Corrections.

Page.	Line.	
2	2	For compact read complete.
19	1	For $a(m-n)$ read $a(m-n,x)$.
19	5	For $T^{m-n}(A)$ read $T^{-(m-n)}(A)$.
20	7	For "closure of its own interior" read "interior of its own closure".
25	-5	There is something missing from the argument here. In addition to the graph of φ being closed it is necessary that for every $x \in X$ there exists a neighbourhood U of x with $\overline{\varphi(U)}$ compact. Let C_i be a sequence of compact sets with $\bigcup_i C_i = G$; then $\overline{O}_a(x_o,0)$ can be expressed as the union of closed sets $\bigcup_i (\overline{O}_a(x_o,0) \cap (X \times C_i))$. It now follows from Baire's Theorem that U exists for some x; the minimality of T proves it for all x.
30	3	For $\pi_X(\overline{O}_a(y)) \cap \ldots$ read $\pi_X(\overline{O}_a(y) \cap \ldots$.
41	-2	For "uncountable collection of cocycles" read "uncountable collection of pairwise non-cohomologous cocycles".
71	-3	For x read $T^m(x)$.

NON-COMPACT EXTENSIONS

OF TRANSFORMATIONS

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SUMMARY

A skew-product extension of a transformation $T:X \to X$ by an abelian group G is a transformation $S:X \times G \to X \times G$ of the form $S(x,g) = (T(x), g + \psi(x))$, where $\psi:X \to G$ is a function. In the first part of this thesis X is a complete metric space, T is a minimal homeomorphism and ψ is a continuous function into a locally compact second countable abelian group. The orbit structure of S is studied with the help of an invariant from ergodic theory, the group of essential values or ratio set of the extending cocycle. Several types of possible orbit structure for S are described; the most interesting occurs when S is topologically transitive. In the special case where X is compact and G is Euclidean it is shown that for any given T there is a residual subset of functions which define topologically transitive extensions. Necessary and sufficient conditions for S to be topologically transitive are obtained for the special case where T is a translation on a torus. This generalises a theorem of Hedelund.

The second part of the thesis studies a collection of examples of measurable extensions by the reals. The space X is the unit circle and T is rotation through the angle $\exp(2\pi i\alpha)$ with $0<\alpha<1$. The function ψ is defined by

$$\psi(\exp(2\pi ix)) = \chi_{[0,\beta)}(x) - \beta$$

where $0 \leqslant x < 1$ and $0<\beta<1$. The set of pairs (α,β)

for which the resulting S is ergodic is proved to be a residual subset of the unit square which has Lebesgue measure one. The special case where α, β and 1 are integrally related is treated separately. Here the extensions are not ergodic but ergodic extensions by subgroups of the reals can be obtained from them.

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Declaration.

The material in Appendix A has been previously published in the form in which it appears here.

INTRODUCTION

The subject of this thesis is the orbit strucure of skew-product extensions of transformations by locally compact abelian groups. Skew-product extensions are also known as cylinder transformations or cylindrical cascades. Although the title mentions only the non-compact case, extensions by compact groups are not excluded. However, none of the results in this thesis amounts to anything new when applied to extensions by compact groups.

Apart from the first chapter the thesis falls naturally into two parts. (Chapter One contains the basic definitions which are common to both parts.) The first of these, consisting of Chapter Two, is about continuous extensions of minimal homeomorphisms. Most of its results could be reformulated to accomodate extensions of topologically transitive homeomorphisms. However, there is little to be gained from this increase in generality. It would merely complicate matters without introducing anything essentially new.

Most writers on continuous skew-products have concentrated on examples of topologically transitive extensions. An exception is the paper of Hedelund ([8]) which contains a complete analysis of real line extensions of minimal translations on the circle. Although these skew-products do not display the full range of possible orbit behavior, extensions of only moderately more complex transformations do so. It would therefore be sufficient to study only extensions of transformations on compact metric spaces.

However, much of our analysis holds for extensions of any minimal transformation on a compact metric space. Accordingly, we begin with this general case and specialise to compact spaces as nescessary.

The first section of Chapter Two contains the definitions of some notation which is used throughout the chapter. In the second section we consider the recurrence properties of continuous extensions. A skew-product extension is always either wholly conservative or wholly dissipative. The first of these possibilities is of greatest interest and for the rest of the chapter we concentrate upon it.

The third section sees the introduction of our main tool, the group of essential values of the extending cocycle. Essential values are defined in [15] for cocycles of a measure-class preserving action on a probability space. Our definition is derived from that of [15] by replacing sets of positive measure with open sets. In cases where both definitions are applicable the two groups may be quite different. This reflects the fact that, even for simple examples, the metric and topological properties of non-compact skew-products can be completely different.

The main results of the third section show how a knowledge of the essential values of the extending cocycle gives a description of most orbit closures under the extension. By "most orbit closures" we mean that the union of the collection of orbit closures to which the description applies is a residual set. The section ends with two examples. They display the two important varieties of orbit behavior which were not described by Hedelund in [8].

In the fourth section we consider only extensions by $\mathbb{R}^n$ of minimal transformations on compact metric spaces. This section is about the existence of topologically transitive extensions. Our first theorem shows that (with a suitable metric) the topologically transitive extensions form a residual subset of the set of all conservative extensions. For the other main result in this section we specialise still further and consider only extensions of a minimal translation on a torus. Here we prove a generalisation of the main theorem of [8] ; every non-trivial conservative $\mathbb{R}^n$-extension of a minimal translation is topologically transitive.

The final section of Chapter Two deals with the details of the orbit structure of extensions. It turns out that the description obtained from the essential values in Section Three is complete only in the most trivial cases. There are usually points with orbits which do not fit this description and they can make the orbit closure structure of an extension very complicated.

The second part of the thesis, which consists of Chapter Three and the appendices, is less general in scope than the first. Chapter Three is about a collection of examples of discontinuous skew-product extensions. They are conservative real line extensions of irrational rotations on the circle and the extending functions take only two values, on complementary intervals. These transformations preserve a natural measure and so provide an example for the theory of metric properties of extensions developed in [15]. They also have connections with the theory of uniform distribution of sequences (for details see [14]). The second section of Chapter Three contains a proof that almost all of these extensions are ergodic. The third section is concerned with a special case that arises naturally in the course of this proof. Here ergodic extensions are again obtained.

Appendix A is a reproduction of a short paper on measurable extensions of ergodic transformations. It contains a theorem which gives a condition for a real line extension to be recurrent. Since publication of this paper I have discovered that a proof of the theorem is inherent in the first lemma of [10], although it is not explicitly stated.

The final part of the thesis, Appendix B, is concerned with the proof of a lemma in Chapter Three. This lemma is not proved because its proof is a slight modification of the proof of a well known theorem. Instead, the details of the modification are given in this appendix.

Finally, a note about the numbering of results and statements: a number of the form a.b.c.d refers to the d'th numbered statement of the c'th result in Section b of Chapter a. Within Chapter a this would be shortened to b.c.d and within Result a.b.c it would be shortened to (d). The numbers of results are similarly shortened so that the c'th result in Section b of Chapter a is refered to as b.c within that chapter and a.b.c outside it.

CHAPTER ONE

Basic Definitions

This chapter contains the definitions and details of notation that are common to both parts of the thesis. The definitions have two alternative readings, one applying to continuous extensions of minimal homeomorphisms and the other applying to the measurable case. Where the two readings differ the text of the first is interrupted by the substitutions necessary to obtain the second. These interruptions are contained within double pointed brackets (≪,≫).

Definition 1.1.

Let (X,d_X) ≪$(X,\mathcal{S},\mu)$≫ be a complete uncountable metric space ≪nonatomic standard probability space≫. Let $T:X\to X$ be a minimal homeomorphism ≪measure preserving automorphism≫. Let $(G,+)$ be a locally compact, second countable, abelian group. A function $a:\mathbb{Z}\times X\to G$ is called a cocycle for T if it is continuous ≪measurable≫ and satisfies the cocycle equation;

$$a(n+m,x) = a(n,x) + a(m,T^n(x)) \qquad (n,m\in\mathbb{Z}, x\in X).$$

For any cocycle a let $a(1,.)$ denote the function whose value at x is $a(1,x)$. It is easy to see from the cocycle equation that the cocycle is completely determined by this function. Indeed if $\psi:X\to G$ is any continuous ≪measurable≫ function we can obtain a cocycle a by setting

$$a(n,x) = \begin{cases} \sum_{i=0}^{n-1} \psi(T^i(x)) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -a(-n, T^n(x)) & \text{for } n < 0. \end{cases}$$

<u>Definition 1.2.</u>

Let $a:\mathbb{Z}\times X\to G$ be a cocycle for $T:X\to X$. The cocycle a is called a coboundary if there exists a continuous <<measurable>> function $\varphi:X\to G$ such that

$$a(n,x) = \varphi(T^n(x)) - \varphi(x) \qquad (n\in\mathbb{Z}, x\in X).$$

In this case a is said to be the coboundary of φ. Similarly, a function $\psi:X\to G$ is said to be a coboundary if the unique cocycle (for T) $a:\mathbb{Z}\times X\to G$ with $a(1,.) = \psi$ is a coboundary. Two cocycles or two functions, whose (pointwise) difference is a coboundary, are called cohomologous.

The nomenclature of Definitions 1.1 and 1.2 arises from the fact that cocycles and coboundaries, as defined here, belong to a cohomology theory. A cocycle, modulo the coboundaries, is an element of the first cohomology group of the integers with coefficients in a certain $\mathbb{Z}$-module. This is the module of continuous <<measurable>> functions $X\to G$, withthe module action induced from the action of the powers of T.

<u>Definition 1.3.</u>

Let $a:\mathbb{Z}\times X\to G$ be a cocycle for $T:X\to X$. Let $Y = X\times G$. The <u>skew-product extension of T by G</u> (or, briefly, <u>G-extension of T</u>) defined by a is the transformation $S_a:Y\to Y$ with

$$S_a(x,g) = (T(x), g + a(1,x)) \qquad (x\in X, g\in G).$$

The cocycle a is called the <u>extending cocycle</u> for S_a and the function $a(1,.)$ is called the <u>extending function</u>. X and T are known as the <u>base space</u> and <u>base transformation</u> respectively.

An extension is clearly a continuous <<measurable>> transformation. If T preserves a measure μ on X then S_a preserves the product measure $\mu\times\lambda$ on Y, where λ is the Haar measure of G.

The reason for using a cocycle rather than just the extending function in Definition 1.3 is that the powers of S_a can be expressed by the cocycle;

$$S_a^n(x,g) = (T^n(x), g + a(n,x)) \qquad (n\in\mathbb{Z},\ x\in X,\ g\in G).$$

The importance of coboundaries in the study of extensions arises from the following lemma.

<u>Lemma 1.4</u>.

Let a and b; $\mathbb{Z}\times X\to G$ be cocycles for $T:X\to X$. Suppose that b is the coboundary of a continuous <<measurable>> function $\varphi:X\to G$. Let a+b be the cocycle which is the pointwise sum of a and b. Let $U:Y\to Y$ be the continuous <<measurable>> transformation with

$$U(x,g) = (x, g + \varphi(x)) \qquad (x\in X,\ g\in G).$$

Then $US_a = S_{a+b}U$.

<u>Proof</u>. For all $x\in X$ and $g\in G$,

$$US_a(x,g) = U(T(x), g + a(1,x))$$

$$\begin{aligned} &= (T(x), g + a(1,x) + \varphi(T(x))); \\ S_{a+b}U(x,g) &= S_{a+b}(x, g + \varphi(x)) \\ &= (T(x), g + \varphi(x) + a(1,x) + b(1,x)) \\ &= (T(x), g + a(1,x) + \varphi(T(x))). \end{aligned}$$

The special case of Lemma 1.4 where a is the zero cocycle is most important. It shows that if b is the coboundary of φ then every orbit under S_b is contained in some translate of the graph of φ. The orbit structure of S_b is then similar to that of the trivial product of T and the identity transformation on G.

Before making the next definition we note that there is no loss of generality in assuming that G is metrisable. From here on we shall assume that the topology of G is derived from a symmetric invariant metric d_G.

The second readings of the next two definitions are specialised forms of Definitions 3.1 and 3.13 of [15], which apply to more general actions.

Definition 1.5.

Let $a:\mathbb{Z}\times X\to G$ be a cocycle for $T:X\to X$. An element g of G is an essential value of a if, for every $\varepsilon > 0$ and every non-empty open set $A\subset X$ <<every measurable set $A\subset X$ with $\mu(A) > 0$ >>, there exists an $n\in\mathbb{Z}$ such that

$$A \cap T^{-n}(A) \cap \{x: d_G(a(n,x),g) < \varepsilon\} \neq \emptyset$$

$$<<\mu(A \cap T^{-n}(A) \cap \{x: d_G(a(n,x),g) < \varepsilon\}) > 0 >>.$$

Let ∞ be the extra point in the one point compactification of G. It is an essential value if for every compact set

$C\subset G$ and every non-empty open set $A\subset X$ $\ll$every measurable set $A\subset X$ with $\mu(A) > 0$ $\gg$, there exists an $n\in\mathbb{Z}$ such that

$$A \cap T^{-n}(A) \cap \{x\colon a(n,x)\notin C\} \neq \emptyset$$

$$\ll \mu(A \cap T^{-n}(A) \cap \{x\colon a(n,x)\notin C\}) > 0 \gg.$$

The set of all essential values of a is denoted by $\bar{E}(a)$. We define: $E(a) = \bar{E}(a) \cap G$.

Definition 1.6.

Let $a\colon\mathbb{Z}\times X\to G$ be a cocycle for $T\colon X\to X$. The cocycle a is called <u>recurrent</u> if, for every $\varepsilon > 0$ and every non-empty open set $A\subset X$ $\ll$every measurable set $A\subset X$ with $\mu(A) > 0$ $\gg$, there exists an $n\in\mathbb{Z}$, $n\neq 0$, such that

$$A \cap T^{-n}(A) \cap \{x\colon d_G(a(n,x),0) < \varepsilon\} \neq \emptyset$$

$$\ll \mu(A \cap T^{-n}(A) \cap \{x\colon d_G(a(n,x),0) < \varepsilon\}) > 0 \gg.$$

A cocycle which is not recurrent is called <u>transient</u>.

Finally, we fix the meaning of some standard notation. The symbols X,G,T,Y,a and G_a will always have the meanings of Definitions 1.1 - 1.3. The symbols π_X and π_G stand for the first and second coordinate projections of Y:

$$\pi_X\colon Y\to X,\ \pi_X(x,g) = x;$$

$$\pi_G\colon Y\to G,\ \pi_G(x,g) = g.$$

For each $h\in G$ the translation on the second coordinate in Y is denoted by L_h;

$$L_h\colon Y\to Y,\ L_h(x,g) = (x,g+h).$$

The norm in $\mathbb{R}^n$ is denoted by single bars $(|.|)$ rather than the usual double bars $(\|.\|)$. This is because double bars are used for another purpose in Chapter Three. The supremum norm in $\mathbb{R}^n$ is used throughout;

$$|(v_1, \ldots, v_n)| = \sup_{1 \leq i \leq n} |v_i|.$$

Finally, we give the usual meanings to the symbols $\mathbb{Z}_+$, $\mathbb{Z}_-$, $\mathbb{R}_+$, $\mathbb{R}_-$; so $\mathbb{Z}_+ = \{n \in \mathbb{Z}: n > 0\}$ etcetera.

CHAPTER TWO

Continuous Extensions of Minimal Homeomorphisms

§1. Introduction and notation.

In this chapter we explore the consequences for continuous extensions of the definitions of Chapter One. The first reading of Chapter One applies throughout and except at the end of §1 all functions and transformations appearing here are continuous.

We now introduce some notation which is used throughout this chapter but not outside it. As X is a complete metric space it is possible to introduce a complete metric on Y;

$$d_Y((x,g),(x',g')) = \text{Sup}\ \{d_X(x,x'),\ d_G(g,g')\}.$$

The open ball of radius ε about a point $y \in Y$ is denoted by $B_Y(y,\varepsilon)$; the meanings of $B_X(x,\varepsilon)$ and $B_G(g,\varepsilon)$ are analagous.

If $a:\mathbb{Z}\times X\to G$ is a cocycle for $T:X\to X$ then the orbit of a point $y\in Y$ under the extension S_a is written $O_a(y)$. The orbit closure is then $\overline{O}_a(y)$ and $O_a^+(y)$, $\overline{O_a^-}(y)$ denote the forward orbit and the closure of the backward orbit respectively.

We conclude by making a remark which is used implicitly at one or two places in this chapter. Nothing of importance is lost if we replace the group G by the closure of the subgroup generated by $\{a(1,x):\ x\in X\}$. When X is compact this means that we may assume that G is separable and compactly generated. The structure theorem for compactly generated, locally compact, abelian groups shows that such a group may be expressed as a product; $G = \mathbb{R}^n\times\mathbb{Z}^m\times C$ $(n,m \geqslant 0)$, where C is a compact group.

§2. Recurrence.

Definition 2.1.

Let Z be a topological space and let $S:Z\to Z$ be a homeomorphism. The S-wandering set, $W(S)$, is the set of points $z\in Z$ for which there exists a neighbourhood U, containing z, such that $U\cap S^n(U)=\emptyset$ for all $n\in\mathbb{Z}$, $n\neq 0$. The S-recurrent set is the set,

$$R(S)=\{z\in Z:\ z\in\overline{\{S^n(z):\ n>0\}}\cap\overline{\{S^n(z):\ n<0\}}\}.$$

The transformation S is called conservative if $W(S)=\emptyset$ and dissipative if $W(S)=Z$.

Lemma 2.2.

Let Z be a complete metric space and let $S:Z\to Z$ be a homeomorphism. Then $R(S)$ and $W(S)$ are invariant under S, $W(S)$ is open and $W(S)\cup R(S)$ is a residual set.

Proof. See Gottschalk and Hedelund ([6]), Theorem 7.24.

Definition 2.1 is standard in Topological Dynamics. The next result gives the connection between the recurrence properties of an extension S_a and the recurrence of the extending cocycle, as defined in 1.1.6.

Proposition 2.3.

Let $T:X\to X$ be a minimal homeomorphism. Let $a:\mathbb{Z}\times X\to G$ be a cocycle for T. Then one of the following statements is true:

(1) The extension S_a is conservative and a is recurrent;

(2) The extension S_a is dissipative and a is transient.

Also, the sets $R(S_a)$, $W(S_a)$ are invariant under S_a and all the translations L_h, $h \in G$.

Proof. Suppose that $W(S_a) \neq \emptyset$; then for some $y \in Y$ and $\varepsilon > 0$, $B_Y(y,\varepsilon) \cap S_a^n(B_Y(y,\varepsilon)) = \emptyset$ for all $n \in \mathbb{Z}$, $n \neq 0$. As all the translations L_h, $(h \in G)$ commute with S_a this implies that

$$S_a^n L_h(B_Y(y,\varepsilon)) \cap S_a^m L_h(B_Y(y,\varepsilon)) = \emptyset \quad (n,m \in \mathbb{Z}, n \neq m, h \in G).$$

But as T is minimal,

$$\bigcup_{n \in \mathbb{Z}} \bigcup_{h \in G} S_a^n L_h(B_Y(y,\varepsilon)) = Y.$$

So $W(S_a) = Y$.

We have shown that if $W(S_a)$ is non-empty, then it is all of Y. To complete the proof we will show that this happens if and only if the cocycle a is transient.

If a is transient, then for some non-empty open set $A \subset X$ and some $\varepsilon > 0$,

$$\pi_X((A \times B_G(0,\varepsilon)) \cap S_a^n(A \times B_G(0,\varepsilon)))$$

$$= T^n(A \cap T^{-n}(A) \cap \{x\colon a(n,x) \in B_G(0,\varepsilon)\}) = \emptyset,$$

for all $n \in \mathbb{Z}$, $n \neq 0$. So $W(S_a) \neq \emptyset$.

Conversely, if $W(S_a) \neq \emptyset$ then it contains an open ball, $B_Y(y,\varepsilon)$. Let $\pi_X(y) = x$, then

$$B_X(x,\varepsilon) \cap T^{-n}(B_X(x,\varepsilon)) \cap \{x\colon a(n,x) \in B_G(0,\varepsilon)\} = \emptyset$$

for all $n \in \mathbb{Z}$, $n \neq 0$ and a is transient.

The final statement of the Lemma is obvious.

Corollary 2.4.

Assume, in addition to the hypotheses of Lemma 2.3, that X is compact. Then Statement (2) of 2.3 holds if and only if , for every compact set $C \subset G$ and every $x \in X$, the set $\{n \in \mathbb{Z}: a(n,x) \in C\}$ is finite.

Proof. If, for some $x \in X$ and some compact set $C \subset G$, the set $\{n: a(n,x) \in C\}$ is infinite then $O_a(x,0) \cap (X \times C)$ has an accumulation point $y \in Y$. For any neighbourhood U of y the set $\{n: U \cap S_a^n(U) \neq \emptyset\}$ is infinite. Therefore $y \notin W(S_a)$ and Statement 2.3.1 holds.

Conversely, suppose that Statement 2.3.1 is true. Then Lemma 2.2 shows that $R(S_a) \neq \emptyset$. As $R(S_a)$ is invariant under all the translations L_h we can choose an $x \in X$ such that $(x,0) \in R(S_a)$. This implies that if C is any compact neighbourhood of the identity in G then $\{n: a(n,x) \in C\}$ is infinite.

Since Statements 2.3.1 and 2.3.2 are mutually exclusive, the corollary is proved.

We now restrict our attention to the special case where X is compact and G is a closed subgroup of $\mathbb{R}$. In this case the theory of ergodic sets makes it possible to determine whether a cocycle a is recurrent in terms of the integrals of $a(1,.)$.

Lemma 2.5.

Let X be a compact metric space and let $a: \mathbb{Z} \times X \to \mathbb{R}$ be a cocycle for $T: X \to X$. Suppose there exists a sequence of positive integers (n_i) such that $n_i^{-1}|a(n_i,x)| \to 0$, for some $x \in X$. Then there exists a T-invariant Borel probability measure μ on X such that $\int a(1,.)\, d\mu = 0$.

