applied for solving the life-cycle problems, is robust optimization developed by El Ghaoui and Lebret [1997], Ben-Tal and Nemirovski

[1998] and Bertsimas and Sim [2004], among others.[2] Robust optimization takes the worst-case perspective to optimize the investor's consumption and bequest utilities in view of uncertain data, and explicitly addresses computational tractability of the underlying problem from the modelling stage. The robust optimization approach assumes that the random variables belong to uncertainty sets that can be constructed from probability distributions of uncertain factors [Gulpinar and Hu, 2016]. Depending on the specification of uncertainty sets, the robust counterpart of the original problem can be formulated as a tractable optimization problem with no random parameters. The robust optimal strategy remains feasible for all realizations of the stochastic data within the pre-specified uncertainty sets, including the worst-case values if they can be found.

In this thesis, we use robust optimization to model two life-cycle problems assuming the investor is ambiguity-averse towards stock return and/or labour income. We also apply a dynamic programming algorithm to solve a life-cycle consumption and housing problem under uncertainty.

1.4 Structure of Thesis and Contributions

In this thesis, we study life-cycle problems that are interlinked with each other in terms of different aspects such as investors' preferences, underlying uncertainties. This thesis consists of three main chapters in which we introduce the underlying life cycle problems and present computational results. Our main contributions are summarised as follows.

In Chapter 2, we introduce a life-cycle consumption and asset allocation problem introducing habit formation preferences and demand for term life insurance. We assume the investor is not only risk-averse but also averse to ambiguity about stock returns. A robust optimization framework is presented and other preferences of the investor such as subjective survival belief and borrowing option on investor's life-cycle decisions are empirically studied. The robust life-cycle consumption and asset allocation model is tractable in handling various model ingredients, investor's subjective preferences and uncertainties. Our main finding is that not only the existence of the ambiguity aversion, but also the degree of it has important consequences for life-cycle decisions, especially on the stock allocation. We also contribute to literature by investigating the relationship between term life insurance demand and investor's ambiguity aversion and habit

[2] Robust optimization in the context of financial economics is similar to the max-min expected utility framework with multiple priors proposed by Gilboa and Schmeidler [1989], Epstein and Wang [1994] and Epstein and Schneider [2003] in finance and economic literature.

formation preferences.

In Chapter 3, we study a life-cycle consumption and asset allocation problem assuming the investor is not only ambiguity-averse about stock returns but also labour income and introduce a robust optimization framework. In the presence of ambiguity aversion, we analyse the effect of correlation between labour income and stock return and stock market predictability on investor's life-cycle decisions. The main contribution of studying this life-cycle problem is to model the ambiguity aversion about labour incomes using the robust optimization framework and show its importance in explaining the saving motive observed in data. We then empirically compare the life-cycle decisions assuming the stock return is predictable or not and show the effect of stock market predictability on not only the stock allocation, but also the consumption and saving decisions.

Chapter 4, focuses on a life-cycle consumption and housing problem under uncertainties of labour income and house price. We incorporate letting decisions along with renting and owning in a life-cycle consumption and housing model and show how letting choices are driven by investor's preferences and hence influence renting and owning decisions. We formulate the problem as a Markov decision process and solved with dynamic programming via backward induction. The model assumes that the investor has recursive preferences and also considers investor's different subjective preferences such as risk aversion, elasticity of intertemporal substitution, bequest motive and the housing weight in the consumption utility. Our computational results illustrate that by including letting decisions, our calibrated life-cycle model performs well in matching the data in terms of life-cycle homeownership rate and investor's living space patterns.

Chapter 5 introduces concluding remarks and summarizes the future work in this research area.

Chapter 2

Life-cycle Asset Allocation of Ambiguity Averse Investors: Habit Formation and Term Life Insurance

2.1 Introduction

One of the fundamental problems individuals face is how to choose consumption and asset allocation optimally to maximize the life-cycle utility, especially in a world where most of the financial decision making is shifting towards individuals, e.g., retirement plans, life insurance choices, etc. This problem is notoriously difficult to solve [e.g., Samuelson, 1969; Merton, 1969, 1971; Cocco et al., 2005] since life-cycle decisions are intertwined, and a small change in one decision at any time has impact on other decisions for the rest of life, not only for the individual herself, but also for other members of the household.

In this chapter, we formulate a life-cycle model that incorporates habit formation in consumption. Habit formation models have been successful in explaining the equity premium puzzle [e.g., Sundaresan, 1989; Abel, 1990; Constantinides, 1990] and understanding the stylized facts about aggregate consumption [e.g., Carroll et al., 2000; Fuhrer, 2000] and asset returns [Chen and Ludvigson, 2009; Grishchenko, 2010]. With habit formation, the investor's utility of consumption is not determined by the absolute amount of consumption but the relative change compared to some reference level.[1] In this setting, we add the bequest motive to study the demand for life insurance. We also deviate from the standard expected utility paradigm, i.e., ambiguity neutrality, and assume that the investor is not only averse to risk but also averse to ambiguity about stock returns within the max-min expected utility framework [Ellsberg, 1961; Gilboa and Schmeidler, 1989]. Thus we analyse investor's life-cycle problem focusing on the interaction between habit formation preferences and different degrees of ambiguity aversion.

Stock market is an important vehicle to transfer consumption over the life-cycle, however it is a risky investment. The investor's consumption, saving and asset allocation strategies crucially depend on the probability structure (e.g., mean, variance and covariance) of stock returns. A classical approach to model stock returns and associated risk is to obtain point estimates from past observations, which inevitably involve errors. Because of the estimation errors, the investor faces ambiguity about the stock return when making life-cycle decisions. Earlier experimental evidence [Ellsberg, 1961] confirms the Ellsberg Paradox which reveals that investors are not neutral but averse to this ambiguity.[2]

Empirical evidence, e.g., stock-holding puzzle, also shows that investors tend to be more conservative in forming portfolios when facing ambiguity [e.g., Cao et al., 2005; Garlappi et al., 2007]. In this chapter, we consider an ambiguity-averse investor who makes decisions that are robust to stock return ambiguity. We model the stock return ambiguity by defining a set around the point estimates which represents alternative paths (priors) for stock returns. The risk-averse investor still evaluates the expected utility when making life-cycle decisions, but chooses the optimal decisions by maximizing the minimal (in other words, worst-case) expected utility over this set due to ambiguity aversion [Gilboa and Schmeidler, 1989].

By studying the life insurance demand in a life-cycle model with habit formation and ambiguity, we fill an important gap in the literature. Human capital [e.g., Yaari, 1965; Chen et al., 2006] and bequest motives [e.g., Bernheim et al., 1985; Bernheim, 1991; Inkmann and Michaelides, 2012] are two main ingredients to study the life insurance demand. Human capital represents the economic value of investor's skills and

[1] There are different kinds of habit formation models. As summarized by Gomes and Michaelides [2003], according to the source of the reference level, the habit formation models can be categorized as external (the consumption of some reference groups) and internal ones (the investor's past consumption). Regarding the measurement of change, the models can be divided into additive (measuring the difference) and multiplicative (measuring the ratio) ones. We use the internal additive version in this chapter based on Chen and Ludvigson [2009] and Grishchenko [2010].

[2] There is also evidence that rhesus monkeys show similar ambiguity aversion preference [Hayden et al., 2010], and it is stronger than risk aversion among chimps and bonobos [Rosati and Hare, 2011].

knowledge, and is often measured by the present value of future labour income [Schultz, 1961]. A sudden death, especially at an early stage of life, deprives the heirs from the human capital and thus life insurance has been proposed to protect the heirs against this mortality risk [Yaari, 1965]. Meanwhile, bequest motive drives the investor to transfer some wealth to the heirs when she dies. Hence, life insurance is also a good tool to guarantee bequest since it provides death benefits. In our model, we consider a one-year renewable term life insurance product, which gives a certain amount of death benefit to the heirs if the investor dies within a year, and the contract is renewable at maturity[3]. The lump-sum amount of death benefit depends both on the premium paid and the mortality rate of the investor. While the term life insurance guarantees the death benefit just for one year, the investor is still able to decide the amount of inheritance by accumulating financial wealth over her life time. Therefore, in our model we examine the investor's optimal choice when both short-term and long-term bequest decisions are available. We find that the optimal choice depends on both habit formation preferences and investor's ambiguity aversion. To the best of our knowledge, this is the first work that incorporates both aspects.

There are two main contributions of this chapter. First, not only the existence of the ambiguity aversion, but also the degree of it has important consequences for life-cycle decisions. The stock allocation is monotonically decreasing in the degree of ambiguity aversion. We also observe that habit formation only has effect on asset allocation when the degree of ambiguity aversion is high. Moreover, in certain parameter range of the ambiguity aversion, the model delivers a hump-shaped stock allocation pattern over the life span. Empirical observations [e.g., Ameriks and Zeldes, 2004; Yao and Zhang, 2005; Fagereng et al., 2017] suggest investors should have such a hump-shaped investment profile. In particular, at the early ages of life, investors who have low labour income and savings, should stay out of the stock market while they increase the stock allocation during the middle ages when they receive higher labour income and accumulate more wealth. After retirement, when income is reduced, the investment strategy should become more conservative, and the investors should switch to less risky investments. Similar advice is given by life-cycle investment theory and asset managers [e.g., Bakshi and Chen, 1994; Campbell and Viceira, 2002]. However, such a hump-shaped investment profile is difficult to obtain in previous life-cycle models [e.g., Gomes and Michaelides,

[3]It is worthwhile to mention that the *whole life insurance* is another type of life insurance contracts and typically provides a protection for the entire life. As stated by Brown [2001], whole life insurance and term life insurance products have different economic implications. The former one may be more attractive to the high-income investors while the latter one is more affordable and thus, is more appealing to broader population.

2003; Cocco et al., 2005].[4]

Second, we contribute to literature by investigating the term life insurance demand of an ambiguity-averse investor with habit formation preferences. While earlier papers show the role of human capital and bequest motive [e.g., Bernheim, 1991; Chen et al., 2006; Inkmann and Michaelides, 2012; Hambel et al., 2017] on life insurance demand, we consider other important characteristics such as ambiguity aversion and habit formation and additional realistic features such as subjective survival belief[5] and borrowing opportunity. Our results suggest that similar to the bequest motive, ambiguity aversion, subjective survival belief and borrowing opportunity also increase the investor's demand on term life insurance while habit formation, which plays a dominant role compared to other factors, has an opposite effect. This finding, which has not been explored before in the context of life-cycle problems, can explain why the demand for term life insurance products remains weak [Chambers et al., 2011], especially among those who are more financially vulnerable [Bernheim et al., 2003], despite being more affordable compared to other insurance products.

Our work is related to three strands of literature. We follow several studies that introduce internal habit formation models in life-cycle problems. Both Diaz et al. [2003] and Polkovnichenko [2007] find a positive relationship between habit formation and wealth accumulation. Under similar settings, Gomes and Michaelides [2003] document a contradictory finding with empirical evidence that habit formation results in an early participation in the stock market. As mentioned above, by varying the degree of the stock market ambiguity in the model, our model shows that the investor prefers not to participate in the stock market at early ages. However, the model fails to generate realistically low stock market participation level while keeping the hump-shape in the absence of additional features such as fixed entry costs, recursive preferences or learning mechanism.

Moreover, we study the investor's life insurance demand in the life-cycle setting similar to earlier studies. Chen et al. [2006] show that human capital increases the term life insurance demand. Inkmann and Michaelides [2012] argue that bequest motive has

[4]Gomes and Michaelides [2005] obtain more realistic life-cycle asset allocation via recursive preferences, fixed entry cost and risk heterogeneity. Benzoni et al. [2007] obtains a hump-shaped profile by assuming that stock return and labour income are co-integrated. Campanale [2011] and Peijnenburg [2016] also build life-cycle models with ambiguity aversion as we do in this chapter. But to get a hump-shape profile, they assume that investors learn about the equity premium. These assumptions are absent in our model.

[5]Recent empirical evidence [e.g., Heimer et al., 2015; Groneck et al., 2016] shows that objective survival beliefs differ from the subjective ones, that is, young investors typically underestimate their life-span while the elderly expect to live longer than the predictions obtained from the mortality table.

a positive correlation with term life insurance. Gao and Ulm [2015] and Koijen et al. [2016] provide suggestions about the optimal portfolio choice among different insurance products such as variable annuities, term life insurance and long-term care insurance. Compared to those papers, we mainly investigate the relationship between the term life insurance and different factors (as ambiguity aversion and habit formation) and analyse how the investor's life insurance demand is affected by these unexplored factors.

Furthermore, our work is based on the max-min expected utility framework with multiple priors proposed by several works such as Gilboa and Schmeidler [1989], Epstein and Wang [1994] and Epstein and Schneider [2003]. Meanwhile, independently to aforementioned papers, El Ghaoui and Lebret [1997], Ben-Tal and Nemirovski [1998] and Bertsimas and Sim [2004] among others, develop a robust optimization paradigm in operations research domain, which shares similar ideas, but is presented in a different terminology. Robust optimization models uncertainty[6] in deterministic sets (i.e., the set of priors) and solves the problem with respect to the worst-case over the set. Unlike the max-min expected utility framework in financial economics, robust optimization aims to solve general decision making problems under uncertainty. Such techniques have been applied in solving various investment problems such as mean-variance portfolio optimization [e.g., Goldfarb and Iyengar, 2003; Garlappi et al., 2007] and multi-period portfolio selection [e.g., Ben-Tal et al., 2000; Bertsimas and Pachamanova, 2008] and asset-liability management [Gulpinar and Pachamanova, 2013; Gulpinar et al., 2016].

In the life-cycle setting, Campanale [2011] models stock return ambiguity by two priors (either higher or lower than the mean return) with equal probability and find that the ambiguity reduces the stock allocation. Peijnenburg [2016] assumes the stock return belongs to a confidence interval symmetrically around the stock return mean and can vary within the length of one-fourth standard deviations. She also finds that stock return ambiguity reduces the market participation and moreover, decreases the wealth accumulation during the life. Both papers assume that the investor can learn about the stock return ambiguity during the life in a sense that the degree of ambiguity is reduced. Only with the learning, these two papers are able get a hump-shaped stock allocation profile, which is closer to the empirical evidence. However, in this chapter, we model the stock return ambiguity by a different set, which is ellipsoid-shaped. The use of such an ellipsoidal uncertainty set is less conservative (in case of more than one risky asset with uncertain returns) and it allows taking into account standard deviations and covariances of random variables. We obtain a hump-shaped stock allocation profile over

[6] Ambiguity and uncertainty have been interchangeably used in finance [e.g., Epstein and Schneider, 2003; Garlappi et al., 2007; Peijnenburg, 2016] and broadly have the same meaning.

the life-cycle (for certain size of the uncertainty sets) without making any assumption on learning. Another difference compared to Campanale [2011] and Peijnenburg [2016] is that we analyze the relationships between stock return ambiguity and other life-cycle decisions such as life insurance demand.

The rest of the chapter is organized as follows. Section 2 describes the life-cycle asset allocation problem for an ambiguity-averse investor. A brief overview on solution approaches is presented in Section 3. We introduce the robust formulation of the life-cycle problem in Section 4. Section 5 summarizes design of numerical experiments in terms of parameter selection and data calibration. We present the results in Section 6. Section 7 summarizes the main findings and concludes the chapter.

Notation: Throughout the thesis, we use tilde ($\tilde{*}$) to denote randomness; e.g., $\tilde{y}$ denotes random variable y. Boldface is used to denote vectors; for example, $\mathbf{a}$ is a vector. In particular, we denote a vector of ones by $\mathbf{1} = [1, \cdots, 1]$ in appropriate dimension and "$\cdot$" displays a vector multiplication.

2.2 The Life-cycle Asset Allocation Problem

In this section, we introduce a stochastic optimization formulation of the life-cycle asset allocation problem for an ambiguity-averse investor. We consider a discrete-time environment that spans T time periods. Life-cycle decisions are made at each time period (i.e., one year) $t = 1, \cdots, T$ and $t = 0$ represents today.

Given the current age of the investor, she is alive for maximum T periods and retires at certain age $K < T$. At each time period t, the investor receives the labour income (the retirement income for $t > K$) and needs to decide the amount of consumption, asset allocation and death benefit of the term life insurance that she is willing to leave to the heirs.

The problem formulation contains random variables on the asset returns $\tilde{\mathbf{r}}_{\mathbf{t}}$ including the return on the risk-free asset and value of the labour income $\tilde{l}_t$ at each point in time $(t = 1, \ldots, T)$.

Investment Products: We assume that the investment portfolio is constructed from M risky assets over a planning horizon T. Securities are denoted by $m = 1, 2, \cdots, M$ and $m = 0$ identifies the risk-free asset. After an initial investment at $t = 0$, the portfolio can be restructured at discrete times $t = 1, \cdots, T-1$ and redeemed at the end of the investment horizon $t = T$.

Apart from financial assets, we assume that the investor has an option to buy one-year renewable life insurance products (abbreviated as *term life insurance* henceforth).

The mechanism of term life insurance products is briefly described as follows. At time t, the investor decides the amount of wealth to buy term life insurance (i.e., premium) which provides amount of death benefit (i.e., face value) to the heirs if the investor dies before $t+1$. The maturity of term life insurance products is one year. In time $t+1$, the investor can renew the insurance contract by paying a new premium. Let q_t and d_t[7] denote the premium and the face value of the term-life insurance contract at time t, respectively. A linear relationship between q_t and d_t is expressed by the pricing formula as $q_t = (1-p_t)d_t$ where p_t represents the conditional probability of being alive at t assuming that she has been alive at time $t-1$.

Constraints: We model constraints in terms of habit formation, asset and cash balance, no short-sale and non-negativity of decisions.

Habit formation: As mentioned before, we use an internal addictive model for habit formation of the investor's preferences as introduced by Polkovnichenko [2007]. Let z_t be a reference point of the consumption amount that the investor him/herself is likely to consume in the past (i.e., the consumption habit). Let c_t denote the amount of consumption at time t. The investor's utility of consumption at time period t is determined by two factors: the current amount of consumption (c_t) and the reference level (z_t) of the consumption habit formation. The investor gains more utility of consumption if the current consumption is higher than the reference point ($c_t > z_t$) and vice versa. We assume that the investor starts with no habit formation; thus the initial reference level is $z_0 = 0$ and the consumption at $t = 0$ is $c_0 = 0$.[8] At any time period $t \geq 1$, z_t is proportional to the reference level z_{t-1} and consumption level c_{t-1} at previous time $t-1$ according to the degree of habit formation persistence λ and formulated as follows;

$$z_t = \lambda z_{t-1} + (1-\lambda)c_{t-1}, \quad t = 1, \cdots, T. \tag{2.1}$$

Balance constraints: At time t, a balance constraint determines the wealth gained from each asset. The investor can adjust the portfolio through trading in risky assets. Let h_t^m, s_t^m and b_t^m denote decision variables representing the amount of asset m to be held, sold and bought at time t, respectively. The holding dynamics for risky asset m at time t are defined in terms of holdings and the gain received at $t-1$ plus the current

[7]Throughout the chapter, the face value (d_t) of a term-life insurance contract is referred as the investor's demand at time t for the term life insurance products.

[8]For the numerical experiments, we have also considered initial reference levels higher than zero. However, the empirical results do not differ much at different age groups apart from the first five years.

trading (for buying b_t^m and selling s_t^m at time t) as:

$$h_t^m = (1 + \tilde{r}_t^m)h_{t-1}^m - s_t^m + b_t^m, \quad m = 1, \cdots, M, \; t = 1, \cdots, T. \tag{2.2}$$

At $t = 0$, both the initial holding of risky asset m and the cash holding are $h_0^m = 0$, $h_0^0 = 0$, respectively. The amount of cash at t consists of value of investment at $t - 1$ plus cash received from the position changes and labour income minus consumption and death benefit to be paid out at time t and is determined as:

$$h_t^0 = (1 + \tilde{r}_t^0)h_{t-1}^0 + (1 - e_s)(\mathbf{1} \cdot \mathbf{s_t}) - (1 + e_b)(\mathbf{1} \cdot \mathbf{b_t}) + \tilde{l}_t - c_t - (1 - p_t)d_t, \quad t = 1, \cdots, T, \tag{2.3}$$

where $\mathbf{s_t}$ and $\mathbf{b_t}$ denote vectors of buying and selling decisions, with fixed transaction costs e_s and e_b, respectively, over all assets $m = 1, \cdots, M$ and $\mathbf{1} \in \mathbb{R}^{\mathbf{M}}$ is a vector of ones.

Total wealth at $t = 1, \cdots, T$ is accumulated over all holdings of risky and risk-free assets;

$$w_t = \sum_{m=1}^{M} h_t^m + h_t^0, \quad t = 1, \cdots, T. \tag{2.4}$$

No short sales: A set of non-negativity constraints on the holdings of asset m at time t is imposed to prevent borrowing (i.e., short sales);

$$h_t^m \geq 0, \quad m = 0, \cdots, M, \; t = 1, \cdots, T. \tag{2.5}$$

Non-negativity constraints: The investor's consumption and demand for term life insurance over the life-cycle are non-negative;

$$c_t, \; d_t \geq 0, \quad t = 1, \cdots, T. \tag{2.6}$$

The amount of asset m to be bought or sold at time t cannot be negative. This implies the following non-negativity constraints;

$$b_t^m, \; s_t^m \geq 0, \quad m = 1, \cdots, M, \; t = 1, \cdots, T. \tag{2.7}$$

Objective Function: The investor's goal is to maximize expected total utilities of consumption and bequest over the life (T periods).

- The utility of consumption at t does not only depend on the current consumption c_t, but also is influenced by previous consumption levels $c_{t-1}, \cdots, c_1$. Let β define importance of habit formation. The utility of consumption at t is represented as $\mathcal{U}_C(c_t - \beta z_t)$.

- If the investor dies before $t+1$, the inheritors receive the financial wealth (w_t) left by the investor and the death benefit of the term life insurance (d_t) bought by the investor. Therefore, the utility of bequest is formulated as $\mathcal{U}_B(w_t + d_t)$.

Given the bequest motive strength (η) and time discount factor (δ), the objective function (OF) of the life-cycle asset allocation problem of an ambiguity-averse investor is formulated as the expected total utilities of consumption and bequest over the life;

$$OF = \mathbb{E}_{\tilde{\boldsymbol{r}}}\Big[\sum_{t=1}^{T}\delta^t\Big(\prod_{i=1}^{t-1}p_i\Big)\Big(p_t\mathcal{U}_C\big(c_t - \beta z_t\big) + \eta(1-p_t)\mathcal{U}_B\big(w_t + d_t\big)\Big)\Big]. \tag{2.8}$$

Note that the expectation is taken over a random vector $\tilde{\boldsymbol{r}}$ of asset returns consisting of $\tilde{r}_t^m$ for all t and m. Following the previous studies [for instance, see Cocco et al., 2005], we use the time-separable power utility function (CRRA) with the coefficient of relative risk aversion (where $\gamma > 0$ and $\gamma \neq 1$) for both utilities of consumption and bequest.

The Stochastic Optimization Model: We can formulate the life-cycle consumption and asset allocation problem in view of the investor's preferences: habit formation, bequest motive and demand for term life insurance as a multi-stage stochastic optimization model in a compact form as follows;

$$\begin{aligned} \mathcal{P}^{stoc}(\tilde{r}): \quad & \max_{\boldsymbol{c},\boldsymbol{d},\boldsymbol{h},\boldsymbol{b},\boldsymbol{s}} \quad \mathbb{E}_{\tilde{\boldsymbol{r}}}\Big[\sum_{t=1}^{T}\delta^t\Big(\prod_{i=1}^{t-1}p_i\Big)\Big[p_t\frac{(c_t-\beta z_t)^{1-\gamma}}{1-\gamma} + \eta(1-p_t)\frac{(w_t+d_t)^{1-\gamma}}{1-\gamma}\Big] \\ & s.t. \qquad \text{Constraints} \quad (2.1), \cdots, (2.7). \end{aligned} \tag{2.9}$$

In this model we assume that the investor lives until time T (age 100) for sure, with monotonically decreasing survival rates with the increase of t (age). This can be interpreted as the average situation for all investors in reality. In other words, on average, the percentage of investors who can live up to age t should be roughly the same as the survival rate given in the mortality table at that age.

Next, we first provide a brief overview on the stochastic optimization approaches to solve the life-cycle asset allocation problem and then introduce a robust optimization approach to the life-cycle asset allocation problem.

A description of the notation used in this chapter is provided in Table 2.1.

Table 2.1: Description of notation

	Parameters
T, M	investment horizon and number of risky assets, respectively
p_t	probability of investor to be alive at t conditional on being alive at $t-1$
η	strength of bequest motive
β	importance of habit formation
λ, γ	degrees of habit formation persistence and risk aversion, respectively
e_b, e_s	transaction costs for buying and selling assets, respectively
χ	interest rate spread
δ	discount factor
f_t	deterministic component in the log labour income
	Decision Variables
c_t	consumption at time t
d_t	term life insurance demand at time t
b_t^m, s_t^m	amount bought and sold of asset m at time t, respectively
h_t^m	holding in asset m at time t
w_t	financial wealth at time t
z_t	habit formation reference level at time t
v_t	amount borrowed at time t
	Random Parameters
$\tilde{r}_t^m$	return on asset m between time $t-1$ and t
$\tilde{l}_t$	labour income at time t
$\tilde{R}_t^m$	cumulative gross return on asset m at time t
ν_t	permanent shock to the log labour income
ε_t	temporary shock to the log labour income

2.3 Brief Overview: Stochastic Optimization Approaches

The life-cycle consumption and asset allocation problem can be solved by traditional approaches based on dynamic programming algorithms that require specific modelling framework using states and actions that correspond to random paths and decisions, respectively [e.g., Merton, 1969, 1971; Richard, 1975; Viceira, 2001; Gomes and Michaelides, 2003; Cocco et al., 2005]. The (stochastic) dynamic programming models suffer from the curse of dimensionality in state and action spaces of the system, especially when applying for such a life-cycle problem that involves a long investment horizon and many life-cycle decisions. Moreover, explicit optimal solutions may not exist with the inclusion of constraints regarding portfolio weight limit, transaction costs, borrowing, as well as the investor's preferences such as habit formation and bequest motive.

Alternatively, the scenario-based stochastic programming approach has been applied for solving the life-cycle problems [e.g., Geyer et al., 2009; Konicz, Pisinger, Rasmussen and Steffensen, 2015]. The stochastic programming methods assume that random variables arising in the underlying real-life problem follow a known probability distribution. The uncertainty can be modelled by a finite number of realizations (i.e. scenarios) given the distribution. The optimal strategy is determined in view of these scenarios [Dantzig and Infanger, 1993]. However, the scenario-based stochastic programming also suffers from the curse of dimensionality. The problem size grows exponentially with the increase in decision periods and number of scenarios, which affect the computational tractability. In particular, the previous studies for the life-cycle problems aim to reduce the decision periods in the model due to tractability issues. For example, Geyer et al. [2009] assume that the investor makes decisions in four particular stages during the life.

In this chapter, we introduce a robust optimization approach to the life-cycle consumption and asset allocation problem of an ambiguity-averse investor with habit formation preferences and term life insurance products. Robust optimization takes the worst-case perspective to optimize the investor's consumption and bequest utilities in view of uncertain data and explicitly addresses computational tractability of the underlying problem from the modelling stage. The robust optimization approach assumes that the random variables belong to uncertainty sets that can be constructed from probability distributions of uncertain factors. Depending on the specification of uncertainty sets, the robust counterpart of the original problem can be formulated as a tractable optimization problem with no random parameters. The robust optimal strategy remains feasible for all realizations of the stochastic data within the pre-specified uncertainty sets, including the worst-case values if they can be found.

Soyster [1973] first presents the robust optimization framework for a linear model and Becker et al. [1986] apply robust optimization to macroeconomic policy under model uncertainty. Then robust optimization is re-developed theoretically and systematically by El Ghaoui and Lebret [1997] and Ben-Tal and Nemirovski [1998] independently. Since then, it has been applied for solving various practical problems in different areas. In particular, robust optimization has been used in finance for asset management (for a comprehensive overview, see Fabozzi et al. [2007], single period mean-variance portfolio management [e.g., Goldfarb and Iyengar, 2003; Gulpinar and Rustem, 2007; Oguzsoy and Güven, 2007; Soyster and Murphy, 2013] and multi-period investment problems [Ben-Tal et al., 2000; Bertsimas and Pachamanova, 2008] and asset-liability management [e.g., Gulpinar and Pachamanova, 2013; Gulpinar et al., 2016]. For further information on robust optimization and recent developments, the reader is referred to Ben-Tal et al. [2009]. For an extensive research on robust optimisation algorithms and its applications in finance, the reader is referred to Rustem and Howe [2002].

The uncertainty sets can be defined as discrete or infinite number of realisations of uncertain parameters around the mean [e.g., El Ghaoui and Lebret, 1997; Ben-Tal and Nemirovski, 1999; Rustem et al., 2000; Gulpinar and Rustem, 2007]. There also exist symmetric (such as interval and polyhedral), asymmetric and data-driven uncertainty sets developed for various applications in the literature. In particular, an interval uncertainty set, which can be regarded as the investor's confidence about the estimated value, is empirically proven to be a conservative set for the robust investment strategies [Goldfarb and Iyengar, 2003]. Natarajan et al. [2009] show that the shape of uncertainty sets also defines a risk measure (e.g., CVaR) on the constraints with uncertain coefficients.

Robust optimization has also been applied for solving the life-cycle investment problems. Recently, Peijnenburg [2016] studies the robust life-cycle portfolio choice problem and modelled stock return ambiguity using an interval set constructed by one-fourth standard deviation around the mean of the asset returns. Campanale [2011] considers an uncertainty set containing two realisations (either low or high) of stock returns.

