Differential Operators On Algebraic Varieties

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Contents

Introduction:

This thesis is dedicated to the investigation of the structure of rings of differential operators on affine algebraic varieties. In particular, two aspects of the theory of differential operators are considered: the theory of primary decomposible subspaces as introduced by [Cannings & Holland], and the work begun in [Smith & Stafford], [Hart & Smith] and [Chamarie & Stafford] about S_2 varieties with smooth injective normalisation. The former topic is used to classify the right ideals of the ring of differential operators on a smooth curve. In Chapter Two, an alternative characterisation of primary decomposible subspaces is given which leads to an easier proof of the classification of the right ideals, and also extends to a certain extent to surfaces. But it is the second of the two topics that most concerns this thesis and the main result of the thesis is the calculation of a counter example to a conjecture which previously seemed to be intractable.

In [Smith & Stafford] it is shown that if $\mathcal{X}$ is a (possibly singular) curve with normalisation $\widetilde{\mathcal{X}}$ and if the normalisation map $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is injective then $\mathcal{D}(\mathcal{X})$ is a simple, noetherian domain with Krull and global dimensions equal to one. The method of proof is find a connection between the two rings $\mathcal{D}(\widetilde{\mathcal{X}})$ and $\mathcal{D}(\mathcal{X})$. This link is Morita equivalence and is afforded by the bimodule $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$. In the papers [Hart & Smith] and [Chamarie & Stafford], a generalisation of this result to two surfaces is given. That is, it is proved that if $\mathcal{X}$ is an S_2 surface with smooth, injective normalisation $\widetilde{\mathcal{X}}$ then $\mathcal{D}(\mathcal{X})$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$ and so is simple and noetherian with Krull and global dimensions equal to two. This result works by proving that the module $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ is a reflexive $\mathcal{D}(\widetilde{\mathcal{X}})$-module then using the fact that in rings of global dimension two, reflexivity is the same as projectivity. In three or more dimensions however, this trick is no longer available but it was conjectured in both [Hart & Smith] and [Chamarie & Stafford] that the result would still hold for higher dimensional varieties (although [Chamarie & Stafford] does suggest that extra conditions may have to be placed on the variety $\mathcal{X}$).

This thesis was motivated by the desire to find examples of high dimensional S_2 varieties with smooth, injective normalisation and to calculate the rings of differential operators on them. When looking for such examples it is natural to consider subrings of polynomial rings and it quickly becomes clear that the easiest set of examples are tensor products of the coordinate rings of curves. In Chapter Three it is shown that the differential operator ring of a tensor product is the tensor product of the differential operator rings. This is a natural result to want to prove and so it is surprising that it has not appeared previously in the literature. Once this result is established, it is used to prove that if $\mathcal{X}$ is a product of curves then $\mathcal{D}(\mathcal{X})$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$. This result might seem to lend weight to the conjecture mentioned in the previous paragraph.

After seeing the result on tensor products, it is natural to want to find an example of an S_2 variety with smooth, injective normalisation which is not a product of curves. This is the subject of Chapter Four where the line of attack is to construct a variety $\mathcal{X}$ whose singular locus is determined by a height one prime of the coordinate ring $\mathcal{O}(\mathcal{X})$. If $\mathcal{X}$ were a product then either its singular locus would be smooth or it would not be irreducible, so by choosing the prime ideal carefully, one may construct an $\mathcal{X}$ which cannot be a product.

The next problem is to calculate the differential operators on the varieties which we have constructed. Fortunately, the construction makes use of a derivation which has close links with the module of differential operators $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$. In the second part of Chapter Four, a criterion is found on this derivation for the $\mathcal{D}(\widetilde{\mathcal{X}})$-module $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ to be projective. Finally, it is shown that there exists an example of an S_2 variety $\mathcal{X}$ with smooth, injective normalisation $\widetilde{\mathcal{X}}$ whose differential operator ring $\mathcal{D}(\mathcal{X})$ is not Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$. This is a counter example to the aforementioned conjecture.

Chapter One contains all the background material necessary for the later chapters and is of a completely expository nature. This should make the thesis self contained, even for the non-specialist in differential operator rings. The only results that are assumed are well-known facts about algebra such as the Dual Basis Lemma and Goldie's Theorem. Chapter Five presents a discussion of the results contained in this thesis and how they could possibly be improved or extended.

I would like to take this opportunity to thank my supervisor Dr.C.Hajarnavis for his help and guidance in writing this thesis and also Dr.M.Holland for his advice and ideas.

Chapter 1

This chapter is primarily concerned with the setting forth of all the background information necessary in order to be able to discuss the results proved in the following chapters. The main subjects we need to look at are differential operators, algebraic varieties and some ring theory (both commutative and non-commutative). In particular we shall examine the interplay between differential operators and affine algebraic curves as studied in [Smith & Stafford] and the generalisations of this to higher dimensions. To do this we first need to know the basic properties of rings of differential operators and also some elementary algebraic geometry. We start by introducing the concept of rings of differential operators on varieties and we analyse their basic ring theoretic structure. In particular we completely determine the structure of the ring of differential operators on a smooth (non-singular) variety. We then study ways of passing from a general singular variety to a smooth one and show how these methods can lead to powerful tools for the examination of the properties of the ring of differential operators on a singular curve.

We assume that the reader is familiar with the correspondence between affine algebraic varieties over a base field k, and commutative, affine (i.e. finitely generated)

k-algebras via the set of maximal ideals (**max-Spec**) of a ring. This correspondence is particularly nice when k is algebraically closed as the Hilbert Nullstellensatz tells us that in this case the points of a variety are in bijection with the **max-Spec** of its coordinate ring. Unless otherwise specified we will always be working with irreducible, affine varieties over an algebraically closed base field k of characteristic zero.

1.1 Derivations and Differentials

Let K be a commutative ring and let $\mathcal{R}$ be a commutative K-algebra. Before defining the ring of differential operators on $\mathcal{R}$ we look at the closely related ring of K-linear derivations which is the subring of $End_K(\mathcal{R})$ generated by the derivations. Notice first of all that $End_K(\mathcal{R})$ does actually form a ring with multiplication being the composition of maps. We may think of $\mathcal{R}$ itself as lying inside $End_K(\mathcal{R})$ since multiplication by an element x of $\mathcal{R}$ is a linear map.

Definition 1 *(i) Let $\mathcal{R}$ be a commutative K-algebra. A map $\theta \in End_K(\mathcal{R})$ is called a* (K-linear) derivation *if $\theta(xy) = \theta(x)y + x\theta(y)$ for all x and y in $\mathcal{R}$. Denote the set of all K-linear derivations on $\mathcal{R}$ by $Der_K(\mathcal{R})$.*
(ii) Let $\mathcal{R}$ be a commutative K-algebra. We define the ring $\Delta_K(\mathcal{R})$ to be the subring of $End_K(\mathcal{R})$ generated by all the elements of $\mathcal{R}$, and the set of K-linear derivations on $\mathcal{R}$. This is called the (K-linear) derivation ring of $\mathcal{R}$.

Of course, almost all of the time we shall simply be looking at commutative k-algebras for a field k, and k-linear derivations. In this case, when the context is clear we shall omit the k subscript from $Der_k(\mathcal{R})$ and $\Delta_k(\mathcal{R})$ and simply talk about derivations and the derivation ring. Notice that the set of derivations $Der(\mathcal{R})$ is closed under multiplication on the left by elements of $\mathcal{R}$ and is therefore a left $\mathcal{R}$-module. When $\mathcal{R} = \mathcal{O}(\mathcal{X})$ is the coordinate ring of a variety $\mathcal{X}$ we shall simply write $Der(\mathcal{X})$ in place of $Der(\mathcal{R})$. The ring $\Delta(\mathcal{R})$ is a subring of the ring of differential operators on $\mathcal{R}$ as defined in the next section and it will turn out that when $\mathcal{R}$ is regular (i.e. when the variety corresponding to $\mathcal{R}$ is smooth), these two rings are actually equal. It would seem sensible therefore to study the derivation ring of a regular ring $\mathcal{R}$ and we will see that $\Delta(\mathcal{R})$ has a number of surprisingly strong properties.

Example: Let $\mathbf{A}^n$ denote affine n-space so that its coordinate ring $\mathcal{O}(\mathbf{A}^n)$ equals $k[x_1, \ldots, x_n]$ for n indeterminates $x_1, \ldots, x_n$. Then the set of derivations $Der(\mathbf{A}^n)$ is a free left $\mathcal{O}(\mathbf{A}^n)$-module with free basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$. Hence the derivation ring $\Delta(\mathcal{O}(\mathbf{A}^n))$ is an iterated Ore extension:

$$\Delta(\mathcal{O}(\mathbf{A}^n)) = k[x_1, \ldots, x_n][\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}].$$

This ring is very well known and is called the nth Weyl algebra $A_n(k)$. It has been studied extensively and is known to have many nice properties, some of which are summarised in the proposition below.

Proposition 2 *Let $A_n(k)$ be the nth Weyl algebra. Then:*

(i) $A_n(k)$ *is a left and right noetherian integral domain.*

(ii) $A_n(k)$ *is a simple ring.*

(iii) $A_n(k)$ *has Krull and global dimensions equal to* n.

(iv) Every right (or left) ideal of $A_n(k)$ *can be generated by two elements.*

Proof: These results can be found in [MR] in sections 1.3, 6.6 and 7.5, and [Stafford].

$\square$

This tells us that $A_n(k)$ is a very nice ring indeed and that although it is highly non-commutative (its centre is just k), the properties stated in parts (i) and (iii) of the proposition mirror those of the commutative ring $\mathcal{O}(\mathbf{A}^n)$ itself. On the other hand, the simplicity of $A_n(k)$ indicates a marked difference from the properties of general commutative rings, as the only simple commutative rings are fields. Now, when $\mathcal{X}$ is a smooth variety its coordinate ring $\mathcal{R} = \mathcal{O}(\mathcal{X})$ is a regular ring whose derivation ring looks very much like $A_n(k)$, where n is the dimension of $\mathcal{X}$. In order to prove this fact, a number of small results are required, most of which we will need in greater generality for the later chapters. We therefore summarise the properties of the derivation ring of a smooth variety in the next proposition and remark that their proofs can be easily adapted from the proofs of the results contained in the remainder of this section.

Proposition 3 *Let* $\mathcal{R}$ *be the coordinate ring of a smooth, n-dimensional variety. Then* $\Delta(\mathcal{R})$ *is a simple, noetherian domain with Krull and global dimensions equal to* n.

Proof: As mentioned above we omit the proofs of these facts as they are very similar to proofs given later in this section. The interested reader is therefore referred to

[MR] Theorem 15.3.7 and Proposition 15.3.6(i). □

Remark: There exist results about numbers of generators of right ideals of $\Delta(\mathcal{R})$ in [Coutinho & Holland], namely that each right ideal needs only three generators. Notice the dichotomy between this and part (iv) of Proposition 2. It is an open question as to whether every right ideal of $\Delta(\mathcal{X})$ can be generated by two elements for an arbitrary smooth variety $\mathcal{X}$. In fact, there is no known example of any simple, noetherian ring with a right ideal which cannot be generated by two elements.

In addition to derivations on a ring $\mathcal{R}$ we may consider the module of derivations between two $\mathcal{R}$-modules, and dual to the notion of derivations is that of the module of differentials. Fortunately we only need to consider derivations between the ring itself and a module. Again, for technical reasons we need to look at K-linear derivations for a commutative ring K. This time we shall be working inside the set $Hom_K(\mathcal{R}, M)$ for a right $\mathcal{R}$-module M where $\mathcal{R}$ is a commutative K-algebra. Since $\mathcal{R}$ is commutative this set forms a bimodule over $\mathcal{R}$ under the following actions: for $\theta \in Hom_K(\mathcal{R}, M)$ and x and $y \in \mathcal{R}$, define $(\theta.x)(y) = \theta(xy)$ and $(x.\theta)(y) = \theta(y).x$.

Definition 4 *Let K be a commutative ring, $\mathcal{R}$ a commutative K-algebra and M a right $\mathcal{R}$-module.*

(i) Let $\theta \in Hom_K(\mathcal{R}, M)$. Then θ is called a (K-linear) derivation *between $\mathcal{R}$ and M if $\theta(xy) = \theta(x)y + \theta(y)x$ for every x and $y \in \mathcal{R}$. Denote the set of all K-linear derivations between $\mathcal{R}$ and M by $Der_K(\mathcal{R}, M)$.*

(ii) Let F be the free right $\mathcal{R}$-module with free basis the set of symbols $\{dx | x \in \mathcal{R}\}$ and let G be the submodule of F generated by all the elements of the form $d\alpha, d(x+y) - dx - dy$ and $d(xy) - d(x)y - d(y)x$ for $\alpha \in K$ and x and $y \in \mathcal{R}$. Then define the module of (Kähler) differentials, $\Omega_K(\mathcal{R})$, *to be the factor module F/G.*

Notice that that map d from $\mathcal{R}$ to $\Omega_K(\mathcal{R})$ given by $d(x) = dx$ is a derivation. This map d is called the *universal derivation* of $\mathcal{R}$. The link between differentials and derivations is explained by the next result. For the rest of this section we will assume that we are in the situation of the previous definition in that K is a commutative ring and $\mathcal{R}$ is a commutative K-algebra.

Lemma 5 *Let M be an $\mathcal{R}$-module. Then:*

(i) Given a derivation $\delta \in Der_K(\mathcal{R}, M)$ there is a unique $\mathcal{R}$-module homomorphism $\phi : \Omega_K(\mathcal{R}) \to M$ such that $\delta = \phi \circ d$.

(ii) The map from $Hom_{\mathcal{R}}(\Omega_K(\mathcal{R}), M)$ to $Der_K(\mathcal{R}, M)$ given by $\phi \mapsto \phi \circ d$ is an isomorphism of left $\mathcal{R}$-modules. In particular $Der_K(\mathcal{R}) \cong \Omega_K(\mathcal{R})^* = Hom(\Omega_K(\mathcal{R}), \mathcal{R})$.

Proof: Part (i) follows by setting $\phi(dx) = \delta(x)$ for $x \in \mathcal{R}$ and part (ii) follows from part (i). $\square$

Example: Referring back to our previous example, let $K = k$ be a field and let $\mathbf{A}^n$ be affine n-space. Let $f \in \mathcal{R} = k[x_1, \ldots, x_n]$. Then it is easy to see, using the relations in Ω, that $df = \sum(\partial f/\partial x_i)dx_i$ so that the dx_i generate Ω over $\mathcal{R}$. Now let M be the free right $\mathcal{R}$-module with basis $m_1, \ldots, m_n$ and define $\delta \in Der(\mathcal{R}, M)$ by $\delta(x_i) = m_i$ for each i. Then by Lemma 5 there exists a unique homomorphism ϕ from $\Omega(\mathcal{R})$ to

M which maps dx_i to m_i for each i. Therefore there can be no relations between the dx_i in $\Omega(\mathcal{R})$ and it is a free $\mathcal{R}$-module on n generators dx_i.

Next we give a result which although easy to prove, is powerful enough to eventually enable us to develop a theory of localisation for the non-commutative ring $\Delta(\mathcal{R})$. In the proof we need the elementary fact that a sequence of $\mathcal{R}$-modules

$$N' \to N \to N'' \to 0$$

is exact if and only if the induced sequence

$$0 \to Hom_{\mathcal{R}}(N'', M) \to Hom_{\mathcal{R}}(N, M) \to Hom_{\mathcal{R}}(N', M)$$

is exact for all $\mathcal{R}$-modules M (see [Atiyah & Macdonald; 2.9] for example).

Lemma 6 *Let $\mathcal{S}$ be a commutative K-algebra, $\psi : \mathcal{R} \to \mathcal{S}$ a K-algebra homomorphism and M an $\mathcal{S}$-module. Then:*

(i) $Hom_{\mathcal{S}}(\mathcal{S} \otimes_{\mathcal{R}} \Omega_K(\mathcal{R}), M) \cong Der_K(\mathcal{R}, M)$.

(ii) The following sequences of $\mathcal{S}$-modules are exact:

(a) $0 \to Der_{\mathcal{R}}(\mathcal{S}, M) \xrightarrow{\sigma} Der_K(\mathcal{S}, M) \xrightarrow{\tau} Der_K(\mathcal{R}, M)$

(b) $\mathcal{S} \otimes_{\mathcal{R}} \Omega_K(\mathcal{R}) \xrightarrow{\alpha} \Omega_K(\mathcal{S}) \xrightarrow{\beta} \Omega_{\mathcal{R}}(\mathcal{S}) \to 0$

with α being a split injection if and only if τ is surjective for all $\mathcal{S}$-modules M.

Proof: (i)

$$Hom_{\mathcal{S}}(\mathcal{S} \otimes_{\mathcal{R}} \Omega_K(\mathcal{R}), M) \cong Hom_{\mathcal{R}}(\Omega_K(\mathcal{R}), Hom_{\mathcal{S}}(\mathcal{S}, M)) \cong Hom_{\mathcal{R}}(\Omega_K(\mathcal{R}), M)$$

which is isomorphic to $Der_K(\mathcal{R}, M)$ by Lemma 5 (ii).

(ii) Sequence (b) is achieved by setting $\alpha(s \otimes dr) = sd(\psi(r))$ and $\beta(sdt) = sdt$ for $r \in \mathcal{R}$ and $s, t \in \mathcal{S}$. Sequence (a) is then yielded by appyling the functor $Hom_{\mathcal{S}}(-, M)$ to (b). The final remark is simply an easy application of homological algebra. $\square$

The exact sequence (b) above is called the *first fundamental exact sequence* for the differentials. The next lemma is called the *second fundamental exact sequence*.

Lemma 7 *Let $\mathcal{S}$ be a commutative K-algebra and let $\mathcal{R} = \mathcal{S}/I$ for I an ideal of $\mathcal{S}$. Then we have an exact sequence of $\mathcal{S}$-modules as follows:*

$$I/I^2 \xrightarrow{\theta} \mathcal{R} \otimes_{\mathcal{S}} \Omega_K(\mathcal{S}) \xrightarrow{\phi} \Omega_K(\mathcal{R}) \rightarrow 0.$$

Proof: Define $\theta(x + I^2) = 1 \otimes dx$ and $\phi(1 \otimes ds) = d(s + I)$ for $x \in I$ and $s \in \mathcal{S}$. It is straightforward to check that these are well defined and that $\phi \circ \theta = 0$. Now let M be an $\mathcal{R}$-module and apply the functor $Hom_{\mathcal{S}}(-, M)$ to the sequence as in the last lemma to give us the following sequence:

$$0 \rightarrow Der_K(\mathcal{R}, \mathcal{S}) \xrightarrow{\theta'} Der_K(\mathcal{S}, M) \xrightarrow{\phi'} Hom_{\mathcal{S}}(I, M).$$

Here θ' is given by composition with the projection of $\mathcal{S}$ onto $\mathcal{R}$ and ϕ' is restriction to I. This sequence is easily seen to be exact, and so the original sequence must be exact also. $\square$

We now must restrict ourselves to the situation that is of most interest to us and will be of most importance in Chapter Two: for the rest of this section let k be a field and let $\mathcal{R}$ be a noetherian, regular, semilocal k-algebra which is an integral domain and contains a second field K over which every factor ring of $\mathcal{R}$ by a maximal ideal is algebraic. That is, if $\mathbf{m}$ is a maximal ideal of $\mathcal{R}$, then $\mathcal{R}/\mathbf{m}$ is algebraic over K (identifying K with its image in the factor ring). The semilocal condition (i.e. $\mathcal{R}$ has only finitely many maximal ideals) is needed to ensure that $\Omega_K(\mathcal{R})$ is a finitely generated $\mathcal{R}$-module. The regular condition is not actually needed for the next few results, but is necessary in order to prove that $\Delta_K(\mathcal{R})$ is simple and has well-behaved Krull and global dimensions. Also, from now on Q will denote the field of fractions of $\mathcal{R}$ and n will be the Krull and global dimensions of $\mathcal{R}$.

One example of a ring which satisfies all these conditions is the coordinate ring of a smooth variety, localised at a maximal ideal, where the field K is just k itself. But the example that we have in mind is the following:

Example: Let $\mathcal{A}$ be the coordinate ring of a smooth n-dimensional variety. Then the field of fractions F of $\mathcal{A}$ has transcendence degree n over k. Let $x_1, \ldots, x_n$ be a transcendence basis for F with each $x_i \in \mathcal{A}$ so that $k[x_1, \ldots, x_n] \subseteq \mathcal{A}$. Let $\mathbf{p}$ be a height one prime ideal of $\mathcal{A}$. Then the length of a maximal regular sequence of $\mathcal{A}$ contained in $\mathbf{p}$ is at most one and we may assume without loss of generality that $k[x_1, \ldots, x_{n-1}] \bigcap \mathbf{p} = 0$. Therefore if we localise $\mathcal{A}$ at $\mathbf{p}$ we invert every element of $k[x_1, \ldots, x_{n-1}]$ and so the field $K = k(x_1, \ldots, x_{n-1})$ is contained in $\mathcal{A}_\mathbf{p}$. The ring $\mathcal{A}_\mathbf{p}$ now has only one maximal ideal $\mathbf{p}\mathcal{A}_\mathbf{p}$ and the field obtained by factoring out this maximal ideal is algebraic over K.

Lemma 8 *Let $\mathcal{R}$, K and k be as above. Then $\Omega_K(\mathcal{R}_{\mathbf{m}})$ is finitely generated for every maximal ideal $\mathbf{m}$ of $\mathcal{R}$.*

Proof: Let $\mathbf{m}$ be a maximal ideal of $\mathcal{R}$, $\mathcal{A} = \mathcal{R}_{\mathbf{m}}$ and $\overline{\mathcal{A}} = \mathcal{A}/\mathbf{m}\mathcal{A}$. Write $\mathbf{M} = \mathbf{m}\mathcal{A}$. Then by Lemma 7 we have the following exact sequence:

$$\mathbf{M}/\mathbf{M}^2 \to \overline{\mathcal{A}} \otimes_{\mathcal{A}} \Omega_K(\mathcal{A}) \to \Omega_K(\overline{\mathcal{A}}) \to 0.$$

Now, each element $f \in \overline{\mathcal{A}}$ is algebraic over K: $\sum f^i c_i = 0$ for some $c_i \in K$. Applying the universal derivation d to this expression yields that $\sum i c_i f^{i-1} df = 0$. Therefore $df = 0$ and $\Omega_K(\overline{\mathcal{A}}) = 0$. Hence the map from $\mathbf{M}/\mathbf{M}^2$ to $\overline{\mathcal{A}} \otimes_{\mathcal{A}} \Omega_K(\mathcal{A})$ is surjective and it follows that $\Omega_K(\mathcal{A})$ is a finitely generated $\mathcal{A}$-module. □

Now we prove some facts about localisation which will in turn yield results about the properties of $\mathcal{R}$.

Lemma 9 *Let S be a multiplicatively closed subset of $\mathcal{R}$, M an $\mathcal{R}_S$-module and let $\delta \in Der_K(\mathcal{R}, M)$. Then δ induces a unique derivation in $Der_K(\mathcal{R}_S, M)$.*

Proof: Let $r \in \mathcal{R}$ and $s \in S$. Define $\overline{\delta}(s^{-1}r) = s^{-1}(s\delta(r) - r\delta(s))$. Then $\overline{\delta}$ is a well defined derivation in $Der_K(\mathcal{R}_S, M)$ which restricts to δ on $\mathcal{R}$. □

Proposition 10 *Let S be a multiplicatively closed subset of $\mathcal{R}$. Then:*

(i) There is an isomorphism $\phi : \Omega_K(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R}_S \to \Omega_K(\mathcal{R}_S)$.

(ii) $\Omega_K(\mathcal{R})$ is a finitely generated $\mathcal{R}$-module.

(iii) $Der_K(\mathcal{R}_S) \cong \mathcal{R}_S \otimes_{\mathcal{R}} Der_K(\mathcal{R})$ and $Der_K(\mathcal{R}) \cong \{\delta \in Der_K(\mathcal{R}_S) \mid \delta(\mathcal{R}) \subseteq \mathcal{R}\}$.

Proof: (i) By Lemma 9 the map from $Der_K(\mathcal{R}_S, M)$ to $Der_K(\mathcal{R}, M)$ is surjective for every $\mathcal{R}_S$-module M. Therefore Lemma 6 $(ii)(a)$ gives us the following exact sequence:

$$0 \to \Omega_K(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R}_S \to \Omega_K(\mathcal{R}_S) \to \Omega_{\mathcal{R}}(\mathcal{R}_S) = 0.$$

(ii) By Lemma 8, $\Omega_K(\mathcal{R}_{\mathbf{m}})$ is finitely generated for every maximal ideal $\mathbf{m}$. But by part (i), $\Omega_K(\mathcal{R}_{\mathbf{m}}) \cong \Omega_K(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R}_S$. For each maximal ideal $\mathbf{m}$, let $\{a_1^{\mathbf{m}}, \ldots, a_{r(\mathbf{m})}^{\mathbf{m}}\}$ be a generating set with each $a_i^{\mathbf{m}} \in \Omega_K(\mathcal{R})$. Define M to be the submodule of $\Omega_K(\mathcal{R})$ generated by all the a_i's. Then the factor module $\Omega_K(\mathcal{R})/M$ goes to zero when localised at every maximal ideal of $\mathcal{R}$ and hence is the zero module itself. Therefore $\Omega_K(\mathcal{R})$ is finitely generated.

(iii) Since $\Omega_K(\mathcal{R})$ is finitely generated we have that:

$$\mathcal{R}_S \otimes_{\mathcal{R}} Hom_{\mathcal{R}}(\Omega_K(\mathcal{R}), M) \cong Hom_{\mathcal{R}_S}(\Omega_K(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R}_S, M) \cong Hom_{\mathcal{R}_S}(\Omega_K(\mathcal{R}_S), M)$$

where the second isomorphism follows by part (i). Hence by lemma $5(ii)$,

$$\mathcal{R}_S \otimes_{\mathcal{R}} Der_K(\mathcal{R}, M) \cong Der_K(\mathcal{R}_S, M).$$

The final statement now follows from lemma 9. □

Theorem 11 *Let S be a multiplicatively closed subset of $\mathcal{R}$. Then S is a left and right Ore set of $\Delta_K(\mathcal{R})$, and $\Delta_K(\mathcal{R})_S = \Delta_K(\mathcal{R}_S)$.*

Proof: By the proposition, $\Delta_K(\mathcal{R}_S)$ is generated by $\mathcal{R}_S$ and the copy of $Der_K(\mathcal{R})$ contained in $Der_K(\mathcal{R}_S)$. The relations $\delta r - r\delta = \delta(r)$ for $\delta \in Der_K(\mathcal{R})$ and $r \in \mathcal{R}$ ensure that every element of $\Delta_K(\mathcal{R}_S)$ can be written in the form θs^{-1} or $s^{-1}\theta$ for

$\theta \in \Delta_K(\mathcal{R})$ and $s \in S$. So $\Delta_K(\mathcal{R}_S)$ must be the left and right localisation of $\Delta_K(\mathcal{R})$ with respect to S. $\square$

Proposition 10 can also be used to prove that $\Delta_K(\mathcal{R})$ is noetherian. In order to do this we have to form the graded ring $\mathrm{gr}\Delta_K(\mathcal{R})$ associated to the grading on $\Delta_K(\mathcal{R})$ given by setting the m^{th} graded part $\Delta_K(\mathcal{R})_m$ of $\Delta_K(\mathcal{R})$ to be the $\mathcal{R}$-submodule generated by all the products of at most m derivations ($\Delta_K(\mathcal{R})_0$ is just $\mathcal{R}$ of course). The relations $\delta r - r\delta = \delta(r)$ and $\delta.\delta' - \delta'.\delta \in Der_K(\mathcal{R})$ for $\delta, \delta' \in Der_K(\mathcal{R})$ and $r \in \mathcal{R}$ ensure that $\mathrm{gr}\Delta_K(\mathcal{R})$ is commutative.

Theorem 12 *The rings* $\mathrm{gr}\Delta_K(\mathcal{R})$ *and* $\Delta_K(\mathcal{R})$ *are noetherian.*

Proof: By Proposition 10(*ii*), $\Omega_K(\mathcal{R})$ is a finitely generated $\mathcal{R}$-module. Hence $Der_K(\mathcal{R}) = \Omega_K(\mathcal{R})^*$ is also finitely generated. Therefore $\mathrm{gr}\Delta_K(\mathcal{R})$ is a factor ring of a polynomial ring over $\mathcal{R}$ and is noetherian. [MR;1.6.9] then shows that $\Delta_K(\mathcal{R})$ is also noetherian. $\square$

Theorem 11 allows us to show that $\Delta_K(\mathcal{R})$ is an integral domain by using the inclusion $\Delta_K(\mathcal{R}) \hookrightarrow \Delta_K(Q)$ where Q is the field of fractions of $\mathcal{R}$. So let us determine the structure of rings of derivations on fields.

Lemma 13 *Let* $F \supseteq k$ *be a field of transcendence degree* n *over* k. *Then* $\Omega_k(F)$ *is a free* F*-module with basis* $dx_1, \ldots, dx_n$ *where* $x_1, \ldots, x_n$ *form a transcendence basis for* F *over* k. *Also,* $Der_k(F)$ *is free with a basis consisting of extensions to* F *of* $\partial/\partial x_1, \ldots, \partial/\partial x_n$.

Proof: Let $G = k[x_1, \ldots, x_n]$. Then since F is algebraic over G, for every $f \in F$ we have a relation of the form $\sum a_i f^i = 0$ for some $a_i \in G$. Applying the universal derivation $d : F \to \Omega_k(F)$ we find that $dx_1, \ldots .dx_n$ span $\Omega_k(F)$. A similar argument to that given in the proof of Lemma 8 shows that $\Omega_G(F) = 0$. Therefore by Lemma 6$(ii)(b)$ there is a surjection from $F \otimes_G \Omega_k(G)$ onto $\Omega_k(F)$ which is an isomorphism provided that each $\delta \in Der_k(G, M)$ extends to a derivation in $Der_k(F, M)$ for every F-module M. Now, $F = G(f)$ for some $f \in F$ which has a minimum polynomial $p = \sum p_i x^i$ say. Then $F \cong G[y]/(p)$ and so we may view M as a $G[y]$-module. Define $\zeta \in Der_k(G[y], M)$ to extend δ and have $\zeta(y) = -p'(f)^{-1} \sum f^i \delta(p_i)$. Then $\zeta(p) = 0$ and ζ induces a derivation in $Der_k(F, M)$ as required. The rest of the lemma follows from the fact that $Der_k(F) = \Omega_k(F)^*$. □

Corollary 14 *If $F \supseteq k$ is a field of transcendence degree n over k with transcendence basis $x_1, \ldots, x_n$ then $\Delta_k(F) = F[\partial/\partial x_1, \ldots, \partial/\partial x_n]$.*

Proof: This is an immediate consequence of the fact that $Der_k(F)$ is a free F-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$. □

Corollary 15 *$\Delta_K(\mathcal{R})$ is an integral domain.*

Proof: By Theorem 11, by setting $S = \mathcal{R}\backslash\{0\}$, we have the following inclusion: $\Delta_K(\mathcal{R}) \hookrightarrow \Delta_K(Q)$. But by the previous corollary, $\Delta_K(Q)$ is an integral domain, and the result follows. □

The results that we have proved so far have not yet used the regularity condition on $\mathcal{R}$ and it is this which we turn our attention to now. The next proposition shows that the ring of derivations on a regular ring is very closely linked to the Weyl algebra.

Proposition 16 *Let* $\mathbf{m}$ *be a maximal ideal of* $\mathcal{R}$ *and set* $\mathcal{A} = \mathcal{R}_{\mathbf{m}}$. *Set* $M = \mathbf{m}\mathcal{A}$ *and let* $y_1, \ldots, y_n$ *be a minimal generating set for* M. *Then* $\Omega_K(\mathcal{A})$ *is a free* $\mathcal{A}$*-module with basis* $dy_1, \ldots, dy_n$ *and* $\Delta_K(\mathcal{A}) = \mathcal{A}[\partial/\partial y_1, \ldots, \partial/\partial y_n]$.

Proof: As in the proof of Lemma 8, we have a surjective homomorphism from M/M^2 onto $\Omega_K(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}/M$ so that $\Omega_K(\mathcal{A})$ is generated by the dy_i's. But by Lemma 12 $\Omega_K(Q)$ is free of rank n and so the dy_i's must freely generate $\Omega_K(\mathcal{A})$. □

So we have seen that $\Delta_K(\mathcal{R})$ looks quite a lot like the Weyl algebra at least locally. The final things to do for derivations are to calculate the Krull and global dimensions for $\Delta_K(\mathcal{R})$ and to prove that it is a simple ring. To this end we need the following lemma.

Lemma 17 *Let* $\{\mathbf{m}_1, \ldots, \mathbf{m}_r\}$ *be the maximal ideals of* $\mathcal{R}$. *Then* $\bigoplus_{i=1}^{r} \Delta_K(\mathcal{R}_{\mathbf{m}_i})$ *is a left and right faithfully flat* $\Delta_K(\mathcal{R})$*-module.*

Proof: Let I be a left ideal of $\Delta_K(\mathcal{R})$ such that $\Delta_K(\mathcal{R}_{\mathbf{m}_i}) \otimes (\Delta_K(\mathcal{R})/I) = 0$ for each i. But $\Delta_K(\mathcal{R}_{\mathbf{m}_i}) \otimes (\Delta_K(\mathcal{R})/I) = \mathcal{R}_{\mathbf{m}_i} \otimes (\Delta_K(\mathcal{R})/I)$. So $\Delta_K(\mathcal{R})/I$ is a left $\mathcal{R}$-module which is zero when localised at every maximal ideal of $\mathcal{R}$. Therefore it must be the zero module itself and $\Delta_K(\mathcal{R}) = I$ as required. The same argument proves the right-handed version also. □

The faithfully flat $\Delta_K(\mathcal{R})$-module constructed in the last lemma provides a way of transferring information known about the localised rings $\Delta_K(\mathcal{R}_{\mathbf{m}},)$ down to $\Delta_K(\mathcal{R})$. Therefore we look at these localised rings in the next propostion.