Proof. It is a standard result in ergodic theory that if X is a compact Hausdorff space and $T:X\to X$ is a minimal homeomorphism then there exists a T-invariant Borel probability measure on X. The usual proof of this theorem (see Oxtoby, [11], Theorem 2.1) shows that for any point $x\in X$ there exists a sequence of positive integers (n_i') and a T-invariant Borel probability measure μ such that

$$(1/n_i')\sum_{j=0}^{n_i'-1}\varphi(T^j(x)) \to \int\varphi\, d\mu,$$

for every continuous function $\varphi:X\to R$. This proof is still valid if (n_i') is chosen as a subsequence of (n_i). he conclusion follows on setting $\varphi = a(1,.)$.

Lemma 2.6.

Let X be a compact metric space with $T:X\to X$ a minimal homeomorphism. Let $a:\mathbb{Z}\times X\to R$ be a cocycle for T. Suppose that μ is a T-invariant Borel probability measure on X and that T is ergodic with resect to μ. Then if $\int a(1,.)\, d\mu = 0$ a is recurrent.

Proof. The theorem in Appendix A shows that a is recurrent in the sense of the second reading of Definition 1.1.6. It is therefore only necessary to show that the second reading implies the first. That is so because the support of μ is necessarily a closed T-invariant subset of X. As T is minimal it must be all of X; but then $\mu(A) > 0$ for every non-empty open set $A\subset X$.

Lemma 2.7.

Let X be a compact metric space, with $T:X\to X$ a minimal homeomorphism. Let $a:\mathbb{Z}\times X\to\mathbb{R}$ be a cocycle for T. For each $r\in\mathbb{R}$ let $a^{(r)}$ be the cocycle with $a^{(r)}(n,x) = a(n,x)+nr$. Then the set $I = \{r\in\mathbb{R}: a^{(r)}$ is recurrent$\}$ is an interval.

Proof. Suppose that $r,t\in I$ and $r<s<t$. We will show that $s\in I$. Proposition 2.3 shows that the extensions $S_a(r)$ and $S_a(t)$ are conservative. By using Lemma 2.2 and the fact that the sets, $R(S_a(r))$ and $R(S_a(t))$ are invariant under all the tranlations L_h, we may choose a point $x\in X$ such that $(x,0)\in R(S_a(r))\cap R(S_a(t))$. Then there exist increasing sequences (n_i) and (m_i) such that $a^{(r)}(n_i,x)\to 0$ and $a^{(t)}(m_i,x)\to 0$. Because $a^{(s)}(n_i,x)-a^{(r)}(n_i,x)\to\infty$ and $a^{(s)}(m_i,x)-a^{(t)}(m_i,x)\to -\infty$, $a^{(s)}(n,x)$ must be positive and negative for infinitely many $n>0$. So if $M=\sup_{x'\in X}|a^{(s)}(1,x')|$ then the set $\{n>0: |a^{(s)}(n,x)|\leq M\}$ must be infinite. Corollary 2.4 shows that $a^{(s)}$ is recurrent.

Theorem 2.8

Let X be a compact metric space and let $T:X\to X$ be a minimal homeomorphism. Let $a:\mathbb{Z}\times X\to\mathbb{R}$ be a cocycle for T. Then the following statements are equivalent:

(1) a is recurrent;

(2) For some $x\in X$, $\operatorname{Lim}_n \operatorname{Inf} n^{-1}|a(n,x)| = 0$;

(3) There exists a T-invariant Borel probability measure μ on X such that $\int a(1,.)\,d\mu = 0$.

Proof. Suppose (1) is true; then Proposition 2.3 shows that $R(S_a)\neq\emptyset$. Hence (2) is satisfied. If (2) holds then Lemma 2.5 proves (3).

Now suppose that (3) is true. Lemma 2.6 of [11] shows that μ can be expressed as an integral of T-invariant ergodic Borel probability measures. (An ergodic measure for T is a measure with respect to which T is ergodic.) It follows that either there exists a T-invariant ergodic probability measure μ_0 (possibly identical to μ) such that $\int a(1,.)\,d\mu_0 = 0$, or there exist two such measures μ_1, μ_2 with

$$r = \int a(1,.)\,d\mu_1 < 0 < \int a(1,.)\,d\mu_2 = t.$$

In the first case Lemma 2.6 shows that a is recurrent. In the second case it shows that $a^{(-r)}$ and $a^{(-t)}$ are recurrent. Statement (1) then follows from Lemma 2.7.

§3. Essential values and extensions.

This section is concerned with the relationship between the essential values of a cocycle and the properties of the extension which it defines. If a is a transient cocycle then it is clear that $\infty \in \overline{E}(a)$. Corollary 3.9 will show that in fact $\overline{E}(a) = \{0,\infty\}$.

When a is a recurrent cocycle a knowledge of $\overline{E}(a)$ and the essential values of a related cocycle, $\tilde{a}$, yields a description of "most" orbit closures under S_a. This description is given by Theorem 3.7, its corollaries and Propositions 3.10, 3.15 and 3.16.

We begin by establishing some elementary properties of the set $E(a)$.

Proposition 3.1

Let $T: X \to X$ be a minimal transformation and let $a: \mathbb{Z} \times X \to G$ be a cocycle for T. Then $E(a)$ is a closed subgroup of G.

Proof. It is clear that $E(a)$ is closed. Suppose that $g, h \in E(a)$; we shall show that $(g-h) \in E(a)$.

Given any $\varepsilon > 0$ and any non-empty open set $A \subset X$, let $n \in \mathbb{Z}$ be such that

$$B = A \cap T^{-n}(A) \cap \{x: a(n,x) \in B_G(h, \varepsilon/2)\} \neq \emptyset.$$

As B is an open set there exists an $m \in \mathbb{Z}$ such that

$$C = B \cap T^{-m}(B) \cap \{x: a(m,x) \in B_G(g, \varepsilon/2)\} \neq \emptyset.$$

If $x \in T^n(C)$ then because $T^{-n}(x) \in C \subset B$,

$$
\begin{aligned}
a(m-n) &= a(m,T^{-n}(x)) + a(-n,x) \\
&= a(m,T^{-n}(x)) - a(n,T^{-n}(x)) \\
&\in B_G(g,\varepsilon/2) - B_G(h,\varepsilon/2) = B_G(g-h,\varepsilon).
\end{aligned}
$$

Also $T^{m-n}(x) \in T^{m}(C) \subset A$. Therefore,

$$x \in (A \cap T^{m-n}(A) \cap \{x:\ a(m-n,x) \in B_G(g-h,\varepsilon)\}) \neq \emptyset.$$

Proposition 3.2.

Let a and $b: \mathbb{Z} \times X \to G$ be cocycles for a minimal transformation $T: X \to X$. Suppose that b is a coboundary; then $\overline{E}(a) = \overline{E}(a+b)$.

Proof. As $-b$ is also a coboundary, it is only necessary to prove that $\overline{E}(a) \subset \overline{E}(a+b)$. Let $\varphi: X \to G$ be a continuous function with $b(n,x) = \varphi(T^n(x)) - \varphi(x)$ for all $n \in \mathbb{Z}$ and $x \in X$. For any $\varepsilon > 0$ and any non-empty open set $A \subset X$ there exists a non-empty open subset B of A such that, for all $x, x' \in B$, $d_G(\varphi(x), \varphi(x')) < \varepsilon/2$. Whenever $n \in \mathbb{Z}$ and $x \in X$ are such that $x \in B \cap T^{-n}(B)$ we have:

$$d_G(b(n,x),0) = d_G(\varphi(x), \varphi T^n(x)) < \varepsilon/2.$$

Hence, for each $g \in E(a)$ there exists $n \in \mathbb{Z}$ such that

$$
\begin{aligned}
&A \cap T^{-n}(A) \cap \{x:\ (a+b)(n,x) \in B_G(g,\varepsilon)\} \\
&\supset B \cap T^{-n}(B) \cap \{x:\ a(n,x) \in B_G(g,\varepsilon/2)\} \neq \emptyset.
\end{aligned}
$$

So $E(a) \subset E(a+b)$.

If $\infty \notin E(a+b)$ then there exists a compact set $C \subset G$ and a non-empty open set $A \subset X$ such that $(a+b)(n,x) \in C$ whenever $x \in A \cap T^{-n}(A)$. Fix $\varepsilon > 0$ so that $\overline{B_G(0,\varepsilon)}$ is compact and choose $B \subset A$ as above. Then $C + \overline{B_G(0,\varepsilon)}$ is a compact subset of G and

$$a(n,x) = (a+b)(n,x) - \varphi(T^n(x)) + \varphi(x)$$
$$\in C + \overline{B_G(0,\varepsilon)}$$

whenever $x \in B \cap T^{-n}(B)$. It follows that $\infty \notin \overline{E}(a)$. So $\overline{E}(a) \subset \overline{E}(a+b)$ and the proof is complete.

Definition 3.3. Let Z be a topological space and let $S: Z \to Z$ be a homeomorphism. A subset U of Z is called S-regular if it is the closure of its own interior and satisfies: $S(U) = U$.

Lemma 3.4.

Let $a: \mathbb{Z} \times X \to G$ be a cocycle for $T: X \to X$. Let S_a be the the G-extension defined by T and a. Let $\mathcal{U}$ be the collection of all S_a-regular sets. Then

$$E(a) = \{g: L_g(U) = U,\ U \in \mathcal{U}\}.$$

Proof. Suppose that $g \in E(a)$ and $U \in \mathcal{U}$. Because $E(a)$ is a group it is only necessary to prove that $L_g(U) \subset U$. For every $y \in U$, there exists an $\varepsilon > 0$ such that $B_Y(y,\varepsilon) \subset U$. Let $y = (x,h)$ and consider the set

$$D = \bigcup_{n=-\infty}^{\infty} (B_X(x,\varepsilon) \cap T^{-n}(B_X(x,\varepsilon)) \cap \{x: a(n,x) \in B_G(-g,\varepsilon/2)\}).$$

This is a dense subset of $B_X(x,\varepsilon)$ because $g \in E(a)$. (If the complement of D in $B_X(x,\varepsilon)$ contained an open set A then we would have

$$A \cap T^{-n}(A) \cap \{x: a(n,x) \in B_G(-g,\varepsilon/2)\} = \emptyset$$

for all $n \in \mathbb{Z}$.) Because D is dense,

$$L_g(B_Y(y,\varepsilon/2)) \subset \overline{\bigcup_{n=-\infty}^{\infty} S_a^n(B_Y(y,\varepsilon))} \subset \overline{U};$$

but $L_g(B_Y(y,\varepsilon/2))$ is open and so lies in the interior, U of $\overline{U}$. In particular, $L_g(y)\in U$. This argument applies to all $y\in U$; so we have $L_g(U)\subset U$ as required.

Conversely, suppose that $L_g(U)\subset U$ for every $U\in\mathcal{U}$. For any $\varepsilon>0$ and non-empty open set $A\subset X$, the interior of

$$\overline{\bigcup_{n=-\infty}^{\infty} S_a^n(A\times B_G(0,\varepsilon/2))}$$

is an S_a-regular set. So for some $n\in\mathbb{Z}$,

$$(A\times B_G(g,\varepsilon/2))\cap S_a^n(B_G(0,\varepsilon/2))\neq\emptyset.$$

Equivalently,

$$A\cap T^{-n}(A)\cap\{x:\ a(n,x)\in B_G(g,\varepsilon)\}\neq\emptyset.$$

<u>Corollary 3.5</u>.

Let $a:\mathbb{Z}\times X\to G$ be a cocycle for $T:X\to X$. Let $g\in G$. Suppose that $L_g(U)\subset U$, for every S_a-regular set U; then $g\in E(a)$.

<u>Lemma 3.6</u>.

Let $a:\mathbb{Z}\times X\to G$ be a cocycle for a minimal transformation $T:X\to X$. Let S_a be the corresponding G-extension of T. Suppose that for some $x,x'\in X$ and $g,h\in G$ the points (x',h) and $(x',h+g)$ both lie in $\overline{O}_a(x,0)$. Then $g\in E(a)$.

<u>Proof</u>. Let U be any S_a-regular set; we shall show that $L_g(U)\subset U$. Let (m_j) and (m_j') be sequences with $S^{m_j}(x,0)\to(x,h)$ and $S_a^{m_j'}(x,0)\to(x,g+h)$. For each $(w,f)\in U$, choose a sequence (n_i) such that $T^{n_i}(x')\to w$ and $(T^{n_i}(x'),f)\in U$ for all $i\geqslant 1$. Then for every $i\geqslant 1$

$$S_a^{n_i+m_j}(x,f-h-a(n_i,x'))\to(T^{n_i}(x'),f)$$

as $j\to\infty$.

Because U is open and S_a-invariant this implies that for each $i \geqslant 1$, $(x, f-h-a(n_i, x')) \in U$. So

$$L_g(w,f) = (w,f+g) = \lim_i \lim_j S_a^{m_j' + n_i}(x, f-h-a(n_i,x')) \in \overline{U}.$$

This argument applies to all $(w,f) \in U$; so $L_g(U) \subset \overline{U}$. But $L_g(U)$ is open and so lies in the interior, U of $\overline{U}$. We have $L_g(U) \subset U$ for all $U \in \mathcal{U}$ and the conclusion follows from Corollary 3.6.

Theorem 3.7.

Let X be a complete metric space and let $T: X \to X$ be a minimal homeomorphism. Suppose that $a: \mathbb{Z} \times X \to G$ is a cocycle for T. Then the following statements are equivalent:

(1) g is an essential value of a;

(2) The set $\{y \in Y: L_g(y) \in \overline{O}_a(y)\}$ is a residual subset of Y which is invariant under S_a and the group of translations $\{L_h: h \in G\}$.

(3) There exists $y \in Y$ with $L_g(y) \in \overline{O}_a(y)$.

Proof. Suppose that Statement (1) holds. Then for every $x \in X$ and $\varepsilon > 0$ there exists an $n \in \mathbb{Z}$ such that

$$B_X(x, \varepsilon/2) \cap T^{-n}(B_X(x, \varepsilon/2)) \cap \{x: a(n,x) \in B_G(g, \varepsilon)\} \neq \emptyset.$$

In terms of S_a, this means that if

$$P_\varepsilon = \{x \in X: \inf_{n \in \mathbb{Z}} d_Y(S_a^n(x,0), (x,g)) < \varepsilon\},$$

then $P_\varepsilon \cap B_X(x, \varepsilon/2) \neq \emptyset$. It follows that for every $\varepsilon > 0$ P_ε is an open dense subset of X. Let

$$P = \bigcap_{i=1}^{\infty} P_{i^{-1}} = \{x: (x,g) \in \overline{O}_a(x,0)\}.$$

P is a residual subset of X. Clearly

$$\{y: L_g(y) \in \overline{O}_a(y)\} = \{(x,h): x \in P, h \in G\};$$

so Statement (2) holds.

The implication from Statement (2) to (3) is trivial. That Statement (1) follows from (3) is a direct consequence of Lemma 3.6.

Corollary 3.8.

Let X be a complete metric space and let $T:X\to X$ be a minimal homeomorphism. Suppose that $a:\mathbb{Z}\times X\to G$ is a cocycle for T. Then the set $\{y: L_g(\overline{O}_a(y)) = \overline{O}_a(y),\ g\in E(a)\}$ is a residual subset of Y which is invariant under S_a and the group of translations $\{L_h: h\in G\}$.

Proof. Because G is second countable, it is separable. Therefore $E(a)$ is separable; let $\{g_i: i\in\mathbb{Z}\}$ be a countable dense subset of $E(a)$ with $g_{-i}=-g_i$ $(i\in\mathbb{Z})$. For any $g\in G$ and $y\in Y$, we have $L_g(\overline{O}_a(y))=\overline{O}_a(y)$ if and only if $L_g(y).L_{-g}(y)\in\overline{O}_a(y)$. So

$$\{y: L_g(\overline{O}_a(y))=\overline{O}_a(y), g\in E(a)\} = \bigcap_{i=-\infty}^{\infty}\{y: L_{g_i}(\overline{O}_a(y))=\overline{O}_a(y)\}$$
$$= \bigcap_{i=-\infty}^{\infty}\{y: L_{g_i}(y)\in\overline{O}_a(y)\}.$$

The conclusion now follows from Statement 3.7.2 and Baire's Theorem.

Corollary 3.9.

Let $a:\mathbb{Z}\times X\to G$ be a cocycle for a minimal transformation $T:X\to X$. Suppose that $E(a)\neq\{0\}$; then a is recurrent.

Proof. Let $g\in E(a)$, $g\neq 0$; then for some $y\in Y$, $L_g(y)\in\overline{O}_a(y)$.

so Statement (2) holds.

The implication from Statement (2) to (3) is trivial. That Statement (1) follows from (3) is a direct consequence of Lemma 3.6.

Corollary 3.8.

Let X be a complete metric space and let $T: X \to X$ be a minimal homeomorphism. Suppose that $a: \mathbb{Z} \times X \to G$ is a cocycle for T. Then the set $\{y: L_g(\overline{O}_a(y)) = \overline{O}_a(y),\ g \in E(a)\}$ is a residual subset of Y which is invariant under S_a and the group of translations $\{L_h: h \in G\}$.

Proof. Because G is second countable, it is separable. Therefore $E(a)$ is separable; let $\{g_i: i \in \mathbb{Z}\}$ be a countable dense subset of $E(a)$ with $g_{-i} = -g_i$ $(i \in \mathbb{Z})$. For any $g \in G$ and $y \in Y$, we have $L_g(\overline{O}_a(y)) = \overline{O}_a(y)$ if and only if $L_g(y), L_{-g}(y) \in \overline{O}_a(y)$. So

$$\begin{aligned}\{y: L_g(\overline{O}_a(y)) = \overline{O}_a(y), g \in E(a)\} &= \bigcap_{i=-\infty}^{\infty} \{y: L_{g_i}(\overline{O}_a(y)) = \overline{O}_a(y)\} \\ &= \bigcap_{i=-\infty}^{\infty} \{y: L_{g_i}(y) \in \overline{O}_a(y)\}.\end{aligned}$$

The conclusion now follows from Statement 3.7.2 and Baire's Theorem.

Corollary 3.9.

Let $a: \mathbb{Z} \times X \to G$ be a cocycle for a minimal transformation $T: X \to X$. Suppose that $E(a) \neq \{0\}$; then a is recurrent.

Proof. Let $g \in E(a)$, $g \neq 0$; then for some $y \in Y$, $L_g(y) \in \overline{O}_a(y)$.

This means that there exists a sequence (n_i) such that $S_a^{n_i}(y) \to L_g(y)$. So, if U is any neighbourhood of $L_g(y)$, the set $\{n: S_a^n(U) \cap U \neq \emptyset\}$ is infinite. This shows that $L_g(y) \notin W(S_a)$ and the conclusion follows from Proposition 2.3.

Corollary 3.10.

Let X be a complete metric space and let $T: X \to X$ be a minimal homeomorphism. Suppose $a: \mathbb{Z} \times X \to G$ is a cocycle for T; let S_a be the corresponding G-extension of T. Then S_a is topologically transitive if and only if $E(a) = G$.

Proof. If S_a is topologically transitive then for some $y \in Y$, $\overline{O}_a(y) = Y$. Theorem 3.7 shows that $E(a) = G$.

Conversely, suppose $E(a) = G$. Corollary 3.8 shows that for some $y \in Y$, $\{L_g(y): g \in G\} \subset \overline{O}_a(y)$. The fact that T is minimal implies that $\overline{O}_a(y) = Y$.

Theorem 3.7 and Corollary 3.8 do not rule out the possibility that for some $g \in E(a)$ and $y \in Y$, $L_g(y) \in \overline{O}_a(y)$ but $L_{-g}(y) \notin \overline{O}_a(y)$. We shall give an example for which this occurs in §5. When G is not compact the existence of this and similar phenomena makes the statements of 3.7 and 3.8 the best possible.

The next result shows how coboundaries are characterised by their essential values. Its two corollaries will be useful in §4 and §5.

Proposition 3.11.

Let X be a complete metric space and let $T:X\to X$ be a minimal homeomorphism. Suppose that $a:\mathbb{Z}\times X\to G$ is a cocycle for T; then a is a coboundary if and only if $\overline{E}(a)=\{0\}$.

Proof. Proposition 3.2 shows that if a is a coboundary then $\overline{E}(a)=\overline{E}(0)=\{0\}$.

Conversely, suppose that $\overline{E}(a)=\{0\}$. In particular, $\infty\notin\overline{E}(a)$ so there exists a compact set $C\subset G$ and a non-empty open set $A\subset X$ such that $a(n,x)\in C$ whenever $x\in A\cap T^{-n}(A)$. Fix a point $x_0\in A$. For every $x\in A$ there is a sequence (n_i) such that $T^{n_i}(x_0)\to x$ with $T^{n_i}(x_0)\in A$ for all $i\geqslant 1$. This means that for all $i\geqslant 1$, $a(n_i,x_0)\in C$. We may assume, by replacing (n_i) with a subsequence if necessary, that $a(n_i,x_0)$ and $S_a^{n_i}(x_0,0)$ converge.

The above argument applies to every $x\in A$, so we have shown that $A\subset\pi_X(\overline{O}_a(x_0,0))$. Clearly $\pi_X(\overline{O}_a(x_0,0))$ is a T-invariant set. As it contains a non-empty open set it must be all of X.

So for each $x\in X$ there exists $\varphi(x)$ such that $(x,\varphi(x))\in\overline{O}_a(x_0)$. Lemma 3.6 shows that each $\varphi(x)$ is unique. This means that $\overline{O}_a(x_0,0)$ is the graph of a function $\varphi:X\to G$. The graph of φ is closed, so the function must be continuous. Also because $\overline{O}_a(x_0,0)$ is invariant under S_a,

$$\begin{aligned} S_a(x,\varphi(x)) &= (T(x),\varphi(x)+a(1,x)) \\ &= (T(x),\varphi(T(x))) \end{aligned}$$

for all $x\in X$. So $a(1,x)=\varphi(T(x))-\varphi(x)$ for all $x\in X$ and a is a coboundary.

