ISOMORPHISM PROBLEMS FOR MARKOV SHIFTS AND EXPANDING ENDOMORPHISMS OF THE CIRCLE

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DECLARATION

I declare that no portion of this thesis has been previously submitted for any degree at any university or institute of learning.

In Chapter 1 the following results are due to me: Theorem 5, Theorem 9, Theorem 10, Proposition 1, Lemma 1, Corollary 5.1 and Corollary 10.1.

. The proof of Lemma 6 is based on a result by W. Parry and R.F. Williams. The other results in §8 closely follow the work of R.F. Williams.

With the exception of Proposition 1 and Theorem 1, the results contained in Chapter 2 are due to me.

SUMMARY

This thesis consists of two independent chapters.

Each chapter has its own detailed introduction and references.

In Chapter one we give new complete topological conjugacy invariants for finite state stationary Markov chains. These new invariants give a classification up to a topological conjugacy which preserves certain equilibrium states. We use our new invariants to investigate Williams' problem. Finally, we generalise the topological classification of one-sided finite state stationary Markov chains to give a classification up to block-isomorphism.

In Chapter two we investigate a problem posed by Shub and Sullivan on the classification of real analytic Lebesgue measure-preserving expanding endomorphisms of the circle. We introduce a new Jacobian invariant that enables us to study the phase factor. Finally, we introduce complete isomorphism invariants but these invariants have a measure-theoretic and topological nature.

CHAPTER 1

ISOMORPHISM PROBLEMS FOR MARKOV CHAINS

§0. INTRODUCTION

The theory of topological Markov chains (or subshifts of finite type) plays an important role in many branches of ergodic theory and dynamical systems see [2], [3], [4], [5], and [19] for example. We will be concerned with the topological and the measure-theoretic classification of these subshifts. In [25] Williams introduced two invariants of topological conjugacy called strong shift-equivalence and shift-equivalence. He showed that strong shift-equivalence is a complete invariant for topological conjugacy. However, for many categories of matrices it is easier to decide whether they are shift-equivalent (cf [1], [9], [16] and [25]). One of the fundamental problems in the theory of subshifts of finite type is to determine whether shift-equivalence is a complete invariant for topological conjugacy. The topological classification was generalised in [15] and [16] to give a classification up to block-isomorphism, i.e. a topological conjugacy between subshifts of finite type which preserves Markov measures.

Let us suppose that we are given two subshifts of finite type, each supporting the equilibrium state of a continuous function with certain properties (these will be defined in §4). We will introduce a new invariant in §4 which gives a necessary and sufficient condition for these subshifts to be topologically conjugate via a homeomorphism which preserves these measures.

We return to Williams' problem in §5 and illustrate some of the difficulties encountered when trying to deduce this new-invariant from a shift-equivalence of matrices.

In §6 we apply our results to the stochastic generalisation of Williams' problem. In §7 we investigate suspension flows over subshifts of finite type. With appropriate assumptions, we give sufficient conditions for two suspension flows to be topologically conjugate - where the conjugacy will preserve flow-invariant measures derived from equilibrium states of continuous functions with certain properties.

Finally, in §8 we extend the topological classification of one-sided subshifts of finite type given by Williams in [25] to a classification up to block-isomorphism.

§1. TOPOLOGICAL MARKOV CHAINS AND WILLIAMS' PROBLEM

Let S be a $n \times n$ irreducible 0-1 matrix. Give $\{1,\ldots,n\}$ the discrete topology and $\Sigma = \prod_{-\infty}^{\infty} \{1,\ldots,n\}$ the product topology. Consider the subspace Σ_S of Σ defined by

$$\Sigma_S = \{x \in \Sigma : S(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\}.$$

The shift σ_S is defined on Σ_S by $(\sigma_S x)_i = x_{i+1}$ for $x = (x_i)$. σ_S is a homeomorphism of the compact metrisable space Σ_S. (Σ_S, σ_S) is called a *topological Markov chain* (or *subshift of finite type*) given by S. For S as above, we define $\Sigma^+ = \prod_{0}^{\infty} \{1,\ldots,n\}$ and let

$$\Sigma_S^+ = \{x \in \Sigma^+ : S(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}^+\}.$$

The shift σ_S^+ is defined on Σ_S^+ by $(\sigma_S^+ x)_i = x_{i+1}$ for $x = (x_i)$. σ_S^+ is a bounded-to-one continuous surjection and (Σ_S^+, σ_S^+) is called a *one-sided topological Markov chain* (or *subshift of finite type*). When the context is clear we will denote σ_S^+ just by σ_S.

Given two topological Markov chains (Σ_S,σ_S) and (Σ_T,σ_T) we say that they are *topologically conjugate* if there exists a homeomorphism ϕ of Σ_S onto Σ_T such that $\phi\sigma_S = \sigma_T\phi$. In [25] Williams defines S and T to be *strong shift-equivalent* if there exists non-negative integral rectangular matrices $U_1,\dots,U_\ell$ and $V_1,\dots,V_\ell$ such that

$$S = U_1\, V_1,\ V_1\, U_1 = U_2\, V_2,\dots,V_\ell\, U_\ell = T.$$

Williams proved the following result:

Theorem 1 [25] (Williams)

(Σ_S,σ_S) and (Σ_T,σ_T) are topologically conjugate if and only if S and T are strong shift-equivalent.

The matrices S and T are said to be *shift-equivalent (with lag ℓ)* if there exists a positive integer ℓ and non-negative integral rectangular matrices U and V such that

$$US = TU \qquad UV = T^\ell$$

$$SV = VT \qquad VU = S^\ell.$$

It is easy to see that strong shift-equivalence implies shift-equivalence and it is conjectured that the converse is true.

This is known as Williams' problem. For many categories of matrices there is a finite procedure for deciding if two matrices are shift-equivalent. Because of this, shift-equivalence remains the best necessary condition known for topological conjugacy.

§2. PRESSURE AND RUELLE OPERATORS

Let $M(\Sigma_S)$ denote the set of Borel probability measures on the space Σ_S and let $C(\Sigma_S)$ denote the set of real-valued continuous functions acting on Σ_S. For $f \in C(\Sigma_S)$ the *pressure* of f is defined by

$$P(f) = \sup \{h_m(\sigma_S) + \int f\, dm : m \in M(\Sigma_S) \text{ is } \sigma_S\text{-invariant}\},$$

where $h_m(\sigma_S)$ is the entropy of σ_S with respect to m. This supremum is always attained and the measures for which $P(f) = h_m(\sigma_S) + \int f\, dm$ are called *equilibrium states* for f (cf. [24]). When f has a unique equilibrium state it will sometimes be denoted by m_f. We can similarly define the pressure and equilibrium states for $f \in C(\Sigma_S^+)$. Since the shift-invariant measures in $M(\Sigma_S)$ and $M(\Sigma_S^+)$ are in bijective correspondence, we will often denote an equilibrium state for $f \in (\Sigma_S^+)$ and its counterpart in $M(\Sigma_S)$ by the same symbol.

For $\phi \in C(\Sigma_S^+)$ the *Ruelle operator* $\mathcal{L}_\phi : C(\Sigma_S^+) \to C(\Sigma_S^+)$ is defined by

$$(\mathcal{L}_\phi f)(x) = \sum_{\sigma_S y = x} e^{\phi(y)} f(y).$$

$\mathcal{L}_\phi$ is positive, linear and continuous with respect to the sup norm on $C(\Sigma_S)$. Denote its spectral radius by $r(\mathcal{L}_\phi)$. Put

$$\mathrm{var}_n\phi = \sup\{|\phi(x) - \phi(y)| : x_0 = y_0, \ldots, x_{n-1} = y_{n-1}\}.$$

In [22] Walters combines a convergence theorem of Ruelle [17] and results by Keane [8] on g-measures to give a proof of:

2. Ruelle's Operator Theorem

Let (Σ_S^+, σ_S) be a topologically mixing one-sided subshift of finite type (i.e. S is aperiodic). Let $\phi \in C(\Sigma_S^+)$ satisfy $\sum_{n=1}^{\infty} \mathrm{var}_n\phi < \infty$. Then there exists $\lambda > 0$, $h \in C(\Sigma_S^+)$ and $\nu \in M(\Sigma_S^+)$ such that $h > 0$, $\nu(h) = 1$, $\mathcal{L}_\phi h = \lambda h$, $\mathcal{L}_\phi^* \nu = \lambda\nu$ and for $f \in C(\Sigma_S^+)$ $\lambda^{-n} \mathcal{L}_\phi^n f \to \nu(f)h$ uniformly in $C(\Sigma_S^+)$. λ , h and ν are uniquely defined by these properties and $\lambda = e^{P(\phi)} = r(\mathcal{L}_\phi)$. Moreover $\mu \in M(\Sigma_S^+)$ defined by $\mu(f) = \nu(hf)$ is the unique equilibrium state for ϕ.

Remark

When S is an irreducible matrix with period d, it is well known (cf. [18])that Σ_S^+ can be represented as a union of d spaces. These are cyclically permuted by σ_S and the restriction of σ_S^d to each one is a topologically mixing subshift. From this decomposition one can show that, apart from the convergence of $\lambda^{-n} \mathcal{L}_\phi^n f$, the conclusions of Ruelle's operator theorem will hold. Although we do not have uniform convergence for the normalised powers of $\mathcal{L}_\phi$, by decomposing S one can show that if

$$\phi \in C(\Sigma^+), \quad \sum_{n=0}^{\infty} \mathrm{var}_n \phi < \infty \quad \text{and } \mathcal{L}_\phi = 1$$

then for $f \epsilon C(\Sigma_S^+)$

$$\frac{1}{N} \sum_{n=0}^{N} \mathcal{L}_\phi^n f \to \int f \, dm_f \text{ uniformly.}$$

§3. THE INFORMATION COCYCLE AND MAXIMAL MEASURES

Given an $n \times n$ matrix S the state partition α_S of Σ_S is the partition of Σ_S into sets $[i]^o$ for $1 \leq i \leq n$ where $[i]^o = \{x \in \Sigma_S : x_0 = i\}$. Let $\alpha_S^- = \bigvee_o^\infty \sigma_S^{-i} \alpha_S$ denote the smallest σ-algebra containing the partitions $\sigma_S^{-i} \alpha_S$, $i \geq 0$ and let m be a σ_S-invariant measure. If β is a countable partition of Σ_S and $\mathcal{C} \subset \bigvee_{-\infty}^{\infty} \sigma_S^{-i} \alpha_S$ is a sub σ-algebra, then the *conditional information of* β *given* $\mathcal{C}$ *is*

$$I_m(\beta|\mathcal{C}) = - \sum_{A\in\beta} \chi_A \log E(\chi_A|\mathcal{C}).$$

The *conditional entropy of* β *given* $\mathcal{C}$ *is* $\int I_m(\beta|\mathcal{C})dm$ and the *information cocycle* is $I_m = I_m(\alpha_S|\sigma_S^{-1} \alpha_S^-)$. For details and notation used see [12] and [15]. Associated to (Σ_S, σ_S) is a natural σ_S-invariant Markov measure, denoted by m_S. This measure is called the *measure of maximal entropy* (cf. [11]).

§4. TOPOLOGICAL CONJUGACY AND RUELLE OPERATORS

In this section we will give a necessary and sufficient condition for topological conjugacy in terms of four identities involving Ruelle operators and positive linear operators between the spaces of continuous functions. Therefore one way of approaching Williams' problem would be to try and construct these relations from a shift-equivalence of matrices. Starting with a shift-equivalence of matrices, in §5 we obtain two of the identities required for a shift-equivalence of Ruelle operators. We show how this can be done by two separate methods.

For $0 < \theta < 1$ and $f \in C(\Sigma_S^+)$ let $\|f\|_\theta = \sup \{\frac{\mathrm{var}_n f}{\theta^n} : n \geq 0\}$.

We will be considering functions in the real Banach space

$$\mathcal{F}_\theta^+(S) = \{f \in C(\Sigma_S^+) : \|f\|_\theta < \infty\}$$

with norm

$$|||f|||_\theta = \max \{\|f\|_\theta, \quad \|f\|_\infty\}.$$

Similarly, $\mathcal{F}_\theta(S) \subset C(\Sigma_S)$ can be defined.

Notice that for $f \in \mathcal{F}_\theta^+(S)$ we have that

$$\sum_{n=0}^{\infty} \mathrm{var}_n\, f \quad \leq \|f\|_\theta \sum_{n=0}^{\infty} \theta^n = \|f\|_\theta \quad \frac{1}{1-\theta} < \infty .$$

Given (Σ_S^+, σ_S), (Σ_T^+, σ_T) and $\phi \in \mathcal{F}_\theta^+(S)$, $\psi \in \mathcal{F}_\theta^+(T)$ we say that the Ruelle operators $\mathcal{L}_\phi$ *and* $\mathcal{L}_\psi$ *are shift-equivalent* if there are positive linear operators

$$U : C(\Sigma_S^+) \to C(\Sigma_T^+) \quad \text{and} \quad V : C(\Sigma_T^+) \to C(\Sigma_S^+)$$

such that

$$U\mathcal{L}_\phi = \mathcal{L}_\psi U \qquad\qquad UV = \mathcal{L}_\psi^\ell$$

$$V\mathcal{L}_\psi = \mathcal{L}_\phi V \qquad\qquad VU = \mathcal{L}_\phi^\ell \text{ for some } \ell > 0.$$

We will show that if the Ruelle operators $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are shift-equivalent (for $\phi \in \mathcal{F}_\theta^+(S)$ and $\psi \in \mathcal{F}_\theta^+(T)$), it is necessary and sufficient that there is a topological conjugacy which preserves the measures m_ϕ and m_ψ.