2.4 Robust Life-cycle Asset Allocation Model

As the investor is ambiguity-averse towards asset returns, it may be desirable to obtain robust life-cycle decisions that remain optimal even under the worst-case realisations of random events (asset returns). In this case, the worst-case approach rather than expected utility maximization would be appropriate to employ. The worst-case approach is based on the max-min utility optimization criteria. Assume that the uncertain asset returns belong to a discrete or continuous uncertainty set, denoted as Θ. The life-cycle asset allocation problem can be formulated as follows;

$$
\begin{array}{llll}
\mathcal{P}^{rob}(\tilde{r}): & \max\limits_{\boldsymbol{c},\boldsymbol{d},\boldsymbol{h},\boldsymbol{b},\boldsymbol{s}} & \min\limits_{\tilde{\boldsymbol{r}}\in\Theta} & \Big[\sum_{t=1}^{T}\delta^{t}\Big(\prod_{i=1}^{t-1}p_i\Big)\Big[p_t\frac{(c_t-\beta z_t)^{1-\gamma}}{1-\gamma}+\eta(1-p_t)\frac{(w_t+d_t)^{1-\gamma}}{1-\gamma}\Big] \\
& s.t. & \text{Constraints} & (2.1),\cdots,(2.7).
\end{array}
\tag{2.10}
$$

In order to derive the robust counterpart of $\mathcal{P}^{rob}(\tilde{r})$, we first reformulate the model in terms of the cumulative returns and then apply a convenient variable transformation as suggested by Ben-Tal et al. [2000]. In this way we reduce the number of constraints with uncertain asset returns. This transformation does not only eliminate cross-constraint correlations of random variables, which is harder to model, but also to reduce the conservativeness of the robust counterpart solution.

Let us introduce the cumulative return $\tilde{R}_t^m$ for $m = 0, 1, \cdots, M$ at $t = 0, 1, \cdots, T$ as $\tilde{R}_0^m = 1$, $\tilde{R}_1^m = 1+\tilde{r}_1^m$, $\tilde{R}_2^m = (1+\tilde{r}_1^m)(1+\tilde{r}_2^m), \cdots, \tilde{R}_t^m = (1+\tilde{r}_1^m)(1+\tilde{r}_2^m)\cdots(1+\tilde{r}_t^m)$. We define new decision variables as $\theta_t^m = \frac{h_t^m}{\tilde{R}_t^m}$, $\varrho_t^m = \frac{b_t^m}{\tilde{R}_t^m}$ and $\zeta_t^m = \frac{s_t^m}{\tilde{R}_t^m}$. When h_t^m, b_t^m and s_t^m in the problem $\mathcal{P}^{stoc}(\tilde{r})$ are replaced by new decision variables, then the balance and total wealth constraints become:

$$
w_t = \sum_{m=0}^{M}\tilde{R}_t^m\theta_t^m, \quad t = 1,\cdots,T, \tag{2.11}
$$

$$
\theta_t^m = \theta_{t-1}^m - \zeta_t^m + \varrho_t^m, \quad m = 1,\cdots,M, \ t = 1,\cdots,T, \tag{2.12}
$$

$$
\theta_t^0 = \theta_{t-1}^0 + \sum_{m=1}^{M}\frac{\tilde{R}_t^m}{\tilde{R}_t^0}(\zeta_t^m - \varrho_t^m) + \frac{\tilde{l}_t}{\tilde{R}_t^0} - \frac{c_t + (1-p_t)d_t}{\tilde{R}_t^0}, \quad t = 1,\cdots,T. \tag{2.13}
$$

Let $\tilde{\alpha}_t = \Big[\tilde{R}_t^0, \tilde{R}_t^1, \cdots, \tilde{R}_t^M\Big]'$ and $\tilde{\mu}_t = \Big[\frac{\tilde{R}_t^1}{\tilde{R}_t^0}, \cdots, \frac{\tilde{R}_t^M}{\tilde{R}_t^0}, \frac{\tilde{l}_t}{\tilde{R}_t^0}, \frac{1}{\tilde{R}_t^0}\Big]'$ denote vectors of random variables in constraints (2.11) and (2.13), respectively. For notational convenience, we also define vectors of decision variables ϵ_t and τ_t such that:

$$
\epsilon_t' = \Big[\theta_t^0, \theta_t^1, \cdots, \theta_t^M\Big] \text{ and } \tau_t' = \Big[\zeta_t^1 - \varrho_t^1, \cdots, \zeta_t^M - \varrho_t^M, 1, -c_t - (1-p_t)d_t\Big]
$$

in corresponding to $\tilde{\alpha}_t$ and $\tilde{\mu}_t$. Let S_t^α and S_t^μ denote pre-specified uncertainty sets for random variables $\tilde{\alpha}_t$ and $\tilde{\mu}_t$, respectively. The life-cycle asset allocation problem $\mathcal{P}^{rob}(\tilde{r})$ becomes:

$$\mathcal{P}^{rob}(\tilde{R}): \quad \max_{\boldsymbol{c},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\varrho},\boldsymbol{\zeta}} \quad \min_{\tilde{\alpha}_t \in S_t^{\alpha},\, \tilde{\mu}_t \in S_t^{\mu}} \quad \left\{ \sum_{t=1}^{T} \delta^t \Big(\prod_{i=1}^{t-1} p_i \Big) \Big[p_t \frac{(c_t - \beta z_t)^{1-\gamma}}{1-\gamma} + \eta(1-p_t) \frac{(w_t + d_t)^{1-\gamma}}{1-\gamma} \Big] \right\}$$

$$s.t. \qquad \text{Constraints} \quad (2.1), (2.5), (2.6), (2.7), (2.11), (2.12), (2.13).$$

Note that the tractability of the optimization model depends on the choice of uncertainty sets around uncertain stock returns and labour income shocks. As described earlier $\tilde{\alpha}_t$ is a non-linear function of asset returns $\tilde{r}_t^m$ for $m = 0, \ldots, M$ whereas $\tilde{\mu}_t$ is also a non-linear function of asset returns $\tilde{r}_t^m$ as well as labour income $\tilde{l}_t$. The labour income $\tilde{l}_t$ is calibrated as a log function of two independent uncertain shocks. The calibration method will be explained in detail in Section 2.5. We assume that the uncertain parameters $\tilde{\alpha}_t$ and $\tilde{\mu}_t$ belong to symmetric (ellipsoidal) uncertainty sets. These sets are specified by mean values ($\hat{\alpha}_t$ and $\hat{\mu}_t$) and covariance matrices (Σ_t^{α} and Σ_t^{μ}) of random variables ($\tilde{\alpha}_t$ and $\tilde{\mu}_t$, respectively) as:

$$S_t^{\alpha} = \left\{ \tilde{\alpha}_t \mid \|(\Sigma_t^{\alpha})^{-\frac{1}{2}}(\tilde{\alpha}_t - \hat{\alpha}_t)\|_2 \leq \Gamma_t^{\alpha} \right\} \text{ and } S_t^{\mu} = \left\{ \tilde{\mu}_t \mid \|(\Sigma_t^{\mu})^{-\frac{1}{2}}(\tilde{\mu}_t - \hat{\mu}_t)\|_2 \leq \Gamma_t^{\mu} \right\},$$

where $\| \cdot \|_2$ represents the Euclidean norm. Parameters Γ_t^{α} and Γ_t^{μ} are so-called the degree of ambiguity aversion (or the budget of robustness).

The robust counterpart of $\mathcal{P}^{rob}(\tilde{R})$ can be obtained from the following proposition.

Proposition 2.4.1. *Given the ellipsoidal uncertainty sets S_t^{α} and S_t^{μ} for random variables $\tilde{\alpha}_t$ and $\tilde{\mu}_t$, the robust counterpart of the problem $\mathcal{P}^{stoc}(\tilde{r})$ is obtained as follows;*

$$\begin{aligned}
\max_{\boldsymbol{c},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\varrho},\boldsymbol{\zeta}} \quad & \sum_{t=1}^{T} \delta^t \Big(\prod_{i=1}^{t-1} p_i \Big) \Big[p_t \frac{(c_t - \beta z_t)^{1-\gamma}}{1-\gamma} + \eta(1-p_t) \frac{(w_t + d_t)^{1-\gamma}}{1-\gamma} \Big] \\
s.t. \quad & \|(\Sigma_t^{\alpha})^{-\frac{1}{2}} \epsilon_t\|_2 \leq \tfrac{1}{\Gamma_t^{\alpha}} (\hat{\alpha}_t' \epsilon_t - w_t), \;\; t = 1, \cdots, T \\
& \theta_t^m = \theta_{t-1}^m - \zeta_t^m + \varrho_t^m, \;\; m = 1, \cdots M, \;\; t = 1, \cdots, T \\
& \|(\Sigma_t^{\mu})^{-\frac{1}{2}} \tau_t\|_2 \leq \tfrac{1}{\Gamma_t^{\mu}} (\theta_{t-1}^0 + \hat{\mu}_t' \tau_t - \theta_t^0), \;\; t = 1, \cdots, T \\
& z_t = \rho z_{t-1} + (1-\lambda) c_{t-1}, \;\; t = 1, \cdots, T \\
& c_t, \; d_t \geq 0, \;\; t = 1, \cdots, T \\
& \theta_t^m \geq 0, \;\; m = 0, \cdots, M, \;\; t = 1, \cdots, T \\
& \varrho_t^m, \; \zeta_t^m \geq 0, \;\; m = 1, \cdots, M, \;\; t = 1, \cdots, T.
\end{aligned}$$

Proof. Consider the life cycle asset allocation model $\mathcal{P}^{rob}(\tilde{R})$:

$$\begin{aligned}
\max_{\boldsymbol{c},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{\varrho},\boldsymbol{\zeta}} \quad & \sum_{t=1}^{T} \delta^t \Big(\prod_{i=1}^{t-1} p_i \Big) \Big[p_t \frac{(c_t - \beta z_t)^{1-\gamma}}{1-\gamma} + \eta(1-p_t) \frac{(w_t + d_t)^{1-\gamma}}{1-\gamma} \Big] \\
\text{s.t.} \quad & w_t \le \min_{\tilde{\alpha}_t \in S_t^{\alpha}} \Big\{ \epsilon_t' \cdot \tilde{\alpha}_t \Big\}, \quad t = 1, \cdots, T \\
& \theta_t^m = \theta_{t-1}^m - \zeta_t^m + \varrho_t^m, \quad m = 1, \cdots, M, \ t = 1, \cdots, T \\
& \theta_t^0 \le \theta_{t-1}^0 + \min_{\tilde{\mu}_t \in S_t^{\mu}} \Big\{ \tau_t' \cdot \tilde{\mu}_t \Big\}, \quad t = 1, \cdots, T \\
& z_t = \rho z_{t-1} + (1-\rho) c_{t-1}, \quad t = 1, \cdots, T \\
& c_t, \ d_t \ge 0, \quad t = 1, \cdots, T \\
& \theta_t^m \ge 0, \quad m = 0, \cdots, M, \ t = 1, \cdots, T \\
& \varrho_t^m, \ \zeta_t^m \ge 0, \quad m = 1, \cdots, M, \ t = 1, \cdots, T.
\end{aligned}$$

The inner minimization problem in the first set of constraints can be rewritten as follows;

$$\begin{aligned}
\max_{\tilde{\alpha}_t} \quad & \Big\{ \epsilon_t' \cdot \tilde{\alpha}_t \Big\} \\
\text{s.t.} \quad & \|(\Sigma_t^{\alpha})^{-\frac{1}{2}} (\tilde{\alpha}_t - \hat{\alpha}_t)\|_2 \le \Gamma_t^{\alpha}.
\end{aligned}$$

Let $\kappa_t \ge 0$ denote the Lagrangian multiplier for the constraint. The Lagrangian function can be constructed as:

$$\mathcal{L}(\tilde{\alpha}_t, \kappa_t) = \Big\{ \epsilon_t' \cdot \tilde{\alpha}_t \Big\} + \kappa_t \Big(\|(\Sigma_t^{\alpha})^{-\frac{1}{2}} (\tilde{\alpha}_t - \hat{\alpha}_t)\|_2 - \Gamma_t^{\alpha} \Big).$$

The first-order optimality and complementary conditions, respectively, are driven as:

$$\frac{\partial \mathcal{L}(\tilde{\alpha}_t, \kappa_t)}{\partial \tilde{\alpha}_t} = 0 \Longrightarrow \epsilon_t + \frac{\kappa_t}{\|(\Sigma_t^{\alpha})^{-\frac{1}{2}} (\tilde{\alpha}_t - \hat{\alpha}_t)\|_2} (\Sigma_t^{\alpha})^{-1} (\tilde{\alpha}_t - \hat{\alpha}_t) = 0, \tag{2.14}$$

$$\kappa_t \Big(\|(\Sigma_t^{\alpha})^{-\frac{1}{2}} (\tilde{\alpha}_t - \hat{\alpha}_t)\|_2 - \Gamma_t^{\alpha} \Big) = 0. \tag{2.15}$$

From (2.14) and (2.15), one can obtain $\kappa_t \neq 0$. In addition, we have

$$\|(\Sigma_t^{\alpha})^{-\frac{1}{2}} (\tilde{\alpha}_t - \hat{\alpha}_t)\|_2 = \Gamma_t^{\alpha}. \tag{2.16}$$

Substituting (2.16) into (2.14), we can compute

$$\tilde{\alpha}_t - \hat{\alpha}_t = \left(-\frac{\Gamma_t^\alpha}{\kappa_t}\right)\Sigma_t^\alpha \epsilon_t. \tag{2.17}$$

The Lagrangian multiplier κ_t can be found from (2.16) and (2.17) as $\kappa_t = \sqrt{\epsilon_t' \Sigma_t^\alpha \epsilon_t}$. Then the optimal objective function value of the inner minimization problem $\min\limits_{\tilde{\alpha}_t \in S_t^\alpha} \left\{\epsilon_t' \cdot \tilde{\alpha}_t\right\}$ becomes $\left\{\epsilon_t' \cdot \tilde{\alpha}_t\right\} = \epsilon_t' \cdot \hat{\alpha}_t - \Gamma_t^\alpha \|(\Sigma_t^\alpha)^{-\frac{1}{2}} \epsilon_t\|_2$. By reinjecting this back into the original constraint, we obtain the robust formulation of the constraint.

Similarly, we can derive the robust counterpart of the third set of constraints where the inner minimization problem $\min\limits_{\tilde{\mu}_t \in S_t^\mu} \left\{\tau_t' \cdot \tilde{\mu}_t\right\}$ needs to be solved. □

The degree of ambiguity aversion (Γ) adjusts the size of the corresponding uncertainty sets.[9] When Γ is selected as zero, the policy corresponds to the nominal investment strategy that is achieved by solving the underlying stochastic program with fixed mean returns of assets. For the increasing values of Γ, the size of uncertainty set increases so that more possible realizations of random variable are included into the set. This implies that the investor is more ambiguity-averse. The choice of Γ also links with the probability guarantee of the corresponding constraint's feasibility. As shown in Ben-Tal et al. [2000] and Bertsimas and Sim [2004], the constraint that contains random variable is feasible with a probability of $1 - \kappa$ if the size of uncertainty set is chosen as $\Gamma = \sqrt{-2 \ln \kappa}$. In other words, the high probabilistic guarantee of feasibility is achieved for the choice of low κ values (at high specification of price of robustness). Since the life cycle asset allocation problem involves multi decision-making stages, the probability guarantee in constraints for $t \geq 1$ is conditional on the previous stages. In order to avoid too conservative strategies, we choose relatively small values of Γ (i.e., $\Gamma \leq 1$) for the computational experiments, which is in line with Ben-Tal et al. [2000] and Bertsimas and Pachamanova [2008].

Note that in the optimization model (2.4.1), the constraints are either linear or second order conic (due to the use of ellipsoidal uncertainty sets) and the objective function contains $2 \cdot T$ power utility functions. As shown by Alizadeh and Goldfarb [2003], when the coefficient γ is a rational number, the power utility function can be reformulated as a second order cone program (SOCP). In our numerical experiments, we used Mosek to solve the underlying non-linear programming problems. This is in line with several other studies [e.g., Brown and Smith, 2011; Konicz, Pisinger, Rasmussen

[9] In the current setting, the only source of ambiguity aversion is uncertain stock market returns, hence $\Gamma_t^\alpha = \Gamma_t^\mu = \Gamma$.

and Steffensen, 2015; Konicz, Pisinger and Weissensteiner, 2015; Haugh et al., 2016] where similar power utility maximization problems are considered.

2.5 Model Setting

The robust life-cycle asset allocation model (2.4.1) contains four main elements: ambiguity aversion towards uncertain asset returns, habit formation preferences, bequest motive and term life insurance. Due to the curse of dimensionality, most previous papers which formulate the life-cycle problem via dynamic programming, study these elements in isolation. In comparison, by formulating the problem in the multi-stage stochastic programming form, we study these elements together in a unified model while keeping it computationally tractable. In particular we are concerned with two main questions which have been only partially answered before:

- How are the investor's life-cycle decisions affected by investor's preferences such as ambiguity aversion and habit formation?
- What factors determine the investor's willingness to buy term life insurance?

2.5.1 Baseline Calibration

We address the first question using a baseline calibration of the model parameter values which are widely used in the literature and are treated as standard choices. Later, we will modify the model setting in order to study the second question.

In all the models, we consider an investor who starts making life-cycle decisions at the age of 20 (i.e., at $t = 1$). The investor retires at age 65, and dies with probability 1 at age 100 (i.e., $T = 80$). We consider an asset allocation problem where wealth can be allocated among two assets: risk-free and risky asset (stock). Following the earlier literature [e.g., Cocco et al., 2005; Peijnenburg, 2016], we set the risk-free return to 2% and assume that the return on risky asset is time-independent, and normally distributed with equity premium 4% and standard deviation 15.7%.

We assume that the labour income (before retirement) is uncertain and formulated as suggested by Cocco et al. [2005]:

$$\log(\tilde{l}_t) = f_t + \tilde{\nu}_t + \tilde{\varepsilon}_t. \tag{2.18}$$

The deterministic component f_t in (4.10) is modeled by a third-order polynomial function of age t as follows;

$$f_t = a_0 + a_1 t + a_2 t^2/10 + a_3 t^3/100,$$

where coefficients a_0, a_1, a_2 and a_3 need to be estimated. Moreover, $\tilde{\nu}_t = \tilde{\nu}_{t-1} + \tilde{u}_t$ and $\tilde{u}_t \sim \mathcal{N}(0, \sigma_\nu^2)$ refers to permanent shock to the labour income whereas $\tilde{\varepsilon}_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is the temporary shock. The retirement income is a constant fraction (e.g., ξ=0.68) of the permanent labour income in the year just before retirement. In other words, there is no temporary shock $\tilde{epsilon}$ during retirement.

In numerical results, we use the same labour income estimates, as given in Cocco et al. [2005], of shock variances (σ_ε, σ_ν), polynomial coefficients (a_0, a_1, a_2, a_3) as well as the retirement income fraction for three different education groups of investors who have no high school degree, high school degree and college degree. Since the choice of groups does not affect the main conclusions reported in this chapter, we only present the computational results obtained by the labour income estimates for the investor with high school degree group.

We use the 2013 mortality rates from US Social Security[10] to calculate the conditional probability of being alive (p_t) in each time period. The results related with amount of consumption, death benefit of term life insurance, holding in assets and total wealth are presented in terms of thousands of 1992 US dollars.

The investor's degree of ambiguity aversion (Γ) towards uncertain asset returns is varied from 0 (that is, ambiguity neutral) to 1. From the empirical study, we observe that when $\Gamma = 1$, the investor allocates almost whole wealth into the risk-free asset. Increasing Γ beyond 1 does not change her investment pattern.

As an usual practice in the literature [e.g., Gomes and Michaelides, 2003; Polkovnichenko, 2007], we also choose two levels of habit formation: i) the investor has no habit formation $\beta = 0$[11], and ii) the investor has a certain degree of habit formation. In particular, the importance of habit formation is defined as $\beta = 0.5$ whereas the degree of habit formation persistence is $\lambda = 0.5$.

The other model parameters are selected as follows. Time discount factor is $\delta = 0.96$ and the degree of risk aversion is fixed at $\gamma = 4$. The investor has a bequest motive $\eta = 1$; this implies that she views the utility of consumption and bequest equally important.

Alternative Parameter Settings: After showing the baseline calibration results, we vary the values of some model parameters to investigate whether investor's life-cycle decisions (especially the willingness to buy term life insurance) are sensitive to chosen parameter values. We particularly focus on the degree of risk aversion (γ), bequest motive (η) and the importance of habit formation (β).

We also modify the model to add borrowing constraints [e.g., Cocco et al., 2005;

[10] The reader is referred to `https://www.ssa.gov/oact/STATS/table4c6.html`.

[11] Note that $\lambda = 1$ and $z_0 = 0$ also corresponds for a case of no habit formation.

Kouwenberg and Zenios, 2006], and consider investor's subjective survival belief [Groneck et al., 2016] to investigate how these features change investor's life cycle decisions.

2.5.2 Model Implementation

We apply a rolling-horizon procedure to find optimal life-cycle decisions by repeatedly solving the robust stochastic optimization model in T numbers of iterations, denoted by $k = 1, 2, \cdots, T-1, T$. The main steps of this procedure in terms of states updated and actions taken at each time period are summarised in Table 2.2.

Table 2.2: Dynamic rolling-horizon procedure

Iteration number	# of time periods in the model	Decisions implemented	Random variables realized	State variables updated
$k = 1$	T	c_1, d_1, $\boldsymbol{b}_1$, $\boldsymbol{s}_1$	$\tilde{\boldsymbol{r}}_1$, $\tilde{l}_1$	$\boldsymbol{h}_1$, z_1, w_1
$k = 2$	$T-1$	c_2, d_2, $\boldsymbol{b}_2$, $\boldsymbol{s}_2$	$\tilde{\boldsymbol{r}}_2$, $\tilde{l}_2$	$\boldsymbol{h}_2$, z_2, w_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k = T-1$	2	c_{T-1}, d_{T-1}, $\boldsymbol{b}_{T-1}$, $\boldsymbol{s}_{T-1}$	$\tilde{\boldsymbol{r}}_{T-1}$, $\tilde{l}_{T-1}$	$\boldsymbol{h}_{T-1}$, z_{T-1}, w_{T-1}
$k = T$	1	c_T, d_T, $\boldsymbol{b}_T$, $\boldsymbol{s}_T$	$\tilde{\boldsymbol{r}}_T$, $\tilde{l}_T$	$\boldsymbol{h}_T$, z_T, w_T

At the beginning, the uncertain stock returns ($\tilde{\boldsymbol{r}}_t$) are unknown to the investor. What the investor knows is the mean and variance information used in the calibration. Based on this information the investor builds uncertainty sets (the size of the uncertainty set is determined by investor's degree of ambiguity aversion Γ) and derive the robust counterpart introduced in Section 2.4. The uncertain labour income ($\tilde{l}_t$) is also unknown and we assume that the investor is ambiguity-neutral to such uncertainty. Thus, we generate labour income for all time periods according to Equation (4.10).

At iteration $k = 1$, the model with all T time periods is solved but only the first-stage decisions at $t = 1$ in terms of consumption c_1, investment decisions b_1^1 and s_1^1 and term life insurance demand d_1 are implemented.

Then the investor observes the actual stock return and the labour income for $t = 1$, which is simulated according to the calibrated mean/variance information of the stock returns and labour income process, respectively. Once the actual asset returns and labour income are observed, the asset holdings $\boldsymbol{h}_1$, total wealth w_1 and the habit formation reference level z_1 are updated.

By going forward in time, the number of time periods in the optimization model is reduced by one at each iteration ($1 < k \leq T$). Then the optimization model is solved again and only the first stage decisions are implemented. The holdings in each asset, total wealth and habit formation reference level (i.e. h_k^1, w_k^1 and z_k) are updated according to the new observed asset returns and labour income. At the last iteration $k = T$, we solve the problem with only one time period.

We run the rolling-horizon procedure 1000 times with different stock returns and labour income processes simulated from the probability distributions introduced in Section 2.5.1. We then report the average and standard deviation (in parentheses) of 1000 simulations as the optimal life-cycle decisions (i.e., consumption, wealth, stock allocation and term life insurance demand) during the life.

2.6 Computational Results

2.6.1 Consumption and Investment Decisions

In order to analyse the life-cycle decisions using the baseline calibration, we present the results in five age groups: '20+', '20-24', '25-44', '45-64' and '65+'. In particular, '20+' represents the entire life span of the investor (from age 20 to 99 represented by $t = 1, ..., T$) while the other four age groups refer to certain periods during the life. For instance, '25-44' corresponds to time periods $t = 6, ..., 25$ in the model.

For each age group, we report the results in terms of average values of the optimal consumption, wealth, stock allocation and term life insurance demand during that specific time period. For example, the average consumption value for age groups '25-44' and '45-64' are computed as follows. First, we run the rolling horizon procedure 1000 times to obtain 1000 sets of c_t for $t = 1, \ldots, T$. Then, the average value $\hat{c}_t$ at time period t is computed over 1000 points of c_t. Finally, we calculate the average consumption values for the age groups '25-44' and '45-64' as $\frac{\hat{c}_6+\hat{c}_7+...+\hat{c}_{25}}{20}$ and $\frac{\hat{c}_{26}+\hat{c}_{27}+...+\hat{c}_{45}}{20}$, respectively.

In particular, the case $\Gamma = 0$ corresponds to the expected value optimization model where average stock returns are used as inputs (i.e. $\bar{\boldsymbol{r}}_t = \mathbb{E}[\tilde{\boldsymbol{r}}_t]$). By varying Γ from 0 to 1, we increase the size of uncertainty sets that uncertain stock returns belong to.

Table 2.3 summarizes the simulation results in terms of different degrees of ambiguity aversion ($\Gamma \in [0, 1]$) towards stock returns under two habit formation levels: $\beta = 0$ (left) and $\beta = 0.5$ (right). The results of $\Gamma = 0, 0.25, 0.55, 0.65, 0.75, 1.0$ are chosen to present due to space limitations. The following observations stand out from Table 2.3.

Table 2.3: Average consumption, wealth and asset allocation with baseline calibration

	Habit level ($\beta = 0$)					Habit level ($\beta = 0.5$)				
Γ	20+	20-24	25-44	45-64	65+	20+	20-24	25-44	45-64	65+
Panel A. Consumption										
0	25.20	17.35	24.91	27.75	25.03	25.57	16.60	25.16	28.05	25.67
	(2.01)	(0.60)	(1.07)	(1.81)	(3.55)	(3.02)	(0.52)	(1.02)	(2.26)	(5.89)
0.25	25.31	17.36	24.89	27.70	25.32	25.79	16.59	25.15	27.93	26.24
	(2.11)	(0.60)	(1.06)	(1.75)	(3.80)	(3.30)	(0.53)	(1.01)	(2.17)	(6.60)
0.55	24.93	17.36	25.10	27.19	24.62	25.28	16.74	25.31	27.14	25.42
	(2.41)	(0.60)	(1.08)	(1.83)	(4.61)	(2.95)	(0.53)	(1.04)	(1.81)	(5.96)
0.65	24.01	17.36	25.46	26.52	22.70	24.17	16.75	25.64	26.56	23.03
	(1.70)	(0.60)	(1.08)	(1.65)	(2.85)	(1.85)	(0.53)	(1.05)	(1.65)	(3.27)
0.75	22.85	17.36	25.56	25.91	20.34	22.88	16.75	25.69	26.01	20.37
	(1.24)	(0.60)	(1.08)	(1.55)	(1.43)	(1.22)	(0.53)	(1.06)	(1.55)	(1.37)
1	22.68	17.36	25.57	25.85	19.99	22.86	16.75	25.70	26.01	20.32
	(1.20)	(0.60)	(1.08)	(1.54)	(1.33)	(1.22)	(0.53)	(1.06)	(1.55)	(1.37)
Panel B. Wealth										
0	46.38	0.27	9.09	72.27	59.48	50.43	5.29	12.46	78.79	62.36
	(14.91)	(0.23)	(3.63)	(19.81)	(25.48)	(27.05)	(1.93)	(5.97)	(33.48)	(47.57)
0.25	48.56	0.23	9.54	72.80	63.88	54.25	5.13	13.05	79.95	70.11
	(16.76)	(0.20)	(3.61)	(20.58)	(29.22)	(31.80)	(1.89)	(6.21)	(35.86)	(57.21)
0.55	44.24	0.22	6.59	62.63	61.53	47.12	4.33	7.60	64.13	66.10
	(21.57)	(0.19)	(3.21)	(25.28)	(38.45)	(29.33)	(1.44)	(3.73)	(29.01)	(54.97)
0.65	32.41	0.22	3.29	45.84	45.98	32.81	4.32	4.31	47.06	45.02
	(13.48)	(0.19)	(1.95)	(17.98)	(22.75)	(17.50)	(1.43)	(2.47)	(21.03)	(30.45)
0.75	20.88	0.22	2.40	33.05	27.43	18.75	4.32	3.55	32.93	21.40
	(3.45)	(0.19)	(1.40)	(7.58)	(4.14)	(3.55)	(1.43)	(1.82)	(8.32)	(3.31)
1	19.56	0.22	2.31	32.13	25.00	18.60	4.32	3.54	32.84	21.11
	(3.12)	(0.19)	(1.40)	(7.42)	(3.13)	(3.53)	(1.43)	(1.82)	(8.31)	(3.28)
Panel C. Stock allocation (%)										
0	99.68	95.96	99.79	99.95	99.46	99.83	99.97	99.98	99.99	99.65
	(0.09)	(8.08)	(0.10)	(0.01)	(0.17)	(0.09)	(0.01)	(0.01)	(0.00)	(0.21)
0.25	98.87	12.38	96.52	99.18	98.88	98.97	61.10	96.92	99.30	99.03
	(0.73)	(11.55)	(1.87)	(0.55)	(0.89)	(0.71)	(13.63)	(1.77)	(0.63)	(0.71)
0.55	93.06	5.91	90.89	95.00	91.98	94.90	1.50	84.19	95.10	96.58
	(2.57)	(11.68)	(7.59)	(3.22)	(2.74)	(3.23)	(5.48)	(12.17)	(3.64)	(3.09)
0.65	73.89	5.75	74.54	80.49	69.57	72.26	0.19	63.13	80.60	67.95
	(5.87)	(11.36)	(17.84)	(9.75)	(4.19)	(6.87)	(0.15)	(24.26)	(10.57)	(4.63)
0.75	13.20	5.64	13.19	9.26	15.93	1.65	0.13	0.91	0.87	2.46
	(5.49)	(11.40)	(6.61)	(4.49)	(6.83)	(0.44)	(0.08)	(0.53)	(0.30)	(0.62)
1	0.44	5.32	3.67	0.49	0.28	0.03	0.09	0.21	0.03	0.02
	(0.13)	(10.94)	(2.00)	(0.18)	(0.09)	(0.01)	(0.04)	(0.10)	(0.01)	(0.01)
Data	40.00	——	38.50	42.00	36.50	40.00	——	38.50	42.00	36.50

This table displays the average consumption, wealth and stock allocation (in Panels A, B, and C, respectively) for different degrees of ambiguity aversion (Γ) and two habit formation levels (β). The average (and standard deviation) values are calculated as the 50 percentile (median) of 1000 simulations using the rolling-horizon procedure. The last row (labelled as 'Data') displays the stock allocation during certain time periods given by the empirical data from Survey of Consumers Finance 2010. Note that unlike the results in Panels A, B and C, the blocks under '20+' and '65+' show the average stock allocation up to age 84 (not 100).