Corollary 3.12.

Let X be a compact metric space and let $T:X\to X$ be a minimal homeomorphism. Let G be a locally compact, second countable abelian group which has no non-trivial compact subgroups. Let $a:\mathbb{Z}\times X\to G$ be a cocycle for T and suppose that for some $x_0\in X$ there exists a compact set $C\subset G$ with $a(n,x_0)\in C$ for all $n>0$. Then a is a coboundary.

Proof. We will prove $\overline{E}(a)=\{0\}$. Because of the special nature of G it is enough to show that $\infty\notin\overline{E}(a)$.

As X is compact we have $\overline{\{T^n(x_0):n>0\}}=X$. Let $F=\bigcap_{i=1}^{\infty}\overline{O}_a^+(S_a^i(x_0,0))$. This set is clearly invariant under S_a. An argument very similar to that in the second paragraph of the proof of 3.11 shows that $\pi_X(F)=X$. Therefore for every $x\in X$ there exists $h(x)\in C$ such that $(x,h(x))\in F$. But then, for all $x\in X$ and $n\in\mathbb{Z}$

$$\begin{aligned} a(n,x) &= \pi_G(S_a^n(x,0)) \\ &= \pi_G(L_{-h(x)}S_a^nL_{h(x)}(x,0)) \\ &\in \pi_G(L_{-h(x)}(F))\subset C-C \end{aligned}$$

which is compact. So $\infty\notin\overline{E}(a)$ and a is a coboundary.

Corollary 3.13.

Let X be a complete metric space and let $T:X\to X$ be a minimal homeomorphism. Suppose that $a:\mathbb{Z}\times X\to\mathbb{R}$ is a cocycle for T which is not a coboundary. Then the set

$$B^+(a)=\{x:\sup_{n\in\mathbb{Z}}a(n,x)<\infty\}$$

is meagre.

Proof. For each $M \in \mathbb{Z}_+$ let $B_M = \{x : a(n,x) \leq M, n \in \mathbb{Z}\}$. Each B_M is closed and $B^+(a) = \bigcup_{M>0} B_M$. Hence, if $B^+(a)$ is not meagre some B_M has non-empty interior A. Suppose this is so and that for some $i \in \mathbb{Z}$ and $x \in X$, $a(i,x) < -M$ with $T^i(x) \in A$. Then for all $n \in \mathbb{Z}$,

$$\begin{aligned} a(n,x) &= a(i,x) + a(n-i, T^i(x)) \\ &< a(i,x) + M < 0. \end{aligned}$$

This is clearly impossible, so we must have $a(n,x) \in [-M,M]$ whenever $x \in A \cap T^{-n}(A)$. This implies that $\infty \notin \bar{E}(a)$ and, as in the proof of 3.12, $\bar{E}(a) = \{0\}$.

To complete our analysis of the relationship between essential values and the properties of extensions we consider the quotient group $G/E(a)$. Because $E(a)$ is closed the quotient is itself a locally compact group. The topology of $G/E(a)$ is that defined by the metric d, where

$$d(h + E(a), g + E(a)) = \inf_{f \in E(a)} d_G(h-g, f).$$

Let $\tilde{a}$ denote the cocycle $\nu a : \mathbb{Z} \times X \to G/E(a)$, where $\nu : G \to G/E(a)$ is the natural projection. The essential values of $\tilde{a}$ give some further information about S_a.

Lemma 3.14.

Let $a : \mathbb{Z} \times X \to G$ be a cocycle for a minimal transformation $T : X \to X$. Let $\tilde{a}$ be the cocycle defined above. Then $E(\tilde{a}) = \{0\}$.

Proof. Suppose that $h+E(a) \in E(\tilde{a})$; we shall show that $h \in E(a)$. For any non-empty open set $A \subset X$ and $\varepsilon > 0$ there exists an $n \in \mathbb{Z}$ such that

$$\bigcup_{g \in E(a)} (A \cap T^{-n}(A) \cap \{x:\ a(n,x) \in B_G(g+h, \varepsilon/2)\})$$

$$= \quad A \cap T^{-n}(A) \cap \{x:\ \tilde{a}(n,x) \in B_{G/E(a)}(h+E(a), \varepsilon/2)\}$$

$$\neq \quad \emptyset.$$

At least one set in this union must be non-empty, so for some $g \in E(a)$,

$$B = A \cap T^{-n}(A) \cap \{x:\ a(n,x) \in B_G(g+h, \varepsilon/2)\} \neq \emptyset.$$

Because $-g \in E(a)$, there exists an $m \in \mathbb{Z}$ such that

$$A \cap T^{-(m+n)}(A) \cap \{x:\ a(m+n,x) \in B_G(h, \varepsilon)\}$$

$$\supset \quad T^{-n}(T^n(B) \cap T^{-m}(T^n(B)) \cap \{x:\ a(m,x) \in B_G(-g, \varepsilon)\})$$

$$\neq \quad \emptyset.$$

This shows that $h \in E(a)$ and the lemma is proved.

Proposition 3.15.

Let $a: \mathbb{Z} \times X \to G$ be a cocycle for a minimal transformation $T: X \to X$. Let $\tilde{a}: \mathbb{Z} \times X \to G/E(a)$ be the cocycle defined above. Then $\infty \in \bar{E}(\tilde{a})$ if and only if $\pi_X(\bar{O}_a(y)) \subsetneq X$ for every $y \in Y$.

Proof. Suppose, to the contrary, that $\pi_X(\bar{O}_a(y)) = X$ for some $y \in Y$. Let $\pi_X(y) = x$; then $\pi_X(\bar{O}_{\tilde{a}}(x,0)) = X$. So for each $x' \in X$, there exists $\varphi(x')$ such that $(x', \varphi(x')) \in \bar{O}_{\tilde{a}}(x,0)$. Lemmas 3.6 and 3.14 show that each $\varphi(x')$ is unique. The argument of the last part of the proof of 3.11 shows that $\tilde{a}$ is a coboundary. So $\infty \notin \bar{E}(\tilde{a})$.

Conversely, suppose that $\infty \notin \overline{E}(\tilde{a})$. By using Corollary 3.8 choose $x_0 \in X$ so that $L_g(\overline{O}_a(x_0,0)) = \overline{O}_a(x_0,0)$ for all $g \in E(a)$. Proposition 3.11 and Lemma 3.14 together show that $\tilde{a}$ is a coboundary. Let $\tilde{\varphi}: X \to G/E(a)$ be a continuous function with $\varphi(x_0) = 0$ and $\tilde{a}(n,x) = \tilde{\varphi}(T^n(x)) - \tilde{\varphi}(x)$ for all $n \in \mathbb{Z}$ and $x \in X$. For each $x \in X$, $(x,\tilde{\varphi}(x)) \in \overline{O}_{\tilde{a}}(x_0,0)$. Choose a sequence (n_i) such that $S_{\tilde{a}}^{n_i}(x_0,0) \to (x,\tilde{\varphi}(x))$ and fix $\varphi(x) \in \tilde{\varphi}(x)$. For each $i \geqslant 1$, there exists $g_i \in E(a)$ such that $a(n_i,x_0) + g_i \to \varphi(x)$. The choice of x_0 implies that, for every $i \geqslant 1$,

$$(T^{n_i}(x_0), a(n_i,x_0)+g_i) = L_{g_i}(T^{n_i}(x_0), a(n_i,x_0)) \in \overline{O}_a(x_0,0).$$

The point $(x,\varphi(x))$ is the limit of this sequence as $i \to \infty$ and so lies in $\overline{O}_a(x_0,0)$. This argument applies to every $x \in X$, so $\pi_X(\overline{O}_a(x_0,0)) = X$.

The next result summarises some facts that are useful in interpreting Proposition 3.15.

<u>Proposition 3.16</u>.

Let X be a complete metric space and let $T: X \to X$ be a minimal homeomorphism. Let $a: \mathbb{Z} \times X \to G$ be a cocycle for T. For each $y \in Y$ either $\pi_X(\overline{O}_a(y)) = X$, or $\pi_X(\overline{O}_a(y))$ is a T-invariant union of nowhere dense closed subsets of X. If the set $\{y : \pi_X(\overline{O}_a(y)) = X\}$ is not empty then it is a dense G_δ-subset of Y which is invariant under S_a and the group of translations $\{L_g : g \in G\}$.

Proof. Because G is locally compact and second countable, it can be expressed as a countable union of compact sets; $G = \bigcup_{i=1}^{\infty} C_i$. For any $y \in Y$, $\pi_X(\overline{O}_a(y)) = \bigcup_{i=1}^{\infty} (\pi_X(\overline{O}_a(y)) \cap (X \times C_i))$, so that $\pi_X(\overline{O}_a(y))$ is a T-invariant union of closed subsets of X. If any of these has non-empty interior then the fact that T is minimal ensures that $\pi_X(\overline{O}_a(y)) = X$. This proves the first assertion.

Now suppose that for some $y_0 \in Y$, $\pi_X(\overline{O}_a(y_0)) = X$. $\overline{O}_a(y_0)$ is a closed subspace of Y and so is itself a complete separable metric space. Let $\{U_i : i \geqslant 1\}$ be a countable basis for its topology. Let $\hat{S}_a$ denote the restriction of S_a to $\overline{O}_a(y_0)$.

Let F be the set $\{y : \overline{O}_a(y) = \overline{O}_a(y_0)\}$; then

$$F = \bigcap_{i=1}^{\infty} \bigcup_{n=-\infty}^{\infty} \hat{S}_a^n(U_i).$$

So F is a G_δ-subset of $\overline{O}_a(y_0)$ and it is clearly dense. The projection $\pi_X(F)$ is a dense G_δ-subset of X. For any $x \in \pi_X(F)$ there exists $y \in F$ with $\pi_X(y) = x$. Then for every $g \in G$, $\pi_X(\overline{O}_a(x,g)) = \pi_X(\overline{O}_a(y)) = X$. Therefore,

$$\{y : \pi_X(\overline{O}_a(y)) = X\} = \{(x,g) : x \in \pi_X(F),\ g \in G\}.$$

and this set is of the required form.

At the end of this section we shall give an example of a recurrent cocycle a with $\overline{E}(a) = \{0,\infty\}$. For such cocycles Theorem 3.7 and Propositions 3.15 and 3.16 give only a partial description of the orbit closures under the corresponding extension. If the extending group G is compactly generated or if X is connected then we may assume that $G = R^n \times \mathbb{Z}^m \times C$ $(n,m \geqslant 0)$, where C is a compact group. The cocycle a can then be regarded as the direct product of $n+m+1$ component cocycles. By applying Corollary 3.13 (and the analagous statement for orbits which are bounded below) to each non-compact component we obtain some additional information about S_a.

The derived cocycle $\tilde{a}$ which was used in Proposition 3.15 is also useful in relating our results to the special case where G is compact. Let K be the unit circle in the complex plane and recall that a character of a topological group G is a continuous homomorphism $G \to K$. If G is a locally compact group then the set of characters is itself a locally compact abelian group which is denoted by $\hat{G}$. When our extending group is compact it is possible to prove a stronger result than Corollary 3.10. In fact we can construct cocycles which give rise to minimal extensions. As this result is usually stated in terms of $\hat{G}$, we first prove a lemma which describes the relationship between essential values and elements of $\hat{G}$.

<u>Lemma 3.17</u>.

Let G be a locally compact, second countable abelian group. Let $a: \mathbb{Z} \times X \to G$ be a cocycle for a minimal transformation $T: X \to X$. Suppose that $\infty \notin \bar{E}(\tilde{a})$; then for any $\gamma \in \hat{G}$ the following statements are equivalent:

(1) The composite cocycle $\gamma a: \mathbb{Z} \times X \to K$ is a coboundary;

(2) $E(a) \subset \mathrm{Ker}(\gamma)$.

Proof. If γ is any character of G then it follows immediately from Definition 1.1.5 that $\gamma(E(a)) \subset E(\gamma a)$. Hence Proposition 3.11 shows that Statement (1) implies (2).

Because $\infty \notin \bar{E}(\tilde{a})$ Proposition 3.11 and Lemma 3.14 show that $\tilde{a}$ is a coboundary. If γ is any character of G for which (2) holds then we can put $\tilde{\gamma}(g + E(a)) = \gamma(g)$ $(g \in G)$ and so define a character $\tilde{\gamma}$ of $G/E(a)$. Clearly $\tilde{\gamma}\tilde{a}: \mathbb{Z} \times X \to K$ is a coboundary. For all $n \in \mathbb{Z}$ and $x \in X$ we have:

$$\gamma a(n,x) = \tilde{\gamma}(a(n,x) + E(a)) = \tilde{\gamma}\tilde{a}(n,x).$$

So γa is a coboundary and Statement (1) holds for γ.

Proposition 3.18.

Let G be a compact group and let $a: \mathbb{Z} \times X \to G$ be a cocycle for a minimal transformation $T: X \to X$. The corresponding extension, S_a, is minimal if and only if there is no non-trivial character $\gamma \in \hat{G}$ such that $\gamma a: \mathbb{Z} \times X \to K$ is a coboundary.

Proof. The Proposition is proved in the case where X is compact as Corollary 2 of [12]. In general, the argument in the second paragraph of 3.11 shows that $\pi_X(\bar{O}_a(y)) = X$ for every $y \in Y$. Choose $x_0 \in X$ so that $L_g(\bar{O}_a(x_0, 0)) = \bar{O}_a(x_0, 0)$ for every $g \in E(a)$. For any $y \in Y$ there exist elements $h, g_0 \in G$ such that $(x_0, h) \in \bar{O}_a(y)$ and $L_{g_0}(y) \in \bar{O}_a(x_0, h)$. Theorem 3.7 shows that $g_0 \in E(a)$; so for any $g \in E(a)$,

$$L_g(y) \in \bar{O}_a(x_0, h + g - g_0) = \bar{O}_a(x_0, h) \subset \bar{O}_a(y).$$

As this holds for every $y \in Y$, the minimality of T implies that if $E(a) = G$ then S_a is minimal. Conversely, if S_a is minimal then Corollary 3.10 shows that $E(a) = G$. Lemma 3.17 shows that $E(a) \neq G$ if and only if there exists a non-trivial character $\gamma \in \hat{G}$ such that γa is a coboundary.

We conclude this section with two illustrative examples. The second of these will show that the assumption, $\infty \notin E(\tilde{a})$, is necessary in Lemma 3.17. The set of characters $\gamma \in \hat{G}$ for which γa is a coboundary is therefore not as useful as the group of essential values in studying non-compact group extensions. This collection of characters is in fact related to a compact group extension - the extension of T by the Bohr compactification of G which arises naturally from the cocycle a.

In both the following examples the extending group is the additive group of real numbers. The first example is an R-extension, S_a, of a minimal transformation of the two-torus with $E(a) = \mathbb{Z}$. This example is significant because the torus is a connected space. There is therefore no possibility that there exists a coboundary b with $a(n,x) + b(n,x) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $x \in X$. This possibility can occur if the base space of an extension is totally disconnected. In that case the extension is merely a disguised form of a topologically transitive $\mathbb{Z}$-extension.

The construction of our examples requires an auxiliary transformation. Let K be the unit circle as before and let $\alpha \in \mathbb{R}$ be any irrational number. Let $T':K \to K$ be the minimal translation: $T'(k) = k.\exp(2\pi i\alpha)$. Let $a':\mathbb{Z} \times K \to R$ be a cocycle for T' with $E(a')=\mathbb{R}$. The existence of such a cocycle will be demonstrated in the next section. We will also require a metric on K; the most suitable is the metric d_K, for which $d_K(\exp(2\pi i\beta), \exp(2\pi i\delta))$ is the distance from $\beta-\delta$ to the nearest integer ($\beta,\delta\in\mathbb{R}$).

Example 3.19.

Let $X=K^2$ and define $T:X\to X$ by the equation:

$$T(k_1,k_2) = (T'(k_1),\ k_2.\exp(2\pi i.a'(1,k_1))).$$

Then T is a skew-product extension of T' and is minimal by Proposition 3.18 and Lemma 3.17. Let $a:\mathbb{Z}\times X\to\mathbb{R}$ be the cocycle for T with $a(n,(k_1,k_2))=a'(n,k_1)$. We will show that $E(a)=\mathbb{Z}$.

Fix any $m\in\mathbb{Z}$. For any non-empty open set $A\subset K$ and any $k\in K$ and $\varepsilon>0$ there exists an $n\in\mathbb{Z}$ such that

$$(A\times B_K(k,\varepsilon))\cap T^{-n}(A\times B_K(k,\varepsilon))\cap\{x:\ a(n,x)\in B_{\mathbb{R}}(m,\varepsilon)\}$$
$$\supset(A\cap T'^{-n}(A)\cap\{k':\ a'(n,k')\in B_{\mathbb{R}}(m,\varepsilon)\})\times\{k\}\ \neq\emptyset.$$

This is enough to show that $m\in E(a)$; so we have $\mathbb{Z}\subset E(a)$. To see that $E(a)=\mathbb{Z}$ observe that for any $((k_1,k_2),t)\in Y$,

$$\overline{O}_a((k_1,k_2),r)\subset\{((k_1',k_2'),r'):\ k_2'=k_2.\exp(2\pi i(r'-r))\} \qquad \text{—(1)}$$

and apply Theorem 3.7. If k_1 is such that $\overline{O}_{a'}(k_1,0)=K\times\mathbb{R}$ then we have equality in (1) for every $k_2\in K$ and $r\in\mathbb{R}$. So "most" orbit closures under S_a are of this form.

Our second example, which is constructed in a similar fashion to the first, is of a recurrent cocycle a with $E(a)=\{0,\infty\}$.

Example 3.20.

This time let $X = K^3$. Choose an irrational number β and define $T: X \to X$ by the equation:

$$T(k_1, k_2, k_3) = (T'(k_1), k_2.\exp(2\pi i.a'(1,k_1)), k_3.\exp(2\pi\beta i.a'(1,k_1))).$$

As in the previous example T is a minimal skew-product extension of T'. Define a cocycle $a: \mathbb{Z} \times X \to \mathbb{R}$ for T by:

$$a(n,(k_1,k_2,k_3)) = a'(n,k_1) \qquad (n \in \mathbb{Z}, k_1, k_2, k_3 \in K).$$

For each $((k_1,k_2,k_3),r) \in Y$ we have:

$$\overline{O}_a((k_1,k_2,k_3),r) \subset$$
$$\{((k_1'.k_2',k_3'),r'): k_2' = k_2.\exp(2\pi i(r'-r)), k_3' = k_3.\exp(2\pi\beta i(r'-r))\}.$$

Together with Theorem 3.7 this implies that if $t \in E(a)$ then $\exp(2\pi i t) = 1 = \exp(2\pi\beta i t)$. Hence $E(a) = \{0\}$. Clearly a is recurrent and is not a coboundary so that $\overline{E}(a) = \{0, \infty\}$.

Every character of $\mathbb{R}$ is of the form $\hat{\theta}$ for some $\theta \in \mathbb{R}$, where $\hat{\theta}(r) = \exp(2\pi i \theta r)$. We shall show that for most $\theta \in \mathbb{R}$, $\hat{\theta}a: \mathbb{Z} \times X \to K$ is not a coboundary even though $E(a) = \{0\} \subset \mathrm{Ker}(\theta)$.

Suppose θ is any real number for which the set $\{1, \theta, \beta\}$ is linearly independent over the rationals. Let $\hat{T}: K^4 \to K^4$ be the transformation with

$$\hat{T}(k_1, k_2, k_3, k_4) =$$
$$(T'(k_1), k_2\exp(2\pi i a(1,k_1)), k_2\exp(2\pi\beta i a'(1,k_1)), k_3\exp(2\pi\theta i a'(1,k_1))).$$

This transformation is a K^3-extension of T'. If we call the extending cocycle $\hat{a}: \mathbb{Z} \times K \to K^3$ then it is easy to see that $E(\hat{a}) \supset \overline{\{(\exp(2\pi i r), \exp(2\pi\beta i r), \exp(2\pi\theta i r)): r \in \mathbb{R}\}} = K^3$. So $\hat{T}$ is minimal. $\hat{T}$ can also be regarded as a K-extension of T. Since $\hat{}$ is minimal, the extending cocycle $\hat{\theta}a: \mathbb{Z} \times X \to K$ cannot be a coboundary.

§1. Topologically transitive extensions.

It will be proved in the next section that an extension can never be minimal if the base space is compact and the extending group is non-compact. However such an extension may be topologically transitive. Examples of topologically transitive real line extensions have been given for particular base transformations by Besicovitch ([1,2]) and Gottschalk and Hedelund ([6]) among others. More recently, Siderov has shown in [16] that such extensions exist for every minimal transformation, where the extending group may be any Banach space. (It is clear that the definitions of Chapter One can be generalised to define extensions by any abelian group.) Siderov and Krygin have also given examples of topologically transitive real line extensions with various special properties ([16] and [9]).

In this section we restrict our attention to $\mathbb{R}^m$-extensions of transformations on compact spaces. With this restriction we can prove results that are stronger than the simple existence of topologically transitive extensions. Theorem 4.4 shows that the collection of cocycles which give rise to topologically transitive extensions is a residual subset of the set of all recurrent cocycles, when the latter is equipped with a suitable metric. For a particular class of transformations, the minimal translations on tori, Theorem 4.14 gives an even stronger statement; all non-trivial conservative extensions are topologically transitive.

The following lemma contains more than is necessary in the proof of Theorem 4.4 but it will be useful in §5. Corollary 4.2 is what is required here.

Lemma 4.1.