Suppose that we are given a shift-equivalence of Ruelle operators $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ as above.

From Ruelle's operator theorem there is an eigenvalue $\lambda > 0$ and eigenfunctions $h \in C(\Sigma_S^+)$, $k \in C(\Sigma_T^+)$ such that $h > 0$, $k > 0$ where $\mathcal{L}_\phi h = \lambda h$ and $\mathcal{L}_\psi k = \lambda k$. In fact $h \in \mathcal{F}_\theta^+(S)$ and $k \in \mathcal{F}_\theta^+(T)$ (cf. [17]).

If we define

$$\phi' = \phi + \log h - \log h\sigma_S - \log \lambda \qquad \text{and}$$

$$\psi' = \psi + \log k - \log k\sigma_T - \log \lambda$$

then $\mathcal{L}_{\phi'} 1 = 1$ and $\mathcal{L}_{\psi'} 1 = 1$. Now Uh and Vk are positive eigenvectors for the eigenvalue λ and $\mathcal{L}_\psi$, $\mathcal{L}_\phi$ respectively hence there exists $\beta_1, \beta_2 > 0$ such that $Uh = \beta_1 k$ and $Vk = \beta_2 h$. If we define new operators $\bar{U}$ and $\bar{V}$ by $\bar{U}f = \frac{U(f\cdot h)}{Uh}$ and $\bar{V}g = \frac{V(g\cdot k)}{Vk}$ then $\bar{U}1 = 1$ and $\bar{V}1 = 1$. Since $\beta_1 \beta_2 = \lambda^\ell$ (from the identity $UV = \mathcal{L}_\psi^\ell$) we have that

$$\bar{U}\mathcal{L}_{\phi'} = \mathcal{L}_{\psi'}\bar{U} \qquad \bar{V}\bar{U} = \mathcal{L}^{\ell}_{\phi'}$$

$$\bar{V}\mathcal{L}_{\psi'} = \mathcal{L}_{\phi'}\bar{V} \qquad \bar{U}\bar{V} = \mathcal{L}^{\ell}_{\psi'}$$

Therefore, given a shift-equivalence of Ruelle operators, there is no loss of generality in assuming that all the operators are normalised. As $P(f + g\sigma - g + c) = P(f) + c$ for $c \in \mathbb{R}$ the equilibrium states for ϕ and ϕ' (and ψ, ψ') are the same.

For normalised positive linear operators we have:

<u>Lemma 1</u>

Let $U : C(\Sigma_S^+) \to C(\Sigma_T^+)$ be a positive linear operator with $U1 = 1$ then $(Uf)^2 \leq Uf^2$.

<u>Proof</u>

For $c \in \mathbb{R}$ we have

$$U(cf + 1)^2 = c^2Uf^2 + 2c\,Uf + 1 \geq 0.$$

Treating $c^2\,Uf^2 + 2c\,Uf + 1$ as a polynomial, by looking at the discriminant we get that $4c^2(Uf)^2 - 4c^2Uf^2 \leq 0$ and the result follows.

Given (Σ_S^+, σ_S) and (Σ_T^+, σ_T) we say that the one-sided shifts σ_S *and* σ_S *are shift-equivalent* if there exists continuous surjections $\tau_1 : \Sigma_S^+ \to \Sigma_T^+$ and $\tau_2 : \Sigma_T^+ \to \Sigma_S^+$ such that

$$\sigma_T \tau_1 = \tau_1 \sigma_S \qquad \tau_2 \tau_1 = \sigma_S^\ell$$

$$\sigma_S \tau_2 = \tau_2 \sigma_T \qquad \tau_1 \tau_2 = \sigma_T^\ell \text{ for some } \ell > 0.$$

To show that a shift-equivalence of Ruelle operators implies that our subshifts of finite type are topologically conjugate, we will use the shift-equivalence to obtain operators that are induced by continuous surjections. These functions will form a shift-equivalence of the one-sided shifts. The result will follow from the following Theorem:

Theorem 3 [25] Williams

(Σ_S, σ_S) and (Σ_T, σ_T) are topologically conjugate if and only if σ_S^+ and σ_T^+ are shift-equivalent.

Let X_1 and X_2 be uncountable Borel spaces equipped with finite measures m_1 and m_2 respectively. We require a result by A. Iwanik which extends Lamperti's theorem for operators on the $\mathcal{L}^p(m)$ spaces $1 \le p < \infty$, $p \ne 2$.

Theorem 4 [7] (Iwanik)

Let $1 \leq p < \infty$. Then an operator $U : \mathcal{L}^p(m_1) \to \mathcal{L}^p(m_2)$ is a non-negative linear isometry (not necessarily onto) with $U1 = 1$ if and only if there exists a measure-preserving transformation $\tau : X_2 \to X_1$ such that $Uf = f\tau$.

Lemma 2

Let S and T be irreducible non-negative integer matrices. If $U : C(\Sigma_S^+) \to C(\Sigma_T^+)$ is a non-negative linear operator with $U(1) = 1$ U is continuous with respect to the sup norms.

Proof

Let $f \in C(\Sigma_S^+)$ such that $|f| \leq 1$.
Then $f \leq 1$ and so

$$\| U(f) \| \leq \| U(1) \| = 1.$$

Therefore U is bounded and the result follows.

We now prove:

Proposition 1

Suppose that $\phi \in \mathcal{F}_\theta^+(S)$, $\psi \in \mathcal{F}_\theta^+(T)$ and $P(\phi) = P(\psi)$. If $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are shift-equivalent then (Σ_S, σ_S) and (Σ_T, σ_T) are topologically conjugate. Moreover, the topological conjugacy preserves the equilibrium states m_ϕ and m_ψ.

Proof

By the observations above, we can assume without loss of generality that $P(\phi) = P(\psi) = 0$ and all the operators in the shift-equivalence are normalised.

Lemma 1 enables us to show that our operators $U : C(\Sigma_S^+) \to C(\Sigma_T^+)$ and $V : C(\Sigma_T^+) \to C(\Sigma_S^+)$ can be extended to act on $\mathcal{L}^2$ spaces. We will then apply Theorem 4 to the dual of these extensions.

By Lemma 2 U is continuous and from the remark following Theorem 2, if $f \in C(\Sigma_S^+)$ then

$$|m_\psi(Uf) - m_\phi(f)|$$

$$\le |m_\psi(Uf) - \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}_\psi^n Uf|$$

$$+ |U(\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}_\phi^n f) - U(m_\phi(f))|$$

$$\to 0 \text{ as } N \to \infty .$$

Thus $m_\psi(Uf) = m_\phi(f)$ and so

$$\|Uf\|^2_{\mathcal{L}^2(m_\psi)} = m_\psi(|Uf|^2)$$

$$\le m_\psi(Uf^2) \quad \text{(by Lemma 1)}$$

$$= \|f\|^2_{\mathcal{L}^2(m_\phi)} .$$

Therefore U acting on $C(\Sigma_S^+)$ is a bounded operator when using the $\mathcal{L}^2(m_\phi)$ and $\mathcal{L}^2(m_\psi)$ norms. Consequently, U has an extension (which will also be denoted by U) $U : \mathcal{L}^2(m_\phi) \to \mathcal{L}^2(m_\psi)$ (cf. [10], p. 100).

Since $\mathcal{L}^2(m_\phi)^* \cong \mathcal{L}^2(m_\phi)$ and $\mathcal{L}^2(m_\psi)^* \cong \mathcal{L}^2(m_\psi)$, the operator U induces a dual operator $U^* : \mathcal{L}^2(m_\psi) \to \mathcal{L}^2(m_\phi)$ given by $m_\psi(Uf.k) = m_\phi(f.U^*k)$ for $f \in \mathcal{L}^2(m_\phi)$ and $k \in \mathcal{L}^2(m_\psi)$. The operator U^* is non-negative and linear. By repeating the argument above we know that $\|U^*k\|_{\mathcal{L}^2(m_\phi)} \leq \|k\|_{\mathcal{L}^2(m_\psi)}$. Similarly $V^* : \mathcal{L}^2(m_\phi) \to \mathcal{L}^2(m_\psi)$, $\mathcal{L}_\phi^* : \mathcal{L}^2(m_\phi) \to \mathcal{L}^2(m_\phi)$ and $\mathcal{L}_\psi^* : \mathcal{L}^2(m_\psi) \to \mathcal{L}^2(m_\psi)$ can be defined.

We now wish to show that U^* and V^* are isometries and apply Theorem 4. Let $U_T : \mathcal{L}^2(m_\psi) \to \mathcal{L}^2(m_\psi)$ denote the operator given by $U_T k = k\sigma_T$. For g, $k \in \mathcal{L}^2(m_\psi)$ we have that

$$m_\psi(\mathcal{L}_\psi g.k) = m_\psi(\mathcal{L}_\psi(g.k\sigma_T)) = m_\psi(g.k\sigma_T),$$

thus $\mathcal{L}_\psi^* = U_T$. From $UV = \mathcal{L}_\psi^\ell$ we obtain that $V^* U^* = U_T^\ell$ and consequently, $\|V^*U^*k\|_{\mathcal{L}^2(m_\psi)} = \|k\|_{\mathcal{L}^2(m_\psi)}$. If there exists $k \in \mathcal{L}^2(m_\psi)$ where $\|U^*k\|_{\mathcal{L}^2(m_\phi)} \leq \|k\|_{\mathcal{L}^2(m_\psi)}$ then

$\|U^*k\|_{\mathcal{L}^2(m_\phi)} \leq \|k\|_{\mathcal{L}^2(m_\psi)} = \|V^*U^*k\|_{\mathcal{L}^2(m_\psi)}$ which is a contradiction. Therefore U^* and similarly V^* are isometries. By Theorem 4 there exists surjective measure-preserving transformations $\tau_1 : \Sigma_S^+ \to \Sigma_T^+$ and $\tau_2 : \Sigma_T^+ \to \Sigma_S^+$ such that

$U^*h = h\tau_1$ and $V^*f = f\tau_2$.

We now show that τ_1 and τ_2 have continuous versions. If $f \in \mathcal{L}^2(m_\phi)$ and $g \in \mathcal{L}^2(m_\psi)$ we have that

$$m_\phi(VU_T^\ell g.f) = m_\phi(VV^*U^*g.f) = m_\psi(V^*U^*g.V^*f)$$

$$= m_\psi(V^*(U^*g.f)) = m_\phi(U^*g.f).$$

Hence $VU_T^\ell k = k\tau_1$ a.e. (m_ϕ) for all $k \in C(\Sigma_T^+)$. Let $c \subset \Sigma_T^+$ be a closed-open set (i.e. a finite union of cylinders) and let 1_c denote its characteristic function. Since $1_c = 1_c.1_c$ we have that $VU_T^\ell 1_c = VU_T^\ell 1_c \,.\, VU_T^\ell 1_c$ a.e. (m_ϕ). Since our operator V sends continuous functions to continuous functions we have that $VU_T^\ell 1_c$ is continuous. Now continuous functions which are equal almost everywhere are equal everywhere. Therefore $VU_T^\ell 1_c = (VU_T^\ell 1_c)^2$ and we conclude that $VU_T 1_c$ takes only the value 0 or 1. Hence VU_T^ℓ maps characteristic functions of closed-open sets to continuous characteristic functions. If we now apply Theorem 4 to the operator $VU_T^\ell : \mathcal{L}^2(m_\psi) \to \mathcal{L}^2(m_\phi)$ we obtain a measure-preserving transformation $\tau_1' : \Sigma_S^+ \to \Sigma_T^+$ such that $\tau_1' = \tau_1$ a.e. (m_ϕ) and $VU_T^\ell 1_c = 1_c\tau_1' = 1_{\tau_1'^{-1}c}$ is continuous. Since the only continuous characteristic functions acting on Σ_S^+ are characteristic functions of closed-open sets, we conclude that $\tau_1'^{-1}c$ is closed-open and so τ_1' is continuous. Therefore there exists a continuous version of τ_1 and without loss of generality, we will assume τ_1 and similarly τ_2 are continuous.

From our operator identities we can simply deduce that σ_S and σ_T are shift-equivalent. For example from $V^*U^* = U_T^\ell$ we see that $k\tau_1\tau_2 = k\sigma_T^\ell$ a.e (m_ψ) for all $k \in C(\Sigma_T^+)$. Therefore $k\tau_1\tau_2 = k\sigma_T^\ell$ and consequently $\tau_1\tau_2 = \sigma_T^\ell$. From Theorem 3 the natural extensions of τ_1 and τ_2 will be topological conjugacies and the Proposition is proved.

The remainder of this section will be concerned with proving the converse of Proposition 1.

We shall need:

<u>Proposition 2 [21]</u>

Suppose $\tau\colon \Sigma_S \to \Sigma_T$ $(\tau\colon\Sigma_S^+ \to \Sigma_T^+)$ is bounded-to-one. For every $f \in \mathcal{F}_\theta(T)$ $(\mathcal{F}_\theta^+(T))$ there is exactly one shift-invariant $\mu \in M(\Sigma_S)$ $(M(\Sigma_S^+))$ with $\mu\tau^{-1} = m_f$. The measure μ is the unique equilibrium state for $f\tau$.

