KAC-MOODY GROUPS AND COMPLETIONS

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In memory of Kay Magaard

Abstract. In this paper we construct a new "pro-p-complete" topological Kac-Moody group and compare it to various known topological Kac-Moody groups. We come across this group by investigating the process of completion of groups with BN-pairs. We would like to know whether the completion of such a group admits a BN-pair. We give explicit criteria for this to happen.

In this paper we study Kac-Moody groups over finite fields.

A Kac-Moody group is a generalisation of the notion of a reductive group to a more general Kac-Moody root datum $\mathcal{D}$ or a closely related group with a BN-pair. Connected reductive groups are classified via a one-to-one correspondence to root data of finite type. A root datum of finite type yields a group scheme, generalised by Tits to a construction of a functor $G_{\mathcal{D}}$ from the category of commutative rings to the category of groups [T2, T3]. For instance, the Steinberg central extension $\widehat{\mathrm{SL}_n(\mathbb{F}_q[z,z^{-1}])}$ is a Kac-Moody group $G_{\mathcal{D}}(\mathbb{F}_q)$ for the simply-connected root datum $\mathcal{D}$ of the affine type $\widetilde{A}_{n-1}$ (see [CKRu, sec. 6] for further details). On the other hand, $\mathrm{SL}_n(\mathbb{F}_q[z,z^{-1}])$ is not of the form $G_{\mathcal{D}}(\mathbb{F}_q)$, yet it is still called a Kac-Moody group.

A topological Kac-Moody group is a locally compact totally disconnected topological group that contains a Kac-Moody group. Often it is obtained by completion of a Kac-Moody group. In the examples of the previous paragraph, one arrives at topological Kac-Moody groups $\widehat{\mathrm{SL}_n(\mathbb{F}_q((z)))}$ and $\mathrm{SL}_n(\mathbb{F}_q((z)))$.

There are several known topological Kac-Moody groups.[1] They are the Mathieu-Rousseau group G^{ma+}, the Carbone-Garland group $G^{c\lambda}$ and the Caprace-Rémy-Ronan group G^{crr}. Each of them contains $G_{\mathcal{D}}(\mathbb{F}_q)$. In this paper we show the existence of a new topological Kac-Moody group $\widehat{G}$.

Main Theorem. *Let A be an irreducible generalised Cartan matrix, $\mathcal{D}$ a simply connected root datum of type A and $\mathbb{F}=\mathbb{F}_q$ a finite field of characteristic p ($q=p^a$, $a\in\mathbb{N}$). Let $G{:=}G_{\mathcal{D}}(\mathbb{F}_q)$ be the corresponding minimal Kac-Moody group. Recall that it has a BN-pair (B,N) with $B=U\rtimes T$ where $U=\langle X_\alpha \mid \alpha\in\Delta_+^{re}\rangle$ (see Section 2, where the notations are introduced).*

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[1]Since this paper a new book by Marquis [M3] was published, which would be a comprehensive source for further reading on the subject.

There exists a locally compact totally disconnected group $\widehat{G}$ satisfying the following conditions:

1. *G is a dense subgroup of $\widehat{G}$.*
2. *$\widehat{G}$ has a BN-pair $(\widehat{B}, N)$ where $\widehat{B} = \widehat{U} \rtimes T$ and $\widehat{U}$ is the full pro-p completion of U.*
3. *If $\overline{G} = G^{crr}$ or $C^{c\lambda}$, or in the case when G is dense in G^{ma+} and $\overline{G} = G^{ma+}$, then there exists an open continuous surjective homomorphism $\widehat{G} \twoheadrightarrow \overline{G}$.*
4. *Let $Z'(\widehat{G}) := Z(G) \times C(\widehat{G})$ where $C(\widehat{G}) = \cap_{g \in \widehat{G}} \widehat{U}^g$.*
   (a) *$\widehat{G}/Z'(\widehat{G})$ is a topologically simple group.*
   (b) *If A is 2-spherical, then $\widehat{G}/Z'(\widehat{G})$ is an abstractly simple group.*

Let us explain the content of the present paper. We investigate the process of completion of a group G with a BN-pair in Chapter 1. The main result is Theorem 1.1 that contains a sufficient condition for the completion $\widehat{G}$ to inherit a BN-pair from G. It relies on Tits' description of groups with BN-pairs by generators and relations [T1]. The remainder of Chapter 1 contains several technical or user-friendly results about these completions. For instance, Theorem 1.2 conveniently constructs the completion $\widehat{G}$ together with its BN-pair.

We put our completion results to good use in Chapter 2. After quickly recalling the definition of $G_{\mathcal{D}}(\mathbb{F}_q)$ we construct the new group $\widehat{G_{\mathcal{D}}(\mathbb{F}_q)}$. We compare it to other known completions and address its topological simplicity in Theorem 2.2 and its algebraic simplicity in Theorem 2.4. Thus, the Main Theorem is a combination of results in Section 2.

There have been previous attempts to compare various topological Kac-Moody groups: Capdeboscq and Rémy [CR], Baumgartner and Rémy [CarERi, 2.6], Marquis [M2], and Rousseau [Rou] all discuss the maps between different completions at length. We address these questions in Section 2 only modulo "congruence kernel" $Z'(\widehat{G})$ whose full computation remains mysterious. We devote the last chapter of the paper to several observations about $Z'(\widehat{G})$. Our major insight into the nature of the congruence kernel is its parabolic decomposition in Theorem 3.5.

Hristova and Rumynin study representations of topological Kac-Moody groups [HrRu]. The groups $\widehat{G}$ are new examples for their theory.

1. Completion Theorem

Let $(G, \mathcal{T})$ be a Hausdorff topological group (G is a group, $\mathcal{T}$ is a topology). The topology determines a right uniformity on G that we now describe following Bourbaki [B1]. Pick a basis $\mathcal{J}$ of the topology at 1. The basis of uniformity $\mathcal{T}^\diamondsuit$ is $\mathcal{J}^\diamondsuit = \{V^\diamondsuit | V \in \mathcal{J}\}$ where $V^\diamondsuit = \{(x, y) \in G^2 | xy^{-1} \in V\}$. The completion $\widehat{G}$ is the set of all minimal Cauchy filters on $(G, \mathcal{T}^\diamondsuit)$. Recall that a filter is a non-empty collection $\mathcal{F}$ of open sets closed under intersections and oversets. A filter is Cauchy if it contains arbitrary "small" subsets, i.e., for each $V \in \mathcal{J}$ there exists $U \in \mathcal{F}$ such that $xy^{-1} \in V$ for all $x, y \in U$. We define the left uniformity ${}^\diamondsuit\mathcal{T}$ in a similar way, using ${}^\diamondsuit V = \{(x, y) \in G^2 | x^{-1}y \in V\}$ instead. The inverse map is an isomorphism of uniform spaces $\mathrm{Inv} : (G, \mathcal{T}^\diamondsuit) \to (G, {}^\diamondsuit\mathcal{T})$ imposing an isomorphism between the right and left completions.

The completion $\widehat{G}$ is always a monoid, although the multiplication $\mathrm{Mult} : (G,\mathcal{T}^{\diamondsuit})\times(G,\mathcal{T}^{\diamondsuit}) \to (G,\mathcal{T}^{\diamondsuit})$ is not uniformly continuous in general. It is a monoid because $\mathrm{Mult}(\mathcal{F},\mathcal{G})$ is a Cauchy filter for two Cauchy filters $\mathcal{F}$, $\mathcal{G}$ on $(G,\mathcal{T}^{\diamondsuit})$ [B1, Prop. III.3.4(6)]. On the other hand, the completion is not necessarily a group: $\mathrm{Inv} : (G,\mathcal{T}^{\diamondsuit}) \to (G, {}^{\diamondsuit}\mathcal{T})$ is uniformly continuous but we have no information about uniform continuity of $\mathrm{Inv} : (G,\mathcal{T}^{\diamondsuit}) \to (G,\mathcal{T}^{\diamondsuit})$. The latter uniform continuity is a sufficient (but not necessary) condition for $\widehat{G}$ to be a group. A necessary and sufficient condition is the following: if $\mathcal{F}$ is a Cauchy filter on $(G,\mathcal{T}^{\diamondsuit})$, then $\mathrm{Inv}(\mathcal{F})$ is a Cauchy filter on $(G,\mathcal{T}^{\diamondsuit})$ [B1, Th. III.3.4(1)].