First we focus on the case with no habit formation (left panel). The average stock allocation decreases from nearly 100% to 0% over the life span (column '20+') with the increase of the degree of ambiguity aversion (Γ) from 0 to 1 (Panel C). Average consumption and wealth follow a similar decreasing pattern, but there is a slight increase in the average values when Γ varies from 0 to 0.25. As we will discuss in detail below, this implies that the investor has two separate strategies when she faces the stock return uncertainty.

When $\Gamma \leq 0.25$, the investor does not immediately reduce the allocation in risky asset (i.e, the stock allocation is still greater than 99%). We observe that she accumulates more wealth before the retirement (as shown in columns 25-44 and 45-64) by sacrificing some amount of consumption. Such saving motive boosts the wealth and consumption after the retirement because of the high stock allocation (i.e., 99.46 and 98.88% as in column 65+).

This results in high average wealth and consumption over the life span. However, there exists a utility loss over the life span despite the increase in average wealth and consumption. In other words, the life-cycle utility (sum of utilities over time, i.e., objective function value) still decreases as Γ increases from 0 to 0.25. This implies that the investor life-cycle strategy is more conservative when she shifts from being ambiguity neutral ($\Gamma = 0$) to ambiguity averse ($\Gamma = 0.25$).

When $\Gamma \geq 0.25$, the investor chooses another strategy to respond to the stock return uncertainty. In this case, the investor prefers to consume more (unlike the decreasing pattern when $\Gamma \leq 0.25$) and accumulates less wealth during the age group of 25-44. This impacts his consumption and saving behaviour during 45-64 and 65+. In other words, she starts consuming less and also saving less wealth due to previous decisions. As ambiguity aversion increases, the investor fears possible stock market downside performance, and its potential damage on her life-cycle utility. Therefore, this induces her to invest less wealth in the stock market (or invest more in risk-free asset). Unless the investor is willing to participate in the stock market, more saving at early ages does not necessarily bring high portfolio return (and high utility) at later ages (due to stock return uncertainty). However, consuming more in early ages can provide immediate utility without facing stock return uncertainty although such a consumption behaviour may negatively affect the wealth accumulation as well as consumption at later ages.

So far we are concerned with the behaviour of the investor at specific age category (by looking at column-wise results) as the degree of ambiguity aversion varies. Similarly, we can analyse the structure of different investment decisions over life span at a certain ambiguity aversion level. In terms of life-cycle decisions in Table 2.3, we observe that both consumption and wealth profiles display a hump-shaped pattern; that is in line with earlier life-cycle models [e.g., Gomes and Michaelides, 2003; Cocco et al., 2005; Peijnenburg, 2016]. However, most the earlier life-cycle models struggle to generate a hump-shaped pattern in stock allocation, an important feature of the life-cycle investment hypothesis [e.g., Bakshi and Chen, 1994; Geanakoplos et al., 2004; Yao and Zhang, 2005] and the tactical asset allocation [Campbell and Viceira, 2002]. Note that with baseline calibration, the model generates the hump-shaped investment pattern for $0.55 \leq \Gamma \leq 0.65$. For example, in case of $\Gamma = 0.65$, the model predicts that a young investor (aged 25-44) on average allocates around 74.5% of her wealth in the stock market. This increases up to 80.5% for the middle-aged investor, and decreases again to 69.6% during the retirement age. While the model captures a realistic hump-shaped pattern (e.g., stock allocation increases (from 74.54% to 80.49%) by 8% during young-middle ages and decreases (from 80.49% to 69.57%) by 13.5% during middle-old ages), the level of average stock allocation is relatively high (73.9%) compared to the empirical data (40%) gathered from the Survey of Consumer Finances in 2010.

Now, let's turn to the case that incorporates moderate level of habit formation ($\beta = 0.5$). The right panel of Table 2.3 shows that the overall patterns of consumption, wealth and stock market allocation are similar to that with no habit formation ($\beta = 0$). But the investor's savings motive is stronger at the expense of lower consumption in the first five years of economic life. Such effect of habit formation on the savings motive is also observed in earlier studies [e.g., Diaz et al., 2003; Gomes and Michaelides, 2003; Polkovnichenko, 2007]. Compared to ambiguity aversion, habit formation plays a secondary role on the asset allocation, having only a significant impact for high values of Γ (≥ 0.7). Highly ambiguity-averse investor allocates less wealth in stocks if she has habit formation preferences.

2.6.2 Term Life Insurance Demand

While the important life-cycle decisions involve consumption, wealth accumulation and asset allocation, our model also incorporates the choice for term life insurance products. As stated in Section 4.2, we assume that the investor has an option to buy one-year renewable life insurance products (abbreviated as *term life insurance*). We start analysing the term life insurance in the same model setting using baseline calibration and then present additional results to illustrate the interactions and tensions between the term life insurance demand and investor's various preferences such as risk/ambiguity aversion and bequest motive/habit formation. Finally, we would like to establish how the introduction of borrowing option and the subjective survival belief into the life-cycle consumption and asset allocation model alters the ambiguity-averse investor's decision regarding term life insurance.

Baseline Setting: in Table 2.4, we first note that the investor purchases more term life insurance if she has a higher degree of ambiguity aversion towards stock market returns. Compared to the stock return uncertainty, the pay-off (i.e., death benefit) from the term life insurance is relatively certain. This feature of the term life insurance becomes attractive for an investor who is highly averse to possible bad outcomes from the stock market. To the best of our knowledge, this relationship has not been studied before in a life-cycle setting.

Table 2.4: Average term life insurance demand with baseline calibration

	Habit level ($\beta = 0$)					Habit level ($\beta = 0.5$)				
Γ	20+	20-24	25-44	45-64	65+	20+	20-24	25-44	45-64	65+
0	6.09	16.99	16.37	0.54	1.83	2.61	8.55	6.66	0.04	0.90
	(0.86)	(0.57)	(2.64)	(0.51)	(0.69)	(0.69)	(1.83)	(2.28)	(0.07)	(0.48)
0.25	6.06	17.00	16.22	0.70	1.75	2.55	8.88	6.44	0.05	0.85
	(0.86)	(0.57)	(2.52)	(0.53)	(0.72)	(0.72)	(1.80)	(2.36)	(0.08)	(0.48)
0.55	7.06	17.03	19.09	1.36	2.01	3.34	9.74	9.30	0.15	0.85
	(1.12)	(0.57)	(2.54)	(1.42)	(1.02)	(0.64)	(1.54)	(1.92)	(0.38)	(0.54)
0.65	8.39	17.03	22.00	3.59	2.12	4.10	9.75	11.64	0.82	0.85
	(1.30)	(0.57)	(1.89)	(2.71)	(1.13)	(0.77)	(1.53)	(1.78)	(1.20)	(0.52)
0.75	9.18	17.02	22.64	4.58	3.01	4.53	9.75	12.00	1.05	1.50
	(1.35)	(0.57)	(1.69)	(2.98)	(1.15)	(0.84)	(1.53)	(1.72)	(1.35)	(0.64)
1	9.53	17.07	22.92	4.75	3.54	4.56	9.76	12.02	1.05	1.57
	(1.43)	(0.58)	(1.73)	(3.07)	(1.25)	(0.85)	(1.53)	(1.72)	(1.36)	(0.68)

This table displays the average term life insurance demand values for different degrees of ambiguity aversion (Γ) and two habit levels (β). The average (and standard deviation) values are calculated as the 50 percentile (median) of 1000 simulations using the rolling-horizon procedure.

When habit formation preferences are introduced into the model (right panel of Table 2.4), the investor spends less on term life insurance, regardless of the level of ambiguity aversion. Recall that the main purpose of life insurance is to protect the heirs from unexpected death and its adverse financial consequences. Therefore, the demand for life insurance mainly stems from altruistic motives. On the hand, habit formation preferences lead to more egoistic choices. In other words, an investor with habit formation preferences would save more for her future self resulting in less wealth allocated to life insurance products.

Regarding the life-cycle choices, the investor's demand for life insurance is highest at the young age (20-44) when she has the highest human capital, that is, expected value of future labour income. While this has been documented in earlier literature [Chen et al., 2006], our results also show that the investor's demand for life insurance is sensitive neither to the degree of ambiguity aversion nor to the habit formation preferences.

Risk versus Ambiguity Aversion: we now investigate the impact of both risk (γ) and ambiguity (Γ) aversion on life-cycle asset allocation decisions, and mainly focus on establishing their interaction in the context of life insurance demand. These factors have been studied in isolation in the literature. Most of asset pricing models are concerned with risk aversion [e.g., Mehra and Prescott, 1985; Ingersoll, 1987; Cochrane, 2009]. There are also other studies establishing the effect of ambiguity aversion on financial decision making [e.g., Gilboa and Schmeidler, 1989; Goldfarb and Iyengar, 2003; Garlappi et al., 2007; Gulpinar et al., 2016; Peijnenburg, 2016].

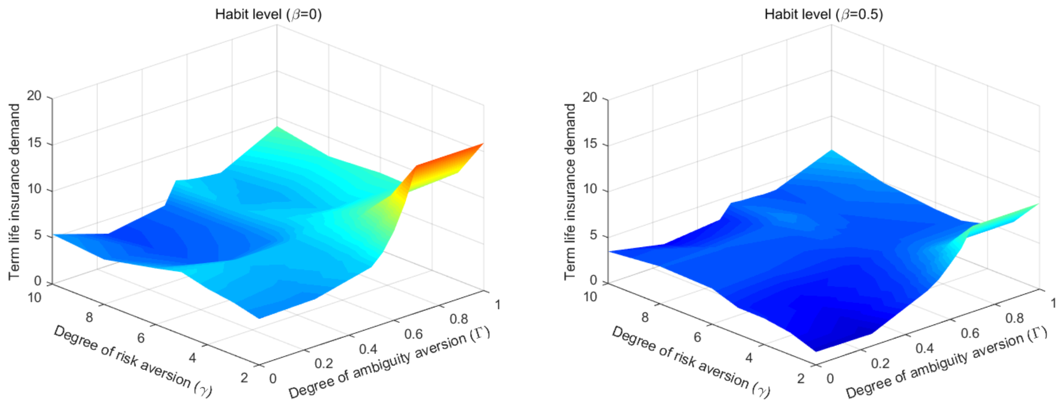

Figure 2.1: Risk versus ambiguity aversion
The two plots compare the average term life insurance demand over the entire life for varying degrees of risk aversion and ambiguity aversion, under two habit formation preferences: either the investor has no habit formation (left) or has a habit formation level of $\beta = 0.5$ (right). The x, y and z axes denote the degree of ambiguity aversion, risk aversion and term life insurance demand, respectively.

Figure 2.1 illustrates the corresponding average term life insurance demand over the entire life for varying degrees of investor's ambiguity aversion ($\Gamma \in [0, 1]$) and risk aversion level ($\gamma \in [2, 10]$). The strength of bequest motive is fixed at unity ($\eta = 1$). We first observe at lower levels of risk aversion, investor's term life insurance demand increases significantly with the degree of ambiguity aversion. The effect is less pronounced for intermediate level of risk aversion. However, at the higher end of risk aversion (i.e., $\gamma = 10$), we see that life the insurance demand increases again with the degree of ambiguity aversion, though to a lesser extent.

Second, as demonstrated in Table 2.4, the average life insurance demand is lower when the investor has habit formation preferences (see the right panel of Figure 2.1). Moreover, the life insurance demand does not monotonically change in the degree of risk aversion: it is concave, that is, higher insurance demand at intermediate levels of risk aversion, for ambiguity neutral investors, and becomes convex, that is, higher insurance demand either at low or high levels of risk aversion, for highly ambiguity-averse investors ($\Gamma = 1$).

Bequest Motive versus Habit Formation: as shown by the baseline case in Figure 2.1, habit formation preferences reduce investor's willingness to buy term life insurance. We also know from earlier literature [e.g., Bernheim, 1991; Inkmann and Michaelides, 2012] that the bequest motive has the opposite effect, which creates a tension between bequest motive and habit formation. In other words, an investor with habit formation places more weight on her own utility of consumption rather than the utility of the heirs (bequest motive). Therefore, in this section we investigate how these two factors jointly affect the term life insurance demand. For this purpose, the risk aversion level is fixed at 4 and the bequest motive strength is varied within a range ($\eta \in [1, 5]$). In the literature, there is little consensus on the choice of bequest motive parameter since it is difficult to estimate [Dynan et al., 2002]. We therefore choose a range of bequest motive parameters $\eta \in [1, 5]$ that is commonly used in earlier literature; for instance, see [Cocco et al., 2005]. One can also consider other settings such as $\eta \in [0, 1]$ and show the impact of this parameter on life-cycle decisions.

Figure 2.2 presents the average term life insurance demand obtained by varying the strength of bequest motive under different degrees of ambiguity aversion with (right) and without (left) habit formation. One can easily observe that the bequest motive increases the term life insurance demand not only for ambiguity-neutral investors as shown before in earlier literature [e.g., Bernheim, 1991; Inkmann and Michaelides, 2012] but also for ambiguity-averse investors. The conclusion from Table 2.4 that the degree of ambiguity aversion increases the term life insurance demand, still holds regardless of the strength of bequest motive. Introducing the habit formation preferences in the model (as in the right panel in Figure 2.2) does not change the general conclusion that habit formation preferences reduce the term life insurance demand.

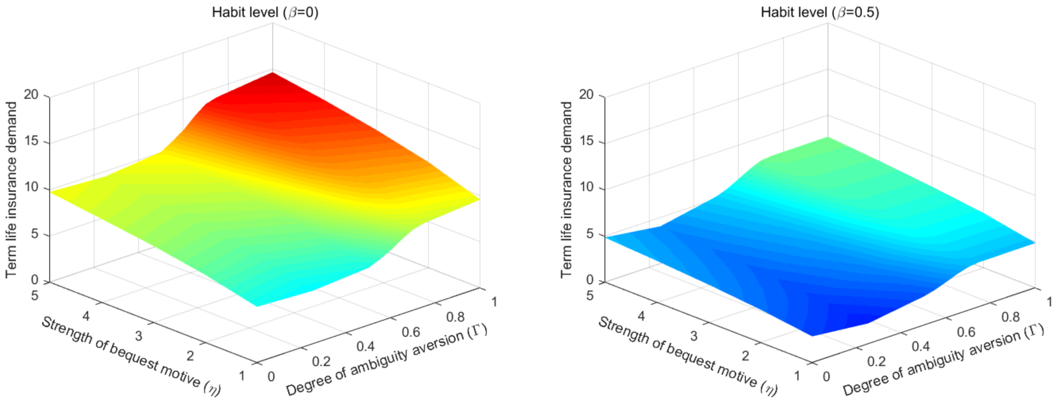

Figure 2.2: Bequest motive versus ambiguity aversion
The two plots compare the average term life insurance demand over the entire life for varying degrees of bequest motive and ambiguity aversion, under two habit formation preferences: either the investor has no habit formation (left) or has a habit formation level of $\beta = 0.5$ (right). The x, y and z axes denote the degree of ambiguity aversion, bequest motive and term life insurance demand, respectively.

So far, we have discussed the implications of incorporating habit formation only at the certain level ($\beta = 0.5$). Next we illustrate how different levels of habit formation ($\beta \in [0, 1]$) and bequest motives ($\eta \in [1, 5]$) affect life insurance demand for investors with two representative degrees of aversion towards stock return uncertainty. As in the benchmark case, the level of risk aversion is fixed at $\gamma = 4$.

We start with a case of an ambiguity-neutral investor ($\Gamma = 0$). As displayed in the left panel in Figure 2.3, when the investor has strong habit formation preferences, the bequest motive does not play an important role on the life insurance demand. In other words, habit formation dominates the bequest motive. In addition, the bequest motive only matters when habit formation preferences are weak. From the right panel in Figure 2.3, we see that introducing the ambiguity aversion ($\Gamma = 1$) does not change the previous observation.

Moreover, while an ambiguity-neutral investor with strong habit formation preferences (i.e. $\beta = 0.6,\ 0.8$) hardly demand life insurance products, an ambiguity-averse investor with strong habit formation preference still purchases substantial amount of life insurance products unless the preference is extremely strong ($\beta = 0.8$).

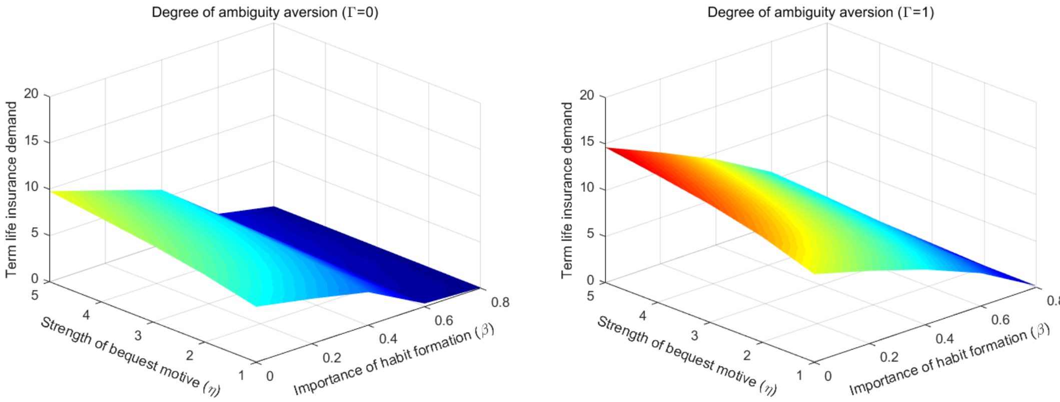

Figure 2.3: Bequest motive versus habit formation
The two plots compare the average term life insurance demand over the entire life with varying bequest motives and habit formation preferences in two degrees of ambiguity aversion. The left plot denotes $\Gamma = 0$ (ambiguity-neutral) while the right one denotes $\Gamma = 1$ (ambiguity-averse). The x, y and z axes denote the importance of habit formation, strength of bequest motive and term life insurance demand, respectively.

2.6.3 Borrowing Option

In the computational experiments so far, we have assumed that the investor is not allowed to borrow (i.e., $w_t \geq 0$ for $t = 1, \cdots, T$). However, in reality, investors often prefer financial markets to borrow capital, so that they can have further flexibility to consume or save over time. We would like to investigate how the borrowing option affects the life-cycle decisions (in particular, the term life insurance demand) using the baseline calibration. Now, we present the mathematical formulation of the robust life cycle asset allocation model in view of the borrowing option.

In order to investigate impact of borrowing option on life-cycle decisions, we extend the life cycle consumption and asset allocation model as suggested by Kouwenberg and Zenios [2006].

Suppose that the investor is allowed to borrow some amount of capital $v_t \geq 0$ at intermediate time periods $t = 1, \cdots, T-1$ and the amount of borrowing at each time period t is restricted by her/his income; that is $v_t \leq \tilde{l}_t$ for $t = 1, \cdots, T-1$. The borrowing option is not available at the final period, $v_T = 0$. The amount of money borrowed at t will be repaid at next time period $t+1$ under fixed interest rate spread

χ. In this case, the cash balance constraint for each time period is modified as follows;

$$h_t^0 = (1+\tilde{r}_t^0)h_{t-1}^0 + \sum_{m=1}^{M}(s_t^m - b_t^m) + \tilde{l}_t - c_t - (1-p_i)d_t + v_t - (1+\chi)v_{t-1},\ t = 1, \cdots, T.$$

Therefore, the total wealth at time t becomes $(w_t - v_t)$, and accordingly the utility of bequest is updated as $\mathcal{U}_B(w_t - v_t + d_t)$.

In order to investigate impact of borrowing on the life-cycle decisions, we conduct experiments with the life cycle asset allocation model with and without borrowing option using baseline calibration. We fix $\chi = 0.025$ as suggested by Kouwenberg and Zenios [2006].

Table 2.5 presents the difference between the average consumption levels obtained by the robust life-cycle optimization model with and without borrowing options under the investor's preferences towards habit formation and ambiguity aversion.

Table 2.5: Differences in average consumption with and without borrowing

	Habit level ($\beta = 0$)					Habit level ($\beta = 0.5$)				
Γ	20+	20-24	25-44	45-64	65+	20+	20-24	25-44	45-64	65+
0	-0.18	3.64	-0.33	-0.56	-0.43	0.14	2.45	0.14	-0.09	-0.06
0.25	-0.06	3.57	-0.35	-0.59	-0.12	0.27	2.34	0.20	-0.20	0.29
0.55	-0.20	3.58	-0.37	-0.84	-0.27	-0.31	2.86	-0.14	-0.85	-0.54
0.65	-0.36	3.58	-0.38	-1.06	-0.51	-0.30	2.86	-0.21	-0.93	-0.45
0.75	-0.40	3.58	-0.49	-0.99	-0.59	-0.31	2.86	-0.27	-0.92	-0.44
1	-0.32	3.58	-0.50	-0.95	-0.42	-0.30	2.86	-0.27	-0.92	-0.40

We observe that the investor tends to consume more in the first five years if she can borrow money. The consumption level decreases slightly in later ages due to the repayment of interest on the amount of capital borrowed. This is because increasing early consumption (at the cost of lower consumption in later ages) can benefit the life-cycle utility.

Next we would like to establish how the wealth of an investor with bequest motive ($\eta = 1$) evolves over the life cycle with or without the borrowing option. Figure 2.4 displays the net wealth (computed as the total wealth minus borrowed amount) of different types of investors: ambiguity neutral ($\Gamma = 0$) versus ambiguity aversion ($\Gamma = 1$) with no habit preference ($\beta = 0$) and habit formation ($\beta = 0.5$).

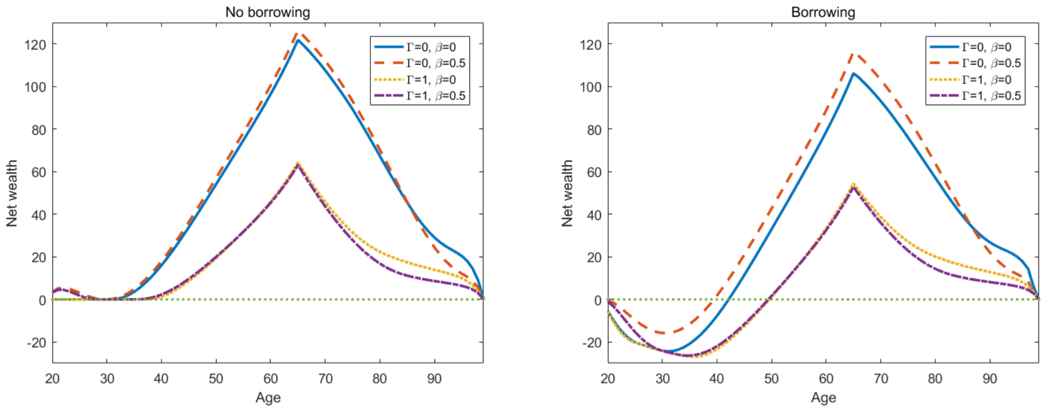

Figure 2.4: Net wealth gained with and without borrowing
The two plots compare the net wealth pattern over the entire life without (left) and with (right) borrowing. Each curve represents different parameter setting regarding with degree of ambiguity aversion and importance of habit formation. The horizontal dashed line indicates the zero wealth.

As displayed in Figure 2.4, the net wealth is negative up to a certain stage of life span with borrowing option (right panel) and the particular age at which it becomes non-negative depends on both ambiguity aversion and habit formation. However, for the ambiguity-averse investor, the effect of habit formation is not visible before the retirement.

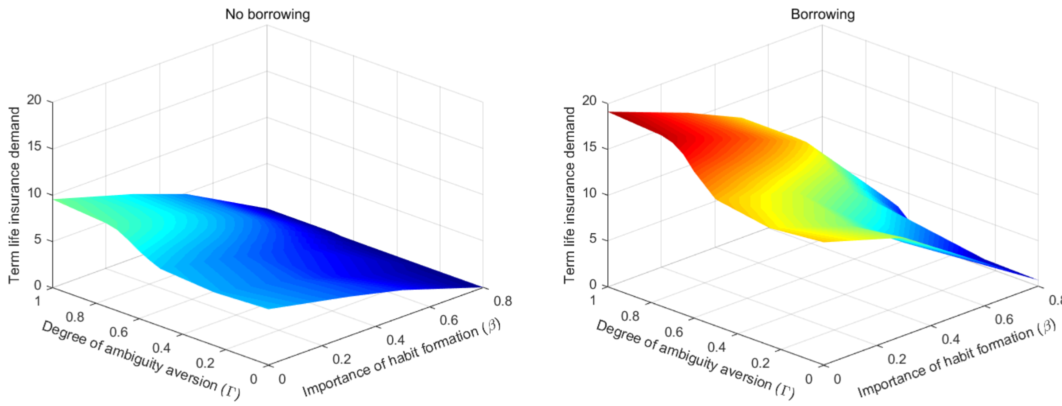

Figure 2.5: Borrowing option
The two plots show the average term life insurance demand during the life without borrowing (left) and with borrowing (right). The x, y and z axes in the right two panels denote the importance of habit formation, degree of ambiguity aversion, and term life insurance demand, respectively.

In Figure 2.5, we vary the degree of ambiguity aversion and importance of habit formation and compare investor's term life insurance demand without (left plot) and with (right plot) borrowing option. We observe that the borrowing option substantially increases term life insurance demand, regardless of investor's preferences with respect to uncertainty and habit formation. The increased amount is more visible when the investor has low habit formation preference (i.e. β is small). Because of the borrowing option, the investor now has more resources at the early stages of life both to consume and to invest in term life insurance products (see Table 2.5 for the evidence). Thus, the investor not only increases the own utility of consumption but also the utility of heirs. In other words, borrowing option is instrumental in improving investor's life-cycle utility.

Table 2.6: Comparison of average borrowing rates (%) at various habit levels

Γ	20+	20-24	25-44	45-64	65+	20+	20-24	25-44	45-64	65+
	Habit level ($\beta = 0$)					Habit level ($\beta = 0.3$)				
0	99.26	99.97	99.94	99.66	98.51	99.84	99.98	99.97	99.88	99.72
0.25	98.20	80.75	99.92	99.29	99.12	98.71	81.92	99.97	99.80	99.80
0.55	64.99	75.31	99.96	84.74	31.28	62.31	50.72	99.44	86.21	28.11
0.65	38.43	75.31	99.87	31.43	0.98	36.57	50.67	99.38	31.21	0.70
0.75	29.89	75.33	91.25	7.34	0.39	27.70	50.65	89.88	6.70	0.11
1	29.66	75.28	90.97	6.78	0.34	27.65	50.62	89.80	6.60	0.09
	Habit level ($\beta = 0.5$)					Habit level ($\beta = 0.8$)				
0	99.96	100.00	99.99	99.97	99.93	100.00	100.00	100.00	100.00	99.99
0.25	99.32	90.00	99.99	99.92	99.94	99.96	100.00	99.92	99.96	99.99
0.55	60.00	34.99	97.50	86.28	26.17	52.78	95.41	99.62	84.34	0.38
0.65	35.22	34.95	97.37	32.16	0.51	25.34	12.86	76.84	20.03	0.01
0.75	25.50	34.94	86.24	5.72	0.02	5.61	12.16	19.11	0.00	0.00
1	25.49	34.91	86.22	5.69	0.02	5.61	12.16	19.10	0.00	0.00

Table 2.6 compares the average borrowing rates (%) of investors within different age groups and varying habit formation levels. The average borrowing rate is calculated by taking the average value of $\frac{v_t}{l_t}$ at each time t within the specific age group. For instance, the average borrowing rate of an ambiguity-neutral investor with no habit formation ($\Gamma = 0$ and $\beta = 0$) over the life is 99.26%. From the results in Table 2.6, one can analyse the investor's willingness to borrow in different circumstances.

We notice that the average borrowing rate over life time decreases as Γ increases. This is because in the case of fixed interest spread (χ), the investors with higher degree of ambiguity aversion are concerned with downside risk of the stock market and the ability to repay the borrowed amount. Moreover, for high Γ values ($\Gamma \geq 0.65$), the investor is much less willing to borrow in later ages, especially after retirement, since she mainly relies on the income from the stock market (as opposed to labour income) to pay back the debt.

2.6.4 Subjective Survival Beliefs

In reality, there is evidence [Groneck et al., 2016] that subjective survival beliefs are different from the objective ones. The latter, obtained from the mortality table, is used for the computational experiments so far. We are now concerned with analysing the effect of subjective survival beliefs on investor's life-cycle decisions using the baseline calibration. We adopt the learning model suggested by Groneck et al. [2016] to calculate subjective survival rates in lieu of objective rates (p_t) in the utility functions. Since we assume that the price of the insurance products is not affected by the investor's subjective survival belief, we still use the objective survival rates to price term life insurance products.

