Electron. Commun. Probab. **23** (2018), no. 8, 1–6.
https://doi.org/10.1214/18-ECP113
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Quasi-invariance of countable products of Cauchy measures under non-unitary dilations

Han Cheng Lie* T. J. Sullivan†

Abstract

Consider an infinite sequence $(U_n)_{n\in\mathbb{N}}$ of independent Cauchy random variables, defined by a sequence $(\delta_n)_{n\in\mathbb{N}}$ of location parameters and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of scale parameters. Let $(W_n)_{n\in\mathbb{N}}$ be another infinite sequence of independent Cauchy random variables defined by the same sequence of location parameters and the sequence $(\sigma_n\gamma_n)_{n\in\mathbb{N}}$ of scale parameters, with $\sigma_n \neq 0$ for all $n \in \mathbb{N}$. Using a result of Kakutani on equivalence of countably infinite product measures, we show that the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent if and only if the sequence $(|\sigma_n| - 1)_{n\in\mathbb{N}}$ is square-summable.

Keywords: Cauchy distribution; change of measure; equivalence of measure; random sequence.
AMS MSC 2010: Primary 60G30, Secondary 60G20; 60E07.
Submitted to ECP on November 30, 2016, final version accepted on January 29, 2018.
Supersedes arXiv:1611.10289v3.

1 Introduction

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent, Cauchy random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where each random variable U_n has the density

$$f_n(x, \delta_n, \gamma_n) = \frac{1}{\pi\gamma_n} \frac{\gamma_n^2}{(x-\delta_n)^2 + \gamma_n^2}. \tag{1.1}$$

The distribution of U_n is parametrised by the location and scale parameters δ_n and γ_n respectively. We shall assume throughout that γ_n is strictly positive for all $n \in \mathbb{N}$. For every U_n, we may define another Cauchy random variable W_n by multiplicatively perturbing the scale parameter γ_n by $\sigma_n \neq 0$, so that the pair $(\delta_n)_{n\in\mathbb{N}}$ and $(\sigma_n\gamma_n)_{n\in\mathbb{N}}$ determines the law of $(W_n)_{n\in\mathbb{N}}$. In this note, we prove the following result:

Theorem 1.1. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of independent Cauchy random variables defined by the sequences $(\delta_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ of location and scale parameters, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence of independent Cauchy random variables with the sequences $(\delta_n)_{n\in\mathbb{N}}$ and $(\sigma_n\gamma_n)_{n\in\mathbb{N}}$ of location and scale parameters, where $\sigma_n \neq 0$ for all $n \in \mathbb{N}$. Then the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent if and only if $(|\sigma_n| - 1)_{n\in\mathbb{N}}$ is square-summable.*

*Institut für Mathematik, Freie Universität Berlin, Germany. E-mail: hlie@math.fu-berlin.de
†Free University of Berlin and Zuse Institute Berlin, Germany.
E-mail: sullivan@zib.de http://www.zib.de/sullivan

Equivalence of laws under additive perturbations to the sequence of locations or means has been studied for Gaussian U_n [1, Example 2.7.6], for sequences of random variables with finite Fisher information [5], and for stable Radon probability distributions on locally convex spaces [2, Theorem 5.2.1]. In contrast, Theorem 1.1 on multiplicative perturbations of scale parameters appears to be new.

2 Proof of Theorem 1.1

We shall use Kakutani's theorem [4, Theorem 1], which we specialise to our context.

Theorem 2.1. *Let $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ be given as in Theorem 1.1. Then the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are either equivalent or singular. The laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent if and only if both the following conditions hold: the laws $\mathbb{P}\circ U_n^{-1}$ and $\mathbb{P}\circ W_n^{-1}$ of U_n and W_n respectively are equivalent for every $n\in\mathbb{N}$, and*

$$\sum_{n\in\mathbb{N}} -\log \mathbb{E}\left[\sqrt{\varphi_n(U_n,|\sigma_n|)}\right] < \infty \tag{2.1}$$

where $\varphi_n(\ \cdot\ ,|\sigma_n|)$ is the Radon–Nikodym derivative of $\mathbb{P}\circ W_n^{-1}$ with respect to $\mathbb{P}\circ U_n^{-1}$.