Let X be a compact metric space and let $T:X\to X$ be a minimal homeomorphism. Let $a:\mathbb{Z}\times X\to \mathbb{R}$ be a cocycle for T and suppose that for some $x_o\in X$, $\operatorname{Lim}_n n^{-1}a(n,x_o)=0$. Then for any $r\in\mathbb{R}$, $\varepsilon>0$ and $\eta>0$, there exists a $\delta\in\mathbb{R}$, $\eta>\delta>0$, a $k\in\mathbb{Z}_+$ and a coboundary $b:\mathbb{Z}\times X\to G$ such that:

(1) $d_X(T^k(x_o),x_o)<\eta$;

(2) $a(k,x_o)+b(k,x_o) = r$;

(3) $\operatorname{Sup}_{x\in X}|b(1,x)|<\varepsilon$;

(4) For all $n\in\mathbb{Z}$, $b(n,x_o)$ lies in the closed interval with endpoints zero and $r-a(k,x_o)$;

(5) $b(n,x_o)=0$ whenever $T^n(x_o)\in B_X(x_o,\delta)$.

Proof. Choose $k\in\mathbb{Z}_+$ so that: (1) is satisfied; $k^{-1}|a(k,x_o)|<\varepsilon/2$; and $k^{-1}|r|<\varepsilon/2$. The points $T^i(x_o)$ $0\leq i\leq 2k-1$ are all distinct, so there exists a δ, $\eta>\delta>0$, such that

$$T^i(B_X(x_o,\delta))\cap T^j(B_X(x_o,\delta)) = \emptyset \qquad (0\leq i,j\leq 2k-1). \quad \text{—(6)}$$

Let $\varphi:X\to\mathbb{R}$ be a continuous function with $\varphi(x_o)=k^{-1}(r-a(k,x_o))$ such that $\varphi(x)$ lies between zero and $\varphi(x_o)$ for all $x\in B_X(x_o,\delta)$ and $\varphi(x)=0$ for all $x\notin B_X(x_o,\delta)$. It is clear that $\operatorname{Sup}_{x\in X}\varphi(x)<\varepsilon$. Let $b:\mathbb{Z}\times X\to\mathbb{R}$ be the cocycle with

$$b(1,x) = \sum_{i=0}^{k-1}\varphi(T^{-i}(x)) - \sum_{i=k}^{2k-1}\varphi(T^{-i}(x)) \qquad (x\in X).$$

Equation (6) ensures that (3) is satisfied. Also for all $x \in X$,

$$b(1,x) = \sum_{i=0}^{k-1} (\varphi - \varphi T^{-k})(T^{-i}(x))$$
$$= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (\varphi T^{-j} - \varphi T^{-(j+1)})(T^{-i}(x))$$
$$= (\sum_{i=0}^{k-1} \sum_{j=1}^{k} \varphi T^{-(i+j)})(T(x)) - (\sum_{i=0}^{k-1} \sum_{j=1}^{k} \varphi T^{-(i+j)})(x).$$

So b is a coboundary and if

$$\psi(x) = \sum_{i=0}^{k-1} \sum_{j=1}^{k} \varphi T^{-(i+j)}(x) \qquad (x \in X)$$

then $b(n,x) = \psi(T^n(x)) - \psi(x)$ for all $n \in \mathbb{Z}$ and $x \in X$. Note that $\psi(x) = 0$ for all $x \in B_X(x_o, \delta)$. This implies that $b(n,x_o) = \psi(T^n(x_o))$ for all $n \in \mathbb{Z}$ so that (4) and (5) hold. Also, $\psi(T^k(x_o)) = r - a(k,x_o)$, so (2) is correct.

<u>Corollary 4.2</u>.

Let X be a compact metric space and let $T: X \to X$ be a minimal homeomorphism. Let $a: \mathbb{Z} \times X \to \mathbb{R}^n$ be a cocycle for T and suppose that for some $x_o \in X$, $\operatorname{Lim}_n n^{-1}|a(n,x_o)| = 0$. Then for any $v \in \mathbb{R}^n$, $\varepsilon > 0$ and $\eta > 0$ there exists a $\delta > 0$, $\eta > \delta > 0$, a $k \in \mathbb{Z}_+$ and a coboundary $b: \mathbb{Z} \times X \to \mathbb{R}^n$ such that:

(1) $d_X(T^k(x_o), x_o) < \eta$;

(2) $a(k,x_o) + b(k,x_o) = v$;

(3) $\sup_{x \in X} |b(1,x)| < \varepsilon$;

(4) $b(n,x_o) = 0$ whenever $T^n(x_o) \in B_X(x_o, \delta)$.

Proof. Apply Lemma 4.1 to each component of a and v, using the same k and δ for each. This can be done because there is no upper bound on the choice of k in the proof of 4.1 and no lower bound (except zero) on the choice of δ.

When X is a compact metric space and $T:X \to X$ is a minimal homeomorphism there always exists at least one T-invariant ergodic probability measure on X. Let μ be such a measure; for each $m \geqslant 1$, $Z^1_\mu(T,\mathbb{R}^m)$ will denote the set of cocycles $a = (a_1, \ldots ,a_m):\mathbb{Z}\times X \to \mathbb{R}^m$ for T with

$$\int a_i(1,.)\, d\mu = 0 \qquad (1 \leqslant i \leqslant m).$$

It is not known under what circumstances it is sufficient that a cocycle belongs to $Z^1_\mu(T,\mathbb{R}^m)$ for it to be recurrent. The ergodic theorem shows that for every $a \in Z^1_\mu(T,\mathbb{R}^m)$,

$$\mu(\{x: \operatorname{Lim}_n n^{-1}|a(n,x)| = 0\}) = 1$$

and this is enough to make Corollary 4.2 useful.

In order to state the next Lemma we need a metric on $Z^1_\mu(T,\mathbb{R}^m)$. It is clear that the set $\{a(1,.): a \in Z^1_\mu(T,\mathbb{R}^m)\}$ is a closed subspace of the space of continuous functions $X \to \mathbb{R}^m$ (with the usual metric). Therefore if we set

$$d(a,a') = \operatorname*{Sup}_{x\in X}|a(1,x) - a'(1,x)| \qquad (a,a' \in Z^1_\mu(T,\mathbb{R}^m),$$

$Z^1_\mu(T,\mathbb{R}^m)$ becomes a complete metric space.

Lemma 4.3.

Let X be a compact metric space and let $T:X \to X$ be a minimal homeomorphism. Let μ be a T-invariant ergodic probability measure on X. For each $v \in \mathbb{R}^m$ $(m \geqslant 1)$ let $F(v)$ be the set $\{a \in Z^1_\mu(T,\mathbb{R}^m): v \in E(a)\}$. Then each $F(v)$ $(v \in \mathbb{R}^m)$ is a residual subset of $Z^1_\mu(T,\mathbb{R}^m)$.

Proof. Fix $v \in \mathbb{R}^m$. We shall first prove that $F(v)$ is a dense subset of $Z^1_\mu(T,\mathbb{R}^m)$. Given any $a \in Z^1_\mu(T,\mathbb{R}^m)$ and $\rho > 0$, we will construct a cocycle $b \in Z^1_\mu(T,\mathbb{R}^m)$ such that $v \in E(a+b)$ and $d(a,a+b) < \rho$.

First fix an $x_0 \in X$ such that $\operatorname{Lim}_n n^{-1}|a(n,x_0)| = 0$. We choose a sequence of coboundaries $(b_i : \mathbb{Z} \times X \to \mathbb{R}^m)$, a sequence of positive integers (k_i), and a sequence of positive real numbers (δ_i) inductively. Let b_0 be the zero coboundary and let $k_0 = 1$ and $\delta_0 = 1$. Now suppose that $b_0, \dots, b_{i-1}$, $k_0, \dots, k_{i-1}$ and $\delta_0, \dots, \delta_{i-1}$ have already been chosen. Using Corollary 4.2 we choose b_i, k_i and δ_i (with $\delta_{i-1} > \delta_i > 0$) such that:

(1) $d_X(T^{k_i}(x_0),\ x_0) < \operatorname{Inf}\{\delta_{i-1},\ 2^{-i}\}$;

(2) $a(k_i,x_0) + b_i(k_i,x_0) = v$;

(3) $\sup_{x \in X} |b_i(1,x)| < \rho/k_{i-1}2^i$;

(4) $b_i(n,x_0) = 0$ whenever $T^n(x_0) \in B_X(x_0,\delta_i)$.

It follows from (3) that the series of functions $\sum_{i=0}^{\infty} b_i(1,.)$ converges uniformly. Let $b : \mathbb{Z} \times X \to \mathbb{R}^m$ be the cocycle with $b(1,.) = \sum_{i=0}^{\infty} b_i(1,.)$; then because b is a limit of coboundaries, $b \in Z^1_\mu(T,\mathbb{R}^m)$. Condition (1) shows that whenever $0 < i < j$, $T^{k_i}(x_0) \in B_X(x_0,\delta_j)$ so that $b(k_i,x_0) = \sum_{j=i}^{\infty} b_j(k_i,x_0)$. Also, Equation 4.1.6 together with (1) ensures that $k_0 < k_1 < k_2 \dots$. Hence

$$|a(k_i,x_0) + b(k_i,x_0) - v| = \Big|\sum_{j=i+1}^{\infty} b_j(k_i,x_0)\Big|$$

$$\leq k_i \sum_{j=i+1}^{\infty} \sup_x |b_j(1,x)|$$

$$< \rho/2^i.$$

Together with (1) this shows that $S_{a+b}^{k_i}(x_0,0) \to (x_0,v)$. Theorem 3.7 now implies that $v \in E(a+b)$.

To complete the proof we will show that $F(v)$ is a G_δ-subset of $Z_\mu^1(T,\mathbb{R}^m)$. Let $\{U_i : i \geq 1\}$ be a countable basis for the topology of X. It follows directly from Definition 1.1.5 that

$$F(v) = \bigcap_{j=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcup_{n=-\infty}^{\infty} \{a \in Z_\mu^1(T,\mathbb{R}^m) : U_i \cap T^{-n}(U_i) \cap \{x : |a(n,x)-v| < 2^{-j}\} \neq \emptyset\}.$$

This is clearly of the required form.

<u>Theorem 4.4</u>.

Let X be a compact metric space and let $T:X \to X$ be a minimal homeomorphism. Let μ be any T-invariant ergodic Borel probability measure on X; then the set $\{a \in Z_\mu^1(T,\mathbb{R}^m) : E(a) = \mathbb{R}^m\}$, is a residual subset of $Z_\mu^1(T,\mathbb{R}^m)$.

<u>Proof</u>. Let $\{v_j : j \geq 1\}$ be a countable dense subset of $\mathbb{R}^m$. If $a \in \bigcap_{j=1}^{\infty} F(v_j)$ then it is clear that $E(a) = \mathbb{R}^m$. This is an intersection of residual sets and so is itself residual.

<u>Corollary 4.5</u>.

Let X be a compact metric space and let $T:X \to X$ be a minimal homeomorphism. Then for each $m \geq 1$, there exists an uncountable collection, H_m of cocycles such that $E(a) = \mathbb{R}^m$ for each $a : \mathbb{Z} \times X \to \mathbb{R}^m$ contained in H_m.

Proof. Let μ be any T-invariant ergodic probability measure on X. Choose a maximal collection of non-cohomologous cocycles H_m from the set

$$F = \{a \in Z_\mu^1(T,\mathbb{R}^m) : E(a) = \mathbb{R}^m\}.$$

All the coboundaries for T lie in F^c, the complement of F in $Z_\mu^1(T,\mathbb{R}^m)$, and this complement is a meagre set. Because H_m is maximal $F = \bigcup_{a \in H_m} \{a + b : b \in F^c\}$; if H_m were countable this would express F as the union of a countable collection of meagre sets. Baire's Theorem shows that this is impossible, so H_m must be uncountable.

Corollary 4.6.

Let X be a compact metric space and let $T: X \to X$ be a minimal homeomorphism. Let μ be any T-invariant ergodic Borel probability measure on X. Then both the set of coboundaries and the set of transient cocycles which lie in $Z_\mu^1(T,\mathbb{R}^m)$ are meagre (as subsets of $Z_\mu^1(T,\mathbb{R}^m)$).

In all cocycle problems of topological dynamics or analysis it is interesting to know whether every recurrent cocycle is a limit of coboundaries. In the present case Theorem 4.4 shows that there may be cocycles which give rise to topologically transitive extensions and are not the pointwise limit of any sequence of coboundaries. This is the case if there are two distinct T-invariant ergodic probability measures μ_1 and μ_2 on X. Every $\mathbb{R}^m$-valued cocycle which is a limit of coboundaries then belongs to $Z_{\mu_1}^1(T,\mathbb{R}^m) \cap Z_{\mu_2}^2(T,\mathbb{R}^m)$, but Theorem 4.4 implies that there are cocycles in $Z^1_{\mu_1}(T,\mathbb{R}^m)\Delta Z^1_{\mu_2}(T,\mathbb{R}^m)$ which define topologically transitive extensions.

The rest of this section is about the $\mathbb{R}^m$-extensions of some special transformations - the minimal translations on a torus. A theorem of Hedelund ([6], Theorem 14.13) states that if a real valued cocycle for one of these transformations is neither transient nor a coboundary then it defines a topologically transitive extension. In Theorem 4.14 we extend this result to $\mathbb{R}^m$-extensions. The main part of the proof is contained in the following sequence of lemmas. The first of these gives an additional criterion for an element of the extending group to be an essential value.

<u>Lemma 4.7</u>.

Let X be a complete metric space and let $T:X\to X$ be a minimal homeomorphism. Let $a:\mathbb{Z}\times X\to G$ be a cocycle for T. Suppose that for some $x_0\in X$ and $g\in G$,

$$g \in \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=-\infty}^{\infty} \{a(n,x):\ x\in B_X(x_0,2^{-(i+1)})\cap T^{-n}(B_X(x_0,2^{-i}))\ \}}.$$

Then $g\in E(a)$.

<u>Proof</u>. For any non-empty open set $A\subset X$ and any $\varepsilon>0$, there exists an integer, k, such that $T^k(x_0)\in A$. Choose $i\in\mathbb{Z}_+$, sufficiently large that $T^k(B_X(x_0,2^{-i}))\subset A$ and $d_G(a(k,x),a(k,x'))<\varepsilon/2$ for all $x,x'\in B_X(x_0,2^{-i})$.

By assumption there exists an $x\in B_X(x_0,2^{-(i+1)})$ and and an $n\in\mathbb{Z}$ such that $T^n(x)\in B_X(x_0,2^{-i})$ and $a(n,x)\in B_G(g,\varepsilon/2)$.

Let $x' = T^k(x)$; then $T^n(x') = T^k(T^n(x)) \in T^k(B_X(x_o, 2^{-i})) \subset A$. Also by applying the cocycle equation twice we have:

$$\begin{aligned} a(n,x') &= a(n,T^k(x)) \\ &= a(n+k,x) - a(k,x) \\ &= a(n,x) + a(k,T^n(x)) - a(k,x). \end{aligned}$$

So $d_G(a(n,x'),g) \leqslant d_G(a(n,x),g) + d_G(a(k,T^n(x)),a(k,x)) < \varepsilon$. Hence, $x' \in A \cap T^{-n}(A) \cap \{x\colon a(n,x) \in B_G(g,\varepsilon)\} \neq \emptyset$ and g is an essential value of a.

From here until the end of this section we will always assume that the base space X is a torus - a finite or countable product of circles. Each torus is a compact abelian group; we will write the group operation multiplicatively and use "e" to denote the identity. The minimal homeomorphism $T\colon X \to X$ will be a translation, so that for some $x_T \in X$, $T(x) = x_T x$ $(x \in X)$. A torus, like any other separable compact group, has a translation-invariant symmetric metric which defines its topology. We shall use such a metric d_X, so that $d_X(T(x),T(x')) = d_X(x,x')$ for all $x,x' \in X$.

<u>Lemma 1.5</u>.

Let $a\colon \mathbb{Z} \times X \to \mathbb{R}^m$ $(m \geqslant 1)$ be a cocycle for $T\colon X \to X$. Suppose that $E(a) = \{0\}$. Then for every $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ such that $|a(n,x)| < \varepsilon$ for all $x \in X$, whenever $d_X(T^n(e)),e) < \eta(\varepsilon)$ and $\{x\colon |a(n,x)| < \varepsilon\} \neq \emptyset$.

Proof. For every $n \in \mathbb{Z}$ and $x \in X$,

$$d_X(T^n(e), e) = d_X(x_T^n, e) = d_X(x_T^n x, x) = d_X(T^n(x), x).$$

So whenever $d_X(T^n(e), e) < 2^{-i}$, $T^n(B_X(x, 2^{-i})) \subset B_X(x, 2^{-(i-1)})$ for every $x \in X$. For each $\varepsilon > 0$, let $S(\varepsilon) = \{v \in \mathbb{R}^m : |v| = \varepsilon\}$. Then, because $S(\varepsilon) \cap E(a) = \emptyset$, Lemma 4.7 implies that for all $x \in X$,

$$\bigcap_{i=1}^{\infty} \left(\overline{\{a(n,x') : x' \in B_X(x, 2^{-i}),\ d_X(T^n(e), e) < 2^{-i}\}} \cap S(\varepsilon) \right) = \emptyset.$$

This is an intersection of compact sets, so the finite intersection property shows that for each $x \in X$ there exists an $i(x) \in \mathbb{Z}_+$ with

$$\overline{\{a(n,x') : x' \in B_X(x, 2^{-i(x)}),\ d_X(T^n(e), e) < 2^{-i(x)}\}} \cap S(\varepsilon) = \emptyset.$$

As X is compact we can choose a finite collection of points $\{x_j : 1 \leq j \leq k\}$ with $\bigcup_{j=1}^{k} B_X(x_j, 2^{-i(x_j)}) = X$ and let $\eta(\varepsilon) = \inf_{1 \leq j \leq k} 2^{-i(x_j)}$.

Now suppose that one of the sets $B_X(x_j, 2^{-i(x_j)})$, $1 \leq j \leq k$ contains a point x with $|a(n,x)| < \varepsilon$ for some $n \in \mathbb{Z}$ for which $d_X(T^n(e), e) < \eta(\varepsilon)$. Then because $\eta(\varepsilon) \leqslant 2^{-i(x_j)}$,

$$\{x \in B_X(x_j, 2^{-i(x_j)}) : |a(n,x)| = \varepsilon\} = \emptyset.$$

But $B_X(x_j, 2^{-i(x_j)})$ is connected and $|a(n,.)|$ is a continuous function. So for all $x \in B_X(x_j, 2^{-i(x_j)})$ we have $|a(n,x)| < \varepsilon$.

The argument of the last paragraph shows that whenever $d_X(T^n(e), e) < \eta(\varepsilon)$ the set $\{x : |a(n,x)| < \varepsilon\}$ is a union of some of the open balls $B_X(x_j, 2^{-i(x_j)})$, $1 \leq j \leq k$. Because X is connected if this union is not empty then it must be all of X.

Corollary 4.9.

Let $a:\mathbb{Z}\times X\to\mathbb{R}$ be a cocycle for $T:X\to X$ and suppose that $r\notin E(a)$, for some $r>0$. Then there exists an $\eta(r)>0$ such that $|a(n,x)|<r$ for all $x\in X$, whenever $d_X(T^n(e),e)<\eta(r)$ and $\{x:\ |a(n,x)|<r\}\neq\emptyset$.

Proof. Set $\varepsilon=r$ in the proof of 4.8.

Lemma 4.10.

Let $a:\mathbb{Z}\times X\to\mathbb{R}^m$ $(m\geqslant 1)$ be a recurrent cocycle for $T:X\to X$. Suppose that $E(a)=\{0\}$. Then for any sequence of integers (n_i) the following statements are equivalent:

(1) For all $y\in Y$, the sequence $(S_a^{n_i}(y))$ converges;

(2) For some $x_o\in X$, the sequence $(S_a^{n_i}(x_o,0))$ converges;

(3) The sequence of functions $(a(n_i,\cdot))$ converges uniformly to a continuous function $\varphi:X\to\mathbb{R}^m$ and the sequence $(T^{n_i}(e))$ converges.

Proof. That Statement (1) implies (2) is trivial. Suppose that Statement (2) is true; then the sequences $(a(n_i,x_o))$ and $(T^{n_i}(x_o))$ converge. In particular they are Cauchy sequences, so for every $\varepsilon>0$ there exists an $N\in\mathbb{Z}_+$ such that $|a(n_i,x_o)-a(n_j,x_o)|<\varepsilon$ and $d_X(T^{n_j}(x_o),T^{n_j}(x_o))<\eta(\varepsilon)$ for all $i,j\geqslant N$. But

$$d_X(T^{n_i}(x_o),T^{n_j}(x_o))=d_X(T^{n_i-n_j}(e),e)$$

and the cocycle equation shows that

$$a(n_i,x_o)-a(n_j,x_o)=a(n_i-n_j,T^{n_j}(x_o));$$

so Lemma 4.8 shows that $|a(n_i-n_j, T^{n_j}(x))| < \varepsilon$ for all $x \in X$. Reversing the argument above now shows that the sequence $(a(n_i,.))$ is uniformly Cauchy and so converges uniformly to some continuous function $\varphi: X \to \mathbb{R}^m$. The convergence of $(T^{n_i}(e))$ is clear. Finally, it clearly follows from the definitions of T and S_a that Statement (3) implies (1).

__Lemma 4.11.__

Let $a: \mathbb{Z} \times X \to \mathbb{R}^m$ be a recurrent cocycle for $T: X \to X$ and suppose that $E(a) = \{0\}$. Then every orbit closure under S_a is __minimal__.

__Proof__. Choose any $x \in X$ and define;

$$W = \{w \in X:\ wx \in \pi_X(\overline{O}_a(x,0))\}.$$

If $w \in W$ then there exists a sequence (n_i) such that $T^{n_i}(e) \to w$ and $(a(n_i,x))$ converges. Lemma 4.10 shows that the sequence of functions $(a(n_i,.))$ converges uniformly; so $wx \in \pi_X(\overline{O}_a(x,0))$ for every $x \in X$. The definition of W is therefore independent of the choice of x.

Suppose that $w_1, w_2 \in W$. Then $w_2x \in \pi_X(\overline{O}_a(x,0))$ and $w_1w_2x \in \pi_X(\overline{O}_a(w_2x,0)) \subset \pi_X(\overline{O}_a(x,0))$. So W is a subsemigroup of X. We shall now show that it is in fact a subgroup.