A brief description of the learning model is as follows. Let j denote the investor's current age. The investor's subjective survival rate ω_t^j from age j to a target age t in the future is determined as:

$$\omega_t^j = \pi_j \cdot \phi + (1 - \pi_j) \cdot \varphi_t^j,$$

where φ_t^j denotes the objective survival rate from age j to t and is calculated by $\varphi_t^j = \prod_{i=1}^{t} p_i$. As introduced before, p_i represents the conditional probability of being alive at t given being alive at $t-1$ and is calibrated from the mortality table. $\phi \in [0, 1]$ denotes the investor's degree of optimism about her survival rate. Given the initial weight $\pi_0 \in [0, 1]$, the age-dependent weight π_j on the optimism at age j is defined as follows;

$$\pi_j = \frac{\pi_0}{\pi_0 + (1 - \pi_0) \cdot \frac{1}{\sqrt{j}}}.$$

Note that $\pi_j \in [0, 1]$ and the formulation has two implications. First, the investor's subjective survival rate (ω_t^j) is governed by investor's initial weight on optimism π and age j. If $\pi_0 = 0$, then $\pi_j = 0$ and subjective survival rates are equal to the objective ones. For $\pi_0 > 0$, π_j is an increasing function on the age j. This implies that as the investor becomes older, her survival belief depends more and more on her own degree of optimism (ϕ) rather than the objective survival rates (φ_t^j). Second, the subjective survival rate is a function of degree of optimism (ϕ) and objective survival rate (φ_t^j). Note that if $\phi > \varphi_t^j$, then subjective survival rate is larger than the objective one. That is, the investor believes she can live longer than expected according to the mortality table.

We choose the same values of degree of optimism ($\phi = 0.418$) and initial weight on the optimism ($\pi = 0.135$) as in Groneck et al. [2016] to calculate the investor's subjective survival rates in our model.

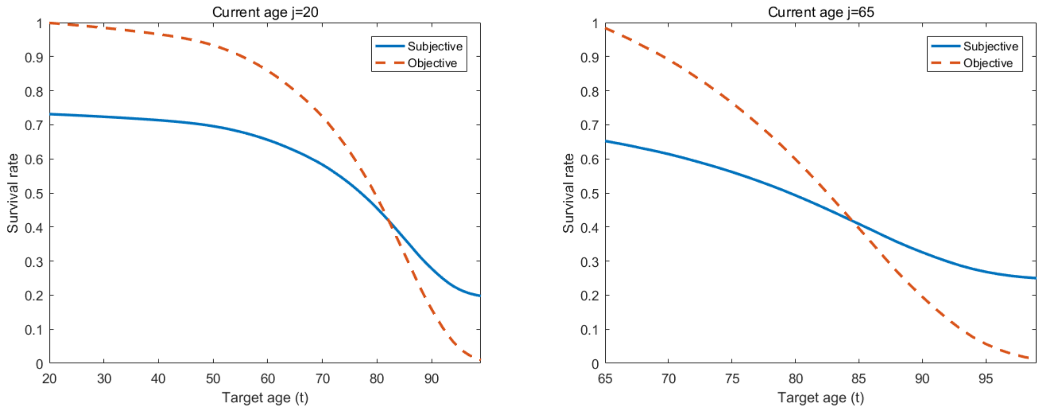

Figure 2.6: Subjective versus objective survival rates
This figure compares subjective (solid blue) and objective (dashed red) survival rates for the investor being at age 20 (left) and age 65 (right).

Figure 2.6 shows the differences between objective and subjective survival rates for an investor at current age of 20 (left) and 65 (right). As we can see, regardless of the current age, the investor believes that she dies sooner than suggested by the mortality table in the near future while she overestimates the life span in distance future.

Next, we are interested in investors' willingness to purchase term life insurance in the presence of subjective survival beliefs. Figure 2.7 compares the average (over the life span, 20+) term life insurance demand of an investor with either objective or subjective survival beliefs, and with varying individual's preferences with respect to ambiguity aversion and habit formation.

We observe that the investor with subjective survival belief purchases substantially more term life insurance products regardless of degree of ambiguity aversion and habit formation. The difference is particularly striking when habit formation is less relevant for the investor. This is an artifact of the evidence presented in Figure 6, that is, the underestimation of survival probability in the near future, since term life insurance products are tightly links to short term bequest decisions. Overall, this implies that, apart from bequest motive and ambiguity aversion, subjective survival belief is also a potential factor that affects the investor's willingness to buy term life insurance.

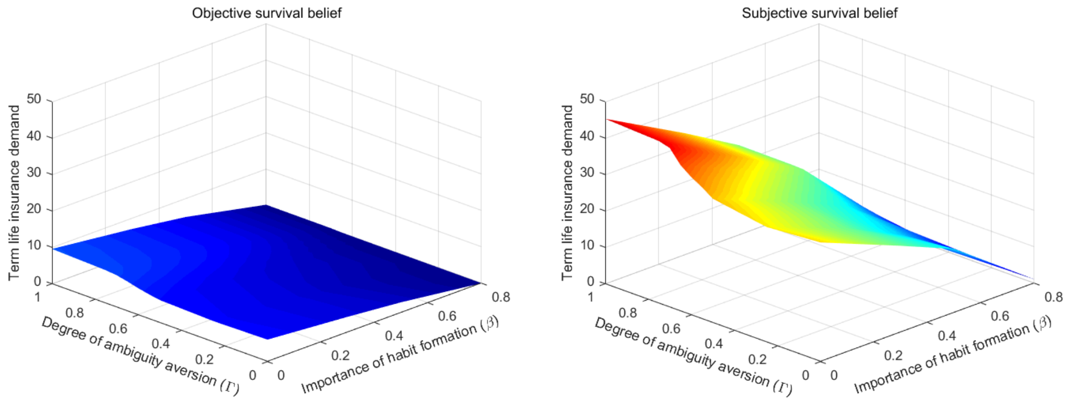

Figure 2.7: Objective versus subjective survival belief

These two plots compare the average term life insurance demand over the entire life with objective (left) or subjective (right) survival beliefs. The x, y and z axes denote importance of habit formation, the degree of ambiguity aversion and term life insurance demand, respectively.

2.6.5 Embedded Models

We are now concerned with comparison of different robust life-cycle optimization models under investor's preferences represented by different degrees of ambiguity aversion and habit formation. In particular, we would like to investigate how bequest motive and the choice to buy term life insurance affect the investor's life cycle decisions. We implement three types of robust life-cycle asset allocation models. A brief description of these models is as follows.

The robust life-cycle asset allocation model presented in Proposition 1 is a generic model (labelled as $M_{(BEQ,\, INS)}$) and involves both options of having bequest motive and life insurance products. The other two models (labelled as $M_{(NBEQ,NINS)}$ and $M_{(BEQ,\, NINS)}$, respectively) that have been widely studied in the literature can be seen as special cases of the general life-cycle asset allocation model $M_{(BEQ,\, INS)}$. More precisely, $M_{(NBEQ,\, NINS)}$ assumes that the investor is neither a bequest motive nor has choice to buy life insurance. On the other hand, $M_{(BEQ,\, NINS)}$ allows the investor to be a bequest motive, but does not give any choice to buy life insurance products.

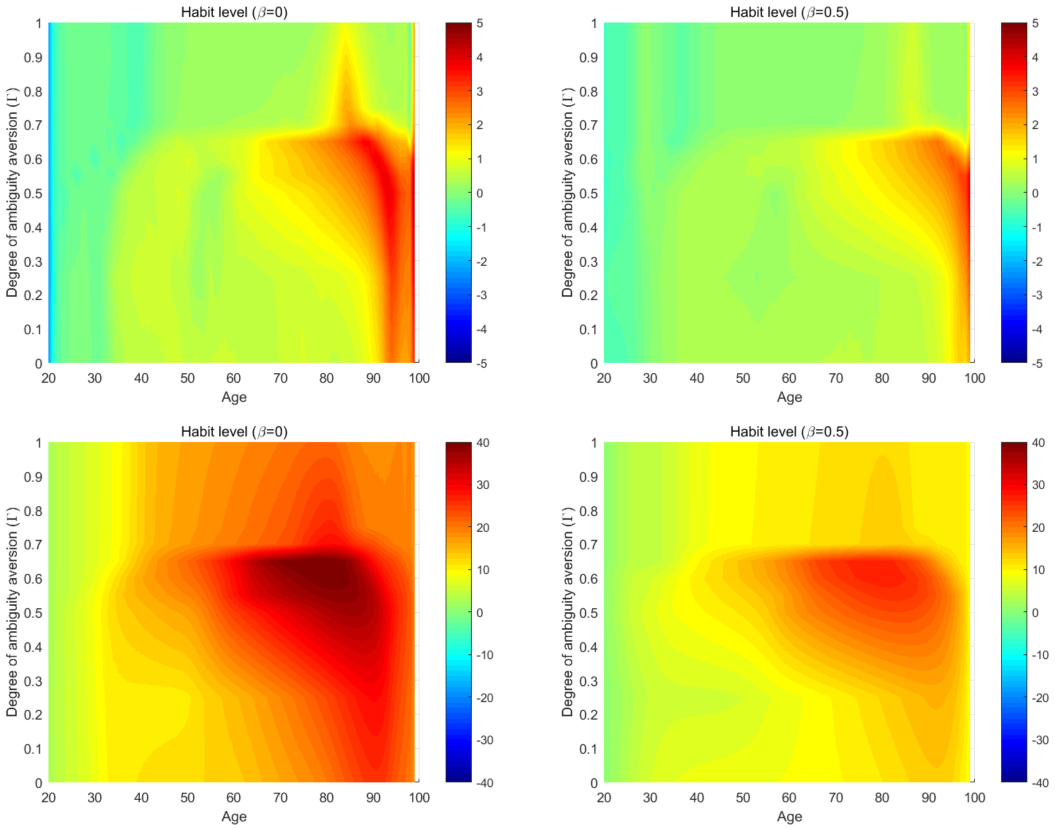

Figure 2.8: Impact of bequest motive on investor's life cycle decisions: consumption (top panel) and wealth (bottom panel).

We run all three models using baseline calibration with and without habit formations (i.e., $\beta = 0$ and 0.5, respectively) and also vary ambiguity aversion parameter between 0 and 1. The empirical experiments show that the main differences between these models are realised in the consumption and wealth decisions. The asset allocation, however, is largely determined by the degree of ambiguity aversion. Therefore, we only present the comparison results of consumption and wealth decisions obtained by these models.

Figure 2.8 shows impact of having bequest motive on life-cycle decisions in terms of the differences between average amount of consumption (above) and wealth (below) obtained by $M_{(BEQ,\, NINS)}$ and $M_{(NBEQ,\, NINS)}$. In four plots, x and y axes represent the investor's age and degree of ambiguity aversion, respectively. A positive number (shown by warm colour) indicates that the model $M_{(BEQ,\, NINS)}$ provides higher consumption (or wealth) values than those produced by the model $M_{(NBEQ,\, NINS)}$. From these results, we observe that the investor has a stronger saving motive if she has bequest motive since he/she gives up more consumption in early ages (as displayed by blue parts in the two plot at the top panel). Such a saving motive increases the wealth during the remaining life and also, increases the consumption in later ages (as highlighted by red parts).

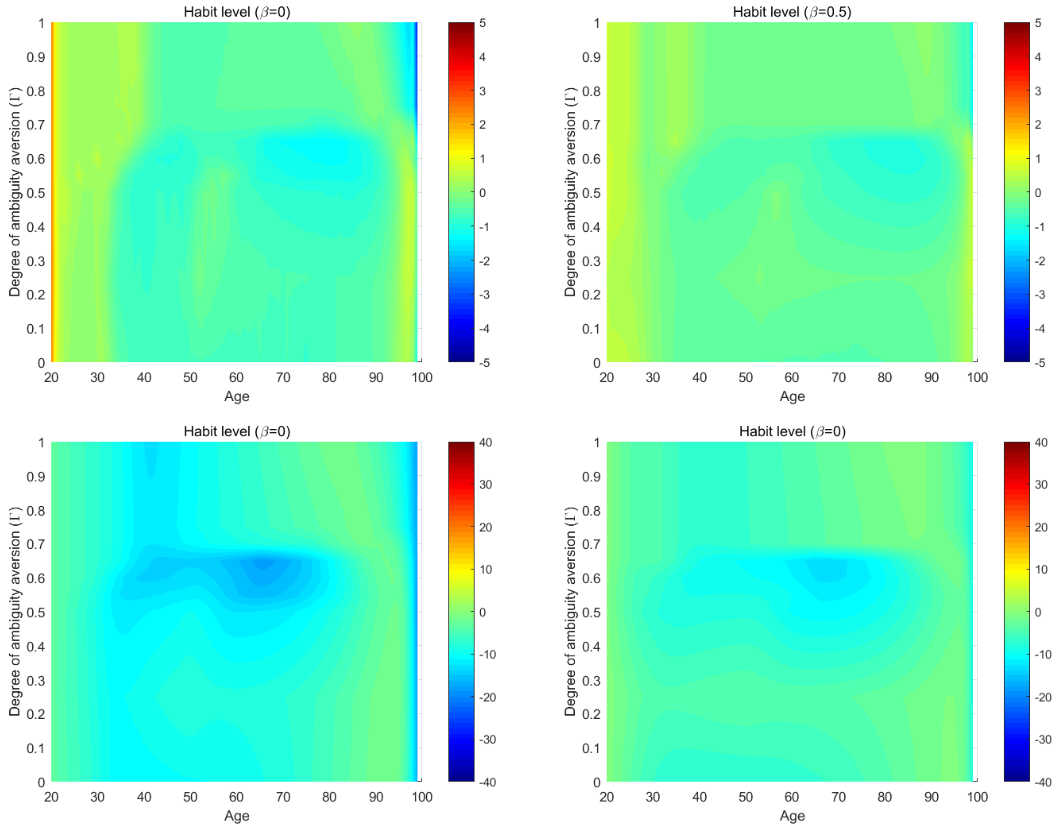

Figure 2.9: Impact of term life insurance investment on life cycle decisions: consumption(top panel) and wealth (bottom panel).

Finally, we would like to illustrate how the choice of life insurance investment impacts on the consumption and wealth decisions of the investor over life span. Figure 2.9 presents results in terms of the differences between the average consumption (top panel) and wealth (bottom panel) obtained by models $M_{(BEQ,\, INS)}$ and $M_{(BEQ,\, NINS)}$ at various age of the investor and degree of ambiguity aversion (at x and y axes, respectively). Note that a positive number (represented by warm colour) describes amount of average consumption or wealth obtained by $M_{(BEQ,\, INS)}$ to be higher than those provided by $M_{(BEQ,NINS)}$.

If the investor buys term life insurance, then he/she consumes slightly more at early ages as shown by red and yellow parts in two plots at the top panel. In this case, the investor uses the term life insurance to ease the concern of leaving bequest to the heirs, but not directly saves more wealth (by consuming less) if she cannot buy term life insurance. Such a behaviour results in a reverse pattern compared to the Figure 2.8. The investor with the choice to buy term life insurance consumes less and accumulates less wealth in middle and later ages.

2.7 Conclusions

In this chapter, we study a life-cycle consumption and asset allocation problem with term life insurance that incorporates realistic features such as ambiguity aversion, habit formation and bequest motive. We model stock return ambiguity using uncertainty sets and formulate the problem in a robust optimization framework.

We obtain several important findings that contribute to the literature. Investor's asset allocation largely depends on the degree of ambiguity aversion. While earlier literature shows that an ambiguity-averse investor allocates less wealth in stock, we add to this evidence that this relation is monotonically decreasing in the degree of ambiguity aversion. This is precisely important, because we only observe an effect of habit formation on asset allocation when the degree of ambiguity aversion is high.

There are several factors that affect the term life insurance demand within a life-cycle model. Similar to the bequest motive, ambiguity aversion also increases the demand for term life insurance. If one takes into account other realistic features such as subjective survival beliefs and borrowing opportunities, one would expect that investors should buy term life insurance more than they actually do. Our model potentially explains why this is the case. We show that habit formation leads to less term life insurance demand and plays a first-order role compared to other factors.

There are several possible extensions of this chapter. One can consider other types of insurance products such as annuities, whole life insurance and long-term care insurance within the life-cycle model. Given the important role of habit formation on term life insurance, it may provide an explanation for other known puzzles in the literature such as annuity puzzle. One can also consider robustifying other uncertain parameters such as labour income uncertainty and its implications for life-cycle choices.

Chapter 3

Life-cycle Portfolio Choice with Labour Income Ambiguity and Stock Market Predictability

3.1 Introduction

The life-cycle consumption and portfolio choice problems have drawn substantial research interest since the pioneering works by Samuelson [1969], Merton [1969] and Merton [1971]. Merton [1971] builds a life-cycle model considering insurable labour income in a complete market setting (in other words, no labour income risk). His striking result is that the investor should have a constant stock allocation rule over the life cycle in contrast with observed data and the general industry advice of a time-varying stock allocation.

Since then, alternative economic mechanisms have been proposed and various ingredients have been added to the life-cycle models to match the empirical evidence (that is, time varing stock allocation pattern over the life cycle). However, there still exist at least two main disagreements between predictions of theoretical models and empirical observations.

First, it is well known that many U.S. households hold limited amount of stocks, which is irreconcilable with theoretical predictions [e.g., Mankiw and Zeldes, 1991; Dimmock et al., 2016]. Moreover, they tend to have a hump-shaped stock allocation pattern over the life time as shown in data (e.g., Survey of Consumer Finances). That is, the investor participates more in the stock market when middle aged and less so when she is young or retired. The low level of stock allocation and hump-shaped life-cycle pattern jointly form the so-called stock holding puzzle [e.g., Viceira, 2001; Cocco et al., 2005; Gomes and Michaelides, 2005].

Second, empirical evidence also indicates that U.S. Households hold more than one-third of the total wealth after retirement [Wolff, 2016] and they decumulate their wealth much slower than the predictions obtained from life-cycle models, even in presence of a strong bequest motive [e.g., De Nardi et al., 2010, 2016]. There should be some other omitted factors that have not been incorporated in previous life-cycle models that explain such strong savings motive. We refer to this as 'retirement saving puzzle', which is becoming more important given the ageing population in a global context.

In this paper, we formulate a life-cycle consumption and portfolio choice model where we assume that the investor can be ambiguity-averse towards both stock return and/or labour income uncertainty. In the presence of ambiguity aversion, we also investigate how the correlation between stock return and labour income shocks affects investor's stock allocation, consumption and wealth accumulation decisions. Moreover, we investigate whether the assumption of market return predictability alters our conclusions. While most of the life-cycle models assume no predictability in market returns (or equity premium) [e.g., Merton, 1971; Cocco et al., 2005; Peijnenburg, 2016], with the exception of few papers [e.g., Campbell and Viceira, 1999; Michaelides and Zhang, 2017], there has been no concensus in the empirical literature on predictability evidence [e.g., Welch and Goyal, 2007; Cochrane, 2011].

Our contribution is twofold. First, we develop a life-cycle model that incorporates both stock return and labour income ambiguity using the max-min expected utility framework [e.g., Gilboa and Schmeidler, 1989; Epstein and Wang, 1994; Epstein and Schneider, 2003], a.k.a. robust optimization in operational research [e.g., El Ghaoui and Lebret, 1997; Ben-Tal and Nemirovski, 1998; Bertsimas and Sim, 2004]. To the best of our knowledge, we are the first paper to investigate the ambiguity aversion towards labour income on investor's life-cycle decisions. Our results show that, when the investor is ambiguity-averse to the labour income uncertainty, she accumulates more wealth during the life time, especially after retirement. We do not need to include other features to explain the retirement saving puzzle such as medical spending and longevity risk [e.g., De Nardi et al., 2010, 2016]. Meanwhile, if the investor is also ambiguity-averse towards the stock return, our results indicate that she has limited stock market allocation in line with the empirical data, and the stock allocation pattern over the life-cycle is hump-shaped. Therefore, we show that the ambiguity aversion towards these uncertainties during the life time is one of the key factors that help reconcile the seemly contradictive results offered by earlier theoretical models and empirical observations in terms of savings and portfolio choices. Moreover, we also find that if there is a positive correlation between stock return and labour income, the level of stock allocation is even lower, especially when the investor is ambiguity-averse towards stock return uncertainty.

Second, we investigate the effect of stock market predictability on the life-cycle decisions, in the presence of ambiguity aversion. We compare the investor's consumption, saving and stock allocation strategies assuming the stock return is either i.i.d. (lack of predictability) or follows a mean reversion process (predictable stock returns). We have two main observations. First, stock market predictability makes the investor's consumption and saving pattern smoother over the life-cycle. We calculate the coefficient of variation (CV) of investor's consumption and saving at all ages. In almost all (35 out of 36) cases (depending on the degree of ambiguity aversion and regardless of the correlation between stock returns and labour income), CV with i.i.d. stock return is higher than that with predictable stock return. Second, we find that overall, whether stock market is predictable or not does not affect the average stock allocation over the life cycle. In other words, stock allocation results are mostly robust to the assumption on the underlying stock return process. This is different from the findings in Michaelides and Zhang [2017], where they find that stock market predictability substantially reduces stock allocation when the investor is young.

Our research is related to three strands of literature. First, to explain the investor's stock allocation pattern over the life cycle, many ingredients have been added to the basic life-cycle model by Merton [1971]. Viceira [2001], Cocco et al. [2005] and Gomes and Michaelides [2005] among others, show that if there is labour income risk, the investor's stock allocation decreases as the age grows. And high correlation between stock return and labour income or disastrous labour income shocks reduce the stock allocation when the investor is young. Benzoni et al. [2007] allow correlation between stock dividend (other than the return) and labour income, and find that such correlation helps to form a hump-shaped life-cycle stock allocation pattern, especially in the young ages. Fagereng et al. [2017] incorporate disastrous stock return shocks and show that along with fixed stock market entry costs (which is also included in Gomes and Michaelides [2005]), they can match the Norwegian data in terms of stock participation and allocation rates. We, along with Campanale [2011], Peijnenburg [2016] and Dimmock et al. [2016] study the ambiguity aversion towards the stock return, which reduces the level of stock allocation over the life cycle.

Second strand of literature is related to the 'retirement saving puzzle'. High medical expenses may be one of the reasons that retired investors decumulate wealth too slowly. De Nardi et al. [2010] find that medical expenses increase quickly as the investor's age grows and is also positively correlated with the income, and these expenses drive the investor to save more. In contrast, some other papers such as Hubbard et al. [1994] and Palumbo [1999] find medical expenses only have minor effects on retirement saving. Longevity risk can be another explanation for strong saving motive after retirement because the investor face the risk that she will live too long but does not have enough income [e.g., Spillman and Lubitz, 2000; Cocco and Gomes, 2012]. But the importance of medical expense and longevity risk can be questionable because if this is the case, investor should buy more annuity or long-term care insurance products than actually observed in data (annuity puzzle, see for example, Lockwood [2012] and Pestieau and Ponthière [2012]). We provide a new perspective, that is, the ambiguity aversion towards labour (retirement) income motivates the investor to save more, especially during retirement.

Third, many papers investigate the effect of stock market predictability on portfolio choice and/or consumption decisions in the absence of labour income risk [e.g., Kim and Omberg, 1996; Brennan et al., 1997; Campbell and Viceira, 1999; Barberis, 2000; Wachter, 2002; Campbell et al., 2013]. Similarly to ours, Michaelides and Zhang [2017] consider both labour income risk and stock market predictability and compare the results with the i.i.d. case. But they do not consider ambiguity aversion towards either labour income or stock return as we do.

The rest of the paper is organized as follows. In Section 2, we introduce the life-cycle consumption and portfolio choice model, the mean reversion and labour income processes. Section 3 explain the robust optimization formulation with ambiguity aversions towards uncertain stock return and labour income. Section 4 focuses on model calibration and parameter settings. We present numerical experiments in Section 5. Our findings and concluding remarks are summarized in Section 6.

3.2 A Multi-stage Stochastic Life-cycle Portfolio Choice Model

In this section we introduce a basic multi-stage stochastic programming formulation of the life-cycle problem for an ambiguity-averse investor who is mainly concerned with uncertainties arising due to labour income and asset returns.

We consider a discrete-time life-cycle model involves T time periods of an investment horizon. Life-cycle decisions are made at each time period (i.e., one year or one month) $t = 1, \cdots, T$ and $t = 0$ represents today. We assume that, given the current age of the investor, s/he is alive for maximum T periods and retires at certain age $K < T$. At each time period t, the investor receives the labour income $t \leq K$ and the retirement income for $t > K$. The life-cycle problem aims to find the optimal allocation of asset allocation and amount of consumption r so that he life-cycle CRRA utilities over consumption gained over the life-cycle is maximized.

We assume that the investor is allowed to allocate the financial wealth in two accounts at each time period t: a stock account h_t^s with uncertain rate of returns, $\tilde{r}_t$, and a cash account h_t^c with fixed risk-free rate of return, r^f. The investor receives labour income (or the retirement income) $\tilde{l}_t$ and make the consumption c_t and stock allocation x_t decisions at each time period t. We define x_t as the capital putting in or taking out from the stock account, which can be either positive or negative, representing the buying or selling transaction of the stock.[1] Then the capital in the stock and cash accounts between time periods are dynamically evolved as follows;

$$h_t^s = \tilde{r}_t h_{t-1}^s + x_t, \ \forall t,$$

$$h_t^c = r^f h_{t-1}^c - x_t + \tilde{l}_t - c_t, \ \forall t.$$

We assume that the investor cannot hold negative positions in the stock and cash accounts to avoid borrowing and short-sales in the model. Therefore, we impose non-negativity constraints for both cash and stock holdings constraints;

$$h_t^c, h_t^s \geq 0, \ \forall t.$$

The investor aims to maximize the life-cycle CRRA utilities over consumption c_t, given the subjective discount factor δ and survival rate p_t at each time period;

$$\sum_{t=1}^{T} \delta^t \left(\prod_{i=1}^{t} p_i \right) \left(\frac{c_t^{1-\gamma}}{1-\gamma} \right),$$

where parameter γ represents the degree of risk aversion. Overall, the multi-stage

[1] Although we ignore transaction cost in the current investment model, it can be easily integrated.

stochastic life-cycle optimization model (P^{MP}) can be formulated as follows;

$$
\begin{aligned}
P^{MP}: \quad \max_{\boldsymbol{c},\boldsymbol{x}} \quad & E_t\Big[\sum_{t=1}^{T} \delta^t \left(\prod_{i=1}^{t} p_i\right) \left(\frac{c_t^{1-\gamma}}{1-\gamma}\right)\Big] \\
s.t. \quad & h_t^s = \tilde{r}_t h_{t-1}^s + x_t, \ \forall t \\
& h_t^c = r^f h_{t-1}^c - x_t + \tilde{l}_t - c_t, \ \forall t \\
& c_t, \ h_t^c, \ h_t^s \geq 0, \ \forall t.
\end{aligned}
$$

Modelling Uncertain Asset Returns and labour Income: Assume that the risk-free rate r^f at time t is known. Following Campbell and Viceira [1999]; Michaelides and Zhang [2017], we consider a mean reversion process to model uncertain stock returns during the life-cycle as follows;

$$
\begin{aligned}
\tilde{r}_t &= r^f + \tilde{y}_t + \tilde{z}_t, \\
\tilde{y}_t &= \mu_y + \phi(y_{t-1} - \mu_y) + \tilde{\epsilon}_t,
\end{aligned}
$$

where $\tilde{y}_t$ is the prediction of future equity premium and follows an AR(1) process. μ_y is the expected mean of the equity premium and ϕ determines the strength of mean reversion (in other words, the speed that the stock return moves to the mean μ_y). In addition, $\tilde{z}_t \sim \mathcal{N}(0, \sigma_z^2)$ and $\tilde{\epsilon}_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ represent the innovations and the correlation $\rho_{z,\epsilon}$ between these two innovations is estimated. Given this generic model, the value of ϕ determines the type of model.

- If $\phi = 0$ and $\sigma_\epsilon = 0$, the prediction of future equity premium $\tilde{y}_t$ becomes deterministic and equals to μ_y. The mean reversion process reduces to an i.i.d. stock return model with mean $\mu_y + r^f$ and standard deviation σ_z.

- For $0 < \phi < 1$, $\tilde{y}_t$ depends on the mean of equity premium and the difference between $\tilde{y}_{t-1}$ and mean of equity premium. Therefore, in the long run, the prediction of future equity premium $\tilde{y}_t$ tends to become mean μ_y.

- On the other hand, if $\phi = 1$, then $\tilde{y}_t$ only depends on the previous observation $\tilde{y}_{t-1}$ and is not mean-reversed.

For the labour income uncertainty, we use standard specification in the literature to model the labour income process. Following Cocco et al. [2005]; Gomes and Michaelides [2005], the logarithm of the labour income $\tilde{l}_t$ at time t has three components: a deterministic function f_t as well as two random variables with respect to permanent and temporary shocks of the labour income, $\tilde{u}_t$ and $\tilde{\varepsilon}_t$, respectively. It is formulated as:

$$\log(\tilde{l}_t) = f_t + \tilde{\nu}_t + \tilde{\varepsilon}_t,\ \forall t \leq K,$$

where $\tilde{\nu}_t = \tilde{\nu}_{t-1} + \tilde{u}_t$ and $\tilde{u}_t \sim \mathcal{N}(0, \sigma_u^2)$ represents to permanent shock to the labour income, whereas $\tilde{\varepsilon}_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ refers the temporary shock. The correlation between permanent shock to the labour income and stock return $\rho_{z,u}$ is determined as in some previous papers [e.g., Barberis, 2000; Cocco et al., 2005; Benzoni et al., 2007; Michaelides and Zhang, 2017]. In addition, the deterministic function f_t depends on the investor's age. According to Cocco et al. [2005], f_t is specified by a third-order polynomial function of the age of the investor as:

$$f_t = a_0 + a_1 t + a_2 t^2/10 + a_3 t^3/100.$$

Finally, we formulate the retirement income for $t > K$ as a constant fraction (ξ) of the permanent labour income in the year just before retirement as:

$$\log(\tilde{l}_t) = \log(\xi) + f_t + \tilde{\nu}_K,\ \forall t > K.$$

In other words, there is no temporary shock $\tilde{\varepsilon}$ after the retirement.