If $\sigma_n \neq 0$, then the laws of U_n and W_n are equivalent for every $n\in\mathbb{N}$, with

$$\varphi_n(x,|\sigma_n|) = |\sigma_n|\,\frac{(x-\delta_n)^2+\gamma_n^2}{(x-\delta_n)^2+\sigma_n^2\gamma_n^2}. \tag{2.2}$$

Let $y_n := (x-\delta_n)/\gamma_n$. From (1.1) and (2.2) it follows that

$$\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}] = \frac{\sqrt{|\sigma_n|}}{\pi}\int_{\mathbb{R}} \frac{1}{\sqrt{y_n^2+\sigma_n^2}\sqrt{y_n^2+1}}\,\mathrm{d}y_n = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{\sqrt{\frac{y_n^2}{|\sigma_n|}+|\sigma_n|}\sqrt{y_n^2+1}}\,\mathrm{d}y_n. \tag{2.3}$$

We use (2.3) in the following lemma.

Lemma 2.2. *For all $|\sigma_n|>0$, $-\log\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]$ is nonnegative, and is strictly positive if and only if $|\sigma_n|\neq 1$. Furthermore, if* (2.1) *holds, then $\lim_{n\to\infty}|\sigma_n| = 1$.*

Proof. Using Jensen's inequality with the concave map $x\mapsto\sqrt{x}$, and using that φ_n is the Radon–Nikodym derivative of $\mathbb{P}\circ W_n^{-1}$ with respect to $\mathbb{P}\circ U_n^{-1}$, we obtain

$$1 = \sqrt{\mathbb{E}[\varphi_n(U_n,|\sigma_n|)]} \geq \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}],$$

and hence $-\log\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]$ is nonnegative for all $|\sigma_n|>0$. Since $x\mapsto\sqrt{x}$ is not affine, equality holds above (and hence $-\log\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]=0$) if and only if $\varphi_n(U_n,|\sigma_n|)$ is $\mathbb{P}$-almost surely constant. By (2.3), $\varphi_n(U_n,|\sigma_n|)$ is $\mathbb{P}$-almost surely constant if and only if $|\sigma_n|=1$. This proves the first statement.

To prove the second statement, observe that (2.3) implies continuity of the map $|\sigma_n|\mapsto\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]$ on $(0,\infty)$. The dominated convergence theorem and the second equality in (2.3) imply that, if $(|\sigma_n|)_{n\in\mathbb{N}}$ decreases to zero or increases to infinity, then $\lim_{n\to\infty}\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]=0$. Summarising, we have that the map $|\sigma_n|\mapsto\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]$ from $(0,\infty)$ to $[0,1]$ is continuous, attains the value 1 if and only if its argument is 1, and does not increase asymptotically to 1 at either end of its domain. It follows that $\lim_{n\to\infty}\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]=1$ if and only if $\lim_{n\to\infty}|\sigma_n|=1$. Since (2.1) implies that $\lim_{n\to\infty}\mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]=1$, the proof is complete. □

Remark 2.3. A consequence of Lemma 2.2 that we shall use repeatedly is the following: if (2.1) holds, and if $|\sigma_n|\neq 1$ for all $n\in\mathbb{N}$, then, for any $0<\alpha<1$, there exists some $N(\alpha)\in\mathbb{N}$ such that

$$0 < ||\sigma_n|-1| < \alpha, \quad \forall n\geq N(\alpha). \tag{2.4}$$

To simplify the subsequent presentation, we shall assume that α is sufficiently small for our purposes, and that $N(\alpha) = 1$. If (2.1) holds, then by Lemma 2.2 we may assume this without any loss of generality, since we may discard finitely many terms in the series (2.1) without losing convergence.

Given (2.3), we may define for $0 < \alpha < 1$, a function $I : (1-\alpha, 1+\alpha) \to (0, \infty)$ by

$$I(|\sigma|) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y^2+\sigma^2}} \frac{1}{\sqrt{y^2+1}} \, \mathrm{d}y. \tag{2.5}$$

Therefore, $I(|\sigma|) = 1$ if and only if $|\sigma| = 1$. Furthermore, for some $0 < \alpha < 1$ (see Remark 2.3), it follows from (2.4) that the integrand on the right-hand side of (2.5) is dominated by an integrable function that depends on α but not on σ. Thus, by the dominated convergence theorem, we may interchange integration and differentiation to compute the derivative of I with respect to $|\sigma|$. By computing the higher-order derivatives of $(y^2+\sigma^2)^{-1/2}$ with respect to $|\sigma|$ and employing the same argument, the assertion holds for higher-order derivatives of I as well. The Taylor expansion of $I(|\sigma|)$ about $|\sigma| = 1$ up to second order is

$$I(|\sigma|) = 1 + a_1(|\sigma|-1) + a_2(|\sigma|-1)^2 + O((|\sigma|-1)^3), \tag{2.6a}$$

$$a_m = \frac{1}{m!} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y^2+1}} \left(\frac{\partial}{\partial |\sigma|} \right)^m \frac{1}{\sqrt{y^2+|\sigma|^2}} \Bigg|_{|\sigma|=1} \mathrm{d}y. \tag{2.6b}$$

Note that the a_m are independent of $|\sigma|$.