For reader's convenience we sketch an example of a group G with non-a-group $\widehat{G}$ following a hint in Bourbaki [B1, Exercise X.3(16)]. Let G be the group of auto-homeomorphisms of $[0,1]$ with the topology of uniform convergence. It suffices to exhibit a uniformly convergent sequence f_m of homeomorphisms such that the sequence of inverses f_m^{-1} is not uniformly convergent. The following sequence fits the bill:

$$f_m(x) = \begin{cases} x^m & \text{if } \ x \le \frac{1}{2} \\ (2-2^{1-m})x + (2^{1-m}-1) & \text{if } \ x \ge \frac{1}{2} \end{cases}$$

Now suppose that G admits a BN-pair (B,N). The key question is whether the completion $\widehat{G}$ admits a BN-pair. Let $\overline{B}$ be the closure of B in $\widehat{G}$. It is a moot point that $\overline{B}$ is isomorphic to the completion of B in the restriction uniformity $\mathcal{T}^{\diamondsuit}|_B$ [B1, Cor. II.3.9(1)]. A candidate BN-pair on $\widehat{G}$ is $(\overline{B},N)$ but it does not work in general. Let $\mathcal{G}$ be a simple split group scheme, $G=\mathcal{G}(\mathbb{F}[z,z^{-1}])$ its points over Laurent polynomials over a finite field, $N \le G$ the group of monomial matrices, $I_- = [\mathcal{G}(\mathbb{F}[z^{-1}]) \xrightarrow{z^{-1}\mapsto 0} \mathcal{G}(\mathbb{F})]^{-1}(B)$ its negative Iwahori. The pair (I_-,N) is a BN-pair on G but $(\overline{I_-},N)=(I_-,N)$ is not a BN-pair on the positive completion $\widehat{G}=\mathcal{G}(\mathbb{F}((z))\,)$: the countable groups I and N cannot generate uncountable $\widehat{G}$. A reader can see that the condition 3 of Theorem 1.1 fails for the negative Iwahori. On the other hand, if $\mathbb{F}$ is finite, all conditions of Theorem 1.1 holds for the positive Iwahori $I_+ = [\mathcal{G}(\mathbb{F}[z]) \xrightarrow{z\mapsto 0} \mathcal{G}(\mathbb{F})]^{-1}(B)$ so that the theorem yields the standard BN-pair on $\widehat{G}$.

Nevertheless, we can prove the following partial affirmative answer, sufficiently general for the study of Kac-Moody groups. Let us explain some notations before stating the theorem. The notations (B,N) and $W=(W,S)$ are standard for the groups with BN-pairs. The homomorphism $\pi : N \to W$ is the natural surjection. For elements $s\in S$, $w \in W$ we choose some liftings $\dot{s}\in\pi^{-1}(s)$, $\dot{w}\in\pi^{-1}(w)$. The minimal parabolic P_s is the subgroup generated by B and $\dot{s}$.

Theorem 1.1. *Let G be a Hausdorff topological group with a BN-pair (B,N) with the Weyl group (W,S) where S is finite. If the following three conditions hold, then $(\overline{B},N)$ is a BN-pair on the completed group $\widehat{G}$.*

1. *The completion $\widehat{G}$ is a group.*
2. *The index $|P_s : B|$ is finite for all $s\in S$.*
3. *B is open in G.*

Proof. We have systems of subgroups: $\mathfrak{A} = (B,N,P_s; s\in S)$ of G and $\mathfrak{B} = (\overline{B},N,\overline{P_s}; s\in S)$ of $\widehat{G}$, where $\overline{X}$ is the closure of X. The system of groups $\mathfrak{A}$ satisfies all conditions of Tits' Theorem as observed by Tits [T1]. We claim that under the assumptions of this theorem the system $\mathfrak{B}$ also satisfies these conditions.

We will verify this claim at the end of the proof. For the reader's convenience we restate Tits theorem (cf. [Ku, Th. 5.1.8]):

Tits Theorem. *Suppose that the system $\mathfrak{B}$ satisfies the following conditions:*

- $(\mathbf{P_1})$ *If $s \neq t \in S$, then $\overline{P_s} \cap \overline{P_t} = \overline{B}$.*
- $(\mathbf{P_2})$ *The subgroup $\overline{B} \cap N$ is normal in N.*
- $(\mathbf{P_3})$ *Given $s \in S$, let $N_s \coloneqq \overline{P_s} \cap N$. Then $N_s/(\overline{B} \cap N)$ is of order 2 for all $s \in S$.*
- $(\mathbf{P_4})$ *$\overline{P_s} = \overline{B} \cup \overline{B}\dot{s}\overline{B}$ for all $s \in S$.*
- $(\mathbf{P_5})$ *The pair $(N/(\overline{B} \cap N), S)$ is a Coxeter group.*
- $(\mathbf{P_6})$ *Let $\pi : N \to W \coloneqq N/(\overline{B} \cap N)$ be the quotient map. For any $n = \dot{s}_1\dot{s}_2 \cdots \dot{s}_t \in N$ with $s_i \in S$ such that $s_1 s_2 \cdots s_t$ is a reduced word in W, the subgroup $\overline{B}(s_1, \ldots, s_t)$ (see (1)) depends only on $w \coloneqq \pi(n) = s_1 \cdots s_t \in W$ and the homomorphism $\overline{\gamma}(\dot{s}_1, \ldots, \dot{s}_t)$ (see (2)) depends only on n. (This justifies the notation $\overline{B}_w$ and $\overline{\gamma}_n$ from now on.)*
- $(\mathbf{P_7})$ *If $w \in W$, $s \in S$ satisfy $l(ws) = l(w) + 1$, then $\overline{B}_w \overline{B}_s = \overline{B}_{ws}$.*
- $(\mathbf{P_8})$ *If $w \in W$, $s, t \in S$ satisfy $wtw^{-1} = s$ and $l(wt) = l(w) + 1$, then for all $x \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and $b \in \overline{B} \setminus \overline{B}_t$, there exist $y \in b\overline{B}_t \cap \overline{B}_n$ and $y', y^\sharp \in \overline{B}_n$ such that*
  (a) *$x'^{-1} y x' = y' x' y^\sharp$ in $\overline{P_t}$ and*
  (b) *$x \overline{\gamma}_n(y) x^{-1} = \overline{\gamma}_n(y') x^{-1} \overline{\gamma}_n(y^\sharp)$ in $\overline{P_s}$.*
  where $x' \coloneqq n^{-1} x^{-1} n \in \pi^{-1}(t)$.
- $(\mathbf{P_9})$ *$\overline{B}$ is not normal in any $\overline{P_s}$.*

Then the canonical map

$$N \cup (\cup_{s \in S} \overline{P_s}) \longrightarrow \widetilde{G} \coloneqq \mathop{*}_{\mathfrak{B}} H$$

to the amalgam $\widetilde{G}$ is injective. The amalgam $\widetilde{G}$ admits a BN-pair $(\overline{B}, N)$ with a set of simple reflections S, where we identify the groups $\overline{B}$ and N with their images in $\widetilde{G}$ under the canonical map.

Furthermore, consider a group G' and an injective function

$$\varphi : N \cup (\cup_{s \in S} \overline{P_s}) \longrightarrow G'$$

such that $\varphi|_N$ and all $\varphi|_{\overline{P_s}}$ are group homomorphisms. If G' is generated by the image of φ, then the canonical homomorphism $\varphi^\sharp : \widetilde{G} \to G'$ is an isomorphism.

Tits' Theorem applies to $G' = \widehat{G}$, allowing us to conclude that the group $\widetilde{G}$ admits a BN-pair $(\overline{B}, N)$ and the natural group homomorphism $f : \widetilde{G} \to \widehat{G}$ is injective.

It remains to check surjectivity of f. The image $f(\widetilde{G})$ contains N and B, which generate G. Hence, $f(\widetilde{G})$ is dense in $\widehat{G}$. On the other hand, $f(\widetilde{G})$ contains $\overline{B}$, which is open in $\widehat{G}$ because B is open in G. Thus, $f(\widetilde{G})$ is open in $\widehat{G}$ but it is a subgroup, hence, $f(\widetilde{G})$ is also closed in $\widehat{G}$. Being closed and dense, $f(\widetilde{G})$ must be equal to $\widehat{G}$.

It only remains to verify all nine conditions in Tits' Theorem for $\mathfrak{B}$. Our starting point is that these nine conditions hold in G for $\mathfrak{A}$ as shown by Tits [T1].

$(\mathbf{P_2})$ We know that $B \cap N$ is normal in N. Clearly, $B \cap N \subseteq \overline{B} \cap N$. In the opposite direction, $\overline{B} \cap N = (\overline{B} \cap G) \cap N$. An element $x \in \overline{B} \cap G$ is a limit of a net

$$x = \lim_{m \in \mathbb{M}} b_m , \quad b_m \in B, \quad \mathbb{M} \text{ is an ordinal.}$$

This limit works in G as well where B is open, hence, closed. Thus, $x \in B$ and $\overline{B} \cap G = B$. Therefore, $\overline{B} \cap N = B \cap N$ is normal in N.