3.3 Robust Optimization Approach to the Life-cycle Portfolio Choice Problem

We assume the investor is ambiguity-averse towards the uncertain stock return and labour income. Instead of solving P^{MP}, the investor takes the worst-case perspective assuming the uncertain stock return $\tilde{r}_t$ and labour incomes $\tilde{l}_t$ belonging to uncertainty sets S_t^r and S_t^l, respectively. We then need to solve the following robust optimization model (or the max-min framework) to maximize the life-cycle utility while minimizing the uncertainties over sets S_t^r and S_t^l as follows;

$$
\begin{aligned}
P^{RO}: \quad \max_{\boldsymbol{c},\boldsymbol{x}} \quad & \sum_{t=1}^{T} \delta^t (\prod_{i=1}^{t} p_i)(\frac{c_t^{1-\gamma}}{1-\gamma}) \\
s.t. \quad & h_t^s \leq \min_{\tilde{r}_t \in S_t^r} \tilde{r}_t h_{t-1}^s + x_t, \ \forall t \\
& h_t^c \leq r^f h_{t-1}^c - x_t + \min_{\tilde{l}_t \in S_t^l} \tilde{l}_t - c_t, \ \forall t \\
& c_t, \ h_t^c, \ h_t^s \geq 0, \ \forall t.
\end{aligned}
$$

Since the uncertain stock return and labour income appear in the constraints of stock and cash accounts, respectively, we can consider two independent interval uncertainty sets for both asset return and labour income uncertainties as follows;

$$
S_t^r = [\mu_t^r - \Gamma_r \sigma_t^r, \ \mu_t^r + \Gamma_r \sigma_t^r] \text{ and } S_t^l = [\mu_t^l - \Gamma_l \sigma_t^l, \ \mu_t^l + \Gamma_l \sigma_t^l],
$$

where the mean (μ_t^r and μ_t^l) and standard deviation (σ_t^r and σ_t^l) of stock return and labour income need to be estimated. Since labour income is a log function of two independent shocks, building uncertainty sets around the shocks will increase the problem complexity. As done in Chapter 2, we directly build uncertainty sets around labour income $\tilde{l}_t$ as a simplification and estimate the corresponding μ_t^l and σ_t^l by using simulation. In addition, Γ_r, $\Gamma_l \geq 0$ represent the investor's degree of ambiguity aversion towards stock return and labour income uncertainties. Note that if $\Gamma_r = \Gamma_l = 0$, then the investor is ambiguity-neutral towards both uncertainties and the uncertainty sets become a single value (that is the mean). On the other hand, if $\Gamma_r, \Gamma_l > 0$, then the investor is more ambiguity-averse towards the uncertainty. In this case, the investment decisions become more conservative.

The optimal solutions of the inner minimization problems in the constraints of P^{RO} can be easily found as the $\mu_t^r - \Gamma_r \sigma_t^r$ and $\mu_t^l - \Gamma_l \sigma_t^l$ respectively. In other words, if the investor's degrees of ambiguity aversion against asset return and labour income uncertainties are specified as Γ_r and $\Gamma_l > 0$, then s/he assumes that the stock return and labour income take the worst-case values to obtain an optimal consumption and stock allocation decisions in the life-cycle problem. Therefore, P^{RO} is equivalent to the following multi-stage optimization problem;

$$
\begin{aligned}
P^{RO-IN}: \quad \max_{\boldsymbol{c},\boldsymbol{x}} \quad & \sum_{t=1}^{T} \delta^t (\prod_{i=1}^{t} p_i)(\frac{c_t^{1-\gamma}}{1-\gamma}) \\
s.t. \quad & h_t^s \leq (\mu_t^r - \Gamma_r \sigma_t^r) h_{t-1}^s + x_t, \ \forall t \\
& h_t^c \leq r^f h_{t-1}^c - x_t + \mu_t^l - \Gamma_l \sigma_t^l - c_t, \ \forall t \\
& c_t, \ h_t^c, \ h_t^s \geq 0, \ \forall t.
\end{aligned}
$$

Note that this optimization model displays the same characteristics in terms of objective function and constraints as in Section 2.4.1. We use Mosek to solve the underlying optimization model.

3.4 Design of Computational Experiments

In the numerical results, we aim to investigate the following several questions.

- How do the stock return ambiguity and/or labour income ambiguity affect the investor's life-cycle decisions?
- How will the life-cycle decisions change, if there is correlation between stock return and labour income, in the presence of ambiguity aversion.
- What is the implication of stock market predictability on the life-cycle decisions?

In order to answer these three questions, we conduct computational experiments in the following ways.

1. We assume there is stock market predictability (i.e., using a mean reversion stock return process) as the benchmark case. We let the degree of ambiguity aversion towards labour income to 0 ($\Gamma_l = 0$) to show the effect of stock return ambiguity on the consumption, saving and stock allocation decisions.

2. We do the reverse and set the degree of ambiguity aversion towards stock return to 0 ($\Gamma_r = 0$) and show the effect of labour income ambiguity ($\Gamma_l > 0$).

3. We assume the investor is ambiguity-averse to both stock return and labour income ambiguities (i.e., $\Gamma_r > 0, \Gamma_l > 0$) and illustrate the results.

4. In the case of both ambiguities, we compare the results of consumption, saving and stock allocation decisions assuming there is or is not correlation between stock return and labour income and the stock market predictability.

3.4.1 Model Calibration

To calibrate the mean reversion process of the stock return over the entire life, we choose the same parameter values as in Michaelides and Zhang [2017]. The risk-free return r^f is 2%. The expected mean of the equity premium μ_y is 4%. the strength of mean reversion $\phi = 0.91$, the volatilities of two innovations z_t and ϵ_t are 0.18 and $\sqrt{0.000034}$ respectively. The correlation is two innovation $\rho_{z,\epsilon}$ equals to -0.8. The baseline calibration of the correlation between permanent shock of labour income and stock return $\rho_{z,u}$ is 0.15 according to Michaelides and Zhang [2017] but we also test the case of no correlation $\rho_{z,u} = 0$.

When choosing the parameters related to the labour income, we follow Cocco et al. [2005], which consider three different labour income processes classified by investor's different education groups. We use the same estimates of parameter values assuming the investor has high school degree, which gives the medium labour incomes compared to other two age groups: no high school degree and college degree. We find that the choice of different education groups does not affect the main findings.

We use a degree of risk aversion of 5 and discount factor of 0.96, which are standard choices in the literature [e.g., Cocco et al., 2005; Gomes and Michaelides, 2005]. We take the 2013 mortality rates from US Social Security[2] to calculate the conditional probability of being alive (p_t) in each time period.

For the degrees of ambiguity aversion towards stock return and labour income, in the numerical experiments, we consider three cases: the investor is ambiguity-averse towards either the stock return or labour income (($\Gamma_r > 0, \Gamma_l = 0$) or ($\Gamma_r = 0, \Gamma_l > 0$)) and also, the investor is ambiguity-averse towards both ambiguities ($\Gamma_r > 0, \Gamma_l > 0$). In each case, we vary Γ_r and/or Γ_l from 0 and 0.4, which is enough to show the effect of ambiguity aversion towards stock return and/or labour income on the life-cycle consumption, saving and stock allocation decisions.

The results related with amount of consumption, holding in assets and total wealth are presented in terms of thousands of 1992 US dollars.

3.4.2 Model Implementation

We apply a rolling-horizon procedure to find optimal life-cycle decisions by repeatedly solving the robust optimization model P^{RO-IN} in T numbers of iterations, denoted by $k = 1, \dots, T$. The main steps of this procedure in terms of states updated and actions taken at each time period are summarised in Table 3.1.

[2] The reader is referred to `https://www.ssa.gov/oact/STATS/table4c6.html`.

At the beginning, the uncertain stock returns ($\tilde{r}_t$) and labour incomes ($\tilde{l}_t$) are unknown to the investor. The investor only knows about the mean and standard deviations as in the calibration (μ_t^r, μ_t^l, σ_t^r and σ_t^l). Based on this information the investor builds uncertainty sets with degrees of ambiguity aversion (Γ_r and Γ_l) and formulate P^{RO-IN} explicitly.

At iteration $k = 1$, the model with all T time periods is solved but only the first-stage decisions at $t = 1$ in terms of consumption c_1 and stock allocation x_1 are implemented.

Then the investor observes the actual stock return and the labour income for $t = 1$, which is simulated according to the calibrated mean/variance information of the stock returns and labour income process, respectively. Once the actual asset returns and labour income are observed, the value in stock and cash accounts ($\boldsymbol{h}_c$, $\boldsymbol{h}_s$) are updated.

By going forward in time, the number of time periods in the optimization model is reduced by one at each iteration ($1 < k \leq T$). Then the optimization model is solved again and only the first stage decisions are implemented. The stock and cash account values (i.e. h_k^c and h_k^s) are updated according to the new observed asset returns and labour incomes. At the last iteration $k = T$, we solve the problem with only one time period.

Table 3.1: Dynamic rolling-horizon procedure

Iteration number	# of time periods in the model	Decisions implemented	Random variables realized	State variables updated
$k = 1$	T	c_1, x_1	$\tilde{r}_1$, $\tilde{l}_1$	h_1^c, h_1^s
$k = 2$	$T - 1$	c_2, x_2	$\tilde{r}_2$, $\tilde{l}_2$	h_2^c, h_2^s
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k = T - 1$	2	c_{T-1}, x_{T-1}	$\tilde{r}_{T-1}$, $\tilde{l}_{T-1}$	h_{T-1}^c, h_{T-1}^s
$k = T$	1	c_T, x_T	$\tilde{r}_T$, $\tilde{l}_T$	h_T^c, h_T^s

We run the rolling-horizon procedure 1000 times with different stock returns and labour income processes simulated from the probability distributions introduced in Section 3.4.1. We then report the average and standard deviation (in parentheses) of 1000 simulations as the optimal life-cycle decisions (i.e., consumption and stock allocation) during the life.

3.5 Numerical Results

3.5.1 Impact of Ambiguity Aversions

As explained in Section 3.4, we first show the results assuming the investor has different degrees of ambiguity aversion towards stock return and/or labour income.

We present the results in the following ways. We define several age groups, each of those represent certain periods during the life. They are: '20-34', '35-44', '45-54','55-64', '65-74', '75-100'. Such classification of age is commonly used in previous literature [e.g., Gomes and Michaelides, 2005; Peijnenburg, 2016] and empirical datasets (e.g., Survey of Consumer Finances). To show the average performance over the entire life, we also report results in age group '20-99'.

In each age group, we report the results in terms of average values of the optimal consumption, wealth and stock allocation. For example, the average consumption value for age group '35-44' is computed as follows. First, we run the rolling horizon procedure 1000 times to obtain 1000 sets of c_t for $t = 1, \dots, T$. Then, the average value $\hat{c}_t$ at time period t is computed over 1000 points of c_t. Finally, we calculate the average consumption values as $\frac{\hat{c}_{16}+\hat{c}_{17}+,\dots,+\hat{c}_{25}}{10}$.

Stock Return Ambiguity Only: Table 3.2 summarizes the investor's consumption, wealth and asset allocation decisions with different degrees of ambiguity aversion towards stock return.

From Panel C in Table 3.2, it is clear that the stock allocation is negatively correlated with the degree of ambiguity aversion in almost all age groups, which means the ambiguity aversion has effect on the stock allocation during the entire life of the investor. Moreover, we obtain a hump-shaped life-cycle stock allocation pattern for all degrees of Γ_r, while only for $\Gamma_r = 0.2$, the highest stock allocation appears at the middle age group '55-64'.

From Panel A and B, we notice that in general, if the investor is ambiguity aversion to the stock return, the average consumption and wealth over the life cycle are lower. When Γ_r increases from 0 (ambiguity-neutral) to 0.2, the investor consumes more in age groups '35-44' and '45-54' and accumulates less wealth, which means her saving motive is weaken. In comparison, if Γ_r increases to 0.4, the investor accumulates more wealth in age group '55-64' (saving motive is stronger) by sacrificing some amount of consumption. As a result, case, the investor consumes more during the retirement period compared to the case when $\Gamma_r = 0.2$. Such effect of ambiguity aversion towards stock return on the saving motive is also documented in Chapter 2.

Table 3.2: Life-cycle decisions under stock return ambiguity

Age group	**20-100**	20-34	35-44	45-54	55-64	65-74	75-100
Γ_r	*Panel A. Consumption*						
0	**27.03**	21.62	29.39	31.29	32.63	30.32	24.07
0.2	**26.16**	21.41	29.97	32.57	32.19	25.87	22.63
0.4	**25.88**	21.52	29.65	30.53	30.58	26.11	23.15
Γ_r	*Panel B. Wealth*						
0	**48.79**	10.29	33.91	59.05	97.72	91.83	36.95
0.2	**36.75**	11.80	33.76	48.33	62.44	45.52	34.51
0.4	**41.11**	8.96	28.94	46.98	80.57	76.99	32.79
Γ_r	*Panel C. Stock allocation (%)*						
0	**69.76**	38.50	50.61	57.68	74.67	76.28	59.85
0.2	**49.96**	39.51	52.34	52.00	56.37	48.41	29.12
0.4	**34.95**	19.94	39.82	39.50	33.54	27.05	25.40

This table displays the average consumption, wealth and stock allocation (in Panels A, B, and C, respectively) for different degrees of ambiguity aversion towards stock return (Γ_r). The average values are calculated as the 50 percentile (median) of 1000 simulations using the rolling-horizon procedure.

Labour Income Ambiguity Only: We now investigate the effect of ambiguity aversion towards labour income on the life-cycle decisions ($\Gamma_l \geq 0$) and assume the investor is ambiguity-neutral to the stock return uncertainty ($\Gamma_r = 0$). Table 3.3 presents our results.

As shown in Panel B of Table 3.3, the investor's saving motive increases at every age group with the increase of Γ_l. It is worth mentioning that, in our model, we assume the degree of ambiguity aversion is constant over the life cycle (e.g., $\Gamma_l = 0.2$ for $t = 1, \ldots, T$). But actually, the investor may have higher degrees of ambiguity aversion especially when the labour income level is high in middle ages or she is only to get retirement income during the retirement. That means the degree of ambiguity aversion may increases as the age grows. So , the ambiguity aversion towards labour income can be one penitential reason to explain the 'retirement saving puzzle'.

Table 3.3: Life-cycle decisions under labour income ambiguity

Age group	**20-100**	20-34	35-44	45-54	55-64	65-74	75-100
Γ_l	*Panel A. Consumption*						
0	**27.03**	21.62	29.39	31.29	32.63	30.32	24.07
0.2	**27.81**	21.18	29.11	31.88	33.74	32.15	25.54
0.4	**28.77**	20.50	28.57	33.08	35.52	34.45	27.11
Γ_l	*Panel B. Wealth*						
0	**48.79**	10.29	33.91	59.05	97.72	91.83	36.95
0.2	**65.14**	14.60	50.22	84.88	128.36	116.45	47.71
0.4	**86.39**	20.77	73.20	121.16	167.18	146.77	60.66
Γ_l	*Panel C. Stock allocation (%)*						
0	**69.76**	38.50	50.61	57.68	74.67	76.28	59.85
0.2	**72.18**	47.92	62.52	67.25	76.82	76.69	63.14
0.4	**73.52**	56.86	72.30	73.31	76.96	74.64	70.16

This table displays the average consumption, wealth and stock allocation (in Panels A, B, and C, respectively) for different degrees of ambiguity aversion towards labour income (Γ_l). The average values are calculated as the 50 percentile (median) of 1000 simulations using the rolling-horizon procedure.

Because of the increase in the saving over the life-cycle, the investor's consumption pattern in young ages or later is different. If the investor is aged below 44, she consumes less if Γ_l is higher because she is accumulating more wealth while for investors aged above 45, the consumption level is higher because of the high level of wealth. In other words, if the investor is ambiguity aversion to the labour income, she shifts part of the consumption to retirement.

In terms of the stock allocation, we find if the investor is ambiguity-averse towards the labour income, she allocates more in the stock. This is because we assume the investor is ambiguity-neutral towards the stock return and therefore, stocks become safer compared to the uncertain labour income.

Stock Return and Labour Income Ambiguities: We assume that i) the investor is ambiguity-averse to both stock return and labour income uncertainties and, ii) the degree of ambiguity aversion is the same for both uncertainties. The results are summarized in Table 3.4.

Table 3.4: Life-cycle decisions under stock return and labour income ambiguities

Age group	**20-100**	20-34	35-44	45-54	55-64	65-74	75-100
Γ_r and Γ_l	*Panel A. Consumption*						
0	**27.03**	21.62	29.39	31.29	32.63	30.32	24.07
0.2	**26.34**	21.08	29.83	32.72	32.34	26.41	23.10
0.4	**26.90**	20.48	27.98	29.68	32.36	29.63	25.93
Γ_r and Γ_l	*Panel B. Wealth*						
0	**48.79**	10.29	33.91	59.05	97.72	91.83	36.95
0.2	**42.46**	14.65	40.18	55.71	71.04	52.49	39.32
0.4	**75.27**	16.47	58.43	99.02	147.14	133.01	55.95
Γ_r and Γ_l	*Panel C. Stock allocation (%)*						
0	**69.76**	38.50	50.61	57.68	74.67	76.28	59.85
0.2	**48.32**	37.05	51.50	53.81	55.57	53.89	26.28
0.4	**26.51**	14.32	29.58	27.44	20.30	28.28	28.36

This table displays the average consumption, wealth and stock allocation (in Panels A, B, and C, respectively) for different degrees of ambiguity aversion towards stock return and labour income (Γ_r and Γ_l). The average values are calculated as the 50 percentile (median) of 1000 simulations using the rolling-horizon procedure.

The results in Panel C shows that stock allocation is mostly influenced by the ambiguity aversion towards stock return but not the labour income because as the degrees of ambiguity aversion towards both uncertainties increase, the stock allocation decreases, which is different from that in Table 3.3. Moreover, by comparing Table 3.2, 3.3 and 3.4, we find if the investor is ambiguity-averse to both uncertainties, the stock allocation is even lower compared to the single ambiguity case.

By analysing the results in Panel A and B with those in Table 3.2 and 3.3, we find that the consumption and wealth decisions are affected by both uncertainties. And the effect of labour income ambiguity may be a bit stronger than that of stock return ambiguity because for $\Gamma_r = \Gamma_l - 0.4$, the wealth pattern is much closer to that in Table 3.3 rather than that in Table 3.2. Overall, the results in Table 3.4 suggest that ambiguity aversion to both stock return and labour income can explain the low level of stock allocation and high saving observed in data.

In the next subsection, we allow the correlation between stock return and permanent shock to the labour income and set $\rho_{z,u} = 0.15$ according to the calibration in Michaelides and Zhang [2017]. We also compare the results assuming the stock return is predictable (mean-reversed) or not (i.i.d.). The results are displayed in Table 3.5.

3.5.2 Degree of Correlation and Stock Market Predictability

The results in Table 3.5 can be divided into four blocks. We first compare the upper two and lower two blocks to investigate the effect of correlation between stock return and labour income. We find that the correlation does not affect the consumption and wealth too much for different degrees of ambiguity aversion. However, when there is correlation, especially for high degree of ambiguity aversion (0.4) and if there is no stock market predictability, the stock allocation is lower.

By comparing the left two and right two blocks, we analyse the effect of stock market predictability. In contrast to the findings in Michaelides and Zhang [2017] where they find stock market predictability substantially reduces the stock allocation, especially when the investor is young, we observe that the stock market predictability does not affect the stock allocation much. Only for high degree of ambiguity aversion (0.4), if the stock return is predictable, the investor allocates more wealth in stocks. To some extent, this is in line with the findings in Barberis [2000], but they claim stock market predictability can substantially (in our case, much weaken) increases the stock allocation in the long run because the mean-reversion property slows the growth of conditional variances of stock return.

Moreover, we find stock market predictability makes the investor's consumption and wealth smoother. We obtain this conclusion by calculate the coefficient of variation (CV) of the consumption and wealth over the life. For instance, the CV for consumption is calculated as the standard deviation of consumption value in 6 age groups divided by the mean consumption over the life (i.e., the value in age group '20-100'). We find that the CV values of consumption and wealth if the stock return is i.i.d. (i.e., no stock market predictability) are all higher than those if the stock return follows a mean reversion process (i.e., stock market predictability) in 35 out of 36 cases with different $\rho_{z,u}$ and Γ values. In Table 3.6, we show the CV values assuming $\Gamma_r = \Gamma_l \geq 0$ as an example.

We notice that for both consumption and wealth, CV values with no stock market predictability are all higher than those with stock market predictability, which implies that consumption and wealth are smoother over the life cycle. Such effect of stock market predictability on the consumption and wealth is hardly documented in the literature. We believe why consumption and wealth become smoother if there is stock market

Table 3.5: Life-cycle decisions (degree of correlation and stock market predictability)

	$\rho_{z,u} = 0$													
	i.i.d.							Mean Reversion						
Age group	20-100	20-34	35-44	45-54	55-64	65-74	75-100	20-100	20-34	35-44	45-54	55-64	65-74	75-100
Γ_r and Γ_l	*Panel A. Consumption*													
0	**27.48**	21.46	30.11	32.37	33.57	31.08	24.20	**27.03**	21.62	29.39	31.29	32.63	30.32	24.07
0.2	**26.95**	21.10	30.59	34.13	34.05	25.71	23.79	**26.34**	21.08	29.83	32.72	32.34	26.41	23.10
0.4	**27.39**	20.42	27.82	30.02	32.36	31.30	26.81	**26.90**	20.48	27.98	29.68	32.36	29.63	25.93
Γ_r and Γ_l	*Panel B. Wealth*													
0	**52.39**	9.60	37.83	66.95	108.36	97.66	37.59	**48.79**	10.29	33.91	59.05	97.72	91.83	36.95
0.2	**45.45**	13.16	42.42	68.27	67.15	49.69	46.55	**42.46**	14.65	40.18	55.71	71.04	52.49	39.32
0.4	**84.66**	15.49	63.99	119.50	167.91	151.73	60.36	**75.27**	16.47	58.43	99.02	147.14	133.01	55.95
Γ_r and Γ_l	*Panel C. Stock Allocation (%)*													
0	**69.75**	38.38	48.74	62.59	75.34	77.04	60.33	**69.76**	38.50	50.61	57.68	74.67	76.28	59.85
0.2	**48.10**	37.48	54.27	53.23	55.78	57.02	27.87	**48.32**	37.05	51.50	53.81	55.57	53.89	26.28
0.4	**24.96**	13.46	22.24	20.61	22.59	25.21	27.80	**26.51**	14.32	29.58	27.44	20.30	28.28	28.36
	$\rho_{z,u} = 0.15$													
Γ_r and Γ_l	*Panel D. Consumption*													
0	**27.03**	21.82	29.61	31.70	31.82	30.16	24.07	**26.85**	22.33	30.63	31.36	31.66	28.70	23.59
0.2	**26.58**	21.41	30.00	33.91	31.13	25.73	23.89	**26.43**	21.89	31.02	32.78	31.54	24.46	23.51
0.4	**26.74**	20.71	27.24	30.43	31.91	29.44	25.52	**26.83**	21.15	28.55	29.44	31.54	28.89	25.78
Γ_r and Γ_l	*Panel E. Wealth*													
0	**50.86**	11.14	37.65	65.64	97.69	95.02	37.66	**47.18**	11.84	39.78	57.95	92.03	83.37	34.61
0.2	**45.13**	15.12	47.31	57.58	60.14	52.67	48.26	**42.66**	16.49	45.36	54.43	57.55	48.47	44.29
0.4	**79.17**	17.49	64.23	115.41	150.55	132.87	57.62	**76.45**	18.48	65.56	107.74	143.36	125.22	56.80
Γ_r and Γ_l	*Panel F. Stock Allocation (%)*													
0	**67.97**	38.23	47.78	55.83	72.74	72.42	55.04	**68.25**	39.78	55.63	63.75	75.39	72.07	55.73
0.2	**47.47**	37.62	48.94	49.80	53.81	55.28	31.23	**45.34**	39.72	49.14	52.09	52.89	52.72	30.02
0.4	**21.21**	8.84	23.30	20.62	18.91	21.79	23.13	**25.29**	10.95	28.08	24.08	25.04	21.11	26.46

This table displays the average consumption, wealth and stock allocation (in Panels A, B, and C, respectively) for different degrees of ambiguity aversion(Γ_r and Γ_l), degrees of correlation between stock return and permanent shock to the labour income and stock market predictability. The average values are calculated as the 50 percentile (median) of 1000 simulations using the rolling-horizon procedure.

predictability is because of the lower variance of stock return compared to the case if stock return is i.i.d.

Table 3.6: Coefficient of variation comparison over the life cycle

		Coefficient of Variation			
		Consumption		Wealth	
	Γ_r and Γ_l	i.i.d.	Mean Reversion	i.i.d.	Mean Reversion
$\rho_{z,u} = 0$	0	**0.18**	0.16	**0.72**	0.70
	0.2	**0.23**	0.20	**0.52**	0.50
	0.4	**0.16**	0.16	**0.68**	0.64
$\rho_{z,u} = 0.15$	0	**0.16**	0.16	**0.67**	0.64
	0.2	**0.21**	0.20	**0.43**	0.40
	0.4	**0.15**	0.14	**0.62**	0.60

This table displays the coefficient of Variation (CV) values for consumption and wealth for different degrees of ambiguity aversion(Γ_r and Γ_l), degrees of correlation between stock return and permanent shock to the labour income assuming there is stock market predictability. The CV for consumption is calculated as the standard deviation of consumption(wealth) value in 6 age groups divided by the mean consumption(wealth) over the life (i.e., the value in age group '20-100').

3.5.3 Comparison with Empirical Data

In this section, we aim to compare the model predictions with those in the empirical. We follow Gomes and Michaelides [2005] and Peijnenburg [2016] that use two criteria: stock allocation and wealth-to-income ratio.

The dataset we refer to is Survey of Consumer Finances, which is a triennial cross-sectional survey of U.S. families on financial assets[3]. For the stock allocation, we collect from the 'Stock holdings as share of group's financial assets' in the Survey of Consumer Finances public data. In terms of the wealth-to-income ratio, we calculated as the median value of 'Family net worth' divided by 'Before-tax family income'. We take the average stock allocation and wealth-to-income ratio values of the recent 6 surveys (from

[3]For further information, the reader is referred to `https://www.federalreserve.gov/econres/aboutscf.htm`.

year 2001 to 2016) as the comparable counterparts, which show the average performance in the 21th centenary in general.

The stock allocation results in our model are just as those showed in Panel C in previous tables. The wealth-to-income ratio at t is calculated as the wealth divided by the labour or retirement income at every t. The average wealth-to-income ratio in each age group is then calculated as the mean wealth-to-income ratios for every t in that age group.

We show the comparison results between empirical data and our model predictions in five cases with different degrees of ambiguity aversion towards stock return and labour income, which is shown in Table 3.7.

Table 3.7: Model predictions versus empirical data

Age group	20-100	20-34	35-44	45-54	55-64	65-74	75-100
Panel A. Survey of Consumer Finances (2001-2016)							
stock allocation	50.70	44.10	53.50	54.24	53.95	50.90	47.54
wealth-to-income ratio	3.38	0.32	1.16	2.19	3.79	5.44	7.40
Panel B. Model: $\Gamma_r = 0.2$, $\Gamma_l = 0.2$							
stock allocation	50.44	37.05	51.50	53.81	55.57	53.89	32.18
wealth-to-income ratio	1.35	0.55	1.04	1.39	1.82	1.97	1.13
Panel C. Model: $\Gamma_r = 0.2$, $\Gamma_l = 0.4$							
stock allocation	49.03	36.69	46.57	51.58	52.41	56.58	40.86
wealth-to-income ratio	1.64	0.71	1.32	1.61	2.25	2.50	1.35
Panel D. Model: $\Gamma_r = 0.2$, $\Gamma_l = 0.6$							
stock allocation	49.79	38.51	45.26	49.70	52.63	55.73	42.51
wealth-to-income ratio	2.16	0.94	1.81	2.05	3.06	3.53	1.73
Panel E. Model: $\Gamma_r = 0.3$, $\Gamma_l = 0.3$							
stock allocation	41.22	30.76	44.31	42.73	40.36	43.11	38.00
wealth-to-income ratio	1.73	1.85	0.57	1.10	1.66	2.75	3.62
Panel F. Model: $\Gamma_r = 0.4$, $\Gamma_l = 0.4$							
stock allocation	26.53	14.32	29.58	27.44	20.30	28.28	28.44
wealth-to-income ratio	2.63	0.61	1.49	2.53	4.07	5.41	2.33

This table shows the comparison results between model predictions with different degrees of ambiguity aversion towards stock return and labour income and data from Survey of Consumer Finances in terms of stock allocation and wealth-to-income ratio.

We find the best match of data in Panel A is the results with $\Gamma_r = 0.2$ and $\Gamma_l = 0.6$ in Panel D. The average stock allocation over the life cycle is almost the same as that in the data and also we obtain a hump-shaped life-cycle pattern. In terms of the wealth-to-income ratio, we obtain close results up to age 54 but still report less wealth-to-income ratio since age 55, mainly after age 75. Recall that we consider a very simple life-cycle consumption and portfolio choice model that only considers ambiguity aversions. One can incorporate other factors such as bequest motive and medical expenses to improve the wealth-to-income ratio during the retirement period. Overall, the results imply that the ambiguity aversion towards stock return can explain the empirical stock allocation data quite well but the ambiguity aversion towards labour income itself is not enough to explain the wealth pattern.

In other panels, we show if we change the degrees of ambiguity aversion, what will happen to the stock allocation and wealth-to-income ratio matching. As shown in Panel F, if both degrees of ambiguity aversion are 0.4, the matching of wealth-to-income ratio improves compared to the results in Panel D but the stock allocation is too low and there is no hump shape. If both degrees of ambiguity aversion reduce to 0.2 (See Panel B), the matching of stock allocation is even better than that in Panel D but the wealth-to-income ratio is too low.

3.6 Conclusions

In this paper, we analyse the effects of ambiguity aversion towards stock return and/or labour income on the life-cycle consumption, wealth and stock allocation decisions. Our main findings are as follows.