For any $w \in W$ let (n_i) be a sequence such that $T^{n_i}(e) \to w$ and $(a(n_i,.))$ coverges uniformly to a continuous function $\varphi: X \to \mathbb{R}^m$. Because X is compact the function φ is uniformly continuous. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varphi(x) - \varphi(x')| < \varepsilon/2$ for all $x, x' \in X$ with $d_X(x,x') < \delta$. For each ε, choose $N \in \mathbb{Z}_+$ such that

$d_X(T^{-n_i}(e), w^{-1}) = d_X(T^{n_i}(e), w) < \delta$ and $|a(n_i, x) - \varphi(x)| < \varepsilon/2$ for all $x \in X$ and all $i \geq N$. Then for all $x \in X$ and $i \geq N$ we have:

$$|\varphi(w^{-1}x) + a(-n_i, x)| < |\varphi(w^{-1}x) - \varphi(T^{-n_i}(x))| + |\varphi(T^{-n_i}(x) - a(n_i, T^{-n_i}(x))|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So the sequence $(a(-n_i, .))$ converges uniformly to the continuous function $-\varphi(w^{-1}.)$. Also $T^{-n_i}(e) \to w^{-1}$. The conditions of Statement (3) of Lemma 4.10 are satisfied so that $(S_a^{-n_i}(x,0))$ converges for all $x \in X$ and $w^{-1} \in W$.

We can now prove the lemma. Let $x, x' \in X$ and $v, v' \in \mathbb{R}^m$. If $(x', v') \in \overline{O}_a(x, v)$ then $x'x^{-1} \in W$; so $x(x')^{-1} \in W$ and there exists $v'' \in \mathbb{R}^m$ such that $(x, v'') \in \overline{O}_a(x', v') \subset \overline{O}_a(x, v)$. Theorem 3.7 implies that $v - v'' \in E(a)$, so $v = v''$. It follows that $\overline{O}_a(x, v) = \overline{O}_a(x', v')$ and that this is a minimal orbit closure.

Our next lemma is a simplified form of Theorem 7.05 of [5].

<u>Lemma 4.12</u>.

Let Z be a locally compact topological space and let $S: Z \to Z$ be a minimal homeomorphism. Suppose that every $z \in Z$ lies in the closure of its own forward orbit under S. Let U be an open set of Z with $\overline{U}$ compact. Then there exists an $N \in \mathbb{Z}_+$ such that $U \cap \{S^{i+j}(z) : 0 \leq j \leq N-1\} \neq \emptyset$ for all $i \in \mathbb{Z}_+$ and $z \in U$.

Proof. Because S is minimal the orbit of any point $z \in Z$ is dense in Z: because z is a limit point of its own forward orbit, that forward orbit must itself be dense in Z. In particular if $z \in \overline{U}$ then the forward orbit of z enters U. This implies that $\overline{U} \subset \bigcup_{j=1}^{\infty} S^{-j}(U)$. Since $\overline{U}$ is compact there exists an $N \in \mathbb{Z}_+$ such that $U \subset \overline{U} \subset \bigcup_{j=1}^{N} S^{-j}(U)$.

Given any $z \in U$ we can use this fact to choose by induction a sequence of positive integers (n_l) such that $S^{n_l}(z) \in U$ for every $l \geqslant 1$. Let $n_0 = 0$. If n_l $(l \geqslant 0)$ has been chosen so that $S^{n_l}(z) \in U$ then there exists j with $1 \leqslant j \leqslant N$ such that $S^{n_l}(z) \in S^{-j}(U)$. Let $n_{l+1} = n_l + j$; then $S^{n_{l+1}}(z) \in U$. If i is any positive integer then at least one of the integers $i, i+1, \ldots, i+N-1$ is a member of the sequence (n_l). This proves the lemma.

Lemma 4.13.

Let $a: \mathbb{Z} \times X \to \mathbb{R}^m$ be a recurrent cocycle for $T: X \to X$ and suppose that $E(a) = \{0\}$. Then a is a coboundary.

Proof. Because a is recurrent Lemma 2.2 and Proposition 2.3 show that for some $x \in X$ there exists a sequence of positive integers (n_i) such that $S_a^{n_i}(x,0) \to (x,0)$. It follows from Lemmas 4.8 and 4.10 that $S_a^{n_i}(y) \to y$ for all $y \in Y$. This fact, together with Lemma 4.11, shows that for any $x \in X$ the restriction of S_a to $\overline{O}_a(x,0)$ satisfies the conditions of Lemma 4.12. Let U be the set: $\overline{O}_a(x,0) \cap (X \times B_{\mathbb{R}^m}(0,1))$. Lemma 4.12 then shows that for all $n \in \mathbb{Z}_+$, $|a(n,x)| < 1 + N. \sup_{x' \in X} |a(1,x')|$. The conclusion of the lemma now follows from Corollary 3.12.

Theorem 4.14.

Let X be a torus with $T:X \to X$ a minimal translation. Let $a:\mathbb{Z}\times X \to \mathbb{R}^m$ $(m \geqslant 1)$ be a recurrent cocycle for T. Let S_a be the skew-product extension of T defined by a. Then S_a is topologically transitive if and only if there is no non-zero linear functional $\psi: \mathbb{R}^m \to \mathbb{R}$ such that ψa is a coboundary

Proof. It is clear that if such a linear functional exists then S_a is not topologically transitive. For if $\psi(v) \neq 0$ then for every $x \in X$, $(x,v) \notin \bar{O}_a(x,0)$ as is shown by applying Theorem 3.7 to ψa.

The converse will be proved by induction; we first prove it in the case where m is equal to one. Suppose in this case that S_a is not topologically transitive. Then there exists an $r > 0$ such that $r \notin E(a)$. Let $\eta(r)$ be the number whose existence was proved in Corollary 4.9. Because T is minimal and X is compact there exists an $N \in \mathbb{Z}_+$ such that $X = \bigcup_{j=0}^{N-1} T^{-j}(B_X(e,\eta(r)))$. So for all $i \in \mathbb{Z}$ there exists an n with $i \leqslant n \leqslant i+N-1$ such that $T^n(e) \in B_X(e,\eta(r))$.

Before we can apply Corollary 4.9 we need to know that $\{x: |a(n,x)| < r\} \neq \emptyset$. The Haar measure μ is the unique T-invariant Borel probability measure on X and it follows from Theorem 2.8 that $\int a(n,.)\, d\mu = 0$ for all $n \in \mathbb{Z}$. Because each $a(n,.)$ is a continuous function this implies that for some $x \in X$, $a(n,x) = 0$. Corollary 4.9 now shows that for every $i \in \mathbb{Z}$ there exists an n with $i \leqslant n \leqslant i+N-1$

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such that $|a(n,x)| < r$ for all $x \in X$. Therefore

$$\sup_{k,x} |a(k,x)| < r + M . \sup_{x} |a(1,x)|$$

and Corollary 3.12 shows that a is a coboundary.

We now prove the induction step. Suppose the theorem is true when m is equal to some positive integer k and let $a: \mathbb{Z} \times X \to \mathbb{R}^{k+1}$ be any recurrent cocycle whose compositions with non-zero linear functionals are all non-trivial. In particular a is not a coboundary, so Lemma 4.13 shows that $E(a) \neq \{0\}$. By choosing, if necessary, a new coordinate basis in $\mathbb{R}^{k+1}$ we may assume that $\{(0, \ldots, 0, n) : n \in \mathbb{Z}\} \subset E(a)$. Let $a_j : \mathbb{Z} \times X \to \mathbb{R}$, $1 \leq j \leq k+1$, be the components of a relative to the chosen coordinate system and consider the cocycle $(a_1, \ldots, a_k): \mathbb{Z} \times X \to \mathbb{R}^k$. It clearly satisfies the hypotheses of the theorem; so the extension $S_{(a_1, \ldots, a_k)}$ is topologically transitive. By using Corollaries 3.8 and 3.10 we can choose an $x \in X$ such that $(x,v) \in \overline{O}_{(a_1, \ldots, a_k)}(x,0)$ for all $v \in \mathbb{R}^k$ and $L_w(\overline{O}_a(x,0)) = \overline{O}_a(x,0)$ for all $w \in E(a)$.

For any $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ there exists a sequence (n_i) such that $S^{n_i}_{(a_1, \ldots, a_k)}(x,0) \to (x,v)$. For each $i \geqslant 1$ let t_i be the residue modulo one of $a_{k+1}(n_i,x)$. Each t_i lies in the interval $[0,1]$ so we may assume (by replacing n_i by a subsequence if necessary) that $t_i \to t$ for some $t \in [0,1]$. It follows from the choice of x that for every $i \geqslant 1$,

$$(T^{n_i}(x), (a_1(n_i,x), \ldots, a_k(n_i,x), t_i)) \in \overline{O}_a(x,0).$$

Therefore by taking the limit as $i \to \infty$,

$$(x, (v_1, \ldots, v_k, t)) \in \overline{O}_a(x,0)$$

and this is enough to show that $(v_1, \ldots, v_k, t) \in E(a)$ by using Theorem 3.7.

We have now proved that for every $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ there exists a $t \in [0,1]$ such that $(v_1, \ldots, v_k, t) \in E(a)$. As $E(a)$ is a closed subgroup of $\mathbb{R}^{k+1}$ it is possible to choose a new coordinate basis in $\mathbb{R}^{k+1}$ so that

$$\{(v_1, \ldots, v_k, n) : v_j \in \mathbb{R},\ 1 \leq j \leq k,\ n \in \mathbb{Z}\} \subset E(a).$$

If the argument of the last two paragraphs is now applied to a with the roles of a_1 and a_{k+1} interchanged, then we have enough information about $E(a)$ to deduce that $E(a) = \mathbb{R}^{k+1}$. Corollary 3.8 then completes the proof of the theorem.

Before Theorem 4.14 can be applied in the case where $m > 2$ it is necessary to know that the cocycle is recurrent. It is not known whether it is sufficient for the Haar integral of each of the components to be zero. However, in the case that $K = K$ Proposition 4.17 does give a sufficient condition for the cocycle to be recurrent. Definition 4.15 and Lemma 4.16, which are required for its proof, are stated in terms of Borel measurable functions because they are needed in Chapter Three.

Definition 4.15.

Let $\phi : K \to \mathbb{R}$ be a Borel measurable function. The variation of ϕ, $\mathrm{Var}(\phi)$ is defined to be the variation of the function $\hat{\phi} : [0,1] \to \mathbb{R}$, with $\hat{\phi}(\alpha) = \phi(\exp(2\pi i \alpha))$ for every $\alpha \in \mathbb{R}$.

Lemma 4.16. (The Denjoy-Koksma inequality.)

Let $\alpha \in \mathbb{R}$ be irrational and let $T:K \to K$ be the minimal homeomorphism with $T(k) = k.\exp(2\pi i \alpha)$ for all $k \in K$. Let $\varphi: K \to \mathbb{R}$ be a Borel measurable function whose variation is finite (a function of bounded variation). Suppose that p and q are integers with $q > 0$ and $|\alpha - p/q| < q^{-2}$. Then for all $k \in K$,

$$\left| \sum_{i=0}^{q-1} \varphi(T^i(k)) - q\int \varphi \, d\lambda \right| \leq \mathrm{Var}(\varphi),$$

where λ is the Haar measure on K.

Proof. See Denjoy ([5]).

Proposition 4.17.

Let $T:K \to K$ be a minimal translation on the circle and let $a: \mathbb{Z} \times X \to \mathbb{R}^m$ $(m \geq 1)$ be a cocycle for T. Let $a_j: \mathbb{Z} \times X \to \mathbb{R}$, $1 \leq j \leq m$ be the components of a. If each of the functions $a(1,.)$ has finite variation and zero Haar integral then a is a recurrent cocycle.

Proof. Let α be a positive irrational number such that $T(k) = k.\exp(2\pi i \alpha)$ for all $k \in K$. Theorem I, Chapter I of [3] shows that there are infinitely many pairs of coprime positive integers (p,q) such that $|\alpha - p/q| < q^{-2}$. Lemma 4.16 when applied to each $a_j(q,.)$ shows that for every $k \in K$ the set

$$\{q: |a_j(q,k)| \leq \mathrm{Var}(a_j(1,.)),\ 1 \leq j \leq m\}$$

is infinite. Corollary 2.4 now shows that a is recurrent.

§5. Orbit closures.

The results of §3 showed how the essential values of a recurrent cocycle describe "most" of the orbit closures under the corresponding extension. A non-compact extension of a transformation on a compact space always has some orbit closures which behave differently, unless its cocycle is a coboundary. In particular this means that these extensions are never minimal. In this section we examine these fine details of orbit structure and the question of whether non-compact extensions possess minimal orbit closures.

We shall assume that the base space X is compact; this means that we may also assume that our extending group G is a product; $G = \mathbb{Z}^n \times \mathbb{R}^m \times C$ $(n,m \geqslant 0)$, where C is a compact group. Because of this Proposition 5.1 is strong enough to prove the existence of the nonconformist orbit closures mentioned above. Both the statement and proof of this proposition are derived from Lemma 14.09 of [6].

Proposition 5.1.

Let X be a compact metric space and let $T:X \to X$ be a minimal homeomorphism. If $a:\mathbb{Z} \times X \to \mathbb{R}$ is a recurrent cocycle for T then there exists a point $x_0 \in X$ such that $a(n,x_0) \leqslant 0$ for all $n \in \mathbb{Z}$.

Proof. If a is a coboundary then there exists a continuous function $\varphi: X \to \mathbb{R}$ such that $a(n,x) = \varphi(T^n(x)) - \varphi(x)$ for all $n \in \mathbb{Z}$ and $x \in X$. Let x_0 be a point at which φ attains its maximum value; then clearly $a(n,x_0) \leq 0$ for all $n \in \mathbb{Z}$.

Now suppose that a is not a coboundary. Proposition 2.3 and Corollary 3.13 show that there exists an $x \in X$ with $(x,0) \in R(S_a)$ and $\operatorname{Sup}_{n \in \mathbb{Z}} a(n,x) = \infty$. Because $(x,0) \in R(S_a)$ there exist increasing sequences $n_i \to \infty$, $n_i' \to \infty$ such that $a(n_i,x) < 1$ and $a(-n_i',x) < 1$ for all $i \geqslant 1$. For each $i \geqslant 1$ let n_i'', $-n_i' < n_i'' < n_i$ be such that $a(n_i'',x) \geqslant a(n,x)$ for all n with $-n_i' < n < n_i$. Let $x_i = T^{n_i''}(x)$; then for all n with $-n_i' - n_i'' < n < n_i - n_i''$, we have $a(n,x_i) = a(n+n'',x) - a(n'',x) \leqslant 0$ because $-n_i' < n-n_i'' < n_i$. Because X is compact some subsequence of (x_i) must converge to a point $x_0 \in X$. Call this subsequence (x_i'); then for every $n \in \mathbb{Z}$ the inequality above implies that $a(n,x_0) = \operatorname{Lim}_i a(n,x_i') \leqslant 0$.

The set of points $B^+(a) = \{x: \operatorname{Sup}_{n \in \mathbb{Z}} a(n,x) < \infty\}$ which Proposition 5.1 shows to be non-empty can have some peculiar properties. (Of course anything which can be said about $B^+(a)$ applies also to $B^-(a) = \{x: \operatorname{Inf}_{n \in \mathbb{Z}} a(n,x) > -\infty\}$ for which a statement similar to Proposition 5.1 can be proved.) When $a: \mathbb{Z} \times X \to \mathbb{R}$ is a cocycle which is not a coboundary Corollary 3.13 shows that $B^+(a)$ is the complement of a residual set. It is therefore not unreasonable to consider that it is essentially negligible. However, because X is compact there is always a T-invariant ergodic probability measure μ on X and this gives another method of assessing the importance of $B^+(a)$. In his paper [16] E.A.Siderov shows how to construct cocycles $a:\mathbb{Z}\times X\to\mathbb{R}$ such that $\mu(B^+(a))=1$ for any minimal $T:X\to X$ with invariant ergodic measure μ. These cocycles are the coboundaries of functions that are measurable (with respect to the μ-completion of the Borel σ-algebra) but not continuous. Siderov also gives examples of cocycles with $\mu(B^+(a))=\mu(B^-(a))=0$.

Another rather odd phenomenon is that it may be possible to deduce the existence of elements of $E(a)$ from the behavior of a point lying in $B^+(a)$. This is shown by the following example which is based on the methods of [16]. The method works even when X is not compact.

Example 5.2.

Let X be a complete metric space and let $T:X\to X$ be a minimal homeomorphism. Fix a point $x_0\in X$ which lies in the closure of its own forward orbit under T. We will describe a cocycle $a:\mathbb{Z}\times X\to\mathbb{R}$ with the property that $x_0\in B^+(a)$ and $(x_0,-1)\in\overline{O}_a(x_0,0)$; it will be defined as the limit of a series of coboundaries.

The coboundaries $b_i:\mathbb{Z}\times X\to\mathbb{R}$ are chosen inductively. Let b_0 be the zero cocycle and set $k_0=1$, $\delta_0=1$. Now suppose that coboundaries $b_0,\ldots,b_{i-1}$ positive integers $k_0,\ldots,k_{i-1}$ and positive real numbers $\delta_0,\ldots,\delta_{i-1}$ have already been chosen. Using Lemma 4.1

choose a coboundary $b_i: \mathbb{Z}\times X \to \mathbb{R}$, $k_i \in \mathbb{Z}_+$ and δ_i with $\delta_{i-1} > \delta_i > 0$ to satisfy the conditions:

(1) $d_X(T^{k_i}(x_0), x_0) < \mathrm{Inf}\{2^{-i}, \delta_{i-1}\}$;

(2) $b_i(k_i, x_0) = -1$;

(3) $\sup_{x\in X} |b_i(1,x)| < 1/2^i k_{i-1}$;

(4) For all $n \in \mathbb{Z}$, $0 \geq b_i(n, x_0) \geq -1$;

(5) $b_i(n, x_0) = 0$, whenever $T^n(x_0) \in B_X(x_0, \delta_i)$.

(Note that Conditions (1) and (5) make it possible to satisfy (4).) Condition (3) ensures that the series of functions $\sum_{i=0}^{\infty} b_i(1,.)$ converges uniformly. Let $a: \mathbb{Z}\times X \to \mathbb{R}$ be the cocycle with $a(1,.) = \sum_{i=0}^{\infty} b_i(1,.)$; then for all $n \in \mathbb{Z}$ and $x \in X$, $a(n,x) = \sum_{i=0}^{\infty} b_i(n,x)$.

It is clear from Condition (4) that $x_0 \in B^+(a)$. Condition (1) implies that whenever $0 < j < i$, $T^{k_i}(x_0) \in B_X(x_0, \delta_j)$ so $a(k_i, x_0) = \sum_{j=i}^{\infty} b_j(k_i, x_0)$. Also if the k_i have all been chosen to be as small as possible then Condition (1) and Equation 4.1.6 together imply that $k_0 \leq k_1 \leq k_2 \ldots$. Therefore using Condition (3) we have:

$$\begin{aligned} |a(k_i, x_0) + 1| &= \Big|\sum_{j=i+1}^{\infty} b_j(k_i, x_0)\Big| \\ &\leq k_i \sum_{j=i+1}^{\infty} \sup_{x\in X} |b_j(1,x)| \\ &< \sum_{j=i+1}^{\infty} 2^{-j} = 2^{-i}. \end{aligned}$$

So $S_a^{k_i}(x_0, 0) \to (x_0, -1)$; we have $(x_0, -1) \in \overline{O}_a(x_0, 0)$ but $(x_0, 1) \notin \overline{O}_a(x_0, 0)$.

We now turn to the question of the existence of minimal orbit closures for non-compact extensions. If the base space X is compact we need only deal with extensions by groups of the form: $G = \mathbb{R}^m \times \mathbb{Z}^n$. In this case Lemma 4.12 yields the following result.

Proposition 5.3.

Let X be a compact metric space and let $T:X \to X$ be a minimal homeomorphism. Let G be a locally compact second countable abelian group which has no non-trivial compact subgroups and let $a:\mathbb{Z} \times X \to G$ be a cocycle for T which is not a coboundary. Then if M is a minimal orbit closure under S_a there exist points $y_+, y_- \in M$ such that $O_a^+(y_-)$ and $O_a^-(y_-)$ are discrete.

Proof. We shall prove the existence of y_+. As S_a^{-1} is a skew-product extension of T^{-1} applying this proof to S_a^{-1} will prove the existence of y_-.

Suppose y_+ does not exist; then for each $y \in M$ the ω-limit set of y $\bigcap_{i=0}^{\infty} \overline{\{S_a^{n+i}(y): n \in \mathbb{Z}_+\}}$ is a non-empty closed S_a-invariant set and is therefore equal to M. So for each $y \in M$, $y \in \overline{O_a^+(y)}$. Fix $y_0 = (x,g) \in M$ and choose $\varepsilon > 0$ such that $\overline{B_G(g,\varepsilon)}$ is compact. Let U be the set $M \cap (X \times B_G(g,\varepsilon))$ and apply Lemma 4.12 to U and the restriction of S_a to M. The lemma shows that there exists an $N \in \mathbb{Z}_+$ such that, for all $i \in \mathbb{Z}_+$, one of the points $S_a^i(y_0), \ldots, S_a^{i+N-1}(y_0)$ lies in U. This implies that for all $i \in \mathbb{Z}_+$, $a(i,x)$ belongs to the compact set

$$\overline{B_G(0,\varepsilon)} + N.\{a(1,x'): x' \in X\}.$$

Corollary 3.12 now shows that a is a coboundary. We have assumed that a is not a coboundary and this contradiction shows that y_+ must exist.

Besicovitch gives an example in [2] of a topologically transitive real line extension which has some discrete orbits. Such orbits clearly constitute examples of minimal orbit closures. There is no known example of a minimal orbit closure of a non-trivial extension by a non-compact group with $y_+ \neq y_-$. It is possible for such an extension to have no minimal orbit closures. The following proposition shows that this occurs for extensions of the cocycles of Proposition 4.17.