($\mathbf{P_1}$) Let $s \neq t \in S$. Since $P_s = B\dot{s}B \cup B$, we conclude that $\overline{P_s} = \overline{B\dot{s}B} \cup \overline{B}$. Let us prove now that $\overline{B\dot{s}B} \cap \overline{B\dot{t}B} = \emptyset$. An element $x \in \overline{B\dot{s}B} \cap \overline{B\dot{t}B}$ is a limit of two nets

$$x = \lim_{m \in \mathbb{M}} a_m \dot{s} b_m = \lim_{m \in \mathbb{M}} c_m \dot{t} d_m \ , \quad a_m, b_m, c_m, d_m \in B.$$

Since $\overline{B}$ is open there exists an ordinal $\mathbb{L} < \mathbb{M}$ such that $a_m \dot{s} b_m x^{-1} \in \overline{B} \ni x(c_m \dot{t} d_m)^{-1}$ and consequently $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1} \in \overline{B}$ for all $m \geq \mathbb{L}$. Clearly $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1} \in G$. It is shown ($\mathbf{P_2}$) that $\overline{B} \cap G = B$, thus, $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1} \in B$ for all $m \geq \mathbb{L}$. On the other hand, these elements $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1}$ lie in $B\dot{s}B\dot{t}B$ equal to the double coset $B\dot{s}\dot{t}B$ since $l(\dot{s}\dot{t}) = 2 = l(\dot{s}) + l(\dot{t})$. Since $B\dot{s}B\dot{t}B \cap B = \emptyset$, no such x exists. Therefore, $\overline{P_s} \cap \overline{P_t} = \overline{B}$.

($\mathbf{P_3}$) The minimal parabolic P_s is a union of cosets of B, hence, open in G. Similarly to the proof in ($\mathbf{P_2}$), $N_s = \overline{P_s} \cap N$ is equal to $(\overline{P_s} \cap G) \cap N = P_s \cap N$. Therefore, $N_s/(\overline{B} \cap N)$ is of order 2 for all $s \in S$.

($\mathbf{P_4}$) By condition (2), B has finite index in P_s. Hence $B\dot{s}B = XB$ for some finite subset $X \subseteq P_s$. Observe now that

$$\overline{B}\dot{s}\overline{B} \subseteq \overline{B\dot{s}B} = \overline{XB} \overset{f}{\supseteq} X\overline{B} \subseteq \overline{B}\dot{s}\overline{B}.$$

Since X is finite, $X\overline{B}$ is closed and the inclusion f is an equality. Thus, $\overline{B\dot{s}B} = \overline{B}\dot{s}\overline{B}$. Therefore, $\overline{P_s} = \overline{B} \cup \overline{B\dot{s}B} = \overline{B} \cup \overline{B}\dot{s}\overline{B}$.

($\mathbf{P_5}$) We have proved in ($\mathbf{P_2}$) that $\overline{B} \cap N = B \cap N$. Therefore, $(N/\overline{B} \cap N, S) = (W, S)$ is a Coxeter group.

($\mathbf{P_6}$) Let us first observe that if $A, K \leq G$ are open subgroups, then $\overline{A} \cap \overline{K} = \overline{A \cap K}$ in $\widehat{G}$. Indeed, the inclusion $\supseteq$ is obvious. To prove the inclusion $\subseteq$ consider $x \in \overline{A} \cap \overline{K}$. It is a limit of two nets

$$x = \lim_{m \in \mathbb{M}} a_m = \lim_{m \in \mathbb{M}} k_m \ , \quad a_m \in A, \ k_m \in K.$$

Then the net $a_m k_m^{-1} = (a_m x^{-1})(x k_m^{-1})$ converges to $1 \in G$. Since $A \cap K$ is open there exists an ordinal $\mathbb{L} < \mathbb{M}$ such that $a_m k_m^{-1} \in A \cap K$, and consequently $a_m, k_m \in A \cap K$ for all $m \geq \mathbb{L}$. Thus, $x \in \overline{A \cap K}$.

The following subgroups are defined recursively for a reduced word $s_1 s_2 \cdots s_t \in W$ and its fixed lift $n = \dot{s}_1 \dot{s}_2 \cdots \dot{s}_t \in N$

$$B(s_1, \ldots, s_i) \coloneqq B \cap \dot{s}_i^{-1} B(s_1, \ldots, s_{i-1}) \dot{s}_i,$$

$$\overline{B}(s_1, \ldots, s_i) \coloneqq \overline{B} \cap \dot{s}_i^{-1} \overline{B}(s_1, \ldots, s_{i-1}) \dot{s}_i. \tag{1}$$

The aforementioned observation implies that $\overline{B}(s_1, \ldots, s_t) = \overline{B(s_1, \ldots, s_t)}$. Consequently, the homomorphism

$$\overline{\gamma}(\dot{s}_1, \ldots, \dot{s}_t) : \overline{B}(s_1, \ldots, s_t) \to \overline{B}, \quad x \mapsto \dot{s}_1 \cdots \dot{s}_t x \dot{s}_t^{-1} \cdots \dot{s}_1^{-1} \tag{2}$$

is uniquely determined by its restriction $\gamma(\dot{s}_1, \ldots, \dot{s}_t) : B(s_1, \ldots, s_t) \to B$. Property ($\mathbf{P_6}$) hold for $\mathfrak{A}$. This means that $B(s_1, \ldots, s_t)$ depends only on the element $w = s_1 \cdots s_t \in W$, not the word or the choice of the liftings $\dot{s}_i \in N$. It also means that $\gamma(\dot{s}_1, \ldots, \dot{s}_t)$ depends only on the element $n = \dot{s}_1 \cdots \dot{s}_t \in N$. Therefore, the subgroup $\overline{B}(s_1, \ldots, s_t)$ and the homomorphism $\overline{\gamma}(\dot{s}_1, \ldots, \dot{s}_t)$ depend only on $w \in W$ and $n \in N$ correspondingly. (We denote these $B_w, \overline{B}_w, \gamma_n, \overline{\gamma}_n$.)

($\mathbf{P_7}$) We begin by proving that all subgroups B_w, $w \in W$ are commensurable. We proceed by induction on the length $l(w)$ to show that B_w has finite index in B.

If $l(w) = 1$, then $w = s$ for some $s \in S$. Since $xB_s \mapsto x\dot{s}B$ is an embedding of quotient sets $B/B_s \hookrightarrow P_s/B$, we conclude that $|B : B_s| \leq |P_s : B| < \infty$ by assumption (2).

Suppose the case of $l(w) = m-1$ is settled. Consider $w \in W$, $s \in S$ with $l(ws) = m$. Then $B_{ws} = B \cap \dot{s}^{-1} B_w \, \dot{s}$. Hence,

$$|B : B_{ws}| = |B : B_s||(B \cap \dot{s}^{-1}B\dot{s}) : (B \cap \dot{s}^{-1}B_w\dot{s})| \leq |B : B_s||B : B_w| < \infty$$

since by induction assumption $|B : B_w| < \infty$.

Property ($\mathbf{P_7}$) for $\mathfrak{A}$ ensures that $B_w B_s = B$ if $l(ws) = l(w)+1$ as before. Hence,

$$\overline{B} = \overline{B_w B_s} \supseteq \overline{B_w}\ \overline{B_s} = \overline{B}_w\, \overline{B}_s$$

because $\overline{XY} \supseteq \overline{X}\ \overline{Y}$ for all subsets X, Y and $\overline{B_w} = \overline{B}_w$ for all $w \in W$ as shown in ($\mathbf{P_6}$). Since B_w and B_s are commensurable, $B_w B_s = X B_s$ for a finite subset $X \subseteq B_w$. Then

$$\overline{B_w B_s} = \overline{XB_s} = X\overline{B_s} \subseteq \overline{B}_w\, \overline{B}_s$$

since $X\overline{B_s}$ is closed as a finite union of closed cosets of $\overline{B_s}$. Therefore, $\overline{B}_w\, \overline{B}_s = \overline{B}$.