<u>Proof</u>

When τ is bounded-to-one, we have $h_\mu(\sigma_S) = h_{\mu\tau^{-1}}(\sigma_T)$

for all shift-invariant $\mu \in M(\Sigma_S)$, so that $P(f\tau) = P(f)$ for every $f \in C(\Sigma_T)$. Let $f \in \mathcal{F}_\theta(T)$ and let μ be the unique equilibrium state for $f\tau$. Then clearly $\mu\tau^{-1}$ is an equilibrium state of f, so it equals m_f. If $\nu \in M(\Sigma_S)$ is shift-invariant and $\nu\tau^{-1} = m_f$, then

$$h_\nu(\sigma_S) + \int f\tau \, d\nu = h_{m_f}(\sigma_T) + \int f \, dm_f = P(f) = P(f\tau),$$

so that $\nu = \mu$.

The proof of the following result is the same as the proof for the case when the shift is topologically mixing and this can be found in [17], [22] and [23].

<u>Proposition 3</u>

Let S be an irreducible 0-1 matrix and let $f \in \mathcal{F}_\theta(S)$ $(\mathcal{F}_\theta^+(S))$. The following statements hold:

(i) The function I_{m_f} has a continuous version and

$-I_{m_f} = f + k\sigma_S - k - P(f)$ for some $k \in \mathcal{F}_\theta(S)$ $(\mathcal{F}_\theta^+(S))$.

(ii) If $f, g \in \mathcal{F}_\theta(S)$ $(\mathcal{F}_\theta^+(S))$ then $m_f = m_g$ iff there exists $c \in \mathbb{R}$, $k \in \mathcal{F}_\theta(S)$ $(\mathcal{F}_\theta^+(S))$ such that $f = g + k\sigma_S - k + c$.

Suppose that $\tau: \Sigma_S \to \Sigma_T$ is a topological conjugacy and $m_\phi \tau^{-1} = m_\psi$ for $\phi \in F_\theta^+(S)$ and $\psi \in F_\theta^+(T)$, where $P(\phi) = P(\psi)$. By Theorem 3 there exists continuous surjections $\tau_1 : \Sigma_S^+ \to \Sigma_T^+$ and $\tau_2 : \Sigma_T^+ \to \Sigma_S^+$ such that

$$\sigma_T \tau_1 = \tau_1 \sigma_S \qquad \tau_2 \tau_1 = \sigma_S^\ell$$

$$\sigma_S \tau_2 = \tau_2 \sigma_T \qquad \tau_1 \tau_2 = \sigma_T^\ell \text{ for some } \ell > 0.$$

There is no loss of generality in assuming that $m_\phi \tau_1^{-1} = m_\psi$ and $m_\psi \tau_2^{-1} = m_\phi$ (cf. [6]). Since τ_1 is continuous the zero-th coordinate $(\tau_1(x))_o$ depends on $(x_o,\dots,x_{m-1})$ for some $m > 0$. Therefore $\mathrm{var}_{m+n}\ \psi\tau_1 = \mathrm{var}_n \psi$ for $n \geq 0$ and consequently $\psi\tau_1 \in F_\theta^+(T)$. As $m_\phi \tau_1^{-1} = m_\psi$, we conclude from Proposition 2 that $m_{\psi\tau_1} = m_\phi$. Thus, by Proposition 3

$$\phi = \psi\tau_1 + f\sigma_S - f + c \text{ for some } f \in F_\theta^+(S)$$

and $c \in \mathbb{R}$. Similarly $\psi = \phi\tau_2 + h\sigma_T - h + d$ for some $h \in F^+(T)$ and $d \in \mathbb{R}$. Since $P(\phi) = P(\psi)$ we have that $c = d = 0$.

Substituting the equation $\psi = \phi\tau_2 + h\sigma_T - h$ into $\phi = \psi\tau_1 + f\sigma_S - f$ we get that

$$\phi = \phi\sigma_S^{\ell} + (h\tau_1 + f)\sigma_S - (h\tau_1 + f).$$

Now, if $S_n f$ denotes the sum $\sum_{i=0}^{n-1} f\sigma^i$ then

$$\phi = \phi\sigma_S^{\ell} + S_\ell\phi - (S_\ell\phi)\sigma_S.$$

As σ_S is ergodic we conclude that $h\tau_1 + f = -S_\ell\phi + c$ for some $c \in R$. By the addition of constants to f and h, we can assume $h\tau_1 + f = -S_\ell\phi$ and similarly $f\tau_2 + h = -S_\ell\psi$. Define

$$U : C(\Sigma_S^+) \to C(\Sigma_T^+) \text{ and } V : C(\Sigma_T^+) \to C(\Sigma_S^+)$$

by

$$Ug(y) = \sum_{\tau_1 x = y} e^{-f(x)} g(x)$$

and

$$Vk(x) = \sum_{\tau_2 y = x} e^{-h(y)} k(y).$$

Then

$$VU\, g(x) = \sum_{\tau_2 y = x} e^{-h(y)} \left(\sum_{\tau_1 x' = y} e^{-f(x')} g(x') \right)$$

$$= \sum_{\tau_2 \tau_1 x' = x} e^{-f(x') - h\tau_1(x')} g(x')$$

$$= \sum_{\sigma_S^\ell x' = x} e^{S_\ell \phi(x')} g(x')$$

$$= \mathcal{L}_\phi^\ell\, g(x).$$

Similarly, we can show that

$$U\, \mathcal{L}_\phi = \mathcal{L}_\psi U \qquad UV = \mathcal{L}_\psi^\ell$$

$$V\, \mathcal{L}_\psi = \mathcal{L}_\phi V \qquad VU = \mathcal{L}_\phi^\ell\ .$$

Combining this with Proposition 1 we have:

<u>Theorem 5</u>

Suppose that $\phi \in \mathcal{F}_\theta^+(S)$, $\psi \in \mathcal{F}_\theta^+(T)$ and $P(\phi) = P(\psi)$. There is a topological conjugacy $\tau\colon \Sigma_S \to \Sigma_T$ such that $m_\phi\, \tau^{-1} = m_\psi$ if and only if $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are shift-equivalent.

Theorem 5 could have been proved using the weaker assumptions that $\phi \in C(\Sigma_S^+)$ and $\psi \in C(\Sigma_T^+)$ satisfy

$$\sum_{n=1}^{\infty} \mathrm{var}_n \phi < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathrm{var}_n \psi < \infty \quad .$$

Since a topological conjugacy preserves the measures of maximal entropy we have:

Corollary 5.1

(Σ_S, σ_S) and (Σ_T, σ_T) are topologically conjugate if and only if $\mathcal{L}_{I_{m_S}}$ and $\mathcal{L}_{I_{m_T}}$ are shift-equivalent.

Remark

Suppose that $\phi \in \mathcal{F}_\theta^+(S)$, $\psi \in \mathcal{F}_\theta^+(T)$ where $P(\phi) = P(\psi) = 0$ and the Ruelle operators $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are shift-equivalent, i.e. there exists positive linear operators $U : C(\Sigma_S^+) \to C(\Sigma_T^+)$ and $V : C(\Sigma_T^+) \to C(\Sigma_S^+)$ such that

$$U\,\mathcal{L}_\phi = \mathcal{L}_\psi\,U \qquad VU = \mathcal{L}_\phi^\ell$$

$$V\,\mathcal{L}_\psi = \mathcal{L}_\phi\,V \qquad UV = \mathcal{L}_\psi^\ell \quad \text{for some } \ell > 0.$$

Then the proof of Theorem 5 shows how to define operators

$\bar{U} : C(\Sigma_S^+) \to C(\Sigma_T^+)$ and $\bar{V} : C(\Sigma_T^+) \to C(\Sigma_S^+)$ where

$\bar{U}g(x) = \sum_{\tau_1 y = x} e^{-f(y)} g(y)$ for some $f \in \mathcal{F}_\theta^+(S)$, $\bar{V}k(x) =$

$\sum_{\tau_2 y = x} e^{-h(y)} k(y)$ for some $k \in \mathcal{F}_\theta^+(T)$ and

$$\bar{U}\,\mathcal{L}_\phi = \mathcal{L}_\psi\,\bar{U} \qquad \bar{V}\,\bar{U} = \mathcal{L}_\phi^\ell$$

$$\bar{V}\,\mathcal{L}_\psi = \mathcal{L}_\phi\,\bar{V} \qquad \bar{U}\,\bar{V} = \mathcal{L}_\psi^\ell .$$

The proof used in Proposition 1 to show that $m_\psi(Ug) = m_\phi(g)$ for all $g \in C(\Sigma_S^+)$ can also be applied to $\bar{U}$ to show that $m_\psi(\bar{U}g) = m_\phi(g)$. Thus $m_\psi(Ug.k) = m_\phi(g.k\tau_1) = m_\psi(\bar{U}(g.k\tau_1)) = m_\psi(\bar{U}g.k)$ for $k \in C(\Sigma_T^+)$. Therefore $Ug = \bar{U}g$ a.e. (m_ψ). Now Ug and $\bar{U}g$ are continuous functions. Consequently, $U = \bar{U}$ and similarly $V = \bar{V}$. Thus, given a shift-equivalence of Ruelle operators, the operators U and V in the shift-equivalence are of a "Ruelle type".

§5. WILLIAMS' PROBLEM REVISITED

We now try and deduce a shift-equivalence of Ruelle operators from a shift-equivalence of matrices S and T. This will be attempted in two different ways. Unfortunately, both methods yield only two out of the four equations required in the definition of shift-equivalent Ruelle operators.

For a 0-1 matrix S let $\theta_n(S)$ be the number of $(i_0,\dots,i_n)$ such that $S(i_0,i_1),\dots,S(i_{n-1},i_n) > 0$. Then $\theta_n(S)$ is the number of allowable words of length n + 1. We can define a $\theta_n(S) \times \theta_n(S)$ 0-1 matrix S_n by

$$S_n(i_0,\dots,i_n;\ j_0,\dots,j_n) = 1 \text{ iff}$$

$$i_1 = j_0,\dots,i_n = j_{n-1}.$$

The topological Markov chains (Σ_S,σ_S) and (Σ_T,σ_T) are called *adapted shift-equivalent* if there exists $\ell \geq 1$ such that S_ℓ and T_ℓ are shift-equivalent with lag ℓ. The following result is due to Parry:

<u>Theorem 6 [13] (Parry)</u>

(Σ_S,σ_S) and (Σ_T,σ_T) are topologically conjugate if and only if they are adapted shift-equivalent.

In [13] Parry made some observations about the construction of adapted shift-equivalence from shift-equivalence which we will briefly summarise. A rectangular 0-1 matrix is called a *division matrix* if its rows are non-trivial and each column contains exactly one non-zero entry. A 0-1 matrix is called an *amalgamation matrix* if its transpose is a division matrix.

Proposition 4 [25]

If M is a non-trivial non-negative integral matrix, then it can be written as $M = DA$, where D is a division matrix and A is an amalgamation matrix. This decomposition into the product of a division matrix with an amalgamation matrix is essentially unique in the sense that, if $M = D'A'$ also, then $D' = DP$, $A' = P^{-1}A$ for some permutation matrix P.

If S and T are shift-equivalent with lag ℓ then for all $n \geq 1$ Parry gives a method for finding non-negative integer matrices U_n and V_n such that $U_nS_n = T_nU_n$, $S_nV_n = V_nT_n$. Although U_nV_n and V_nU_n do not necessarily equal $T_n{}^n$ and $S_n{}^n$ respectively there are matrices u_n and v_n such that

$$U_nS_n = T_nU_n \qquad u_nv_n = T_n{}^n$$

$$V_nT_n = S_nV_n \qquad v_nu_n = S_n{}^n.$$

Moreover, if $U_n = D_1A_1$ and $u_n = D_2A_2$ then $D_1 = D_2P$ and $A_1 = QA_2$ for some permutations P and Q. That is, U_n and u_n only differ by a permutation inside the splitting and similarly for V_n and v_n.

Let $W_n(S)$ be the vector space of all real-valued functions on Σ_S^+ which are dependent on the first n+1 variables ($f(x) = f(x_0,\ldots,x_n)$). In [13] it was pointed out that the transpose of S (denoted by S*) has an interpretation as a homomorphism acting on $W_0(S)$ to itself:

$$S^*f(x) = \sum_{\sigma_S y = x} f(y).$$

Similarly the transpose S_n^* of S_n can be viewed as a homomorphism of $W_n(S)$ into $W_{n-1}(S) \subset W_n(S)$:

$$S_n^*f(x) = \sum_{\sigma_S y = x} f(y).$$

Therefore if $U_nS_n = T_nU_n$ and $S_nV_n = V_nT_n$ we have homomorphisms $U_n^* : W_n(T) \to W_n(S)$ and $V_n^* : W_n(S) \to W_n(T)$ such that $S_n^*\, U_n^* = U_n^*T_n^*$ and $V_n^*S_n^* = T_n^*V_n^*$ where S_n^*, T_n^*, V_n^*, U_n^* are extensions of S_{n-1}^*, T_{n-1}^*, V_{n-1}^*, U_{n-1}^* respectively. We now amend our homomorphisms to make them simultaneously stochastic. To do this we require a result in the theory of matrices that has played an important role in the study of topological Markov chains.

7. Perron-Frobenius Theorem [18]

Suppose S is an $n \times n$ irreducible non-negative integer matrix. Then there exists an eigenvalue r such that

(i) r is real, $r > 0$.

(ii) With r there are associated strictly positive left and right eigenvectors.

(iii) $r \geq |\lambda|$ for any eigenvector $\lambda \neq r$

(iv) r is a simple root of the characteristic equation of S.