<u>Proposition</u> <u>5.4</u>.

Let $T:K \to K$ be a minimal translation on the circle and let $a:\mathbb{Z} \times K \to \mathbb{R}^m$ $(m \geqslant 1)$ be a cocycle for T which is not a coboundary. Let $a_j:\mathbb{Z} \times K \to \mathbb{R}$, $1 \leqslant j \leqslant m$ be the components of a. If each of the functions $a_j(1,.)$ has finite variation and zero Haar integral then S_a has no minimal orbit closures.

<u>Proof</u>. It is shown in the course of the proof of Proposition 4.17 that for every $k \in K$ the set

$$\{q > 0: \ |a_j(q,k)| \leqslant \mathrm{Var}(a_j(1,.)), \ 1 \leqslant j \leqslant m\}$$

is infinite. It follows that for every $y \in Y$ the ω-limit set of y is non-empty. Consequently the point y_+ of Proposition 5.3 cannot exist.

CHAPTER THREE

A Class of Measurable Real Line Extensions

§1. Introduction.

In this chapter we adopt the second reading of the definitions of Chapter One and use the results of [15] to investigate a collection of concrete examples of real line extensions. Before we can describe these transformations it is necessary to introduce some notation.

For each real number x let $\langle x \rangle$ denote the fractional part of x; the difference between x and the largest integer which is less than or equal to x. Let $\| x \|$ denote the distance from x to the nearest integer, so that $\| x \| = \operatorname{Inf}\{\langle x \rangle, 1 - \langle x \rangle\}$. We will use the Greek letters λ and ν to stand for one and two-dimensional Lebesgue measure respectively. These will be the only measures appearing in this chapter, so all words and phrases such as "ergodic" and "almost all" should be interpreted as referring to one of them. The Greek letters α, β, and θ will be used for real numbers, usually lying in the interval $[0,1)$. As is usual the notation $\chi_{[0,\beta)}$ is used to indicate the characteristic function of the half-open interval $[0,\beta)$. We will frequently need to refer to the half-open interval $[0,1)$ and the open interval $(0,1)$; they will therefore be denoted by X and X' respectively.

We are now ready to define the subject matter of this chapter.

Definition 1.1.

For each $\alpha \in X'$ let $T_\alpha : X \to X$ be the transformation with $T_\alpha(x) = \langle x + \alpha \rangle$ $(x \in X)$. Each T_α is an automorphism of the standard probability space $(X, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra. Accordingly we define $a : \mathbb{Z} \times X \times X' \times X' \to \mathbb{R}$ to be the unique function which satisfies the conditions:

(1) $a(1, x, \alpha, \beta) = \chi_{[0,\beta)}(x) - \beta \quad (x \in X,\ \alpha, \beta \in X')$;

(2) For every pair $(\alpha, \beta) \in X' \times X'$ the function, $a(\cdot, \cdot, \alpha, \beta) : \mathbb{Z} \times X \to \mathbb{R}$ is a cocycle for T_α.

For each pair $(\alpha, \beta) \in X' \times X'$ we will use the notation $S_{\alpha,\beta}$ as an abbreviation of $S_{a(\cdot,\cdot,\alpha,\beta)}$, the skew-product extension of T defined by $a(\cdot,\cdot,\alpha,\beta)$.

It is clear that for each $\alpha \in X'$, the transformation T_α is isomorphic with the translation on the circle by $\exp(2\pi i \alpha)$. When α is irrational T_α is ergodic; the extensions $S_{\alpha,\beta}$ $(\beta \in X')$ therefore provide an illusration for the theory of extensions of ergodic automorphisms contained in [15]. Our first two lemmas show that they are sufficiently non-trivial to warrant further investigation.

Lemma 1.2.

Let α and β be elements of X' with α irrational; then the cocycle $a(\cdot,\cdot,\alpha,\beta)$ is a coboundary if and only if there exists a $k \in \mathbb{Z}$, $k \neq 0$, such that $\beta = \langle k\alpha \rangle$.

Proof. See Petersen ([13]).

Lemma 1.3.

Let α and β be elements of X' with α irrational; then the cocycle $a(.,.,\alpha,\beta)$ is recurrent.

Proof. This is an immediate consequence of the theorem in Appendix A.

The cocycles $a(.,.,\alpha,\beta)$, $(\alpha,\beta \in X')$ have been studied in [4], [10] and [15] in the special case where β is a rational fraction. The most general results are those of Conze; they will be described in §3. In this chapter we are concerned with the more general case where β is irrational. In §2 we prove two theorems which show that the set of pairs (α,β) for which $S_{\alpha,\beta}$ is ergodic contains almost all of X'×X', in both the metric and the topological sense. The results of §3 are concerned with the special case where there exist integers k and l such that $\langle k\alpha\rangle = \langle l\beta\rangle$. In this case we again obtain ergodic extensions, but only after the cocycles have been modified.

In proving these results we use two main tools. The first of these is the theory of skew-product extensions of an ergodic automorphism contained in [15]; the second is the theory of Diophantine approximation.

For each positive integer q it is possible to approximate an irrational $\alpha \in X'$ by choosing the integer p $(0 \leqslant p \leqslant q)$ which minimises $|\alpha - p/q|$. If this is done then $q|\alpha - p/q| = |q\alpha - p| = \|q\alpha\|$. It is not difficult to see that for every irrational α, $\operatorname{Inf}_{q \leqslant n} \|q\alpha\| \to 0$ as $n \to \infty$. The speed of convergence gives a measure of how well α is approximated by rationals. The following result gives an estimate of this speed for almost all $\alpha \in X'$.

Lemma 1.4.

Let W be the set $\{\alpha \in X' : \operatorname{Lim}_q \operatorname{Inf} q\|q\alpha\| = 0\}$; then $\lambda(W) = 1$.

Proof. This follows immediately from Theorem I, Chapter VII of [3] if the function ψ which appears there is defined by $\psi(q) = 1/q.\log(q)$.

The definition of the set W implies that if $\alpha \in W$ then there exists a sequence of pairs of coprime positive integers (p_n, q_n) such that $q_n^2|\alpha - p_n/q_n| = q_n\|q_n\alpha\| \to 0$ as $n \to \infty$. We shall make use of this fact in the next section.

§2. "Almost everywhere" theorems.

The aim of this section is to prove the following two theorems.

Theorem 2.1.

For each $(\alpha,\beta)\in X'\times X'$ let $S_{\alpha,\beta}$ be the skew-product extension of T_α described in Definition 1.1; then the set of points (α,β) for which $S_{\alpha,\beta}$ is ergodic has measure one.

Theorem 2.2.

For each $(\alpha,\beta)\in X'\times X'$ let $S_{\alpha,\beta}$ be the skew-product extension of T_α described in Definition 1.1; then the set of points (α,β) for which $S_{\alpha,\beta}$ is ergodic is a residual subset of the complete metric space $[0,1]\times[0,1]$.

Corollary 5.4 of [15] shows that $S_{\alpha,\beta}$ is ergodic if and only if $E(a(.,.,\alpha,\beta)) = \mathbb{R}$. We begin the proof of Theorem 2.1 with a series of lemmas which will show that $\nu(\{(\alpha,\beta): 1\in E(a(.,.,\alpha,\beta))\}) = 1$. In the first four of these we shall work with a fixed but arbitrary irrational α belonging to the set W of Lemma 1.4.

Lemma 2.3.

Let p and q be coprime positive integers and suppose that for some real number η with $0<\eta<1/2$, $q^2|\alpha - p/q| < \eta$. For each integer k with $0 \leqslant |k| \leqslant q-1$ let $\bar{k}$ be the residue of kp modulo q; then $|<k\alpha> - \bar{k}/q| < \eta/q$ for all k with $0<|k|<q-1$.

For each integer i with $0 \leqslant i \leqslant q-1$ let k_i $(0 \leqslant k_i \leqslant q-1)$ be the unique solution of the equation: $\bar{k}_i = i$ and let k_i' $(-(q-1) \leqslant k_i' \leqslant 0)$ be the unique solution of the equation: $\bar{k}_i' = i$. Then

$$0 < \begin{Bmatrix} <k_1\alpha> \\ <k_1'\alpha> \end{Bmatrix} < \begin{Bmatrix} <k_2\alpha> \\ <k_2'\alpha> \end{Bmatrix} < \ldots < \begin{Bmatrix} <k_{q-1}\alpha> \\ <k_{q-1}'\alpha> \end{Bmatrix} < 1.$$

Proof. For any k with $0 \leqslant |k| \leqslant q-1$ we have

$$\begin{aligned} |<k\alpha> - \bar{k}/q| &= |<k\alpha> - <kp/q>| \\ &\leqslant |k||\alpha - p/q| \\ &< q|\alpha - p/q| < \eta/q \end{aligned}$$

and this proves the first assertion. The second assertion follows from the combination of this with the trivial inequalities: $0 < 1/q < 2/q < \ldots < (q-1)/q < 1$.

The next lemma is the cornerstone of the proof of Theorem 2.1. Before stating it we need to introduce an auxiliary transformation $\hat{T}_\alpha : X \times X' \to X \times X'$ defined by: $\hat{T}_\alpha(x,\beta) = (T_\alpha(x),\beta)$. It is clear that $\hat{T}_\alpha$ preserves the Lebesgue measure on $X \times X'$ and that $a(.,.,\alpha,.) : \mathbb{Z} \times X \times X' \to \mathbb{R}$ is a cocycle for $\hat{T}_\alpha$.

Lemma 2.4.

Let R be a closed rectangle of the form $[u_1,v_1] \times [u_2,v_2]$ which is contained in $X \times X'$. Let (p_n,q_n) be a sequence of pairs of coprime positive integers with the property that $q_n^2|\alpha - p_n/q_n| \to 0$ as $n \to \infty$. Then for every ε with $1/2 > \varepsilon > 0$,

$$\operatorname{Lim\,Inf}_n \ \nu(R \cap \hat{T}_\alpha^{-q_n}(R) \cap \{(x,\beta): \ |1 - |a(q_n.x,\alpha,\beta)|\ | < \varepsilon\}) > \frac{\varepsilon^2\nu(R)}{64}.$$

Proof. Suppose that n is sufficiently large that the following conditions are satisfied:

(1) $|\alpha - p_n/q_n| < \varepsilon/2q_n^2$;

(2) $\| q_n\alpha \| < (v_1 - u_1)/4$;

(3) $q_n > 32/(v_1 - u_1)$;

(4) $q_n > 16/(v_2 - u_2)$.

We shall show that

$$\nu(R \cap \hat{T}_\alpha^{-q_n}(R) \cap \{(x,\beta): \ |1 - |a(q_n,x,\alpha,\beta)|\ | < \varepsilon\}) > \frac{\varepsilon^2\nu(R)}{64}. \qquad -(5)$$

We begin by considering the discontinuities of the function $a(q_n,.,\alpha,.):X \times X' \to \mathbb{R}$. The domain of this function may be extended to $\mathbb{R} \times X'$ by defining

$a(q_n,x,\alpha,\beta) = a(q_n,<x>,\alpha,\beta)$ for every $(x,\beta) \in \mathbb{R} \times X'$.

Let D be the set of points in $X \times X'$ at which the extended function is discontinuous. Then $D = B \cup C$ where

$$B = \bigcup_{i=0}^{q_n-1} \{(x,\beta) \in X \times X': \ T_\alpha^i(x) = 0\}$$

and $$C = \bigcup_{j=0}^{q_n-1} \{(x,\beta) \in X \times X': \ T_\alpha^j(x) = \beta\}.$$

This is because $a(q_n,x,\alpha,\beta) = \sum_{i=0}^{q_n-1} \chi_{[0,\beta)}(T_\alpha^i(x)) - q_n\beta$ $\qquad -(6)$

for all $(x,\beta) \in X \times X'$.

The intersection of B and C contains exactly q_n^2 points which we will name $c_{0,0}, \ldots, c_{q_n-1,q_n-1}$. Potentially there are two different orderings on $B \cap C$; the ordering inherited from the lexicographic ordering of $\mathbb{R}^2$ and the lexicographic ordering on the subscripts. We assign the names $c_{i,j}$ $(0 \leqslant i,j \leqslant q_n-1)$ in such a way that these two orderings coincide. For each pair of subscripts (i,j) there exists a pair of integers $(i',j'(i))$ with $0 \leqslant i', j'(i) \leqslant q_n-1$ such that

$$c_{i,j} = \{(x,\beta): T_\alpha^{i'}(x) = 0\} \cap \{(x,\beta): T_\alpha^{j'(i)}(x) = \beta\}$$

$$= (<-i'\alpha>, <(j'(i)-i')\alpha>).$$

It follows from the second statement of Lemma 2.3 that $\overline{i'} = i$ and $\overline{j'(i)-i'} = j$ (with $p = p_n$ and $q = q_n$). For each pair of subscripts (i,j) $(0 \leqslant i,j \leqslant q_n-1)$ let $b_{i,j}$ be the point $(<-i'\alpha>, j/q_n)$; then the distance between $c_{i,j}$ and $b_{i,j}$ is

$$|<(j'(i)-i')\alpha> - j/q_n| = |<(j'(i)-i')\alpha> - \overline{(j'(i)-i')}/q_n|$$

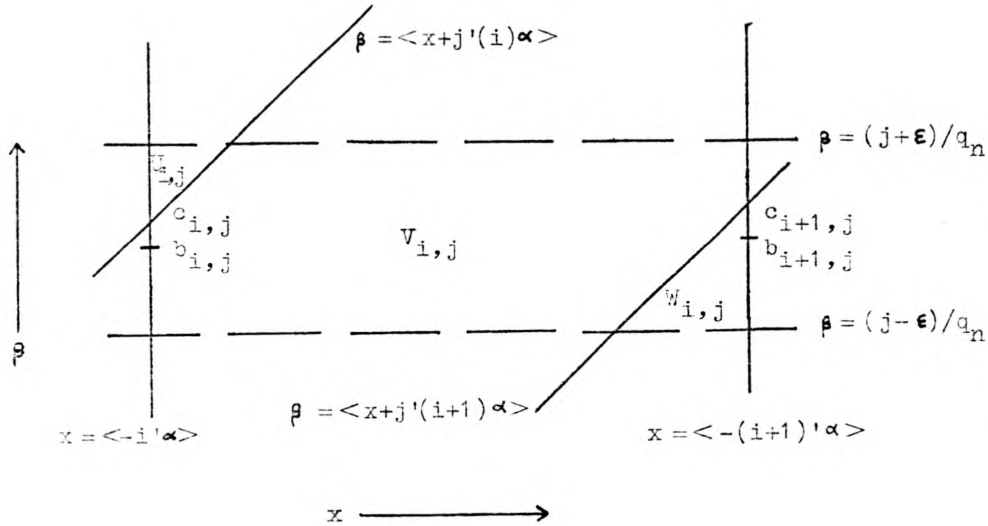
$$< \varepsilon/2q_n \qquad -(7)$$

as is shown by the first assertion of Lemma 2.3 and Condition (1).

Now let $H \subset X \times X'$ be the set:

$$\bigcup_{i=0}^{q_n-1} \{(x, <\beta>): x \in X, \beta \in \mathbb{R}, |\beta - i/q_n| < \varepsilon/q_n\}.$$

The diagram on the next page shows a neighbourhood of a typical element $c_{i,j}$ of $B \cap C$. It is assumed that $1 \leqslant i \leqslant q_n - 2$ and $1 \leqslant j \leqslant q_n - 1$ so that $c_{i,j}$ does not lie close to the boundary of $X \times X'$. The vertical lines represent part of B, the diagonal lines represent part of C and the horizontal lines represent part of the boundary of H.



Let $U_{i,j}$, $V_{i,j}$ and $W_{i,j}$ be the interiors of the two triangles and the irregular hexagon formed by the intersections of these lines, as shown. These sets are well defined whenever $0 \leqslant i \leqslant q_n - 2$ and $1 \leqslant j \leqslant q_n - 1$; our next step is to estimate their measures. $U_{i,j}$ and $W_{i,j}$ are the interiors of right-angled isoceles triangles, so Inequality (7) implies that

$$\varepsilon^2/8q_n^2 < \nu(U_{i,j}) , \nu(W_{i,j}) < 9\varepsilon^2/8q_n^2 . \qquad -(8)$$

We also have

$$\nu(U_{i,j} \cup V_{i,j} \cup W_{i,j}) = |<-i'\alpha> - <-(i+1)'\alpha>|.2\varepsilon/q_n$$

and

$$||<-i'\alpha> - <-(i+1)'\alpha>| - 1/q_n| = |i'-(i+1)'||\alpha - p_n/q_n|$$

$$< q_n|\alpha - p_n/q_n|$$

$$< \varepsilon/2q_n .$$

Therefore

$$\varepsilon(2-\varepsilon/2)/q_n^2 < \nu(U_{i,j} \cup V_{i,j} \cup W_{i,j}) < \varepsilon(2+\varepsilon/2)/q_n^2 .$$

Combining this with Inequality (8) we have

$$\nu(U_{i,j}) , \nu(V_{i,j}) , \nu(W_{i,j}) > \varepsilon . \nu(U_{i,j} \cup V_{i,j} \cup W_{i,j})/32. \qquad -(9)$$

Now let I be the collection of all index pairs (i,j) with $0 \leqslant i \leqslant q_n - 2$ and $1 \leqslant j \leqslant q_n - 1$ for which $U_{i,j} \cup V_{i,j} \cup W_{i,j} \subset R \cap \hat{T}_\alpha^{-q_n}(R)$. Inequality (2) ensures that

$$\nu(R \cap \hat{T}_\alpha^{-q_n}(R)) > 3\nu(R)/4$$

and Inequalities (3) and (4) then ensure that the subdivision of $R \cap \hat{T}_\alpha^{-q_n}(R)$ (by the collection of lines consisting of D and the boundary of H) is sufficiently fine that

$$\nu\Big(\bigcup_{(i,j) \in I} (U_{i,j} \cup V_{i,j} \cup W_{i,j})\Big) > \varepsilon . \nu(R)/2 . \qquad -(10)$$

To complete the proof of the lemma we examine the values taken by $a(q_n, ., \alpha, .)$ on the sets $U_{i,j}$, $V_{i,j}$ and $W_{i,j}$ when $(i,j) \in I$.

The set H was chosen so that

$$\|a(q_n,x,\alpha,\beta)\| = \|q_n\beta\| < \varepsilon \qquad -(11)$$

for all $(x,\beta)\in H$. Because $a(q_n,.,\alpha,.)$ is continuous on each $U_{i,j}$ $((i,j)\in I)$ we can unambiguously define $u_{i,j}$ to be the nearest integer to $a(q_n,x,\alpha,\beta)$ for any $(x,\beta)\in U_{i,j}$. The quantities $v_{i,j}$ and $w_{i,j}$ may be similarly defined for all $(i,j)\in I$. It follows from Equation (6) that

$$u_{i,j}-1 = v_{i,j} = w_{i,j}+1 \qquad -(12)$$

for every $(i,j)\in I$. For each $\beta\in X'$ we now apply Lemma 2.4.16 to T_α and the function which sends $\exp(2\pi ix)$ to $\chi_{[0,\beta)}(x)-\beta$ $(x\in X)$. The result shows that $|a(q_n,x,\alpha,\beta)|\leqslant 2$ for all $(x,\beta)\in X\times X'$. Equation (12) now implies that at least one of the quantities $|u_{i,j}|$, $|v_{i,j}|$, $|w_{i,j}|$ is equal to one for every $(i,j)\in I$. Inequality (5) follows from the combination of this fact with (9),(10) and (11).

Let $\{J_i:\ 1\leqslant i<\infty\}$ be the collection of all closed subintervals of X which have rational endpoints. If $J\subset X$ is any interval then for every $\varepsilon>0$ there exists an i such that $\lambda(J\Delta J_i)<\varepsilon$. We shall make use of this property of $\{J_i\}$ in proving Lemma 2.5

Lemma 2.5.

For all $i\in\mathbb{Z}_+$, all ε with $1/2>\varepsilon>0$ and almost all $\beta\in X'$ there exists an $n\in\mathbb{Z}_+$ such that

$$\lambda(J_i\cap T_\alpha^{-q_n}(J_i)\cap\{x:\ |1-|a(q_n,x,\alpha,\beta)||<\varepsilon\}) > \frac{\varepsilon^2\lambda(J_i)}{128}. \qquad -(1)$$

Proof. Suppose this is false. Then for some $i \in \mathbb{Z}_+$ and $\varepsilon > 0$ there exists a measurable set $N \subset X'$ with $\lambda(N) > 0$ such that Inequality (1) fails whenever $\beta \in N$ and $n \in \mathbb{Z}_+$. The Lebesgue density theorem shows that there exists a closed interval $J \subset X$ with $\lambda(J \setminus N)/\lambda(J) < \varepsilon^2/128$. When applied to the rectangle $J_i \times J$ Lemma 2.4 shows that for all sufficiently large n the measure of the set

$$(J_i \times J) \cap \hat{T}_\alpha^{-q_n}(J_i \times J) \cap \{(x,\beta) : \big|\, |a(q_n, x, \alpha, \beta)| - 1 \,\big| < \varepsilon\}$$

is greater than $\varepsilon^2 \lambda(J_i)\, \lambda(J)/64$. This implies that for each such n the set of points $\beta \in J$ for which Inequality (1) holds has measure greater than $\varepsilon^2 \lambda(J)/64$. So Inequality (1) must hold for some $\beta \in N$ and $n \in \mathbb{Z}_+$; but this contradicts the definition of N. So no such set can exist; this proves the lemma.

Lemma 2.6.