The stock allocation is negatively correlated with the ambiguity aversion towards stock return and if the investor is also ambiguity-averse towards the labour income, she allocates even less wealth in stock. The correlation between stock return and permanent shock to the labour income has a similar effect on the stock allocation. In contrast, if the investor is only ambiguity-averse towards the labour income, the stock allocation increases over the life cycle. We also find that the stock market predictability is almost irrelevant to the stock return. Only if the investor is highly ambiguity-averse towards the stock return, she allocates more in the stocks if there is stock return is predictable.

The ambiguity aversion towards labour income substantially increases investor's saving motive. In other words, the investor accumulates more wealth (by sacrificing early consumption). The results still hold if the investor is also ambiguity-averse towards the stock return. We also find that the presence of stock market predictability make the investor's life-cycle consumption and wealth pattern smoother, which is hardly documented in the literature.

By incorporating the ambiguity aversion towards stock return and labour income, we manage to obtain a stock allocation pattern closer to real data in terms of the low level and the hump shape over the life cycle. We can also explain part of the strong (retirement) saving motive in data, although we do not incorporate any other ingredients such as bequest motive and medical expenses, which have been reported in the literature that can increases the investor's saving motive.

Chapter 4

Dynamic Life-cycle Consumption and Housing Problem under Uncertainty

4.1 Introduction

Although real estate is one of the most important asset classes in an investor's portfolio, it has been absent in many life-cycle portfolio selection problems [e.g., Samuelson, 1969; Merton, 1969, 1971; Gourinchas and Parker, 2002; Cocco et al., 2005]. Households face important decisions regarding housing investment such as how to choose between owning and renting a house, and how to decide the proportion of wealth put in the housing asset (e.g., the size of the owned or rented house) during her lifetime. Moreover, conditional on homeownership, one can also decide to let (i.e., downsize the living space) to others to obtain rental income. The latter aspect is the main focus of this chapter.

Some recent papers study the housing decisions in a life-cycle setting. These papers mainly focus on investor's housing choices between owning and renting, and/or how the housing assets act as a part of the investment portfolio [e.g., Cocco, 2004; Yao and Zhang, 2005; Li and Yao, 2007; Kraft and Munk, 2011; Attanasio et al., 2012; Cooper and Zhu, 2016]. But to the best of our knowledge, none of them considers the letting choice along with the homeownership and renting decisions.

In this chapter, we formulate a consumption and housing model that incorporates three types of housing decisions: owning, renting and letting (conditional on house ownership). We show that letting, owning and renting decisions are intertwined in such a way that letting decision also affects investor's choices between owning and renting.

This is because with the letting option, homeownership not only provides the investor the opportunity to do housing investments during her lifetime, but also generates cash flows, which can be used for future housing and non-durable consumption and hence affects housing and liquid wealth accumulation. Therefore, we argue that it is important to include the letting decision when studying investor's housing problem in a life-cycle setting.

Another important feature of our a life-cycle model is that it does not only cover the retirement period (i.e., wealth-decumulation period) as most other papers do [e.g., Yogo, 2016; Xu et al., 2017], but also the wealth-accumulation period, when the investor earns stochastic labour income. Using this life-cycle model, we investigate how an investor chooses the optimal owning, renting and letting decisions as well as non-durable consumption and saving strategies during her lifetime. In particular, by studying the life-cycle pattern of letting, we can explore some related questions such as when is the optimal time to start letting? Or, how do the optimal letting choices affect investor's decisions on living space profile (e.g., downsizing) over the lifetime?

Our model assumes that the investor has recursive preferences, that is, Epstein-Zin-Weil-type utility over consumption and bequest [e.g., Epstein and Zin, 1989, 1991; Weil, 1989]. We then study how investor's different subjective preferences such as risk aversion, elasticity of intertemporal substitution (EIS), bequest motive and the housing weight in the consumption utility[1] affect the life-cycle housing decisions in the presence of letting choice. This helps us explain the primary factors that affect investor's letting decisions, as well as other consumption and housing choices.

There are two main contributions of this chapter. First, we show how letting choices are driven by investor's preferences and hence influence housing decisions. In particular, we find that the homeowner's willingness to let is negatively correlated with the housing weight in the utility of consumption but positively correlated with the bequest motive and EIS. Not surprisingly, the housing weight in the utility of consumption is one of the key factors that drives the homeowner's letting decisions. For a homeowner with relatively low housing weight in the utility of consumption (e.g., housing weight = 0.2 [as in Yao and Zhang, 2005]). In other words, non-durable consumption contributes 80% of the total utility of consumption.), she opts for letting in almost the entire life. However, when the housing weight increases to 0.4 [as in Yogo, 2016], the homeowner is only willing to let during the retirement period. The bequest motive, on the other hand, determines the homeowner's life-cycle letting pattern, especially after retirement. For the

[1]The housing weight in the utility of consumption determines the investor's preference of housing consumption over non-durable consumption.

homeowner with no bequest motive, the overall life-cycle letting pattern is hump-shaped. The letting willingness reaches the peak at around age 70, and decreases since then, no matter how large the housing weight is in the utility of consumption. In contrast, for the homeowner with some bequest motives, she has a similar letting pattern up to the age 70 as in the case without the bequest motive. However, after the age 70, the homeowner keeps letting during the remaining lifetime, with a slightly increased letting willingness as she ages. The EIS plays a similar role as the bequest motive.

Second, our calibrated life-cycle model (using a choice of housing weight = 0.3) performs well in matching the data in terms of life-cycle homeownership rate and investor's living space patterns. We are not the first ones who offer a model that aims at matching the life-cycle homeownership data. However, many previous models [e.g., Li and Yao, 2007; Díaz and Luengo-Prado, 2008; Attanasio et al., 2012; Iacoviello and Pavan, 2013] find it difficult to match the homeownership rate pattern in the late time periods after retirement (e.g, after about age 75). Our model, along with some others [e.g., Yang, 2009; Chambers et al., 2009], manages to match the homeownership rate pattern after age 75 but these two papers do not consider letting as ours. Moreover, while matching the life-cycle homeownership rate pattern, we are also able to get close results of the average living space[2] compared to the data, using the same calibrated model and we also find that letting plays an important role in matching the living space. That is, if letting is not allowed, the living space in most periods during the life (around age 40-80) will be too high compared to data. This means that the letting choice is an essential ingredient in the life-cycle consumption and housing models to explain investor's housing decisions.

Our work is related to a broad literature studying investor's life-cycle consumption and asset allocation decisions. In this literature most papers consider non-durable consumption and asset allocation among liquid financial assets such as bonds and stocks, but abstract from housing consumption and housing assets [e.g., Viceira, 2001; Gomes and Michaelides, 2003; Polkovnichenko, 2007; Koijen et al., 2009; Peijnenburg, 2016; Michaelides and Zhang, 2017]. Some others include the housing asset,s and investigate investor's asset allocation choices among liquid and housing assets. But these studies still assume that investor does not gain any utility of consumption from housing [e.g., Cocco, 2004; Cooper and Zhu, 2016; Yogo, 2016; Xu et al., 2017]. One difference from those above-mentioned studies, our work does not include equity as an asset class but only considers a bond with a stochastic return over the life-cycle. This is partly motivated by empirical studies [e.g., Mankiw and Zeldes, 1991; Garlappi et al., 2007; Dimmock et al.,

[2]For the homeowner, the living space is calculated as the total size of the house owned minus the size to let out while for the tenant, the living space is simply the size of the house rented.

2016] which find that investors have limited stock market participation.

In the life-cycle setting, some other papers study the investor's consumption choices between housing and non-durable consumption and/or housing choices between owning and renting [e.g., Yao and Zhang, 2005; Piazzesi et al., 2007; Li and Yao, 2007; Díaz and Luengo-Prado, 2008; Chambers et al., 2009; Yang, 2009; Attanasio et al., 2012]. Compared to these papers, we not only incorporate the housing choices between renting and owning, but also assume the investor can let if she is an owner of the house and investigate how investor chooses the life-cycle consumption and housing decisions in the presence of letting choice.

The rest of the chapter is organized as follows. In Section 2, we introduce our dynamic programming formulation of the life-cycle consumption and housing problem. Section 3 focuses on model calibration and parameter settings. We present the results of numerical experiments in Section 4. Our findings and concluding remarks are summarized in Section 5.

4.2 Problem Statement

In this section, we introduce a dynamic programming formulation of the life-cycle consumption and housing problem for an investor who has bequest motive. We consider a discrete-time environment that spans T time periods. Life-cycle decisions are made at each time period (i.e., one year) $t = 1, \cdots, T$ and $t = 0$ represents today. Given the current age of the investor, s/he is alive for maximum T periods and retires at certain age $K < T$. At each time period t, the investor receives the labour income (the retirement income for $t > K$) and needs to decide the amount of consumption of non-durable goods, and housing decisions during the remaining life-time to maximize the expected life-cycle utilities. In a standard form, the investor is assumed to retire at age 65 and die at age 100.

The problem formulation contains random variables on real interest rate $\tilde{r}_t$, labour income $\tilde{l}_t$, and house prices $\tilde{q}_t$ at each point in time $(t = 1, \ldots, T)$.

Housing Decisions: The life cycle problem is concerned with three types of housing decisions, namely renting, owning and letting. While the investor has incompatible housing choices as renting and owning some units of house, s/he can let some units of house to others to gain rental income once she owns the house. We consider a broad interpretation for the size of the house, which does not only refer to the physical space of the house, but also the quality and location of the house [Cocco, 2004]. Let h_t denote a decision variable for displaying house status as renting or owning at time t. The sign of h_t determines the status of homeownership. If $h_t < 0$ (or $h_t > 0$), then this implies that the investor rents (or owns) the house. We display the floor area (square meter) of the house to be rented or owned by the investor as $|h_t|$.

Let h_{min} represent the minimum square metre of the house. Following previous studies [e.g., Cocco, 2004; Yogo, 2016], we assume that the investor can only own a house that is larger than h_{min}. where $h_t \geq h_{min} > 0$. On the other hand, there is no minimum requirement for the size of house to be rented; therefore, $h_t \leq 0$. A feasible set of housing decision variables for owning and renting becomes $\mathcal{F}_t = \{h_t \mid -\infty \leq h_t \leq 0, \text{ or } h_{min} \leq h_t \leq \infty\}$.

Let $x_t \geq 0$ at time period t be a decision variable that represents size of house that the owner lets to others. The maximum floor area that owner of the house is allowed to let is expressed by a constraint as $x_t \leq h_t - h_{min}$.

Cash Flow: Housing transactions provide flow of cash at each time period. Let $g(h_t \mid h_{t-1})$ be a function that represents amount of cash received from owning ($h_t > 0$) or renting ($h_t < 0$) the house at time t given the house status h_{t-1} determined at time $t-1$. Similarly, we define another function $\bar{g}(x_t)$ on the basis of letting decisions x_t at time t. Table 4.1 summarizes the corresponding cash flow functions for different house status under specific conditions.

Table 4.1: Cash flow received/paid for transformations in house status (renting, owning and letting) under specific conditions

House status	Owning/Renting Conditions (h_{t-1}, h_t)	Received/Paid Cash $g(h_t \mid h_{t-1})$	Letting Conditions (x_t)	Cash $\bar{g}(x_t)$
Keep renting	$h_{t-1},\ h_t \leq 0$	$\varphi\tilde{q}_t h_t$	$x_t = 0$	0
From renting to owning	$h_{t-1} \leq 0,\ h_t \geq h_{min}$	$-(1+\lambda+\xi)\tilde{q}_t h_t$	$0 \leq x_t \leq h_t - h_{min}$	$\varphi\tilde{q}_t x_t$
Keep owning & upsize	$h_{t-1}, h_t \geq h_{min},\ h_t \geq h_{t-1}$	$-(1+\lambda+\xi)\tilde{q}_t h_t + (1+\lambda)\tilde{q}_t h_{t-1}$	$0 \leq x_t \leq h_t - h_{min}$	$\varphi\tilde{q}_t x_t$
Keep owning & downsize	$h_{t-1},\ h_t \geq h_{min},\ h_t < h_{t-1}$	$-(1-\lambda+\xi)\tilde{q}_t h_t + (1-\lambda)\tilde{q}_t h_{t-1}$	$0 \leq x_t \leq h_t - h_{min}$	$\varphi\tilde{q}_t x_t$
From owning to renting	$h_{t-1} \geq h_{min},\ h_t \leq 0$	$(1-\lambda)\tilde{q}_t h_{t-1} + \varphi\tilde{q}_t h_t$	$x_t = 0$	0

House-renting: If the investor rents a house at time $t-1$ (i.e., $h_{t-1} < 0$), she can either keep renting (but allowed to change size of the rented house) or buy a new house at time t. If the investor chooses to rent the house at time t (i.e., $h_t \leq 0$), s/he pays a rent that is determined as a certain percentage (φ) of the current market price ($\tilde{q}_t$) of the house as $\varphi\tilde{q}_t h_t$. On the other hand, if the investor decides to buy a new house to become a homeowner at time t (i.e., $h_t \geq h_{min}$), then s/he needs to pay a transaction cost $\lambda\tilde{q}_t h_t$ at the top of the house price $\tilde{q}_t h_t$ where parameter λ is a fraction paid for buying the house. Moreover, at the end of time period t, the homeowner needs to pay a fraction ξ of the house price as the annual maintenance cost $\xi \tilde{q}_t h_t$.

House-owning: If the investor owns a house at time $t-1$, then there are two options to implement at time t. First, the investor can use the house-ownership as further housing investment opportunities and upsize or downsize her/his owned house by receiving or paying the price differences, respectively. Of course, s/he can also carry on living in the same house as before, which does not generate any cash flows. Second, the investor may prefer selling the house and become a tenant. This leads to the house status to be changed from owning to renting. In this case, s/he receives $(1-\lambda)\tilde{q}_t h_t$ after paying the transaction cost and will pay rental payments according to the size of the house to be rented.

House-letting: Once the investor owns a house, then any remaining space beyond the minimum required living area can be let out to others. The investor who decides to let $x_t \geq 0$ floor areas of the house receives the rental income $\varphi \tilde{q}_t x_t$. Following Iacoviello and Pavan [2013], we assume that no-arbitrage condition holds such that the rental payments per house floor area equals to the rental incomes from letting.

Investor's Preferences: We assume that the investor has Epstein-Zin-Weil-type preferences over consumption and bequest [Epstein and Zin, 1989, 1991; Weil, 1989]. We denote the utility of housing goods in consumption at time t as a function $f(h_t, x_t)$ that is defined as

$$f(h_t, x_t) = \begin{cases} h_t - x_t & \text{if } h_t \geq h_{min} \text{ (owning)} \\ -\zeta h_t & \text{if } h_t \leq 0 \text{ (renting)} \end{cases}$$

This implies that in both renting and owning housing situations, the investor obtains utility of housing consumption unlike in case of c_t that only gives utility of non-durable consumption. If the investor is a homeowner, then s/he receives $h_t - x_t$ as the utility of housing consumption. On the other hand, if he is a tenant, we assume that s/he experiences utility loss compared to owning a house, which is modelled by multiplying h_t by a discount factor ζ. Such an assumption is common in previous economic literature; for instance, see Rosen [1985], Poterba [1992], Iacoviello and Pavan [2013].

The investor's consumption preference over non-durable and durable goods is expressed by the Cobb–Douglas-type utility function as follows;

$$U(c_t, h_t, x_t) = f(h_t, x_t)^{\theta}(c_t)^{1-\theta}, \tag{4.1}$$

where $\theta \in (0, 1)$ determines the investor's utility weight over durable or non-durable consumption.

Let $e(h_t)$ denote the utility of housing wealth in bequest at time t that is defined as follows;

$$e(h_t) = \begin{cases} (1-\xi)\tilde{q}_t h_t & \text{if } h_t \geq h_{min}, \\ 0 & \text{if } h_t \leq 0. \end{cases}$$

This implies that the investor who is bequest motive leaves $(1-\xi)\tilde{q}_t h_t$ as bequest (after paying the maintenance cost). On the other hand, a tenant does not leave any house wealth as bequest. Following Yogo [2016], we assume that the investor does not view the financial wealth w_t and housing wealth $e(h_t)$ equally in the bequest, which is specified by the weight θ defined in the Cobb–Douglas-type utility function. For the investor with bequest motive $\eta > 0$, she also has the utility of bequest, which is measured by the total wealth upon death as:

$$U(w_t, h_t) = (w_t + e(h_t)) \left(\frac{\theta}{(1-\theta)\,\tilde{q}_t} \right)^{\theta}. \tag{4.2}$$

In other words, financial and housing wealth are not perfectly substitutable in the bequest. Under our parameter calibration, $(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta}$ is negatively correlated with unit house price $\tilde{q}_t$. This means for the same size of the house, the increase of utility of bequest is slower than housing wealth and vice versa.

Budget and Borrowing Constraints: At each time t, the investor receives labour income $\tilde{l}_t$ (and the retirement income after the age of 65) and makes (housing and non-durable) consumptions as well as housing transactions. If the investor is a homeowner, she can borrow up to a fraction ϖ of the market price of the house. This is a simplified model of the mortgage, which is commonly used in previous studies such as Cocco [2004], Yao and Zhang [2005] and Yogo [2016].

We now define two variables representing wealth accumulation. Let w_t denote the (real) *cash-on-hand* at time t and a_t be the *remaining wealth* at time t. The remaining wealth at time t is obtained by reducing the consumption (c_t) of non-durable goods from the total amount of cash that is consisting of cash-on-hand (w_t), labour income $(\tilde{l}_t)$ and total cash flow received from housing transactions (in renting, owning and letting as $g(h_t \mid h_{t-1}) + \bar{g}(x_t)$). If the remaining wealth is positive then it will be accumulated by the interest rate $\tilde{r}_t$ to obtain the cash-on-hand w_{t+1} at next time period $t+1$. In the case of negative remaining wealth, we assume that the investor also needs to pay a mortgage premium with fixed rate κ [Campbell and Cocco, 2015], at the top of the interest rate payment. These budget and borrowing conditions can be expressed by a set of linear constraints as follows;

$$w_{t+1} = a_t\tilde{r}_t + \kappa \min\{a_t, 0\}, \tag{4.3}$$

$$a_t = w_t - c_t + g(h_t \mid h_{t-1}) + \bar{g}(x_t) + \tilde{l}_t, \tag{4.4}$$

$$a_t \geq -\varpi_h \frac{\mathbb{E}[e(h_t)]}{1-\xi}. \tag{4.5}$$

The Dynamic Life-cycle Optimization Model: The dynamic life-cycle consumption and housing problem finds the optimal strategies consisting of consumption and housing decisions at each time period so that the investor's total life-cycle utilities of consumption and bequest are maximized subject to budget and borrowing constraints. Given any state of the system in cash-on-hand w_t and housing status h_{t-1} in previous time period $t-1$, we define a value function at time t as $J_t(w_t, h_{t-1})$. Let p_t denote the investor's survival rate at time t conditional on being alive at $t-1$. We can formulate $J_t(w_t, h_{t-1})$ in a compact form as follows;

$$\begin{aligned} J_t(w_t, h_{t-1}) = \max_{c_t, h_t, x_t} & \Bigg\{ (1-\delta) U(c_t, h_t, x_t)^{1-1/\psi} + \\ & \delta \mathbb{E}_t \Big[p_{t+1} J_{t+1}(w_{t+1}, h_t)^{1-\gamma} + \eta(1-p_{t+1}) U(w_t, h_t)^{1-\gamma} \Big]^{\frac{1-1/\psi}{1-\gamma}} \Bigg\}^{\frac{1}{1-1/\psi}}, \end{aligned} \tag{4.6}$$

that can be also recursively rewritten as:

$$\begin{aligned} J_t(w_t, h_{t-1}) = \max_{c_t, h_t, x_t} & \Bigg\{ (1-\delta)\Big(f(h_t, x_t)^{\theta} c_t^{1-\theta}\Big)^{1-1/\psi} + \\ & \delta \mathbb{E}_t \Bigg[p_{t+1} J_{t+1}(w_{t+1}, h_t)^{1-\gamma} + \eta(1-p_{t+1})\Big(w_{t+1} + e(h_t)\Big)^{(1-\gamma)} \Big(\frac{\theta}{(1-\theta)\bar{q}_t}\Big)^{\theta(1-\gamma)} \Bigg]^{\frac{1-1/\psi}{1-\gamma}} \Bigg\}^{\frac{1}{1-1/\psi}} \\ s.t. \quad & w_{t+1} = a_t\tilde{r}_t + \kappa \min\{a_t,\ 0\}, \\ & a_t = w_t - c_t + g(h_t \mid h_{t-1}) + \bar{g}(x_t) + \tilde{l}_t, \\ & a_t \geq -\varpi_h \tfrac{\mathbb{E}[e(h_t)]}{1-\xi}, \\ & x_t \leq h_t - h_{min}, \\ & h_t \in \mathcal{F}_t,\ x_t \geq 0,\ c_t \geq 0. \end{aligned} \tag{4.7}$$

where δ denotes the subjective discount factor, ψ measures the elasticity of intertemporal substitution and γ represents the relative risk aversion.

4.3 Solution of the Life-cycle Consumption and Housing Problem

A closed-form solution leading the optimal policy is obtained for limited cases of the life-cycle problems such as Merton [1969]. Numerical approaches have been used for more general cases including grid methods with backward induction, simulation optimization and scenario-based stochastic programming. The common drawback for these approaches is the curse of dimensionality. The size of state space exponentially increases as the number of decisions and the decision epochs increase [e.g., Cocco et al., 2005; Gao and Ulm, 2015]. In the grid methods, the state space for each time period is discretized into many nodes while the simulation based approach as well as the scenario-based stochastic programming approach require large simulation steps [e.g., Chen et al., 2006] and large number of scenarios, respectively, to achieve more accurate approximate strategies (close to the optimal policy if possible). However, in practice, the modeler needs to reduce the number of scenarios or decision stages to manage the curse of dimensionality of dynamic programs (see for example, Geyer et al. [2009] and Konicz, Pisinger, Rasmussen and Steffensen [2015]).

4.3.1 Backward Induction Method

In order to solve the dynamic life-cycle consumption and housing problem (4.7), we consider the backward induction method. There are three control variables at each time period t: namely, consumption (c_t), housing floor area (h_t) and letting floor area (x_t). The state variables are the cash-on-hand (w_t) and housing floor area in the previous time period (h_{t-1}). Note that h_{t-1} is used as a state variable to compute the value function at time t. In addition, we need to compute the function $g(h_t \mid h_{t-1})$ as amount of cash received/paid at time t.

The backward induction method starts from terminal states at the end of planning horizon T towards to the initial states at time $t = 1$. For each grid point of discretised state spaces, we find the optimal consumption and housing decisions. This leads to $N * M$ number of states where $m = 1, \dots M$ and $n = 1, \dots, N$. For those points that are not lying on the grid specified, we firstly interpolate the housing value on N grids and then look up the cash-on-hand value on M grids.

Next, we will describe the main steps of the algorithm below.

Step 1: Reformulation of the Model

We first reformulate the dynamic programming formulation of the life-cycle consumption and housing problem. Notice that, solving the maximization problem (4.7) is equivalent to solving the following minimization problem;

$$
\begin{aligned}
J_t(w_t, h_{t-1})^{1-1/\psi} = \min_{c_t, h_t, x_t} & \Bigg\{ (1-\delta)\Big(f(h_t, x_t)^{\theta} c_t^{1-\theta}\Big)^{1-1/\psi} + \\
& \delta \mathbb{E}_t \left[p_{t+1} J_{t+1}^{1-\gamma} + \eta(1-p_{t+1})\Big(w_{t+1} + e(h_t)\Big)^{(1-\gamma)} \left(\frac{\theta}{(1-\theta)\tilde{q}_t}\right)^{\theta(1-\gamma)} \right]^{\frac{1-1/\psi}{1-\gamma}} \Bigg\} \\
& s.t. \quad w_{t+1} = a_t r^f + \kappa \min\{a_t,\ 0\} + \tilde{l}_{t+1}, \\
& \qquad a_t = w_t - c_t + g(h_t \mid h_{t-1}) + \bar{g}(x_t), \\
& \qquad a_t \geq -\varpi_h \tfrac{\mathbb{E}[e(h_t)]}{1-\xi}, \\
& \qquad x_t \leq h_t - h_{min}, \\
& \qquad h_t \in \mathcal{F}_t = \{h_t \mid -\infty \leq h_t \leq 0, \text{ or } h_{min} \leq h_t \leq \infty\}, \\
& \qquad x_t \geq 0,\ c_t \geq 0.
\end{aligned}
\tag{4.8}
$$

Then we introduce an auxiliary decision variable z_t to re-state the $\min\{a_t,\ 0\}$ as:

$$
\begin{aligned}
J_t(w_t, h_{t-1})^{1-1/\psi} = \min_{c_t, h_t, x_t, z_t} & \Bigg\{ (1-\delta)\Big(f(h_t, x_t)^{\theta} c_t^{1-\theta}\Big)^{1-1/\psi} + \\
& \delta \mathbb{E}_t \left[p_{t+1} J_{t+1}^{1-\gamma} + \eta(1-p_{t+1})\Big(w_{t+1} + e(h_t)\Big)^{(1-\gamma)} \left(\frac{\theta}{(1-\theta)\tilde{q}_t}\right)^{\theta(1-\gamma)} \right]^{\frac{1-1/\psi}{1-\gamma}} \Bigg\} \\
& s.t. \quad w_{t+1} = a_t r^f + z_t \kappa + \tilde{l}_{t+1}, \\
& \qquad a_t = w_t - c_t + g(h_t \mid h_{t-1}) + \bar{g}(x_t), \\
& \qquad a_t \geq z_t, \\
& \qquad a_t \geq -\varpi_h \tfrac{\mathbb{E}[e(h_t)]}{1-\xi}, \\
& \qquad x_t \leq h_t - h_{min}, \\
& \qquad h_t \in \mathcal{F}_t = \{h_t \mid -\infty \leq h_t \leq 0, \text{ or } h_{min} \leq h_t \leq \infty\}, \\
& \qquad x_t \geq 0,\ c_t \geq 0,\ z_t \leq 0.
\end{aligned}
\tag{4.9}
$$

Step 2: Construction of Grids and Formulation of Sub-problems

We set up M number of grids for cash-on-hand (w_t) at each time period w_t and N grids for home size h_{t-1} at time $t-1$ that leads to form a grid space with size of $M * N$ in total. The whole grid space can be divided into two blocks: $h_{t-1} \leq 0$ and $h_{t-1} \geq h_{min}$ that represent being a tenant and a homeowner at previous time period t-1, respectively.

For each block of grid space, instead of solving the problem $J_t(w_t, h_{t-1})^{1-1/\psi}$, we solve the following three (incompatible) sub-problems, differing from the range of current home size h_t:

- currently being tenant: $h_t \leq 0$

- currently being homeowner and upsizing the home size: $h_t \geq h_{t-1}$ and $h_t \geq h_{min}$
- currently being homeowner and downsizing the home size: $h_t \leq h_{t-1}$ and $h_t \geq h_{min}$

In order to find the optimal solutions of subproblems, we use the KKT optimality conditions. The derivation of the first order optimality conditions are presented in Appendix 4.A.

The solution of three sub-problems provides three sets of optimal decisions. Then we calculate the objective function value of these three sub-problems and pick the set of optimal decisions which give the highest objective function value and nominate it as the set of optimal decisions of the whole problem $J_t(w_t, h_{t-1})^{1-1/\psi}$. In this way, we avoid introducing extra binary state variables to represent the house status as either renting or owning [Yao and Zhang, 2005].

4.4 Model Setting and Calibration

We design computational experiments to investigate performance of dynamic life-cycle consumption and housing model. In particular we are concerned with two main questions which have been only partially answered before:

- How are the investors life-cycle consumption and housing decisions affected by investor's preferences?
- What factors affect the housing strategies in terms of renting, owning and letting?

We first explain our calibration and parameter setting used in empirical study and then present our computational results.

4.4.1 Model Calibration

As mentioned in formulation of the life-cycle consumption and housing decision-making problem, the investor faces three kinds of uncertainties including interest rates, labour incomes and house prices. We now describe how to model these uncertain parameters. For real interest rate, the AR(1) processes is used to model real interest rate $\tilde{r}_t$ as presented by Campbell and Cocco [2015]. Let $\tilde{\varsigma}_t$ denote white noise that is normally distributed: $\tilde{\varsigma}_t \sim \mathcal{N}(0, \sigma_\varsigma^2)$. Given coefficients μ_r and ϕ_r, the log of interest rate at time t can be formulated as follows;

$$\log(\tilde{r}_t) = \mu_r(1 - \phi_r) + \phi_r \log(\tilde{r}_{t-1}) + \tilde{\varsigma}_t.$$

We assume that the labour income (before retirement) is uncertain and formulated as suggested by Cocco et al. [2005]. The log of labour income at t consists of three components: a deterministic term f_t, permanent shock $\tilde{\nu}_t$ and temporary shock $\tilde{\varepsilon}_t$ as:

$$\log(\tilde{l}_t) = f_t + \tilde{\nu}_t + \tilde{\varepsilon}_t. \tag{4.10}$$

The deterministic component f_t in (4.10) is modelled by a third-order polynomial function of age t as follows;

$$f_t = a_0 + a_1 t + a_2 t^2/10 + a_3 t^3/100,$$

where coefficients a_0, a_1, a_2 and a_3 need to be estimated. Moreover, the permanent shock follows a one-period auto-regression process

$$\tilde{\nu}_t = \tilde{\nu}_{t-1} + \tilde{u}_t,$$

where $\tilde{u}_t \sim \mathcal{N}(0, \sigma_u^2)$ refers to permanent shock to the labour income whereas $\tilde{\varepsilon}_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is the temporary shock. The retirement income is a constant fraction (e.g., 0.68) of the permanent labour income in the year just before retirement. In other words, there is no temporary shock during the retirement.