We shall use the following lemma to calculate the values of a_m for $m = 1, 2$.

Lemma 2.4. *Let $r, s \in \mathbb{N}_0$ with $r \le s-1$. Then*

$$\int_{\mathbb{R}} \frac{x^{2r}}{(x^2+1)^s} \mathrm{d}x = \pi \frac{(2r)!(2(s-r-1))!}{4^{s-1} r!(s-r-1)!(s-1)!}$$

Proof. Letting $y := x^2$, we have $\mathrm{d}x = 2y^{-1/2}\mathrm{d}y$, so

$$\int_{\mathbb{R}} \frac{x^{2r}}{(x^2+1)^s} \mathrm{d}x = \int_0^\infty \frac{y^{r-1/2}}{(y+1)^s} \mathrm{d}y = \int_0^\infty \frac{y^{(r+1/2)-1}}{(y+1)^s} \mathrm{d}y.$$

Using [6, Equation (2)] and properties of the gamma function $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \mathrm{d}x$ (for Re $t > 0$), we have, for $0 <$ Re $(r+1/2) <$ Re s,

$$\int_0^\infty \frac{y^{(r+1/2)-1}}{(y+1)^s} \mathrm{d}y = \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(s-r-\frac{1}{2}\right)}{\Gamma(s)} = \frac{(2r)!}{4^r r!}\sqrt{\pi} \frac{(2(s-r-1))!}{4^{s-r-1}(s-r-1)!}\sqrt{\pi} \frac{1}{(s-1)!}.$$

Simplifying the right-hand side yields the desired conclusion. □

Thus we have

$$a_1 = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y_n^2+1}} \frac{(-1)}{(y_n^2+1)^{3/2}} \mathrm{d}y_n = -\frac{1}{\pi}\left(\pi \frac{0!\,2!}{4 \cdot 1! \cdot 1!}\right) = -\frac{1}{2}, \tag{2.7a}$$

$$a_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y_n^2+1}} \frac{2-y_n^2}{(y_n^2+1)^{5/2}} \mathrm{d}y_n = \frac{1}{2\pi}\left(\frac{9\pi}{8} - \frac{\pi}{2}\right) = \frac{5}{16}. \tag{2.7b}$$

Given the function I, a sequence $(\sigma_n)_{n\in\mathbb{N}}$ satisfying (2.4), and (2.7b), define

$$\epsilon_n := \frac{1}{|\sigma_n|-1} \left(I(|\sigma_n|) - 1 - a_1(|\sigma_n|-1) \right), \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Lemma 2.5. *Let ϵ_n be as in* (2.8)*. If* (2.1) *holds, then for some $0 < \alpha < 1$ and some $C > 0$ that do not depend on n,*

$$c\,||\sigma_n| - 1| \le |\epsilon_n| \le C\,||\sigma_n| - 1|\,, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Proof. First, note that by Remark 2.3, we may bound the constant in the $O(||\sigma_n| - 1|^3)$ term in (2.5) by some $C > 0$ that may depend on α, but not on $|\sigma|$, i.e.

$$I(|\sigma|) = 1 + \sum_{m=1}^{2} a_m(|\sigma| - 1)^m + C(|\sigma| - 1)^3, \quad \forall\, |\sigma| \in (1-\alpha, 1+\alpha).$$

Thus by (2.8), the triangle inequality, and (2.4), we obtain

$$|\epsilon_n| = \left|a_2(|\sigma_n| - 1) + C(|\sigma_n| - 1)^2\right| \le \max\{C, a_2\}\,||\sigma_n| - 1|\,(1 + ||\sigma_n| - 1|)\,,$$

which yields the upper bound in (2.9) since $||\sigma_n| - 1| < \alpha$ by (2.4). For the lower bound, we have for the same $C > 0$ as above that

$$|\epsilon_n| \ge |a_2(|\sigma_n| - 1)| - \left|C(|\sigma_n| - 1)^2\right| = ||\sigma_n| - 1|\,(|a_2| - C\,||\sigma_n| - 1|)\,.$$

and for sufficiently small α, we can make $|a_2| - C\,||\sigma_n| - 1|$ strictly positive, by (2.4). □