From the Perron-Frobenius Theorem we know that there is a $\lambda > 0$ and unique (up to multiplication by a constant) strictly positive vectors r,p such that $S^*r = \lambda r$ and $T^*p = \lambda p$. For a vector t, the matrix with t down the diagonal and zero elsewhere will be denoted by Δ_t. As $S^* U^* = U^* T^*$ we obtain that

$$\left(\frac{\Delta_r^{-1} S^* \Delta_r}{\lambda}\right)(\Delta_r^{-1} U^* \Delta_p) = (\Delta_r^{-1} U^* \Delta_p)\left(\frac{\Delta_p^{-1} T^* \Delta_p}{\lambda}\right)$$

where

$$\left(\frac{\Delta_r^{-1} S^* \Delta_r}{\lambda}\right) 1 = 1 \quad \text{and} \quad \left(\frac{\Delta_p^{-1} T^* \Delta_p}{\lambda}\right) 1 = 1.$$

Now

$$\left(\frac{\Delta_r^{-1}\ S^*\ \Delta_r}{\lambda}\right)(\Delta_r^{-1}\ U^*\ \Delta_p)1 = (\Delta_r^{-1}\ U^*\ \Delta_p)1$$

and so $(\Delta_r^{-1}\ U^*\ \Delta_p)1 = \beta.1$ for some $\beta > 0$. Hence the four homomorphisms in

$$\left(\frac{\Delta_r^{-1}\ S^*\ \Delta_r}{\lambda}\right)\left(\frac{\Delta_r^{-1}\ U^*\ \Delta_p}{\beta}\right) = \left(\frac{\Delta_r^{-1}\ U^*\ \Delta_p}{\beta}\right)\left(\frac{\Delta_p^{-1}\ T^*\ \Delta_p}{\lambda}\right)$$

are stochastic.

Let q be the $\theta_n(S) \times 1$ vector defined by $q_{i_0,\ldots,i_n;1} = r_{i_0}$ and let t be the $\theta_n(T) \times 1$ vector given by $t_{j_0,\ldots,j_n;1} = p_{j_0}$. Then the four homomorphisms in

$$\left(\frac{\Delta_q^{-1}\ S_n^*\ \Delta_q}{\lambda}\right)\left(\frac{\Delta_q^{-1}\ U_n^*\ \Delta_t}{\beta}\right) = \left(\frac{\Delta_q^{-1}\ U_n^*\ \Delta_t}{\beta}\right)\left(\frac{\Delta_t^{-1}\ T_n^*\ \Delta_t}{\lambda}\right)$$

are stochastic. In a similar way $V_n^*\ S_n^* = T_n^*\ V_n^*$ can be amended.

Note that for $f \in W_n(S)$ we have

$$\left(\frac{\Delta_q^{-1}\ S_n^*\ \Delta_q}{\lambda}\right)f(x) = \sum_{\sigma_S y = x} \phi(y)f(y) \quad \text{where } \phi(x_0,x_1) = \frac{r_{x_0}}{\lambda\ r_{x_1}} = e^{-I_{m_S}}.$$

Since $\bigcup_{n\geq 0} W_n(S)$ is dense in $C(\Sigma_S^+)$, we have shown how the observations made in [13] lead to a partial result along the lines of constructing a shift-equivalence of Ruelle operators by obtaining the equations $U\,\mathcal{L}_{-I_{m_S}} = \mathcal{L}_{-I_{m_T}} U$ and $V\,\mathcal{L}_{-I_{m_T}} = \mathcal{L}_{-I_{m_S}} V$.

We now attempt to deduce a shift-equivalence of Ruelle operators from a shift-equivalence of matrices by a different method. If S and T are shift-equivalent with lag ℓ then S^ℓ and T^ℓ are strong shift-equivalent. This implies that $(\Sigma_S, \sigma_S^\ell)$ and $(\Sigma_T, \sigma_T^\ell)$ are topologically conjugate (cf. [25]) and by a similar argument to the one used in the proof of Theorem 3, it can be shown that there are continuous surjections $\tau_1 : \Sigma_S^+ \to \Sigma_T^+$ and $\tau_2 : \Sigma_T^+ \to \Sigma_S^+$ such that

$$\sigma_T^\ell\,\tau_1 = \tau_1\,\sigma_S^\ell \qquad \tau_2\,\tau_1 = \sigma_S^{\rho\ell}$$

$$\sigma_S^\ell\,\tau_2 = \tau_2\,\sigma_T^\ell \qquad \tau_1\,\tau_2 = \sigma_T^{\rho\ell} \text{ for some } \rho \ge 1.$$

By following a similar method to the one used in §4, we can derive from these equations positive linear operators

$$U : C(\Sigma_S^+) \to C(\Sigma_T^+) \quad \text{and} \quad V : C(\Sigma_T^+) \to C(\Sigma_S^+)$$

such that

$$U \ \mathcal{L}^{\ell}_{I_{m_S}} = \mathcal{L}^{\ell}_{I_{m_T}} \ U \qquad UV = \mathcal{L}^{\rho\ell}_{I_{m_T}}$$

$$V \ \mathcal{L}^{\ell}_{I_{m_T}} = \mathcal{L}^{\ell}_{I_{m_S}} \ V \qquad VU = \mathcal{L}^{\rho\ell}_{I_{m_S}} .$$

Thus, by this process we can deduce two of the identities required in the definition of a shift-equivalence of Ruelle operators.

§6. THE STOCHASTIC PROBLEM

We will apply the conclusions of Theorem 5 to the problem of classifying finite state stationary Markov chains up to *block-isomorphism* i.e. a topological conjugacy which preserves the Markov measures. Williams' problem can be generalised to the Stochastic case and this was investigated in [14] and [16]. The topological problem is a special case of the block-isomorphism problem because a conjugating homeomorphism preserves the measures of maximal entropy.

Let P be a stochastic matrix and denote the matrix obtained from P by raising every non-zero entry to the power t, $t \in \mathbb{R}$, by P^t. Let p denote the unique probability vector such that $pP = p$. From the stochastic matrix P we can define a unique (Markov) probability measure $m_p \in (\Sigma_{P^o})$, where m_p is σ_{P^o}-invariant. This is defined on the Borel subsets of Σ_{P^o} and assigns $p(i_o)\ P(i_0,i_1)\ \ldots\ P(i_{n-1},i_n)$ to the cylinder

$$[i_0,\ldots,i_n]^m = \{x \in \Sigma_{P^o} : x_m = i_0,\ldots,x_{m+n} = i_n\}.$$

Irreducible stochastic matrices P and Q are said to be *strong stochastic shift-equivalent* if there are stochastic rectangular matrices $U_1,\ldots,U_\ell$, $V_1,\ldots,V_\ell$ such that for every $t \in R$

$$P^t = U_1^t V_1^t,\ V_1^t U_1^t = U_2^t V_2^t,\ldots,V_\ell^t U_\ell^t = Q^t.$$

If stochastic matrices P and Q are strong stochastic shift-equivalent then (by putting $t = 0$) it is clear that P^o and Q^o are strong shift-equivalent. The classification of Markov chains up to block-isomorphism is given by the following result - the proof can be found in [14] and [16].

<u>Theorem 8 (Parry and Williams)</u>

The Markov chains $(\Sigma_{P^o}, \sigma_{P^o}, m_P)$ and $(\Sigma_{Q^o}, \sigma_{Q^o}, m_Q)$ are block-isomorphic if and only if P and Q are strong stochastic shift-equivalent.

We say P and Q are *stochastic shift-equivalent* if there exists matrices U(t) and V(t) whose entries are non-negative integral combinations of exponential functions c^t for $c > 0$, such that

$$U(t)P^t = Q^t\,U(t) \qquad V(t)U(t) = [P^t]^{(\ell)}$$

$$V(t)Q^t = P^t\,V(t) \qquad U(t)V(t) = [Q^t]^{(\ell)}$$

for some positive integer ℓ, where $[P^t]^{(\ell)}$ $([Q^t]^{(\ell)})$ denotes the ℓ-th power of $P^t(Q^t)$ in the usual sense.

Strong stochastic shift-equivalence clearly implies stochastic shift-equivalence but it is an open problem whether stochastic shift-equivalence is a complete invariant for block-isomorphism. This conjecture is known as the generalised Williams' problem. If the generalised Williams' problem was solved then the (topological) Williams' problem would also be solved. The reverse implication is not true since a topological conjugacy will not necessarily preserve the Markov measures.

Now $m_{\underline{I}_{m_P}} = m_P$ and $m_{\underline{I}_{m_Q}} = m_Q$. An immediate consequence of this fact and Theorem 5 is:

<u>Theorem 9</u>

The Markov chains $(\Sigma_{P^o}, \sigma_{P^o}, m_P)$and $(\Sigma_{Q^o}, \sigma_{Q^o}, m_Q)$ are block-isomorphic if and only if the Ruelle operators $\mathcal{L}_{\underline{I}_{m_P}}$ and $\mathcal{L}_{\underline{I}_{m_Q}}$ are shift-equivalent.

Therefore one way of approaching the generalised Williams' problem would be to try and deduce the shift-equivalence of $\mathcal{L}_{I_{m_P}}$ and $\mathcal{L}_{I_{m_Q}}$ from the stochastic shift-equivalence of P and Q. If P and Q are stochastic shift-equivalent (with lag ℓ) then $(\Sigma_{P^O}, \sigma^{\ell}_{P^O}, m_P)$ and $(\Sigma_{Q^O}, \sigma^{\ell}_{Q^O}, m_Q)$ are block-isomorphic. By following a method similar to the one used in the proof of Theorem 5 we can show that there are positive linear operators $U : C(\Sigma^+_{P^O}) \to C(\Sigma^+_{Q^O})$ and $V : C(\Sigma_{Q^O}) \to C(\Sigma^+_{P^O})$ such that $VU = \mathcal{L}^{\ell\rho}_{I_{m_P}}$ and $UV = \mathcal{L}^{\ell\rho}_{I_{m_Q}}$ for some $\rho \geq 1$. Hence, just as in the topological case, one can deduce two of our required equations by this method. Alternatively, we could have used the stochastic version of adapted shift-equivalence defined in [14] and followed a procedure similar to the one used in §5.

§7. SUSPENSION FLOWS

Let (Σ_S, σ_S) be a subshift of finite type and let $f \in C(\Sigma_S)$ be strictly positive. We can define a compact metric space $\Sigma^f_S = \{(x,s) \in \Sigma_S \times R^+ : x \in \Sigma_S, 0 \leq s \leq f(x)\}$ where $(x,f(x))$ and $(\sigma_S x, 0)$ are identified. The *f-suspension*

σ_t^f of σ_S is the vertical flow defined on Σ_S^f by the local flow $\sigma_t^f(x,s) = (x,s+t)$ when $0 \le s \le f(x)$, $0 \le s+t \le f(x)$ for $t \in \mathbb{R}$. Σ_S^f can be transformed into a measure space by taking the σ-algebra generated by the sets

$$\Sigma_S^f \cap \{A \times B : A \text{ is a cylinder, } B \text{ is Lebesgue measurable}\}.$$

The σ_t^f-invariant probability measures on Σ_S^f all have the form $(\mu \times \ell)/\mu(f)$ where ℓ is Lebesgue measure and $\mu \in M(\Sigma_S)$ is σ_S-invariant (here $\mu(f)$ denotes $\int f\, d\mu$).

For strictly positive functions $f \in C(\Sigma_S)$ and $h \in (\Sigma_T)$ we say that σ_t^f and σ_t^h are *topologically conjugate* if there exists a homeomorphism $\Phi : \Sigma_S^f \to \Sigma_T^h$ such that $\sigma_t^h\Phi = \Phi\sigma_t^f$ for all $t \in \mathbb{R}$. In due course we will give sufficient conditions for two suspension flows to be topologically conjugate. To this end we need the following:

<u>Lemma 3 [15]</u>

If f, $g \in C(\Sigma_S)$ are strictly positive functions such that $f = g + k\sigma_S - k$ for some $k \in C(\Sigma_S)$ then σ_t^f and σ_t^g are topologically conjugate.

If $\mu \in M(\Sigma_S)$ is σ_S-invariant then the topological conjugacy constructed in Lemma 3 preserves the measures $(\mu \times \ell)/\mu(f)$ and $(\mu \times \ell)/\mu(g)$.

Lemma 4 [20]

Let $f \in \mathcal{F}_\theta(S)$, then there exists $f' \in \mathcal{F}_{\theta^{\frac{1}{2}}}(S) \supset \mathcal{F}_\theta(S)$ and $g \in C(\Sigma_S)$ such that $f = f' + g\sigma_S - g$ and $f'(x) = f'(y)$ whenever $x_i = y_i$ for $i \geq 0$.

If the function f in Lemma 4 is bounded away from zero, $\frac{1}{n}(S_n f')$ is strictly positive for large enough n. This function differs from f' by a function of the form $k\sigma_S - k$ for some $k \in C(\Sigma_S)$ and so f' may be taken to be a strictly positive function which acts on Σ_S and Σ_S^+.