Proposition 18 *Let* $\mathbf{m}$ *be a maximal ideal of* $\mathcal{R}$. *Then:*

(i) $\Delta_K(\mathcal{R}_{\mathbf{m}},)$ *is simple.*

(ii) $\mathcal{K}(\Delta_K(\mathcal{R}_{\mathbf{m}},)) = n$.

(iii) $\mathrm{gld}\Delta_K(\mathcal{R}_{\mathbf{m}},) = n$.

Proof: (i) Let $\mathcal{A} = \mathcal{R}_{\mathbf{m}}$, let I be a non-zero ideal of $\Delta_K(\mathcal{A})$ and let $0 \neq x \in \mathcal{A}$. Recall from Proposition 16 that $\Delta_K(\mathcal{A}) = \mathcal{A}[\partial/\partial y_1, \ldots, \partial/\partial y_n]$ where $y_1, \ldots, y_n$ generate $\mathbf{m}\mathcal{A}$. Now, when we take the commutator of x with y_1 we get another element of I, and repeating this process enough times for each y_i we arrive at a non-zero element of $I \bigcap \mathcal{A}$. Hence the Krull dimension of $\mathcal{A}/(I \bigcap \mathcal{A})$ is strictly less than that of $\mathcal{A}$ so that $(I \bigcap \mathcal{A}) \bigcap k[y_1, \ldots, y_n] \neq 0$ (otherwise there would be a copy of $k[y_1, \ldots, y_n]$ inside $\mathcal{A}/(I \bigcap \mathcal{A})$ which would then have to have Krull dimension greater or equal to n). Taking commutators of an element of $I \bigcap k[y_1 \ldots, y_n]$ with a suitable number of $\partial/\partial y_i$'s shows that $I \bigcap k \neq 0$ and so $I = \Delta_K(\mathcal{A})$ as required.
(ii) & (iii) Since $\Delta_K(\mathcal{A}) = \mathcal{A}[\partial/\partial y_1] \ldots [\partial/\partial y_n]$, a repeated application of [MR;9.1.14] gives the result. □

We are now ready to finish this section with as complete a description of $\Delta_K(\mathcal{R})$ as we shall need for the later chapters.

Theorem 19 *(i)* $\Delta_K(\mathcal{R})$ *is a simple, noetherian domain.*

(ii) $\mathcal{K}(\Delta_K(\mathcal{R})) = n$.

(iii) $\mathrm{gld}(\Delta_K(\mathcal{R})) = n$.

Proof: (i) We already know from Theorem 12 and Corollary 15 that $\Delta_K(\mathcal{R})$ is a noetherian integral domain. Let I be a non-zero ideal of $\Delta_K(\mathcal{R})$. Then $I_{\mathbf{m}} = I \otimes \Delta_K(\mathcal{R}_{\mathbf{m}}) \lhd \Delta_K(\mathcal{R}_{\mathbf{m}})$ for every maximal ideal $\mathbf{m}$ of $\mathcal{R}$. But $\Delta_K(\mathcal{R}_{\mathbf{m}})$ is simple by Proposition 18, hence $I_{\mathbf{m}} = \Delta_K(\mathcal{R}_{\mathbf{m}})$ and

$$\bigoplus_{i=1}^{r} \Delta_K(\mathcal{R}_{\mathbf{m}_i}) \otimes (\Delta_K(\mathcal{R})/I) = 0.$$

But by Lemma 17 $\bigoplus \Delta_K(\mathcal{R}_{\mathbf{m}_i})$ is faithfully flat, and so $I = \Delta_K(\mathcal{R})$ as required.

(ii) By [MR; Lemma 6.5.3(i)], $\mathcal{K}(\Delta_K(\mathcal{R})) \leq \sup\{\mathcal{K}(\Delta_K(\mathcal{R}_{\mathbf{m}_i})\}$. But by Proposition 18, the Krull dimension of each $\mathcal{K}(\Delta_K(\mathcal{R}_{\mathbf{m}_i})$ equals n. Also, $n = \mathcal{K}(\Delta_K(\mathcal{R}_{\mathbf{m}_i}) \leq \mathcal{K}(\Delta_K(\mathcal{R}))$ so that $\mathcal{K}(\Delta_K(\mathcal{R})) = n$.

(iii) By Proposition 18, $\mathrm{gld}\Delta_K(\mathcal{R}_{\mathbf{m}_i}) = n$ for each maximal ideal $\mathbf{m}_i$ of $\mathcal{R}$. Given a left $\Delta_K(\mathcal{R})$-module M, we shall show that the flat dimension $\mathrm{fd}M \leq n$. It will then follow by [MR;7.1.5] that the projective dimension of M is less than or equal to n also. For each i and each module M, denote by M_i the kernel of the map $M \to \mathcal{R}_{\mathbf{m}_i} \otimes M$. Call M_i the *i-torsion submodule* and say that M is i-torsion if $M = M_i$. Note that if M is i-torsion then $\mathcal{R}_{\mathbf{m}_j} \otimes M$ is i-torsion also. Suppose that there exist modules M with $\mathrm{fd}M > n$. Out of all such modules choose an M which is i-torsion for as many i as possible (after renumbering we may assume that M is i-torsion for $i = 1, \ldots, s$). The faithful flatness of $\bigoplus_{i=1}^{r} \Delta_K(\mathcal{R}_{\mathbf{m}_i})$ ensures that $s < r$. Let $\overline{M} = M/M_{s+1}$ and consider the following exact sequences:

$$0 \to M_{s+1} \to M \to \overline{M} \to 0 \text{ and}$$

$$0 \to \overline{M} \to \mathcal{R}_{s+1} \otimes M \to (\mathcal{R}_{s+1} \otimes M)/\overline{M} \to 0.$$

Notice that both M_{s+1} and $(\mathcal{R}_{s+1} \otimes M)/\overline{M}$ are i-torsion for $i = 1, \ldots, s+1$ and so have flat dimension less than or equal to n. Also,

$$\mathrm{fd}_{\Delta_K(\mathcal{R})}(\mathcal{R}_{s+1} \otimes M) = \mathrm{fd}_{\Delta_K(\mathcal{R}_{\mathfrak{m}_s})}(\mathcal{R}_{s+1} \otimes M) \leq n.$$

The second exact sequence now shows that $\mathrm{fd}\overline{M} \leq n$ and then the first that $\mathrm{fd}M \leq n$ which is a contradiction.

Therefore $\mathrm{gld}\Delta_K(\mathcal{R}) = n$. $\square$

This concludes our study of rings of derivations and we now turn our attention to the principal objects of interest in this thesis, the rings of differential operators.

1.2 Differential Operators

In this section we begin our study of the rings of differential operators on algebraic varieties. We start off by defining what we mean by the ring of differential operators, then develop some of the tools which are used in their study and finally show that the ring of differential operators on a smooth variety coincides with its ring of derivations which we already know to be a very well behaved ring.

Definition 1 *Let K be a commutative ring and let $\mathcal{R}$ be a commutative K-algebra. The ring of K-linear differential operators $\mathcal{D}_K(\mathcal{R})$ is defined inductively as follows:*

$$Set\ \mathcal{D}_K^0(\mathcal{R}) = \mathcal{R},\ \mathcal{D}_K^n(\mathcal{R}) = \{\theta \in End_K(\mathcal{R}) | \theta r - r\theta \in \mathcal{D}_K^{n-1}(\mathcal{R})\}\ for\ n \in \mathbf{N},$$

$$and\ \ \mathcal{D}_K(\mathcal{R}) = \bigcup_{n \in \mathbf{N}} \mathcal{D}_K^n(\mathcal{R}).$$

When θ lies in $\mathcal{D}_K^n(\mathcal{R})$ but not in $\mathcal{D}_K^{n-1}(\mathcal{R})$, we say that θ is an n^{th} order differential operator.

This definition may seem quite intractable, but the object $\mathcal{D}_K(\mathcal{R})$ carries a lot of information about $\mathcal{R}$ and it is the object of the theory of differential operators to find out as much as possible about $\mathcal{D}_K(\mathcal{R})$ by studying the algebraic structure of it without having to use the definition very often. From now on, when the context is clear we shall omit the K subscript from $\mathcal{D}_K(\mathcal{R})$, but the reader should be warned that we will come across situations where there are two or more base rings that come into play at one time. In these cases though we will replace the subscripts for clarity.

The first thing to notice about $\mathcal{D}(\mathcal{R})$ is that it forms a ring under the composition of maps inside $End_K(\mathcal{R})$. Indeed, a simple induction argument shows that $\mathcal{D}^n(\mathcal{R}).\mathcal{D}^m(\mathcal{R}) \subseteq \mathcal{D}^{n+m}(\mathcal{R})$. Of course, $\mathcal{D}(\mathcal{R})$ is a non-commutative ring. The case that will be of most interest to us is when the commutative ring $\mathcal{R}$ is the coordinate ring of a variety, $\mathcal{X}$ say. In this case we will write $\mathcal{D}(\mathcal{X})$ in place of $\mathcal{D}_k(\mathcal{R})$ and will call this *the ring of differential operators on $\mathcal{X}$*. As this section progresses we will see that the behaviour of the ring of differential operators on a variety $\mathcal{X}$ is closely linked to the geometry of $\mathcal{X}$. In the best possible case (when $\mathcal{X}$ is smooth) the ring of differential operators on $\mathcal{X}$ coincides with the ring of derivations on $\mathcal{X}$ which, as we saw in the previous section, is a very nice ring indeed. The more singular that $\mathcal{X}$ becomes, the more badly behaved $\mathcal{D}(\mathcal{X})$ is.

Let us start off by developing one the major tools which we shall need for our investigation of rings of differential operators: localisation. We are able to prove results which are analogous to the ones for derivation rings as in Theorem 1.11.

Proposition 2 *Let K be a commutative ring, $\mathcal{R}$ be a commutative K-algebra which is an integral domain and S a multiplicatively closed subset of $\mathcal{R}$. Then $\mathcal{D}_K(\mathcal{R}) \cong \{\delta \in \mathcal{D}_K(\mathcal{R}_S) | \delta(\mathcal{R}) \subseteq (\mathcal{R})\}$.*

Proof: Set $D^n(\mathcal{R}) = \{\delta \in \mathcal{D}_K^n(\mathcal{R}_S) | \delta(\mathcal{R}) \subseteq \mathcal{R}\}$. I claim that the map from $D^n(\mathcal{R})$ to $\mathcal{D}_K^n(\mathcal{R})$ given by restriction to $\mathcal{R}$ is an isomorphism. We prove this by induction on n, noting that $D^0(\mathcal{R}) = \mathcal{R} = \mathcal{D}_K^0(\mathcal{R})$ for the $n = 0$ case. So suppose that $n > 0$ and that the claim is true for all $i < n$. Let $\delta \in D^n(\mathcal{R})$ be such that $\delta|_{\mathcal{R}} = 0$. Then $[\delta, s](x) = \delta(sx) - s\delta(x) = 0$ for all $s \in S$ and $x \in \mathcal{R}$. But $[\delta, s] \in D^{n-1}(\mathcal{R})$ and so by induction, since $[\delta, s]|_{\mathcal{R}} = 0$, $[\delta, s] = 0$ on the whole of $\mathcal{R}_S$. But $s\delta(s^{-1}x) = -[\delta, s](s^{-1}x) + \delta(x) = 0$. Therefore $\delta(s^{-1}x) = 0$ and $\delta = 0$. So the restriction map is injective.

To show that the restriction map is surjective, let $\delta \in \mathcal{D}_K^n(\mathcal{R})$ and define $\hat{\delta} \in D^n(\mathcal{R})$ by:

$$\hat{\delta}(s^{-1}x) = s^{-1}(\delta(x) - \widehat{[\delta, s]}(s^{-1}x))$$

for $s \in S$ and $x \in \mathcal{R}$, where $\widehat{[\delta, s]}$ is the unique element of $D^{n-1}(\mathcal{R})$ extending $[\delta, s]$ (which exists by induction). This $\hat{\delta}$ is unique since if there exist $t \in S$ and $y \in \mathcal{R}$

such that $s^{-1}x = t^{-1}y$, then we have the following:

$$\begin{aligned}
s^{-1}(\delta(x) - \widehat{[\delta, s]}(s^{-1}x) &= (st)^{-1}(t\delta(x) - t\widehat{[\delta, s]}(s^{-1}x)) \\
&= (st)^{-1}(\delta(tx) - [\delta, t](x) - t\widehat{[\delta, s]}(s^{-1}x)) \\
&= (st)^{-1}(\delta(tx) - ([\delta, t]s - t\widehat{[\delta, s]})(s^{-1}x)) \\
&= (st)^{-1}(\delta(sy) - \widehat{[\delta, st]}(t^{-1}y))
\end{aligned}$$

where the last equality follows from the uniqueness of $\widehat{[\delta, st]}$. Finally, this last line equals $t^{-1}(\delta(y) - \widehat{[\delta, t]}(t^{-1}y)$ by symmetry.

It remains to be shown that $\hat{\delta} \in \mathcal{D}_K^n(\mathcal{R}_S)$. We prove this by showing that $[\hat{\delta}, x] = \widehat{[\delta, x]}$ which lies in $\mathcal{D}_K^{n-1}(\mathcal{R}_S)$ by induction. So let $x, y \in \mathcal{R}$ and $s \in S$. Then:

$$\begin{aligned}
[\hat{\delta}, x](s^{-1}y) &= \hat{\delta}(s^{-1}xy) - x\hat{\delta}(s^{-1}y) \\
&= s^{-1}(\delta(xy) - [\widehat{[\delta, s]}(s^{-1}xy)) - xs^{-1}(\delta(y) - \widehat{[\delta, s]}(s^{-1}y) \\
&= s^{-1}([\delta, x](y) - [\widehat{[\delta, x]}, s](s^{-1}y)) \\
&= \widehat{[\delta, x]}(s^{-1}y).
\end{aligned}$$

This completes the proof. □

Corollary 3 *If $\mathcal{R}$, K and S are as in Proposition 2 and if $\mathcal{R}$ is affine over K then S is a left and right Ore set of $\mathcal{D}_K(\mathcal{R})$ and $\mathcal{D}_K(\mathcal{R})_S \cong \mathcal{D}_K(\mathcal{R}_S)$.*

Proof: By Proposition 2, we may think of $\mathcal{D}_K(\mathcal{R})$ as lying inside $\mathcal{D}_K(\mathcal{R}_S)$. We must show that every element of $\mathcal{D}_K(\mathcal{R}_S)$ can be written in the form $s^{-1}\partial$ for $s \in S$ and $\partial \in \mathcal{D}_K(\mathcal{R})$. Let Λ be a finite subset of $\mathcal{R}$, containing 1, that generates $\mathcal{R}$ as a K-algebra. Denote by Λ^m the subset of $\mathcal{R}$ given by the collection of all finite sums of products of at most m elements of Λ. We claim that given a $\delta \in \mathcal{D}_K(\mathcal{R}_S)$ such that $\delta(\Lambda) \subseteq \mathcal{R}$, then $\delta(\Lambda^i) \subseteq \mathcal{R}$ for all $i \geq 0$. The proof of this claim is by induction on the order of δ. If δ has order zero then $\delta \in \mathcal{R}_S$ and $\delta(\Lambda^i) = \Lambda^{i-1}\delta(\Lambda) \subseteq \mathcal{R}$ as required. So suppose that δ has order n and that the claim is true for all differential operators of order less than n. We now have to use induction on i. We are given that $\delta(\Lambda) \subseteq \mathcal{R}$, so the case $i = 1$ is trivial. Suppose that $\delta(\Lambda^i) \subseteq \mathcal{R}$ for some $i > 1$. Then:

$$\delta(\Lambda^{i+1}) \subseteq [\delta, \Lambda](\Lambda^i) + \Lambda\delta(\Lambda^i) \subseteq \mathcal{R}.$$

This proves the claim.

Now let δ be any element of $\mathcal{D}_K(\mathcal{R}_S)$. Then $\delta(\Lambda)$ is a finite set so that there exists an element $s \in S$ such that $s\delta(\Lambda) \subseteq \mathcal{R}$. Then by the claim, $s\delta(\Lambda^i) \subseteq \mathcal{R}$ for all i. But Λ is a generating set for $\mathcal{R}$, so $s\delta(\mathcal{R}) \subseteq \mathcal{R}$. Therefore, by Proposition 2, $\delta \in \mathcal{D}_K(\mathcal{R})_S$ as required. □

Corollary 3 is of fundamental importance to us and we shall use it again and again, usually without mention. Propostion 2 is of interest in itself. In particular, when $S = \mathcal{R}\backslash\{0\}$, it says that $\mathcal{D}_K(\mathcal{R}) = \{\delta \in \mathcal{D}_K(Q) | \delta(\mathcal{R}) \subseteq \mathcal{R}\}$, where Q is the field of fractions of $\mathcal{R}$. If we were then able to prove the analogous statement for derivations, we would have that the ring of derivations and the ring of differential operators on an algebra coincide. This turns out not to be the case though, except when $\mathcal{R}$ is a regular ring (such as the coordinate ring of a smooth variety).

Let us prove then, as promised that for a smooth variety $\mathcal{X}$, $\mathcal{D}(\mathcal{X}) = \Delta(\mathcal{X})$. As in the previous section, we also need a slight variation of this result to cover the case of when $\mathcal{R}$ is a regular, noetherian, semilocal domain which contains a field K over which $\mathcal{R}/\mathbf{m}$ is algebraic for each maximal ideal $\mathbf{m}$. Since the proofs of these two results are very similar and the first is contained in [MR; 15.5.6], we just prove the second. So let $\mathcal{R}$ be as above, and denote the field of fractions of $\mathcal{R}$ by Q.

Proposition 4 *Let $\mathcal{R}$ be as in the preceding paragraph. Then* $\Delta_K(\mathcal{R}) = \{\delta \in \Delta_K(Q) | \delta(\mathcal{R}) \subseteq \mathcal{R}\}$.

Proof: Let $\mathbf{m}$ be a maximal ideal of $\mathcal{R}$ and let $\delta \in \Delta_K(Q)$ be such that $\delta(\mathcal{R}) \subseteq \mathcal{R}$. Then I claim that $\delta(\mathcal{R}_{\mathbf{m}}) \subseteq \mathcal{R}_{\mathbf{m}}$. To see this, let $s \in \mathcal{R}\backslash\mathbf{m}$. Then $\delta s - s\delta$ is an element of the derivation ring of degree strictly less than that of δ and so by induction we may asssume that $(\delta s - s\delta)(s^{-1}r) \in \mathcal{R}_{\mathbf{m}}$ for all $r \in \mathcal{R}$. Hence $\delta(r) - s\delta(s^{-1}r) \in \mathcal{R}_{\mathbf{m}}$ and so $\delta(s^{-1}r) \in \mathcal{R}_{\mathbf{m}}$ as required.

Next we claim that the same δ actually lies in $\Delta_K(\mathcal{R}_{\mathbf{m}})$. We proceed by induction on the order of δ, noticing that the claim is trivial for the order zero case. So let m be the order of δ and suppose that the claim is true for derivations of order $m-1$ or less. Recall from Proposition 1.16 that $Der_K(\mathcal{R}_{\mathbf{m}})$ is a free $\mathcal{R}_{\mathbf{m}}$-module with basis $\partial/\partial y_1, \ldots, \partial/\partial y_n$ where $y_1, \ldots, y_n$ are a minimal generating set for $\mathbf{m}\mathcal{R}_{\mathbf{m}}$. If we set $\delta_i = \delta y_i - y_i\delta$ for each i then by induction, $\delta_i \in \Delta_K(\mathcal{R}_{\mathbf{m}})$. Again by Proposition 1.16, each δ_i may be viewed as a polynomial in the $\partial/\partial y_i$'s with coefficients in $\mathcal{R}$ written on the left. Therefore, we may think of $[[\delta, y_i], y_j]$ as follows:

$$[[\delta, y_i], y_j] = \frac{\partial}{\partial \frac{\partial}{\partial y_j}}(\delta_i).$$

Now, since the y_i's commute with each other we also have that $[[\delta, y_i], y_j] = [[\delta, y_j], y_i]$. Therefore, by formally integrating $[\delta_i, y_i]$ twice with respect to $\partial/\partial y_i$ and calling the result $F \in \Delta_K(\mathcal{R}_{\mathbf{m}})$, we see that $\partial F/\partial \frac{\partial}{\partial y_i} = \delta_i$ for each i. Now let $g = \delta - F \in \Delta_K(Q)$.

Then $g(\mathcal{R}_{\mathbf{m}}) \subseteq \mathcal{R}_{\mathbf{m}}$ and $[g, y_i] = 0$ for all i. It follows that the order of g must be zero so that $g \in \mathcal{R}_{\mathbf{m}}$ and $\delta \in \Delta_K(\mathcal{R}_{\mathbf{m}})$ as claimed.

Finally, let $\mathbf{m}_1, \ldots, \mathbf{m}_r$ be the set of maximal ideals of $\mathcal{R}$ so that $\delta \in \Delta_K(\mathcal{R}_{\mathbf{m}_i})$ for each i by the above. Thus if $M = (\mathcal{R}\delta + \Delta(\mathcal{R}))/\Delta(\mathcal{R})$, then $\Delta(\mathcal{R}_{\mathbf{m}_i}) \otimes M = 0$ for each i. But the faithful flatness of the module $\bigoplus \Delta_K(\mathcal{R}_{\mathbf{m}_i})$ proved in 1.17 implies that $M = 0$. □

Theorem 5 *(i) Let $\mathcal{X}$ be a smooth variety. Then $\mathcal{D}_k(\mathcal{X}) = \Delta_k(\mathcal{X})$.*

(ii) If $\mathcal{R}$ is as in Proposition 4 then $\mathcal{D}_K(\mathcal{R}) = \Delta_K(\mathcal{R})$.

Proof: (ii) All we have to prove is that $\mathcal{D}_K(Q) = \Delta_K(Q)$ and then Propositions 2 and 4 will give us the result. Once again, the proof is by induction on the order of differential operators. We shall show that $\mathcal{D}^r_K(Q) = \Delta_K(Q)_r$, the r^{th} filtered part of $\Delta_K(Q)$. The case when $r = 0$ is easy and also, when $r = 1$, it is easy to see that $\mathcal{D}^1_K(Q) = \Delta_K(Q)_1$. So suppose that $r > 1$ and that $\mathcal{D}^s_K(Q) = \Delta_K(Q)_s$ for all $s < r$. Let $y_1, \ldots, y_n$ be a transcendence basis for Q over K and let $\delta \in \mathcal{D}^r_K(Q)$. Then as in the proof of Proposition 4 and using the fact that $\mathcal{D}^{r-1}_K(Q) = \Delta_K(Q)_{r-1}$, we may find an $F \in \Delta_K(Q)_r$ such that $[g, y_i] = 0$ for all i, where $g = \delta - F$. We prove that this cannot happen unless $g \in Q$. Now, if $q \in Q$, then $[g, q]$ has the same property as g, but has lower order. Therefore, if the order of g is greater than 1, commutate it with enough elements of Q so that the resultant differential operator, h say, has order 1. By induction, $h \in \Delta_K(Q)_1 = Q + Der_K Q$ so that $h = a + \theta$ for some $a \in Q$ and $\theta \in Der_K(Q)$. The fact that $[h, y_i] = 0$ for all i ensures that θ is zero on $K(y_1, \ldots, y_n)$. But by Lemma 1.13, $Der_K(Q)$ is free on the $\partial/\partial y_i$'s, so θ must be zero. Therefore the order of h is zero, contradicting the fact that we chose h to have order 1. Hence $\delta - F \in Q$, and so $\delta \in \Delta_K(Q)$ as required. □

Occasionally, in what follows we shall be in the following situation: $\mathcal{R}$ is a commutative domain, $\mathcal{S}$ is a subring of $\mathcal{R}$ and $\mathcal{D}(\mathcal{R}, \mathcal{S})$ is the subset of differential operators δ on $\mathcal{R}$ whose image $\delta(\mathcal{R})$ is contained in $\mathcal{S}$. An alternative way of viewing this is to view $\mathcal{R}$ as an $\mathcal{S}$-module and consider the 'module of differential operators' from $\mathcal{R}$ into $\mathcal{S}$. Let us formalise this notion in a definition.

Definition 6 *Let K be a commutative ring, let $\mathcal{R}$ be a commutative K-algebra which is an integral domain and let M and N be $\mathcal{R}$-modules. Then define the* module of differential operators $\mathcal{D}_K(M, N)$ *from M into N as follows:*

$$\begin{aligned} \textit{Set} \quad \mathcal{D}_K^0(M,N) &= Hom_{\mathcal{R}}(M,N), \textit{ and for } n \geq 1, \\ \mathcal{D}_K^n(M,N) &= \{\theta \in Hom_K(M,N) \big| \theta r - r\theta \in \mathcal{D}_K^{n-1}(M,N)\ \forall r \in \mathcal{R}\}. \end{aligned}$$

Finally, define $\mathcal{D}_K(M,N) = \bigcup_{n \in \mathbb{N}} \mathcal{D}_K^n(M,N)$.

It is evident that if M and N both equal $\mathcal{R}$ then we recover the usual definition of $\mathcal{D}_K(\mathcal{R})$. Notice that $\mathcal{D}_K(M, N)$ is a right and left $\mathcal{R}$-module under composition of homomorphisms, so we may consider localisations of $\mathcal{D}_K(M, N)$. Exactly in the same way as for differential operator rings, modules of differential operators behave well with respect to localisation. In order to avoid complications we shall always assume that M and N are torsion free $\mathcal{R}$-modules.

Lemma 7 *Let K, $\mathcal{R}$ be as above with $\mathcal{R}$ being K-affine and suppose that S is a multiplicatively closed subset of $\mathcal{R}$. Suppose that M and N are torsion free $\mathcal{R}$-modules* *and that M is finitely generated over $\mathcal{R}$. Then:*

(i) an element of $\mathcal{D}_K(M,N)$ extends to a unique differential operator in $\mathcal{D}_K(M_S,N_S)$, where M_S and N_S may be regarded as modules over either $\mathcal{R}$ or $\mathcal{R}_S$;

(ii) $\mathcal{R}_S\mathcal{D}_K(M,N) \cong \mathcal{D}_K(M_S,N_S) \cong \mathcal{D}_K(M,N)\mathcal{R}_S$.

Proof: The proofs of (i) and the first part of (ii) follow in almost exactly the same way as in Propostion 2 and Corollary 3, using the fact that M is finitely generated. To see that $\mathcal{D}_K(M_S,N_S) \cong \mathcal{D}_K(M,N)\mathcal{R}_S$, notice that by the first part of (ii), we have that $\mathcal{D}_K(M,N) \subseteq \mathcal{D}_K(M_S,N_S)$, and so $\mathcal{D}_K(M,N)\mathcal{R}_S \subseteq \mathcal{D}_K(M_S,N_S)$.

For the opposite inclusion, suppose that $\mathcal{D}_K(M_S,N_S) \neq \mathcal{D}_K(M,N)\mathcal{R}_S$ and let $\theta \in \mathcal{D}_K(M_S,N_S)$ have the smallest possible degree as a differential operator with respect to not lying inside $\mathcal{D}(M,N)\mathcal{R}_S$. Then by the first part of (ii) there exists an $s \in S$ such that $s\theta \in \mathcal{D}_K(M,N)$. Also, if $\phi = s\theta - \theta s$ then ϕ has degree strictly less than θ, so must lie in $\mathcal{D}_K(M,N)\mathcal{R}_S$. Hence there is some $t \in S$ with $\phi t \in \mathcal{D}_K(M,N)$. Then $\theta st = -\phi t + s\theta t \in \mathcal{D}_K(M,N)$ and $\theta \in \mathcal{D}_K(M,N)\mathcal{R}_S$ as required. □

When $M = \mathcal{R}$, it is possible to rephrase Definition 6 in terms of a generalisation of the module of differentials $\Omega_K(\mathcal{R})$. This will enable us to prove things about the module $\mathcal{D}(\mathcal{R},N)$ by using the properties of $\Omega_K(\mathcal{R})$. It is actually possible to give a definition of differentials on a module, but we shall not need to do this.

Now, give $\mathcal{R}\otimes_K\mathcal{R}$ the structure of a ring in the obvious way and define the multiplication map $\mu : \mathcal{R}\otimes\mathcal{R} \to \mathcal{R}$ by $\mu(x\otimes y) = xy$. Define J to be the kernel of the multiplication map so that J is an ideal of $\mathcal{R}\otimes\mathcal{R}$. Notice that J is generated by all the elements of the form $1 \otimes x - x \otimes 1$ for $x \in \mathcal{R}$. Now, we may make $Hom_K(\mathcal{R}, M)$ into an $\mathcal{R}\otimes\mathcal{R}$-module by defining $(x\otimes y)(\theta) = x\theta y$. Then given a $\theta \in \mathcal{D}^n_K(\mathcal{R}, M)$ and an $x \in \mathcal{R}$, we can see that $(1\otimes x - x\otimes 1)(\theta) \in \mathcal{D}^{n-1}_K(\mathcal{R}, M)$. It is easy to see from this that for each $n \in \mathbf{N}$,

$$\mathcal{D}^n_K(\mathcal{R}, M) = \{\theta \in Hom_K(\mathcal{R}, M) \big| J^{n+1}\theta = 0\}.$$

It now makes sense to make the following definition:

Definition 8 *For $n \geq 1$, define the* module of n^{th} order differentials $\Omega^n_K(\mathcal{R})$ *by*

$$\Omega^n_K(\mathcal{R}) = (\mathcal{R}\otimes_K\mathcal{R})/J^{n+1}.$$

Define $d_n : \mathcal{R} \to \Omega^n_K(\mathcal{R})$ to be the composition of the following two maps:

$$\begin{array}{ccccccc} d_n & : & \mathcal{R} & \to & \mathcal{R}\otimes_K\mathcal{R} & \to & \Omega^n_K(\mathcal{R}) \\ & & x & \mapsto & 1\otimes x & \mapsto & 1\otimes x + J^{n+1}. \end{array}$$

The next lemma shows that $\Omega^n_K(\mathcal{R})$ has certain universal properties which, together with the universal property of $\Omega_K(\mathcal{R})$ imply that $\Omega^1_K(\mathcal{R})$ coincides with $\Omega_K(\mathcal{R})$. Hence the n^{th} order differentials really are a generalisation of our usual notion of differentials. Also, the n^{th} order differentials arise in a natural way from the n^{th} order differential operators.

Lemma 9 *Let K, $\mathcal{R}$ and M be as above. Then:*

(i) d_n is a differential operator of order n (i.e. $d_n \in \mathcal{D}^n_K(\mathcal{R}, \Omega^n_K(\mathcal{R}))$).

(ii) Suppose that $\partial \in \mathcal{D}^n_K(\mathcal{R}, M)$. Then there is a unique $\mathcal{R}$-module homomorphism $\partial^\perp$ from $\Omega^n_K(\mathcal{R})$ into M such that $\partial = \partial^\perp \circ d_n$.

(*iii*) *The map given by* $\partial \mapsto \partial^{\perp}$ *induces an isomorphism of* $\mathcal{R}$-$\mathcal{R}$*-bimodules between* $\mathcal{D}^n_K(\mathcal{R}, M)$ *and* $Hom_{\mathcal{R}}(\Omega^n_K(\mathcal{R}), M)$.

Proof: (*i*) Clearly $J^{n+1}.d_n = 0$ so that $d_n \in \mathcal{D}^n_K(\mathcal{R}, \Omega^n_K(\mathcal{R}))$.

(*ii*) Since ∂ is a K-linear map, we may choose an $\mathcal{R}$-module homomorphism ψ : $\mathcal{R} \otimes_K \mathcal{R} \to M$ such that $\partial = \psi \circ d$ where $d(x) = 1 \otimes x$. Now because $\partial \in \mathcal{D}^n_K(\mathcal{R}, M)$, we have that $0 = J^{n+1}\partial = J^{n+1}\psi d = \psi J^{n+1} d$. Therefore, $0 = \psi J^{n+1} d * \mathcal{R} = \psi J^{n+1}$ so that $J^{n+1} \subseteq \psi$. Hence the map $\partial^{\perp} : \Omega^n_K(\mathcal{R}) \to M$ given by $\partial^{\perp}(1 \otimes x + J^{n+1}) = \partial * x$ is well-defined.

(*iii*) This follows from (*i*) and (*ii*). □

As one would expect, the differentials also have their version of localisation.

Lemma 10 *Let* S *be a multiplicatively closed subset of* $\mathcal{R}$. *Then:*

$$\Omega^n_K(\mathcal{R}_S) \cong \mathcal{R}_S \otimes \Omega^n_K(\mathcal{R}).$$

Proof: By Lemma 7, each $d_n : \mathcal{R} \to \Omega^n_K(\mathcal{R})$ extends uniquely to a map $\widehat{d_n} : \mathcal{R}_S \to \mathcal{R}_S \otimes \Omega^n_K(\mathcal{R})$, so it suffices to show that $\mathcal{R}_S \otimes \Omega^n_K(\mathcal{R})$ with $\widehat{d_n}$ has the universal property ascribed to $\Omega^n_K(\mathcal{R}_S)$ in Lemma 9.