For all $i \in \mathbb{Z}_+$, all ε with $1/2 > \varepsilon > 0$ and almost all $\beta \in X'$ there exists an $m \in \mathbb{Z}$ such that

$$\lambda(J_i \cap T_\alpha^{-m}(J_i) \cap \{x : |a(m,x,\alpha,\beta) - 1| < \varepsilon\}) > \varepsilon^2 \lambda(J_i)/256 .$$

Proof. Suppose that for some $i \in \mathbb{Z}_+$, $\varepsilon > 0$, $\beta \in X'$ and $m \in \mathbb{Z}_+$ there exists $x \in J_i \cap T_\alpha^{-m}(J_i)$ with $\big|\, |a(m,x,\alpha,\beta)| - 1 \,\big| < \varepsilon$. Then either $|a(m,x,\alpha,\beta) - 1| < \varepsilon$ or $|a(m,x,\alpha,\beta) + 1| < \varepsilon$. In the second case $x \in J_i \cap T_\alpha^{m}(J_i)$ and $|a(-m,x,\alpha,\beta) - 1| < \varepsilon$, so we can substitute $-m$ for m. The conclusion now follows from Lemma 2.5.

Lemma 2.7.

Let (α,β) be any element of $X'\times X'$ and let $c: \mathbb{R}_+ \to \mathbb{R}_+$ be any function. Let $r \in \mathbb{R}$ and suppose that for every $i \in \mathbb{Z}_+$ and $\varepsilon > 0$ there exists an $m \in \mathbb{Z}$ such that

$$\lambda(J_i \cap T_\alpha^{-m}(J_i) \cap \{x: |a(m,x,\alpha,\beta) - r| < \varepsilon\}) > c(\varepsilon)\lambda(J_i).$$

Then $r \in E(a(.,.,\alpha,\beta))$.

Proof. Fix $\varepsilon > 0$ and let $A \subset X$ be any measurable set with $\lambda(A) > 0$. Choose $i \in \mathbb{Z}_+$ such that $\lambda(J_i \setminus A)/\lambda(J_i) < c(\varepsilon)/3$; this is possible by the Lebesgue density theorem. There exists an $m \in \mathbb{Z}$ such that

$$\lambda(A \cap T_\alpha^{-m}(A) \cap \{x: |a(m,x,\alpha,\beta) - r| < \varepsilon\})$$

$$\geq \lambda((A \cap J_i) \cap T_\alpha^{-m}(A \cap J_i) \cap \{x: |a(m,x,\alpha,\beta) - r| < \varepsilon\})$$

$$\geq \lambda(J_i \cap T_\alpha^{-m}(J_i) \cap \{x: |a(m,x,\alpha,\beta) - r| < \varepsilon\})$$

$$- \lambda(J_i \setminus A) - \lambda(T_\alpha^{-m}(J_i \setminus A))$$

$$> c(\varepsilon)\lambda(J_i) - 2\lambda(J_i \setminus A) > c(\varepsilon)\lambda(J_i)/3.$$

Hence $r \in E(a(.,.,\alpha,\beta))$.

Lemma 2.8.

Let F be the set: $\{(\alpha,\beta) \in X'\times X': 1 \in E(a(.,.,\alpha,\beta))\}$. Then F is measurable and $\nu(F) = 1$.

Proof. For each $\varepsilon > 0$ let $\varphi_\varepsilon : X' \times X' \to \mathbb{R}$ be the function with $\varphi_\varepsilon(\alpha, \beta) =$

$$\inf_{i \geq 1} \sup_{m \in \mathbb{Z}} \lambda(J_i \cap T_\alpha^{-m}(J_i) \cap \{x : |a(m,x,\alpha,\beta) - 1| < \varepsilon\}) / \lambda(J_i) .$$

It is not difficult to show that each φ_ε is a measurable function and this implies that the set

$$F' = \bigcap_{k \geq 1} \{(\alpha,\beta) : \varphi_{2^{-k}}(\alpha,\beta) > 0\}$$

is measurable. Lemma 2.6 applies to every $\alpha \in W$ and Lemma 1.4 shows that $\lambda(W) = 1$; so Fubini's theorem shows that $\nu(F') = 1$. Lemma 2.7 shows that $F' \subset F$ and because Lebesgue measure is complete this implies that F is measurable and $\nu(F) = 1$.

For each $l \in \mathbb{Z}_+$ let $l^{-1}.\mathbb{Z} = \{m/l : m \in \mathbb{Z}\}$; then each $l^{-1}.\mathbb{Z}$ is a closed subgroup of $\mathbb{R}$ and every non-trivial closed subgroup which contains one is of this form. We make use of this fact in proving the next two lemmas, which will enable us to deduce Theorem 2.1 from Lemma 2.8.

Lemma 2.9.

Let T be an ergodic automorphism of the standard probability space $(X, \mathcal{S}, \lambda)$ and let $\bar{a} : \mathbb{Z} \times X \to \mathbb{R}$ be a cocycle for T. Suppose that $1 \in E(\bar{a}) \neq \mathbb{R}$; then there exists an $l \in \mathbb{Z}_+$ such that $E(\bar{a}) = l^{-1}.\mathbb{Z}$ and $\exp(2\pi i l.\bar{a}(.,.)) : \mathbb{Z} \times X \to K$ is a coboundary.

<u>Proof</u>. Lemma 3.3 of [15] shows that $E(\overline{a})$ is a closed subgroup of $\mathbb{R}$: so there exists an $l \in \mathbb{Z}_+$ with $E(\overline{a}) = l^{-1}.\mathbb{Z}$. Proposition 3.12 of [15] proves the existence of a coboundary $b:\mathbb{Z}\times X \to \mathbb{R}$ such that $\overline{a}(n,x) + b(n,x) \in E(\overline{a})$ for all $n \in \mathbb{Z}$ and $x \in X$. But then

$$\exp(2\pi i l.\overline{a}(.,.)) = \exp(2\pi i l.b(.,.))$$

and this is a coboundary

<u>Lemma 2.10</u>.

Let l be any positive integer and let $(\alpha,\beta) \in X' \times X'$ with α irrational. Then $\exp(2\pi i l.a(.,.,\alpha,\beta)):\mathbb{Z}\times X \to K$ is a coboundary for T_α if and only if there exists a $k \in \mathbb{Z}$ such that $\langle k\alpha\rangle = \langle l\beta\rangle$.

<u>Proof</u>. Suppose $\exp(2\pi i l.a(.,.,\alpha,\beta))$ is the coboundary of a measurable function $\psi: X \to \mathbb{R}$. Then for all $x \in X$

$$\psi(T_\alpha(x))/\psi(x) = \exp(2\pi i l(\chi_{[0,\beta)}(x) - \beta))$$

$$= \exp(-2\pi i l\beta).$$

This shows that $\exp(-2\pi i l\beta)$ is an eigenvalue of the unitary operator that T_α induces on $L^2(X,\mathcal{S},\lambda)$. It is well known that the spectrum of this operator is $\{\exp(2\pi i k\alpha): k \in \mathbb{Z}\}$, so for some $k \in \mathbb{Z}$, $\langle k\alpha\rangle = \langle l\beta\rangle$.

Conversely if $\langle k\alpha\rangle = \langle l\beta\rangle$ then

$$\exp(2\pi i l.a(n,x,\alpha,\beta)) = \exp(-2\pi i l\beta)$$

$$= \exp(-2\pi i k.T_\alpha(x))/\exp(-2\pi i kx)$$

for all $n \in \mathbb{Z}$ and $x \in X$.

Proof of Theorem 2.1.

Lemmas 2.9 and 2.10 show that if $(\alpha,\beta)\in X'\times X'$ and $1\in E(a(.,.,\alpha,\beta))\neq \mathbb{R}$ then there exists a $k\in\mathbb{Z}$ and an $l\in\mathbb{Z}_+$ with $<k\alpha> = <l\beta>$. A simple application of Fubini's theorem shows that the set of points (α,β) for which this is true has measure zero. The theorem now follows directly from our Lemma 2.8 and Corollary 5.4 of [15].

We now turn to the task of proving Theorem 2.2; most of the work has already been done.

Lemma 2.11.

For each $i\in\mathbb{Z}_+$ and ε with $1/2>\varepsilon>0$ let $\varphi_{i,\varepsilon}:\mathbb{Z}\times X'\times X'\to\mathbb{R}$ be the function with

$$\varphi_{i,\varepsilon}(m,\alpha,\beta) = \lambda(J_i\cap T_\alpha^{-m}(J_i)\cap\{x: |a(m,x,\alpha,\beta)-1|<\varepsilon\}).$$

Suppose that for some $(\alpha_0,\beta_0)\in X'\times X'$ there exist i,ε and $m\in\mathbb{Z}_+$ such that $\varphi_{i,\varepsilon}(m,\alpha_0,\beta_0)>0$. Then $\varphi_{i,\varepsilon}(m,.,.):X'\times X'\to\mathbb{R}$ is continuous at (α_0,β_0).

Proof. For all $x\in X$

$$a(m,x,\alpha_0,\beta_0) = \sum_{j=0}^{\infty}\chi_{[0,\beta_0)}(<x+j\alpha_0>) - m\beta_0.$$

So for all $x\in X$, $\|a(m,x,\alpha_0,\beta_0)\| = \|-m\beta_0\|<\varepsilon$. Let k be the nearest integer to $-m\beta_0$ and suppose that $\beta\in X'$ satisfies the inequality:

$$|\beta-\beta_0| < (\varepsilon - \|-m\beta_0\|)/m \; ; \qquad -(1)$$

then k is the nearest integer to $-m\beta$ and $\|-m\beta\|<\varepsilon$.

This implies that for all $\alpha \in X'$ and all $\beta \in X'$ which satisfy Inequality (1),

$$\varphi_{i,\varepsilon}(m,\alpha,\beta) = \lambda(J_i \cap T_\alpha^{-m}(J_i) \cap \{x: \sum_{j=0}^{m-1} \chi_{[0,\beta)}(<x+j\alpha>) = k+1\}).$$

So, for these (α,β),

$$|\varphi_{i,\varepsilon}(m,\alpha,\beta) - \varphi_{i,\varepsilon}(m,\alpha_0,\beta_0)|$$

$$\leqslant \lambda(\{x: \sum_{j=0}^{m-1} \chi_{[0,\beta)}(<x+j\alpha>) \neq \sum_{j=0}^{m-1} \chi_{[0,\beta_0)}(<x+j\alpha_0>)\})$$

$$+ \lambda(T_\alpha^{-m}(J_i) \Delta T_{\alpha_0}^{-m}(J_i))$$

$$\leqslant 2m|\alpha - \alpha_0| + \sum_{j=0}^{m-1} |<-j\alpha>-<-j\alpha_0>| + |<\alpha-j\beta>-<\alpha_0-j\beta_0>|$$

$$\leqslant 2m|\alpha-\alpha_0| + \sum_{j=0}^{m-1} 2j|\alpha-\alpha_0| + |\beta-\beta_0|$$

$$\leqslant 2m(m+1)|\alpha-\alpha_0| + m|\beta-\beta_0|.$$

The set of points (α,β) for which this last inequality holds is an open neighbourhood of (α_0,β_0). The lemma is proved.

<u>Lemma 2.12</u>.

Let F be the set $\{(\alpha,\beta) \in X' \times X': 1 \in E(.,.,\alpha,\beta)\}$; then F is residual subset of $[0,1] \times [0,1]$.

<u>Proof</u>. For each $i \in \mathbb{Z}_+$ and ε with $1/2 > \varepsilon > 0$ let $G_{i,\varepsilon}$ be the set

$$\{(\alpha,\beta): \sup_{m \in \mathbb{Z}} \varphi_{i,\varepsilon}(m,\alpha,\beta) > \varepsilon^2 \lambda(J_i)/256\}.$$

It follows from Lemma 1.4 and Lemma 2.6 that each $G_{i,\varepsilon}$

is a dense subset of $[0,1] \times [0,1]$ and Lemma 2.11 shows that it is an open set. Therefore the intersection

$$\bigcap_{i=1}^{\infty} \bigcap_{k=2}^{\infty} G_{i,2^{-k}}$$

is a dense G_δ-subset of $[0,1] \times [0,1]$. Lemma 2.7 shows that this intersection is contained in F.

Proof of Theorem 2.2.

It follows from Lemma 2.9 and Lemma 2.10 that

$$\{(\alpha,\beta) : 1 \in E(a(.,.,\alpha,\beta)) \neq \mathbb{R}\} \subset \bigcup_{k=-\infty}^{\infty} \bigcup_{l=1}^{\infty} \{(\alpha,\beta) : \langle k\alpha \rangle = \langle l\beta \rangle\}.$$

This is clearly a countable union of nowhere dense sets so Lemma 2.12 shows that $\{(\alpha,\beta) : E(.,.,\alpha,\beta) = \mathbb{R}\}$ is a residual subset of $[0,1] \times [0,1]$. The proof is completed by Corollary 5.4 of [15].

§3. The special case: $\langle k\alpha\rangle = \langle l\beta\rangle$.

Although Theorems 2.1 and 2.2 show that most of the extensions $S_{\alpha,\beta}$ are ergodic they do not yield a single concrete example of an ergodic transformation. In this section we study the cocycle $a(.,.,\alpha,\beta)$ in the special case where $\langle k\alpha\rangle = \langle l\beta\rangle$, for some $k\in\mathbb{Z}$ and $l\in\mathbb{Z}_+$. Lemma 2.10 shows that the corresponding extension is never ergodic but the reason for this is simple; it is possible to modify the cocycle by adding a suitable coboundary so as to obtain a new cocycle $a'_{\alpha,\beta}:\mathbb{Z}\times X\to l^{-1}.\mathbb{Z}$. We shall show that in "most" cases the new extension $S'_{\alpha,\beta}:X\times l^{-1}.\mathbb{Z}\to X\times l^{-1}.\mathbb{Z}$, which corresponds to this cocycle, is ergodic. We also obtain an explicit description of the quantities α and β, so it is possible to obtain concrete examples.

We shall use the following terminology: if α and β are two elements of X' which satisfy an equation $\langle k\alpha\rangle = \langle l\beta\rangle$ then this equation is called <u>reduced</u> if there is no l' properly dividing l and k' dividing k such that $\langle k'\alpha\rangle = \langle l'\beta\rangle$.

<u>Lemma 3.1.</u>

Let $(\alpha,\beta)\in X'\times X'$ be a pair of irrational numbers which satisfy the reduced equation $\langle k\alpha\rangle = \langle l\beta\rangle$ for some non-zero $k\in\mathbb{Z}$ and $l\in\mathbb{Z}_+$. Let c be the highest common factor of k and l; let $k' = k/c$ and let $l' = l/c$. Then there exists an integer s with $-l<s<c$ and an irrational

number $\theta \in X'$ such that $\beta' = \langle k'\theta \rangle$ and $\alpha = \langle l'\theta \rangle$ where $\beta' = \beta + s/l$. Also if s is not zero then it and c are coprime.

<u>Proof</u>. Clearly $(k\alpha - l\beta) \in \mathbb{Z}$; let s' be the residue of $(k\alpha - l\beta)$ modulo c and let s be either s' or s' − 1, whichever ensures that $\beta' = \beta + s/l \in X'$. Then

$\langle l'\beta' \rangle = \langle l'(\beta + s/l) \rangle = \langle l' \langle \beta + s'/l \rangle \rangle = \langle l'\beta + s'/c \rangle$,

so that

$$\begin{aligned} |\langle k'\alpha \rangle - \langle l'\beta' \rangle| &= \langle\langle k'\alpha \rangle - \langle l'\beta' \rangle\rangle \\ &= \langle\langle k'\alpha \rangle - \langle l'\beta + s'/c \rangle\rangle \\ &= \langle\langle k'\alpha - l'\beta \rangle - s'/c \rangle \\ &= 0 . \end{aligned}$$

So $d = k'\alpha - l'\beta' \in \mathbb{Z}$. Because k' and l' are coprime there exist integers u and v such that $ul' - vk' = d$. Then

$$l'(\beta' + u) = d + vk' + l'\beta'$$

and $$(\beta' + u)/k' = (l'\beta' + d)/l'k' + v/l' .$$

But $$(l'\beta' + d)/k' = \alpha ,$$

so $$(\beta' + u)/k' = (\alpha + v)/l' .$$

Let $\theta = \langle (\alpha + v)/l' \rangle$; then $\langle l'\theta \rangle = \langle \alpha + v \rangle = \alpha$ and $\langle k'\theta \rangle = \langle \beta' + u \rangle = \beta'$.

Finally, suppose that $j > 1$ and j divides both s and c. Then

$$\begin{aligned} \langle (k/j)\alpha \rangle = \langle (c/j)k'\alpha \rangle &= \langle (c/j)l'\beta' \rangle = \langle (c/j)(l'\beta + s/c) \rangle \\ &= \langle (c/j)l'\beta \rangle = \langle (l/j)\beta \rangle . \end{aligned}$$

This is contrary to our assumption that the equation $\langle k\alpha \rangle = \langle l\beta \rangle$ is reduced. Therefore s and c are coprime.

By using Lemma 3.1 we can break the problem down into four separate cases. We assume that α and β are elements of X' which satisfy the reduced equation $\langle k\alpha \rangle = \langle l\beta \rangle$ and that α is irrational.

Case 1: $l = 1$.

In this case Lemma 1.2 shows that $a(.,.,\alpha,\beta)$ is a coboundary.

Case 2: $k = 0$.

Here $\langle l\beta \rangle = 0$ so that $\beta = s/l$ where s and l are coprime and $1 \leqslant s \leqslant l-1$. The cocycle $a(.,.,\alpha,\beta)$ clearly takes values only in the group $l^{-1}.\mathbb{Z}$. Let $S'_{\alpha,\beta} : X \times l^{-1}.\mathbb{Z} \to X \times l^{-1}.\mathbb{Z}$ be the corresponding $l^{-1}.\mathbb{Z}$-extension of T_α. It follows from a theorem of Conze ([4], Theorem 5) that for each rational $\beta \in X'$ the extension $S'_{\alpha,\beta}$ is ergodic for almost all $\alpha \in X'$. A stronger result, which is proved in both [4] and [15], holds when $k = 0$ and $l = 2$. Here $\beta = 1/2$ and $S'_{\alpha,1/2}$ is ergodic for every irrational $\alpha \in X'$.

Case 3: $l \geqslant 2$, $k \neq 0$, l divides k.

Here we have $c = l$ and $\theta = \alpha$ in Lemma 3.1; so $\beta' = \langle k'\alpha \rangle$ and $\beta = \langle k'\alpha \rangle - s/l$. Therefore for all $x \in X$,

$$a(1,x,\alpha,\beta) = \chi_{[0,\beta)}(x) - \beta$$

$$= \begin{cases} (\chi_{[0,\langle k'\alpha\rangle)}(x) - \langle k'\alpha\rangle) - (\chi_{[\beta,\langle k'\alpha\rangle)}(x) - s/l) & \text{if } s > 0, \\ (\chi_{[0,\langle k'\alpha\rangle)}(x) - \langle k'\alpha\rangle) + (\chi_{[\langle k'\alpha\rangle,\beta)}(x) + s/l) & \text{if } s < 0, \end{cases}$$

$$=\begin{cases} a(1,x,\alpha,<k'\alpha>) - a(1,<x-\beta>,\alpha,s/l) & \text{if } s>0, \\ a(1,x,\alpha,<k'\alpha>) + a(1,<x-k'\alpha>,\alpha,-s/l) & \text{if } s<0. \end{cases}$$

The cocycle $a(.,.,\alpha,<k'\alpha>)$ is an example of Case 1 and is therefore a coboundary. Let $a'_{\alpha,\beta}:\mathbb{Z}\times X \to l^{-1}.\mathbb{Z}$ be the cocycle with

$$a'_{\alpha,\beta}(n,x) = \begin{cases} -a(n,<x-\beta>,\alpha,s/l) & \text{if } s>0, \\ a(n,<x-k'\alpha>,\alpha,-s/l) & \text{if } s<0, \end{cases}$$

for all $n\in\mathbb{Z}$ and $x\in X$. Let $S'_{\alpha,\beta}$ be the $l^{-1}.\mathbb{Z}$-extension of T which is defined by $a'_{\alpha,\beta}$. It is easy to see that $S'_{\alpha,\beta}$ is conjugate to $S'_{\alpha,|s|/l}$ and so has the same ergodic properties. The study of the present case therefore reduces to that of Case 2.

Case 4: $l\geq 2$, $k\neq 0$, l does not divide k.

This is the most important case and the main subject of this section.

Let $\varphi:\mathbb{Z}\times X\to\mathbb{R}$ be the function with $\varphi(x)=-kx/l$ and let $b:\mathbb{Z}\times X\to\mathbb{R}$ be the corresponding coboundary for T_α. Let $a'_{\alpha,\beta} = a(.,.,\alpha,\beta) - b$; then for all $n\in\mathbb{Z}$ and $x\in X$,

$$\begin{aligned} <l.a'_{\alpha,\beta}(n,x)> &= <l(-n\beta - \varphi(<x+n\alpha>) + \varphi(x))> \\ &= <-ln\beta + k(x+n\alpha) - kx> \\ &= <-ln\beta + kn\alpha> \\ &= <n(l\beta - k\alpha)> = 0. \end{aligned}$$

So $a'_{\alpha,\beta}$ takes values only in $l^{-1}.\mathbb{Z}$. Let $S'_{\alpha,\beta}$ be the $l^{-1}.\mathbb{Z}$-extension defined by $a'_{\alpha,\beta}$; we shall show that $S'_{\alpha,\beta}$ is ergodic under suitable conditions on α and β.

Lemma 3.2.

Let $\alpha, \beta \in X'$ be two irrationals which satisfy the reduced equation $\langle k\alpha \rangle = \langle l\beta \rangle$ where $k \neq 0$, $l \geqslant 2$ and l does not divide k. Let s, c, k', l', β' and θ be the quantities defined in Lemma 3.1. Suppose that $\operatorname{Lim\,Inf}_q q\|ll'q\theta\| = 0$ so that there exists a sequence of pairs of coprime integers (p_n, q_n) such that ll' divides each q_n and $q_n^2|\theta - p_n/q_n| \to 0$. For each $n \geqslant 1$ let $r_n = q_n/l$ and for each $\varepsilon > 0$ define $D_{n,\varepsilon}$ to be the set:

$\{x: |a(cr_n, x, \alpha, \beta) - a(cr_n, x, \langle p_n/cr_n \rangle, \langle (k'p_n - sr_n)/q_n \rangle)| \geqslant \varepsilon\}$.