($\mathbf{P_8}$) Let $s, t \in S$, $w \in W$ such that $wtw^{-1} = s$ and $l(wt) = l(w)+1$. Let us fix arbitrary $x \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and define $x' \coloneqq n^{-1}x^{-1}n \in \pi^{-1}(t)$. Now pick any $b \in \overline{B} \setminus \overline{B}_t$. As shown in ($\mathbf{P_7}$), $B = XB_t$ for a finite set X. Without loss of generality, $1 \in X$ and $B \setminus B_t = (X \setminus \{1\})B_t$. By the argument as above ($X\overline{B_t}$ is closed etc.), $\overline{B} \setminus \overline{B}_t = (X \setminus \{1\})\overline{B}_t$. Hence, $b = b_1 b_2$ for some $b_1 \in X \setminus \{1\}$ and $b_2 \in \overline{B}_t$. This brings property ($\mathbf{P_8}$) down to the system $\mathfrak{A}$ where we know it [T1]. Therefore, there exist elements $y \in b_1 B_t \cap B_w \subseteq b\overline{B}_t \cap \overline{B}_w$ and $y', y^\sharp \in B_w \subseteq \overline{B}_w$ satisfying $x'^{-1}yx' = y'x'y^\sharp \in P_t \subseteq \overline{P_t}$ and $x\gamma_n(y)x^{-1} = \gamma_n(y')x^{-1}\gamma_n(y^\sharp) \in P_s \subseteq \overline{P_s}$.

($\mathbf{P_9}$) Recall that $\overline{B} \cap G = B$ and $\overline{P_s} \cap G = P_s$ as shown in ($\mathbf{P_2}$). If $\overline{B}$ were normal in $\overline{P_s}$, then B would be normal in P_s, contradicting ($\mathbf{P_9}$) for $\mathfrak{A}$. Therefore, $\overline{B}$ is not normal in $\overline{P_s}$. □

A shortcoming of Theorem 1.1 is that it requires the group $\widehat{G}$ to exist first. It would be useful to tweak the theorem to enable construction of new groups. The next theorem addresses this issue. If $\mathcal{T}$ is a topology on a group B, we denote $\mathcal{T}_1 \coloneqq \{A \in \mathcal{T} \mid 1 \in A\}$.

Theorem 1.2. *Let G be a group with a BN-pair (B, N) with the Weyl group (W, S) where S is finite. Suppose further that a topology $\mathcal{T}$ on B is given such that the four conditions (1)–(4) hold.*

(1) *$(B, \mathcal{T})$ is a topological group.*
(2) *The completion $\widehat{B}$ is a group.*
(3) *$\mathcal{T}_1$ is a basis at 1 of topology on each minimal parabolic P_s, $s \in S$ that defines a structure of topological group on P_s.*
(4) *The index $|P_s : B|$ is finite for each $s \in S$.*

Under these conditions the following statements hold:

(a) *$\mathcal{T}_1$ is a basis at 1 of topology on G that defines a structure of topological group on G.*
(b) *The completion $\widehat{G}$ is a group and $\widehat{B} = \overline{B}^{\subseteq \widehat{G}}$.*

(c) *The completion* $\widehat{G}$ *is isomorphic to the amalgam* $\underset{\mathfrak{B}}{*} H$ *where* $\mathfrak{B} = \{\overline{B}, N, \overline{P_s}; s \in S\}$.

(d) *The pair* $(\overline{B}, N)$ *is a BN-pair on the completed group* $\widehat{G}$.

Proof. **(a)** We already know that $\mathcal{T}_1$ is a filter of neighbourhoods of 1 in a topological group B. To verify (a) it suffices to show that for all $g \in G$, $A \in \mathcal{T}_1$ it holds that $gAg^{-1} \in \mathcal{T}_1$ [B1, Prop. III.1.2(1)]. By (3) we know this property for all $g \in P_s$. Since G is generated by all P_s, we conclude the proof.

(b) Denote the aforementioned topology on G by $\mathcal{T}_G$. We need to show that the monoid $\widehat{(G, \mathcal{T}_G)}$ is a group. Consider $x \in \widehat{(G, \mathcal{T}_G)}$ and a convergent net $x_m \longrightarrow x$, $m \in \mathbb{M}$, $x_m \in G$. Since $B \in \mathcal{T}_1$, B is open in $(G, \mathcal{T}_G)$. The net x_m is Cauchy, so there exists an ordinal $\mathbb{L}$ such that $x_m x_{\mathbb{L}}^{-1} \in B$ for all $m \geq \mathbb{L}$. Let

$$y_m = \begin{cases} 1 & \text{if } \ m < \mathbb{L}, \\ x_m x_{\mathbb{L}}^{-1} & \text{if } \ m \geq \mathbb{L}. \end{cases}$$

Since $y_m y_l^{-1} = (x_m x_{\mathbb{L}}^{-1})(x_l x_{\mathbb{L}}^{-1})^{-1} = x_m x_l^{-1}$, the net y_m is a Cauchy net in B. Let $y = \lim y_m \in \widehat{B}$. The inverse $y^{-1} \in \widehat{B}$ exists because $\widehat{B}$ is a group. Then

$$(x_{\mathbb{L}}^{-1} y^{-1}) \cdot x = x_{\mathbb{L}}^{-1} \cdot \lim_m y_m^{-1} \cdot \lim x_m = x_{\mathbb{L}}^{-1} \cdot \lim_m y_m^{-1} x_m = x_{\mathbb{L}}^{-1} \cdot \lim_m x_{\mathbb{L}} = 1.$$

Similarly, $x \cdot (x_{\mathbb{L}}^{-1} y^{-1})$. Thus, we have found the inverse $x^{-1} = x_{\mathbb{L}}^{-1} y^{-1}) \in \widehat{G}$ so that $\widehat{B}$ is a group. Coincidence of the completion and the closure is standard.

(c+d) These follow immediately from Theorem 1.1. □

Suppose that a group G admits two topological group structures $(G, \mathcal{S})$ and $(G, \mathcal{T})$ such that $\mathcal{S} \subseteq \mathcal{T}$. Then the identity map $\mathrm{Id} : (G, \mathcal{T}) \to (G, \mathcal{S})$ is a homomorphism of topological groups that admits a unique extension $\widehat{\mathrm{Id}} : \widehat{(G, \mathcal{T})} \to \widehat{(G, \mathcal{S})}$ [B1, Prop. III.3.4(8)]. This extension $\widehat{\mathrm{Id}}$ may or may not be injective in general. Ditto for surjective [B1, Exercise III.3(12)]. However, we can give nice criteria for surjectivity and injectivity for the topologies we are interested in.

Corollary 1.3. *Consider a group* G *with a BN-pair* (B, N) *that admits two topological group structures* $(G, \mathcal{S})$ *and* $(G, \mathcal{T})$ *such that* $\mathcal{S} \subseteq \mathcal{T}$. *Suppose* $B \in \mathcal{S}$. *Then the kernel of* $\widehat{\mathrm{Id}} : \widehat{(G, \mathcal{T})} \to \widehat{(G, \mathcal{S})}$ *is equal to the kernel of* $\widehat{\mathrm{Id}_B} : \widehat{(B, \mathcal{T}_B)} \to \widehat{(B, \mathcal{S}_B)}$.

Proof. Clearly, $\ker(\widehat{\mathrm{Id}}) \supseteq \ker(\widehat{\mathrm{Id}_B})$. In the opposite direction, consider $x \in \ker(\widehat{\mathrm{Id}})$. This element is a limit of a Cauchy net $x_m \in G$, $m \in \mathbb{M}$ in $\mathcal{T}$ such that $x_m \longrightarrow 1$ in $\mathcal{S}$. Since $B \in \mathcal{S}$ there exists an ordinal $\mathbb{L} < \mathbb{M}$ such that $x_m \in B$ for all $m \geq \mathbb{L}$. Thus, $x \in \widehat{(B, \mathcal{T}_B)}$ and $x \in \ker(\widehat{\mathrm{Id}_B})$. □

Corollary 1.4. *Consider a group* G *with a BN-pair* (B, N) *that admits two topological group structures* $(G, \mathcal{S})$ *and* $(G, \mathcal{T})$ *such that* $\mathcal{S} \subseteq \mathcal{T}$. *Suppose* $(G, \mathcal{S})$ *satisfies the conditions of Theorem 1.1 or Theorem 1.2. If* $\widehat{\mathrm{Id}_B} : \widehat{(B, \mathcal{T}_B)} \to \widehat{(B, \mathcal{S}_B)}$ *is surjective, then* $\widehat{\mathrm{Id}} : \widehat{(G, \mathcal{T})} \to \widehat{(G, \mathcal{S})}$ *is surjective.*

Proof. This holds because $\widehat{(G, \mathcal{S})}$ is generated by N and $\overline{B}$. □

Surjectivity of $\widehat{\mathrm{Id}}$ has a very interesting consequence as pointed out to us by Guy Rousseau. Note that the map $\widehat{\mathrm{Id}} : \widehat{(G, \mathcal{T})} \to \widehat{(G, \mathcal{S})}$ defines an injective map of Tits buildings $\mathcal{TB}(\widehat{(G,\mathcal{T})}) \to \mathcal{TB}(\widehat{(G,\mathcal{S})})$. Surjectivity of $\widehat{\mathrm{Id}}$ implies that this map of Tits buildings is bijective.