Suppose that M is an $\mathcal{R}_S$-module and that $\partial \in \mathcal{D}^n_K(\mathcal{R}_S, M)$. Then ∂ restricted to $\mathcal{R}$ lies in $\mathcal{D}^n_K(\mathcal{R}, M)$ so that there exists a unique $\mathcal{R}$-module homomorphism $\partial^{\perp}$: $\Omega^n_K(\mathcal{R}) \to M$ such that $\partial = \partial^{\perp} \circ d_n$. We may extend $\partial^{\perp}$ to $\mathcal{R}_S \otimes \Omega^n_K(\mathcal{R})$ and then $\mathcal{R}_S \otimes \Omega^n_K(\mathcal{R})$ has the required universal property. □

As an example, let us calculate the differentials on a regular ring. We start off with a regular local ring and use Lemma 10 to cover the general case.

Lemma 11 *Suppose that $\mathcal{R}$ is a regular local ring with maximal ideal* $\mathbf{m}$ *and suppose that $\mathcal{R}$ contains a field K over which $\mathcal{R}/\mathbf{m}$ is integral. Let $y_1, \ldots, y_n$ be a minimal generating set for* $\mathbf{m}$. *Then $\Omega^r_K(\mathcal{R})$ is a free $\mathcal{R}$-module with basis $\{d_r\mathbf{y}^{\mathbf{i}} \| |\mathbf{i}| \leq r\}$, where* $\mathbf{i}$ *is an n-tuple of postive integers$(i_1, \ldots, i_n)$, $|\mathbf{i}| = i_1 + \ldots i_n$ and $\mathbf{y}^{\mathbf{i}} = y_1^{i_1} \ldots y_n^{i_n}$.*

Proof: We already know from Proposition 1.16 that $\mathcal{D}^r_K(\mathcal{R})$ is the free $\mathcal{R}$-module with basis $\{(\partial/\partial y_1)^{i_1} \ldots (\partial/\partial y_n)^{i_n} \| |\mathbf{i}| \leq r\}$. Therefore the subset $\sum_{|\mathbf{i}| \leq \mathbf{r}} d_r\mathbf{y}^{\mathbf{i}}$ of $\Omega^r_K(\mathcal{R})$ is a direct sum since any relations in $\Omega^r_K(\mathcal{R})$ would translate into relations inside $\mathcal{D}^r_K(\mathcal{R})$.

It is clear that the $d_r\mathbf{x}^{\mathbf{i}}$'s generate $\Omega^r_K(\mathcal{R})$ for $|\mathbf{i}| \leq \mathbf{r}$ since if $d_r f \in \Omega^r_K(\mathcal{R})$ for some f of degree $s > r$, then we would get a relation inside $\mathcal{D}^s_K(\mathcal{R})$. □

Corollary 12 *Let $\mathcal{R}$ be a regular domain containing a field K over which $\mathcal{R}/\mathbf{m}$ is integral for every maximal ideal* $\mathbf{m}$. *Then $\Omega^r_K(\mathcal{R})$ is a projective $\mathcal{R}$-module for every $r \in \mathbf{N}$.*

Proof: Clear. □

We conclude this section with two results about differential operator rings on smooth varieties that will be useful when considering factor rings. The first result says that when $\mathcal{R}$ is a regular ring and I is an ideal of $\mathcal{R}$ then the differential operator ring of the factor ring $\mathcal{R}/I$ can be identified with another factor ring. There is a problem in that as we have seen, if $\mathcal{R}$ is the coordinate ring of a smooth variety then

$\mathcal{D}(\mathcal{R})$ is simple and has no proper factor rings. The solution is to choose a certain right ideal of $\mathcal{D}(\mathcal{R})$ and take its *idealiser*.

Definition 13 *Let A be a non-commutative ring and let J be a right ideal of A. Define the* idealiser *of J in A to be the set:*

$$\mathbf{I}_A(J) = \{a \in A \big| aJ \subseteq J\}.$$

$\mathbf{I}_A(J)$ is the largest subring of A such that it has J as an ideal. We can therefore form the factor ring $\mathbf{I}_A(J)/J$. This has the property that $\mathbf{I}(J)/J \cong End_A(A/J)$ as is easily seen. Now, in our situation, the ring $\mathcal{D}(\mathcal{R})$ will take the place of A, where $\mathcal{R}$ is as before. The right ideal J of $\mathcal{D}(\mathcal{R})$ will be replaced by the module of differential operators $\mathcal{D}(\mathcal{R}, I)$, viewing I as an $\mathcal{R}$-module. Although $\mathcal{D}(\mathcal{R}, I)$ has so far only be defined to be a right $\mathcal{D}(\mathcal{R})$-module, it can be identified with a right ideal of $\mathcal{D}(\mathcal{R})$ in the obvious way. In fact, $\mathcal{D}(\mathcal{R}, I)$ is precisely the set $\{\partial \in \mathcal{D}(\mathcal{R}) \big| \partial * \mathcal{R} \subseteq I\}$, where $\delta * \mathcal{R}$ denotes the ideal of $\mathcal{R}$ generated by all elements of the form $\delta(r)$ for $r \in \mathcal{R}$.

One benefit of introducing the generalised differentials is that we may now use Corollary 12 to obtain a neater description of $\mathcal{D}(\mathcal{R}, I)$: recall that each $\mathcal{D}^r(\mathcal{R}, I) \cong Hom_{\mathcal{R}}(\Omega^r(\mathcal{R}), I)$. Now, because each $\Omega^r(\mathcal{R})$ is projective (Corollary 12), we see that $Hom_{\mathcal{R}}(\Omega^r(\mathcal{R}), I) \cong I \otimes_{\mathcal{R}} Hom_{\mathcal{R}}(\Omega^r(\mathcal{R}), \mathcal{R})$. Therefore, after taking unions (i.e. direct limits) of both sides, we get that:

$$\mathcal{D}(\mathcal{R}, I) \cong I \otimes_{\mathcal{R}} \mathcal{D}(\mathcal{R}) \cong I\mathcal{D}(\mathcal{R}).$$

We are now in a position to prove that $\mathcal{D}(\mathcal{R}/I)$ may be represented as a factor ring.

Proposition 14 *Let $\mathcal{R}$ be a regular domain containing a field K over which every factor of $\mathcal{R}$ by a maximal ideal is integral. Suppose that I is an ideal of $\mathcal{R}$. Then:*

$$\mathcal{D}_K(\mathcal{R}/I) \cong \mathbf{I}_{\mathcal{D}_K(\mathcal{R})}(I\mathcal{D}_K(\mathcal{R}))/I\mathcal{D}_K(\mathcal{R}).$$

Proof: Define a map $\phi : \mathbf{I}(I\mathcal{D}(\mathcal{R})) \to \mathcal{D}(\mathcal{R}/I)$ as follows. Given $\partial \in \mathbf{I}(I\mathcal{D}(\mathcal{R}))$, $\partial I \subseteq I\mathcal{D}(\mathcal{R})$ so that $\partial * I = \partial I * \mathcal{R} \subseteq I\mathcal{D}(\mathcal{R}) * \mathcal{R} = I$. Therefore ∂ induces a differential operator $\phi(\partial) : \mathcal{R}/I \to \mathcal{R}/I$. The kernel of ϕ is the set $\{\partial \in \mathcal{D}(\mathcal{R}) | \partial * \mathcal{R} \subseteq I\} = I\mathcal{D}(\mathcal{R})$. Hence ϕ induces an injective map $\overline{\phi} : \mathbf{I}(I\mathcal{D}(\mathcal{R}))/I\mathcal{D}(\mathcal{R}) \to \mathcal{D}(\mathcal{R}/I)$.

To prove that $\overline{\phi}$ is surjective, consider the right $\mathcal{D}(\mathcal{R})$-module $\mathcal{D}(\mathcal{R}, \mathcal{R}/I)$. In the same way we proved that $\mathcal{D}(\mathcal{R}, I) \cong I\mathcal{D}(\mathcal{R})$, we can see that:

$$\mathcal{D}(\mathcal{R}, \mathcal{R}/I) \cong \mathcal{R}/I \otimes_{\mathcal{R}} \mathcal{D}(\mathcal{R}) \cong \mathcal{D}(\mathcal{R})/I\mathcal{D}(\mathcal{R}).$$

Hence given a $\delta \in \mathcal{D}(\mathcal{R}/I)$ which gives rise to a $\overline{\delta} \in \mathcal{D}(\mathcal{R}, \mathcal{R}/I)$ by composition with the projection $\mathcal{R} \to \mathcal{R}/I$, we can find an element $\delta' \in \mathcal{D}(\mathcal{R})/I\mathcal{D}(\mathcal{R})$ which corresponds to $\overline{\delta}$. This map $\delta \mapsto \delta'$ is an inverse to $\overline{\phi}$ and so $\overline{\phi}$ is surjective. □

What this result is really telling us is that if $\mathcal{R}$ is regular then a differential operator $\delta : \mathcal{R}/I \to \mathcal{R}/I$ may be lifted to a differential operator $\partial \in \mathcal{D}(\mathcal{R})$.

Finally, we prove the second result which completely determines the structure of differential operator rings on certain factor rings.

Proposition 15 *Suppose that $\mathcal{R} = k + \mathbf{m}$ for some maximal ideal $\mathbf{m}$ of a commutative ring $\mathcal{R}$ and that $\mathbf{m}^n = 0$ for some $n \in \mathbf{N}$. Then:*

$$\mathcal{D}_k(\mathcal{R}) = End_k(\mathcal{R}).$$

Proof: Let $\theta \in End_k(\mathcal{R})$ and let $x_1, \ldots, x_{2n-1} \in \mathcal{R}$. Write each $x_i = a_i + y_i$ for some $a_i \in k$ and $y_i \in \mathbf{m}$. Then:

$$[\ldots[[\theta, x_1], x_2], \ldots, x_{2n-1}] = [\ldots[[\theta, y_1], y_2], \ldots, y_{2n-1}] \in \sum_{i=0}^{2n-1} \mathbf{m}^i \theta \mathbf{m}^{2n-1-i}.$$

But for all $0 \leq i \leq 2n-1$, either i or $2n-1-i$ is greater or equal to n so that one of $\mathbf{m}^i$ and $\mathbf{m}^{2n-1-i}$ is zero. Therefore $\theta \in \mathcal{D}_k(\mathcal{R})$. □

1.3 Differential Operators On Curves

We now know that the ring of differential operators on a smooth variety is a very well-behaved ring since it looks very much like the Weyl algebra. The next step is to allow the variety to have singularities and see what happens to the differential operators. In particular, we are interested in which varieties have the properties that their rings of differential operators are simple and noetherian with Krull and global dimensions equal to the dimension of the variety. It turns out that there is quite a simple answer to this question in the case of curves, but in higher dimensions the situation is a lot less clear. We will present the facts known about differential operators on curves à la [Smith & Stafford] and then discuss possible generalisations of these to the higher dimensional case.

The key to the answer in the curves case is a link between the ring of differential operators on a singular curve and the ring of differential operators on a smooth curve associated in some way to the original curve. The smooth curve associated to the original curve will be the normalisation, and the link between the two rings will be afforded by Morita equivalence. We formalise these two concepts before returning to the differential operators. Since this thesis is concerned with generalising the theory of differential operators on curves to higher dimensions, we will define normalisation for varieties of arbitrary dimension.

Definition 1 *Let $\mathcal{X}$ be a possibly singular variety and let Q be the field of fractions of its coordinate ring $\mathcal{R}$. Let $\widetilde{\mathcal{R}}$ be the integral closure of $\mathcal{R}$ in Q. Then by* [Matsumura; ch1.2.3.], *$\widetilde{\mathcal{R}}$ is a finitely generated $\mathcal{R}$-module so is affine over k. Define $\widetilde{\mathcal{X}}$ to be the variety associated to $\widetilde{\mathcal{R}}$ by $\widetilde{\mathcal{X}} = \mathbf{max}-\mathbf{Spec}(\widetilde{\mathcal{R}})$. Call $\widetilde{\mathcal{X}}$ the* normalisation *of $\mathcal{X}$. There is a map π from $\widetilde{\mathcal{X}}$ to $\mathcal{X}$ arising from the map $\pi : \mathbf{Spec}(\widetilde{\mathcal{R}}) \to \mathbf{Spec}(\mathcal{R})$ given by intersecting prime ideals of $\widetilde{\mathcal{R}}$ with $\mathcal{R}$:*

$$P \in \mathbf{Spec}(\widetilde{\mathcal{R}}) \mapsto P \cap \mathcal{R} \in \mathbf{Spec}(\mathcal{R}).$$

This map π is called the normalisation map *of $\mathcal{X}$.*

An important feature of the normalisation map is that it is a well-behaved map. In particular, for a given prime ideal of $\mathcal{R}$, there are only finitely many prime ideals of $\widetilde{\mathcal{R}}$ which map onto it. Also, it is easy to see that normalisation respects localisation (i.e. if S is a multiplicatively closed subset of $\mathcal{R}$ then $\widetilde{\mathcal{R}}_S = (\widetilde{\mathcal{R}_S})$), and the localisation of $\widetilde{\mathcal{R}}$ at a prime ideal of $\mathcal{R}$ is especially nice.

Lemma 2 *Let $\mathcal{X}$, Q and $\mathcal{R}$ be as in Definition 1 and let $\mathbf{p}$ be a prime ideal of $\mathcal{R}$. Then:*

(i) There exists a prime ideal P of $\widetilde{\mathcal{R}}$ such that $\pi(P) = P \cap \mathcal{R} = \mathbf{p}$.

(ii) There are no inclusions between prime ideals of $\widetilde{\mathcal{R}}$ which map onto $\mathbf{p}$.

(iii) There exist only finitely many prime ideals P of $\widetilde{\mathcal{R}}$ with $\pi(P) = \mathbf{p}$.

(iv) *Suppose that there is just one prime ideal P of $\overline{\mathcal{R}}$ with $\pi(P) = \mathbf{p}$ and let $S = \mathcal{R}\backslash\mathbf{p}$. Then $\mathcal{R}_{\mathbf{p}} = \overline{\mathcal{R}}_S = \overline{\mathcal{R}}_P$.*

Proof: Parts (i) and (ii) are [Matsumura; Theorem 9.3]. Part (iii) follows by setting $I = \mathbf{p}\overline{\mathcal{R}}$. Then since $\overline{\mathcal{R}}$ is noetherian, I has only finitely many minimal primes and by part (ii), no non-minimal primes of I can map onto $\mathbf{p}$. Finally, part (iv) is [Matsumura; Exercise 9.1]. $\square$

Another key property of normalisation is that the inclusion $\mathcal{R} \subseteq \overline{\mathcal{R}}$ makes $\overline{\mathcal{R}}$ into a finitely generated $\mathcal{R}$-module.

Lemma 3 *Let $\mathcal{X}$, Q and $\mathcal{R}$ be as in Definition 1. Then $\overline{\mathcal{R}}$ is a finitely generated $\mathcal{R}$-module.*

Proof: This is the subject of [Matsumura; Section 33]. $\square$

Although $\overline{\mathcal{X}}$ may not be smooth in general, it is the case that if $\mathcal{X}$ is one dimensional then $\overline{\mathcal{X}}$ is smooth.

Lemma 4 *Let $\mathcal{R}$ be a one dimensional commutative noetherian domain. Then $\mathcal{R}$ is normal if and only if $\mathcal{R}$ is regular.*

Proof: First suppose that $\mathcal{R}$ is normal and let $\mathbf{p}$ be a maximal ideal of $\mathcal{R}$. Then by [Matsumura; Theorem 11.2], $\mathcal{R}_{\mathbf{p}}$ is a principal ideal domain. Therefore, if we write $P = \mathbf{p}\mathcal{R}_{\mathbf{p}}$, $dim_k(P/P^2) = 1$ and $\mathcal{R}_{\mathbf{p}}$ is regular. Hence $\mathcal{R}$ is regular also.

The converse is just [Matsumura; Theorem 19.4]. $\square$

In the higher dimensions, since we are interested in linking $\mathcal{X}$ to a variety whose differential operator ring is nice, we shall often restrict ourselves to the case of when $\widetilde{\mathcal{X}}$ is smooth. Although this is quite a strong condition to impose, it seems to be necessary since [Bernstein, Gelfand & Gelfand] gives an example of a variety whose coordinate ring $\mathcal{R}$ is normal, but $\mathcal{D}(\mathcal{R})$ is not simple and not even noetherian on either side. Also, we must restrict ourselves to the study of varieties whose normalisation maps are injective since, apart from the curves case, almost nothing is known about differential operators on varieties with non-injective normalisation. For curves, the situation is a little nicer since in [Smith & Stafford] and [Brown] it is shown that if $\mathcal{X}$ is an arbitrary curve then $\mathcal{D}(\mathcal{X})$ has a unique minimal ideal, and the factor ring obtained on factoring out this ideal is a finite dimensional k-algebra. Since we are only interested in those facts about curves which generalise though, we shall only look at curves with injective normalisation.

Recall from Corollary 2.3 that $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}(\widetilde{\mathcal{R}})$ are both contained in $\mathcal{D}(Q)$. Since $\mathcal{R} \subseteq \widetilde{\mathcal{R}}$ it is natural to compare $\mathcal{D}(\mathcal{R})$ with $\mathcal{D}(\widetilde{\mathcal{R}})$, but it turns out that in general neither one is contained within the other. There is however a subset of both rings which is of major interest.

Definition 5 *Let $\mathcal{R}$ and $\mathcal{S}$ be commutative k-algebras with the properties that both are integral domains with the same field of fractions Q. Define the subset $\mathcal{D}(\mathcal{S},\mathcal{R})$ of $\mathcal{D}(Q)$ as follows:*

$$\mathcal{D}(\mathcal{S},\mathcal{R}) = \{\delta \in \mathcal{D}(Q) \big| \delta(\mathcal{S}) \subseteq \mathcal{R}\}.$$

When $\mathcal{R}$ and $\mathcal{S}$ are the coordinate rings of two varieties, $\mathcal{X}$ and $\mathcal{Y}$ say, then we will write $\mathcal{D}(\mathcal{Y},\mathcal{X})$ for $\mathcal{D}(\mathcal{S},\mathcal{R})$. Notice that $\mathcal{D}(\mathcal{S},\mathcal{R})$ is a right $\mathcal{D}(\mathcal{S})$-module and a left $\mathcal{D}(\mathcal{R})$-module. Now, when $\mathcal{S}=\widetilde{\mathcal{R}}$ so that $\mathcal{R}\subseteq\mathcal{S}$, we have that $\mathcal{D}(\widetilde{\mathcal{R}},\mathcal{R})$ is contained inside both $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}(\widetilde{\mathcal{R}})$. This bimodule $\mathcal{D}(\widetilde{\mathcal{R}},\mathcal{R})$ affords us an equivalence of categories between $\mathcal{D}(\widetilde{\mathcal{R}})$-modules and $\mathcal{D}(\mathcal{R})$-modules which will allow us to transfer over many properties of $\mathcal{D}(\widetilde{\mathcal{R}})$ to $\mathcal{D}(\mathcal{R})$. Once we have established this equivalence, called *Morita equivalence*, we will be able to say everything we want to about $\mathcal{D}(\mathcal{R})$.

Definition 6 *Let A and B be two arbitrary, non-commutative rings and suppose that there exists a B-A-bimodule M with the following properties:*

(i) M is finitely generated as a right A-module,

(ii) M is a projective right A-module,

(iii) M is a generator (i.e. $M^*(M)=A$ where $M^*=Hom_A(M,A)$),

(iv) $B\cong End_A(M)$.

Then B is said to be Morita equivalent *to A. M is said to be a (right)* progenerator *for A if it satisfies properties (i), (ii) and (iii) of the above.*

There may appear to be a lack of symmetry in this definition in that B could be Morita equivalent to A without A being Morita equivalent to B. Also, if we had defined Morita equivalence in terms of a left A-module, would we get a different equivalence? The answer to both of these questions is 'no' however. Suppose that B is Morita equivalent to A and that M is as in the definition. Then we may view M^* as a left A-module via $(a\theta)(m) = a.\theta(m)$, and as a right B-module via $(\theta.b)(m) = \theta(b(m))$ for $a \in A$, $b \in B$, $m \in M$ and $\theta \in M^*$. Using these actions, we have the following result:

Lemma 7 *Suppose B is Morita equivalent to A and M is the progenerator affording the equivalence. Then:*

(i) $M \cong Hom_B(M^*, B)$ and $A \cong End_B(M^*)$,

(ii) M^* is a right progenerator for B so that A is Morita equivalent to B,

(iii) M^* is a left progenerator for A and M is a left progenerator for B,

(iv) $M \cong Hom_A(M^*, A)$ and $M^* \cong Hom_B(M, B)$,

(v) $A \cong End_B(M)$ and $B \cong End_A(M^*)$.

Proof: (i) Identifying B with $End_A(M)$, we may define a homomorphism $\alpha : M \to Hom_B(M^*, B)$ by $\alpha(m)(\theta) = m.\theta \in End_A(M)$ for $m \in M$ and $\theta \in M^*$, where $(m.\theta)(n) = m(\theta(n))$ for $n \in M$. Since M is a generator, we have that $M.M^* = A$ and we may write $1 = \sum_{i=1}^{r} \theta_i(m_i)$ for some $m_i \in M$ and $\theta_i \in M^*$. Suppose that $m \in Ker(\alpha)$. Then $m\theta(M) = 0$ for all $\theta \in M^*$. But $m = m.1 = \sum_{i=1}^{n} m\theta_i(m_i) = 0$, so α is injective.

Now let $\psi \in Hom_B(M^*, B)$ and define $m = \sum_{i=1}^{r}(\psi(\theta_i))(m_i)$. Then:

$$\begin{aligned}
\psi(\phi) &= \psi(\textstyle\sum \theta_i(m_i)\phi) = \psi(\sum \theta_i.(m_i.\phi)) \\
&= \textstyle\sum \psi(\theta_i).(m_i.\phi) = (\sum \psi(\theta_i)(m_i)).\phi \\
&= m.\phi.
\end{aligned}$$

Therefore α is surjective.

To prove the second statement, we already have a map $\beta : A \to End_B(M^*)$ since M^* is a left A-module and multiplication by elements of A form B-module homomorphisms. β is clearly injective and can be seen to be surjective by the same method that was used to prove that α is surjective in the last paragraph.

$(ii) - (v)$ The rest of the lemma follows by very similar arguments to those used in the proof of (i). See for example [MR; 3.5.4]. □

Lemma 7 establishes the complete symmetry of Definition 6 and we may now talk about two rings A and B being Morita equivalent without worry. The Morita equivalence of two rings is a very strong relation to have since if one of the rings has a certain property (such as being simple), then the chances are that the other ring will have it too. The reason this happens is that Morita equivalence sets up a correspondence between the modules of the two rings. In more detail, suppose A and B are Morita equivalent and M is the progenerator affording the equivalence. Let P be a left A-module. Then $M \otimes_A P$ is a left B-module. Conversely, let Q be a left B-module. Then $M^* \otimes_B Q$ is a left A-module. So we have two maps between the category of left A-modules, ${}_A\mathbf{mod}$ and the category of left B-modules, ${}_B\mathbf{mod}$:

$$\Theta : {}_A\,\mathbf{mod} \to {}_B\,\mathbf{mod} \quad \text{and} \quad \Phi : {}_B\,\mathbf{mod} \to {}_A\,\mathbf{mod}$$

$$\text{via} \qquad P \mapsto M \otimes_A P \qquad \text{and} \qquad Q \mapsto M^* \otimes_B Q.$$

The progenerator properties of M and M^* ensure that $\Theta \circ \Phi$ and $\Phi \circ \Theta$ are naturally equivalent to the identity functors on the relevant categories so that the two categories are in bijection. Two modules P and Q which correspond to each other under this bijection share many properties.

Definition 8 *A property of a module P over a ring A is called a* Morita invariant *if whenever A is Morita equivalent to a ring B via a progenerator M then $M \otimes_A P$ also has that property. Similarly, a property of the ring A is called a* Morita invariant *if whenever A is Morita equivalent to B then B also has that property.*

Examples of Morita invariants are given by the following:

Lemma 9 *The following properties of a module are Morita invariant:*

(i) being right (or left) artinian,

(ii) being right (left) noetherian,

(iii) being finitely generated,

(iv) being projective,

(v) being a generator,

(vi) having projective dimension n.

Proof: These are all straightforward, using the equivalence of categories described above. □

Lemma 10 *The following properties of a ring are Morita invariant:*

(i) being Artinian,

(ii) being noetherian,

(iii) having right Krull dimension n

(iv) having right global dimension n.

(v) being simple.

Proof: These all follow easily from Lemma 9 apart from (v) which is proved as follows: Let A be a simple ring which is Morita equivalent to B via the progenerator

M. Suppose that B is not simple and that I is a non-trivial ideal of B. Then $M^*\otimes_B I\otimes_B M$ is an A-A-bimodule and is contained in $M^*\otimes B\otimes M \cong M^*\otimes M \cong A$ so that A is not simple, a contradiction. $\square$

We now have all the tools we need to be able to study the ring of differential operators on a singular curve $\mathcal{X}$: we can associate a smooth curve $\widetilde{\mathcal{X}}$ to $\mathcal{X}$, we have a bimodule $\mathcal{D}(\widetilde{\mathcal{X}},\mathcal{X})$ with which to establish a Morita equivalence between $\mathcal{D}(\mathcal{X})$ and $\mathcal{D}(\widetilde{\mathcal{X}})$, and we know a lot of nice properties of $\mathcal{D}(\widetilde{\mathcal{X}})$. All we need to do now is show that $\mathcal{D}(\widetilde{\mathcal{X}},\mathcal{X})$ is a progenerator over $\mathcal{D}(\widetilde{\mathcal{X}})$ and that $\mathcal{D}(\mathcal{X}) \cong End_{\mathcal{D}(\widetilde{\mathcal{X}})}(\mathcal{D}(\widetilde{\mathcal{X}},\mathcal{X}))$. This is where the curves case detaches itself from the general case.

If $\mathcal{X}$ is a curve and $\mathcal{R}$ is its coordinate ring then $Dim(\widetilde{\mathcal{R}}) = 1$ so that the global dimension of $\mathcal{D}(\widetilde{\mathcal{R}})$ is also 1 (by Theorem 1.19). Therefore, since $\mathcal{D}(\widetilde{\mathcal{R}},\mathcal{R})$ is a right ideal of $\mathcal{D}(\widetilde{\mathcal{R}})$, $\mathcal{D}(\widetilde{\mathcal{R}},\mathcal{R})$ must be projective. Again by Theorem 1.19, $\mathcal{D}(\widetilde{\mathcal{R}})$ is a simple ring so that $\mathcal{D}(\widetilde{\mathcal{R}},\mathcal{R})$ is a generator by necessity. Finally, since $\mathcal{D}(\widetilde{\mathcal{R}})$ is noetherian (Theorem 1.19), $\mathcal{D}(\widetilde{\mathcal{R}},\mathcal{R})$ is finitely generated and hence is a progenerator. So the main part of the argument is to show for a curve $\mathcal{X}$ that $\mathcal{D}(\mathcal{X}) \cong End_{\mathcal{D}(\widetilde{\mathcal{X}})}(\mathcal{D}(\widetilde{\mathcal{X}},\mathcal{X}))$. In higher dimensions this fact will follow fairly easily from the one dimensional case, but then we will have lost the projectivity of $\mathcal{D}(\widetilde{\mathcal{X}},\mathcal{X})$.

From now on, until otherwise specified, $\mathcal{X}$ will be a singular curve, $\mathcal{R}$ will be its coordinate ring and Q the field of fractions of $\mathcal{R}$. Since $\mathcal{D}(Q)$ is a noetherian integral domain, Goldie's theorem implies that it has a quotient division ring Δ say, and since $\mathcal{D}(Q)$ is a localisation of both $\mathcal{D}(\widetilde{\mathcal{R}})$ and $\mathcal{D}(\mathcal{R})$, Δ is the quotient division ring of both of these as well. Therefore, we may identify $\mathcal{S} = End_{\mathcal{D}(\widetilde{\mathcal{R}})}(\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}))$ with the set $\{\partial \in \Delta | \partial\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}) \subseteq \mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})\}$. Write $P = \mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$ and let I be the conductor of $\widetilde{\mathcal{R}}$ into $\mathcal{R}$: $I = Ann_{\mathcal{R}}(\widetilde{\mathcal{R}}/\mathcal{R})$. It follows from Lemma 3 and the fact that $\widetilde{\mathcal{R}}/\mathcal{R}$ is a torsion $\mathcal{R}$-module that I is non-zero. Observe that $I\mathcal{D}(\widetilde{\mathcal{R}}) \subseteq P$ so that if $\partial \in \mathcal{S}$, then $\partial I \subseteq P \subseteq \mathcal{D}(\widetilde{\mathcal{R}})$. In particular, if $0 \neq x \in I$ then we have $\partial \in \mathcal{D}(\widetilde{\mathcal{R}})x^{-1} \subseteq \mathcal{D}(Q)$ so that $\mathcal{S} \subseteq \mathcal{D}(Q)$. Therefore, since P is a left $\mathcal{D}(\mathcal{R})$-module we have the following inclusions:

$$\mathcal{D}(\mathcal{R}) \subseteq \mathcal{S} = End_{\mathcal{D}(\widetilde{\mathcal{R}})}(\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})) \subseteq \mathcal{D}(Q).$$

We must show that the first inclusion is actually an equality. To this end we would like to use the technique of localisation and so the next lemma will prove useful.

Lemma 11 *Let S be a multiplicatively closed subset of $\mathcal{R}$. Then:*

$$\mathcal{R}_S\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}) = \mathcal{D}(\widetilde{\mathcal{R}}_S, \mathcal{R}_S) = \mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})\widetilde{\mathcal{R}}_S \subseteq \mathcal{D}(Q).$$

Proof: Recall the definition of the module of differential operators between two modules as given in Definition 2.6. If we can show that $\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$, as defined above, is the same object as the module of differential operators between $\widetilde{\mathcal{R}}$ and $\mathcal{R}$, regarding $\widetilde{\mathcal{R}}$ as an $\mathcal{R}$-module, then Lemma 2.7 will give us the result. Denote the module of differential operators between $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ by $\mathcal{D}^{\mathcal{R}}(\widetilde{\mathcal{R}}, \mathcal{R})$ and let T be the set $\mathcal{R}\backslash 0$ so that $\mathcal{R}_T = \widetilde{\mathcal{R}}_T = Q$. Then using Lemma 2.7, it is easy to see that $\mathcal{D}^{\mathcal{R}}(\widetilde{\mathcal{R}}, \mathcal{R}) \cong \{\partial \in \mathcal{D}(\widetilde{\mathcal{R}}_T, \mathcal{R}_T) | \partial(\widetilde{\mathcal{R}}) \subseteq \mathcal{R}\}$. But $\mathcal{D}(\widetilde{\mathcal{R}}_T, \mathcal{R}_T) = \mathcal{D}(Q)$ and $\{\partial \in \mathcal{D}(Q) | \partial(\widetilde{\mathcal{R}}) \subseteq \mathcal{R}\}$ is precisely the definition of $\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$. $\square$

Next we present a result that although simple, is the stepping stone towards proving that $\mathcal{D}(\mathcal{R}) = \mathcal{S} = End_{\mathcal{D}(\tilde{\mathcal{R}})}(\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}))$. In order to make our notation legible, if $\partial \in \mathcal{D}(Q)$ and $q \in Q$, we will write $\partial * q$ in place of $\partial(q)$. Then the symbol $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}}$ means the set of all finite sums of elements of the form $\partial * x$ for $x \in \tilde{\mathcal{R}}$ and $\partial \in \mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})$.

Lemma 12 *Suppose that* $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}} = \mathcal{R}$. *Then* $\mathcal{D}(\mathcal{R}) = \mathcal{S} = End_{\mathcal{D}(\tilde{\mathcal{R}})}(\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}))$.

Proof: We already know that $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{S}$ since $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})$ is a left $\mathcal{D}(\mathcal{R})$-module. So let $\partial \in \mathcal{S}$ and let $x \in \mathcal{R}$. Then $x \in \mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}}$. Hence:

$$\partial * x \in \partial\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}} \subseteq \mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}} \subseteq \mathcal{R}.$$

Therefore $\partial \in \mathcal{D}(\mathcal{R})$ as required. □

We are now in a position to prove the main result of this section. It is here that we must assume the injectivity of the normalisation map in order that we may use Lemma 2(iv).

Theorem 13 *Let* $\mathcal{X}$ *be a curve and suppose that the normalisation map* $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ *is injective. Then* $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}} = \mathcal{R}$ *and* $\mathcal{D}(\mathcal{R})$ *is Morita equivalent to* $\mathcal{D}(\tilde{\mathcal{R}})$.

Proof: Notice that $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R}) * \tilde{\mathcal{R}}$ is a left $\mathcal{D}(\mathcal{R})$-module and hence is an ideal of $\mathcal{R}$. Suppose that it is contained in some maximal ideal $\mathbf{m}$ of $\mathcal{R}$. Then by Lemma 2(iv), $\tilde{\mathcal{R}}_{\mathbf{m}}$ is a regular local ring and we may write M for the unique maximal ideal $\mathbf{m}\tilde{\mathcal{R}}_{\mathbf{m}}$ of $\tilde{\mathcal{R}}_{\mathbf{m}}$.