Suppose that we are given a f-suspension flow $\sigma_t^f : \Sigma_S^f \to \Sigma_S^f$, where $f \in \mathcal{F}_\theta(S)$. From Lemma 3 and Lemma 4 we can assume that f is a function of the future $(f(x) = f(x_0 x_1 \ldots))$ belonging to $\mathcal{F}^+_{\theta^{\frac{1}{2}}}(S)$. Our main result in this section is the following:

Theorem 10

Let $f \in \mathcal{F}_\theta^+(S)$ and $h \in \mathcal{F}_\theta^+(T)$ be strictly positive functions where $P(f) = P(h)$. If $\mathcal{L}_f$ and $\mathcal{L}_h$ are shift-equivalent, there is a topological conjugacy $\phi: \Sigma_S^f \to \Sigma_T^h$ such that

$$(m_f \times \ell)/m_f(f)\ \phi^{-1} = (m_h \times \ell)/m_h(h).$$

Proof

If $\mathcal{L}_f$ and $\mathcal{L}_h$ are shift-equivalent, we know from Theorem 5 that there is a topological conjugacy $\tau: \Sigma_S \to \Sigma_T$ such that $m_f \tau^{-1} = m_h$. By Proposition 2 $m_{h\tau}\tau^{-1} = m_h$ and so $m_f = m_{h\tau}$. Thus from Proposition 3 and the fact that $P(f) = P(h)$, there is a $k \in \mathcal{F}_\theta(S)$ such that $f = h\tau + k\sigma_S - k$.

Define $\phi: \Sigma_S^f \to \Sigma_T^h$ by

$$\phi(x,s) = (\tau(x),\ s + k(x)),$$

then

$$\begin{aligned}\phi(x,f(x)) &= (\tau(x),\ f(x) + k(x)) \\ &= (\tau(x),\ h\tau(x) + k\sigma_S(x)) \\ &= (\sigma_T\ \tau(x),\ k\sigma_S(x))\end{aligned}$$

and

$$\begin{aligned}\phi(\sigma_S x,0) &= (\tau\ \sigma_S(x),\ k\sigma_S(x)) \\ &= (\sigma_T \tau(x),\ k\sigma_S(x)).\end{aligned}$$

Therefore Φ preserves identifications and it is easy to check that the homeomorphism Φ conjugates the flows σ_t^f and σ_t^h.

Since $m_f \tau^{-1} = m_h$, then it is clear that

$$m_f(f) = m_h(h) \text{ and}$$

$$(m_f \times \ell)/m_f(f) \quad \Phi^{-1} = (m_h \times \ell)/m_h(h).$$

This completes the proof.

Suppose that we are given two suspension flows where the suspending functions are in $\mathcal{F}_\theta^+$ and that these suspension flows support a flow-invariant measure derived from the equilibrium state of a function in $\mathcal{F}_\theta^+$. We will now give a sufficient condition for the suspension flows to be topologically conjugate by a conjugacy that preserves these flow-invariant measures.

<u>Corollary 10.1</u>

Let $f, g \in \mathcal{F}_\theta^+(S)$ and $h, k \in \mathcal{F}_\theta^+(T)$ where $P(f) = P(h)$. Suppose that $\mathcal{L}_f$ and $\mathcal{L}_h$ are shift-equivalent (with lag ℓ) and for the operator $V : C(\Sigma_T^+) \to C(\Sigma_S^+)$ in the shift-equivalence there exists $c \in \mathbb{R}$ and $w \in \mathcal{F}_\theta^+(S)$ such that

$VU_T^{\ell}(k) = g + w\sigma_S - w + c$. Then there is a topological conjugacy $\phi: \Sigma_S^f \to \Sigma_T^h$ such that

$$(m_g \times \ell)/m_g(f)\ \phi^{-1} = (m_k \times \ell)/m_k(h).$$

Proof

Let ϕ and τ be defined as in Theorem 10. The proof of Proposition 1 shows that τ is the natural extension of a map $\tau_1 : \Sigma_S^+ \to \Sigma_T^+$ where $VU_T^0 k = k\tau_1$. Consequently, $k\tau_1 = g + w\sigma_S - \omega + c$ and by Proposition 3 $m_{k\tau_1} = m_g$. From Proposition 2, $m_{k\tau_1}\tau_1^{-1} = m_k$ and so $m_g\ \tau_1^{-1} = m_k$, hence $m_g\tau^{-1} = m_k$. Now $m_g(f) = m_g(h\tau) = m_k(h)$ and it is easy to check that

$$(m_g \times \ell)/m_g(f)\ \phi^{-1} = (m_k \times \ell)/m_k(h).$$

§8. CLASSIFICATION OF ONE-SIDED MARKOV CHAINS

The topological classification of one-sided subshifts of finite type is much simpler than the classification of the two-sided subshifts. For the subshifts of finite type (Σ_S^+, σ_S) and (Σ_T^+, σ_T) there is a finite procedure for determining whether they are topologically conjugate i.e. whether there exists a homeomorphism $\phi: \Sigma_S^+ \to \Sigma_T^+$ such that $\phi\sigma_S = \sigma_T\phi$.

For matrices S and T we say that S *is a division of* T and write $S < T$ if there exists non-negative integer matrices D and R, where D is of division shape, such that

$$S = DR \quad \text{and} \quad T = RD.$$

Given a matrix S we say that S_o *is a total division of* S provided:

(i) $S_0 < A_1 < \ldots < A_r = S$ for some sequence of matrices.

(ii) S_0 has no repeated row.

Williams proved the following classification theorem:

<u>Theorem 11 [25] (Williams)</u>

Every square matrix S over Z^+ has a total division S_0. (Σ_S^+, σ_S) and (Σ_T^+, σ_T) are topologically conjugate if and only if their total divisions S_0, T_0 are conjugate by a permutation $(S_0 = P\, T_0\, P^{-1})$.

We will extend this classification up to *block-isomorphism* i.e. a topological conjugacy that preserves Markov measures. Let P(t) and Q(t) be square matrices with no trivial rows or columns whose entries are non-negative integral combinations of exponential functions c^t for $c > 0$. Also suppose that $P(1)$ and $Q(1)$ are stochastic. We say that $Q(t)$ *is a division of* $P(t)$ and write $Q(t) < P(t)$ if there exists rectangular matrices $R(t)$ and $D(t)$ whose entries are non-negative integral combinations of exponential functions c^t for $c > 0$, such that $R(1)$ and $D(1)$ are stochastic, $D(0)$ is a division matrix and

$$P(t) = R(t)\,D(t)$$

$$Q(t) = D(t)\,R(t).$$

If there exists matrices $P_0(t),\ldots,P_n(t)$ such that $P_0(t) = P(t)$, $P_n(t) = Q(t)$ and for each $1 \le i \le n-1$ either $P_i(t) < P_{i+1}(t)$ or $P_{i+1}(t) < P_i(t)$ we say that $P(t)$ *and* $Q(t)$ *are related.* Given a matrix $P(t)$ a *total division* is a matrix $P_0(t)$ satisfying

(i) $P_0(t) < P_1(t) < \ldots < P_n(t) = P(t)$ for some sequence of matrices.

(ii) $P_0(t)$ has no column which is some exponential c^t times another column.

At the end of this section we will prove:

Theorem 12

Every stochastic matrix P has a total division $P_0(t)$, $(\Sigma^+_{P^0}, \sigma_{P^0}, m_P)$ and $(\Sigma^+_{Q^0}, \sigma_{Q^0}, m_Q)$ are block-isomorphic if and only if the total divisions $P_0(t)$ and $Q_0(t)$ are conjugate by a permutation $(P_0(t) = S^{-1}Q_0(t)S)$.

The proof of Theorem 12 will closely follow the proof of Theorem 11 given by Williams in [25].

Suppose that α and η are partitions of $\Sigma^+_{P^0}$, then we write $\alpha \leq \eta$ if every element of the partition α is a union of elements of η. For $n \geq 0$ let

$$\alpha \vee \sigma^{-1}_{P^0}\alpha \vee \ldots \vee \sigma^{-n}_{P^0} = \{A_1 \cap \ldots \cap A_n : A_i \in \sigma^{-i}_{P^0}\alpha \quad 0 \leq i \leq n\}$$

and denote this partition by α^n. We shall need the following Lemma:

Lemma 5 [16]

Suppose that P is an irreducible stochastic matrix, let β and η be partitions of $(\Sigma^+_{P^0}, \sigma_{P^0}, m_P)$ into closed-open sets and suppose $\alpha \leq \eta \leq \alpha^1$. Define two stochastic matrices indexed by $\alpha \times \eta$:

$$[\alpha,\eta]\ (K,E) = \frac{m_P(K \cap E)}{m_P(K)}$$

and

$$[\alpha,\eta]_{\sigma_P^o}(K,E) = \frac{m_P(K \cap \sigma_{Po}^{-1} E)}{m_P(K)}$$

for

$$(K,E) \in \alpha \times \eta \quad ,$$

then

$$[\alpha,\eta]\ [\eta,\alpha]_{\sigma_P^o} = [\alpha,\alpha]_{\sigma_P^o}$$

and

$$[\eta,\alpha]_{\sigma_P^o}\ [\alpha,\eta] = [\eta,\eta]_{\sigma_P^o}.$$

Note that $[\alpha,\eta]$ has division shape and that the products $[\alpha,\eta]^t\ [\eta,\alpha]^t_{\sigma_P^o}$ and $[\eta,\alpha]^t_{\sigma_P^o}\ [\alpha,\eta]^t$ are 0-1 matrices when $t = 0$.

<u>Lemma 6</u>

If P and Q are stochastic matrices and $\phi\colon \Sigma^+_{P^o} \to \Sigma^+_{Q^o}$ is a block-isomorphism then P^t and Q^t are related.

Proof

Let $\eta = \phi^{-1}\alpha_{Q^o}$ and choose n such that $\eta \le \alpha^n_{P^o}$ and $\alpha_{P^o} \le \eta^n$. Consider the following sequence of partitions:

$$\begin{array}{ccccc}
\alpha_{P^o} \vee \eta^{n-1} & \le & \eta^n & \le & (\alpha_{P^o} \vee \eta^{n-1})^1 \\
\alpha_{P^o} \vee \eta^{n-2} & \le & \alpha_{P^o} \vee \eta^{n-1} & \le & (\alpha_{P^o} \vee \eta^{n-2})^1 \\
\vdots & & \vdots & & \vdots \\
\alpha_{P^o} \vee \eta & \le & \alpha_{P^o} \vee \eta^1 & \le & (\alpha_{P^o} \vee \eta)^1 .
\end{array}$$

By raising each of the matrices defined in Lemma 4 to the power t, we have that $[\eta^n,\eta^n]^t_{\sigma_{P^o}}$ and $[\alpha_{P^o} \vee \eta,\ \alpha_{P^o} \vee \eta]^t_{\sigma_{P^o}}$ are related. Similarly $[\alpha^n_{P^o},\ \alpha^n_{P^o}]^t_{\sigma_{P^o}}$ and $[\alpha_{P^o} \vee \eta,\ \alpha_{P^o} \vee \eta]^t_{\sigma_{P^o}}$ are related. Now $[\alpha_{P^o},\alpha_{P^o}]^t_{\sigma_{P^o}} = P^t$ and $[\eta,\eta]^t_{\sigma_{P^o}} = Q^t$. These matrices are clearly related to $[\alpha^n_{P^o},\alpha^n_{P^o}]^t_{\sigma_{P^o}}$ and $[\eta^n,\eta^n]^t_{\sigma_{P^o}}$ respectively. Hence P^t and Q^t are related.

We now show how total divisions can always be found.

Lemma 7

Let $P(t)$ be a square matrix with no trivial rows or columns such that $P(1)$ is stochastic and whose entries are non-negative integral combinations of exponential functions. Then we can find a total division $P_0(t)$ of $P(t)$.

Proof

Let $P(t)$ be a $n \times n$ matrix and suppose column $j = c^t$ column i. Let the integer k vary over the set $1,\ldots,i,\ldots,j-1,j+1,\ldots,n$. Define a $n \times (n-1)$ matrix $R(t)$ where column k of $R(t)$ equals column k of $P(t)$ if $k \neq i$. When $k = i$ let column k of $R(t)$ equal $(1+c)$. column i. Now let D' be the $(n-1) \times n$ division matrix that partitions the standard row vectors that generate Z^n, $\{y_1,\ldots,y_n\}$ into $n-1$ sets $U_1,\ldots,U_{n-1}$ where $U_k = \{y_k\}$ for $k \neq i$ and $U_k = \{y_i,y_j\}$ for $k = i$. We now construct $D(t)$ by altering the unique non-zero entry of column i in D' to $\dfrac{1}{(1+c)^t}$ and changing the unique non-zero entry of column j of D' to $\dfrac{c^t}{(1+c)^t}$. Then $D(1)$ and $R(1)$ are stochastic matrices with $P(t) = R(t)D(t)$. We now repeat this procedure if necessary on $D(t)\ R(t)$. Since the size of our matrices are being reduced every time this procedure is followed, we will eventually obtain a total division of $P(t)$. We shall need the following Lemma:

Lemma 8

Suppose $B(t)$ and $C(t)$ are divisions of $P(t)$, then there exists $A(t)$ which is a division of both $B(t)$ and $C(t)$.

Proof

Let $P(t)$ be $n \times n$, $B(t)$ $m \times m$ and $C(t)$ $r \times r$. Let q be the smallest number such that the columns of $P(t)$ can be partitioned into two sets W_1 and W_2 of q and $n-q$ columns respectively, where each column of W_2 is some exponential (c^t, for $c > 0$) times one of the columns in W_1. Express $P(t)$ as a product $R_1(t)\ D_1(t)$ where $R_1(t)$ is $n \times q$ and $D_1(t)$ is $q \times n$ by the method used in Lemma 7 and put $A(t) = D_1(t)\ R_1(t)$.