In numerical results, we use the same estimates of parameters of the labour income process as given in Cocco et al. [2005]. They are the deterministic component f_t, shock variances (σ_ε, σ_u), and polynomial coefficients (a_0, a_1, a_2, a_3) as well as the retirement income fraction for three different education groups of investors who have no high school degree, high school degree and college degree. Since the choice of groups does not affect the main conclusions reported in this chapter, we only present the computational results obtained by the labour income estimates for the investor with high school degree group.

We model the log of real house price growth by a random walk with drift as described in Campbell and Cocco [2015]:

$$\log(\tilde{q}_t) = \log(\tilde{q}_{t-1}) + g + \tilde{\tau}_t,$$

where constant parameter d represents the average yearly growth of the house price. In addition, $\tilde{\tau}_t \sim \mathcal{N}(0, \sigma_\tau^2)$ is normally distributed. In addition, we assume that the innovation to house price growth ($\tilde{\tau}_t$) is correlated with permanent shock ($\tilde{u}_t$) of labour income as well as innovations ($\tilde{\varsigma}_t$) of real interest rate. The parameter settings for the mean, standard deviation, AR(1) coefficient as well as the correlation values of the interest rate and house price progresses ($\rho_{\tau,u},\ \rho_{\tau,\zeta}$) are all estimated as in Campbell and Cocco [2015]. The other parameters used for numerical experiments using the baseline setting are summarized in Table 4.2.

Table 4.2: Calibration and parameter settings used in empirical study

Notation	**Uncertainty-related parameters**	**Fixed values**
μ_r	Mean log real rate	0.012
ϕ_r	Log real rate AR(1) coefficient	0.825
σ_ς	Std. of the log real rate shock	0.018
d	Mean log real house price growth	0.003
σ_τ	Std. of the log real house price shock	0.162
$\rho_{\tau,u}$	Correl. perm. inc. and house price shocks	0.191
$\rho_{\tau,\varsigma}$	Correl. real int. rate and house price shocks	0.3
σ_u	Std. of the log real labour income permanent shock	0.103
σ_ε	Std. of the log real labour income temporary shock	0.272
	Other model parameters	
η	bequest motive	1
γ	relative risk aversion	5
ψ	elasticity of intertemporal substitution (EIS)	0.5
δ	time discount factor	0.96
ϖ	borrowing limit	20%
θ	housing weight in the utility of consumption	0.3
ξ	annual housing maintenance cost	1.5%
φ	rental payment/income gain per unit of house	6%
λ	transaction cost of housing trades	6%
ζ	utility discount in housing consumption when renting	0.832
h_{min}	minimum square metre of the house to own	40

Data from US social security 2013[3] is used to calculate investor's conditional probability of alive p_t. The time discount factor δ is chosen as 0.96. We set the investor's bequest motive η as 1, which implies the investor views the utility of consumption and utility of bequest equally important. The relative risk aversion γ and elasticity of intertemporal substitution ψ are fixed as 5 and 0.5, respectively.

In order to calibrate the h_{min} value, data from US Census Bureau[4] is considered. This provides the distribution of the floor area in new single-family houses over the whole US. The data implies that around 90% of the new single-family houses in US is at least 100 square metre. Moreover, according to Household Size and Composition Around the World 2017 data booklet by United Union, the average household size in

[3]The reader is referred to `https://www.ssa.gov/oact/STATS/table4c6.html`.
[4]The reader is referred to `https://www.census.gov/construction/chars/pdf/squarefeet.pdf`.

US is 2.6 people. Therefore, we set $h_{min} = 40$ since in the baseline setting we solve the problem for an individual investor).

To standardize the initial house price per square metre of the house, we follow Iacoviello and Pavan [2013], which assumes that the minimum size of the house to purchase costs two times the average income. Under our calibration of labour income process, the average income at age 20 is 16.04 (thousands U.S. dollar). Then the initial house price per square metre is $16.04 * 2/40 = 0.802$.

We find that in the literature, the choice of housing weight in the consumption utility θ is varied between 0.2 [Yao and Zhang, 2005] and 0.4 [Yogo, 2016]. We choose an intermediate value $\theta = 0.3$ and find that our model performs well in matching the data in terms of life-cycle homeownership rate and investor's living space patterns.

Following Yogo [2016], we set the borrowing limit ϖ as 20%, which indicates that a homeowner is allowed to borrow 20% against her housing equity. According to Iacoviello and Pavan [2013], the utility discount ζ in housing consumption (when renting is compared to owning) is fixed at 0.832. This implies that renting a house with size 100 gives the same utility of housing consumption as owning a house with size 83.2. In line with Yao and Zhang [2005], the annual housing maintenance cost ξ is 1.5%, rental payment (when renting a house) or income (when letting a house) per unit of house φ is 6%. Moreover, the transaction cost (λ) for upsizing or downsizing the owned house is fixed as 6%. In the baseline setting, we also assume the investor starts with 0 financial wealth and 0 housing wealth (i.e., the investor does not own a house).

4.4.2 Model Implementation

In order to solve the dynamic programming model with traditional backward induction approach, we choose $M = 200$ grids of cash-on-hand and $N = 34$ grids of previous housing status. For the uncertain parameters including interest rate, labour income and house price, we first generate $K = 40,000$ simulations and then solve the dynamic programming problem for each simulation. We present our results in terms of average values of the investor's house size rented, owned and let, as well as consumption and saving decisions over all simulations.

We are mainly concerned with three kinds of trade-off in the life-cycle consumption and housing model. First, how does the investor choose between owning and renting a house? We answer this question by showing the well-known homeownership rates throughout investor's entire life. Second, conditional on being a homeowner, how does one choose between living in the house and letting it out? In the case of letting, the homeowner gains rental payments as the cost of reducing the living space. Therefore, we will illustrate the homeowner's letting willingness and living spaces over the life cycle. Finally, we empirically investigate the tradeoff between non-durable consumption and saving once the investor has made her housing decision. We report the results of numerical experiments in the following section.

4.5 Numerical Results

4.5.1 Homeownership versus Rental Property

As mentioned before, we first investigate how the investor chooses her housing decisions during the lifetime using the dynamic programming formulation of the life-cycle consumption and housing problem. To be more specific, we illustrate what factors drive the investor to choose between owning or renting a house.

We first focus on the effect of bequest motive on homeownership rates. In Figure 4.1, we illustrate the homeownership rate with four different degrees of bequest motive (η) under two settings of initial housing wealth. That is, we assume the investor either has no initial housing wealth (i.e., the investor does not own a house, which is our benchmark case) or has the minimum housing wealth (i.e., the investor owns a house of size h_{min}). The homeownership rate, by definition, is the proportion of investors that own houses divided by the total number of investors. Therefore, at every age, the homeownership rate is calculated by the number of positive h_t value (i.e., representing that the investor owns a house) in all K simulations divided by K. For instance, a homeownership rate of 0.6 at age 40 means that out of $K = 40,000$ simulations, there are $24,000$ incidents where the investor owns a house at age 40.

As shown in the left plot of Figure 4.1, for all four degrees of bequest motive (η), the homeownership rate for the investors with zero initial housing wealth starts at 0 (Recall that we assume the minimum-sized house requires on average two years' labour income.) and then increases until investor ages around 40. For investors with no bequest motive (i.e., $\eta = 0$), the homeownership rate keeps increasing until retirement and then decreases after retirement. While for those with some degrees of bequest motive, the homeownership rate maintains almost the same level until retirement and decreases moderately for only 20 more years. Such difference in the life-cycle pattern of homeownership rate with different degrees of bequest motive is mainly due to the differences in the motivation to leave housing wealth to the heirs. We also note that the higher the degree of bequest motive the higher is the homeownership rate at every age.

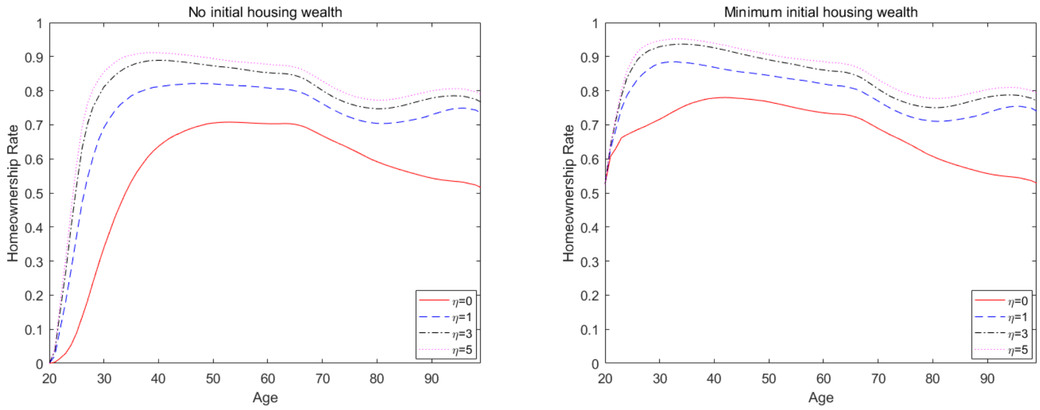

Figure 4.1: Homeownership rates (bequest motive and initial housing wealth)
The two plots compare the homeownership rates over the entire life for varying degrees of bequest motive (η) under two settings of initial housing wealth: (left) 0 housing wealth (i.e., the investor does not own a house) and (right) minimum housing wealth (i.e., the investor owns a house with size h_{min}). At every age, the house ownership rate is calculated by the number of positive h_t in all simulations divided by the total number of simulations.

By comparing the left and right plots of Figure 4.1, we notice that the difference in the initial housing wealth only affects the homeownership rate before retirement. If the investor has some initial housing wealth, the homeownership rate starts at around 0.55 at age 20 and then keeps increasing for another 10 (investor with bequest motive) or 20 (investor without bequest motive) years, and then slightly decreases until retirement. Since we assume the investor has the minimum housing wealth in all simulations. A rate of 0.55 means that in about 45% of the simulations, the investor immediately sells the house at age 20 and become tenants. After retirement, the homeownership rate is almost the same as the case with no initial housing wealth suggesting that the role of initial wealth in housing preferences disappears in the later stages of life.

We also compare the homeownership rates implied by our model with the recent data from Housing Vacancies and Homeownership (CPS/HVS), US Census Bureau[5] in the Table 4.3. The homeownership rate value from our model in each age group is calculated as the mean of homeownership rates in all ages within that age group.

[5]See `https://www.census.gov/housing/hvs/data/q118ind.html` for more information

Table 4.3: Homeownership rate: data versus model predictions

Age group	Data	Model predictions							
		No initial housing wealth (η)				Minimum initial housing wealth (η)			
		0	1	3	5	0	1	3	5
Average	63.9	55.2	71.3	77.3	80.0	68.1	79.3	83.7	85.7
<25	22.6	1.5	10.2	15.3	17.8	63.1	68.8	71.4	72.6
25-29	32.1	18.4	54.5	70.3	76.2	69.3	85.0	90.7	92.9
30-34	45.7	42.8	74.1	84.5	88.3	73.4	88.5	93.5	95.1
35-39	56.4	58.3	79.9	88.4	91.0	76.9	87.7	93.4	94.7
40-44	61.8	66.0	81.6	88.8	91.0	78.0	86.2	91.9	93.3
45-49	67.5	69.5	82.1	87.9	90.1	77.4	85.1	90.0	91.6
50-54	71.1	70.8	81.8	87.0	89.0	76.0	83.9	88.5	90.1
55-59	73.8	70.5	81.3	85.8	88.1	74.3	82.8	86.9	89.0
60-64	76.9	70.3	80.5	85.1	87.5	73.2	81.5	85.7	88.2
65-69	79.1	69.2	78.9	83.2	85.7	71.6	79.7	83.7	86.2
70-74	81.4	65.2	74.4	78.2	81.0	67.2	75.1	78.5	81.4
>75	76.8	56.3	72.4	76.6	78.9	57.7	73.0	76.9	79.3

This table compares the homeownership rates for different degrees of bequest motive and initial housing wealth derived from our model with latest data from Housing Vacancies and Homeownership (CPS/HVS), US Census Bureau. The homeownership rate value in each age group from our model is calculated as the mean of homeownership rates in all ages within that age group.

We first notice that in age group '<25', our model predictions are lower (higher) than the data for investors with no (minimum) initial housing wealth. For example, the data indicate an average homeownership rate of 22.6% for the investor aged lower than 25, while our model predicts a homeownership rate of 1.5% if in all $40,000$ simulations, the investors have no initial housing wealth and no bequest motive, but a much higher rate of 63.4% if all investors have some initial housing wealth. This makes sense because in reality, the proportion of the investors aged 20 who have initial housing wealth (received from others such as the family) must be somewhere in between 0 and 1 rather than the two extreme cases we consider. As noted in Figure 4.1, the homeownership rate increases as the degree of bequest motive increases.

Second, we observe that for the investors before retirement, the data is mostly in line with the model predictions with no bequest motive while for the investor during retirement, the data is closer to the model predictions if investors have some degrees of bequest motive. This is reasonable that investor's bequest motive may increase with the increase of age and mortality risk as argued in the literature [e.g., Menchik and David, 1983; Hurd, 1989].

Another factor that affects the homeownership rate is the housing weight in the utility of consumption (θ). θ measures the investor's preference of consuming housing relative to non-durable goods. A higher θ means that the investor is more willing to consume housing other than non-durable goods. In Figure 4.2, we show the life-cycle homeownership rate with $\theta = 0.2$ [Yao and Zhang, 2005] and 0.4 [Yogo, 2016] assuming that the investor can choose between letting or maintaining the same living space, in the absence and presence of bequest motive. (To make the figure more readable, we have omitted our benchmark choice $\theta = 0.3$, which is the median value.)

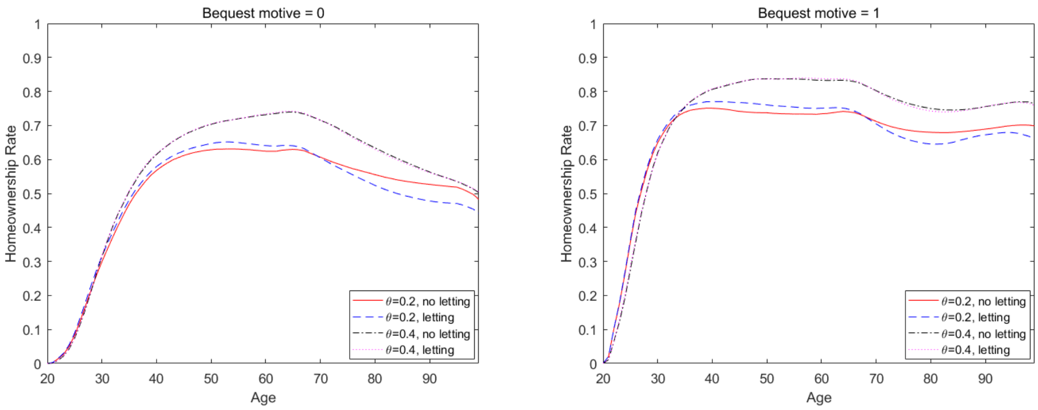

Figure 4.2: Homeownership rates (housing weight in utility of consumption and letting options)

The two plots compare the homeownership rates over the entire lifetime for varying degrees of housing weights in the utility of consumption (θ) and with or without the letting option under two settings of bequest motive: 0 (left) and 1 (right).

We first notice that the model including an investor with higher θ imply higher homeownership rates during most of the lifetime, especially after age 40. This is intuitive because a lower θ means that the investor prefers non-durable consumption rather than the housing consumption. In other words, the investor gains more utility from consuming non-durable goods instead of maintaining the ownership of the same living space.

We also observe that if $\theta = 0.4$, there is almost no difference in the homeownership rates with or without letting option. As we will show in Section 4.5.2, this is because the investor does not choose to let much. On the contrary, if $\theta = 0.2$, for both degrees of bequest motive, the homeownership rate is higher (lower) before (after) retirement if the investor can let. This is related to the amount of labour (retirement) incomes investor receives. Before retirement, the investor receives labour income, which motivates the investor to become the homeowner because she can let out part of the house to gain extra rental income. As we will show in Section 4.5.3, this rental income is mainly used in the form of non-durable consumption rather than liquid or housing wealth accumulation. After retirement, however, since the investor receives much less retirement income compared to labour income, in order to keep the consumption level, some homeowners become tenants, which thus reduces the homeownership rate.

Besides the degree of bequest motive and housing weight in the consumption utility, we also find that investor's life-cycle homeownership pattern is related to the borrowing limit of home equity. In the baseline setting, we choose a borrowing limit of 20%, which means that the investor can borrow up to 20% of her home equity. In Figure 4.3, we vary the borrowing limit value (ϖ) from 0 up to 0.4 for investors with either no bequest motive or a bequest motive equal to 1.

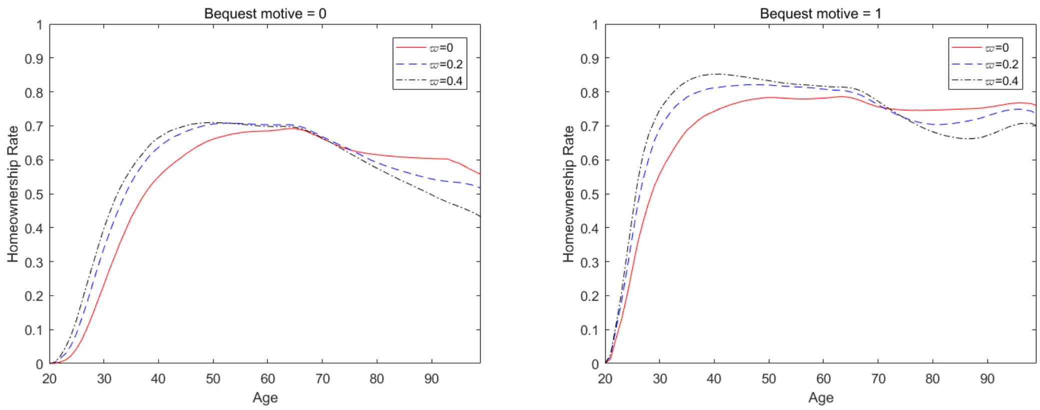

Figure 4.3: Homeownership rates (borrowing limit)
The two plots compare the homeownership rates over the entire life for varying degrees of borrowing limits (ϖ) under two settings of bequest motive: 0 (left) and 1 (right).

We notice that overall, the borrowing limit does no affect the average homeownership rate of the entire life much but only has impact on the life-cycle pattern before and after retirement. That is, for both degrees of bequest motive (0 or 1), the homeownership rate is lower (higher) before (after) retirement if the borrowing limit is lower. This is because before retirement, the investor with higher borrowing limit has more capital (from mortgage) to own larger houses. While after retirement, the investor receives much less (retirement) income. In order to keep the consumption level and repay the borrowed amount of capital, some of the previous homeowners become tenants, which results in a lower homeownership rate if the investor's borrowing limit is higher.

In addition, we illustrate the relationship between homeownership rate and elasticity of intertemporal substitution, that is, preference for the early resolution of uncertainty, in Figure 4.4. Our benchmark choice of EIS is 0.5 [as in Xu et al., 2017]. If EIS =

0.2, then the Epstein-Zin-Weil utility reduces to the power utility (since our benchmark choice of risk aversion is 5). Yogo [2016] also uses higher EIS value of 0.7. Therefore, we vary EIS between 0.2 and 0.7 and plot the homeownership rates for each age group.

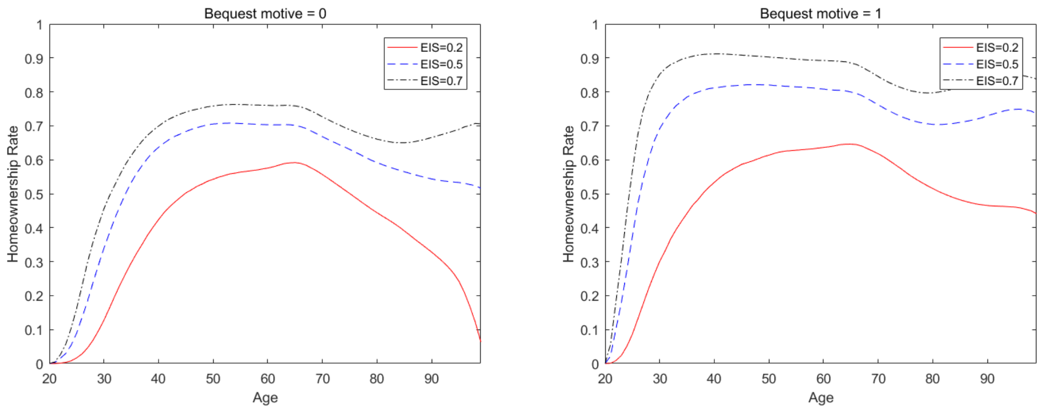

Figure 4.4: Homeownership rates (elasticity of intertemporal substitution (EIS))
The two plots compare the homeownership rates over the entire life for varying degrees of EIS under two settings of bequest motive: 0 (left) and 1 (right).

It is clear that homeownership rate is positively correlated with EIS regardless of the bequest motive. If EIS = 0.2, which means the investor has the power utility over consumption and/or bequest, the homeownership rate is lower, especially in the retirement period and is not in line with the data (See Table 4.3). That means, in order to match the homeownership rate in real data, Epstein-Zin-Weil utility with some degrees of EIS is essential.

4.5.2 Living Space versus Letting

After studying the investor's choice between owning and renting, we then focus on the homeowner's choice between maintaining the living space and letting (that is, downsizing the living space). That is, conditional on homeownership, what is the optimal balance between housing consumption and rental income obtained by letting out (part of) the living space to others?

We first investigate the relationship between the letting choice and bequest motive in Figure 4.5, assuming different levels of initial housing wealth. The criterion we use the measure the homeowner's willingness to let is the letting allocation rate. At every age for every homeowner, the letting allocation rate is calculated as the house size to let divided by the total house sizes which can be used for letting ($h_t - h_{min}$[6]). Then the letting allocation rate at every age we plot in Figure 4.5 is calculated as the mean of all homeowners' letting allocation rates.

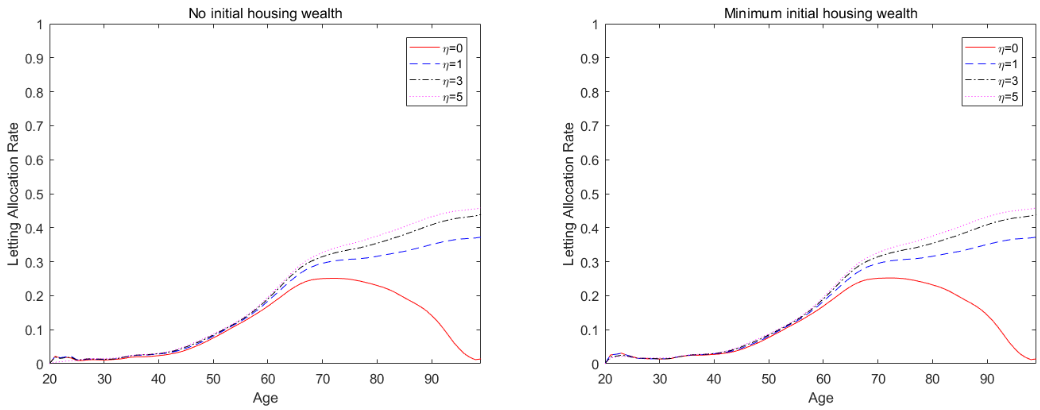

Figure 4.5: Letting allocation rates (bequest motive and initial housing wealth)
The two plots compare the letting allocation rates over the entire life for varying degrees of bequest motive (η) under two settings of initial housing wealth: (left) 0 housing wealth (i.e., the investor does not own a house at age 20) and (right) minimum housing wealth (i.e., the investor owns a house with size h_{min} at age 20). At every age for every homeowner, the letting allocation rate is calculated as the house size to let divided by the total house sizes which can be used for letting ($h_t - h_{min}$. Then the letting allocation rate at every age is calculated as the mean of all homeowners' letting allocation rates.

We first notice that with our benchmark parameter value $\theta = 0.3$, the homeowner almost does not let before age 40, for all degrees of bequest motive and initial housing wealth. That means the homeowner has a strong motive to live in the house (contributing to the housing consumption utility) rather than let out before retirement. Then the homeowner starts to downsize the living space and keep increasing the letting part to gain rental income she ages. For the homeowner with no bequest motive, the letting peaks around age 70, and then starts to decline later in life. As we will show below in Figure 4.9, meanwhile, the homeowner also slightly downsizes in terms of homeownership. This means that instead of letting and owning a large house, the homeowner chooses to live in a smaller house and hold more liquid wealth when she is old.

In contrast, the homeowner with some degree of bequest motive keeps increasing the letting allocation rate while slightly downsizing the living space (See Figure 4.9 below).

[6]Recall that h_{min} is defined as the minimum size to own. In the case of letting, we assume this is the part of the house which should be left for homeowner's living space and cannot be let to others

At age 100, the letting allocation rate is about 40%, which means the homeowner only lives in 60% of her owned house. This implies that for the homeowner with bequest motive, rental income from letting during the retirement period becomes part of her bequest.

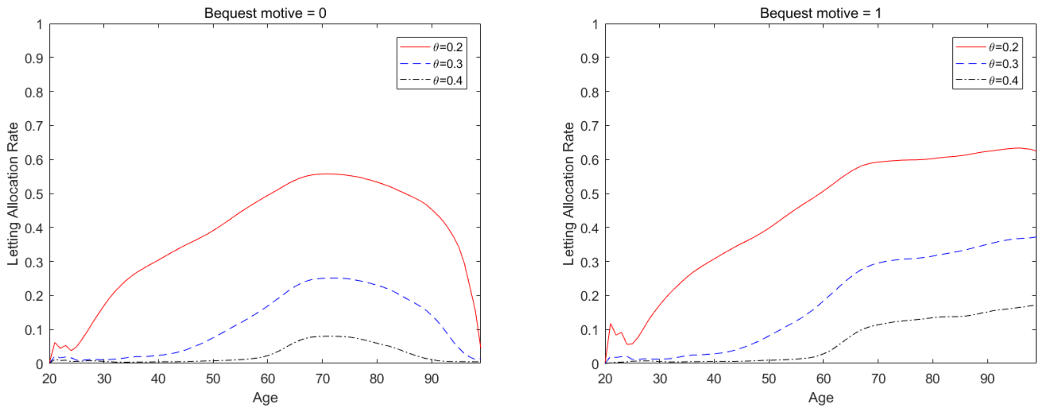

Figure 4.6: Letting allocation rates (housing weight in utility of consumption)
The two plots compare the letting allocation rates over the entire life for varying degrees of bequest motive (η) under two settings of initial housing wealth: (left) 0 housing wealth (i.e., the investor does not own a house at age 20) and (right) minimum housing wealth (i.e., the investor owns a house with size h_{min} at age 20).

Besides the bequest motive, another factor that affects the homeowner's choice between living and letting is the housing weight in the utility of consumption (θ). On one hand, letting reduces the homeowner's housing consumption from living in the owned space. On the other hand, letting gives the homeowner more capital for future use, which can benefit non-durable consumption. So in principle, a homeowner with higher θ should let less. We show such effect of θ on letting allocation rates in Figure 4.6.

It is clear that the housing weigh in utility of consumption is negatively correlated with the letting decisions. A higher θ reduces the letting allocation rate. In comparison, as shown in Figure 4.2, a higher θ is positively correlated with the homeownership rate. For the homeowner with no bequest motive, the life-cycle pattern of letting is hump-shaped. When the homeowner is young, she prefers living in the owned space rather than letting. Then she increases her living space in middle ages. If $\theta = 0.2$, which means the utility of housing consumption is one-fourth of the non-durable consumption in the utility of consumption, the homeowner uses up to 60% of her living space in letting. As we will show below, meanwhile, she slightly reduces the living space. This means at the middle ages, the homeowner prefers letting rather than living in the owned space.

After retirement, mainly after age 80, the letting allocation rate significantly reduces and declines to 0, for all degrees of θ.

For the homeowner with bequest motive, the letting pattern before retirement is quite similar to that without bequest motive. However, during the retirement period, the homeowner keeps allocating her living space in her house in letting in order to gain rental income for the bequest purpose. This confirms the observations we have found in Figure 4.5.

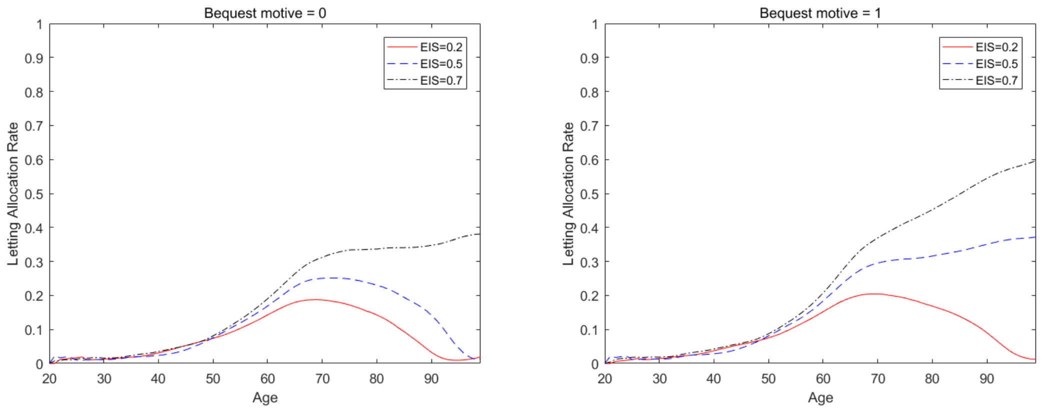

Figure 4.7: letting allocation rates (elasticity of intertemporal substitution (EIS))
The two plots compare the letting allocation rates over the entire life for varying degrees of EIS under two settings of bequest motive: 0 (left) and 1 (right).

We also study the relationship between letting allocation rate and EIS in Figure 4.7, where we vary the EIS value from 0.2 and 0.7 as before. We find the elasticity of intertemporal substitution has a similar effect as the bequest motive. That is, a homeowner with higher EIS preference is more willing to let, mainly after retirement. However, the EIS does not affect the homeowner's letting decision before retirement, which is different from the effect of housing weight in consumption utility.