Set $I = Ann_{\mathcal{R}}(\widetilde{\mathcal{R}}/\mathcal{R})$, the conductor of $\widetilde{\mathcal{R}}$ into $\mathcal{R}$. Since M is the unique minimal prime of $I\widetilde{\mathcal{R}}_{\mathbf{m}}$, some power of M is contained in $I\widetilde{\mathcal{R}}_{\mathbf{m}}$. But I is contained in $\mathcal{R}_{\mathbf{m}}$ and hence some power of M lies inside $\mathcal{R}_{\mathbf{m}}$. Let n be the smallest integer such that $M^n \in \mathcal{R}_{\mathbf{m}}$. If we write $A = \widetilde{\mathcal{R}}_{\mathbf{m}}/M^n$ and set $\overline{M}$ to be the unique maximal ideal of A then we are in the situation of Proposition 2.15: $A = k + \overline{M}$ and $\overline{M}^n = 0$. Hence $\mathcal{D}(A) = End_k(A)$. Now, since $M^n \subseteq \mathcal{R}_{\mathbf{m}}$, we may think of $B = \mathcal{R}_{\mathbf{m}}/M^n$ as a subset of A. Therefore we may find a θ in $\mathcal{D}(A)$ such that $\theta * A \subseteq B$ and $\theta * 1 = 1$.

Now we use Proposition 2.14: since $\widetilde{\mathcal{R}}_{\mathbf{m}}$ is regular, we may lift $\theta : \widetilde{\mathcal{R}}_{\mathbf{m}}/M^n \to \widetilde{\mathcal{R}}_{\mathbf{m}}/M^n$ to a differential operator $\partial \in \mathcal{D}(\widetilde{\mathcal{R}}_{\mathbf{m}})$. Notice that ∂ actually lies in $\mathcal{D}(\widetilde{\mathcal{R}}_{\mathbf{m}}, \mathcal{R}_{\mathbf{m}}) = \mathcal{R}_{\mathbf{m}}\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$. Therefore there exists some $s \in \mathcal{R}\backslash\mathbf{m}$ such that $s\partial \in \mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$. But by the construction of ∂, $(s\partial) * 1 = s \notin \mathbf{m}$. This contradicts the fact that $\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}) * \widetilde{\mathcal{R}} \subseteq \mathbf{m}$ and we have proved the result. $\square$

Corollary 14 *If $\mathcal{X}$ is a curve with injective normalisation then $\mathcal{D}(\mathcal{X})$ is a simple, noetherian, hereditary domain of Krull dimension one.*

Proof: All the stated properties are Morita invariants by Lemma 10. But $\mathcal{D}(\mathcal{X})$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$ which has all of these properties and hence $\mathcal{D}(\mathcal{X})$ has them also. $\square$

In fact, we can actually characterise the varieties $\mathcal{X}$ whose differential operator rings are Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$ as those which have injective normalisation. In order to prove this we need to use the completion of a ring (see [Matsumura; Section 8] for details of completions). Recall that if I is an ideal of a commutative domain A then the completion $\hat{A}$ of A at I is defined to be the set of sequences of the form $(a_n + I^n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A/I^n$ where $a_n - a_{n+r} \in I^n$ for all integers $r \geq 0$. The ring A itself can be identified with the subring of $\hat{A}$ consisting of all the constant sequences. Similarly to localisation, differential operators behave well with respect to completions.

Proposition 15 *Let $\mathcal{R}$ be a commutative noetherian domain and k-algebra and let I be an ideal of $\mathcal{R}$. Then writing $\widehat{\mathcal{R}}$ for the completion of $\mathcal{R}$ at I, $\mathcal{D}(\mathcal{R}) \cong \{\partial \in \mathcal{D}(\widehat{\mathcal{R}}) | \partial * \mathcal{R} \subseteq \mathcal{R}\}$.*

Proof: Write $\phi : \{\partial \in \mathcal{D}(\widehat{\mathcal{R}}) | \partial * \mathcal{R} \subseteq \mathcal{R}\} \to \mathcal{D}(\mathcal{R})$ for the map which restricts differential operators on $\widehat{\mathcal{R}}$ to $\mathcal{R}$. Suppose that there exists a $\partial \in \mathcal{D}(\widehat{\mathcal{R}})$ with $\partial|_{\mathcal{R}} = 0$. Let $x = (a_n + I^n)_{n=1}^{\infty} \in \widehat{\mathcal{R}}$. We shall show that $\partial * x = 0$. By [Matsumura; Theorems 8.10 and 8.11], if $\hat{I}$ is the image of I in $\widehat{\mathcal{R}}$ then $\widehat{(I^r)} = (\hat{I})^r$ and $\bigcap \hat{I}^r = 0$. Hence it suffices to show that $\partial * x \in I^r$ for all $r \geq 0$. Now if ∂ has order m then:

$$\partial * x = \partial * (a_{r+m} + (a_n - a_{r+m} + I^n)_{n=1}^{\infty}) = \partial * (a_n - a_{r+m} + I^n)_{n=1}^{\infty} \in \partial * \hat{I}^{r+m}.$$

But taking commutators of ∂ with $\hat{I}$, we see that $\partial * \hat{I}^{r+m} \subseteq \hat{I}^r$. Therefore $\partial * x = 0$ and ϕ is injective.

To see that ϕ is surjective we must show how to extend differential operators on $\mathcal{R}$ to ones on $\widehat{\mathcal{R}}$, so let $\delta \in \mathcal{D}^m(\mathcal{R})$. Define $\hat{\delta} : \widehat{\mathcal{R}} \to \widehat{\mathcal{R}}$ by $\hat{\delta} * (a_n + I^n)_{n=1}^{\infty} = (\delta * a_{n+m} + I^n)_{n=1}^{\infty}$. This is well-defined since if $r \geq 0$ then:

$$\delta * a_{n+m+r} - \delta * a_{n+m} = \delta * (a_{n+m+r} - a_{n+m}) \in \partial * I^{n+m} \subseteq I^n$$

so that $\tilde{\delta} * \widetilde{\mathcal{R}} \subseteq \widetilde{\mathcal{R}}$. A straightforward calculation shows that $\tilde{\delta}$ is a differential operator and it clearly extends δ, hence ϕ is surjective. □

Theorem 16 *Let $\mathcal{X}$ be a curve. Then the following are equivalent:*

(i) The normalisation map $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is injective,

(ii) $\mathcal{D}(\mathcal{X})$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$,

(iii) $\mathcal{D}(\mathcal{X})$ is a simple ring.

Proof: $(i) \Rightarrow (ii)$ is Theorem 13 and $(ii) \Rightarrow (iii)$ follows from the fact that simplicity is a Morita invariant. So it remains to prove that $(iii) \Rightarrow (i)$. So suppose that $\mathcal{X}$ is a curve with $\mathcal{D}(\mathcal{X})$ simple but whose normalisation is not injective. Let $\mathbf{m}$ be a maximal ideal of $\mathcal{R}$ and let M_1 and M_2 be maximal ideals of $\widetilde{\mathcal{R}}$ such that $M_1 \bigcap \mathcal{R} = M_2 \bigcap \mathcal{R} = \mathbf{m}$. Since $\widetilde{\mathcal{R}}/\mathcal{R}$ is an artinian $\mathcal{R}$-module there exists an integer r such that $M_i^{r+s} + \mathcal{R} = M_i^r + \mathcal{R}$ for $i = 1, 2$ and for all $s \in \mathbf{N}$. Define $\mathcal{S} = (M_1^r + \mathcal{R}) \bigcap (M_2^r + \mathcal{R})$. I claim that $M_1 \bigcap \mathcal{S} \neq M_2 \bigcap \mathcal{S}$. If this is not true then we have the following inclusions:

$$\mathcal{R} \subseteq \mathcal{S} \subseteq k + M_1 M_2 \subset \widetilde{\mathcal{R}}.$$

Notice that $k + M_1 M_2 \neq \widetilde{\mathcal{R}}$ and that $M_1 \bigcap M_2 \subseteq Ann_{k+M_1M_2}(\widetilde{\mathcal{R}}/(k + M_1 M_2))$. Since $\widetilde{\mathcal{R}}$ is a dedekind domain we have that $M_1 \bigcap M_2 = M_1 M_2$ and $\widetilde{\mathcal{R}}/M_1 M_2 \cong \widetilde{\mathcal{R}}/M_1 \bigoplus \widetilde{\mathcal{R}}/M_2$. It is not hard to show that $[(k + M_1 M_2)/M_1 M_2]$ has trivial intersection with $[\widetilde{\mathcal{R}}/M_i] = 0$ inside $\widetilde{\mathcal{R}}/M_1 M_2$ for $i = 1, 2$ but that:

$$(\mathcal{S} + M_1 M_2)/M_1 M_2 = [(\mathcal{S} + M_1 M_2)/M_1] \bigoplus [(\mathcal{S} + M_1 M_2)/M_2].$$

Therefore $\mathcal{S} \subseteq M_1 M_2 \subseteq Ann(\widetilde{\mathcal{R}}/(k + M_1 M_2))$. But $1 \in \mathcal{S}$ so that $\widetilde{\mathcal{R}} = k + M_1 M_2$ which is clearly not true. Hence we have a contradiction and the claim is proved.

Next I claim that $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{S})$. Let $\mathbf{m}_1$ and $\mathbf{m}_2$ denote $M_1 \bigcap \mathcal{S}$ and $M_2 \bigcap \mathcal{S}$ respectively and let $r \in \mathbf{N}$ be such that $(\mathbf{m}_1\mathbf{m}_2)^r \subseteq \mathcal{R}$. Write $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{S}}$ for the completions of $\mathcal{R}$ and $\mathcal{S}$ at $(\mathbf{m}_1\mathbf{m}_2)^r$. We may think of $\widehat{\mathcal{R}}$ as lying inside $\widehat{\mathcal{S}}$. By [Matsumura; Theorem 8.15], $\widehat{\mathcal{S}} = \widehat{\mathcal{S}_{\mathbf{m}_1}} \bigoplus \widehat{\mathcal{S}_{\mathbf{m}_2}}$ and by the construction of $\mathcal{S}$, the projections p_i of $\widehat{\mathcal{R}}$ onto each $\widehat{\mathcal{S}_{\mathbf{m}_i}}$ are both surjective. Therefore, since differential operators commute with direct sums (they are homomorphisms), given a differential operator δ on $\widehat{\mathcal{R}}$ we may extend it to $\bar{\delta}$ on $\widehat{\mathcal{S}}$ by $\bar{\delta} = p_1 \circ \delta + p_2 \circ \delta$. Finally, Proposition 15 tells us that given a differential operator on $\mathcal{R}$ we may extend it to $\widehat{\mathcal{R}}$, then extend it again to $\widehat{\mathcal{S}}$ by the above and then restrict it back down to $\mathcal{S}$. So $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{S})$ as claimed.

Now, if $\mathcal{S}$ does not have injective normalisation we may repeat the above process to $\mathcal{S}$ and since $\widetilde{\mathcal{R}}/\mathcal{R}$ is artinian, this must stop after a finite number of steps until we have arrived at the following situation: $\mathcal{R} \subseteq \mathcal{S}$, $\mathcal{S}$ has injective normalisation and $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{S})$. Let $\mathcal{D}(\mathcal{S},\mathcal{R})$ denote the set $\{\partial \in \mathcal{D}(\mathcal{S}) \big| \partial * \mathcal{S} \subseteq \mathcal{R}\}$. Since $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{S})$, $\mathcal{D}(\mathcal{S},\mathcal{R})$ is an ideal of $\mathcal{D}(\mathcal{R})$. It is non-zero because it contains the conductor of $\mathcal{S}$ into $\mathcal{R}$ and it is not equal to $\mathcal{D}(\mathcal{R})$ since if it were then 1 would lie inside it and $\mathcal{S}$ would equal $\mathcal{R}$. But $\mathcal{R}$ does not have injective normalisation whilst $\mathcal{S}$ does. Therefore $\mathcal{D}(\mathcal{S},\mathcal{R})$ is a proper ideal of $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}(\mathcal{R})$ is not simple, a contradiction. $\square$

1.4 Differential Operators On Surfaces

In this, the last section of Chapter One, we show how to generalise the results of Section 3 about curves to two dimensional varieties. There are more problems here than for the curves case and ultimately the whole thing rests upon a trick that only works for varieties of dimension two or less. No-one has been able to replace this trick with a method which works in all dimensions and it is the general purpose of this thesis to find new tools with which to attack the problem from a different angle.

Most of the material presented in this section is an amalgamation of the two papers [Hart & Smith] and [Chamarie & Stafford], although we have done some of the necessary work in the earlier sections. The line of attack is to try to reduce the situation down to the curves case by localising at height one prime ideals of the coordinate ring. The crux of the problem is the globalisation of the results obtained back to the coordinate ring again.

If we rush in and try to generalise the results of Section 3, we straight away run into the problem that if $\mathcal{X}$ is a surface then $\widetilde{\mathcal{X}}$ need not be smooth. For example if $\mathcal{X}$ is the cubic cone $\{(x, y, z) \in \mathbf{C} \big| x^3 + y^3 = z^3\}$ then the coordinate ring $\mathcal{R}$ of $\mathcal{X}$ is already normal, but $\mathcal{X}$ has a singularity at the origin. In order then to use the properties of differential operators on smooth varieties that we proved in Sections 1 and 2, we shall always insist that the varieties that we work with should have smooth normalisation. Also, Theorem 3.16 indicates that we ought to have injective normalisation.

In order to prove the local case it is not necessary to insist that $\mathcal{X}$ be merely a surface. As mentioned above, this restriction is only needed to globalise the local results to the general case. We therefore state the following theorem in full generality in order to inform the reader of what is and what is not true about the higher dimensional varieties.

Proposition 1 *Let X be an n-dimensional variety with smooth normalisation and let $\mathcal{R}$ be its coordinate ring. If P is a height one prime ideal of $\mathcal{R}$ then $\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)$ is a projective right ideal of $\mathcal{D}(\widetilde{\mathcal{R}}_P)$, where $\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) = \{\partial \in \mathcal{D}(\widetilde{\mathcal{R}}_P) | \partial * \widetilde{\mathcal{R}}_P \subseteq \mathcal{R}_P\}$.*

Proof: Since X has smooth normalisation, $\widetilde{\mathcal{R}}_P$ is a regular, semi-local ring. Now, by Noether normalisation ([Matsumura; Lemma 33.2]), $\mathcal{R}$ contains a polynomial ring $k[t_1, \ldots, t_n]$ over which it is integral. Since P has height one, the length of a maximal regular sequence of $\mathcal{R}$ in P is one and therefore we may reorder the t_i's so that $k[t_1, \ldots, t_{n-1}] \bigcap P = 0$. Then setting $K = k(t_1, \ldots, t_{n-1})$, we find that when we localise at P we get $K \subseteq \mathcal{R}_P$ and $\widetilde{\mathcal{R}}_P / M$ is algebraic over K for every maximal ideal M of $\widetilde{\mathcal{R}}_P$. We are therefore in the situation of Theorem 1.19 so that $\mathcal{D}_K(\widetilde{\mathcal{R}}_P)$ is a simple, noetherian, hereditary domain of Krull dimension one.

Define the set $\mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)$ to be the set $\{\partial \in \mathcal{D}_K(\widetilde{\mathcal{R}}_P) | \partial * \widetilde{\mathcal{R}}_P \subseteq \mathcal{R}_P\}$. Since K-linear homomorphisms are also k-linear, we may regard $\mathcal{D}_K(\mathcal{R}_P)$ and $\mathcal{D}_K(\widetilde{\mathcal{R}}_P)$ as lying inside $\mathcal{D}_k(\mathcal{R}_P)$ and $\mathcal{D}_k(\widetilde{\mathcal{R}}_P)$ respectively. Hence we have that $\mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) \subseteq \mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)$. Since $\mathcal{D}_K(\widetilde{\mathcal{R}}_P)$ is hereditary, $\mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)$ is a projective right ideal of $\mathcal{D}_K(\widetilde{\mathcal{R}}_P)$. Therefore, by the Dual Basis Lemma, $1 \in \mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)[\mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)]^*$.

Notice that the set $\mathcal{D}_K(\mathcal{R}_P, \widetilde{\mathcal{R}}_P) = \{\partial \in \mathcal{D}_K(Q) | \partial * \mathcal{R}_P \subseteq \widetilde{\mathcal{R}}_P\}$ lies inside $[\mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)]^*$, since if $\partial \in \mathcal{D}_K(\mathcal{R}_P, \widetilde{\mathcal{R}}_P)$ then:

$$\partial \mathcal{D}_K(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) * \widetilde{\mathcal{R}}_P \subseteq \partial * \mathcal{R}_P \subseteq \widetilde{\mathcal{R}}_P.$$

In fact the reverse inclusion also holds for if $\partial \in \mathcal{D}_K(\overline{\mathcal{R}}_P, \mathcal{R}_P)$ then:

$$[\mathcal{D}_K(\overline{\mathcal{R}}_P, \mathcal{R}_P)]^* \partial * \overline{\mathcal{R}}_P \subseteq \mathcal{D}_K(\overline{\mathcal{R}}_P) * \overline{\mathcal{R}}_P \subseteq \overline{\mathcal{R}}_P.$$

Hence $1 \in \mathcal{D}_K(\overline{\mathcal{R}}_P, \mathcal{R}_P)\mathcal{D}_K(\mathcal{R}_P, \overline{\mathcal{R}}_P)$. Therefore $1 \in \mathcal{D}(\overline{\mathcal{R}}_P, \mathcal{R}_P)\mathcal{D}(\mathcal{R}_P, \overline{\mathcal{R}}_P)$ also (with the obvious meaning for $\mathcal{D}(\mathcal{R}_P, \overline{\mathcal{R}}_P)$). So by the Dual Basis Lemma again, $\mathcal{D}(\overline{\mathcal{R}}_P, \mathcal{R}_P)$ is projective. □

This is the first of the problems out of the way with for the local case. The next step is to show that $\mathcal{D}(\mathcal{R}_P) = End_{\mathcal{D}(\tilde{\mathcal{R}}_P)}(\mathcal{D}(\overline{\mathcal{R}}_P, \mathcal{R}_P))$ and the method of proof follows that of the curves case. In particular, we start off by showing that $\mathcal{D}(\overline{\mathcal{R}}_P, \mathcal{R}_P) * \overline{\mathcal{R}}_P = \mathcal{R}_P$.

Proposition 2 *Let $\mathcal{X}$, $\mathcal{R}$ and P be as in Proposition 1 and suppose that $\mathcal{X}$ has injective normalisation. Then:*

$$\mathcal{D}(\overline{\mathcal{R}}_P, \mathcal{R}_P) * \overline{\mathcal{R}}_P = \mathcal{R}_P.$$

Proof: Since the normalisation map $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ is injective, the residue fields of $\overline{\mathcal{R}}_P$ and $\mathcal{R}_P$ are equal. The easiest way to see this is geometrically: the prime ideal P defines a codimension one subvariety $\mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{R}_P$ is the set of functions on $\mathcal{X}$ which are regular on $\mathcal{Y}$. Since π is injective, $\tilde{\mathcal{X}}$ and $\mathcal{X}$ are isomorphic on a dense open subset of each. But dense open subsets are all that are needed to define regular functions on so that the residue fields are the same.

Let M and $\mathbf{m}$ be the maximal ideals of $\overline{\mathcal{R}}_P$ and $\mathcal{R}_P$ respectively. Since $\overline{\mathcal{R}}_P/\mathcal{R}_P$ has a non-zero annihilator in $\mathcal{R}_P$, it is a finitely generated module over a factor ring of

$\mathcal{R}_P$ and so must be artinian. Hence there exists an integer $r \in \mathbf{N}$ such that $M^r \subseteq \mathcal{R}_P$ so that we may think of $\mathcal{R}_P/M^r$ as being a subset of $\widetilde{\mathcal{R}}_P/M^r$. By Cohen's Theorem ([Matsumura A; Theorem 60]), since $\widetilde{\mathcal{R}}_P/M^r$ is a complete local ring it contains a copy F of its own residue field $\widetilde{\mathcal{R}}_P/M$ and furthermore, F may also be chosen so that it contains k. By the previous paragraph, F is also the residue field of $\mathcal{R}_P$ so that $F \subseteq \mathcal{R}_P/M^r$.

Now, by Proposition 2.15, $\mathcal{D}_F(\widetilde{\mathcal{R}}_P/M^r) = End_F(\widetilde{\mathcal{R}}_P/M^r)$. Therefore there exists some $\partial \in \mathcal{D}_F(\widetilde{\mathcal{R}}_P/M^r)$ with the property that $\partial * \widetilde{\mathcal{R}}_P/M^r = \mathcal{R}_P/M^r$. Because ∂ is F-linear, it is also k-linear. But by Proposition 2.14, ∂ may be lifted to a differential operator $\overline{\partial}$ on $\widetilde{\mathcal{R}}_P$ with $1 \in \overline{\partial} * \widetilde{\mathcal{R}}_P \subseteq \mathcal{R}_P$. Hence the result holds. □

Corollary 3 $\mathcal{D}(\mathcal{R}_P) = End_{\mathcal{D}(\widetilde{\mathcal{R}}_P)}(\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P))$.

Proof: Lemma 3.12 holds even when $\mathcal{X}$ is not a curve. □

The next thing we want to do is to 'globalise' the results proved so far in this section. That is, we want $\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$ to be a projective $\mathcal{D}(\widetilde{\mathcal{R}})$-module, and $\mathcal{D}(\mathcal{R}) = End_{\mathcal{D}(\widetilde{\mathcal{R}})}(\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}))$. In order to do this we must insist that $\mathcal{X}$ is essentially determined by what happens at the codimension one level. The restriction that we need is Serre's S_2 condition.

Definition 4 *If A is a commutative noetherian ring then A is called S_2 if A is the intersection of its localisations at height one primes:*

$$A = \bigcap_{htP=1} A_P.$$

When X is a variety we shall say that X is S_2 if its coordinate ring is S_2.

In the two dimensional (i.e. surfaces) case, being S_2 is the same as being Cohen-Macaulay as is evidenced by the following lemma.

Lemma 5 *Let A be a commutative noetherian domain. Then A is S_2 if and only if the depth of each prime ideal P is at least $min\{2, height(P)\}$.*

Proof: Suppose first that A is S_2 and without loss of generality assume that P is a height two prime ideal. Let $x \in P$. Then since $A = \bigcap_{htP=1} A_P$, xA is an intersection of height one primary ideals. Therefore if $y \in A$ is a zero divisor in A/xA then y is contained in a union of finitely many height one prime ideals. It is a well known fact (see e.g [Matsumura; Exercise 1.6]) that if an ideal is contained in a finite union of prime ideals, then it is contained inside at least one of them. Hence if P consisted of zero divisors in A/xA then it would be contained in a height one prime ideal which contradicts the fact that P has height two. Therefore there exists a regular sequence of length two in P.

Next suppose that if P is a prime ideal of A then $depth(P) \geq min\{2, height(P)\}$, and set $B = \bigcap_{htP=1} A_P$. If $B \neq A$, choose $x \in B \backslash A$ and write $x = a/b$ for $a, b \in A$. Then $a \in bA_P$ for every height one prime P. Suppose that Q is a P-primary ideal in the primary decomposition of bA. Then P consists entirely of zero divisors in A/bA so that the depth of P is one. Therefore P has height one as well. Hence $a \in bA_P \bigcap A = Q$. Thus a lies in each primary ideal in the primary decomposition of bA and so $a \in bA$. It follows that $x \in A$ which contradicts the fact that $B \neq A$. $\square$

The S_2 condition is precisely what we need in order to be able to globalise the results we haved proved so far.

Lemma 6 *If $\mathcal{X}$ is an S_2 variety with smooth, injective normalisation then $\mathcal{D}(\mathcal{X}) = End_{\mathcal{D}(\tilde{\mathcal{X}})}(\mathcal{D}(\tilde{\mathcal{X}},\mathcal{X}))$.*

Proof: If P is a height one prime ideal of $\mathcal{R}$ then $\mathcal{D}(\tilde{\mathcal{R}}_P,\mathcal{R}_P) * \tilde{\mathcal{R}}_P = \mathcal{R}_P$ by Proposition 2. Therefore, by Corollary 3, $\mathcal{D}(\mathcal{R}_P) = End_{\mathcal{D}(\tilde{\mathcal{R}}_P)}(\mathcal{D}(\tilde{\mathcal{R}}_P,\mathcal{R}_P))$. Since $\mathcal{R}$ is S_2, $(\bigcap_{htP=1}\mathcal{D}(\mathcal{R}_P)) * \mathcal{R} \subseteq \bigcap_{htP=1}(\mathcal{D}(\mathcal{R}_P) * \mathcal{R}_P) \subseteq \bigcap_{htP=1}\mathcal{R}_P = \mathcal{R}$. Hence $\bigcap_{htP=1}\mathcal{D}(\mathcal{R}_P) = \mathcal{D}(\mathcal{R})$. Therefore:

$$\mathcal{D}(\mathcal{R}) \subseteq \bigcap_{htP=1} End_{\mathcal{D}(\tilde{\mathcal{R}}_P)}(\mathcal{D}(\tilde{\mathcal{R}}_P,\mathcal{R}_P)) = \mathcal{D}(\mathcal{R})$$

and the result holds. $\square$

Unfortunately, S_2 does not seem to be strong enough to ensure that $\mathcal{D}(\tilde{\mathcal{X}},\mathcal{X})$ is projective in general, but we do have the following:

Lemma 7 *Let $\mathcal{X}$ be an S_2 variety with smooth, injective normalisation. Then: $\mathcal{D}(\tilde{\mathcal{X}},\mathcal{X})$ is a reflexive right ideal of $\mathcal{D}(\tilde{\mathcal{X}})$.*

Proof: Denote $\mathcal{D}(\tilde{\mathcal{X}},\mathcal{X})$ by I so that:

$$I^{**} = Hom_{\mathcal{D}(\tilde{\mathcal{X}})}(Hom_{\mathcal{D}(\tilde{\mathcal{X}})}(I,\mathcal{D}(\tilde{\mathcal{X}})),\mathcal{D}(\tilde{\mathcal{X}})) \subseteq \mathcal{D}(\tilde{\mathcal{X}}).$$

It is easy to see that $I \subseteq I^{**}$ so it is the reverse inclusion that we need to prove. Let P be a prime ideal of $\mathcal{R}$. Then $(I^*)_P = ({}_PI)^*$ and ${}_PI = \mathcal{D}(\tilde{\mathcal{R}}_P,\mathcal{R}_P)$ is a projective $\mathcal{D}(\tilde{\mathcal{R}}_P)$-module. Thus ${}_PI = ({}_PI)^{**} = ((I^*)_P)^* =_P (I^{**})$. Hence $(I^{**}) * \tilde{\mathcal{R}} \subseteq ({}_P(I^{**})) *$

$\overline{\mathcal{R}}_P \subseteq \mathcal{R}_P$. But since $\mathcal{R}$ is S_2, $\mathcal{R} = \bigcap_{htP=1} \mathcal{R}_P$ and so $(I^{**}) * \overline{\mathcal{R}} \subseteq \mathcal{R}$. Therefore $I^{**} \subseteq I$ and I is reflexive. □

Now comes the trick (due to Bass) which enables us to improve reflexivity to projectivity when X is a surface.

Lemma 8 *Let A be a non-commutative noetherian ring of global dimension 2 and suppose that I is a reflexive right ideal of A. Then I is projective.*

Proof: Let K be a submodule of a finitely generated, projective left A-module P. Then we have a short exact sequence as follows:

$$0 \longrightarrow K \longrightarrow P \longrightarrow P/K \longrightarrow 0.$$

Since A has global dimension two, K must have projective dimension one or less. Therefore, given any short exact sequence as follows:

$$0 \longrightarrow G \longrightarrow G' \longrightarrow K \longrightarrow 0,$$

with G' free, then G must be projective.

Let H be any finitely generated left A-module and let F be some finitely generated free left A-module which maps onto H. Applying the functor $Hom_A(-, A)$ to the exact sequence $F \to H \to 0$, we get another exact sequence $0 \to H^* \to F^*$. Thus H^* and all of its submodules are submodules of a finitely generated free module.

Now, since A is noetherian, $I = I^{**}$ is finitely generated and hence so is I^*. Let G be a finitely generated free left A-module which maps onto I^*. Then we have a short exact sequence as follows:

$$0 \longrightarrow H \longrightarrow G \longrightarrow I^* \longrightarrow 0,$$

with H some submodule of G. Applying $Hom_A(-, A)$ to this, we get another exact sequence:

$$0 \longrightarrow I \longrightarrow G^* \longrightarrow H^*.$$

The first two paragraphs now show that I is projective. □

Corollary 9 *If $\mathcal{X}$ is a two dimensional S_2 variety with smooth, injective normalisation then $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ is a projective right ideal of $\mathcal{D}(\widetilde{\mathcal{X}})$.*

Proof: Combine Lemmas 7 and 8 together with the fact that $\mathcal{D}(\widetilde{\mathcal{X}})$ has global dimension two (Proposition 1.3). □

We now have all the pieces of information we need, so it just remains to put them all together.

Theorem 10 *Let $\mathcal{X}$ be a two dimensional S_2 variety with smooth, injective normalisation. Then $\mathcal{D}(\mathcal{X})$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{X}})$.*

Proof: By Corollary 9, $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ is a projective right ideal of $\mathcal{D}(\widetilde{\mathcal{X}})$ and by Lemma 6, $\mathcal{D}(\mathcal{X}) = End_{\mathcal{D}(\widetilde{\mathcal{X}})}(\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X}))$. Hence the result is true. □

We also have an equivalence as in Theorem 3.16 to show that our conditions are really necessary. This equivalence applies even to varieties that are three dimensional or more and characterises varieties that are S_2 and have smooth, injective normalisation as the varieties whose differential operator rings are maximal orders. Recall the definition of a maximal order:

Definition 11 *Let A be a non-commutative order in a division ring D. Then A is said to be a* maximal order *if given any ring B with $A \subseteq B \subseteq D$ such that there exist elements a and b in A with $aBb \subseteq A$, then $A = B$.*

Notice that if $A \subseteq B$ contains a right (or left) ideal I of B then $x.B.1 \subseteq A$ for any $0 \neq x \in I$ and so we are in the situation of Definition 11. We shall often use this fact later on.

Now, in our characterisation of varieties with smooth, injective normalisation, we must show that if $\mathcal{R}$ is the coordinate ring of a variety whose differential operator ring is a maximal order then $\mathcal{R}$ has injective normalisation. In order to do this it is only necessary to show that if P is a height one prime ideal of $\mathcal{R}$ then $\overline{\mathcal{R}}_P$ is local and the natural inclusion of the residue field of $\mathcal{R}_P$ into the residue field of $\overline{\mathcal{R}}_P$ is an isomorphism. The proof of this fact uses a process called 'Henselisation' which would take too long to describe here. Briefly, Henselisation is a process which allows a form of the implicit function theorem to be used for algebraic functions and is therefore useful for reducing problems about algebraic functions to ones on analytic functions. The idea behind the proof of the result stated above is that the result is easy to show for germs of analytic functions and one uses Henselisation for the general result. We summarise these facts in the following lemma:

Lemma 12 *Let $\mathcal{R}$ be the coordinate ring of a variety X. Then $\mathcal{R}$ has injective normalisation if $\overline{\mathcal{R}}_P$ is local for every height one prime P of $\mathcal{R}$, and $\mathcal{R}_P + M = \overline{\mathcal{R}}_P$, where M is the maximal ideal of $\overline{\mathcal{R}}_P$.*

Proof: For the details of the proof see [Ferrand]. □

The next result will be useful in order to be able to apply Lemma 12.

Lemma 13 *Let $\mathcal{R}$ be the coordinate ring of a variety with smooth normalisation and let P be a height one prime ideal of $\mathcal{R}$. If M is a prime ideal of $\overline{\mathcal{R}}$ which is minimal over P and $P\overline{\mathcal{R}}_M = M\overline{\mathcal{R}}_M$ then $\mathcal{D}(\mathcal{R}_P) \subsetneq \mathcal{D}(\overline{\mathcal{R}}_M)$.*

Proof: As in the proof of Proposition 1, there is a field $K = k[t_1, \ldots, t_r]$ contained in $\mathcal{R}_P$ over which $\overline{\mathcal{R}}_P/M$ is integral. Also, since $\overline{\mathcal{R}}_M$ is regular local of dimension one, we may choose a regular parameter, t_0 say, for M and since $P\overline{\mathcal{R}}_M = M\overline{\mathcal{R}}_M$ we may assume that $t_0 \in P$. Then $t_0, t_1, \ldots, t_r$ form a transcendence basis for Q over k and $\mathcal{D}_k(Q) = Q[\partial/\partial t_0, \ldots, \partial/\partial t_r]$. The derivations $\{\partial/\partial t_0, \ldots, \partial/\partial t_r\}$ form a basis for the free $\overline{\mathcal{R}}_M$-module $Der_k(\overline{\mathcal{R}}_M)$.