Again by making α small enough, we have from (2.8) and (2.7a) that

$$\log I(|\sigma_n|) = \sum_{m\in\mathbb{N}} \frac{(-1)^{m-1}}{m}\,(I(|\sigma_n|) - 1)^m = \sum_{m\in\mathbb{N}} \frac{(-1)^{m-1}}{m}\left(\epsilon_n - \frac{1}{2}\right)^m (|\sigma_n| - 1)^m. \tag{2.10}$$

Proposition 2.6. *If the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent, then $(|\sigma_n| - 1)_{n\in\mathbb{N}}$ is square-summable.*

Proof. If the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent, then by Theorem 2.1, (2.1) holds. Without loss of generality, we may assume that $|\sigma_n| \ne 1$ for all $n \in \mathbb{N}$, since otherwise the corresponding summand vanishes, by Lemma 2.2. Furthermore, (see Remark 2.3), we may also assume that (2.4) holds for some $0 < \alpha < 1$ that we can set as small as needed for (2.10) to hold and such that we can rewrite $\log|\sigma_n|$ as a Mercator series in $|\sigma_n| - 1$, for all $n \in \mathbb{N}$. Using these conditions, (2.3), (2.5), and (2.10), we obtain

$$-\log \mathbb{E}[\sqrt{\varphi_n(U_n, |\sigma_n|)}] = -\sum_{m\in\mathbb{N}} \frac{(-1)^{m-1}}{m}\left[\frac{1}{2} + \left(\epsilon_n - \frac{1}{2}\right)^m\right](|\sigma_n| - 1)^m. \tag{2.11}$$

The summand for $m = 1$ is, by (2.8) and (2.5),

$$-\epsilon_n(|\sigma_n| - 1) = (a_1(|\sigma_n| - 1) + 1 - I(|\sigma_n|)) = -a_2(|\sigma_n| - 1)^2 + C(|\sigma_n| - 1)^3,$$

where $C > 0$ may depend on α but not on $|\sigma_n|$; see the proof of Lemma 2.5. Expanding the coefficient of $(|\sigma_n| - 1)^2$ for the summand in (2.11) corresponding to $m = 2$, we obtain that its value is

$$\frac{1}{2}\left[\frac{1}{2} + \left(\epsilon_n^2 - \epsilon_n + \frac{1}{4}\right)\right] = \frac{3}{8} + \frac{1}{2}\left(\epsilon_n^2 - \epsilon_n\right).$$

By Lemma 2.5, $\epsilon_n^2 - \epsilon_n$ converges to zero linearly in $|\sigma_n| - 1$. Therefore, combining this observation with the preceding two equations, we obtain from (2.11) and (2.7b) that, for some C' independent of $|\sigma_n|$,

$$\begin{aligned} -\log \mathbb{E}[\sqrt{\varphi_n(U_n, |\sigma_n|)}] &= \left(-\frac{5}{16} + \frac{3}{8}\right)(|\sigma_n| - 1)^2 + C'((|\sigma_n| - 1)^3) \\ &= \frac{1}{16}(|\sigma_n| - 1)^2 + C'((|\sigma_n| - 1)^3). \end{aligned}$$

Therefore, by making α sufficiently small, we may ensure that for any $0 < \delta < 1/16$,

$$\frac{1}{(|\sigma_n|-1)^2}\left|-\log \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}] - \frac{1}{16}(|\sigma_n|-1)^2\right| = |C'|\,||\sigma_n|-1| < \delta.$$

Thus, for any $0 < \delta < 1/16$,

$$0 < \left(\frac{1}{16}-\delta\right)(|\sigma_n|-1)^2 < \frac{1}{16}(|\sigma_n|-1)^2 - \left|-\log \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}] - \frac{1}{16}(|\sigma_n|-1)^2\right|.$$

Since $-|x| < x$ if $x > 0$, and $-|x| = x$ if $x < 0$, it follows that the right-hand side of the inequality above is less than or equal to $-\log \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}]$. Summing over n yields

$$0 < \left(\frac{1}{16}-\delta\right)\sum_{n\in\mathbb{N}}(|\sigma_n|-1)^2 < \sum_{n\in\mathbb{N}} -\log \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}] < \infty,$$

and therefore the summability of Kakutani's series implies that $(|\sigma_n|-1)_{n\in\mathbb{N}}$ is square-summable. □