We finish this section with a convenient corollary of Theorem 1.2 whose proof is straightforward.

Corollary 1.5. *Let G be a group with a BN-pair (B,N) with the Weyl group (W,S) where S is finite. Suppose further that a system $\mathfrak{S}$ of subgroups of B is given such that the following three conditions hold.*

1. *$\mathfrak{S}$ forms a topology basis at 1 of a topological group $(B,\mathcal{T})$.*
2. *Each minimal parabolic P_s is split as a semidirect product $P_s = L_s \ltimes U_s$ where L_s is a finite group and U_s is a subgroup of B.*
3. *L_s acts continuously on $(U_s, \mathcal{T}_{U_s})$.*

Then the four conclusions of Theorem 1.2 hold.

2. Completions of Kac-Moody Groups

Let $A=(a_{ij})_{n\times n}$ be a generalised Cartan matrix, $\mathcal{D}=(I,A,\mathcal{X},\mathcal{Y},\Pi,\Pi^\vee)$ a root datum of type A. Recall that this means

- $I=\{1,2,...,n\}$,
- $\mathcal{Y}$ is a free finitely generated abelian group,
- $\mathcal{X}=\mathcal{Y}^*=\hom(\mathcal{Y},\mathbb{Z})$ is its dual group,
- $\Pi=\{\alpha_1,\dots\alpha_n\}$ is a set of simple roots, where $\alpha_i\in\mathcal{X}$,
- $\Pi^\vee=\{\alpha_1^\vee,\dots\alpha_n^\vee\}$ is a set of simple coroots, where $\alpha_i^\vee\in\mathcal{Y}$,
- for all $i,j\in I$, $\alpha_j(\alpha_i^\vee)=a_{ij}$.

Recall that $\mathcal{D}$ is *simply connected* if $\Pi^\vee$ is a basis of $\mathcal{Y}$. We call A 2-spherical if for each $J\subseteq I$ with $|J|=2$, the submatrix $A_J{:=}(a_{ij})_{i,j\in J}$ is a Cartan matrix of finite type. Let Δ^{re} be the set of real roots. Recall that $\Delta^{re}=W(\Pi)$ where W is the Weyl group. Note that $\Delta^{re}=\Delta^{re}_+\cup\Delta^{re}_-$.

Let $\mathbb{F}=\mathbb{F}_q$ be a finite field of $q=p^a$ elements ($a\geq 1$ and p a prime). Tits [T2, T3] gives a definition of a Kac-Moody group $G{:=}G_{\mathcal{D}}(\mathbb{F})$, which is generated by the torus $T=\hom(\mathcal{X},\mathbb{F}^\times)$ and root subgroups $X_\alpha\cong\mathbb{F}^+$, $\alpha\in\Delta^{re}$. For all $i\in I$, set

$$M_i \coloneqq \langle X_{\alpha_i}\cup X_{-\alpha_i}\rangle.$$

The group G admits a BN-pair (B,N) where $B=U\rtimes T$ with $U{:=}\langle X_\alpha \mid \alpha\in\Delta^{re}_+\rangle$ and $N{:=}N_G(T)$, the normaliser of T. If $\mathcal{D}$ is simply connected, $T\cong(\mathbb{F}^\times)^n$. Moreover, the Coxeter group $(N/T, S=\{s_i, i\in I\})$ and (W,S) are isomorphic. We denote by $P_i{:=}P_{s_i}$, $i\in I$, a minimal parabolic subgroup of G. It is known [CaR2, 6.2] that $P_i=U_i\rtimes L_i$ where $L_i{:=}M_iT$ and $U_i=U\cap s_iUs_i^{-1}$. In particular, $|P_i:B|$ is finite for all $i\in I$.

Consider the set of subgroups $\mathcal{F}$ of B where

$$\mathcal{F}=\{A \mid A\trianglelefteq U\leq B,\ |U:A|=p^a \text{ for some } a\in\mathbb{N}\}.$$

Elements of $\mathcal{F}$ form a basis at 1 of a topology on B. Then the completion $\widehat{B}$ of B with respect to this topology is a group. Since $|P_s:U|<\infty$, conditions (1)–(4) of Theorem 1.2 are satisfied, thus its conclusions hold. In particular, $\widehat{G}$ is a topological group with an open subgroup $\widehat{B}$ and $(\widehat{B},N)$ is a BN-pair of $\widehat{G}$.

Since U is a residually finite-p group [E, Remark after Th. 4.1] and $|B:U| \mid (q-1)^n$, our completion $\widehat{B}$ is equal to $\widehat{U}\rtimes T$ where $\widehat{U}$ is the full pro-p completion of U.

Let us recall other known topological Kac-Moody groups. They are the Mathieu-Rousseau group G^{ma+}, the Carbone-Garland group $G^{c\lambda}$ and the Caprace-Rémy-Ronan group G^{crr}. Each of them contains a quotient $G^{\dagger} := G/Z$ by a central subgroup Z, which depends on the completion and could be trivial. In fact, $G^{\dagger}$ is always dense in $G^{c\lambda}$ and G^{crr}. Let $\overline{G}^{+}$ be the closure of $G^{\dagger}$ in G^{ma+}. Rousseau [Rou, 6.10] investigates whether $\overline{G}^{+}$ equals G^{ma+} and show that this happens when $p > max\{|a_{ij}|, i \neq j\}$ [Rou, 6.11]. Rousseau and later Marquis give examples when it does not happen [Rou], [M2].

There are two further known completions of G where the closure $\overline{U}$ is compact totally disconnected [ReW]. *The Belyaev group* G^{b} is the "largest" such completion. *The Schlichting group* G^{s} is the "smallest" such completion. Our completion admits a characterisation similar to the Belyaev group: $\widehat{G}$ is the "largest" completion where the closure $\overline{U}$ is a pro-p-group.

Let $\overline{U}$ and $\overline{B}$ be the closures of U and B correspondingly in either of the topological groups $\overline{G}^{+}$, $G^{c\lambda}$ or G^{crr}. The group homomorphism $\widehat{U} \twoheadrightarrow \overline{U}$ extends to $\widehat{B} \twoheadrightarrow \overline{B}$ and $\widehat{G} \twoheadrightarrow \overline{G}$ (cf. Section 6.3 of [Rou]). Using this and the universal properties of the Belyaev and Schlichting completions, we have open continuous homomorphisms [Rou, 6.3]:

$$G^{b} \twoheadrightarrow \widehat{G} \twoheadrightarrow \overline{G}^{+} \twoheadrightarrow G^{c\lambda} \twoheadrightarrow G^{crr} \xrightarrow{\cong} G^{s}. \qquad (3)$$

It is known that for $\overline{G} \in \{\overline{G}^{+}, G^{c\lambda}, G^{crr}\}$, $\overline{G}/Z'(\overline{G})$ is topologically simple (where $Z'(\overline{G}) = \bigcap_{g\in\overline{G}} g\overline{B}g^{-1}$). What about our new group $\widehat{G}$?

Recall the following criterion of Bourbaki [B2].

Proposition 2.1. *Let (G, B, N, S) be a Tits system with Weyl group $W = N/(B \cap N)$. Let U be a subgroup of B. We set $Z'(G) = \bigcap_{g\in G} gBg^{-1}$. Assume that G is a topological group topologically generated by the conjugates of U in G. Assume further B a closed subgroup of G, and the following conditions hold:*

1. *We have $U \lhd B$ and $B = UT$ where $T = B \cap N$.*
2. *For any proper normal closed subgroup $V \lhd U$, we have $[U/V, U/V] \subsetneq U/V$.*
3. *Subgroup $[G, G]$ is dense in G.*
4. *The Coxeter system (W, S) is irreducible.*

Then for any normal closed subgroup K in G, $K \leq Z'(G)$. In particular, $G/Z'(G)$ is topologically simple.

We now prove the following statement.

Theorem 2.2. *If A is an irreducible generalised Cartan matrix, then $\widehat{G}/Z'(\widehat{G})$ is topologically simple.*

Proof. If V is a closed normal subgroup of $\widehat{U}$, then $[\widehat{U}/V, \widehat{U}/V] \neq \widehat{U}/V$ as shown in [CarERi, 4.4]. Now Proposition 2.1 finishes the proof. □

There is a similar criterion for the abstract simplicity [B2].