Since $B(t)$ is a division of $P(t)$ there are matrices $D_2(t)$ and $R_2(t)$ such that

$$P(t) = R_2(t)\ D_2(t)$$

$$B(t) = D_2(t)\ R_2(t).$$

We claim that there exists a $q \times m$ matrix $D_3(t)$ such that $D_3(0)$ is division, $D_3(1)$ is stochastic and $D_1(t) = D_3(t)D_2(t)$. Let the standard row vectors which generate $\mathbb{Z}^n$, $\mathbb{Z}^q$ and $\mathbb{Z}^m$ be $\{x_1,\dots,x_n\}$, $\{y_1,\dots,y_q\}$ and $\{z_1,\dots,z_m\}$ respectively. The division matrix $D_1(0)$ gives a partition $U_1,\dots,U_q$ of $\{x_1,\dots,x_n\}$ and similarly $D_2(0)$ gives a paritition $V_1,\dots,V_m$ of $\{x_1,\dots,x_n\}$. Now for each $1 \le j \le m$, the columns of $P(t)$ corresponding to all the x_k's in V_j are an exponential times each other. But $U_1,\dots,U_q$ is the smallest partitioning of the x_k's into sets whose corresponding columns are related by being an exponential times each other. Thus $\{V_1,\dots,V_m\}$ refines $\{U_1,\dots,U_q\}$. Let D_3 be the division matrix that partitions $\{z_1,\dots,z_m\}$ into sets $Y_1,\dots,Y_q$ where $Z_j \in Y_i$ if $V_j \subset U_i$, then $D_1(0) = D_3\, D_2(0)$. If $D_3(i,j) = 1$ and $x_k \in V_j \subset U_i$, then $D_1(t)\,(i,k) \neq 0$, $D_2(t)(j,k) \neq 0$ and we can define $D_3(t)$ by

$$D_3(t)\,(i,j) = \frac{D_1(t)\,(i,k)}{D_2(t)\,(j,k)} \quad .$$

We must check that this definition is unambiguous so suppose $x_k, x_\ell \in V_j \subset U_i$. Choose s such that $P(t)\,(s,k) \neq 0$, as $P(t)\,(s,k) = R_1(t)\,(s,i)D_1(t)(i,k)$ we have that $R_1(t)(s,i) \neq 0$. Now

$$D_1(t)(i,k)\,D_2(t)(j,\ell) = \frac{P(t)(s,k)\,D_2(t)(j,\ell)}{R_1(t)\,(s,i)}$$

$$= \frac{R_2(t)(s,j)\,D_2(t)(j,k)\,D_2(t)(j,\ell)}{R_1(t)\,(s,i)}$$

$$= \frac{P(t)(s,\ell)\,D_2(t)(j,k)}{R_1(t)\,(s,i)}$$

$$= D_1(t)(i,\ell)\,D_2(t)(j,k).$$

Hence

$$\frac{D_1(t)(i,k)}{D_2(t)(j,k)} = \frac{D_1(t)(i,\ell)}{D_2(t)(j,\ell)}$$

and $D_3(t)$ is defined unambiguously. Clearly $D_1(t) = D_3(t)\,D_2(t)$ and $D_3(1)$ is stochastic since if $1 \le i \le q$ then

$$\sum_{j=1}^{m} D_3(1)(i,j) = \sum_{j=1}^{m} D_3(1)(i,j) \left(\sum_{k=1}^{n} D_2(1)(j,k) \right)$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} D_3(1)(i,j)\, D_2(1)(j,k)$$

$$= \sum_{k=1}^{n} D_1(1)(i,k)$$

$$= 1.$$

Define $R_3(t) = D_2(t)\, R_1(t)$ and $A(t) = D_3(t) R_3(t)$,

then

$$(D_2(t) R_2(t))\, D_2(t) = D_2(t)\, R_1(t)\, D_1(t)$$

$$= (D_2(t)\, R_1(t)\, D_3(t))\, D_2(t).$$

Since each column of $D_2(t)$ contains only one non-zero entry we conclude that $D_2(t)\, R_2(t) = D_2(t)\, R_1(t)\, D_3(t)$.

Hence $A(t) < B(t)$, similarly $A(t) < C(t)$ and the Lemma is proved.

We may now prove Theorem 12:

<u>Proof of Theorem 12</u>

By Lemma 7 total divisions $P_0(t)$ and $Q_0(t)$ can always be found. If $P_0(t) = S^{-1} Q_0(t)S$ for some permutation matrix S, then $P_0(t) < Q_0(t)$ and so P^t and Q^t are related. Suppose that for matrices $P_1(t)$ and $P_2(t)$ we have that $P_1(t) < P_2(t)$ and

$$P_2(t) = R(t)\, D(t)$$

$$P_1(t) = D(t)\, R(t).$$

The division matrix D(0) defines a topological conjugacy between the one-sided subshifts of finite type $(\Sigma^+_{P_2(0)}, \sigma_{P_2(0)})$ and $(\Sigma^+_{P_1(0)}, \sigma_{P_1(0)})$ (cf. [25]). This topological conjugacy will preserve the measures given by $P_1(1)$ and $P_2(1)$ on $P_1(0)$ and $P_2(0)$ respectively (cf. [14]). By composing all the block-isomorphisms given by the division matrices, we conclude that there exists a block-isomorphism from $\Sigma^+_{P^o}$ onto $\Sigma^+_{Q^o}$.

Conversely, let ϕ be a block-isomorphism from $\Sigma^{+}_{P^{o}}$ onto $\Sigma^{+}_{Q^{o}}$. Then by Lemma 6 and Lemma 7, $P_0(t)$ and $Q_0(t)$ are related by a string of matrices $P_0(t), P_1(t), \ldots, P_r(t) = Q_0(t)$. These can be thought of as vertices of a polygonal line (see Fig. 1.) with a side joining $P_i(t)$ to $P_{i+1}(t)$ up to the right if $P_i(t) < P_{i+1}(t)$ and down to the right if $P_{i+1}(t) < P_i(t)$. If $P_i(t)$ and $P_{i+1}(t)$ are conjugate by a permutation matrix then we draw a horizontal line.

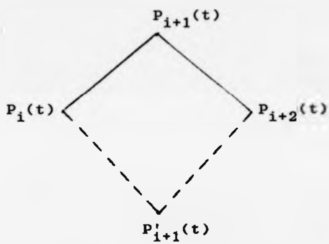

Figure 1

Using Lemma 8 any peak vertex of this graph can be lowered to obtain a lowest graph connecting $P_0(t)$ to $P_r(t) = Q_0(t)$. This lowest graph cannot contain a local minima for then there would be a strictly smaller total division of $P_0(t)$ and $Q_0(t)$. Hence $P_0(t)$ and $Q_0(t)$ are related by a permutation matrix.

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CHAPTER 2

ON THE CLASSIFICATION OF ENDOMORPHISMS OF THE CIRCLE

§0. INTRODUCTION

We will be investigating a problem in the measure-theoretic classification of real analytic Lebesgue measure-preserving expanding endomorphisms of the circle posed by Shub and Sullivan in [6]. It was shown in [5] that any two expanding maps of the same degree are topologically conjugate by a homeomorphism of the circle but in general these homeomorphisms do not preserve Lebesgue measure.

For $i = 1,2$ let f_i be endomorphisms of the Lebesgue spaces $(X_i, \mathcal{B}_i, \mu_i)$. We say that the two systems $(X_1,\mathcal{B}_1,\mu_1,f_1)$ and $(X_2,\mathcal{B}_2,\mu_2,f_2)$ are *isomorphic* if there are sets of measure zero $A_1 \subset X_1$, $A_2 \subset X_2$ and a one-to-one onto map $\phi: X_1 \setminus A_1 \to X_2 \setminus A_2$ such that $\phi f_1 = f_2 \phi$ on $X_1 \setminus A_1$ and $\mu(\phi^{-1}E) = \mu_2(E)$ for all measurable $E \subset X_2 \setminus A_2$. The classification problem in Ergodic Theory is to determine when two given endomorphisms are isomorphic. As usual in measure theory, we do not distinguish between functions which coincide a.e. and so functions need only be defined a.e.

Let $1 \leq r \leq \omega$ and $f : S^1 \to S^1$ be a C^r Lebesgue measure-preserving endomorphism. Then if Df denotes the derivative of f, we say that f is *expanding* if there exists $\lambda \in \mathbb{R}$ such that $|Df(z)| > \lambda > 1$ for all $z \in S^1$. If f and g are Lebesgue measure-preserving endomorphisms of S^1 and there exists a Borel measurable bijection ϕ of S^1 which is non-singular with respect to Lebesgue measure and satisfies $\phi f = g\phi$, then we say that ϕ is an *absolutely continuous conjugacy* between f and g. The following result tells us that under certain conditions, if two endomorphisms are isomorphic, they are isomorphic by an isometry. The analytic case was proved by F. Przytycki in [4].

Proposition 1. [6]

Let f and g be C^2 Lebesgue measure-preserving expanding endomorphisms of S^1 such that $\phi f = g\phi$ for an absolutely continuous conjugacy ϕ, then there is an isometry R of the circle such that $\phi = R$ a.e.

Countable-to-one positively measurable non-singular maps have Jacobian derivatives (see [2], [3] and [10] for details) which we denote by $|D|$. For C^1 Lebesgue measure-preserving endomorphisms the Jacobian derivative is simply the absolute value of the derivative of the endomorphism. We say that the Jacobian derivatives $|Df|$ and $|Dg|$ are isomorphic if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$. If ϕ is a Lebesgue measure-preserving automorphism of S^1 then $|D\phi| = 1$. Therefore, if ϕ is an isomorphism between f and g, we have by the chain rule that $|Df| = |Dg|\phi$ and so the Jacobians will be isomorphic. When our endomorphisms are real analytic and expanding, the following theorem of Shub and Sullivan shows that this invariant is nearly complete.

Theorem 1. [6] (Shub and Sullivan)

Let f and g be real analytic expanding endomorphisms of S^1 which preserve Lebesgue measure. Suppose that the Jacobian derivatives of f and g are isomorphic, then there are isometries R_1 and R_2 of S^1 such that $R_1^{-1} g R_1 = R_2 f$.

Hence if f and g have the same degree they are isomorphic up to a phase factor if their Jacobians are isomorphic. The problem posed by Shub and Sullivan was to determine complete measure-theoretic isomorphism invariants. Although we were not able to settle this completely we will introduce new measure-theoretic isomorphism invariants in §1 and §2 which enable us to study the phase factor and to obtain some related classification results. In §3 we give complete invariants for f and g to be isomorphic but these invariants have a mixed measure-theoretic and topological nature.

§1. THE PHASE GROUP

We now wish to investigate a certain group that can be associated with a continuous surjection $f : S^1 \to S^1$. We will give examples of this group and show that it is a measure-theoretic isomorphism invariant for C^2 Lebesgue measure-preserving expanding endomorphisms. Let

$$G_f = \{\alpha \in S^1 : \exists\beta \in S^1 \text{ such that } f(\alpha z) = \beta f(z)\ \forall z \in S^1\},$$

then G_f is a group where the multiplication of group elements is given by normal multiplication of complex numbers and we call G_f a *phase group*. G_f is never empty since $1 \in G_f$ and as G_f is a closed subgroup of S^1 it is either all of S^1 or the p-th roots of unity for some integer $p \geq 1$.

Lemma 1

If $f : S^1 \to S^1$ is a continuous surjection with degree d then $f(z) = cz^d$ for some constant $c \in S^1$ if and only if $G_f = S^1$.

Proof

Suppose $f(z) = cz^d$ and $\alpha \in S^1$, then if $\beta = \alpha^d$ we have that $f(\alpha z) = \beta f(z)$ for all $z \in S^1$, hence $G_f = S^1$.

Conversely, if $G_f = S^1$ then we can define a transformation $h : S^1 \to S^1$ by $h(\alpha) = \beta$ if $f(\alpha z) = \beta f(z)$ for all $z \in S^1$. h is a group homomorphism since if $f(\alpha_1 z) = \beta_1 f(z)$ and $f(\alpha_2 z) = \beta_2 f(z)$ we have that

$$h(\alpha_1\alpha_2) = \frac{f(\alpha_1\alpha_2 z)}{f(z)} = \frac{h(\alpha_1)f(\alpha_2 z)}{f(z)} = h(\alpha_1)h(\alpha_2).$$

Now the group homomorphisms of S^1 all take the form $z \to z^\ell$ for some $\ell \in \mathbb{Z}$ and so $f(\alpha z) = \alpha^\ell f(z)$. Putting $z = 1$ and allowing α to vary, we see that $f(z) = f(1)z^\ell$ for all $z \in S^1$ and so $\ell = d$.

We will need the following Lemma:

<u>Lemma 2</u>

Suppose that $f : S^1 \to S^1$ is a continuous surjection with degree d, then if $\alpha \in G_f$, we have that $f(\alpha z) = \alpha^d f(z)$ for all $z \in S^1$.

<u>Proof</u>.

When $G_f = S^1$ this Lemma is an immediate consequence of Lemma 1 so we are reduced to dealing with the case when $G_f \neq S^1$.

Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f, then we can write $\tilde{f}$ as $\tilde{f}(x) = q + dx + \theta(x)$ where $q \in \mathbb{Z}$ and $\theta(x+1) = \theta(x)$ for all $x \in \mathbb{R}$. As $G_f \neq S^1$, we know that G_f consists of the p-th roots of unity for some integer $p \geq 1$. Let $\ell \in \mathbb{Z}$ and $b \in \mathbb{R}$ such that if $\alpha = e^{2\pi i\ell/p}$, $\beta = e^{2\pi ib}$ then $f(\alpha z) = \beta f(z)$ for all $z \in S^1$. Since $f(e^{2\pi ix}) = e^{2\pi i\tilde{f}(x)}$ for all $x \in \mathbb{R}$ we have that

$$\tilde{f}(x + \frac{\ell}{p}) - (b + \tilde{f}(x)) : \mathbb{R} \to \mathbb{Z}$$

is a continuous function and so is equal to a constant $m \in \mathbb{Z}$. Consequently,

$$\theta(x + \frac{\ell}{p}) = b + m - \frac{d\ell}{p} + \theta(x)$$

and iterating we have that

$$\theta(x + 1) = pb + mp - d\ell + \theta(x).$$

Therefore $pb + mp - d\ell = 0$ and we conclude that $\beta = \alpha^d$.

We now give examples of real analytic expanding Lebesgue measure-preserving endomorphisms of the circle with degree d whose phase group has order p for integers $d \geq 2$ and $p \geq 1$.

Let $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $B : \bar{D} \to \bar{D}$ be the Blaschke product

$$B(z) = z^{d-1} \frac{(z-a)}{(1-\bar{a}z)}$$

where $|a| < 1$. The restriction of B to the circle will be denoted by f_0. The function f_0 is real analytic, expanding and preserves Lebesgue measure on S^1 (cf. [1]). If $\alpha \in G_{f_0}$ then by Lemma 2 $f_0(\alpha z) = \alpha^d f_0(z)$ and by comparing coefficients of z^d we have that $\alpha = 1$ and so G_{f_0} is trivial.

Let $f : S^1 \to S^1$ be a p-fold cover of f_0, i.e. f is a real analytic Lebesgue measure-preserving endomorphism of S^1 with degree d such that the following diagram commutes.

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ {\scriptstyle z^p}\downarrow & & \downarrow{\scriptstyle z^p} \\ S^1 & \xrightarrow[f_0]{} & S^1 \end{array}$$

If $\alpha \in G_f$ then by Lemma 2 $f(\alpha z) = \alpha^d f(z)$. By raising all the terms in this equation to the power p we have that $f_0(\alpha^p z^p) = \alpha^{dp} f_0(z^p)$ and so $\alpha^p \in G_{f_0}$. Therefore the order of G_f divides p. Conversely, since G_{f_0} is trivial, the order of G_f is greater than or equal to p. This is because if $\omega = e^{2\pi i/p}$ then $(f(\omega z))^p = (f(z))^p$ and as f is continuous, there is an integer k such that $f(\omega z) = \omega^k f(z)$. Hence $k = d$ and $\omega \in G_f$. Consequently, the phase group G_f has order p.

By starting with an orientation reversing Blaschke product and following the method described above, we can give examples of orientation reversing endomorphisms whose phase groups have order p.

For a continuous surjection f of S^1 and an isometry R of S^1 it is not hard to see that $G_f = G_{Rf} = G_{fR}$. An immediate consequence of this fact and Proposition 1 is:

<u>Proposition 2</u>

Suppose that f and g are isomorphic C^2 Lebesgue measure-preserving expanding endomorphisms of S^1, then $G_f = G_g$.

In view of the above remark and Theorem 1 we also have:

<u>Proposition 3</u>

If f and g are real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with isomorphic Jacobians then $G_f = G_g$.

§2. THE PHASE FACTOR AND THE PHASE GROUP

We now return to the problem posed by Shub and Sullivan. We will show (Theorem 2) that if ϕ is a real analytic Lebesgue measure-preserving automorphism of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$ then the phase factor of Theorem 1 is an element of the phase group of f. This Jacobian condition is a measure-theoretic isomorphism invariant since if ϕ is an isomorphism between f and g then $\phi f^n = g^n\phi$ for all $n \geq 1$ and by the chain rule $|Df^n| = |Dg^n|\phi$. When $G_f = S^1$ it easily follows from Theorem 1 and Lemma 1 that the isomorphism of the Jacobians $|Df|$ and $|Dg|$ is a complete measure-theoretic isomorphism invariant. When $G_f \neq S^1$ we show how Theorem 2 can be used to obtain complete measure-theoretic isomorphism invariants for certain classes of endomorphisms. At the end of this section we investigate how this new invariant ties in with other invariants discussed in [3].

Theorem 2

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with the same degree. Suppose that there exists a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$, then there is an isometry R of S^1 such that $R^{-1}g\,R = \alpha f$ for some $\alpha \in G_f$.

Proof

If $|Df^2| = |Dg^2|\phi$ then by the chain rule $|Df|f.|Df| = |Dg|g\phi.|Dg|\phi$. Now since $|Df| > 1$ and $|Df| = |Dg|\phi$, we have that $|Df|f = |Dg|g\phi$. From Theorem 1 there is an isometry R of S^1 and a constant $\alpha \in S^1$ such that $R^{-1}gR = \alpha f$. The proof of Theorem 1 shows that there is a set X of positive measure such that $R = \phi$ on X. As $R^{-1}g^2R = \alpha f \alpha f$ we use the chain rule to obtain that

$$|Dg|gR.\ |Dg|R = |Df|\alpha f.|Df|.$$

Now $|Df| = |Dg|R$ and so $|Dg|gR = |Df|\ \alpha f$, therefore

$$|Df|f = |Dg|g\phi = |Dg|gR = |Df|\alpha f \text{ on } X.$$

In other words $|Df|(z) = |Df|(\alpha z)$ for all z contained in a set of positive measure and by analytic continuation $|Df|(z) = |Df|(\alpha z)$ for all $z \in S^1$. Therefore, there is a constant $\beta \in S^1$ such that $f(\alpha z) = \beta f(z)$ for all $z \in S^1$, thus $\alpha \in G_f$ and the theorem is proved.

As corollaries to Theorem 2, we now give complete measure theoretic isomorphism invariants for various classes of real analytic measure-preserving expanding endomorphisms of S^1.

By Proposition 3 the isomorphism of the Jacobians implies that the *phase* groups are the same and we therefore have:

<u>Corollary 2.1</u>

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with the same degree and such that G_f or G_g is trivial. Then there is a isometry R of S^1 such that Rf = gR if and only if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$.

Suppose that f has degree d and G_f has finite order p where p and d-1 are coprime. If $\omega = e^{2\pi i/p}$ and there is an integer ℓ such that $R^{-1}gR = \omega^{\ell}f$ for some isometry R of S^1, then there is an isometry R_1 of S^1 with $R_1^{-1}\,gR_1 = f$. This is because if m and q are integers satisfying m(d-1) = qp + 1 then defining the isometry R_1 by $R_1(z) = R(\omega^{m\ell}z)$ and using Lemma 2, we have that:

$$\begin{aligned} R_1^{-1}\,gR_1 &= \omega^{m\ell(d-1)}\,R^{-1}gR \\ &= \omega^{m\ell(d-1)+\ell}\,f \\ &= f. \end{aligned}$$

Combining this with Theorem 2 we have:

Corollary 2.2

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with degree d and suppose that the order of G_f or G_g is coprime to d-1. Then there is a isometry R of S^1 such that Rf = gR if and only if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$.

In particular we have:

Corollary 2.3

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with degree d where d-1 is a prime number and the order of G_f or G_g is not equal to d-1. Then there is a isometry R of S^1 such that Rf = gR if and only if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$.

Corollary 2.4

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with degree d where $|d| = 2$, then there is an isometry R of S^1 such that Rf = gR if and only if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$.

Suppose that f and g are C^2 Lebesgue measure-preserving endomorphisms of S^1 and let $H_f = \{\alpha f^n : \alpha \in G_f,\ n \in \mathbb{Z}^+\}$, then H_f can be made into a semi-group where the group operation is simply composition of functions i.e. $\alpha_1 f^n * \alpha_2\ f^m = \alpha_1 f^n\ (\alpha_2 f^m)$. Note that if f has degree $d \in \mathbb{Z}^+$ then from Lemma 2 $\alpha_1 f^n\ (\alpha_2 f^m) = \alpha_1\ \alpha_2{}^{dn}\ f^{n+m}$. We say that H_f *is isomorphic to* H_g if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that the map $\Psi : H_f \to H_g$ given by $\alpha f^n \to \phi\ \ \alpha f^n\ \phi^{-1}$ is a bijection where $\phi\ \ \alpha f^n \phi^{-1}$ need only equal an element of H_g a.e. If αf is isomorphic to g for some $\alpha \in G_f$, we have by Proposition 1 that there is an isometry R of S^1 such that $R^{-1}gR = \alpha f$ and clearly this implies that H_f is isomorphic to H_g. We therefore have:

<u>Corollary 2.5</u>

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1, then H_f is isomorphic to H_g if and only if there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$.

We will now briefly describe two other measure-theoretic isomorphism invariants for Lebesgue measure-preserving endomorphisms of S^1 (cf. [3] and [10]). Let f and g be countable-to-one Lebesgue measure-preserving endomorphisms of S^1. Using Rohlin's theory of measurable partitions (cf. [9]) we can assume that our endomorphisms have been modified so that they are positively measurable and non-singular, i.e. there are null sets A_1, $A_2 \subset S^1$ such that $f|_{S^1 \setminus A_1}$ and $g|_{S^1 \setminus A_2}$ map measurable sets to measurable sets and null sets to null sets.

Let $\beta(f)$ denote the smallest σ-algebra such that $f^{-1}\,\beta(f) \subset \beta(f)$ and $|Df|$ is measurable with respect to $\beta(f)$. If there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $\beta(g) = \phi\beta(f)$ then we say that $\beta(f)$ *and* $\beta(g)$ *are isomorphic*. If ϕ is an isomorphism between f and g then $\beta(g) = \phi\beta(f)$ and so this is a measure-theoretic isomorphism invariant. We say that f and g are *sequentially equivalent* if there are automorphisms $\phi_0, \phi_1, \ldots.$ of S^1 such that $\phi_{n+1} f = g\phi_n$ for $n \geq 0$. Clearly this is a measure-theoretic isomorphism invariant. This invariant is closely related to the notion of isomorphic sequences of σ-algebras studied by Versik in [7] and [8].

Now suppose that f and g are Lebesgue measure-preserving endomorphisms of S^1 with degree d such that $R^{-1}gR = \alpha f$ for an isometry R and $\alpha \in G_f$. Since $f(\alpha z) = \alpha^d f(z)$ we have that $|Df|(\alpha z) = |Df|(z)$ and if $\mathcal{L}$ denotes the Lebesgue σ-algebra of R then

$$R^{-1}\beta(g) = R^{-1} \bigvee_{n=0}^{\infty} g^{-n}\, |Dg|^{-1}\,(\mathcal{L})$$

$$= R^{-1}\,|Dg|^{-1}\,(\mathcal{L}) \vee \bigvee_{n=1}^{\infty} f^{-n}\, \alpha^{-\left(\frac{d^n-1}{d-1}\right)}\, R^{-1}\,|Dg|^{-1}\,(\mathcal{L})$$

$$= |Df|^{-1}\,(\mathcal{L}) \vee \bigvee_{n=1}^{\infty} f^{-n}\, \alpha^{-\left(\frac{d^n-1}{d-1}\right)}\, |Df|^{-1}\,(\mathcal{L})$$

$$= |Df|^{-1}\,(\mathcal{L}) \vee \bigvee_{n=1}^{\infty} f^{-n}\, |Df|^{-1}\,(\mathcal{L})$$

$$= \beta(f).$$

Therefore $\beta(f)$ and $\beta(g)$ are isomorphic. To show that f and g are sequentially equivalent define $\phi_0(z) = R(z)$ and $\phi_i(z) = R(\alpha^{\frac{d^i-1}{d-1}} z)$ for $i \geq 1$. Using Lemma 2 and the fact that α is an element of the phase groups of f and g we have that $\phi_{n+1} f = g\phi_n$ for all $n \geq 0$. Since $R^{-1}gR = \alpha f$ for $\alpha \in G_f$ we get that $R^{-1}g^nR = \alpha^{\frac{d^n-1}{d-1}} f$ for all $n \geq 1$. Using the chain rule, we conclude that $|Df^n| = |Dg^n|R$ for all $n \geq 1$. Combining all these observations with Theorem 2 gives:

Proposition 4

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with the same degree. If there is a Lebesgue measure-preserving automorphism ϕ of S^1 such that $|Df| = |Dg|\phi$ and $|Df^2| = |Dg^2|\phi$ then:-

(i) f and g are sequentially isomorphic.

(ii) $\beta(f)$ is isomorphic to $\beta(g)$.

(iii) There is a Lebesgue measure-preserving automorphism ψ of S^1 such that

$$|Df^n| = |Dg^n|\psi \text{ for all } n \geq 1.$$

§3. COMPLETE INVARIANTS FOR ISOMETRIC ISOMORPHISM

Throughout this section we will assume that f and g are real analytic Lebesgue measure-preserving expanding endomorphisms with degree d. We will give complete invariants for f and g to be isomorphic but these invariants will have a measure-theoretic and topological nature. By Proposition 1, to give complete invariants for f and g to be isomorphic, it is enough to give complete invariants for f and g to be isomorphic by an isometry and this is what we will do.

We begin by showing that when the order of G_f is finite, f is isomorphic to a endomorphism defined on the product of the circle and the phase group.