Let $\delta \in \mathcal{D}(\mathcal{R}_P) \subseteq \mathcal{D}(Q)$ and write:

$$\delta = \sum a_{i_0,\ldots,i_r} \frac{\partial}{\partial t_0}^{i_0} \cdots \frac{\partial}{\partial t_r}^{i_r}.$$

Then if $j_0, \ldots, j_r$ are natural numbers we have that

$$\delta * t_0^{j_0} \ldots t_r^{j_r} = \sum N_{i_0,\ldots,i_r} a_{i_0,\ldots,i_r} t_0^{j_0 - i_0} \ldots t_r^{j_r - i_r}$$

for some $N_{i_0,\ldots,i_r} \in \mathbf{N}$. This implies that each $a_{i_0,\ldots,i_r} \in \mathcal{R}_P$ and that δ therefore lies in $\mathcal{D}(\overline{\mathcal{R}}_M)$. □

Corollary 14 *Let $\mathcal{R}$ be the coordinate ring of a variety with smooth normalisation and let P be a height one prime ideal of $\mathcal{R}$. If $M\widetilde{\mathcal{R}}_M = P\widetilde{\mathcal{R}}_M$ for each prime ideal M of $\widetilde{\mathcal{R}}$ minimal over P then $\mathcal{D}(\mathcal{R}_P) \subseteq \mathcal{D}(\widetilde{\mathcal{R}}_P)$.*

Proof: Let $M_1, \ldots, M_r$ be the prime ideals of $\widetilde{\mathcal{R}}$ minimal over P and let $\partial \in \mathcal{D}(\mathcal{R}_P)$. Then by Lemma 13, ∂ extends to each $\widetilde{\mathcal{R}}_{M_i}$ so that we have:

$$\partial * \widetilde{\mathcal{R}}_P \subseteq \bigcap_{i=1}^{r} \partial * \widetilde{\mathcal{R}}_{M_i} \subseteq \bigcap_{i=1}^{r} \widetilde{\mathcal{R}}_{M_i} = \widetilde{\mathcal{R}}_P.$$

Therefore ∂ extends to $\widetilde{\mathcal{R}}_P$. □

Theorem 15 *If $\mathcal{X}$ is a variety with smooth normalisation then the following are equivalent:*

(i) $\mathcal{X}$ is S_2 and has injective normalisation.

(ii) $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ is a reflexive right $\mathcal{D}(\widetilde{\mathcal{X}})$-module and $\mathcal{D}(\mathcal{X}) = End_{\mathcal{D}(\widetilde{\mathcal{X}})}(\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X}))$.

(iii) $\mathcal{D}(\mathcal{X})$ is a maximal order.

Proof: $(i) \Rightarrow (ii)$ is just Lemmas 6 and 9.

$(ii) \Rightarrow (iii)$ This is just [MR; Proposition 5.1.11].

$(iii) \Rightarrow (i)$ Writing $\mathcal{S}$ in place of $\bigcap_{htP=1} \mathcal{R}_P$, we see that $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{S})$. Now, regular rings are Cohen-Macaulay and hence S_2 by Lemma 5. Therefore $\bigcap \widetilde{\mathcal{R}}_P = \widetilde{\mathcal{R}}$ where the intersection runs over all height one primes P of $\mathcal{R}$. This implies that $\mathcal{S} \subseteq \widetilde{\mathcal{R}}$, so there exists a non-zero ideal I of $\mathcal{R}$ such that $I\mathcal{S} \subseteq \mathcal{R}$. Thus $0 \neq I\mathcal{D}(\mathcal{S}) \subseteq \mathcal{D}(\mathcal{S}, \mathcal{R})$ which is an ideal of $\mathcal{D}(\mathcal{R})$. Hence $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}(\mathcal{S})$ are equivalent orders. But $\mathcal{D}(\mathcal{R})$ is a maximal order which implies that $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{S})$. A comparison of the differential operators of degree zero then shows that $\mathcal{R} = \mathcal{S}$ and is S_2.

By Lemma 12, in order to prove that $\mathcal{X}$ has injective normalisation, it is sufficient to show that the ring $\widetilde{\mathcal{R}}_P$ is local and that $\widetilde{\mathcal{R}}_P = \mathcal{R}_P + J(\widetilde{\mathcal{R}}_P)$ for every height one prime ideal P of $\mathcal{R}$. Notice that $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X}) \subseteq \mathcal{D}(\mathcal{X}) \subseteq End_{\mathcal{D}(\widetilde{\mathcal{X}})}(\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X}))$ so that the fact that $\mathcal{D}(\mathcal{X})$ is a maximal order implies that $\mathcal{D}(\mathcal{X}) = End_{\mathcal{D}(\widetilde{\mathcal{X}})}(\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X}))$. Therefore if P is a height one prime ideal of $\mathcal{R}$, $\mathcal{D}(\mathcal{R}_P) = End_{\mathcal{D}(\widetilde{\mathcal{R}}_P)}(\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P))$, since if $\theta \in End(\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P))$ then $\theta(\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})) \subseteq \mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)$. But $\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) = \mathcal{R}_P\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$ is a finitely generated right $\mathcal{D}(\widetilde{\mathcal{R}}_P)$-module so that there exists some $s \in \mathcal{R}\backslash P$ with $s\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) \subseteq \mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$. Hence $s\theta \in End(\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})) = \mathcal{D}(\mathcal{R})$ and $\theta \in \mathcal{D}(\mathcal{R}_P)$. Also, by Proposition 1, $\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P)$ is projective. Consequently $\mathcal{D}(\mathcal{R}_P)$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{R}}_P)$ and must be simple. In particular, $1 \in \mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) * \widetilde{\mathcal{R}}_P$ otherwise $\mathcal{R}_P/(\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) * \widetilde{\mathcal{R}}_P)$ would be a $\mathcal{D}(\mathcal{R}_P)$-module with a non-zero annihilator, contradicting the fact that $\mathcal{D}(\mathcal{R}_P)$ is simple.

Now, suppose that $\mathcal{S} = \mathcal{R}_P + J(\widetilde{\mathcal{R}}_P)$ and that $\mathcal{S} \neq \widetilde{\mathcal{R}}_P$. Then Lemma 13 implies that $\mathcal{D}(\mathcal{S}) \subsetneq \mathcal{D}(\widetilde{\mathcal{R}}_P)$ so that $\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{S})$ is a proper ideal of $\mathcal{D}(\mathcal{S})$. Therefore $\mathcal{D}(\mathcal{S})$ is not simple. But we know that $1 \in \mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{R}_P) * \widetilde{\mathcal{R}}_P \subseteq \mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{S}) * \widetilde{\mathcal{R}}_P$. The usual argument (Lemma 3.12) implies that $\mathcal{D}(\mathcal{S}) = End_{\mathcal{D}(\widetilde{\mathcal{R}}_P)}\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{S})$ and by Proposition 1, $\mathcal{D}(\widetilde{\mathcal{R}}_P, \mathcal{S})$ is projective. Therefore $\mathcal{D}(\mathcal{S})$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{R}}_P)$ and must be simple. Therefore $\widetilde{\mathcal{R}}_P = \mathcal{R}_P + J(\widetilde{\mathcal{R}}_P)$ and the result holds. □

Chapter 2

Classification of Right Ideals

This chapter is concerned with the right ideals of the ring of differential operators of a smooth variety. These were classified for curves in [Cannings & Holland] by means of certain vector subspaces of the coordinate ring of the curve in question. In more detail, if $\mathcal{R}$ is the coordinate ring of a smooth curve and D is a right ideal of $\mathcal{D}(\mathcal{R})$ which contains an ideal of $\mathcal{R}$ then we may define a vector subspace V of $\mathcal{R}$ as follows: set $V = D * \mathcal{R}$, the subspace of $\mathcal{R}$ generated by all elements of the form $\partial * r$ for $\partial \in D$ and $r \in \mathcal{R}$. [Cannings & Holland] characterises the subspaces that arise in this way (as so-called *primary decomposible subspaces*) and shows that the map $D \mapsto D * \mathcal{R}$ is a bijection. In this chapter we are able to give a different characterisation of these vector subspaces which has two advantages: firstly, the proof of the classification of the right ideals of $\mathcal{D}(\mathcal{R})$ is greatly simplified; and secondly, the method partially generalises to two dimensional varieties to give a classification of projective right ideals of $\mathcal{D}(\mathcal{X})$ where $\mathcal{X}$ is a surface. Although this classification is only partial at present, it is to be hoped that the additional restrictions required to make the method work may be unnecessary.

We begin by stating the definition of a primary decomposible vector space as given in [Cannings & Holland]. We then present an alternative definition of primary decomposible and show that the two definitions are equivalent. Next we prove the classification of the right ideals of the ring of diferential operators on a smooth curve as mentioned above and finally we generalise these results to results about surfaces. Throughout this chapter, $\mathcal{R}$ will be the coordinate ring of a smooth variety $\mathcal{X}$. As usual, the base field k will be algebraically closed of characteristic zero. Also, whenever we talk about vector spaces, we will almost always mean k-vector spaces.

2.1 Primary Decomposible Vector Spaces

In this section we present the two definitions of primary decomposible subpaces of $\mathcal{R}$ and show that they are equivalent. In order to get some finiteness conditions we must insist that most of the objects we work with contain an ideal of the ring $\mathcal{R}$.

Definition 1 *Let V be a k-vector subspace of $\mathcal{R}$. Then V is called* dense *if V contains a non-zero ideal of $\mathcal{R}$. Also, if D is a right ideal of $\mathcal{D}(\mathcal{R})$ then D is called* dense *if D contains a non-zero ideal of $\mathcal{R}$.*

Containing an ideal of $\mathcal{R}$ is a particularly important property to have in the case that $\mathcal{X}$ is a curve. This is because if $\mathcal{X}$ is a curve then $\mathcal{R}$ is regular of Krull dimension one so is a Dedekind domain. Therefore, if I is a non-zero ideal of $\mathcal{R}$ then it is a product of maximal ideals. Furthermore, if $I = \mathbf{m}_1^{r_1} \ldots \mathbf{m}_n^{r_n}$, then:

$$\mathcal{R}/I \cong [\mathcal{R}/\mathbf{m}_1^{r_1}] \oplus \ldots \oplus [\mathcal{R}/\mathbf{m}_n^{r_n}].$$

Hence $\mathcal{R}/I$ is finite dimensional over k and is artinian. Containing a power of a maximal ideal is even nicer so we give vector spaces with this property special consideration. For our purposes we are able to weaken this condition slightly and this is where the primary decomposible subspaces come in. As the next definition only really makes sense when $\mathcal{X}$ is a curve, we impose that restriction.

Definition 2 *Let $\mathcal{X}$ be a curve and let V be a dense subspace of $\mathcal{R}$. Then V is called* primary *if it contains a power of a maximal ideal of $\mathcal{R}$. V is said to be* primary decomposible *if it is an intersection of primary subspaces.*

The notions of *primary* and *primary decomposible* as subspaces coincide with the usual meanings for ideals of $\mathcal{R}$. Thus every ideal of $\mathcal{R}$ is a primary decomposible subspace. Notice that if V is primary decomposible then it is an intersection of finitely many primary subspaces. This is because V is dense and so $\mathcal{R}/V$ is a finite dimensional k-vector space. In fact, the primary vector spaces which intersect to give V are essentially unique.

Lemma 3 *Let $\mathcal{X}$ be a curve and let V be a primary decomposible subspace of $\mathcal{R}$. Suppose that I is the largest ideal of $\mathcal{R}$ contained in V and that $\mathbf{m}$ is a maximal ideal of $\mathcal{R}$ containing I. Then $V(\mathbf{m}) = \bigcap_{n=1}^{\infty}(\mathbf{m}^n + V)$ is the smallest $\mathbf{m}$-primary subspace containing V and:*

$$V = \bigcap_{I \subseteq \mathbf{m} \in \mathbf{Spec}(\mathcal{R})} V(\mathbf{m}).$$

Proof: Note first that $V(\mathbf{m})$ is a primary subspace of $\mathcal{R}$ since $\mathcal{R}/I$ is finite dimensional and $V(\mathbf{m}) = \mathbf{m}^n + V$ for some $n \in \mathbf{N}$. Then, if W were some $\mathbf{m}$-primary subspace containing V, it would also contain $\mathbf{m}^r + V$ for some $r \in \mathbf{N}$. But $V(\mathbf{m}) \subseteq \mathbf{m}^r + V$.

Finally, since V is primary decomposible it is the intersection of the primary vector spaces which contain it and since $V(\mathbf{m})$ is contained in every $\mathbf{m}$-primary subspace for each maximal ideal $\mathbf{m}$ which contains I, $V = \bigcap_{I \subseteq \mathbf{m}} V(\mathbf{m})$. □

The problem with the definition of primary decomposible which we have just given is that it relies heavily on the fact that $\mathcal{X}$ is one dimensional for it to be of any use: lots of right ideals of $\mathcal{D}(\mathcal{X})$ for a surface $\mathcal{X}$ contain ideals of $\mathcal{R}$ which are of height one. It is therefore not clear how to give a definition of primary and primary decomposible for subpaces of two dimensional rings which will be of much use in classifying right ideals of $\mathcal{D}(\mathcal{X})$. Instead, we give a new definition which although is not as intrinsic as Definition 2, has the advantage that it gives a vector space V a module structure over some ring. This module structure does not then rely the dimension of the variety in order to yield positive results.

From now on, if V is a dense vector subspace of $\mathcal{R}$ then we will use the terminology $S(V)$ to denote the set $\{q \in Q \big| qV \subseteq V\}$, where Q is the field of fractions of $\mathcal{R}$. When the context is clear, we will often just write S instead of $S(V)$. The set $S = S(V)$ is a ring and is, in fact, a subring of $\mathcal{R}$. To see this, suppose that $s \in S$ and let I be the largest ideal of $\mathcal{R}$ in V. Then $sI \subseteq V$ is an ideal of $\mathcal{R}$ and therefore lies inside I. Thus I is an ideal of both $\mathcal{R}$ and S. Let $x \in I$. Then $\mathcal{R}[s] \cong x\mathcal{R}[s] \subseteq \mathcal{R}$ so that $\mathcal{R}[s]$ is a finitely generated $\mathcal{R}$-module. Hence s is integral over $\mathcal{R}$ and must lie in $\mathcal{R}$

because $\mathcal{R}$ is integrally closed. So $S \subseteq \mathcal{R}$ as claimed.

The construction of $S = S(V)$ makes V into an S-module in the obvious way and we may therefore localise V (and $\mathcal{R}$) at prime ideals of S. This leads us to a new definition of primary decomposible which is quite similar to the S_2 condition of Definition 1.4.4. Since S_2 was exactly what we needed to work with in Section 1.4, it is plausible that this is the definition we need.

Definition 4 *Let $\mathcal{R}$ be the coordinate ring of a smooth variety $\mathcal{X}$. Then a dense vector subspace V of $\mathcal{R}$ is called* primary decomposible *if the following conditions all hold:*

(i) $S = S(V)$ is a noetherian ring,

(ii) The map $\pi : \mathbf{Spec}\mathcal{R} \to \mathbf{Spec}S$ is bijective,

(iii) $V = \bigcap_{htP=1} V_P$, where the intersection runs over all the height one primes of S.

It is easy to see that condition (iii) of the above gives that S is S_2. In the case that $\mathcal{X}$ is a curve, we automatically have conditions (i) and (iii). Condition (iii) is because height one prime ideals of a one dimensional ring are maximal ideals, and every module is the intersection of its localisations at maximal ideals. For condition (i), we even have that S is affine over k. This is because of the Artin-Tate lemma below. Hence if $\mathcal{X}$ is a curve, a dense vector subspace V of $\mathcal{R}$ is primary decomposible (in the sense of Definition 4) if and only if the map $\pi : \mathbf{Spec}\mathcal{R} \to \mathbf{Spec}S(V)$ is injective.

Lemma 5 (Artin-Tate) *Let $A \subseteq B$ be commutative k-algebras and suppose that B is affine over k and a finitely generated A-module. Then A is also affine over k.*

Proof: Let $x_1, \ldots, x_m$ generate B over k and $y_1, \ldots, y_n$ generate B as an A-module. Each x_i can be expressed as a sum $x_i = \sum_{j=1}^{n} a_{i,j} y_j$ for some $a_{i,j} \in A$. Also, each product $y_i y_j$ equals $\sum_{k=1}^{n} c_{i,j,k} y_k$ for some $c_{i,j,k} \in A$. If we set $C = k[a_{i,j}, c_{i,j,k}]$, then $B = \sum_{j=1}^{n} y_j C$ and so is a finitely generated C-module. But C is affine over k so that both B and A are Noetherian C-modules. Hence A is a finitely generated module over an affine k-algebra and so must be affine itself. $\square$

Now, we have given two definitions of primary decomposible subspaces of the coordinate ring of a smooth curve so we had better check that they agree with one another.

Proposition 6 *Let $\mathcal{X}$ be a curve and let V be a dense subspace of $\mathcal{R}$. Then V is primary decomposible in the sense of Definition 2 if and only if it is in the sense of Definition 4.*

Proof: Suppose first that V is the intersection of the primary subspaces which contain it and set $S = S(V)$. As mentioned above, conditions (i) and (iii) of Definition 4 hold automatically since if I is the largest ideal of $\mathcal{R}$ which sits inside V then I also lies in S. Thus $\mathcal{R}/S$ is a finite dimensional k-vector space and $\mathcal{R}$ must be a finitely generated S-module. Therefore we may apply the Artin-Tate lemma (Lemma 5) to find that S is affine over k.

So the main thing to prove is that the map $\pi : \mathbf{Spec}\mathcal{R} \to \mathbf{Spec}S$ is bijective. Since S is noetherian, $\mathcal{R}$ is a finitely generated S-module because if $0 \neq x \in I$ then $\mathcal{R} \cong x\mathcal{R} \subsetneq S$. Hence given an element y of $\mathcal{R}$ then $S[y]$ is a finitely generated S-

module and y must be integral over S. So $\mathcal{R}$ must be the normalisation of S in its field of fractions Q. The 'lying over' properties of integral extensions then ensures that π is surjective.

To show that π is injective, notice that S is itself the intersection of primary subspaces which contain it. To see this, notice that if $V = \bigcap_{\mathbf{m}} V(\mathbf{m})$ then

$$[\bigcap_{\mathbf{m}} S(V(\mathbf{m}))].V \subseteq \bigcap_{\mathbf{m}}[S(V(\mathbf{m})).V(\mathbf{m})] \subseteq \bigcap_{\mathbf{m}} V(\mathbf{m}) = V.$$

So we have the following:

$$S \subseteq \bigcap_{\mathbf{m}} S(\mathbf{m}) \subseteq \bigcap_{\mathbf{m}} S(V(\mathbf{m})) \subseteq S.$$

Therefore $S = \bigcap S(\mathbf{m})$ and we can now see that π is injective. In more detail, suppose that $M_1, \ldots, M_r$ are the maximal ideals of $\mathcal{R}$ which contain I. Let $s \in \mathbf{N}$ be large enough so that $M_1^s \ldots M_r^s \subseteq I$. Then we may choose idempotents $e_1, \ldots, e_r$ with the properties that $1 = e_1 + \ldots + e_r$ and each $e_i \in M_j^s$ for $j \neq i$ so that the e_i's lie in S. Recall that $\mathcal{R}/M_1^s \ldots M_r^s \cong \mathcal{R}/M_1^s \oplus \ldots \oplus \mathcal{R}/M_r^s$. Using the e_i's we find that $S/M_1^s \ldots M_r^s \cong (S + M_1^s)/M_1^s \oplus \ldots \oplus (S + M_r^s)/M_r^s$. Therefore the maximal ideals of S which contain I are in one to one correspondence with $M_1, \ldots, M_r$. Also, it is clear that if M is a maximal ideal of S which does not contain I then there is a unique maximal ideal of $\mathcal{R}$ containing M. Hence π is injective as claimed.

Now suppose that V and S satisfy properties (i), (ii) and (iii) of Definition 4 and let P be a maximal ideal of S. The injectivity of π ensures that $\mathcal{R}_P$ is local. Therefore V_P contains a power of the maximal ideal of $\mathcal{R}_P$ and if $\mathbf{m}$ is the unique maximal ideal of $\mathcal{R}$ lying over P then $\mathbf{m}^r \subseteq V_P \bigcap \mathcal{R}$ for some $r \in \mathbf{N}$. Hence $V(\mathbf{m}) \subseteq V + \mathbf{m}^r \subseteq V_P \bigcap \mathcal{R}$.

But the fact that V is the intersection of its localisations at maximal ideals of S implies that $\bigcap_{\mathfrak{m}} V(\mathfrak{m}) \subseteq \bigcap_{htP=1} V_P = V$. $\square$

2.2 The Classification For Curves

In this section we will show how to construct a map from the set of dense right ideals of $\mathcal{D}(\mathcal{X})$ to the set of dense subspaces of $\mathcal{R}$, where $\mathcal{X}$ is a curve. We will then prove that this map is injective and that its image is precisely the set of primary decomposible subspaces of $\mathcal{R}$. So the set of dense right ideals of $\mathcal{D}(\mathcal{X})$ is in bijection with the set of primary decomposible subspaces of $\mathcal{R}$. This is achieved by finding a map from the set of dense vector subspaces to the set of dense right ideals which is the inverse to the first map when restricted to the primary decomposible subspaces. Although this section is dedicated to the case of when $\mathcal{X}$ is a smooth curve, the methods we use will often be general enough for when $\mathcal{X}$ is a surface and so we prove most of the results in full generality. Throughout this section therefore, $\mathcal{X}$ will be any smooth surface with coordinate ring $\mathcal{R}$, and we will specify which results only work curves.

Now, given a dense right ideal D of $\mathcal{D}(\mathcal{X})$ we may define a vector subspace $D * \mathcal{R}$ of $\mathcal{R}$ by setting $D * \mathcal{R}$ to be the vector space spanned by elements of the form $\partial * x$ where $\partial \in D$, and $x \in \mathcal{R}$. Notice that $D * \mathcal{R}$ contains any ideals of $\mathcal{R}$ that D does and so is dense.

Conversely, given a dense vector subspace V of $\mathcal{R}$ we can define a right ideal $\mathcal{D}(\mathcal{R}, V)$ of $\mathcal{D}(\mathcal{X})$ by $\mathcal{D}(\mathcal{R}, V) = \{\partial \in \mathcal{D}(\mathcal{X}) \big| \partial * \mathcal{R} \subseteq V\}$. Again, $\mathcal{D}(\mathcal{R}, V)$ contains any ideals of $\mathcal{R}$ that V does and must be dense. The next result shows that $\mathcal{D}(\mathcal{R}, V)$

behaves well with respect to localisation.

Lemma 1 *Let V be a dense subspace of $\mathcal{R}$ and suppose that $S = S(V)$ is noetherian. Let T be a multiplicatively closed subset of $S = S(V)$. Then $S_T\mathcal{D}(\mathcal{R}, V) = \mathcal{D}(\mathcal{R}_T, V_T) = \mathcal{D}(\mathcal{R}, V)\mathcal{R}_T$, where $\mathcal{D}(\mathcal{R}_T, V_T) = \{\partial \in \mathcal{D}(\mathcal{R}_T) | \partial * \mathcal{R}_T \subseteq V_T\}$.*

Proof: Since S is dense and noetherian, V must be a finitely generated S-module. Now, the same argument as given in the proof of Lemma 1.3.11 shows that $\mathcal{D}(\mathcal{R}, V)$ as defined above is the same as the module of differential operators between $\mathcal{R}$ and V as S-modules. Therefore Lemma 1.2.7 gives us the result. □

Proposition 2 *Let V be a primary decomposible subspace of $\mathcal{R}$. Then we have that $\mathcal{D}(\mathcal{R}, V) * \mathcal{R} = V$.*

Proof: Set $S = S(V)$ and let P be a height one prime ideal of S. The fact that V is primary decomposible implies that $\mathcal{R}_P$ is a local ring with maximal ideal M say. Then V_P contains some power M^r of M for some $r \in \mathbf{N}$. By Proposition 1.2.15, $\mathcal{D}_k(\mathcal{R}_P/M^r) = End_k(\mathcal{R}_P/M^r)$. Thus there exists some $\delta \in \mathcal{D}(\mathcal{R}_P/M^r)$ with $\delta * (\mathcal{R}_P/M^r) = V_P/M^r$. But by Proposition 1.2.14, differential operators on factor rings of regular rings lift to give differential operators on the whole ring. Therefore we may lift δ to a differential operator $\partial \in \mathcal{D}(\mathcal{R}_P)$ with $(\partial * \mathcal{R}_P) + M^r = V_P$. This shows that $\mathcal{D}(\mathcal{R}_P, V_P) * \mathcal{R}_P = V_P$.

Since $\mathcal{D}(\mathcal{R}_P, V_P) * \mathcal{R}_P = V_P$, the same argument as in the proof of Lemma 1.3.12 shows that $\mathcal{D}(V_P) = End_{\mathcal{D}(\mathcal{R}_P)}\mathcal{D}(\mathcal{R}_P, V_P)$, where $\mathcal{D}(V_P) = \{\partial \in \mathcal{D}(Q) | \partial * V_P \subseteq V_P\}$.

$$\text{So } [\bigcap_{htP=1} End_{\mathcal{D}(\mathcal{R}_P)}\mathcal{D}(\mathcal{R}_P, V_P)] * V \subseteq \bigcap_{htP=1} [\mathcal{D}(\mathcal{R}_P, V_P) * V_P] \subseteq \bigcap_{htP=1} V_P = V.$$

Also, by Lemma 1, $\mathcal{D}(\mathcal{R}_P, V_P) = \mathcal{D}(\mathcal{R}, V)\mathcal{R}_P$ which implies that $End_{\mathcal{D}(\mathcal{R})}\mathcal{D}(\mathcal{R}, V) \subseteq End_{\mathcal{D}(\mathcal{R}_P)}\mathcal{D}(\mathcal{R}_P, V_P)$ for each height one prime ideal P of S. Hence if $\mathcal{D}(V) = \{\partial \in \mathcal{D}(Q) | \partial * V \subseteq V\}$, then we have the following inclusions:

$$\mathcal{D}(V) \subseteq End_{\mathcal{D}(\mathcal{R})}\mathcal{D}(\mathcal{R}, V) \subseteq \bigcap_{htP=1} End_{\mathcal{D}(\mathcal{R}_P)}\mathcal{D}(\mathcal{R}_P, V_P) \subseteq \mathcal{D}(V)$$

Therefore $\mathcal{D}(V)$ must equal $End_{\mathcal{D}(\mathcal{R})}\mathcal{D}(\mathcal{R}, V)$.

We now have that $\mathcal{D}(V) = End_{\mathcal{D}(\mathcal{R})}\mathcal{D}(\mathcal{R}, V)$ and we already know that $\mathcal{D}(\mathcal{R}, V)$ is a projective $\mathcal{D}(\mathcal{R})$-module since $\mathcal{D}(\mathcal{R})$ has global dimension one. Therefore $\mathcal{D}(V)$ is Morita equivalent to $\mathcal{D}(\mathcal{R})$ and in particular, is a simple ring. Now, notice that $\mathcal{D}(\mathcal{R}, V) * \mathcal{R}$ is a left $\mathcal{D}(V)$-module. Therfore $\mathcal{D}(V, \mathcal{D}(\mathcal{R}, V) * \mathcal{R})$ is a two sided ideal of $\mathcal{D}(V)$. But $\mathcal{D}(V)$ is simple and so $1 \in \mathcal{D}(V, \mathcal{D}(\mathcal{R}, V) * \mathcal{R}) = \mathcal{D}(V)$. Consequently, $\mathcal{D}(\mathcal{R}, V) * \mathcal{R} = V$ as required. □

Proposition 3 *Suppose D is a dense, projective right ideal of $\mathcal{D}(\mathcal{R})$. Then we have that $D = \mathcal{D}(\mathcal{R}, D * \mathcal{R})$.*

Proof: Writing D^* for the dual of D as a right $\mathcal{D}(\mathcal{R})$-module, we have the following:

$$D^* = \{\partial \in \mathcal{D}(Q) | \partial D \subseteq \mathcal{D}(\mathcal{R})\} = \{\partial \in \mathcal{D}(Q) | \partial D * \mathcal{R} \subseteq \mathcal{R}\}.$$

So setting $V = D * \mathcal{R}$, we have that $D^* = \{\partial \in \mathcal{D}(Q) | \partial * V \subseteq \mathcal{R}\} = \mathcal{D}(V, \mathcal{R})$. Now, $\mathcal{D}(V, \mathcal{R})\mathcal{D}(\mathcal{R}, V) * \mathcal{R} \subseteq \mathcal{R}$ so that $\mathcal{D}(V, \mathcal{R}) \subseteq [\mathcal{D}(\mathcal{R}, V)]^*$. In other words, $D^* \subseteq [\mathcal{D}(\mathcal{R}, V)]^*$. Thus $[\mathcal{D}(\mathcal{R}, V)]^{**} \subseteq D^{**}$. But every D is projective and hence reflexive. So we have proved that $\mathcal{D}(\mathcal{R}, V) \subseteq D$. But since $V = D * \mathcal{R}$, it is trivial that $D \subseteq \mathcal{D}(\mathcal{R}, V)$ and the assertion is true. □

Proposition 3 tells us that the map $D \mapsto D * \mathcal{R}$ is injective, and Proposition 2 says that the map $V \mapsto \mathcal{D}(\mathcal{R}, V)$ is injective when restricted to the set of primary decomposible subspaces of $\mathcal{R}$. It now remains to be shown that if D is a dense right ideal of $\mathcal{D}(\mathcal{R})$ then $D * \mathcal{R}$ is primary decomposible. There are (at least) two ways of doing this: one is quite short but only works for curves, and the other is longer but will also be of use for the surfaces case which we treat in this next section. We therefore choose the longer proof. In particular, we need some facts about completion of S-modules and how they relate to one another.

Now, recall that the process of completion applies equally as well to modules as it does to rings. We are particularly interested in the following situation: V is a dense subspace of $\mathcal{R}$ and P is a height one prime ideal of $S = S(V)$. Then V_P is an S_P-module and we may complete V_P at the largest ideal of $\mathcal{R}_P$ which it contains. Thus $V_P \subseteq \widehat{V_P} \subseteq \widehat{\mathcal{R}_P}$. Assuming that S is noetherian, the usual proof shows that $\mathcal{D}(V_P)$ is the module of differential operators from V_P to itself and we have the following proposition:

Proposition 4 *Let V, S and P be as above. Then* $\mathcal{D}(V_P) \cong \{\partial \in \mathcal{D}(\widehat{V_P}) | \partial * V_P \subseteq V_P\}$.

Proof: This proof is an easy adaptation of the proof of Proposition 1.3.15. $\square$

Under the assumptions of Proposition 4, let V_P^+ be the subspace of $\mathcal{R}_P$ defined as follows:

$$V_P^+ = \bigcap_{i=1}^{r} \bigcap_{n=1}^{\infty} (V_P + M_i^n),$$

where $M_1, \ldots, M_r$ are the maximal ideals of $\mathcal{R}_P$. Since $\mathcal{R}_P$ has Krull dimension one, if I is the largest ideal of $\mathcal{R}_P$ inside V_P then $\mathcal{R}_P/I$ is artinian, so there is an integer $s \in \mathbf{N}$ with the property that $V_P + M_i^s = V_P + M_i^{s+n}$ for each i and every $n > 0$. The aim of the next few results is to show that $\mathcal{D}(V_P) \subseteq \mathcal{D}(V_P^+)$.

By [Matsumura; Theorem 8.10], $\widehat{\mathcal{R}_P} \cong (\widehat{\mathcal{R}_P})_1 \oplus \ldots \oplus (\widehat{\mathcal{R}_P})_r$ where $(\widehat{\mathcal{R}_P})_i$ means $\mathcal{R}_P$ completed at the maximal ideal M_i. Now, for each $n \geq 1$ and each $1 \leq i \leq r$, we may choose elements $e_{i,n} \in \mathcal{R}_P$ such that:

(i) $e_{1,n} + \ldots + e_{r,n} = 1$ and

(ii) $e_{i,n} \in M_j^n$ for $j \neq i$.

Define $e_i = (e_{i,n} + I^n)_{n=1}^{\infty} \in \widehat{\mathcal{R}_P}$. Then the e_i are the idempotents corresponding to the decomposition of $\widehat{\mathcal{R}_P}$ mentioned above.

Proposition 5 *Let V_P^+ and the e_i be as above. Then $\widehat{V_P^+} = \sum_i e_i \widehat{V_P}$.*

Proof: Firstly notice that since $\hat{I}$ is an ideal of $\widehat{\mathcal{R}_P}$, $\hat{I} = \sum_i e_i \hat{I}$. Hence we may work modulo $\hat{I}$. But $\widehat{V_P^+}/\hat{I} \cong V_P^+/I$, as is easy to see from the definition of the completion. Now, we may find integers $s_1, \ldots, s_r$ such that $\mathcal{R}_P/I \cong \mathcal{R}_P/M_1^{s_1} \oplus \ldots \oplus \mathcal{R}_P/M_r^{s_r}$. For clarity, let us write W for V_P^+/I and $W(i)$ for $(V_P^+ + M_i^s)/I$. Then since $W(i)$ contains the image of M_i^s in $\mathcal{R}_P/I$, it also contains $\mathcal{R}_i^{\perp} = \bigoplus_{j \neq i} \mathcal{R}_P/M_j^{s_r}$. Thus $W(i) = e_{i,1}W(i) + \mathcal{R}_i^{\perp}$. Hence we have that:

$$W = \bigcap_{i=1}^{r} W(i) = \bigcap_{i=1}^{r} (e_{i,1}W(i) + \mathcal{R}_i^{\perp}) = \sum_{i=1}^{r} e_{i,1}W.$$

It is now easy to see that multiplying $\widehat{V_P^+}/\hat{I}$ by e_i corresponds under the isomorphism to multiplying W by $e_{i,1}$. The result follows. □

Corollary 6 *With the above notation,* $\mathcal{D}(\widehat{V_P}) \subseteq \mathcal{D}(\widehat{V_P^+})$.