Proposition 2.3. *Let (G, B, N, S) be a Tits system with Weyl group $W = N/(B \cap N)$. Let U be a subgroup of B such that G is generated by the conjugates of U. Assume that the following holds.*

1. *We have $U \lhd B$ and $B = UT$ where $T = B \cap N$.*
2. *For any proper normal subgroup $V \lhd U$, we have $[U/V, U/V] \subsetneq U/V$.*

(3) *We have* $G = [G, G]$.
(4) *The Coxeter system* (W, S) *is irreducible.*

Then for any normal subgroup K *in* G, $K \leq Z'(G)$. *In particular,* $G/Z'(G)$ *is abstractly simple.*

This allows us to prove the following statement.

Theorem 2.4. *Suppose* $q \geq 4$. *If* A *is irreducible and* 2*-spherical, then* $\widehat{G}/Z'(\widehat{G})$ *is abstractly simple, and there are natural isomorphisms*

$$\widehat{G}/Z'(\widehat{G}) \xrightarrow{\cong} \overline{G}^{+}/Z'(\overline{G}^{+}) \xrightarrow{\cong} G^{c\lambda}/Z'(G^{c\lambda}) \xrightarrow{\cong} G^{crr}/Z'(G^{crr}).$$

Proof. Let us first show that $\widehat{G}/Z'(\widehat{G})$ is abstractly simple. To do that it suffices to check the conditions of Proposition 2.3 for the Tits system $(\widehat{G}, \widehat{B}, N, S)$.

By construction of $\widehat{G}$, condition (1) holds because it holds in (G, B, N, S).

Abramenko proves that for $q \geq 4$, U is finitely generated if and only if A is 2-spherical [A]. Thus, $\widehat{U}$ is topologically finitely generated. By [CarERi, Lemma 4.4], condition (2) of Proposition 2.3 holds for $\widehat{U}$ and any proper normal subgroup V of $\widehat{U}$.

Moreover, $[\widehat{G}, \widehat{G}] \geq [G, G][\widehat{U}, \widehat{U}]$. Since $q \geq 4$, for every $\alpha \in \Delta^{re}$, the subgroup $M_\alpha := \langle X_\alpha, X_{-\alpha} \rangle$ is perfect (in fact, it is $\mathrm{PSL}_2(\mathbb{F})$ or $\mathrm{SL}_2(\mathbb{F})$), and thus $[M_\alpha, M_\alpha] = M_\alpha \supseteq X_\alpha$. Hence, $G = [G, G]$. Now the argument of Carbone, Ershov and Ritter [CarERi, 4.3(b)] shows that $[\widehat{U}, \widehat{U}]$ is an open subgroup of $\widehat{G}$, and so $\widehat{G} = [\widehat{G}, \widehat{G}]$.

Finally, condition (4) holds since A is irreducible. Therefore, $\widehat{G}/Z'(\widehat{G})$ is abstractly simple.

Observe that the homomorphisms (3) yield open surective homomorphisms

$$\widehat{G}/Z'(\widehat{G}) \twoheadrightarrow \overline{G}^{+}/Z'(\overline{G}^{+}) \twoheadrightarrow G^{c\lambda}/Z'(G^{c\lambda}) \twoheadrightarrow G^{crr}/Z'(G^{crr})$$

that are isomorphisms of abstract groups due to simplicity of $\widehat{G}/Z'(\widehat{G})$. They are isomorphisms of topological groups because they are open. $\square$

3. Congruence Kernel

We finish the paper with some observations on the structure of $Z'(\widehat{G})$. To facilitate our discussion we use the following notation for arbitrary groups $K \leq H$:

- $\widehat{H}$ – the completion of H in the pro-p topology on H or its canonical (such as U) subgroup,
- $\widetilde{H}$ – the completion of H in some other topology,
- $\overline{K}^{\subseteq H}$ (or simply $\overline{K}$) – the closure of K in H,
- $C(H, K) := \cap_{g \in H} gKg^{-1}$ – the normal core of K in H.

The group $Z'(\widehat{G})$ contains two commuting subgroups: the centre (before completion) $Z(G)$ and the normal core $C(\widehat{G}, \widehat{U})$. In fact, as $\widehat{U} \cong \overline{U}^{\subseteq \widehat{G}}$ is a Sylow pro-p subgroup of $\widehat{G}$, $C(\widehat{G}, \widehat{U}) = C(\widehat{G}, V)$ for any Sylow pro-p subgroup V of $\widehat{G}$. Therefore, we may use the notation $C(\widehat{G})$ instead of $C(\widehat{G}, \widehat{U})$. Sometimes it is convenient to use the full notation $C(\widehat{G}, \widehat{U})$. We will use both notations depending on circumstances.

Following the argument of Rousseau [Rou, Prop. 6.4], we can prove that

$$Z'(\widehat{G}) = Z(G) \times C(\widehat{G}).$$

We can compute the centre $Z(G)$ from the Cartan matrix but we see no efficient way of computing the normal core $C(\widehat{G})$. Observe that in the Caprace-Rémy-Ronan completion, $Z'(G^{crr}) = Z(G)$. Hence, Theorem 2.4 implies that $C(\widehat{G}) = \ker(\phi)$ where $\phi : \widehat{G} \to G^{crr}$ is the natural continuous open surjective homomorphism. The kernels of the natural maps between two different completions of the same groups are commonly known as *congruence kernels*, the term used later in the paper. Can we describe $C(\widehat{G})$ explicitly?

Let $\mathcal{P}$ be the collection of all normal index p^n, $n \in \mathbb{N}$, subgroups of U so that

$$\widehat{U} \cong \{(x_H H) \mid x_H \in U,\ H \in \mathcal{P},\ x_H H = x_{H'} H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}} U/H\,.$$

Let us examine the action of U on the Tits building $\mathcal{TB}(\widehat{G})$. Let U_n be the pointwise stabiliser of the ball of radius n around the simplex $\widehat{B}$ in $\mathcal{TB}(\widehat{G})$. Then

$$\mathcal{P}^0 := \{H \in \mathcal{P} \mid \exists n : H \geq U_n\}$$

is a basis of topology on U. We can describe the completion of U in this topology as

$$U^{crr} \cong \{(x_H H) \mid x_H \in U,\ H \in \mathcal{P}^0,\ x_H H = x_{H'} H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}^0} U/H\,.$$

Clearly, $C(G^{crr}, U^{crr}) = 1$ because it consists of those elements $(x_H H)$ that act trivially on $\mathcal{TB}(\widehat{G})$. This forces $x_H \in U_n$ for all n and $(x_H H) = 1$. The natural map $\widehat{U} \to U^{crr}$ is the projection whose kernel is exactly $C(\widehat{G}, \widehat{U})$ that we can describe now as

$$C(\widehat{G}) = \{(x_H H) \mid x_H \in H^\star,\ H \in \mathcal{P},\ x_H H = x_{H'} H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}} H^\star/H$$

where $H^\star := \cap_{H \subseteq K \in \mathcal{P}^0} K$. This description tells us that one of the three following statements holds:

(1) $\mathcal{P} \setminus \mathcal{P}^0$ is finite. Then $C(\widehat{G}, \widehat{U})$ is a finite group.
(2) $\mathcal{P} \setminus \mathcal{P}^0$ is infinite but $\{H^\star \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is finite. Then $C(\widehat{G}, \widehat{U})$ is a finitely generated pro-p group.
(3) $\{H^\star \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is infinite. Then $C(\widehat{G}, \widehat{U})$ may be an infinitely generated pro-p group.

A natural question to address is whether $C(\widehat{G})$ is central. We can do it under some strong assumptions.

Lemma 3.1. *If A is irreducible of indefinite type and $q \geq n > 2$, then at least one of the following statements holds:*

(1) *$C(\widehat{G})$ is not a finitely generated pro-p group,*
(2) *$C(\widehat{G}) \leq Z(\widehat{G})$.*

In particular, if $C(\widehat{G})$ is finite, then $C(\widehat{G}) \leq Z(\widehat{G})$.

Proof. If A is irreducible of indefinite type and $q \geq n > 2$, then $G/Z(G)$ is a simple non-linear group as shown by Caprace and Rémy [CaR].

Let us assume that $C := C(\widehat{G})$ is a finitely generated pro-p group. In this case the Frattini quotient $C/\Phi(C)$ is a finite elementary abelian p-group. Since $\Phi(C) \triangleleft \widehat{G}$, it follows immediately that $\widehat{G}/Z'(\widehat{G})$ acts on $C/\Phi(C)$. Now $\widehat{G}/Z'(\widehat{G})$ contains a dense subgroup isomorphic to $G/Z(G)$. This subgroup is simple non-linear, hence, it must act trivially on the finite group $C/\Phi(C)$. Since the subgroup is dense, the whole $\widehat{G}/Z'(\widehat{G})$ acts trivially. Since the action is given by conjugation $g\cdot(c\Phi(C)) = (gcg^{-1})\Phi(C)$, we can say that $\widehat{G}/Z'(\widehat{G})$ centralises $C/\Phi(C)$.