Proposition 2.7. *If $(|\sigma_n|-1)_{n\in\mathbb{N}}$ is square-summable, then the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent.*

Proof. Without loss of generality, we may assume that the ℓ^2 norm $\|(|\sigma_n|-1)_{n\in\mathbb{N}}\|_2$ of $(|\sigma_n|-1)_{n\in\mathbb{N}}$ is strictly less than 1. Then, for $1 < p \le q \le +\infty$, the ℓ^p-norm of $(|\sigma_n|-1)_{n\in\mathbb{N}}$ is larger than or equal to the ℓ^q-norm of $(|\sigma_n|-1)_{n\in\mathbb{N}}$. Using this fact, the sum formula for a geometric progression, and the hypothesis on the ℓ^2 norm of $(|\sigma_n|-1)_{n\in\mathbb{N}}$, we have

$$\sum_{m\ge 2}\sum_{n\in\mathbb{N}} ||\sigma_n|-1|^m \le \sum_{m\ge 2}\left(\sum_{n\in\mathbb{N}} ||\sigma_n|-1|^2\right)^{m/2} = \frac{\|(|\sigma_n|-1)_{n\in\mathbb{N}}\|_2^2}{1-\|(|\sigma_n|-1)_{n\in\mathbb{N}}\|_2^{1/2}} < \infty. \quad (2.12)$$

Since $(|\sigma_n|-1)_{n\in\mathbb{N}}$ converges to zero, we may choose α in (2.4) so small that, for ϵ_n defined in (2.4), it holds that $|\epsilon_n| < 1/8$ for all $n \in \mathbb{N}$, by Lemma 2.5. Since

$$\left|\frac{1}{2}+\left(\epsilon_n-\frac{1}{2}\right)^m\right| \le \begin{cases} |\epsilon_n| & m=1 \\ \frac{1}{2}+\left(|\epsilon_n|+\frac{1}{2}\right)^m & m\ge 2, \end{cases}$$

the condition that $|\epsilon_n| < 1/8$ implies that we may bound the left-hand side by 1 for all $m \in \mathbb{N}$. Using this observation and the triangle inequality, we may bound the right-hand side of (2.11) according to

$$-\sum_{m\in\mathbb{N}}\frac{(-1)^{m-1}}{m}\left[\frac{1}{2}+\left(\epsilon_n-\frac{1}{2}\right)^m\right](|\sigma_n|-1)^m \le \sum_{m\in\mathbb{N}} ||\sigma_n|-1|^m,$$

and hence from (2.11) it follows that $-\log \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}] \le \Sigma_{m\in\mathbb{N}}\,||\sigma_n|-1|^m$. Summing this inequality over n yields

$$\sum_{n\in\mathbb{N}} -\log \mathbb{E}[\sqrt{\varphi_n(U_n,|\sigma_n|)}] \le \sum_{n\in\mathbb{N}}\sum_{m\ge 2} ||\sigma_n|-1|^m,$$

where the right-hand side is finite, by changing the order of summation in (2.12). Therefore, (2.1) holds, and by Theorem 2.1, the laws of $(U_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are equivalent. □

Remark 2.8. An anonymous referee pointed out that one can express the function I defined in (2.5) in terms of complete elliptic integrals (see, e.g., [3, Section 3.152, Formula 1] and [3, Section 8.112, Formula 1]), and that the series representations of these elliptic integrals in [3, Section 8.113] may provide an alternative method for obtaining the results that we presented above.

References

[1] V. I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR-1642391

[2] V. I. Bogachev, *Differentiable measures and the Malliavin calculus*, Mathematical Surveys and Monographs, vol. 164, American Mathematical Society, Providence, RI, 2010. MR-2663405

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, eighth ed., Elsevier/Academic Press, Amsterdam, 2015, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010]. MR-3307944

[4] S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. (2) **49** (1948), 214–224. MR-0023331

[5] L. A. Shepp, *Distingunishing a sequence of random variables from a translate of itself*, Ann. Math. Statist. **36** (1965), 1107–1112. MR-0176509

[6] F. G. Tricomi and A. Erdélyi, *The asymptotic expansion of a ratio of gamma functions*, Pacific J. Math. **1** (1951), 133–142. MR-0043948

Acknowledgments. The authors are supported by the Freie Universität Berlin within the Excellence Initiative of the German Research Foundation. The authors thank the editors and referees who read the article for their helpful comments.