Suppose that G_f has order $p \in \mathbb{Z}^+$, let $\omega = e^{2\pi i/p}$ and denote $G_f = \langle\omega\rangle$ by G. Define the measurable function $k : S^1 \to G$ by $k(z) = \omega^r$ if z can be written as $z = z_0\, \omega^r$ for $0 \leq \arg(z_0) < \frac{2\pi}{p}$ and $r \in \mathbb{Z}$. Then if ℓ denotes Lebesgue measure on S^1 and h_G denotes the Haar probability measure on G, we have that $\Psi : S^1 \to S^1 \times G$ given by $z \to (z^p, k(z))$ is a measurable bijection such that $\ell\Psi^{-1} = \ell \times h_G$. Consider the endomorphism $\bar{f} : S^1 \times G \to S^1 \times G$ given by $\bar{f} : \Psi f \Psi^{-1}$. If f_0 is the unique endomorphism of S^1 such that the following diagram commutes

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "z^p"'] & S^1 \arrow[d, "z^p"] \\ S^1 \arrow[r, "f_0"'] & S^1 \end{tikzcd}$$

then $G_{f_0} = \{1\}$ and it is easy to check that $\bar{f}(z,y) = (f_0(z), \alpha_f(z)y^d)$ for some measurable $\alpha_f : S^1 \to G$.

In the same way, if the order of G_g is p we can define $\bar{g} : S^1 \times G \to S^1 \times G$ where $\bar{g}(z,y) = (g_0(z), \alpha_g(z)y^d)$ for some measurable $\alpha_g : S^1 \to G$.

We will also need to represent isometries of S^1 as skew-products. Suppose that $R(z) = cz^m$ is an isometry of S^1 where $c \in S^1$, $m \in \{-1,1\}$ and let R_0 be the unique isometry which makes the following diagram commute

$$\begin{tikzcd} S^1 \arrow[r, "R"] \arrow[d, "z^p"'] & S^1 \arrow[d, "z^p"] \\ S^1 \arrow[r, "R_0"'] & S^1 \end{tikzcd} .$$

If $\bar{R} : S^1 \times G \to S^1 \times G$ denotes the isomorphism $\Psi R \Psi^{-1}$ then

$$\bar{R}(z,y) = (R_0(z), \alpha_R(z)y^m)$$

for some measurable $\alpha_R : S^1 \to G$.

When G_f has finite order p, we can define two factors of $\bar{f}$ that depend on a certain decomposition of p. By writing p as a product of powers of prime numbers, it is not hard to see that p has a unique decomposition into a product of two positive integers p_1 and p_2 where p_1 is coprime to d-1 and p_2 divides $(d-1)^k$ for some $k \geq 1$. For ω as above let $G_1 = \langle \omega^{p_1} \rangle$, $G_2 = \langle \omega^{p_2} \rangle$ and define $f_1 : S^1 \times G_1 \to S^1 \times G_1$ and $f_2 : S^1 \times G_2 \to S^1 \times G_2$ by $\bar{f}_1(z,y) = (f_0(z), \alpha_f^{p_1}(z)y^d)$, $\bar{f}_2(z,y) = (f_0(z), \alpha_f^{p_2}(z)y^d)$.

Now the following diagrams commute

$$\begin{tikzcd}
S^1 \times G_1 \arrow[r, "\bar{f}"] \arrow[d, swap, "\substack{(z,y) \\ \downarrow \\ (z,y^{p_1})}"] & S^1 \times G_1 \arrow[d, "\substack{(z,y) \\ \downarrow \\ (z,y^{p_1})}"] \\
S^1 \times G_1 \arrow[r, swap, "\bar{f}_1"] & S^1 \times G_1
\end{tikzcd}$$

$$\begin{tikzcd}
S^1 \times G_2 \arrow[r, "\bar{f}"] \arrow[d, swap, "\substack{(z,y) \\ \downarrow \\ (z,y^{p_2})}"] & S^1 \times G_2 \arrow[d, "\substack{(z,y) \\ \downarrow \\ (z,y^{p_2})}"] \\
S^1 \times G_2 \arrow[r, swap, "\bar{f}_2"] & S^1 \times G_2
\end{tikzcd}$$

therefore, with respect to the measures $\ell \times h_{G_1}$ and $\ell \times h_{G_2}$ on $S^1 \times G_1$ and $S^1 \times G_2$ respectively, we have that $\bar{f}_1$ and $\bar{f}_2$ are factors of $\bar{f}$.

Similarly define $\bar{g}_1 : S^1 \times G_1 \to S^1 \times G_1$, $\bar{g}_2 : S^1 \times G_2 \to S^1 \times G_2$, $\bar{R}_1 : S^1 \times G_1 \to S^1 \times G_1$ and $\bar{R}_2 : S^1 \times G_2 \to S^1 \times G_2$.

We will now describe a new measure-theoretic isomorphism invariant. This invariant consists of a Jacobian condition and something which resembles a coboundary equation.

Suppose G_f and G_g have finite order $p = p_1 p_2$ and $R(z) = cz^m$ is an isometry of S^1 such that $Rf = gR$, then $\bar{R}\,\bar{f} = \bar{g}\,\bar{R}$ and consequently $R_0 f_0 = g_0 R_0$ and

$$\alpha_f^m \cdot \alpha_R (f_0) = \alpha_g(R_0) \cdot \alpha_R^d .$$

In particular

$$\alpha_f^{mp_1} \cdot \alpha_R^{p_1} (f_0) = \alpha_g^{p_1}(R_0) \cdot \alpha_R^{dp_1}$$

Observe that if $p_2 = 1$ then $G_1 = \{1\}$ and this last equation becomes trivial.

Proposition 5

Let f and g be real analytic expanding Lebesgue measure-preserving endomorphisms of S^1 with degree d. Then if f and g are isomorphic, there is an isometry $R(z) = cz^m$ such that $|Df| = |Dg|R$, $|Df^2| = |Dg^2|R$ and if $G_f (= G_g)$ has order $p \in \mathbb{Z}^+$ where $p = p_1 p_2$ is the decomposition of p given above, there is a measurable map $\zeta : S^1 \to G_1$ such that $\alpha_f^{p_1 m} \cdot \zeta(f_0) = \alpha_g^{p_1} (R_0) . \zeta^d$.

To prove that this invariant is complete, we will need the following Lemma which tells us when we can build up from isomorphic factors $\bar{f}_i$, $\bar{g}_i$ $(i = 1,2)$ to an isomorphism of $\bar{f}$ and $\bar{g}$.

Lemma 3

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with degree d such that G_f and G_g have order $p \in \mathbb{Z}^+$. For f_0, $\bar{f}_1$, $\bar{f}_2$ and g_0, $\bar{g}_1$, $\bar{g}_2$ defined above we have that f is isomorphic to g if and only if

(i) There is an isomorphism $\phi_0 : S^1 \to S^1$ such that $\phi_0 f_0 = g_0 \phi_0$.

(ii) For $i = 1,2$ there are isomorphisms $\bar{\phi}_i : S^1 \times G_i \to S^1 \times G_i$ such that $\bar{\phi}_i \bar{f}_i = \bar{g}_i \bar{\phi}_i$ and $\pi\bar{\phi}_i = \phi_0\pi$ where π is the projection onto the first coordinate.

Proof

Suppose that f and g are isomorphic, then by Proposition 1 there is an isometry R of S^1 such that $Rf = gR$. By letting $\phi_0 = R_0$, $\bar{\phi}_1 = \bar{R}_1$ and $\bar{\phi}_2 = \bar{R}_2$ it is easy to check that conditions (i) and (ii) are satisfied.

Conversely, let

$$X = \{((z_1,y_1),(z_2,y_2)) \in (S^1 \times G_1) \times (S^1 \times G_2) : z_1 = z_2\}$$

and define

$$\Lambda : X \to S^1 \times G$$

by

$$((z_1,y_1), (z_2,y_2)) \to (z_1,y_1y_2).$$

The group G is isomorphic to the group $\langle y_1y_2 \rangle$ where $y_1 \in G_1$, $y_2 \in G_2$ and consequently using the product measure $(\ell \times h_{G_1}) \times (1 \times h_{G_2})$ on X we have that Λ is an isomorphism. Define endomorphisms $F : X \to X$, $G : X \to X$ by

$$F((z_1,y_1),(z_2,y_2)) = (\bar{f}_1(z_1,y_1),\ \bar{f}_2(z_2,y_2))$$

and

$$G((z_1,y_1),(z_2,y_2)) = (\bar{g}_1(z_1,y_1),\ \bar{g}_2(z_2,y_2)),$$

then $\Lambda^{-1}\bar{f}\Lambda = F$ and $\Lambda^{-1}\ \bar{g}\Lambda = G$. Let Φ: $X \to X$ be the isomorphism

$$\Phi((z_1,y_1),(z_2,y_2)) = (\bar{\phi}_1(z_1,y_1),\ \bar{\phi}_2(z_2,y_2)).$$

To see that this is well defined, we need to check that $\pi\bar{\phi}_1(z_1,y_1) = \pi\bar{\phi}_2(z_2,y_2)$ when $z_1 = z_2$. This will immediately follow from the identities $\pi\bar{\phi}_1 = \phi_0\pi$ and $\pi\bar{\phi}_2 = \phi_0\pi$.
Now $\Phi F((z_1,y_1),\ (z_2,y_2)) = (\bar{\phi}_1\bar{f}_1(z_1,y_1),\ \bar{\phi}_2\bar{f}_2(z_2,y_2))$

$$= (\bar{g}_1\bar{\phi}_1(z_1,y_1),\ \bar{g}_2\bar{\phi}_2(z_2,y_2))$$

$$= G\ \Phi\ ((z_1,y_1),\ (z_2,y_2)).$$

Thus F and G are isomorphic and consequently f and g will be isomorphic.

We now make use of our decomposition of p into factors p_1 and p_2 by showing that if the Jacobian condition of Proposition 5 holds, we do not have to worry about the isomorphism of $\bar{f}$ and $\bar{g}_2$.

Suppose that G_f and G_g have finite order $p = p_1p_2$ and there is an isometry $R(z) = cz^m$ such that $R^{-1}gR = \alpha f$ for $\alpha \in G_f$, where $\alpha = \omega^\ell$ for $\omega = e^{2\pi i/p}$ and $\ell \in \mathbb{Z}$. Then $R_0f_0 = g_0R_0$ and $\alpha_g(R_0).\ \alpha_R^d = \alpha_f^m \cdot \alpha_R(f_0).\omega^{\ell m}$, in particular $\alpha_g^{p_2}(R_0).\alpha_R^{dp_2} = \alpha_f^{mp_2} .\alpha_R^{p_2}(f_0).\omega^{\ell mp_2}$. As p_1 and d-1 are coprime, there are integers n and q such that $np_1 = q(d-1) + 1$. Therefore

$$\ell n\ p = \ell n\ p_1p_2 = (d-1)\ \ell p_2q + \ell p_2$$

and if we define $\eta: S^1 \to G_2$ by $\eta = \omega^{\ell mp_2q}\ \alpha_R^{p_2}$ we have that

$$\begin{aligned}
\alpha_g^{p_2}\ (R_0).\eta^d &= \alpha_g^{p_2}(R_0).\omega^{d\ell mp_2q} \cdot \alpha_R^{dp_2} \\
&= \omega^{d\ell mp_2q\ +\ \ell mp_2} \cdot \alpha_f^{mp_2} \cdot \alpha_R^{p_2}(f_0) \\
&= \omega^{(d-1)\ell mp_2q+\ell mp_2} \cdot \alpha_f^{mp_2} \cdot \eta(f_0) \\
&= \alpha_f^{mp_2} .\eta(f_0).
\end{aligned}$$

Thus defining $\bar{\phi}_2 : S^1 \times G_2 \to S^1 \times G_2$ by $\bar{\phi}_2(z,y) = (R_0(z),\eta(z)y^m)$ we get that $\bar{\phi}_2 \bar{f}_2 = \bar{g}_2 \bar{\phi}_2$.

If in addition there is a measurable map $\zeta : S^1 \to G_1$ such that $\alpha_f^{p_1 m} \cdot \zeta(f_0) = \alpha_g^{p_1}(R_0).\zeta^d$ then defining $\bar{\phi}_1 : S^1 \times G_1 \to S^1 \times G_1$ by $\bar{\phi}_1(z,y) = (R_0(z), \zeta(z)y^m)$, we obtain that $\bar{\phi}_1 \bar{f}_1 = \bar{g}_1 \bar{\phi}_1$.

Combining these observations with Proposition 4 and Lemma 3, we have completed the proof of:

<u>Theorem 3</u>

Let f and g be real analytic Lebesgue measure-preserving expanding endomorphisms of S^1 with degree d. Then f and g are isomorphic if and only if there is an isometry $R(z) = cz^m$ such that $|Df| = |Dg|R$, $|Df^2| = |Dg^2|R$ and if $G_f (= G_g)$ has order $p \in \mathbb{Z}^+$ where $p = p_1 p_2$ is the decomposition of p given above, then for f_0 and R_0 defined above, there is a measurable map $\zeta : S^1 \to G$ such that

$$\alpha_f^{p_1 m} . \zeta(f_0) = \alpha_g^{p_1}(R_0).\zeta^d.$$

Unfortunately, the complete invariant described in Theorem 3 is a mixed topological and measure theoretic invariant and so we have not fully solved the problem posed by Shub and Sullivan.

When $p_2 = 1$ we can give complete measure theoretic isomorphism invariants for real analytic Lebesgue measure-preserving expanding endomorphisms of the circle and this was done in Corollary 2.2.

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