Now let T be the torus of G, defined at the start of Section 2. Then $[T, C/\Phi(C)] = 1$. Let $C_i{:=}\Phi_i(C)$, the i-th Frattini subgroup. Since C is finitely generated, C/C_i is a finite p-group, $\Phi(C/C_i) = \Phi(C)/C_i$ and $\{C_i, i \in \mathbb{N}\}$ is a fundamental system of open neighbourhoods of 1 in C [RibZ, 2.8.13]. It follows that T acts on C/C_i and centralises $(C/C_i)/\Phi(C/C_i)$. A theorem of Burnside states that a p'-automorphism of a p-group P, inducing the identity automorphism on $P/\Phi(P)$, is the identity itself [G, 5.1.4]. It follows that $[T, C/C_i] = 1$. Since $\{C_i, i \in \mathbb{N}\}$ is a fundamental system in C, it follows that $[T, C] = 1$. Obviously $[T^g, C] = 1$ for all $g \in \widehat{G}$. Therefore, $[\langle T^g, g \in \widehat{G}\rangle, C] = 1$. Since $Z(G)$ is a subgroup of T [CaR2, Cor. 5.14], $G = \langle T^g, g \in G\rangle \leq \langle T^g, g \in \widehat{G}\rangle$. The result now follows. □

In some cases we can describe $C(\widehat{G})$ fully.

Proposition 3.2. *Suppose that the generalised Cartan matrix $A = (a_{ij})_{n\times n}$ is irreducible, of untwisted affine type and $n \geq 3$. Then $C(\widehat{G}) = 1$. In particular,*

$$\widehat{G} \cong G^{ma+} \cong G^{c\lambda} \cong G^{crr}$$
$$\text{and} \qquad \mathcal{G}(\widehat{\mathbb{F}_q[t,t^{-1}]}\,) \cong \mathcal{G}(\mathbb{F}_q((t))\,)\,.$$

Proof. The root datum $\mathcal{D}$ changes only Cartan subgroup and has no effect on U or $C(\widehat{G}) = C(\widehat{G}, \widehat{U})$. Thus we may choose $\mathcal{D}$ so that $G \cong \mathcal{G}(\mathbb{F}_q[t, t^{-1}])$ for the corresponding Chevalley group scheme $\mathcal{G}$. Now Lemma 7 of [CLR] gives us that $\widehat{U}$ is a Sylow pro-p subgroup of $\mathcal{G}(\mathbb{F}_q((t))$. This implies that $\widehat{G} \cong \mathcal{G}(\mathbb{F}_q((t)))$ which gives the desired result. □

We expect Proposition 3.2 to hold for a twisted affine A as well. As pointed out by the referee, it would be interesting to establish whether $\widehat{G} \cong G^{crr}$ implies that G is of affine type. For instance, the isomorphism fails in rank 2 as shown in the next proposition.

Proposition 3.3. *Suppose that the generalised Cartan matrix $A = (a_{ij})_{2\times 2}$ is not of finite type and $p > \max\{|a_{12}|, |a_{21}|\}$. Then $C(\widehat{G})$ is an infinitely-generated pro-p-group and $\{H^\star \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is infinite.*

Proof. For such A, Morita [Mo, 3(6)] gives the description of U in $G_{\mathcal{D}}(\mathbb{F})$ for a field $\mathbb{F}$ of characteristic 0. His description extends to the case $\mathbb{F} = \mathbb{F}_q$, as an interested reader can verify. If $\min\{|a_{12}|, |a_{21}|\} \geq 2$, then $U = U_1 * U_2$, where $U_i \cong \mathbb{F}_q[t]$ for $i = 1, 2$. If $\min\{|a_{12}|, |a_{21}|\} = 1$, then $U = U_1 * U_2$ and each U_i is a metabelian infinitely generated group.

Consider $\widehat{U}$. By [RibZ, 9.1.1], $\widehat{U} = \widehat{U}_1 \amalg \widehat{U}_2$, the free pro-$p$ product of the pro-p groups $\widehat{U}_1$ and $\widehat{U}_2$. Since each $\widehat{U}_i$ is an infinitely-generated pro-p group, [RibZ, 9.1.15] implies that $\widehat{U}$ is infinitely-generated.

On the other hand, as $p > \max\{|a_{12}|, |a_{21}|\}$, the results of [CR, 2.2 and 2.4] give us that U^{crr} is a finitely generated pro-p group. The proposition follows immediately. □

Corollary 3.4. *Let $A = (a_{ij})_{n\times n}$ be an irreducible generalised Cartan matrix whose Dynkin diagram contains an infinite edge, i.e., there exists $1 \leq i \neq j \leq n$*

with $a_{ij}a_{ji} \geq 4$. Suppose that $p > \max\{|a_{ij}|, i \neq j\}$. Then $C(\widehat{G}, \widehat{U})$ is an infinitely-generated pro-p-group and $\{H^\star \mid H \in \mathcal{P}^0\}$ is infinite.

Proof. Let P be a parabolic subgroup of G whose Levi complement corresponds to the subdiagram of the Dynkin diagram Δ of G based on α_i and α_j. Then $P = U_P \rtimes L$ where

$$L = \langle X_\alpha, T \mid \alpha \in \Delta^{re} \cap \mathrm{Span}_{\mathbb{Z}}\{\alpha_i, \alpha_j\}\rangle$$

is a Levi complement of P and $U_P = \cap_{g \in P} U^g$ [R, 6.2.2]. Hence, $U = U_P \rtimes U_L$ where $U_L = U \cap L$. It follows that $\widehat{U_L} \leq \widehat{U}$. Moreover, the natural isomorphism $U/U_P \xrightarrow{\cong} U_L$ yields an exact sequence

$$1 \longrightarrow U_P \xrightarrow{a} U \xrightarrow{b} U_L \to 1. \tag{4}$$

Since pro-p-completion is a right exact functor, the sequence

$$\widehat{U_P} \xrightarrow{\widehat{a}} \widehat{U} \xrightarrow{\widehat{b}} \widehat{U_L} \to 1 \tag{5}$$

is exact as well. Therefore, $\widehat{U_L}$ is a homomorphic image of $\widehat{U}$. Since $\widehat{U_L}$ is an infinitely-generated pro-p group, so is $\widehat{U}$.

As $p > \max\{|a_{ij}|, i \neq j\}$, the results of [CR] imply that U^{crr} is a finitely generated pro-p group. Note that $C(\widehat{G}, \widehat{U})$ is the kernel of the homomorphism $\widehat{U} \to U^{crr}$. This finishes the proof. □

It is possible to relate the calculations of the congruence kernel of a Levi factor and of the unipotent radical of a parabolic. Let $J \subseteq I$ and $P{:=}P_J$ a parabolic in $G_{\mathcal{D}}(\mathbb{F})$ with the unipotent radical U_P and a Levi complement $L = \langle X_\alpha, T \mid \alpha \in \Delta^{re} \cap \mathrm{Span}_{\mathbb{Z}}(J)\rangle$. Then $P = U_P \rtimes L$ [R, 6.2.2] and we have a natural isomorphism $\widehat{L} \xrightarrow{\cong} \overline{L}^{\subseteq \widehat{G}} = \overline{L}^{\subseteq \overline{P}}$ where $\overline{P} := \overline{P}^{\subseteq \widehat{G}}$. Let $U_L{:=}U \cap L$. Then U_L is the unipotent radical of a Borel subgroup of L. Two "parabolic" congruence kernels "approximate" $C(\widehat{G}, \widehat{U})$:

Theorem 3.5. *There exists an exact sequence of topological groups*

$$1 \longrightarrow C(\widehat{G}, \overline{U_P}) \longrightarrow C(\widehat{G}, \widehat{U}) \longrightarrow C(\widehat{L}, \widehat{U_L}) \to 1\ . \tag{6}$$

Moreover, $C(\widehat{L}, \widehat{U_L})$ is a subgroup of $C(\widehat{G}, \widehat{U})$. In particular, if $C(\widehat{L}) \neq 1$ then $C(\widehat{G}) \neq 1$.

Proof. Notice that $\widehat{L}$ is a topological Kac-Moody group on its own letting us talk about $C(\widehat{L}) = C(\widehat{L}, \widehat{U_L})$.