Proof: If $\partial \in \mathcal{D}(\widehat{V_P})$ then:

$$\partial * \widehat{V_P^+} = \partial * \sum_i e_i \widehat{V_P} \subseteq \sum_i e_i \partial * \widehat{V_P} \subseteq \sum_i e_i \widehat{V_P} = \widehat{V_P^+}.$$

Thus $\mathcal{D}(\widehat{V_P}) \subseteq \mathcal{D}(\widehat{V_P^+})$. □

Proposition 7 *With* V_P *and* V_P^+ *as defined above, we have that* $\mathcal{D}(V_P) \subseteq \mathcal{D}(V_P^+)$.

Proof: By Proposition 4 and Corollary 6, we already know that $\mathcal{D}(V_P) \subseteq \mathcal{D}(\widehat{V_P}) \subseteq \mathcal{D}(\widehat{V_P^+})$. Also, by Proposition 4, $\mathcal{D}(V_P^+) \cong \{\partial \in \mathcal{D}(\widehat{V_P^+}) | \partial * V_P^+ \subseteq V_P^+\}$. But the construction of the extensions of differential operators on V_P to $\widehat{V_P^+}$ ensures that if $\partial \in \mathcal{D}(V_P)$ then ∂ takes V_P^+ to itself inside $\widehat{V_P^+}$. Therefore $\mathcal{D}(V_P) \subseteq \mathcal{D}(V_P^+)$. □

This completes all of the preparatory material we need in order to be able to complete the classification of right ideals of $\mathcal{D}(\mathcal{R})$. So let us return to the case of when $\mathcal{X}$ is one dimensional.

Proposition 8 *Let* $\mathcal{X}$ *be a smooth curve. If* D *is a dense right ideal of* $\mathcal{D}(\mathcal{R})$ *then* $D * \mathcal{R}$ *is primary decomposible.*

Proof: Set $V = D * \mathcal{R}$ and $S = S(V)$. We must check that the normalisation map $\pi : \mathcal{R} \to S$ is injective. So let $\mathbf{m}$ be a maximal ideal of S. Since $\mathcal{D}(\mathcal{R}_\mathbf{m}, V_\mathbf{m}) = \mathcal{S}_\mathbf{m}\mathcal{D}(\mathcal{R}, V)$, the fact that $\mathcal{D}(\mathcal{R}, V) * \mathcal{R} = V$ implies that $\mathcal{D}(\mathcal{R}_\mathbf{m}, V_\mathbf{m}) * \mathcal{R}_\mathbf{m} = V_\mathbf{m}$. Therefore $\mathcal{D}(V_\mathbf{m}) = End_{\mathcal{D}(\mathcal{R}_\mathbf{m})}\mathcal{D}(\mathcal{R}_\mathbf{m}, V_\mathbf{m})$. But $\mathcal{D}(\mathcal{R}_\mathbf{m})$ has global dimension one so that $\mathcal{D}(\mathcal{R}_\mathbf{m}, V_\mathbf{m})$ is projective. Hence $\mathcal{D}(V_\mathbf{m})$ is Morita equivalent to $\mathcal{D}(\mathcal{R}_\mathbf{m})$ and is a simple ring.

Now, Proposition 7 shows that $\mathcal{D}(V_{\mathbf{m}}) \subseteq \mathcal{D}(V_{\mathbf{m}}^+)$ and if $V_{\mathbf{m}} \neq V_{\mathbf{m}}^+$ then $\mathcal{D}(V_{\mathbf{m}}^+, V_{\mathbf{m}})$ is a proper ideal of $\mathcal{D}(V_{\mathbf{m}})$. This contradicts the fact that $\mathcal{D}(V_{\mathbf{m}})$ is simple. Hence $V_{\mathbf{m}} = V_{\mathbf{m}}^+$. I claim that $S_{\mathbf{m}} = S(V_{\mathbf{m}})$. Clearly $S_{\mathbf{m}} \subseteq S(V_{\mathbf{m}})$. For the reverse inclusion, let $s \in S(V_{\mathbf{m}})$. Then $sV \subseteq V_{\mathbf{m}}$. But V is a finitely generated S-module so that there exists a $t \in S \backslash \mathbf{m}$ such that $stV \subseteq V$. Therefore $st \in S$ and $s \in S_{\mathbf{m}}$. Thus $S_{\mathbf{m}} = S(V_{\mathbf{m}})$ and the fact that $V_{\mathbf{m}} = V_{\mathbf{m}}^+$ implies that $S_{\mathbf{m}} = S_{\mathbf{m}}^+$, since $S_{\mathbf{m}}^+$ is easily shown to lie inside $S(V^+)$.

The argument given in the proof of Proposition 1.6 now shows that S has injective normalisation. $\square$

Proposition 8 is the final fact needed in order to be able to prove the classification of the dense right ideals of $\mathcal{D}(\mathcal{R})$.

Theorem 9 *Let $\mathcal{X}$ be a smooth curve with coordinate ring $\mathcal{R}$. Then the dense right ideals of $\mathcal{D}(\mathcal{R})$ are in bijection with the primary decomposible subspaces of $\mathcal{R}$, with the bijection being given by $D \mapsto D * \mathcal{R}$.*

Proof: Proposition 3 shows that $D \mapsto D * \mathcal{R}$ is injective, Proposition 8 shows that $D * \mathcal{R}$ is primary decomposible and Proposition 2 shows that the map is surjective. $\square$

2.3 The Classification For Surfaces

In this section we attempt to extend the results of the last section to the case of when $\mathcal{R}$ is the coordinate ring of a smooth surface $\mathcal{X}$. Instead of trying to classify all of the dense right ideals of $\mathcal{D}(\mathcal{R})$ however, it turns out that it is the projective right ideals that we need to concentrate on. Of course, when $\mathcal{X}$ is just a curve, all right ideals of $\mathcal{D}(\mathcal{R})$ are projective, so the results of this section really are generalisations of those given in Section 2.

Throughout this section we make the assumption that $\mathcal{X}$ is a smooth surface with coordinate ring $\mathcal{R}$. In doing this we immediately run into two problems: firstly, not every right ideal of $\mathcal{D}(\mathcal{R})$ is projective; and secondly, if V is a dense subspace of $\mathcal{R}$ then $S = S(V)$ is not necessarily noetherian. For an example of the second problem, suppose that $\mathcal{R} = k[x, y]$ and S is the subring of $\mathcal{R}$ given by $S = k + xk[x, y]$. Then S is dense in $\mathcal{R}$ but is not noetherian since the ideal $xk[x, y]$ of S is not finitely generated.

If $S = \mathcal{D}(\mathcal{R}, S) * \mathcal{R}$ however, and $\mathcal{D}(\mathcal{R}, S)$ is a projective right ideal of $\mathcal{D}(\mathcal{R})$ then S is noetherian. This is because $\mathcal{D}(S)$ is Morita equivalent to $\mathcal{D}(\mathcal{R})$ and so is noetherian. Therefore, if $I_1 \subseteq I_2 \subseteq \ldots$ is an ascending chain of ideals of S then the chain of right ideals $I_1\mathcal{D}(S) \subseteq I_2\mathcal{D}(S) \subseteq \ldots$ of $\mathcal{D}(S)$ must stop after a finite number of steps. So there exists an integer n such that $I_n\mathcal{D}(S) = I_{n+i}\mathcal{D}(S)$ for every $i \geq 0$. But $I_{n+i}\mathcal{D}(S) * S = I_{n+i}$ which gives that $I_n = I_{n+i}$. Hence S is noetherian.

Our problem is though that if D is a projective right ideal of $\mathcal{D}(\mathcal{R})$ and $V = D * \mathcal{R}$ then $\mathcal{D}(V)$ is noetherian. It is not at all clear in this case if either S is noetherian or V is a noetherian S-module. In order to get anywhere then we must restrict ourselves to projective right ideals D of $\mathcal{D}(\mathcal{R})$ for which $S = S(D * \mathcal{R})$ is noetherian. The Artin-Tate lemma (Lemma 1.5) then shows that S is actually affine.

Now, we already know by Proposition 2.2 that if V is a primary decomposible subspace of $\mathcal{R}$ then $V = \mathcal{D}(\mathcal{R}, V) * \mathcal{R}$ and an easy adaption to Lemmas 1.4.7 and 1.4.8 shows that $\mathcal{D}(\mathcal{R}, V)$ is projective. We also know, by Proposition 2.3 that if D is a dense, projective right ideal of $\mathcal{D}(\mathcal{R})$ then $D = \mathcal{D}(\mathcal{R}, D * \mathcal{R})$. So it remains to prove that if D is a dense, projective right ideal of $\mathcal{D}(\mathcal{R})$ then $V = D * \mathcal{R}$ is primary decomposible. As mentioned above, we must assume that $S = S(V)$ is noetherian. The only things to prove therefore are that S has injective normalisation and that $V = \bigcap V_P$ where the intersection runs over the height one prime ideals of S. The latter problem is the easiest to sort out and is covered by the next result. That S has injective normalisation is harder to show and it is this which causes us most difficulties.

Lemma 1 *Suppose that D is a dense, projective right ideal of $\mathcal{D}(\mathcal{R})$ and set $V = D * \mathcal{R}$. Suppose also that $S = S(V)$ is noetherian. Then $V = \bigcap_{htP=1} V_P$.*

Proof: Since $\mathcal{R}$ is regular, $\mathcal{R} = \bigcap_{htP=1} \mathcal{R}_P$ where each P is a height one prime ideal of S. Therefore $V \subseteq \bigcap V_P \subseteq \mathcal{R}$. Now, since S is affine, V is a finitely generated S-module and so Lemma 1.2.7 shows that $\mathcal{D}(V) \subseteq \mathcal{D}(V_P)$ for every height one prime ideal P of S. Also, let $\partial \in \bigcap \mathcal{D}(V_P)$. Then $\partial * \bigcap V_P \subseteq \bigcap V_P$ so that $\mathcal{D}(\bigcap V_P) = \bigcap \mathcal{D}(V_P)$. Hence $\mathcal{D}(V) \subseteq \mathcal{D}(\bigcap V_P)$. So if $V \neq \bigcap V_P$ then $\mathcal{D}(\bigcap V_P, V)$ is a proper ideal of $\mathcal{D}(V)$. But $V = \mathcal{D}(\mathcal{R}, V) * \mathcal{R}$ and the usual argument shows that $\mathcal{D}(V)$ is Morita equivalent to $\mathcal{D}(\mathcal{R})$. Thus $\mathcal{D}(V)$ is simple which contradicts the fact that $\mathcal{D}(\bigcap V_P, V)$ is a proper ideal. Therefore $V = \bigcap V_P$. □

Recall that by Lemma 1.4.12, in order to prove that S has injective normalisation it is enough to show that $\bar{S}_P$ is local for each height one prime ideal of S, and that S_P has the same residue field as $\bar{S}_P$. Since $S \subseteq \mathcal{R}$ and contains an ideal of $\mathcal{R}$, $\bar{S}_P = \mathcal{R}_P$. The next result shows that $\mathcal{R}_P$ must be local.

Proposition 2 *Let D be a dense, projective right ideal of $\mathcal{R}$ and set $V = D * \mathcal{R}$. Then if $S = S(V)$ is noetherian, $\mathcal{R}_P$ is local for every height one prime P of S.*

Proof: By Lemma 2.3, $D = \mathcal{D}(\mathcal{R}, V)$ so since $\mathcal{D}(\mathcal{R}_P, V_P) = S_P\mathcal{D}(\mathcal{R}, V)$, $\mathcal{D}(\mathcal{R}_P, V_P)$ is projective. Also, $\mathcal{D}(\mathcal{R}_P, V_P) * \mathcal{R}_P = V_P$ so that $\mathcal{D}(V_P)$ is Morita equivalent to $\mathcal{D}(\mathcal{R}_P)$ and is simple.

Now, by Proposition 2.7, if $V_P \neq V_P^+$ then $\mathcal{D}(V_P) \subseteq \mathcal{D}(V_P^+)$ and $\mathcal{D}(V_P^+, V_P)$ is a proper ideal of $\mathcal{D}(V_P)$. But $\mathcal{D}(V_P)$ is simple and so $V_P = V_P^+$. This implies that $S_P = S_P^+$ also. The argument given in the proof of Proposition 1.6 now shows that $\mathcal{R}_P$ must be local. □

We want to use Lemma 1.4.12 to prove that S has injective normalisation. We therefore need some form of Lemma 1.4.13 in order to satisfy the conditions of Lemma 1.4.12 and the following result is aimed towards this end.

Lemma 3 *Let S be a dense, noetherian subring of $\mathcal{R}$ and let P be a height one prime ideal of S. Suppose that $\mathcal{R}_P$ is local with maximal ideal M and that W is an S_P-submodule of $\mathcal{R}_P$ containing $S_P + M$. Then $\mathcal{D}(W) \subseteq \mathcal{D}(\mathcal{R}_P)$.*

Proof: The proof is almost exactly the same as that given for Lemma 1.4.13. Since $S_P \subseteq W$, we can find a field $K = (t_1, \ldots, t_r)$ in W over which $\mathcal{R}_P/M$ is algebraic, and we may choose a regular parameter t_0 of M which lies inside W. We may then write any $\delta \in \mathcal{D}(W)$ as follows:

$$\delta = \sum a_{i_0,\ldots,i_r} \frac{\partial}{\partial t_0}^{i_0} \cdots \frac{\partial}{\partial t_r}^{i_r},$$

and just as in Lemma 1.4.13, we find that each $a_{i_0,\ldots,i_r}$ lies in $W \subseteq \mathcal{R}_P$. Hence $\delta \in \mathcal{D}(\mathcal{R}_P)$ as required. □

Let M be the maximal ideal of $\mathcal{R}_P$. We must show that the residue field of S_P coincides with that of $\mathcal{R}_P$. Let I be the largest ideal of $\mathcal{R}_P$ contained in V_P. Then since $\mathcal{R}_P/I$ is a complete local ring, [Matsumura A; Theorem 60] shows that $\mathcal{R}_P/I$ contains a copy of its own residue field, K say. If we show that V_P/I is a K-vector space then it will follow that S_P/I must contain K. Since $\mathcal{R}_P$ is regular local, we may choose a regular parameter t for M and $I = t^r\mathcal{R}_P$ for some $r \in \mathbf{N}$. Then we may write $\mathcal{R}_P/I$ as $K[t]/(t^r)$. For each $n \in \{1,\ldots,r-1\}$ we may define a k-linear map θ_n from V_P/I to K by expanding each $v \in V_P/I$ out as $v = v_0 + v_1t + \ldots + v_{r-1}t^{r-1}$ and setting $\theta_n(v) = v_n$. The next result shows that $\theta_n(V_P/I)$ equals either K or 0 for each n.

Proposition 4 *With θ_n as defined above, $\theta_n(V_P/I)$ is either K or 0 for each n.*

Proof: Fix $n \in \{1,\ldots,r-1\}$ and assume that $\theta_n(V_P/I) \neq 0$. Set $W = \partial^n/\partial t^n * V_P$ so that V_P and W are isomorphic (as S_P-modules). If we define $\phi : W \to K$ by $\phi(w) = w + M$ then it is clear that $\phi(W) = \theta(n)(V_P/I)$. Also, we may assume that $1 \in W$. This is because if $c + m \in W$ with $m \in M$ and $c \in K\backslash 0$ then

$W \cong c^{-1}W$ and $1 \in c^{-1}W$ (such a c exists since $\phi(W) \neq 0$). The upshot of all this is that (after possibly replacing W with $c^{-1}W$) we have $S = S(W)$, $S \subseteq W$ and $W = \mathcal{D}(\mathcal{R}_P, W) * \mathcal{R}_P$. This last fact holds because $V_P = \mathcal{D}(\mathcal{R}_P, V_P) * \mathcal{R}_P$ and $W = c^{-1}\partial^n/\partial t^n(V)$.

Now, Lemma 3 applies to the vector space $W + M$ to give us that $\mathcal{D}(W + M) \subseteq \mathcal{D}(\mathcal{R}_P)$. Therefore, if $W + M \neq \mathcal{R}_P$ we must have that $\mathcal{D}(W + M)$ is not simple (it contains the ideal $\mathcal{D}(\mathcal{R}_P, W + M)$. But we may deduce from the fact that $W = \mathcal{D}(\mathcal{R}_P, W) * \mathcal{R}_P$ that $W + M = \mathcal{D}(\mathcal{R}_P, W + M) * \mathcal{R}_P$. The usual argument now yields that $\mathcal{D}(W + M)$ is Morita equivalent to $\mathcal{D}(\mathcal{R}_P)$ and is simple. This contradiction tells us that $W + M$ must be equal to $\mathcal{R}_P$ or, in other words, $\theta_n(V_P/I) = \phi(W) = K$. $\square$

What Proposition 4 is telling us is that for each $n \in \{1, \ldots, r-1\}$, if $c \in K$ and $v \in V_P$ then there exists some $v' \in V_P$ with $(cv)_n = v'_n$ where $cv = (cv)_0 + \ldots + (cv)_{r-1}t^{r-1}$ modulo I. Unfortunately, this is not quite enough to show that V_P/I is a K-linear vector space since what we need to show is that given such a c and v then there exists such a v' that works for all $n \in \{1, \ldots, r-1\}$. It would seem that we need some further consequence of the fact that $V_P = \mathcal{D}(\mathcal{R}_P, V_P) * \mathcal{R}_P$ to finish off, but it is not clear what this might be. So let us just sum up what we have proved in this chapter in a theorem.

Theorem 5 *Let $\mathcal{R}$ be the coordinate ring of a smooth, two dimensional variety and let V be a primary decomposible subspace of $\mathcal{R}$. Then $V = \mathcal{D}(\mathcal{R}, V) * \mathcal{R}$ and $\mathcal{D}(\mathcal{R}, V)$ is a projective right ideal of $\mathcal{D}(\mathcal{R})$.*

Conversely, let I be a dense projective right ideal of $\mathcal{D}(\mathcal{R})$ with the property that $S(V)$ is noetherian where $V = I * \mathcal{R}$. Then $I = \mathcal{D}(\mathcal{R}, V)$ and $V = \bigcap_{htP=1} V_P$ where the intersection runs over all the height one prime ideals of S. Also, $\mathcal{R}_P$ is local for every height one prime ideal of S and $V_P + M = \mathcal{R}_P$ where M is the maximal ideal of $\mathcal{R}_P$ and P is any height one prime of S.

See the final chapter for a discussion of the problems with this theorem and some indications of how it could possibly be generalised.

Chapter 3

Differential Operators On Tensor Products

In this chapter we show that the results of Section 1.4 are not restricted to surfaces. That is, we produce a large class of examples of varieties of any dimension whose differential operator rings are Morita equivalent to the differential operator rings on their normalisations. The method of attack is to prove that differential operators behave well with respect to tensor products and use the properties of differential operators on curves and surfaces to build up differential operators on higher dimensional varieties in a natural way. This answers a question put forward in [Chamarie & Stafford] which asks if there exist varieties of high dimension whose differential operators rings are simple.

The technical result that we need is presented in Section One. That is, it is proved that differential operators commute with tensor products. Although this is a natural result and not too hard to prove, it surprisingly has not appeared in the literature except in a few special case (e.g. [Smith]). Section Two utilises the results of Section One to show that it is possible to build up examples of varieties with well-behaved differential operator by taking the tensor products of the coordinate rings of curves.

3.1 Tensor Products

Throughout this section $\mathcal{X}$ and $\mathcal{Y}$ will be varieties with coordinate rings $\mathcal{R}$ and $\mathcal{S}$ respectively. We will denote the fields of fractions of $\mathcal{R}$ and $\mathcal{S}$ by $Q(\mathcal{R})$ and $Q(\mathcal{S})$ respectively. We may define multiplication on the set $\mathcal{R}\otimes_k\mathcal{S}$ by $(r \otimes s)(r' \otimes s') = rr' \otimes ss'$ for $r, r' \in \mathcal{R}$ and $s, s' \in \mathcal{S}$. Then the ring $\mathcal{R}\otimes_k\mathcal{S}$ is the coordinate ring of the variety $\mathcal{X} \times_k \mathcal{Y}$. Similarly, we may make the set $\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S})$ into a ring. Our aim is to show that $\mathcal{D}(\mathcal{R}\otimes_k\mathcal{S}) \cong \mathcal{D}(\mathcal{R})\otimes_k\mathcal{D}(\mathcal{S})$. From now on, we shall drop the subscript k since we shall not be tensoring over anything else.

Notice that we have inclusions:

$$\begin{array}{rcllrcl} \mathcal{D}(Q(\mathcal{R})) & \hookrightarrow & \mathcal{D}(Q(\mathcal{R}))\otimes\mathcal{D}(Q(\mathcal{S})) & \text{and} & \mathcal{D}(Q(\mathcal{S})) & \hookrightarrow & \mathcal{D}(Q(\mathcal{R}))\otimes\mathcal{D}(Q(\mathcal{S})) \\ \text{given by } \theta & \mapsto & \theta \otimes 1 & \text{and} & \phi & \mapsto & 1 \otimes \phi \end{array}$$

for $\theta \in \mathcal{D}(Q(\mathcal{R}))$ and $\phi \in \mathcal{D}(Q(\mathcal{S}))$. The tensoring together of elements θ and ϕ from $\mathcal{D}(Q(\mathcal{R})$ and $\mathcal{D}(Q(\mathcal{S}))$ should then be thought of as the multiplication of $\theta \otimes 1$ and $1 \otimes \phi$.

By localising at the multplicatively closed set $\{r \otimes s \in \mathcal{R}\otimes\mathcal{S} \,|\, r \neq 0 \text{ and } s \neq 0\}$, we may identify $\mathcal{D}(\mathcal{R}\otimes\mathcal{S})$ with a subset of $\mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$. Also, since $\mathcal{D}(\mathcal{R}) \subseteq \mathcal{D}(Q(\mathcal{R}))$ and $\mathcal{D}(\mathcal{S}) \subseteq \mathcal{D}(Q(\mathcal{S}))$, $\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S})$ may be thought of as lying inside $\mathcal{D}(Q(\mathcal{R}))\otimes\mathcal{D}(Q(\mathcal{S}))$. The following lemma shows that $\mathcal{D}(Q(\mathcal{R}))\otimes\mathcal{D}(Q(\mathcal{S}))$ is the same as $\mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$ so that everything we do can be thought of as happening inside $\mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$.

Lemma 1 *Let $\mathcal{R}$ and $\mathcal{S}$ be as above. Then $\mathcal{D}(Q(\mathcal{R})\otimes\mathcal{D}(Q(\mathcal{S})) \cong \mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$.*

Proof: Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be transcendence bases for $Q(\mathcal{R})$ and $Q(\mathcal{S})$ over k. Then by Corollary 1.1.14, $\mathcal{D}(Q(\mathcal{R})) = Q(\mathcal{R})[\partial/\partial x_1, \dots, \partial/\partial x_m]$ and $\mathcal{D}(Q(\mathcal{S})) = Q(\mathcal{S})[\partial/\partial y_1, \dots, \partial/\partial y_n]$. Given a $\theta \in \mathcal{D}(Q(\mathcal{R}))$ we may extend θ to all of $Q(\mathcal{R})\otimes Q(\mathcal{S})$ by $\theta(r \otimes s) = \theta(r) \otimes s$, and similarly for elements of $\mathcal{D}(Q(\mathcal{S}))$. Thus:

$$\mathcal{D}(Q(\mathcal{R}))\otimes\mathcal{D}(Q(\mathcal{S})) = Q(\mathcal{R})\otimes Q(\mathcal{S})[\partial/\partial x_1, \dots, \partial/\partial x_m, \partial/\partial y_1, \dots, \partial/\partial y_n].$$

Now, $Q(\mathcal{R})\otimes Q(\mathcal{S})$ is not a field, but its field of fractions $Q(Q(\mathcal{R})\otimes Q(\mathcal{S}))$ is a field with transcendence basis $x_1, \dots, x_m, y_1, \dots, y_n$. So

$$\mathcal{D}(Q(Q(\mathcal{R})\otimes Q(\mathcal{S}))) = Q(Q(\mathcal{R})\otimes Q(\mathcal{S}))[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}].$$

The derivations $\partial/\partial x_i$ and $\partial/\partial y_j$ restrict to derivations on $Q(\mathcal{R})\otimes Q(\mathcal{S})$ and so

$$Q(\mathcal{R})\otimes Q(\mathcal{S})[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}] \subseteq \mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S})).$$

Conversely, any differential operator on $Q(\mathcal{R})\otimes Q(\mathcal{S})$ is a sum of products of the $\partial/\partial x_i, \partial/\partial y_j$'s with coefficients in $Q(Q(\mathcal{R})\otimes Q(\mathcal{S}))$. But it is easy to see that these coefficients must actually lie in $Q(\mathcal{R})\otimes Q(\mathcal{S})$.

Hence $\mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S})) = Q(\mathcal{R})\otimes Q(\mathcal{S})[\partial/\partial x_i, \partial/\partial y_j]$ and the result holds. □

Next we prove the result for $\mathcal{R}\otimes Q(\mathcal{S})$.

Proposition 2 *Let $\mathcal{R}$ and $\mathcal{S}$ be as above. Then:*

$$\mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S})) = \mathcal{D}(\mathcal{R})\otimes\mathcal{D}(Q(\mathcal{S})).$$

Proof: By Lemma 1, both $\mathcal{D}(Q(\mathcal{R}))$ and $\mathcal{D}(Q(\mathcal{S}))$ lie inside $\mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$ and elements of $\mathcal{D}(Q(\mathcal{S}))$ and $\mathcal{D}(\mathcal{R})$ map $\mathcal{R}$ into $\mathcal{R}$. Therefore $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}(Q(\mathcal{S}))$ both lie in $\mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S}))$. Hence $\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(Q(\mathcal{S}))$, which is the set of products of elements from $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}(Q(\mathcal{S}))$, lies inside $\mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S}))$.

So it remains to prove the reverse inclusion. Choose a basis $\{\mathbf{e}_j | j \in B\}$ for $Q(\mathcal{S})$ as a vector space over k. By Corollary 1.1.14, if $y_1, \ldots, y_n$ is a transcendence basis for $Q(\mathcal{S})$ over k, every element ϕ of $\mathcal{D}(Q(\mathcal{S}))$ can be written as:

$$\phi = \sum_{\mathbf{i}\in\mathbf{N}^n}\sum_{j\in B} c_{\mathbf{i},j}\mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}},$$

where $\mathbf{i}$ is an n-tuple of positive integers $(i_1, \ldots, i_n)$, each $c_{\mathbf{i},j} \in k$ (with only finitely many non-zero), and

$$\left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}} = \frac{\partial}{\partial y_1}^{i_1} \cdots \frac{\partial}{\partial y_n}^{i_n}.$$

By localising the non-zero elements of $\mathcal{R}$, we have the following inclusion:

$$\mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S})) \subseteq \mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S})).$$

Let $\Delta \in \mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S})) \subseteq \mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$. Then by Lemma 1 we may write Δ as follows:

$$\Delta = \sum_{\mathbf{i}\in\mathbf{N}^n}\sum_{j\in B} \theta_{\mathbf{i},j}\mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}},$$

where each $\theta_{\mathbf{i},j}$ belongs to $\mathcal{D}(Q(\mathcal{R}))$ and only finitely many are non-zero. Lexicographically order $\mathbf{N}^n$ and choose $\mathbf{k} \in \mathbf{N}^n$ to be maximal with the property that $\theta_{\mathbf{k},j}$

is non-zero for some j. Let δ equal $\sum_j \theta_{\mathbf{k},j} \mathbf{e}_j \frac{\partial}{\partial \mathbf{y}}^{\mathbf{k}}$. Since $y_1 \in Q(\mathcal{S})$, we have that $[\Delta, y_1]$ lies in $\mathcal{D}(\mathcal{R} \otimes Q(\mathcal{S}))$. But for any natural number r, $\frac{\partial}{\partial y_1}^r y_1 - y_1 \frac{\partial}{\partial y_1}^r = r \frac{\partial}{\partial y_1}^{n-1}$, so $[\delta, y_1] = k_1 \sum_j \theta_{\mathbf{k},j} \mathbf{e}_j \frac{\partial}{\partial \mathbf{y}}^{\mathbf{k}'}$, where $\mathbf{k}'$ is the same as $\mathbf{k}$ apart from the the fact that $k'_1 = k_1 - 1$. Repeating this process a further k'_1 times, and also repeating for each k_i, we arrive at the following:

$$[\dots[[\Delta, y_1], y_1] \dots y_n] = N \sum_{j \in B} \theta_{\mathbf{k},j} \mathbf{e}_j \in \mathcal{D}(\mathcal{R} \otimes Q(\mathcal{S})),$$

where N is some positive natural number. It is now easy to see that each $\theta_{\mathbf{k},j}$ actually lies inside $\mathcal{D}(\mathcal{R})$. Thus for each j, $\theta_{\mathbf{k},j} \mathbf{e}_j \in \mathcal{D}(\mathcal{R}) \otimes \mathcal{D}(Q(\mathcal{S}))$, and the operator $\Delta - \sum_j \theta_{\mathbf{k},j} \mathbf{e}_j$ still lies in $\mathcal{D}(\mathcal{R} \otimes Q(\mathcal{S}))$. This resulting differential operator now has degree strictly less than Δ in $\left[\frac{\partial}{\partial \mathbf{y}}\right]$, so by induction lies in $\mathcal{D}(\mathcal{R}) \otimes \mathcal{D}(Q(\mathcal{S}))$. Hence $\Delta \in \mathcal{D}(\mathcal{R}) \otimes \mathcal{D}(Q(\mathcal{S}))$ also. □

We can now proceed with the general case. The proof proceeds in essentially the same way as in Proposition 2, but a little extra work is needed at several of the steps.

Theorem 3 *Let $\mathcal{R}$ and $\mathcal{S}$ be as above, then:* $\mathcal{D}(\mathcal{R} \otimes \mathcal{S}) = \mathcal{D}(\mathcal{R}) \otimes \mathcal{D}(\mathcal{S})$.

Proof: Fix bases $\{\mathbf{d}_{j'} | j' \in A\}$ and $\{\mathbf{e}_j | j \in B\}$ of $Q(\mathcal{R})$ and $Q(\mathcal{S})$ as vector spaces over k respectively. By Corollary 1.1.14, we can choose elements $x_1, \dots, x_m$ of $\mathcal{R}$ so that $\mathcal{D}(Q(\mathcal{R})) = Q(\mathcal{R})[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}]$. Similarly, we can choose elements $y_1, \dots, y_n$ in $\mathcal{S}$ so that $\mathcal{D}(Q(\mathcal{S})) = Q(\mathcal{S})[\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}]$. Then as before $\mathbf{i}$ will denote an n-tuple of positive integers and we will use $\mathbf{i}'$ to denote an m-tuple of positive integers so that $\left[\frac{\partial}{\partial \mathbf{x}}\right]^{\mathbf{i}'}$ has an analogous meaning to that which $\left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}}$ had in Proposition 2.

Let $\Delta \in \mathcal{D}(\mathcal{R}\otimes\mathcal{S})$. Since $\mathcal{D}(\mathcal{R}\otimes\mathcal{S}) \subseteq \mathcal{D}(Q(\mathcal{R})\otimes Q(\mathcal{S}))$, we can write Δ thus:

$$\Delta = \sum_{\mathbf{i}\in\mathbf{N}^n} \sum_j \theta_{\mathbf{i},j}\mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}},$$

where each $\theta_{\mathbf{i},j} \in \mathcal{D}(Q(\mathcal{R}))$. Also, $\mathcal{D}(\mathcal{R}\otimes\mathcal{S}) \subseteq \mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S}))$ and by Proposition 2, $\mathcal{D}(\mathcal{R}\otimes Q(\mathcal{S})) = \mathcal{D}(\mathcal{R})\otimes\mathcal{D}(Q(\mathcal{S}))$. From this it can be seen that the $\theta_{\mathbf{i},j}$'s actually belong to $\mathcal{D}(\mathcal{R})$.

This is precisely the conclusion we arrived at in the Proposition 2, but this time we cannot just subtract a $\theta_{\mathbf{i},j}\mathbf{e}_j\left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}}$ from Δ as the $\mathbf{e}_j\left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}}$ part might not lie in $\mathcal{D}(\mathcal{S})$. What we shall do is prove the existence of a differential operator Δ' in $\mathcal{D}(\mathcal{S})$ which, when written as a sum as follows:

$$\Delta' = \sum_{\mathbf{i}} \sum_j \lambda_{\mathbf{i},j}\mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}},$$

with each $\lambda_{\mathbf{i},j} \in k$, has $\lambda_{\mathbf{i},j}$ equal to zero if and only if the corresponding $\theta_{\mathbf{i},j}$ is zero. Then we may complete the proof by subtracting $(1/\lambda_{\mathbf{a},b})\theta_{\mathbf{a},b}\Delta'$ from Δ for some pair $(\mathbf{a}, b)$ with $\theta_{\mathbf{a},b}$ non-zero. This will give us a differential operator in $\mathcal{D}(\mathcal{R}\otimes\mathcal{S})$ with fewer terms than Δ and, after repeating this process a finite number of times, we will eventually arrive at a situation where, by subtracting elements of $\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S})$ from Δ, we get zero. Hence Δ must actually have been inside $\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S})$ to start with, and the proof will be finished.