After investigating the letting allocation rate, we then study the homeowner's living space during the lifetime. Throughout this chapter, we present the mean values for the average living space for the investors. Other statistical metrics (such as median and/or any percentile) do not change our observations and findings. We first look at the living space of tenants in Figure 4.8, who do not own the house (and hence cannot let). At every age, the living space of a tenant is $-h_t$.

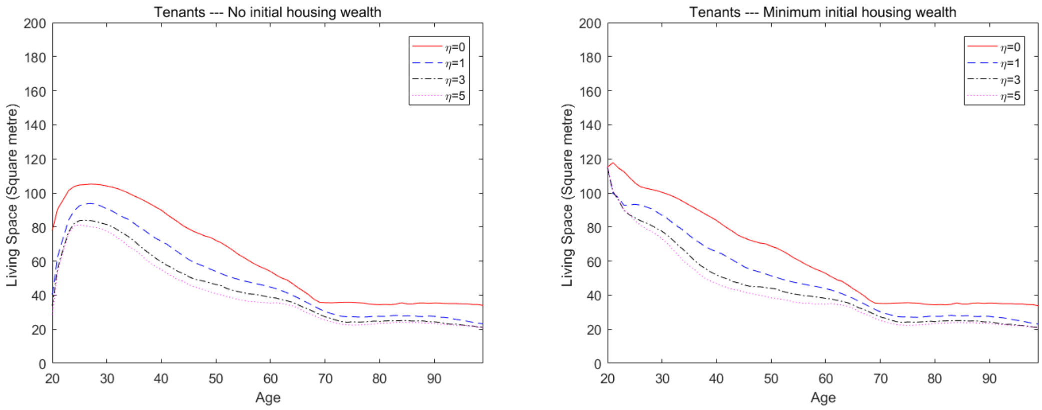

Figure 4.8: Living space (tenants, bequest motive, initial housing wealth)
The two plots compare the average living area (square metre) over the entire life for tenants for different degrees of bequest motive and initial housing wealth. At every age, the living space of a tenant is $-h_t$.

We find that initial housing wealth plays a role for the living space of a tenant. If the investor does not have any housing wealth at age 20, the only choice for her is to rent a house and become a tenant. Recall that in our calibration, the minimum sized house requires two years' of labour income to buy. Then the tenant with no bequest motive starts with a living space of 80 square metre (and around 45 square metre if she has bequest motive). In comparison, if the investor has minimum housing wealth at age 20, she is already a homeowner. Then she has two options, either keep on being a homeowner or sell the house and become a tenant. For those who chooses to become tenants, they live in considerably larger houses than those without initial housing wealth.

After a short increase during the first five years (only for the tenant without initial housing wealth), the living space of a tenant keeps decreasing up to age 70 and remains at a low level until age 100. This is because as the age grows, where the tenants gain labour income. More and more of them are able to buy housing and become homeowners. The remained tenants are those who experience low labour income due to the uncertainty (in other words, they are unlucky). Then they can only rent a small-sized house.

We then focus on the living space of homeowners ad then show the average living spaces among all investors including homeowners and tenants in Figure 4.9. The living space is calculated as $h_t - x_t$, which is the net space of the house owned minus the part let out. The average living space of all investors is calculated as the mean living space regardless of the homeownership.

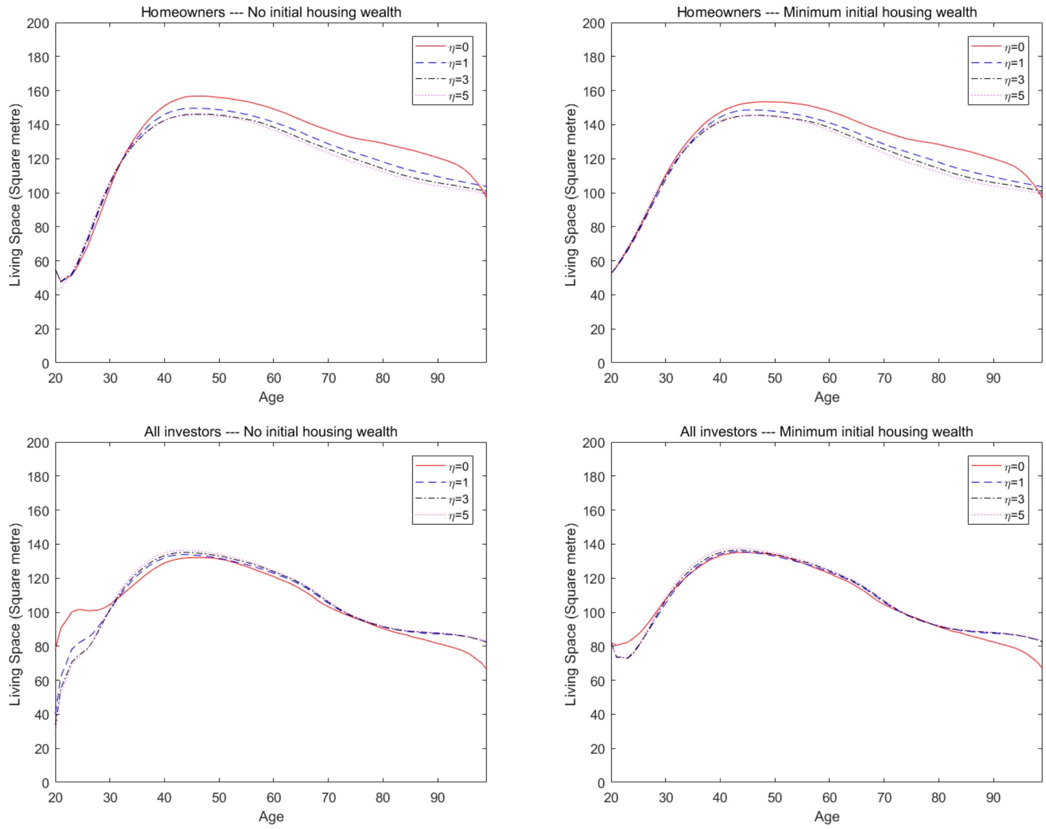

Figure 4.9: Living space (homeowners/all investors, bequest motive, initial housing wealth)

The four plots compare the average living area (square metre) over the entire life for homeowners/all investors for different degrees of bequest motive and initial housing wealth. At every age, the living space of a homeowner is $h_t - x_t$, which is the net space of the house owned minus the part let out. The average living space of all investors is calculated as the mean living space regardless of the homeownership.

Our first observation from the top two plots is that the initial housing wealth does not affect the living space considerably. Regardless of the initial housing wealth, homeowners always start with living spaces of around 50 square metres. That means, if the investor initially own a house with minimum size of 40 square metre, she will not upsize the living space too much and also not will not upsize the letting space as shown in Figure 4.5. Instead, she will use this amount of extra capital (compared to the one with no initial housing wealth) in non-durable consumption and accumulating liquid wealth.

Second, we find that on average, the homeowners without bequest motive have larger living spaces then those with bequest motive. This is mainly because there are more investors who are homeowners if they have bequest motive then those who have not. The increase in number of homeowners reduces the average value.

Combining the living spaces of tenants and homeowners, we get the bottom two plots in Figure 4.9. As we can see, the average living spaces of tenants and homeowners who have bequest motive are less than those without bequest motive. However, since investors with bequest motive have higher homeownership rates, the average living spaces of all investors are almost the same regardless of the bequest motive, except the first (last) ten years, where investors without bequest motive have higher (lower) living spaces. This indicates that overall, investor's living space is not so correlated with the degree of bequest motive during most time in life.

We also compare the living space predicted by our model with the latest data from US Census Bureau[7]. In 2017, the average living space per person in the US is around 96 square metre, where our model predicts around 11% larger average living space per person of 106 to 108 square metre during the entire life, depending on the degrees of bequest motive.

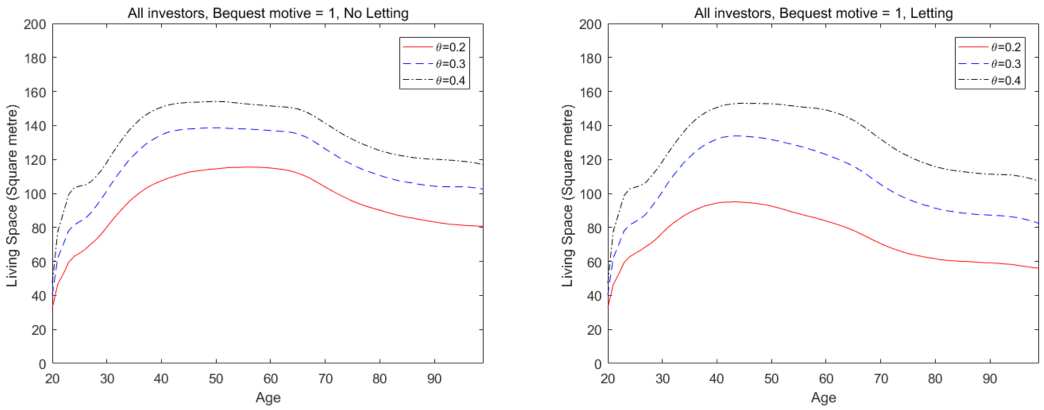

Figure 4.10: Living space (all investors, housing weight in utility of consumption, letting)

The four plots compare the average living area (square metre) over the entire life for all investors including homeowners and tenants with or without letting, and varying degrees of housing weight in the utility of consumption.

Next, we investigate the relationship between house weight in the consumption utility θ and living space over the life cycle. Moreover, we compare the results both with

[7]For further information, the reader is referred to `https://www.census.gov/construction/chars/pdf/c25ann2017.pdf`.

and without letting option. Since from Figure 4.9 we know that bequest motive does not affect the investor's living space much, we only show the results of homeowners with bequest motive ($i.e., \eta = 1$) in Figure 4.10.

We observe a clear positive correlation between the housing weight in the consumption utility θ and living space, which is straightforward to understand. Because the investor with higher θ prefers more housing consumption rather than non-durable consumption.

We also notice that with letting, the investor has less living space compared to the case without letting. The higher θ is, the less living space the investor has. Therefore, letting helps us to match the data in terms of the living space. Without letting, the average living space of an investor during the life is 116 square metre with our benchmark choice of $\theta = 0.3$, while if the investor can let, the average living space reduces to 105 square metre, which is closer to the data of 96 square metre.

In addition to this, we conduct several paired t-tests for different θ values with or without letting to investigate whether our findings are statistically significant. The results show that under 95% confidence level, the investor has a larger (smaller) living space if θ is higher (lower) at all time periods $t = 1, \dots, T$. It is also statistically significant under 95% confidence level that the investor has a larger living space if she cannot let during the life except in some particular time periods in early ages. For instance, $t = 4, 6$ when $\theta = 0.2$, $t = 1$ when $\theta = 0.3$ and $t = 1, 2, 3, 5, 12$ when $\theta = 0.4$. The reason for these exceptions is that in early ages, the investor cannot let much due to the limited size of house owned.

4.5.3 Non-durable Consumption versus Saving

Finally, we would like to investigate investor's non-durable consumption and saving strategies given the investor has made the optimal housing decisions. In particular, we want to see what letting can improve investor's non-durable consumption and saving strategies. In Figure 4.11, we show how non-durable consumption and liquid wealth (cash-on-hand) vary over the life cycle, given the letting option, different degrees of bequest motive and housing weights in consumption utility.

The top left figure shows how letting affects investor's non-durable consumption. First, if letting is not allowed, the increase in the housing weight in consumption utility θ mainly reduces investor's non-durable consumption before retirement (by comparing the cases of '$\theta = 0.2$, no letting' and '$\theta = 0.4$, no letting'). However, if the investor is allowed to let, the investor's non-durable consumption with lower θ dominates that with higher θ (by comparing the cases of '$\theta = 0.2$, letting' and '$\theta = 0.4$, letting'). This implies that letting mainly benefits investor's non-durable consumption during the retirement period, where the investor cannot earn labour income.

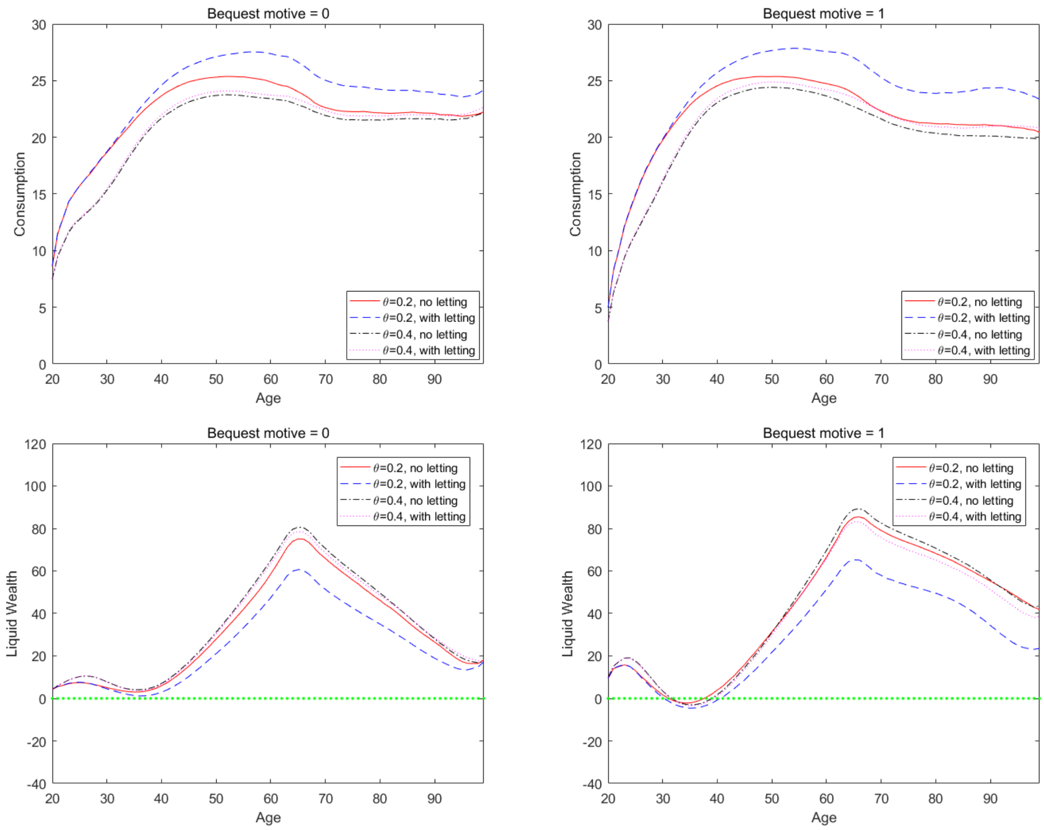

Figure 4.11: Non-durable consumption and liquid wealth (letting, bequest motive, housing weight in consumption utility

The four plots compare the non-durable consumption and liquid wealth over the entire life with/without bequest motive or letting option and varying degrees of housing weight in utility of consumption.

By comparing the top two figures, we also have two observations. First, in the case of no letting, the investor consumes less non-durable goods during the retirement period if she has bequest motive. This is because for the investor who has bequest motive, she wants to accumulate more liquid or housing wealth during the retirement as a bequest. Second, if the investor can let and if $\theta = 0.2$ (which means the investor is willing to let), the investors consume almost the same amount of non-durable goods during the retirement period, no matter whether she has or has not the bequest motive (by comparing the two lines of '$\theta = 0.2$, letting' in left and right two figures). This indicates that letting is more beneficial to the investor who has bequest motive. Because they can keep the non-durable consumption level during retirement although the investor is accumulating wealth for bequest purpose at that time.

From the bottom two figures, we notice that letting reduces the accumulation of liquid wealth during most periods in life, especially during the retirement period for the investor who has bequest motive. This also shows that although the investor gains extra incomes from letting, she will not use this amount of capital to accumulate liquid wealth but mainly consume non-durable goods.

4.6 Conclusions

In this chapter, we formulate a life-cycle consumption and housing model that incorporates three kinds of housing decisions: renting, owning and letting (conditional on the homeownership). Besides the housing decisions, the investor also needs to choose the amount to consume non-durable goods and the remaining wealth is saved as liquid wealth (cash-on-hand).

We mainly investigate three kinds of tradeoffs among those decisions. First, how does the investor choose between renting and owning a house? Second, if the investor owns a house, how does she balance between living in the housing and letting a portion of it (i.e, downsizing the living space) to gain extra rental income? At last, after the investor has made the housing decisions, how does she make non-durable consumption and saving decisions, both in the absence and presence of letting option?

We find that the investor with higher bequest motive and EIS is more willing to own a house rather than rent one, while the housing weight in the consumption utility has the opposite effect. On the other hand, letting and the borrowing limit from house equity helps to make up the life-cycle pattern of homeownership rate. The investor who can let or has a higher borrowing limit is more (less) willing to own a house before (after) retirement, compared to the one who cannot let or has a lower borrowing limit.

The homeowner is more willing to let out rather than living in the home during the life if she has a small preference of housing in the utility of consumption while the degree of bequest motive and EIS mainly affect the letting decision during the retirement period. The homeowner with no bequest motive has hump-shaped life-cycle letting pattern. She has the highest letting willingness at around age 70, just after the retirement age 65. In comparison, the homeowner who has bequest motive still keeps increasing the letting willingness after age 70. Similar effect is seen from a higher EIS value.

The homeowner mainly uses the rental income from letting in consuming non-durable goods rather than accumulating liquid wealth, regardless of bequest motive. To be more specific, letting can benefit the investor's non-durable consumption during the retirement period, where the investor does not have labour income. Moreover, for the investor who has bequest motive, letting helps to keep the non-durable consumption level while she is accumulating liquid wealth for the purpose of bequest.

Appendix

4.A Derivation of the First-order Optimality Conditions

In this section, we derive the first-order optimality conditions for the objective function with respect to each control variable c_t, h_t, x_t, z_t (first-order conditions for the linear inequality constraints are straightforward) at each time period t.

Time t=T:

$$\begin{aligned}
\frac{\partial J_T}{\partial c_T} =&(1-1/\psi)\Big\{(1-\theta)(1-\delta)f(h_T,x_T)^{\theta(1-1/\psi)}c_T^{(1-\theta)(1-1/\psi)-1} \\
&-\delta\eta\mathbb{E}_T\big[r^f(\frac{\theta}{(1-\theta)\tilde{q}_T})^{\theta(1-1/\psi)}(w_{T+1}+e(h_{T+1}))^{-1/\psi}\big]\Big\}, \\
\frac{\partial J_T}{\partial h_T} =&(1-1/\psi)\Big\{\theta(1-\delta)f(h_T,x_T)^{\theta(1-1/\psi)-1}c_T^{(1-\theta)(1-1/\psi)}\frac{\partial f(h_T,x_T)}{\partial h_T} \\
&+\delta\eta\mathbb{E}_T\big[(\frac{\theta}{(1-\theta)\tilde{q}_T})^{\theta(1-1/\psi)}(w_{T+1}+e(h_{T+1}))^{-1/\psi}(r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_T}+\frac{\partial e(h_{T+1})}{\partial h_T})\big]\Big\}, \\
\frac{\partial J_T}{\partial x_T} =&(1-1/\psi)\Big\{\theta(1-\delta)f(h_T,x_T)^{\theta(1-1/\psi)-1}c_T^{(1-\theta)(1-1/\psi)}\frac{\partial f(h_T,x_T)}{\partial x_T} \\
&+\delta\eta\mathbb{E}_T\big[(\frac{\theta}{(1-\theta)\tilde{q}_T})^{\theta(1-1/\psi)}(w_{T+1}+e(h_{T+1}))^{-1/\psi}(r^f(\frac{\partial g(h_t\mid h_{t-1})}{\partial x_T}+\frac{\partial\bar{g}(x_t)}{\partial x_T}))\big]\Big\}, \\
\frac{\partial J_T}{\partial z_T} =&\delta\eta\mathbb{E}_T\big[\kappa(\frac{\theta}{(1-\theta)\tilde{q}_T})^{\theta(1-1/\psi)}(w_{T+1}+e(h_{T+1}))^{-1/\psi}\big]\Big\}.
\end{aligned}$$

Time t=1, . . ., T − 1:

$$
\begin{aligned}
\frac{\partial J_t}{\partial c_t} =&(1-1/\psi)\Big\{(1-\theta)(1-\delta)f(h_t)^{\theta(1-1/\psi)}c_t^{(1-\theta)(1-1/\psi)-1} \\
&-\delta r^f\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}\frac{\partial J_{t+1}}{\partial w_{t+1}}+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}\big]\Big\}, \\
\frac{\partial J_t}{\partial h_t} =&(1-1/\psi)\Big\{\theta(1-\delta)f(h_t)^{\theta(1-1/\psi)-1}c_t^{(1-\theta)(1-1/\psi)}\frac{\partial f(h_t)}{\partial h_t} \\
&+\delta\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}\frac{\partial J_{t+1}}{\partial w_{t+1}}r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_t} \\
&+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}(r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_t}+\frac{\partial e(h_{t+1})}{\partial h_t})\big]\Big\}, \\
\frac{\partial J_t}{\partial x_t} =&(1-1/\psi)\Big\{\theta(1-\delta)f(h_t)^{\theta(1-1/\psi)-1}c_t^{(1-\theta)(1-1/\psi)}\frac{\partial f(h_t)}{\partial h_t} \\
&+\delta\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}\frac{\partial J_{t+1}}{\partial w_{t+1}}r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_t} \\
&+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}(r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial x_t}+\frac{\partial \bar{g}(x_t)}{\partial x_t})\big]\Big\}, \\
\frac{\partial J_t}{\partial z_t} =&\delta\kappa\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}\frac{\partial J_{t+1}}{\partial w_{t+1}}+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}\big]\Big\}.
\end{aligned}
\tag{4.11}
$$

By taking the partial derivative with respect to w_t of Model 4.9, we get:

$$
\begin{aligned}
\frac{\partial J_t}{\partial w_t} =&(1-1/\psi)\Big\{\delta r^f\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}\frac{\partial J_{t+1}}{\partial w_{t+1}}+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}\big]\Big\}.
\end{aligned}
$$

Then we obtain the envelop condition as:

$$
\frac{\partial J_t}{\partial w_t} =(1-1/\psi)\Big\{(1-\theta)(1-\delta)f(h_t)^{\theta(1-1/\psi)}c_t^{(1-\theta)(1-1/\psi)-1}\Big\}. \tag{4.12}
$$

Substitute (4.12) into (4.11), first order conditions can be rewritten as ($\frac{\partial J_{t+1}}{\partial w_{t+1}}$ is eliminated):

$$
\begin{aligned}
\frac{\partial J_t}{\partial c_t} =&(1-1/\psi)\Big\{(1-\theta)(1-\delta)f(h_t)^{\theta(1-1/\psi)}c_t^{(1-\theta)(1-1/\psi)-1} \\
&-\delta r^f\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}(1-\theta)(1-\delta)f(h_{t+1})^{\theta(1-1/\psi)}c_{t+1}^{(1-\theta)(1-1/\psi)-1} \\
&+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}\big]\Big\}, \\
\frac{\partial J_t}{\partial h_t} =&(1-1/\psi)\Big\{\theta(1-\delta)f(h_t)^{\theta(1-1/\psi)-1}c_t^{(1-\theta)(1-1/\psi)}\frac{\partial f(h_t)}{\partial h_t} \\
&+\delta\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}(1-\theta)(1-\delta)f(h_{t+1})^{\theta(1-1/\psi)}c_{t+1}^{(1-\theta)(1-1/\psi)-1}r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_t} \\
&+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}(r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_t}+\frac{\partial e(h_{t+1})}{\partial h_t})\big]\Big\}, \\
\frac{\partial J_t}{\partial x_t} =&(1-1/\psi)\Big\{\theta(1-\delta)f(h_t)^{\theta(1-1/\psi)-1}c_t^{(1-\theta)(1-1/\psi)}\frac{\partial f(h_t)}{\partial h_t} \\
&+\delta\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}(1-\theta)(1-\delta)f(h_{t+1})^{\theta(1-1/\psi)}c_{t+1}^{(1-\theta)(1-1/\psi)-1}r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial h_t} \\
&+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}(r^f\frac{\partial g(h_t\mid h_{t-1})}{\partial x_t}+\frac{\partial \bar{g}(x_t)}{\partial x_t})\big]\Big\}, \\
\frac{\partial J_t}{\partial z_t} =&\delta\kappa\mathbb{E}_t\big[p_{t+1}J_{t+1}^{1-\gamma}+\eta(1-p_{t+1})((w_{t+1}+e(h_{t+1}))(\frac{\theta}{(1-\theta)\tilde{q}_t})^{\theta})^{1-\gamma}\big]^{\frac{1-1/\psi}{1-\gamma}-1} \\
&\cdot\mathbb{E}_t\big[p_{t+1}J_{t+1}^{-\gamma}(1-\theta)(1-\delta)f(h_{t+1})^{\theta(1-1/\psi)}c_{t+1}^{(1-\theta)(1-1/\psi)-1} \\
&+\eta(1-p_{t+1})(\frac{\theta}{(1-\theta)\tilde{q}_{t+1}})^{\theta(1-\gamma)}(w_{t+1}+e(h_{t+1}))^{-\gamma}\big]\Big\}.
\end{aligned}
$$

Notice that, the first-order optimality conditions presented above involve calculations of the expected values $\mathbb{E}_t\big[\dots\big]$. In order to compute these expectations approximately, one can follow a regression-based method introduced by Koijen et al. [2009].

Chapter 5

Summary of Thesis and Future Work

In this thesis, we study three main life-cycle problems focusing on different aspects in terms of model ingredients, investor's subjective preferences and uncertainties.

We first investigate a life-cycle consumption and asset allocation problem introducing habit formation preferences and demand for term life insurance. We consider an investor who is not only risk-averse but also averse to ambiguity about stock returns in a robust optimization framework. Our key findings are as follows. There are several factors that affect the term life insurance demand. Similar to the bequest motive, ambiguity aversion also increases the demand for term life insurance. If one takes into account other realistic features such as subjective survival beliefs and borrowing opportunities, one would expect that investors should buy term life insurance more than they actually do. Our model potentially explains why this is the case. We show that habit formation leads to less term life insurance demand and plays a first-order role compared to other factors.

Moreover, we find that investor's stock allocation largely depends on the degree of ambiguity aversion. While earlier literature shows that an ambiguity-averse investor allocates less wealth in stock, we add to this evidence by showing that this relation is monotonically decreasing in the degree of ambiguity aversion. This is precisely important, because we only observe an effect of habit formation on asset allocation when the degree of ambiguity aversion is high.

Then we study a life-cycle consumption and asset allocation problem with labour income ambiguity. We also analyse the effects of the correlation between stock return and labour income and stock market predictability on investor's life-cycle decisions, in the presence of ambiguity aversion. We obtain several important findings that contribute to the literature. The ambiguity aversion towards labour income substantially increases investor's saving motive. In other words, the investor accumulates more wealth (by sacrificing early consumption). The results still hold if the investor is also ambiguity-averse towards the stock return.

By incorporating the ambiguity aversion towards both uncertainties, we obtain a stock allocation pattern closer to real data in terms of the low level and the hump shape over the life cycle. We can also explain part of the strong (retirement) saving motive in data. Moreover, our computational results confirm that the presence of stock market predictability makes the investor's life-cycle consumption and wealth pattern smoother, which is hardly documented in literature.

In the third life-cycle problem, we focus on the investor's consumption and housing decisions over the life cycle. We introduce the letting decisions (given the homeownership) into the model, along with renting and housing, which have been incorporated in previous life-cycle models. We mainly investigate three kinds of tradeoffs among life-cycle decisions and our findings can be summarised as follows. First, we find that the investor with higher bequest motive and EIS is more willing to own a house rather than rent one, while the housing wealth in the consumption utility has the opposite effect. On the other hand, letting and the borrowing limit from house equity helps to make up the life-cycle pattern of homeownership rate. The investor who can let or has a higher borrowing limit is more (less) willing to own a house before (after) the retirement, compared to the one who cannot let or has a lower borrowing limit.

Second, the homeowner is more willing to let out rather than living in the house/flat during the life if she has a small preference of housing in the utility of consumption while the degree of bequest motive and EIS mainly affect the letting decision during the retirement period. The homeowner with no bequest motive has hump-shaped life-cycle letting pattern. She has the highest letting willingness at around age 70, just after the retirement age 65. In comparison, the homeowner who has bequest motive still keeps increasing the letting willingness after age 70. Similar effect is seen from a higher EIS value.

Furthermore, the homeowner mainly uses the rental income from letting in consuming non-durable goods rather than accumulating liquid wealth, regardless of bequest motive. To be more specific, letting can benefit the investor's non-durable consumption during the retirement period, where the investor does not have labour income. Moreover, for the investor who has bequest motive, letting helps to keep the non-durable consumption level while she is accumulating liquid wealth for the purpose of bequest.

There are several other research questions that we have not investigated in this thesis but remain interesting and valuable for future work. For instance, in the presence of letting, how will the investor optimally choose between investing in housing wealth and financial assets such as stocks and annuities, especially after retirement. What are the implications of ambiguity aversion and habit formation in housing consumption on housing decisions including renting, owning and letting?

Throughout the thesis, we use calibrations widely applied in the literature to define the economic-related parameters and processes such as stock return and labour income. Alternatively, one can collect data related to different groups of the population and estimate the corresponding parameters and processes. This allows us to investigate the life-cycle decision making problems of different population coming from different backgrounds (e.g., different countries and groups of investors).

Moreover, besides the traditional dynamic programming, scenario-based stochastic programming and robust optimization, some recent developed methods such as decision rule approach [e.g., Ben-Tal et al., 2004; Calafiore, 2008; Kuhn et al., 2011], distributionally robust optimization [e.g., Calafiore and Ghaoui, 2006; Delage and Ye, 2010; Goh and Sim, 2010], approximate dynamic programming [e.g., Powell, 2007] may be also suitable in solving the life-cycle problems.

Finally, some other uncertainties such as interest rates and mortgage rates are often assumed to be certain or follow distributions (e.g., normal distribution) and time-independent. One may consider more realistic models for describing these uncertainties and show the implications on life-cycle decisions.

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