Let us examine the exact sequence (5) in the proof of Corollary 3.4. The image $\widehat{a}(\widehat{U_P})$ is a closed subgroup containing $a(U_P)$. Since U_P is dense in $\widehat{U_P}$, $a(U_P)$ is dense in $\widehat{a}(\widehat{U_P})$. This yields another exact sequence

$$1 \longrightarrow \overline{U_P} \xrightarrow{\overline{a}} \widehat{U} \xrightarrow{\widehat{b}} \widehat{U_L} \to 1. \tag{7}$$

The same argument applied to the semidirect decomposition $P = U_P \rtimes L$ gives an exact sequence with $\widehat{c}|_{\widehat{U}} = \widehat{b}$:

$$1 \longrightarrow \overline{U_P} \xrightarrow{\overline{a}} \overline{P} \xrightarrow{\widehat{c}} \widehat{L} \to 1 \tag{8}$$

where the closure $\overline{P} = \overline{P}^{\subseteq \widehat{G}}$ is the completion of P in the uniformity induced from $\widehat{G}$. Loosely speaking, both $\overline{P}$ and $\widehat{G}$ are obtained by pro-p-completion of U. Both $\overline{U_P}$

and $\widehat{U}$ are subgroups of $\overline{P}$. The map $\overline{a}$ is the inclusion of subgroups. Conjugating them by all $g \in \overline{P}$ and then intersecting yields the inclusion $\overline{a} : C(\overline{P}, \overline{U_P}) \hookrightarrow C(\overline{P}, \widehat{U})$. Moreover, the sequence (7) restricts to a new sequence

$$1 \longrightarrow C(\overline{P}, \overline{U_P}) \xrightarrow{\overline{a}} C(\overline{P}, \widehat{U}) \xrightarrow{\widehat{b}} C(\widehat{L}, \widehat{U_L}) \ . \tag{9}$$

Observe that $p \cdot y := \widehat{c}(p) y \widehat{c}(p)^{-1}$, $p \in \overline{P}$, $y \in \widehat{L}$, gives a $\overline{P}$-action on $\widehat{L}$. It follows that $\widehat{b}(C(\overline{P}, \widehat{U})) \subseteq C(\widehat{L}, \widehat{U_L})$, so that the sequence (9) is well-defined.

We can conjugate $C(\overline{P}, \overline{U_P})$ and $C(\overline{P}, \widehat{U})$ by all $g \in \widehat{G}$ and intersect further. This yields a subsequence of the sequence (9):

$$1 \longrightarrow C(\widehat{G}, \overline{U_P}) \xrightarrow{\overline{a}} C(\widehat{G}, \widehat{U}) \xrightarrow{\widehat{b}} C(\widehat{L}, \widehat{U_L}) \ . \tag{10}$$

This sequence is precisely the sequence (6) in the statement of the theorem. It remains to establish surjectivity of $\widehat{b}$ in the sequence (10).

Consider the restriction of the continuous homomorphism $\phi : \widehat{G} \to G^{crr}$ to $\widehat{L}$. Clearly, $\phi(\widehat{L}) = L^{crr}$. In particular, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq L^{crr}$. We have an $\widehat{L}$-equivariant map

$$\eta : \mathcal{TB}(\widehat{L})_m = \widehat{L}/(\overline{B} \cap \widehat{L}) \to \mathcal{TB}(\widehat{G})_m = \widehat{G}/\overline{B}, \quad g(\overline{B} \cap \widehat{L}) \mapsto g\overline{B},$$

where by $\mathcal{TB}(\widehat{G})_m$ and $\mathcal{TB}(\widehat{L})_m$ we denote the set of simplices of maximal dimension in the corresponding Tits buildings. As a subset of $\mathcal{TB}(\widehat{G}) = \mathcal{TB}(G^{crr})$, the image of η consists of those simplices that have P as a face because, corestricted to its image, η can be identified with the natural map $\widehat{L}/(\overline{B} \cap \widehat{L}) \to \overline{P}/\overline{B}$

Since $C(\widehat{L}, \widehat{U_L})$ acts trivially on $\mathcal{TB}(\widehat{L})$, it follows that $C(\widehat{L}, \widehat{U_L})$ fixes the image of η. Since the stabiliser of an individual simplex is a Borel subgroup, the fixator of all these simplices is $C(\overline{P}, \overline{B})$. It follows from [CaR2, Th 6.3] that $C(\overline{P}, \overline{B})$ is equal to $U_P^{crr} T'$, where T' is a subgroup of a torus in $\overline{B}$. Therefore, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq U_P^{crr} T'$.

Now $C(\widehat{L}, \widehat{U_L})$ and U_P^{crr} are pro-p-groups, while T' is a finite p'-group, i.e., a group of order coprime to p. Thus, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq U_P^{crr}$. Furthermore, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq L^{crr} \cap U_P^{crr} = 1$. It follows that $C(\widehat{L}, \widehat{U_L})$ is contained in the kernel of ϕ that is equal to $Z'(\widehat{G}) = Z(G) \times C(\widehat{G}, \widehat{U})$. Since $Z(G)$ is a subgroup of the torus [CaR2, Cor. 5.14], it is a finite p'-group, while $C(\widehat{G}, \widehat{U})$ is a pro-p-group. It follows that $C(\widehat{L}, \widehat{U_L})$ is contained in $C(\widehat{G}, \widehat{U})$. This inclusion splits the sequence (10) proving surjectivity of $\widehat{b}$. □

References

[A] P. Abramenko, *Finiteness properties of groups acting on twin buildings*, Groups: topological, combinatorial and arithmetic aspects, 21–26, LMS Lecture Note Ser. 311, Cambridge Univ. Press, 2004.

[B1] N. Bourbaki, *General Topology, parts 1 and 2*, Hermann, 1966.

[B2] N. Bourbaki, *Lie Algebras, Ch. IV–VI*, Hermann, 1968.

[CKRu] I. Capdeboscq, K. Kirkina, D. Rumynin, *Presentations of affine Kac-Moody groups*, Forum of Mathematics, Sigma, 6 (2018), e21, 35 pp.

[CLR] I. Capdeboscq, A. Lubotzky, B. Rémy, *Presentations: from Kac-Moody groups to profinite and back*, Transform. Groups 21 (2016), 929–951.

[CR] I. Capdeboscq, B. Rémy, *On some pro-p groups from infinite-dimensional Lie theory*, Math. Z. 278 (2014), 39–54.

[CaR] P.-E. Caprace, B. Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. 176 (2009), 169–221.

[CaR2] P.-E. Caprace, B. Rémy, *Groups with a root datum*, Innov. Incidence Geom. 9 (2009), 5–77.

[CarERi] L. Carbone, M. Ershov, G. Ritter, *Abstract simplicity of complete Kac-Moody groups over finite fields*, J. Pure Appl. Algebra 212 (2008), 2147–2162.

[E] M. Ershov, *Golod-Shafarevich groups with property (T) and Kac-Moody groups*, Duke Math. J. 145 (2008), no. 2, 309–339.

[G] D. Gorenstein, *Finite groups*, Harper & Row, 1968.

[HRo] E. Hewitt, K. Ross, *Topological groups*, Springer, 1997.

[HrRu] K. Hristova, D. Rumynin, *Kac-Moody Groups and Cosheaves on Davis Building*, J. Algebra 515 (2018), 202–235.

[Ku] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Birkhäuser, 2002.

[M1] T. Marquis, *Abstract simplicity of locally compact Kac-Moody groups*, Compos. Math. 150 (2014), no. 4, 713–728.

[M2] T. Marquis, *Around the Lie correspondence for complete Kac-Moody groups and Gabber-Kac simplicity*, arXiv:1509.01976.

[M3] T. Marquis, *An introduction to Kac-Moody groups over fields*, European Mathematical Society (EMS), 2018.

[Mo] J. Morita, *Root strings with three or four real roots in Kac-Moody root systems*, Tohoku Math. J. (2) 40 (1988), no. 4, 645–650.

[ReW] C. Reid, P. Wesolek, *Homomorphisms into totally disconnected, locally compact groups with dense image*, Forum Math. 31 (2019), no. 3, 685–701.

[R] B. Rémy, *Groupes de Kac-Moody déployés et presque déployés*, Astérisque No. 277, 2002.

[RibZ] L. Ribes, P. Zalesskii, *Profinite Groups*, Springer, 2010.

[Rou] G. Rousseau, *Groupes de Kac-Moody déployés sur un corps local II. Masures ordonnées*, Bull. Soc. Math. France 144 (2016), 613–692.

[T1] J. Tits, *Définition par générateurs et relations de groups avec BN-paires*, C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 6, 317–322.

[T2] J. Tits, *Groups and group functors attached to Kac-Moody data*, Workshop Bonn 1984, 193–223, Lecture Notes in Math. 1111, Springer, 1985.

[T3] J. Tits, *Uniqueness and presentation of Kac-Moody groups over fields*, J. Algebra 105 (1987), no. 2, 542–573.

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