Now, as Δ lies in $\mathcal{D}(Q(\mathcal{R})\otimes\mathcal{S})$ which equals $\mathcal{D}(Q(\mathcal{R}))\otimes\mathcal{D}(\mathcal{S})$ by Proposition 2, we may write it as follows:

$$\Delta = \sum_{k=1}^{r} \alpha_k \otimes \beta_k \quad \text{for } \alpha_k \in \mathcal{D}(Q(\mathcal{R})) \text{ and } \beta_k \in \mathcal{D}(\mathcal{S}).$$

We must now expand each β_k out as follows:

$$\beta_k = \sum_{\mathbf{i}\in\mathbf{N}^n} \sum_j \lambda_{\mathbf{i},j,k} \mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}} \text{ for some } \lambda_{\mathbf{i},j,k} \in k.$$

Substituting this into the formula for Δ we have that:

$$\Delta = \sum_{\mathbf{i}} \sum_{j,k} \lambda_{\mathbf{i},j,k} \alpha_k \otimes \mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}}.$$

Now, the following relation must hold:

$$(\star) \quad \sum_{k=1}^{r} \lambda_{\mathbf{i},j,k} \alpha_k = \theta_{\mathbf{i},j} \text{ for each } \mathbf{i}, j.$$

Choose a pair $(\mathbf{a}, b)$ such that $\theta_{\mathbf{a},b}$ is non-zero and expand $\theta_{\mathbf{a},b}$ out in terms of the $\mathbf{d}_{j'}$'s and $\left[\frac{\partial}{\partial \mathbf{x}}\right]^{\mathbf{i}'}$'s. Pick a term $\mu \mathbf{d}_{j'} \left[\frac{\partial}{\partial \mathbf{x}}\right]^{\mathbf{i}'}$ in this expansion with μ non-zero and let μ_k be the coefficient of $\mathbf{d}_{j'} \left[\frac{\partial}{\partial \mathbf{x}}\right]^{\mathbf{i}'}$ in each α_k. Define Δ' by:

$$\Delta' = (1/\mu) \sum_{\mathbf{i}} \sum_{j,k} \mu_k \lambda_{\mathbf{i},j,k} \mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}}.$$

Then Δ' actually lies in $\mathcal{D}(\mathcal{S})$ since:

$$\Delta' = (1/\mu) \sum_k \mu_k \sum_{\mathbf{i},j} \lambda_{\mathbf{i},j,k} \mathbf{e}_j \left[\frac{\partial}{\partial \mathbf{y}}\right]^{\mathbf{i}} = (1/\mu) \sum_k \mu_k \beta_k.$$

By the relation $(\star)$, Δ' has no '$(\mathbf{i}, j)$' term if $\theta_{\mathbf{i},j}$ equals zero and, also by $(\star)$, the '$(\mathbf{a}, b)$' term is 1. This completes the proof. $\square$

We may easily extend Theorem 3 to cover the cases of when $\mathcal{R}$ and $\mathcal{S}$ are localisations of coordinate rings of varieties.

Corollary 4 *Let $\mathcal{R}$ and $\mathcal{S}$ be as in Theorem 3, and let* $\mathbf{c}$ *and* $\mathbf{d}$ *be multiplicatively closed sets in $\mathcal{R}$ and $\mathcal{S}$ respectively. Then $\mathcal{D}(\mathcal{R}_{\mathbf{c}} \otimes \mathcal{S}_{\mathbf{d}}) = \mathcal{D}(\mathcal{R}_{\mathbf{c}}) \otimes \mathcal{D}(\mathcal{S}_{\mathbf{d}})$.*

Proof: If we consider $\mathcal{R}$ and $\mathcal{S}$ as subsets of $\mathcal{R}\otimes\mathcal{S}$, we have that $\mathcal{D}(\mathcal{R}_{\mathbf{c}}\otimes\mathcal{S}_{\mathbf{d}}) = \mathcal{D}(\mathcal{R}\otimes\mathcal{S})_{\mathbf{c},\mathbf{d}}$ and $\mathcal{D}(\mathcal{R}_{\mathbf{c}})\otimes\mathcal{D}(\mathcal{S}_{\mathbf{d}}) = (\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S}))_{\mathbf{c},\mathbf{d}}$. But by Theorem 3, we know that $\mathcal{D}(\mathcal{R}\otimes\mathcal{S})_{\mathbf{c},\mathbf{d}} = (\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S}))_{\mathbf{c},\mathbf{d}}$. Putting these facts together yields that

$$\mathcal{D}(\mathcal{R}_{\mathbf{c}}\otimes\mathcal{S}_{\mathbf{d}}) = (\mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S}))_{\mathbf{c},\mathbf{d}}$$

as required. □

Also, Theorem 3 and Corollary 4 may be extended to cover the case of when we have more than two varieties under consideration.

Corollary 5 *Let $\mathcal{R}_i$ for $i = 1, \ldots, n$ be localisations of the coordinate rings of varieties $\mathcal{X}_i$. Then $\mathcal{D}(\otimes_{i=1}^{n}\mathcal{R}_i) = \otimes_{i=1}^{n}\mathcal{D}(\mathcal{R}_i)$.*

Proof: Use induction. □

3.2 Differential Operator Rings On Products Of Varieties

In this section we use the conclusion of Theorem 1.3 to build up varieties whose differential operator rings are Morita equivalent to the differential operator rings on their normalisations, thus extending the results of Sections 1.3 and 1.4. Recall that for a curve $\mathcal{X}$, every right ideal of $\mathcal{D}(\widetilde{\mathcal{X}})$ is projective, and if the normalisation map $\pi : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ is injective then $\mathcal{D}(\mathcal{X}) = End_{\mathcal{D}(\widetilde{\mathcal{X}})}\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ so that $\mathcal{D}(\mathcal{X})$ and $\mathcal{D}(\widetilde{\mathcal{X}})$ are Morita equivalent. Also, if $\mathcal{X}$ is a higher dimensional S_2 variety with smooth, injective normalisation then Lemma 1.4.6 shows that $\mathcal{D}(\mathcal{X}) = End_{\mathcal{D}(\tilde{\mathcal{X}})}\mathcal{D}(\tilde{\mathcal{X}}, \mathcal{X})$ from the one dimensional case and it is the projectivity of $\mathcal{D}(\tilde{\mathcal{X}}, \mathcal{X})$ that is in question. We prove that that if $\mathcal{X}$ is a product of lower dimensional varieties then $\mathcal{D}(\tilde{\mathcal{X}}, \mathcal{X})$ splits up in a natural way and we are able to use this fact to show that $\mathcal{D}(\tilde{\mathcal{X}}, \mathcal{X})$ is projective.

Throughout this section, if $\mathcal{R}$ is the coordinate ring of a variety then we will denote the field of fractions of $\mathcal{R}$ by $Q(\mathcal{R})$. As usual, $\tilde{\mathcal{R}}$ will be the normalisation (i.e. integral closure) of $\mathcal{R}$ in $Q(\mathcal{R})$. Then if $\mathcal{R}$ and $\mathcal{S}$ are the coordinate rings of two varieties, it is easy to check that $Q(\mathcal{R}\otimes\mathcal{S}) = Q(\mathcal{R})\otimes Q(\mathcal{S})$ and $(\widetilde{\mathcal{R}\otimes\mathcal{S}}) = \tilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}$.

Recall that by Theorem 1.3, $\mathcal{D}(\mathcal{R}\otimes\mathcal{S}) = \mathcal{D}(\mathcal{R})\otimes\mathcal{D}(\mathcal{S})$. We begin by proving a similar result to Theorem 1.3. As the proof is so similar to that of Theorem 1.3, we shall leave the reader to check some of the details.

Proposition 1 *Suppose that $\mathcal{R}$ and $\mathcal{S}$ are the coordinate rings of two varieties and let $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})$ and $\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S})$ be the two modules of differential operators between $\tilde{\mathcal{R}}$ and $\mathcal{R}$, and $\tilde{\mathcal{S}}$ and $\mathcal{S}$ respectively. Then $\mathcal{D}(\tilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S})$ and $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S})$ are isomorphic as $\mathcal{D}(\mathcal{R}\otimes\mathcal{S})$-modules.*

Proof: As mentioned above, the proof of this is almost exactly the same as the proof of Theorem 3. One starts by proving that:

$$\mathcal{D}(\tilde{\mathcal{R}}\otimes Q(\mathcal{S}), \mathcal{R}\otimes Q(\mathcal{S})) = \mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})\otimes\mathcal{D}(Q(\mathcal{S})).$$

Then, given an element δ of $\mathcal{D}(\tilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S}) \subseteq \mathcal{D}(\tilde{\mathcal{R}}\otimes Q(\mathcal{S}), \mathcal{R}\otimes Q(\mathcal{S}))$, we may write it as a sum of products of elements taken from $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})$ and $\mathcal{D}(Q(\mathcal{S}))$. Then one may show that the elements taken from $\mathcal{D}(Q(\mathcal{S}))$ actually add up to lie inside $\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S})$. Hence δ lies inside $\mathcal{D}(\tilde{\mathcal{R}}, \mathcal{R})\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S})$. □

Of course, an almost trivial induction argument extends Proposition 1 to the general case as in Corollary 1.5.

Corollary 2 *Suppose that $\mathcal{R}_i$ for $i = 1, \ldots, n$ are the coordinate rings of varieties $\mathcal{X}_i$. Then:*

$$\mathcal{D}(\otimes_{i=1}^{n}\widetilde{\mathcal{R}}_i, \otimes_{i=1}^{n}\mathcal{R}_i) \cong \otimes_{i=1}^{n}\mathcal{D}(\widetilde{\mathcal{R}}_i, \mathcal{R}_i)$$

as $\mathcal{D}(\otimes_{i=1}^{n}\widetilde{\mathcal{R}}_i)$-modules.

Proof: Clear. □

Now that we have shown how to split up $\mathcal{D}(\widetilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S})$, we need to be able to use the properties of $\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$ and $\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S})$ in order to study $\mathcal{D}(\widetilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S})$.

Lemma 3 *If $\mathcal{R}_i$ are coordinate rings of varieties such that $\mathcal{D}(\widetilde{\mathcal{R}}_i, \mathcal{R}_i)$ are projective $\mathcal{D}(\widetilde{\mathcal{R}}_i)$-modules for $i = 1, \ldots, n$, then $\mathcal{D}(\otimes_{i=1}^{n}\widetilde{\mathcal{R}}_i, \otimes_{i=1}^{n}\mathcal{R}_i)$ is a projective $\mathcal{D}(\otimes_{i=1}^{n}\widetilde{\mathcal{R}}_i)$-module.*

Proof: We prove the case of when $n = 2$ and notice that induction proves the general case. Write $\mathcal{R}$ and $\mathcal{S}$ for $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. By Theorem 1.3, $\mathcal{D}(\widetilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S})$ is also a right $\mathcal{D}(\widetilde{\mathcal{R}})\otimes\mathcal{D}(\tilde{\mathcal{S}})$-module. Choose a right $\mathcal{D}(\widetilde{\mathcal{R}})$-module P and a right $\mathcal{D}(\tilde{\mathcal{S}})$-module Q such that:

$$\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}) \bigoplus P = \mathcal{D}(\widetilde{\mathcal{R}})^r \text{ and } \mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) \bigoplus Q = \mathcal{D}(\tilde{\mathcal{S}})^s$$

for some integers r and $s \in \mathbf{N}$, where $\mathcal{D}(\widetilde{\mathcal{R}})^r$ means the free $\mathcal{D}(\widetilde{\mathcal{R}})$-module of rank r. Using Proposition 1 we find that:

$$\begin{aligned}
\mathcal{D}(\widetilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S}) \oplus P\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) &= \mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) \oplus P\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) \\
&= (\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R}) \oplus P)\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) \\
&= \mathcal{D}(\widetilde{\mathcal{R}})^r\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}). \\
\text{Also} : \mathcal{D}(\widetilde{\mathcal{R}})^r\otimes\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) \oplus \mathcal{D}(\widetilde{\mathcal{R}})^r\otimes Q &= \mathcal{D}(\widetilde{\mathcal{R}})^r\otimes(\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S}) \oplus Q) \\
&= \mathcal{D}(\widetilde{\mathcal{R}})^r\otimes\mathcal{D}(\tilde{\mathcal{S}})^s.
\end{aligned}$$

It is easy to see that $\mathcal{D}(\widetilde{\mathcal{R}})^r\otimes\mathcal{D}(\tilde{\mathcal{S}})^s$ is a free $\mathcal{D}(\widetilde{\mathcal{R}})\otimes\mathcal{D}(\tilde{\mathcal{S}})$-module of rank rs. Hence $\mathcal{D}(\widetilde{\mathcal{R}}\otimes\tilde{\mathcal{S}}, \mathcal{R}\otimes\mathcal{S})$ is a direct summand of a free module and is projective. □

In particular, Lemma 3 tells us that if the $\mathcal{R}_i$'s are either curves or S_2 surfaces, with smooth, injective normalisations then we may use the results of Chapter One to get that $\mathcal{D}(\otimes_{i=1}^n\widetilde{\mathcal{R}}_i, \otimes_{i=1}^n\mathcal{R}_i)$ is a projective $\mathcal{D}(\otimes_{i=1}^n\widetilde{\mathcal{R}}_i)$-module. It is now easy to finish the argument off to find that $\mathcal{D}(\otimes_{i=1}^n\mathcal{R}_i)$ is Morita equivalent to $\mathcal{D}(\otimes_{i=1}^n\widetilde{\mathcal{R}}_i)$.

Theorem 4 *Let $\mathcal{R}_i$ for $i = 1, \ldots, n$ be the coordinate rings of a set of S_2 varieties with smooth, injective normalisations and suppose that each $\mathcal{D}(\widetilde{\mathcal{R}}_i, \mathcal{R}_i)$ is a projective $\mathcal{D}(\widetilde{\mathcal{R}}_i)$-module. Then $\mathcal{D}(\otimes_{i=1}^n\mathcal{R}_i)$ is a simple noetherian ring with Krull and global dimensions equal to the sum of the dimensions of the $\mathcal{X}_i$'s.*

Proof: Let $\mathcal{S} = \otimes_{i=1}^n\mathcal{R}_i$. Then $\tilde{\mathcal{S}} = \otimes_{i=1}^n\widetilde{\mathcal{R}}_i$ and by Lemma 3, $\mathcal{D}(\tilde{\mathcal{S}}, \mathcal{S})$ is a projective $\mathcal{D}(\tilde{\mathcal{S}})$-module. Since each $\mathcal{R}_i$ is S_2, by Lemma 1.4.6, $\mathcal{D}(\mathcal{R}_i) = End_{\mathcal{D}(\widetilde{\mathcal{R}}_i)}\mathcal{D}(\widetilde{\mathcal{R}}_i, \mathcal{R}_i)$ for each i. Therefore, since $\mathcal{D}(\widetilde{\mathcal{R}}_i, \mathcal{R}_i)$ is projective, $\mathcal{D}(\mathcal{R}_i)$ is Morita equivalent to $\mathcal{D}(\widetilde{\mathcal{R}}_i)$ for every i. Now, by Theorem 1.3, $\mathcal{D}(\mathcal{S}) = \otimes_{i=1}^n\mathcal{D}(\mathcal{R}_i)$ and so, being a tensor product of simple rings, must itself be simple. Therefore it is a maximal order and must be equal to $End_{\mathcal{D}(\bar{S})}\mathcal{D}(\bar{S},S)$. Finally, as $\mathcal{D}(\bar{S})$ is a simple ring, $\mathcal{D}(\bar{S},S)$ is a generator and hence a progenerator so that $\mathcal{D}(S)$ is Morita equivalent to $\mathcal{D}(\bar{S})$. All the assertions of the statement then follow from the Morita invariance of the required properties.□

Theorem 4 equips us with a large class of varieties whose differential operator rings are Morita equivalent to the differential operator rings on their normalisations. In fact, it is difficult to find S_2 varieties with smooth, injective normalisations which are not tensor products of lower dimensional varieties. It is not true however that every variety with these properties is a product, and this question is treated in the next chapter where it is shown how to write down examples of S_2 varieties with smooth, injective normalisations which are not products of lower dimensional varieties. By chance, the method of construction of these varieties also gives a handhold on the differential operator rings and we are able to give examples of varieties whose differential operator rings are well-behaved and varieties whose differential operator rings are not.

Chapter 4

Higher Dimensional Varieties

In this chapter we produce a class of S_2 varieties with smooth, injective normalisation which includes some varieties which are not products of lower dimensional varieties. Therefore the differential operator rings on these varieties are not subject to the results proved in the previous chapter. We are however able to prove by a different method that the differential operator rings on these varieties *are* Morita equivalent to the rings of differential operators on their normalisations, but only when we insist on imposing a certain restriction on the variety in question. Crucially, when this restriction is dropped the argument fails and we are able to present an example of a three dimensional, S_2 variety with smooth, injective normalisation whose differential operator ring is *not* Morita equivalent to the differential operator ring on the normalisation. This is a counter-example to the conjectures posed in the papers [Hart & Smith] and [Chamarie & Stafford], although to be fair, [Chamarie & Stafford] suggests that extra conditions might have to be imposed on the variety over and above just the S_2 condition.

In Section One we introduce the class of varieties which is of interest to us and in Section Two we prove that the rings of differential operators on these varieties behave as expected when we insist on certain restrictions. That is, they are Morita equivalent to the differential operator rings on the normalisations of the varieties. We then give an example to show that when these restrictions are dropped the differential operator rings behave badly.

4.1 Constructing Varieties

The aim of this section is to find a method of building S_2 varieties with smooth, injective normalisation whilst at the same being able to detect if the variety is a product of lower dimensional varieties or not. The way we do this is to focus our attention on the 'singular locus' of the variety in question. It turns out that if a variety is a product then its singular locus must take a particular form. So what we do is to find a way of building varieties with particular singular loci and then those varieties whose singular loci are not of the special form cannot be products.

Definition 1 *Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be the normalisation map of a variety $\mathcal{X}$. Define the* singular locus *of $\mathcal{X}$ in $\widetilde{\mathcal{X}}$ to be the set of points $x \in \widetilde{\mathcal{X}}$ such that $\pi(x)$ is a singular point of $\mathcal{X}$.*

We shall usually just write 'the singular locus' in place of the 'the singular locus of $\mathcal{X}$ in $\widetilde{\mathcal{X}}$'.

Now, let $\mathcal{X}$ be an S_2 variety with smooth, injective normalisation and let $\mathcal{R}$ be its coordinate ring. We need to impose certain conditions on $\mathcal{X}$ in order to be able to detect if $\mathcal{X}$ is a product of lower dimensional varieties or not. What we do is to insist that the singular locus of $\mathcal{X}$ is precisely determined by a height one prime ideal of $\overline{\mathcal{R}}$, and also that the coordinate ring of $\mathcal{X}$ only depends on what happens 'in codimension one'. That is, if we know the structure of the singular locus, we know everything about $\mathcal{X}$. It turns out that the conditions we impose will also allow us to calculate the properties of the differential operator ring on $\mathcal{X}$.

In more detail, suppose that the singular locus of $\mathcal{X}$ is precisely the set of points in $\overline{\mathcal{X}}$ defined by a height one prime ideal P of $\overline{\mathcal{R}}$ and that $\pi : \overline{\mathcal{X}} \to \mathcal{X}$ has ramification index two along the singular locus. In other words, the singular locus Γ is a prime divisor of $\overline{\mathcal{X}}$, and $\mathcal{X}$ is 'pinched' along Γ. This is equivalent to P^2 being the conductor I of $\mathcal{R}$ in $\overline{\mathcal{R}}$. Moreover, suppose that $(P \bigcap \mathcal{R}) = P^2$. This ensures that $\mathcal{R}$ equals P^2 plus the coordinate ring of a codimension one subvariety of $\mathcal{X}$. The ring $\overline{\mathcal{R}}_P$ is a regular local ring with maximal ideal M say, and $\overline{\mathcal{R}}_P/M^2$ is a complete local ring. Therefore, by [Matsumura A; Theorem 60], $\overline{\mathcal{R}}_P/M^2$ contains a copy of its own residue field, K. Also, the fact that $\overline{\mathcal{R}}_P$ is a regular local ring of Krull dimension one implies that M is a principal ideal with generator $t \in \overline{\mathcal{R}}_P$. Therefore we may write $\overline{\mathcal{R}}_P/M^2$ as $K[t]/(t^2)$.

Write D for the image of the singular locus Γ in $\mathcal{X}$ under π. Intuitively we may think of π as mapping two copies of Γ onto D. Let $C = 2\Gamma$ be the variety with the same underlying topological space as Γ but with coordinate ring $\mathcal{O}_C = \overline{\mathcal{R}}/P^2 \subseteq K[t]/(t^2)$. Then C is a non-reduced scheme and the corresponding reduced variety is Γ. As mentioned above, the behaviour of $\mathcal{X}$ is entirely determined by the behaviour of D. Write $\mathcal{O}_\Gamma$ and $\mathcal{O}_D$ for the coordinate rings of Γ and D respectively. The map π

restricted to Γ induces an inclusion of $\mathcal{O}_D$ into $\mathcal{O}_C$ and it is this inclusion which holds all of the information about what is going on.

Example: Let $\mathcal{X}$ be the variety with coordinate ring $\mathcal{R} = k[x, y^2, y^3]$ so that $\mathcal{X}$ is the product of a cusp with a line. Then $\widetilde{\mathcal{R}} = k[x, y]$ and, with the above notation, Γ is the subset of $\widetilde{\mathcal{X}}$ corresponding to the prime ideal $P = (y)$ of $\widetilde{\mathcal{R}}$. Γ has coordinate ring $\mathcal{O}_\Gamma = \widetilde{\mathcal{R}}/P = k[x, y]/(y) \cong k[x]$ and hence must be isomorphic to the line. On the other hand, $C = 2\Gamma$ has coordinate ring $\mathcal{O}_C = \widetilde{\mathcal{R}}/P^2 = k[x, y]/(y^2) = k[x][y]/(y^2)$. Thus C is a double line.

We shall show that there are at least two ways of constructing varieties of the type described above, and both methods rely on using differential operators. One would hope therefore that the differential operator rings on such varieties might be well-behaved and indeed, this is what we prove in the next section. The next result locates the position of $\mathcal{O}_D$ inside $\mathcal{O}_C$ by means of a derivation on K.

Proposition 2 *Let $\mathcal{X}$ be as described before the above example. Then there exists a (rational) derivation δ from $\mathcal{O}_C = \widetilde{\mathcal{R}}/P^2$ to K such that $\mathcal{O}_D = Ker\ \delta$.*

Proof: We define δ as follows: let $c \in \mathcal{O}_C$ and write $c = k + k't \in K[t]/(t^2)$. Set S to be the set $\mathcal{R} \backslash (P \bigcap \mathcal{R})$ so that $K = \widetilde{\mathcal{R}}_S / P\widetilde{\mathcal{R}}_S$. Since $\mathcal{X}$ has injective normalisation, the residue field of $\mathcal{R}_S$ is also K or in other words, $\mathcal{R}_S + P\widetilde{\mathcal{R}}_S = \widetilde{\mathcal{R}}_S$. Regarding S as a subset of $\mathcal{O}_D$, we may rewrite this as $(\mathcal{O}_D)_S + tK = K[t]/(t^2)$. Therefore we may find an element of $(\mathcal{O}_D)_S$ of the form $k + xt$ for some $x \in K$.

We have a natural projection of $\mathcal{O}_D$-modules from $\mathcal{O}_C$ to $\mathcal{O}_C/\mathcal{O}_D$ and we may localise the $\mathcal{O}_D$-module $\mathcal{O}_C/\mathcal{O}_D$ at S to get $(\mathcal{O}_C/\mathcal{O}_D)_S$. So we have a map from $\mathcal{O}_C$ into $(\mathcal{O}_C/\mathcal{O}_D)_S$ given by $c \mapsto c+(\mathcal{O}_D)_S$. By the previous paragraph, if $c = k + k't$, we can find an element $k + xt \in (\mathcal{O}_D)_S$ so that c is congruent to $(k' - x)t$ modulo $(\mathcal{O}_D)_S$. Set $\delta(c) = k' - x \in K$. If there is another element of $(\mathcal{O}_D)_S$ of the form $k + x't$ then $(x - x')t$ must lie in $(\mathcal{O}_D)_S$. But $\mathcal{O}_D \bigcap Kt = 0$ and so $x - x'$ must equal 0. Hence δ is well-defined.

Let $d = l + l't \in K[t]/(t^2)$ and suppose that $l + yt \in (\mathcal{O}_D)_S$ so that $\delta(d) = l' - y$. Then $cd = kl + (lk' + kl')t$. But $(k + xt)(l + yt) = kl + (ky + lx)t \in (\mathcal{O}_D)_S$ and $\delta(cd)$ must be $l(k' - x) + k(l' - y)$. But $\delta(c)d + \delta(d)c = (k' - x)l + (l' - y)k$. Hence $\delta(cd) = \delta(c)d + \delta(d)c$ and δ is a derivation. □

Corollary 3 *If $\mathcal{X}$ is as above then there exists a derivation Δ from $\overline{\mathcal{R}}$ to K such that $\mathcal{R} = Ker\ \Delta$.*

Proof: Let δ be as in Proposition 2 and let p be the natural projection from $\overline{\mathcal{R}}$ to $\overline{\mathcal{R}}/P^2$. Set Δ to be the compostion of maps $\delta \circ p$. Let $\mathbf{p}$ denote $P \bigcap \mathcal{R}$. Then by the assumption on $\mathcal{X}$, we have that $\mathbf{p} = P^2$. Therefore $\mathcal{R} = P^2 + \mathcal{O}_D$. But $P^2 = Ker\ p$ and $\mathcal{O}_D = Ker\ \delta$. Hence $\mathcal{R} = Ker\ \Delta$. □

Corollary 3 will be very important in the next section, but it is not clear which derivations from $\overline{\mathcal{R}}$ to K give rise to a noetherian subring of $\overline{\mathcal{R}}$. The solution is to find a way of determining more precisely the structure of Δ. So let us examine in more detail how $\mathcal{O}_D$ lies inside $\mathcal{O}_C$.

The map π, when restricted to Γ, may be thought of as taking two copies of Γ (i.e. C) into D. In other words, functions in the coordinate ring $\mathcal{O}_D$ of D give rise to functions in $\mathcal{O}_C$. Therefore, the induced ring homomorphism $\pi^* : K \hookrightarrow K[t]/(t^2)$ takes an element $f \in K$ to $f + t.\delta(f) \in K[t]/(t^2)$, where $\delta(f) \in K$. Now, maps of this type are characterised by k-linear derivations from K into itself. To see this, let f and g be elements of K. Then as π^* is a ring homomorphism, $\pi^*(f.g) = \pi^*(f)\pi^*(g)$. Hence

$$fg + t.\delta(fg) = (f + t.\delta(f)).(g + t.\delta(g)) = fg + t.(f\delta(g) + g\delta(f))$$

since $t^2 = 0$ in $K[t]/(t^2)$. Therefore, $\delta(fg) = f\delta(g) + g\delta(f)$ and δ is a k-linear derivation on K. Conversely, any k-linear derivation on K gives rise to such a ring homomorphism.

We may now locate $\mathcal{O}_D$ inside K. An element $f \in K$ lies inside $\mathcal{O}_D$ if the induced function on C is regular on both of the copies of Γ. That is, f and $\delta(f)$ must be regular when considered as functions on Γ. But f itself is automatically regular, so $\mathcal{O}_D$ is isomorphic to a subset of $\{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$. In fact, the S_2 condition actually forces $\mathcal{O}_D$ to be isomorphic to the whole of this set. The reason for this is that if an element of $\{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$ is left out of $\mathcal{O}_D$ then this element creates an obstruction to finding a regular sequence of length two in $\mathcal{O}_D$. This is best demonstrated by an example.

Example: Let $\mathcal{R}$ be the ring $k[x^2, x^3] + y^2k[x, y]$. This ring comes from the inclusion of $\mathcal{O}_D$ into $\mathcal{O}_C$ via the associated derivation $\delta = 0$. Here, the set $\{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$

is actually all of $k[x] \subseteq K = k(x)$. So by setting $\mathcal{O}_D = k[x^2, x^3]$ as we have done, we are missing out the element x. Now, if we try to make a regular sequence of length two in $\mathcal{R}$, such as $\{y^2, x^2\}$, we run into trouble. This is because the ideal generated by y^2 in $\mathcal{R}$ does not contain the element xy^2. But it does contain $x^2.xy^2 = x^3y^2$, so that x^2 is a zero divisor in $\mathcal{R}/y^2\mathcal{R}$. The same problem arises whichever two elements we choose for a sequence. So this ring $\mathcal{R}$ is not S_2.

In general, suppose that $\mathcal{R}$ is S_2 and let $\mathbf{q}$ be a height two prime ideal of $\mathcal{R}$ containing $\mathbf{p} = P \bigcap \mathcal{R}$. Since $\overline{\mathcal{R}}_{\mathbf{q}}$ is regular local, we may choose a generator, t say, for the ideal $P\overline{\mathcal{R}}_{\mathbf{q}}$. Suppose that $\mathcal{O}_D \neq \{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$ and let $\overline{\mathbf{q}}$ be the image of $\mathbf{q}$ in $\mathcal{O}_D$ under the projection of $\mathcal{R}$ into $\mathcal{R}/P^2$. Set $\mathcal{S} = \{g + t\delta(g) | g, \delta(g) \in \mathcal{O}_\Gamma\} \subseteq K[t]/t^2$. Then $S_{\overline{\mathbf{q}}}$ is a finitely generated $(\mathcal{O}_D)_{\overline{\mathbf{q}}}$-module and there exists some integer $r \in \mathbf{N}$ with the property that $Q^r \subseteq (\mathcal{O}_D)_{\overline{\mathbf{q}}}$, where Q is the unique maximal ideal of $S_{\overline{\mathbf{q}}}$. Therefore we can find an element f in Q with the properties that $(f + t\delta(f)) \notin (\mathcal{O}_D)_{\overline{\mathbf{q}}}$ and $(f + t\delta(f))Q \subseteq (\mathcal{O}_D)_{\overline{\mathbf{q}}}$. Hence $t^2(f + \delta(f))$ lies inside $\mathcal{R}_{\mathbf{q}}$, but does not lie in $t^2\mathcal{R}_{\mathbf{q}}$. But given any other element $g \in \mathbf{q}\mathcal{R}_{\mathbf{q}}$, $g.t^2(f + t\delta(f))$ does lie inside $t^2\mathcal{R}_{\mathbf{q}}$ so that every element of $\mathbf{q}\mathcal{R}_{\mathbf{q}}$ is a zero divisor in $\mathcal{R}_{\mathbf{q}}/t^2\mathcal{R}_{\mathbf{q}}$. Thus we can find no sequence of the form t^2, g which is a regular sequence in $\mathcal{R}_{\mathbf{q}}$. But this contradicts the fact that $\mathcal{R}$ is S_2 and so $\mathcal{O}_D$ must equal $\{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$.

Conversely, if $\mathcal{O}_D$ equals $\{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$, then $\mathcal{R}$ must be S_2. To see this, let $f + t\delta(f)$ be in $\mathcal{R}_{\mathbf{q}}$ and suppose that there exists some $b + ct$ in $\overline{\mathcal{R}}_{\mathbf{q}}$ with the properties that $t^2(b + ct) \notin t^2\mathcal{R}_{\mathbf{q}}$ but $(f + t\delta(f))t^2(b + ct) \in t^2\mathcal{R}_{\mathbf{q}}$. Then $(f + t\delta(f))(b + ct) \in \mathcal{R}_{\mathbf{q}}$

which implies that $fb + (fc + b\delta(f))t \in \mathcal{R}_{\mathbf{q}}$. Therefore we must have that:

$$fb + (fc + b\delta(f))t = fb + \delta(fb)t = fb + (f\delta(b) + b\delta(f))t.$$

Hence $c = \delta(b)$ and $b + ct = b + t\delta(b)$. But $\mathcal{O}_D$ consists of all the elements of this type, and so $b + t\delta(b) \in \mathcal{R}$ and $t^2(b + ct) \in t^2\mathcal{R}_{\mathbf{q}}$, a contradiction. Thus the sequence $\{t^2, f + t\delta(f)\}$ is a regular sequence in $\mathcal{R}_{\mathbf{q}}$. Hence $\mathcal{R}$ must be S_2.

It is not true however that if $\mathcal{X}$ is of dimension three or greater then an arbitrary choice of δ yields a noetherian subring of $\widetilde{\mathcal{R}}$. This is because if one starts with a ring $\widetilde{\mathcal{R}}$ and constructs a second ring $\mathcal{R}$ by the above method then $\widetilde{\mathcal{R}}$ might not be finitely generated over $\mathcal{R}$. That is, $\widetilde{\mathcal{R}}$ might not be a finitely generated $\mathcal{R}$-module. So in dimensions three or greater, we are restricted in our choice of δ.

Example: Let $\widetilde{\mathcal{R}} = k[x, y, z]$, let $P = (z)$ and set $\delta : K \to K$ to be $\delta = x^{-1}\partial/\partial y$, where $K = k(x, y)$. Then the set $S = \{f \in k[x, y] | \delta(f) \in k[x, y]\}$ is the following:

$$S = k[x] + xyk[x, y].$$

Thus $\mathcal{R} = k[x] + xyk[x, y] + z^2k[x, y, z]$. It is easy to see that $k[x, y, z]$ is not finitely generated over $\mathcal{R}$. For example, whichever set of generators one tries, there will be some y^n which does not lie in the module generated by this set.

On the other hand, if $\mathcal{X}$ is two dimensional and Γ is smooth then any choice of δ will do. To see this, let $\mathbf{n}$ be a maximal ideal of $\mathcal{O}_\Gamma$ and let $x \in \mathbf{n}$ generate the local ring $(\mathcal{O}_\Gamma)_{\mathbf{n}}$. Then every derivation on K has the form $\delta = ux^m\partial/\partial x$, for some unit u in $(\mathcal{O}_\Gamma)_n$ and some (possibly negative) integer m. It is easy to see that the set $S = \{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$ contains the ideal of $\mathcal{O}_\Gamma$ generated by $x^{|m|+1}$. Therefore, since $\mathcal{O}_\Gamma$ is a one dimensional affine k-algebra, $\mathcal{O}_\Gamma/S$ must be a finite dimensional vector space over k and so the Artin-Tate Lemma (Lemma 2.1.5) shows that $S \cong \mathcal{O}_D$ is also affine.

Let us summarise what we have proved so far in the following result:

Lemma 4 *Let $\mathcal{X}$ be an S_2 variety with smooth, injective normalisation and with the properties that the conductor of $\overline{\mathcal{R}}$ into $\mathcal{R}$ is precisely P^2 for some height one prime ideal P of $\overline{\mathcal{R}}$, and $P \bigcap \mathcal{R} = P^2$. If Γ is smooth and if D denotes the image of Γ in $\mathcal{X}$ then there exists a derivation δ on the function field K of D such that $\mathcal{O}_D \cong \{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$.*

Conversely, if $\overline{\mathcal{X}}$ is a smooth surface and P gives rise to a smooth subvariety then any δ gives rise to such a variety in this way.

Proof: See above. □

Examples: (i) Set $\overline{\mathcal{R}} = k[x,y]$ and let P be the prime ideal of $\overline{\mathcal{R}}$ generated by y. Let $K = k(x)$ and define $\delta : K \to K$ by $\delta = \partial/\partial x$. Then in this case we find that $\mathcal{O}_D \cong k[x]$, and $\mathcal{R} = P^2 + \{f + y\delta(f) | f \in k[x]\} = k[x+y, y^2, y^3]$. It is easy to see that the map $x \mapsto x - y$ is a k-algebra isomorphism of $k[x,y]$ which maps $k[x+y, y^2, y^3]$ onto $k[x, y^2, y^3]$. Therefore $\mathcal{X}$ is isomorphic to the product of a cusp with a line.

(ii) The image D of the singular locus Γ of $\mathcal{X}$ need not be non-singular, even if Γ itself is regular. Indeed, as in the previous example, take $\overline{\mathcal{R}} = k[x,y]$ and $P = (y)$, but this time set $\delta = x^{-1}\partial/\partial x$. Then $\mathcal{O}_D \cong k[x^2, x^3]$ so that $\mathcal{O}_D$ is isomorphic to the coordinate ring of a cusp. Therefore D itself is singular. In this case $\mathcal{R}$ is the set

$$\mathcal{R} = k[x^2 + x^2y, x^3 + x^3y] + y^2k[x, y].$$

It is this example which we shall show is not a product of lower dimensional varieties.

In order to decide whether a variety $\mathcal{X}$ is a product of varieties of smaller dimension or not, we must look at the image D of the singular locus Γ in $\mathcal{X}$. Since we want to show that Example (*ii*) above is not a product, we need only consider whether such surfaces are products of two curves or not. The argument that we give is easily seen to extend to the general case. So let $\mathcal{X}$ be an S_2 surface with smooth, injective normalisation $\tilde{\mathcal{X}}$ and let D be the image of the singular locus Γ of $\mathcal{X}$ in $\tilde{\mathcal{X}}$. Suppose that $\mathcal{X}$ is the product of two curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $\tilde{\mathcal{X}}$ must be the product of the curves $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$.

Now, suppose that $\mathcal{C}_1$ is singular at a point $c \in \mathcal{C}_1$. Then $\mathcal{X}$ must be singular all along a copy of $\mathcal{C}_2$ in $\mathcal{X}$. This is because $\mathcal{X}$ is singular along all points of the form $c \times d$ for all $d \in \mathcal{C}_2$. Next suppose that $\mathcal{C}_2$ is also singular at a point $d \in \mathcal{C}_2$. Then $\mathcal{X}$ is also singular along a copy of $\mathcal{C}_1$ in $\mathcal{X}$. Therefore, if C and D are the unique points of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ lying over c and d respectively, then Γ contains all points of the form $C \times \tilde{\mathcal{C}}_2$ and $\tilde{\mathcal{C}}_1 \times D$. Hence Γ is not irreducible.

Example: If $\mathcal{R} = k[x^2, x^3, y^2, y^3]$ then $\mathcal{X}$ is the product of two cusps, and Γ is the union of the two axes in k^2.

Now, in our Example (ii) above, Γ is irreducible. Therefore we need only consider the case of when one of the curves is singular. So suppose that C_1 is singular at $c \in C_1$ and that C_2 is non-singular. Now we are in the situation that D contains a copy of C_2. But in the case of Example (ii), D is a cusp and this forces C_2 to be a cusp also. But this is absurd since C_2 is smooth. Therefore Example (ii) cannot be a product of curves.

Of course, Example (ii) is just a surface and the results of Section 1.4 apply to give that $\mathcal{D}(\mathcal{X})$ is simple and noetherian in this case. But we may easily write down an analogous example for when $\mathcal{X}$ is three or more dimensional with singular locus being a prime divisor of $\widetilde{\mathcal{X}}$ such that the image of the singular locus in $\mathcal{X}$ is itself singular. The arguments given above then show that $\mathcal{X}$ cannot be a product of lower dimensional varieties. So the question remains: is $\mathcal{D}(\mathcal{X})$ well-behaved for such a variety? This is the question that is tackled in the next section where some positive results are obtained (and also the important negative one!).

4.2 The Differential Operators

In this section we begin the study the rings of differential operators on the type of varieties constructed in Section One. The method of attack that we use has its roots based on an argument that appears in [Hart] and depends on a consideration of which derivations Δ as in Corollary 1.3 give rise to the sort of varieties that we are looking at. We obtain a criterion for the module $\mathcal{D}(\widetilde{\mathcal{R}}, \mathcal{R})$ to be projective and use this to determine conditions on the derivation Δ which turn the balance one way or the other.

Throughout this section, we retain the notation that was used in Section One. That is, $\mathcal{X}$ is an S_2 variety with smooth, injective normalisation and singular locus Γ defined by a prime ideal of $\overline{\mathcal{R}}$. For this section we must insist that Γ is smooth as this will allow us to explicitly write down differential operators. The conductor of $\overline{\mathcal{R}}$ into $\mathcal{R}$ is precisely $\mathbf{p} = P \bigcap \mathcal{R} = P^2$ and $K = \overline{\mathcal{R}}_P / P\overline{\mathcal{R}}_P$. Also, for the rest of this section, we may as well assume that $\mathcal{R}$ and $\overline{\mathcal{R}}$ have been localised at a maximal ideal of $\mathcal{R}$ containing P^2. Then P is automatically a principal ideal with generator t say. Write M for the maximal ideal of $\overline{\mathcal{R}}$.

By Corollary 1.3, there exists a derivation Δ from $\overline{\mathcal{R}}$ to K such that $\mathcal{R} = Ker\ \Delta$. It is this derivation Δ which will give us a hold on the differential operator ring on $\mathcal{R}$. But first of all, we need to alter Δ slightly so that we only need to consider derivations between finitely generated $\mathcal{R}$-modules.

Lemma 1 *Given $\mathcal{R}$ and Δ as above, there exists a derivation $\overline{\Delta}$ from $\overline{\mathcal{R}}$ into $\overline{\mathcal{R}}/P^2$ with $\mathcal{R} = Ker\ \overline{\Delta}$.*

Proof: Composing with the projection of $\overline{\mathcal{R}}$ onto $\overline{\mathcal{R}}/P^2$, we may assume that Δ is a derivation from $\mathcal{O}_C = \overline{\mathcal{R}}/P^2$ into K. Let S be the set $\overline{\mathcal{R}} \setminus P$ and regard S as a subset of both $\mathcal{O}_C$ and $\mathcal{O}_\Gamma$. Then Δ lies in the module of differential operators $\mathcal{D}((\mathcal{O}_C)_S, (\mathcal{O}_\Gamma)_S)$. Therefore, by Proposition 1.2.7, there exists an $s \in S$ such that $s\Delta \in \mathcal{D}(\mathcal{O}_C, \mathcal{O}_\Gamma)$. Set $\overline{\Delta}(f) = t.s\Delta(p(f)) \in \mathcal{O}_C$ for each $f \in \overline{\mathcal{R}}$, where p is the projection of $\overline{\mathcal{R}}$ onto $\mathcal{O}_C$. Clearly $\mathcal{R} = Ker\ \overline{\Delta}$ as required. □

We now have that $\overline{\Delta}$ lies in $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{O}_C)$ which is a right $\mathcal{D}(\overline{\mathcal{R}})$-module. We therefore have a homomorphism of $\mathcal{D}(\overline{\mathcal{R}})$-modules from $\mathcal{D}(\overline{\mathcal{R}})$ into $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{O}_C)$ given by multiplication on the left by $\overline{\Delta}$, and the right ideal $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R})$ of $\mathcal{D}(\overline{\mathcal{R}})$ plays a particularly important rôle.

Lemma 2 *$\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R})$ is the kernel of the homomorphism of $\mathcal{D}(\overline{\mathcal{R}})$ into $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{O}_C)$. In other words, $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R}) = \{\partial \in \mathcal{D}(\overline{\mathcal{R}}) | \overline{\Delta}\partial = 0\} = Ann_{\mathcal{D}(\overline{\mathcal{R}})}\overline{\Delta}$.*

Proof: Since $\mathcal{R} \subseteq Ker\overline{\Delta}$, we have that $(\overline{\Delta}.\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R})) * \overline{\mathcal{R}} \subseteq \overline{\Delta} * \mathcal{R} = 0$. Hence $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R}) \subseteq Ann(\overline{\Delta})$. Conversely, if $\partial \in \mathcal{D}(\overline{\mathcal{R}})$ is such that $\overline{\Delta}.\partial = 0$ then $\partial * \overline{\mathcal{R}} \subseteq Ker\overline{\Delta} = \mathcal{R}$. Thus $\partial \in \mathcal{D}(\overline{\mathcal{R}}, \mathcal{R})$. □

Lemma 2 gives us the following exact sequence of $\mathcal{D}(\overline{\mathcal{R}})$-modules:

$$0 \longrightarrow \mathcal{D}(\overline{\mathcal{R}}, \mathcal{R}) \longrightarrow \mathcal{D}(\overline{\mathcal{R}}) \xrightarrow{\overline{\Delta}.} \overline{\Delta}.\mathcal{D}(\overline{\mathcal{R}}) \longrightarrow 0.$$

Our aim therefore is to investigate $\overline{\Delta}.\mathcal{D}(\overline{\mathcal{R}})$. If we can show that $\overline{\Delta}.\mathcal{D}(\overline{\mathcal{R}})$ has projective dimension one or less, then $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R})$ must be projective by Schanuel's Lemma.

Now, by Proposition 1.2.14 we have that $\overline{\Delta}$ extends to a differential operator $\Delta^* \in \mathcal{D}(\overline{\mathcal{R}})$. The next lemma allows us to explicitly write down Δ^*.

Lemma 3 *Given $\overline{\mathcal{R}}$, P and t as above, we may extend t to a regular sequence $t, x_1, \ldots, x_{n-1}$ in $\overline{\mathcal{R}}$ which generates the maximal ideal. Then we may write $\mathcal{D}(\overline{\mathcal{R}})$ as $\overline{\mathcal{R}}[\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}]$.*

Proof: Since the subvariety of $\mathcal{X}$ that P generates is smooth, t must lie in $M\backslash M^2$. This is because $\widetilde{\mathcal{R}}/(t)$ must be a regular local ring. In other words, if N is the maximal ideal of $\widetilde{\mathcal{R}}/(t)$, then the dimension of N/N^2 as a k-vector space must be the same as the Krull dimension of $\widetilde{\mathcal{R}}/(t)$. But if t lies in M^2, then factoring out t does not affect the dimension of M/M^2 whereas the Krull dimension of the ring drops by one. So t lies in $M\backslash M^2$ as claimed.

Now we may extend $t + M^2$ to a basis of M/M^2 and choose $x_1, \ldots, x_{n-1}$ to be representatives in $\widetilde{\mathcal{R}}$ of these basis elements. Then it is easy to see that $t, x_1, \ldots, x_{n-1}$ have the required properties. □

If we now complete $\widetilde{\mathcal{R}}$ at its maximal ideal, then (abusing notation and writing $\widetilde{\mathcal{R}}$ for this completion) [Matsumura; Theorem 8.12] shows that

$$\widetilde{\mathcal{R}} = k[\![t, x_1, \ldots, x_{n-1}]\!].$$

From now on, we shall assume that $\widetilde{\mathcal{R}}$ has been completed at its maximal ideal. This will simply allow us to write down elements of $\widetilde{\mathcal{R}}$ as k-linear combinations of monomials in t and the x_i's. Since differential operators behave well with respect to completion, $\mathcal{D}(\widetilde{\mathcal{R}})$ is still equal to $\widetilde{\mathcal{R}}[\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}]$.

Recall that $\mathcal{O}_D \cong \{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$. Since $\mathcal{O}_\Gamma$ is a regular local ring and $x_1, \ldots, x_{n-1}$ form a minimal generating set for the maximal ideal of $\mathcal{O}_\Gamma$, we can write δ as a sum of $\partial/\partial x_i$'s with coefficients in $\mathcal{O}_\Gamma$. Hence we may consider δ as actually acting on $\widetilde{\mathcal{R}}$ itself. From this, we can see that Δ must look like $p \circ (\partial/\partial t - \delta)$, where p is the natural projection of $\widetilde{\mathcal{R}}_P$ onto $\widetilde{\mathcal{R}}_P/P\widetilde{\mathcal{R}}_P$. Therefore, Δ^* must be $f\partial/\partial t - ft\delta$, where $f \in \mathcal{O}_\Gamma$ is such that $f\delta$ is a derivation on $\widetilde{\mathcal{R}}$.

Lemma 4 $\overline{\Delta}.\mathcal{D}(\widetilde{\mathcal{R}}) \cong (\Delta^*\mathcal{D}(\widetilde{\mathcal{R}}) + t^2\mathcal{D}(\widetilde{\mathcal{R}}))/t^2\mathcal{D}(\widetilde{\mathcal{R}})$.

Proof: We have that $\overline{\Delta}$ lies in $\mathcal{D}(\widetilde{\mathcal{R}}, \widetilde{\mathcal{R}}/t^2\widetilde{\mathcal{R}})$. It is easy to see that the map from $\mathcal{D}(\widetilde{\mathcal{R}})/\mathcal{D}(\widetilde{\mathcal{R}}, t^2\widetilde{\mathcal{R}})$ to $\mathcal{D}(\widetilde{\mathcal{R}}, \widetilde{\mathcal{R}}/t^2\widetilde{\mathcal{R}})$ is an isomorphism. This isomorphism takes $(\Delta^*\mathcal{D}(\widetilde{\mathcal{R}}) + t^2\mathcal{D}(\widetilde{\mathcal{R}}))/t^2\mathcal{D}(\widetilde{\mathcal{R}})$ into $\overline{\Delta}.\mathcal{D}(\widetilde{\mathcal{R}})$. $\square$

So all we need to show now is that $(\Delta^*\mathcal{D}(\widetilde{\mathcal{R}}) + t^2\mathcal{D}(\widetilde{\mathcal{R}}))/t^2\mathcal{D}(\widetilde{\mathcal{R}})$ has projective dimension one or less. In order to do this, we need to find out which derivations δ are allowed. Write δ as follows:

$$\delta = \sum_{i=1}^{n-1} \frac{g_i}{h_i}\frac{\partial}{\partial x_i}$$

where each g_i and h_i lie in $\mathcal{O}_\Gamma$ and each pair (g_i, h_i) are coprime (which we may do since $\mathcal{O}_\Gamma$ is a regular local ring and hence a unique factorisation domain). Let S be the set $S = \{f \in \mathcal{O}_\Gamma | \delta(f) \in \mathcal{O}_\Gamma\}$. We claim that each h_i must be a power of x_i (up to multiplication of a unit in $\mathcal{O}_\Gamma$. So suppose that $h_1 = x_1^r.f$ with f not divisible by x_1 and not a unit. Then the only combinations of monomials in $x_1, \ldots, x_{n-1}$ in S which depend on x_1 must be divisible by f also. So in order to find a generating set for $\mathcal{O}_\Gamma$ over S, we need to include every power of x_1. Hence $\mathcal{O}_\Gamma$ is not a finitely generated S-module and δ cannot be of the supposed form.

Therefore δ must take the following form:

$$\delta = \sum_{i=1}^{n-1} g_i x_i^{-r_i}\frac{\partial}{\partial x_i},$$

where each g_i lies in $\mathcal{O}_\Gamma$, each r_i is a positive integer and no g_i is divisible by x_i. Set f equal to $x_1^{r_1} \dots x_{n-1}^{r_{n-1}}$ so that $f.\delta$ is a derivation on $\widetilde{\overline{\mathcal{R}}}/P\widetilde{\overline{\mathcal{R}}}$. Then Δ^* equals $ft\partial/\partial t - ft\delta$. Recall that $\mathcal{D}(\widetilde{\overline{\mathcal{R}}}, \mathcal{R})$ is projective if and only if $(\Delta^*.\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2)\mathcal{D}(\widetilde{\overline{\mathcal{R}}})/t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$ has projective dimension one or less. The next result will help us to build a projective resolution for $(\Delta^*.\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2)\mathcal{D}(\widetilde{\overline{\mathcal{R}}})/t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$.

Proposition 5 *With Δ^* defined as above,* $\Delta^*.\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) = ft\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + ft\delta\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$.

Proof: Since $\Delta^* = ft\partial/\partial t - ft\delta$, we have that

$$\Delta^*\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) \supseteq [\Delta^*, t]\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}}).$$

But $[\Delta^*, t] = [ft\partial/\partial t, t] \in ft + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$. Hence $\Delta^*\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$ contains $ft\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$. Therefore $\Delta^*\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$ is generated by $ft, ft\delta$ and t^2. That is,

$$\Delta^*\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) + t^2\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) = (ft, ft\delta)\mathcal{D}(\widetilde{\overline{\mathcal{R}}})$$

as required. $\square$

In the special case of when the image D of Γ in $\mathcal{X}$ is itself smooth, Proposition 5 allows us to show quite easily that $\mathcal{D}(\widetilde{\overline{\mathcal{R}}}, \mathcal{R})$ is projective.

Proposition 6 *If the image of Γ is smooth in $\mathcal{X}$ then $\mathcal{D}(\widetilde{\overline{\mathcal{R}}}, \mathcal{R})$ is projective.*

Proof: Since the image of Γ, D is smooth, δ must be a regular derivation. In other words, the denominator f must equal 1. Then Proposition 5 shows that $t \in \Delta^*\mathcal{D}(\widetilde{\overline{\mathcal{R}}}) +$

$t^2\mathcal{D}(\overline{\mathcal{R}})$. Hence $(\Delta^*\mathcal{D}(\overline{\mathcal{R}}) + t^2\mathcal{D}(\overline{\mathcal{R}}))/t^2\mathcal{D}(\overline{\mathcal{R}}) = t\mathcal{D}(\overline{\mathcal{R}})/t^2\mathcal{D}(\overline{\mathcal{R}})$ so is a factor of one projective module by another and is therefore of projective dimension one or less. $\square$

Now suppose that D is not smooth. Then since $(\Delta^*\mathcal{D}(\overline{\mathcal{R}}) + t^2\mathcal{D}(\overline{\mathcal{R}}))/t^2\mathcal{D}(\overline{\mathcal{R}})$ is generated by the two elements $ft + t^2\mathcal{D}(\overline{\mathcal{R}})$ and $ft\delta + t^2\mathcal{D}(\overline{\mathcal{R}})$, we have a map from the direct sum of two copies of $\mathcal{D}(\overline{\mathcal{R}})$ into it given by $(x, y) \mapsto ftx + ft\delta y + t^2\mathcal{D}(\overline{\mathcal{R}})$, where $(x, y) \in \mathcal{D}(\overline{\mathcal{R}})^2$. Writing K for the kernel of this map we arrive at the following exact sequence:

$$0 \longrightarrow K \longrightarrow \mathcal{D}(\overline{\mathcal{R}})^2 \longrightarrow (\Delta^*\mathcal{D}(\overline{\mathcal{R}}) + t^2\delta\mathcal{D}(\overline{\mathcal{R}}))/t^2\mathcal{D}(\overline{\mathcal{R}}) \longrightarrow 0.$$

By Schanuel's lemma again, $(\Delta^*\mathcal{D}(\overline{\mathcal{R}}) + t^2\delta\mathcal{D}(\overline{\mathcal{R}}))/t^2\mathcal{D}(\overline{\mathcal{R}})$ has projective dimension one or less if and only if K is projective. Therefore we have that $\mathcal{D}(\overline{\mathcal{R}}, \mathcal{R})$ is projective if and only if K is projective.

We can identify K as the following set:

$$K = \{(x, y) \in \mathcal{D}(\overline{\mathcal{R}})^2 | ftx + ft\delta y \in t^2\mathcal{D}(\overline{\mathcal{R}})\} \cong \{(x, y) \in \mathcal{D}(\overline{\mathcal{R}})^2 | fx + f\delta y \in t\mathcal{D}(\overline{\mathcal{R}})\}.$$

It is now easy to find explicit examples of varieties $\mathcal{X}$ for which the $\mathcal{D}(\overline{\mathcal{R}})$-module K is not projective.

Example: Set $\overline{\mathcal{R}} = k[x, y, z]$, let P be the prime ideal generated by z and define $\delta : K \to K$ to be $\delta = xy^{-1}\partial/\partial y$. Therefore $\Delta = z\partial/\partial z - zxy^{-1}\partial/\partial y$ and $\Delta^* = yz\partial/\partial z - xz\partial/\partial y$. This gives us that $\mathcal{R} = Ker(\Delta^*)$ is the following:

$$\mathcal{R} = k[x] + \{f(y) + xzy^{-1}\frac{\partial f}{\partial y} | f(y) \in k[y^2, y^3]\} + z^2k[x, y, z].$$

It follows that $\mathcal{R}$ is affine, since $\widetilde{\mathcal{R}}$ is generated as an $\mathcal{R}$-module by the elements $1, y, z$ and yz. Also, the argument given before Lemma 1.4 shows that since we have constructed $\mathcal{O}_D$ as the set of regular functions on Γ which stay regular under δ, $\mathcal{R}$ must be S_2. Indeed, it is easy to see that x, y^2+2xz, z^2 is a regular sequence of length three and therefore $\mathcal{R}$ is even Cohen-Macaulay.

Now, in this example we can simplify the module K considerably by slightly changing the exact sequence. Recall that by Proposition 5, we have that yz lies inside $\Delta^{*}\mathcal{D}(\widetilde{\mathcal{R}}) + z^2\mathcal{D}(\widetilde{\mathcal{R}})$. Therefore we can see that $[\Delta^{*}, y^2]$ also lies inside $\Delta^{*}\mathcal{D}(\widetilde{\mathcal{R}}) + z^2\mathcal{D}(\widetilde{\mathcal{R}})$. But $[\Delta^{*}, y^2] = xyz\partial/\partial y + 2xz$, and since yx lies in $\Delta^{*}\mathcal{D}(\widetilde{\mathcal{R}}) + z^2\mathcal{D}(\widetilde{\mathcal{R}})$, we must have that xz lies in there too. This means that the module $(\Delta{*}\mathcal{D}(\widetilde{\mathcal{R}}) + z^2\mathcal{D}(\widetilde{\mathcal{R}}))/z^2\mathcal{D}(\widetilde{\mathcal{R}})$ is generated by the two element xz and yz. Thus we have that:

$$\begin{aligned}\frac{\Delta^{*}\mathcal{D}(\widetilde{\mathcal{R}}) + z^2\mathcal{D}(\widetilde{\mathcal{R}})}{z^2\mathcal{D}(\widetilde{\mathcal{R}})} &\cong \frac{xz\mathcal{D}(\widetilde{\mathcal{R}}) + yz\mathcal{D}(\widetilde{\mathcal{R}}) + z^2\mathcal{D}(\widetilde{\mathcal{R}})}{z^2\mathcal{D}(\widetilde{\mathcal{R}})} \\ &\cong \frac{x\mathcal{D}(\widetilde{\mathcal{R}}) + y\mathcal{D}(\widetilde{\mathcal{R}}) + z\mathcal{D}(\widetilde{\mathcal{R}})}{z\mathcal{D}(\widetilde{\mathcal{R}})}.\end{aligned}$$

Set $M = (x\mathcal{D}(\widetilde{\mathcal{R}}) + y\mathcal{D}(\widetilde{\mathcal{R}}) + z\mathcal{D}(\widetilde{\mathcal{R}}))/z\mathcal{D}(\widetilde{\mathcal{R}})$. We may build an exact sequence of $\mathcal{D}(\widetilde{\mathcal{R}})$-modules as follows:

$$0 \longrightarrow K = Ker(\theta) \longrightarrow \mathcal{D}(\widetilde{\mathcal{R}}) \oplus \mathcal{D}(\widetilde{\mathcal{R}}) \xrightarrow{\theta} M \longrightarrow 0$$

by setting $\theta(a, b) = xa + yb + z\mathcal{D}(\widetilde{\mathcal{R}})$ for $a, b \in \mathcal{D}(\widetilde{\mathcal{R}})$.

There are now several ways now to see that the module K does not have projective dimension one or less. One is to calculate a projective resolution for K and the other is to notice that:

$$M = \frac{x\widetilde{\mathcal{R}} + y\widetilde{\mathcal{R}} + z\widetilde{\mathcal{R}}}{z\widetilde{\mathcal{R}}} \otimes_{\widetilde{\mathcal{R}}} \mathcal{D}(\widetilde{\mathcal{R}}).$$

Define N to be the right $\widetilde{\mathcal{R}}$-module $(x\widetilde{\mathcal{R}} + y\widetilde{\mathcal{R}} + z\widetilde{\mathcal{R}})/z\widetilde{\mathcal{R}}$. Then the projective dimension of N is two. To see this, suppose that N has projective dimension one or less and assume that $\widetilde{\mathcal{R}}$ and N have been localised at the maximal ideal (x, y, z) of $\widetilde{\mathcal{R}}$. Then the Auslander-Buchsbaum Theorem ([Matsumura; Theorem 19.1]) states that

$$proj.dim.(N) + depth_{\widetilde{\mathcal{R}}}(N) = depth(\widetilde{\mathcal{R}}).$$

But it is easy to see that N has depth one. This is because N does not contain the element 1. Therefore, if we start a regular sequence with the element $f \in \widetilde{\mathcal{R}}$, then f does not lie in Nf and $f + Nf$ kills every element of (x, y, z) in N/Nf. Hence we cannot extend any regular sequence of length one to one of length two. Thus N has depth one and it follows that it must have projective dimension two.

In fact, we may construct a minimal free resolution of N as follows: since N is generated by x and y, we may map two copies of $\widetilde{\mathcal{R}}$ into N by setting $\phi(a, b) = xa + yb$ for $a, b \in \widetilde{\mathcal{R}}$. The kernel of ϕ is evidently the submodule of $\widetilde{\mathcal{R}}^2$ generated by the elements $(z, 0)$, $(0, z)$ and $(y, -x)$. This kernel is clearly not a free module as we have the relation $(y, -x)z = (z, 0)y - (0, z)x$, but this is the only relation and so generates a free module of rank one. The projective resolution of N is then:

$$0 \longrightarrow \widetilde{\mathcal{R}} \longrightarrow \widetilde{\mathcal{R}}^3 \longrightarrow \widetilde{\mathcal{R}}^2 \xrightarrow{\phi} N \longrightarrow 0.$$

Now, $\mathcal{D}(\widetilde{\mathcal{R}})$ is the free left $\widetilde{\mathcal{R}}$-module generated by all the products of the k-linear derivations on $\widetilde{\mathcal{R}}$, and so must be a flat $\widetilde{\mathcal{R}}$-module. Hence we can tensor the above exact sequence on the right by $\mathcal{D}(\widetilde{\mathcal{R}})$ to obtain the corresponding exact sequence of

$\mathcal{D}(\overline{\overline{\mathcal{R}}})$-modules:

$$0 \longrightarrow \mathcal{D}(\overline{\overline{\mathcal{R}}}) \longrightarrow \mathcal{D}(\overline{\overline{\mathcal{R}}})^3 \longrightarrow \mathcal{D}(\overline{\overline{\mathcal{R}}})^2 \xrightarrow{\theta} M \longrightarrow 0.$$

This shows that M has projective dimension two since if M had projective dimension less than two then the kernel of θ would be projective (by Schanuel's Lemma). But since $\overline{\overline{\mathcal{R}}}$ is regular so that $\mathcal{D}(\overline{\overline{\mathcal{R}}})$ is a free $\mathcal{R}$-module, $Ker(\theta)$ is isomorphic (as a $\mathcal{D}(\overline{\overline{\mathcal{R}}})$-module) to $Ker(\phi)\otimes\mathcal{D}(\overline{\overline{\mathcal{R}}})$. Hence $Ker(\theta)$ is minimally generated by the same elements as $Ker(\phi)$ and contains the same relations as $Ker(\phi)$. So $Ker(\theta)$ cannot be a direct summand of a free module. Thus M has projective dimension two and hence $\mathcal{D}(\overline{\overline{\mathcal{R}}}, \mathcal{R})$ cannot be projective. $\square$

It would seem then that a lot of work needs to be done in order to determine which conditions on $\mathcal{X}$ force $\mathcal{D}(\overline{\overline{\mathcal{R}}}, \mathcal{R})$ to be projective. The above example shows that even Cohen-Macaulay is not enough. It may be possible though to impose some restriction on the derivations Δ and δ which we have been using in order to make K nicer. A discussion of possible avenues of continuation is presented in the final chapter.

Chapter 5

Conclusions and Conjectures

In this chapter we discuss the results of the preceding chapters and suggest how they might be improved, starting with Chapter Two since the results of this chapter are far from being complete. The main aim of the chapter is to try to classify the dense, projective right ideals of $\mathcal{D}(\mathcal{R})$ where $\mathcal{R}$ is the coordinate ring of a smooth surface. The idea is to map such a right ideal, I say, to a subspace $I * \mathcal{R}$ of $\mathcal{R}$ and then try to identify the subspaces of $\mathcal{R}$ of this form with the primary decomposible subspaces. Proposition 2.2.2 shows that every primary decomposible subspace of $\mathcal{R}$ is of this form, but the converse, that $I * \mathcal{R}$ is primary decomposible, remains to be shown. The two obstructions to this are that to get anywhere we have to assume that $S(I*\mathcal{R})$ is noetherian, and also that we need a certain vector space to be a vector space over a certain field. I am certain that a simple trick is all that is required to overcome the latter problem.

Now, by examining closely the proofs of the various results in Chapter One about localising differential operators, it may be possible to get a form of localisation working for $S(I * \mathcal{R})$ without the noetherian hypothesis. This is also the main stumbling block for any attempt to extend the results to higher dimensions since here, Theorem 1.4.15 suggests that one may have to replace the projective right ideals with the reflexive right ideals. In this case, there would certainly be no Morita equivalence to afford any noetherian conditions. Therefore, if it proves to be the case that the noetherian condition placed on primary decomposible subspaces can either be omitted or replaced with a weaker restriction, then I would put forward the following conjecture:

Conjecture 1 *With a possibly altered definition of primary decomposible, if $\mathcal{R}$ is the coordinate ring of a smooth variety then the dense, reflexive right ideals of $\mathcal{D}(\mathcal{R})$ are classified by the primary decomposible subspaces of $\mathcal{R}$ via the maps $I \mapsto I * \mathcal{R}$ and $V \mapsto \mathcal{D}(\mathcal{R}, V)$ where I is a dense, reflexive right ideal of $\mathcal{D}(\mathcal{R})$ and V is a primary decomposible subsapce of $\mathcal{R}$.*

Chapter Three is complete as it stands and it is hard to see that any generalisation of the results here could be made. However, if looked at in conjunction with the results of Chapter Four, some questions do arise. Since Chapter Four shows that it is not true that if $\mathcal{X}$ is an S_2 variety with smooth, injective normalisation then $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ must be projective, then which varieties do have this property? Chapter Three says that all products of curves and S_2 surfaces with smooth, injective normalisations have $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ projective, so there must be something about products which makes the differential operators behave well. In particular, is there a property which products have which is enough to force $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ to be projective in general. For example, are they Gorenstein, and do Gorenstein varieties with smooth, injective normalisations have $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$ projective?

One possible line of investigation is to examine the derivations which are used in Chapter Four to construct varieties. It may be possible to tell which derivations lead to the projectivity of $\mathcal{D}(\widetilde{\mathcal{X}}, \mathcal{X})$, and then see what sort of properties the resulting varieties have. I believe that the class of varieties presented in Chapter Four gives a good indication of what happens in the general case. Indeed, a general result might be obtained as follows: take any S_2 variety $\mathcal{X}$ with smooth, injective normalisation. Then since $\mathcal{X}$ is singular in codimension one, we may pull the singular locus apart into pieces which are given by height one prime ideals. Then the results of Chapter Four might be applied to each part, and then glued together to give information about the whole picture. The details are left to the reader!

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