Then for every $\varepsilon > 0$, $\lambda(D_{n,\varepsilon}) \to 0$ as $n \to \infty$.

Proof. For every $\varepsilon > 0$, $n \geqslant 1$ and $x \in X$,

$$a(cr_n, x, \alpha, \beta) = \sum_{i=0}^{cr_n - 1} \chi_{[0,\beta)}(\langle x + i\alpha \rangle) - cr_n\beta$$
$$= \sum_{i=0}^{cr_n - 1} \sum_{m=-\infty}^{0} \chi_{[-i\alpha, \beta - i\alpha)}(x + m) - cr_n\beta \quad -(1)$$

and when n is sufficiently large,

$$a(cr_n, x, \langle p_n/cr_n \rangle, \langle (k'p_n - sr_n)/q_n \rangle)$$
$$= \sum_{i=0}^{cr_n - 1} \chi_{[0, \langle (k'p_n - sr_n)/q_n \rangle)}(\langle x + i\langle p_n/cr_n \rangle \rangle) - cr_n \langle (k'p_n - sr_n)/q_n \rangle$$
$$= \sum_{i=0}^{cr_n - 1} \sum_{m=-\infty}^{0} \chi_{[-i\langle p_n/cr_n \rangle, \langle k'p_n/q_n \rangle - s/l - i\langle p_n/cr_n \rangle)}(x + m)$$
$$- cr_n \langle (k'p_n - sr_n)/q_n \rangle. \quad -(2)$$

The identity $\langle (k'p_n - sr_n)/q_n \rangle = \langle k'p_n/q_n \rangle - s/l$, which is used in the third line, may be justified as follows: when n is large $\langle k'p_n/q_n \rangle$ is a good approximation to β' and the right hand side of the identity approximates β. Now both sides of the identity lie in X' and they must be equal, as their difference is necessarily an integer.

Comparing the constant terms of (1) and (2) gives:

$$\begin{aligned} |cr_n(\langle k'p_n/q_n \rangle - s/l) - cr_n\beta| &= cr_n|\langle k'p_n/q_n \rangle - \beta'| \\ &= cr_n|\langle k'p_n/q_n \rangle - \langle k'\theta \rangle| \\ &\leqslant cr_n|k'||\theta - p_n/q_n| . \end{aligned}$$

This can be made less than any $\varepsilon > 0$ by making n large. When this is done $D_{n,\varepsilon}$ is prescisely the set of points $x \in X$ for which the sums in Equations (1) and (2) differ. So for large n,

$$\begin{aligned} \lambda(D_{n,\varepsilon}) &\leqslant \sum_{i=0}^{cr_n-1}(|i\alpha - i\langle p_n/cr_n \rangle| + |\beta - i\alpha - \langle k'p_n/q_n \rangle + s/l - i\langle p_n/cr_n \rangle|) \\ &\leqslant \sum_{i=0}^{cr_n-1}(2i|\alpha - \langle p_n/cr_n \rangle| + |\beta' - \langle k'p_n/q_n \rangle|) \\ &\leqslant cr_n(cr_n-1)l'|\theta - p_n/q_n| + cr_n|k'||\theta - p_n/q_n| \\ &< (q_n^2 + q_n|k'|)|\theta - p_n/q_n| . \end{aligned}$$

This tends to zero as $n \to \infty$.

<u>Theorem 3.3</u>.

Let $\alpha, \beta \in X'$ be two irrationals which satisfy the reduced equation $\langle k\alpha \rangle = \langle l\beta \rangle$ where $k \neq 0$, $l > 2$ and l does not divide k. Let l' and θ be the quantities defined in Lemma 3.1 and suppose that $\operatorname{Lim\,Inf}_q q\|ll'q\theta\| = 0$; then the $l^{-1}.\mathbb{Z}$-extension of T_α which is defined by $a'_{\alpha,\beta}$ is ergodic.

<u>Proof</u>. We will first show that $E(a(.,.,\alpha,\beta)) = l^{-1}.\mathbb{Z}$.
Let c, k' and s be the quantities of Lemma 3.1 and let (p_n), (q_n) and (r_n) be the sequences of Lemma 3.2. For each $n \geqslant 1$ let h_n be the residue of $k'p_n - sr_n$ modulo l'. This number is never zero because $k'p_n$ and l' are always coprime whereas l' always divides r_n. Following Equation 3.2.2 we have, for sufficiently large n and all $x \in X$,

$$a(cr_n, x, <p_n/cr_n>, <(k'p_n - sr_n)/q_n>)$$

$$= \sum_{i=0}^{cr_n-1} \sum_{m=-\infty}^{0} \chi_{[-i<p_n/cr_n>, <k'p_n/q_n> - s/l - i<p_n/cr_n>)}(x+m)$$

$$- cr_n<(k'p_n - sr_n)/q_n>$$

$$= \sum_{j=0}^{cr_n-1} \sum_{m=0}^{1} \chi_{[j/cr_n, j/cr_n + <(k'p_n - sr_n)/q_n>)}(x+m)$$

$$- cr_n<(k'p_n - sr_n)/q_n>$$

$$= \sum_{j=0}^{cr_n-1} \chi_{[jl'/q_n, (jl' + h_n)/q_n)}(x) - h_n/l'. \qquad -(1)$$

Here the penultimate equality follows from the fact that the sets $\{<-i<p_n/cr_n>>: 0 \leqslant i \leqslant cr_n - 1\}$ and $\{j/cr_n: 0 \leqslant j \leqslant cr_n - 1\}$ are identical. The final equality holds because, when both sides are considered as defining functions of x, they are both locally constant except for $2cr_n$ identical discontinuities and both have zero integral.

For each n for which Equation (1) holds the quantity h_n has one of the values: $1, 2, \ldots, l'-1$. By replacing

(p_n), (q_n) and (r_n) with suitable subsequences we may assume that h_n is independent of n and has some constant value, h.

Now let J_i be any member of the set $\{J_i : 1 \leq i < \infty\}$ which was used in §2. Given any $\epsilon > 0$, choose $n \in \mathbb{Z}_+$ sufficiently large that Equation (1) holds and the following inequalities are satisfied:

(2) $\lambda(D_{n,\epsilon}) < \lambda(J_i)/3$;

(3) $cr_n > 4\lambda(J_i)$;

(4) $\lambda(J_i \cap T_\alpha^{-cr_n}(J_i)) > \lambda(J_i)/2$.

Lemma 3.2 shows that it is possible to satisfy Inequality (2) and the fact that $\| cr_n\alpha \| = \| q_n\theta \| \to 0$ shows that (4) may be satisfied.

Equation (1) shows that the function

$$a(cr_n, \cdot, < p_n/cr_n >, < (k'p_n - sr_n)/q_n >) : X \to \mathbb{R}$$

is periodic, with period $1/cr_n$. In each cycle it takes the value $(l'-h)/l'$ on an interval of length $h/l'cr_n$ and the value $-h/l'$ on an interval of length $(l'-h)/l'cr_n$. Inequalities (3) and (4) imply that $J_i \cap T_\alpha^{-cr_n}(J_i)$ is an interval whose length is at least $2/cr_n$ and which therefore contains at least two complete cycles. A simple argument now shows that the measure of the set

$$J_i \cap T_\alpha^{-cr_n}(J_i) \cap \{x : a(cr_n, x, < p_n/cr_n >, < (k'p_n - sr_n)/q_n >) = -h/l'\}$$

is greater than $2(l' - h)\lambda(J_i)/3l'$. Together with Lemma 3.2 and Inequality (2) this implies that

$$\lambda(J_i \cap T_\alpha^{-cr_n}(J_i) \cap \{x : |a(cr_n, x, \alpha, \beta) + h/l'| < \epsilon\}) > \frac{(l'-h)\,\lambda(J_i)}{3l'}$$

Since this is true for every $i \in \mathbb{Z}_+$ and $\varepsilon > 0$, Lemma 2.7 shows that $-h/l' \in E(a(.,.,\alpha,\beta))$. A similar argument shows that $(l'-h)/l' \in E(a(.,.,\alpha,\beta))$ and it follows from Lemma 3.3 of [15] that $1 \in E(a(.,.,\alpha,\beta))$. Lemma 2.9 and Lemma 2.10 now show that $E(a(.,.,\alpha,\beta)) = l^{-1}.\mathbb{Z}$, for otherwise the equation $<k\alpha> = <l\beta>$ would not be reduced.

As the cocycles $a'_{\alpha,\beta}$ and $a(.,.,\alpha,\beta)$ differ only by a coboundary Lemma 3.2 of [15] shows that they have the same essential values. Corollary 5.4 of [15] completes the proof.

It is not difficult to construct concrete examples which satisfy the conditions of Theorem 3.3. In the case where $k = 4$ and $l = 20$ we have $ll' = 100$ and it is easy to find suitable irrationals α and β in terms of their decimal expansions. Let (i_n) be a sequence of positive integers with $i_{n+1} \geqslant 3i_n + 2$ $(n \geqslant 1)$ and let $\theta = \sum_{n=1}^{\infty} 10^{-i_n}$. Then if $q_n = 10^{i_n}$,

$$q_n \| 100 q_n \theta \| = 10^{i_n} \| 10^{i_n+2} \sum_{j=1}^{\infty} 10^{-i_j} \| = 10^{2i_n} \sum_{j=n+1}^{\infty} 10^{-i_j+2} < 2.10^{-i_n}.$$

So if $\alpha = <l'\theta> = 5\theta$ and $\beta = <k'\theta> + s/c = \theta + 1/4$ then α, β, k and l satisfy the conditions of Theorem 3.3 and $S'_{\alpha,\beta}$ is ergodic.

The condition $\operatorname{Lim\,Inf}_q q \| ll' q\theta \| = 0$ in Theorem 3.3 may appear to be rather restrictive but in fact this is not so. We conclude this chapter by proving an "almost everywhere" version of Theorem 3.3. By using the results of Conze it is possible to weaken the conditions on k and l.

Lemma 3.4.

Let W' be the set $\{\theta \in X' : \operatorname*{Lim\,Inf}_q q\|fq\theta\| = 0,\ f \in \mathbb{Z}_+\}$; then $\lambda(W') = 1$.

A proof of Lemma 3.4 can be obtained by modifying the proof of Theorem I, Chapter VII of [3], the theorem that was used to prove Lemma 1.4. Rather than reproduce part of [3] here we give only the details of the modifications. They are contained in Appendix B.

Lemma 3.5.

Let V be the set of all irrational numbers $\alpha \in X'$ for which the following statement holds: if $l^* \in \mathbb{Z}_+$ and $\theta \in X'$ satisfy the equation $\langle l^*\theta \rangle = \alpha$ then $\theta \in W'$.
Then $\lambda(V) = 1$.

Proof. Suppose that the lemma is false. Then there exists a measurable set $A \subset X'$ with $\lambda(A) > 0$ and an integer $l^* \in \mathbb{Z}_+$ such that for every $\alpha \in A$ there exists $\theta \in X' \setminus W'$ with $\langle l^*\theta \rangle = \alpha$. But this implies that

$$\lambda(X' \setminus W') \geqslant \lambda(\{\langle l^*\theta \rangle : \theta \in X' \setminus W'\})/l^* \geqslant \lambda(A)/l^* > 0$$

in contradiction to Lemma 3.4. This contradiction proves the lemma.

Theorem 3.6.

For almost all irrational numbers $\alpha \in X'$ the following statement is true: let k and l be two integers with $l \geqslant 2$ and let $\beta \in X'$ be any solution of the reduced equation $\langle k\alpha \rangle = \langle l\beta \rangle$; then the $l^{-1}.\mathbb{Z}$-extension of T_α which is defined by the cocycle $a'_{\alpha,\beta}$ is ergodic.

Proof. As the union of a countable collection of sets of measure zero has measure zero itself, it is permissible to consider each pair (k,l) separately. If $k = 0$ or if l divides k then we are dealing with an example of our Case 2 or Case 3. The conclusion then follows from Theorem 5 of [4]. If $k \neq 0$ and l does not divide k then Lemma 3.5 shows that the conditions of Theorem 3.3 are satisfied for almost all $\alpha \in X'$.

APPENDIX A

RECURRENCE OF CO-CYCLES AND RANDOM WALKS

GILES ATKINSON

Skew-product extensions of ergodic transformations by non-compact groups can be regarded as generalising the idea of random walks on such groups. The purpose of this paper is to consider the generalisations of two results on random walks on the real line. One of these carries over to skew-product extensions, the other does not.

DEFINITION. Let $(X, \mathscr{B}, \mu)$ be a probability space, $f : X \to \mathbb{R}$ a measurable function and $T : X \to X$ an ergodic automorphism of $(X, \mathscr{B}, \mu)$. We define the *co-cycle for* T given by f to be the function $a_f : \mathbb{Z} \times X \to \mathbb{R}$ with

$$a_f(n, x) = \sum_{i=0}^{n-1} f(T^i(x)) \qquad \text{for } n > 0,$$

$$a_f(0, x) = 0 \qquad \text{for all } x \in X,$$

$$a_f(n, x) = -a_f(-n, T^n(x)) \text{ for } n < 0.$$

The *skew-product extension* of T, determined by f, is the transformation $S_f : X \times \mathbb{R} \to X \times \mathbb{R}$;

$$S_f(x, t) = (T(x), t + f(x)).$$

The powers of S_f may be expressed in terms of the co-cycle a_f;

$$S_f{}^n(x, t) = (T^n(x), t + a_f(n, x)).$$

Finally we say that a_f is *recurrent* if and only if, for every $A \in \mathscr{B}$ with $\mu(A) > 0$ and every $\varepsilon > 0$, there exists an integer $n \neq 0$ such that

$$\mu(A \cap T^{-n}(A) \cap \{x : |a_f(n, x)| < \varepsilon\}) > 0.$$

Co-cycles which are not recurrent are called *transient*.

The structure of skew-product extensions into general locally compact abelian groups has been analysed by Schmidt in [1]. It is shown there that a co-cycle a_f is recurrent if and only if the corresponding extension S_f is conservative (Theorem 4.3).

A random walk on $\mathbb{R}$, which is defined by a probability measure λ on $\mathbb{R}$, may be realised as a skew-product extension in the following way: let X be the space $\prod_{i=-\infty}^{\infty} \mathbb{R}_i$ (each $\mathbb{R}_i = \mathbb{R}$), with the product Borel structure; let μ be the product of measures identical to λ in each factor; let T be the shift

$$T(\ldots x_0, x_1, \ldots x_i, \ldots) = (\ldots x_1, x_2, \ldots x_{i+1}, \ldots).$$

Finally we choose the function f to be the 0th co-ordinate projection;

$$f(\ldots x_0, x_1, \ldots x_i, \ldots) = x_0.$$

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The correspondence between this skew-product extension and the random walk is completed by observing that successive values of $a_f(n, x)$ describe the positions of points performing the walk. By Proposition 4.6 of [1], the recurrence of a_f implies that almost all points return arbitrarily close to zero, infinitely often, under the action of S_f. This is equivalent to the usual definition of the persistence of random walks. The following theorem is an extension to co-cycles of [2; Theorem 4, p. 203].

THEOREM. *Suppose $(X, \mathcal{B}, \mu)$ is a non-atomic probability space, $T : X \to X$ is an ergodic measure-preserving automorphism and $f : X \to \mathbb{R}$ is an integrable function. Then a_f is recurrent if and only if $\int f\,d\mu = 0$.*

Proof. Suppose $\int f\,d\mu \neq 0$. The ergodic theorem, applied to f, shows that for almost all $x \in X$

$$\lim_{|n| \to \infty} n^{-1}\, a_f(n, x) = \int f\,d\mu.$$

Therefore, for some $N > 0$, $\mu(B) > 3/4$, where B is the set

$$\{x : |a_f(n, x)| \geq 1 \text{ for all } |n| \geq N\}.$$

An application of Rohlin's Theorem to T proves the existence of a set $E \in \mathcal{B}$ such that

$$\mu\left(\bigcup_{n=0}^{4N} T^n(E)\right) > 3/4 \text{ and } E \cap T^n(E) = \emptyset,\ 1 \leq n \leq 4N.$$

At least one of the sets $T^n(E)$, with $N \leq n \leq 3N$, must intersect B in a set of positive measure. Let A be such an intersection. Clearly, $A \cap T^{-n}(A) = \emptyset$ for all $|n| \leq N$. As $A \subset B$, this implies that a_f cannot be recurrent.

Conversely suppose a_f is not recurrent. By Proposition 4.6 of [1], the set $M_x = \{n : |a_f(n, x)| \leq 1\}$ is finite for almost all $x \in X$. For each $n > 0$ let $A_n = \{x : \text{Card}\,(M_x) \leq n\}$; then there exists an N with $\mu(A_N) > 1/2$. The ergodic theorem applied to the characteristic function of A_N shows that, for almost all $x \in X$, and for sufficiently large n, over half the points $T^i(x)$, $1 \leq i \leq n$, are in A_N. Each of the corresponding $a_f(i, x)$ has at most N others within distance one of it. Suppose there are r of these values; then any interval containing them must have length greater than $[(r-1)/(N+1)]$. As $r > n/2$, this is greater than $(n-2)/(2(N+1))$. Hence for almost all $x \in X$, if n is sufficiently large

$$\sup_{0 \leq i \leq n} |a_f(i, x)| > \frac{n-2}{4(N+1)}.$$

This implies that for infinitely many $i > 0$, $|a_f(i, x)| > (i-2)/(4(N+1))$. An application of the ergodic theorem to f shows that

$$\left|\int f\,d\mu\right| \geq 1/(4(N+1)).$$

The theorem is proved.

For a random walk on the reals, defined by a probability measure λ, the series $\sum_{n=0}^{\infty} \lambda^{*n}(I)$ is important, where λ^{*n} is the n-th convolution power of λ and I is the interval $[-1, 1]$. In particular, the walk is recurrent in the sense of there being infinitely many returns to zero with probability one, if and only if the series diverges (see [2; pp. 200-203]). The following example shows that this result does not extend to co-cycles.

Let $\tilde{T} : (0, 1] \to (0, 1]$ be a Borel automorphism which is ergodic with respect to Lebesgue measure. Put $X = \bigcup_{i=1}^{\infty} (0, i^{-2}] \times \{i\}$ and let μ be the product of the Lebesgue and counting measures, normalised so that $\mu(X) = 1$. Let $T : X \to X$ be the ergodic transformation given by;

$$T(x, i) = \begin{cases} (x, i+1) & \text{if } 0 < x \leqslant (i+1)^{-2}, \\ (\tilde{T}(x), 1) & \text{if } (i+1)^{-2} < x \leqslant 1. \end{cases}$$

Let a_f be the co-cycle defined by the function;

$$f(x, i) = \begin{cases} 0 \text{ if } 0 < x \leqslant (i+1)^{-2}, \\ 2 \text{ if } (i+1)^{-2} < x \leqslant 1. \end{cases}$$

Since $\int f \, d\mu = 12/\pi^2$, a_f is transient. The series which corresponds to $\sum_{n=0}^{\infty} \lambda^{*n}$,

$$\begin{aligned} \sum_{n=0}^{\infty} \mu(\{x : |a_f(n, x)| \leqslant 1\}) &= \sum_{n=0}^{\infty} \mu(\{x : a_f(n, x) = 0\}) \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \mu((0, (i+n)^{-2}] \times \{i\}) \\ &= (6/\pi^2) \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} (i+n)^{-2}. \end{aligned}$$

The term i^{-2} occurs exactly i times in this sum, which may therefore be written as $\sum_{i=1}^{\infty} i^{-1}$. The series diverges in spite of the fact that a_f is transient.

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APPENDIX B

Lemma 3.3.4: A do-it-yourself kit.

To prove Lemma 3.3.4 it is enough to show that for any $f \in \mathbb{Z}_+$

$$\lambda(\{\theta \in X' : \operatorname{Lim}_q \operatorname{Inf} q\|fq\theta\| = 0\}) = 1 .$$

This follows immediately from setting $\psi(q) = 1/q.\log(q)$ in the following modified version of Theorem I, Chapter VII of [3].

Theorem.

Let f be any positive integer and let $\psi(q)$ be a monotone decreasing function of the integer variable $q > 0$ with $0 < \psi(q) < 1/2$. Then the set of inequalities

$$\|fq\theta\| < \psi(q)$$

has infinitely many integer solutions $q > 0$ for almost no or almost all numbers θ according as $\Sigma\,\psi(q)$ coverges or diverges.

We shall only be concerned with proving the theorem in the divergent case. The proof of Cassels' original formulation consists of five lemmas (Lemmas 1-3 and 5-8) and a section entitled "Proof of Theorem I (divergence, $n = 1$)" which contains two more lemmas (Lemmas 8 and 9). The alterations which must be made to these in order to obtain a proof of the above statement are set out below. They include the insertion of one entirely new lemma, Lemma 5*.

Lemma 1.

Substitute fq for q wherever it appears with the exceptions of the first line of the proof and the expression $\psi(q)$.

Lemmas 2, 3 and 5.

No change.

Before Lemma 6 insert:

Lemma 5*.

Let $\phi(q)$ be the number of integers p, $0 < p < q$ which are prime to q. Then for all positive integers f,

$$\phi(qf) \geqslant \phi(q) .$$

Proof. Theorem 62 of [7] states that

$$\phi(m) = m \prod_{p \mid m} (1 - 1/p) ,$$

where the product is taken over prime numbers only. (As is usual, $p \mid m$ means that p divides m.) Therefore, taking products over prime p, we have:

$$\begin{aligned} \phi(qf) &= qf \prod_{p \mid qf} (1 - 1/p) \\ &\geqslant f \prod_{\substack{p \mid f \\ p \nmid q}} (1 - 1/p)\, \phi(q) \\ &\geqslant \prod_{\substack{p \mid f \\ p \mid q}} p(1 - 1/p)\, \phi(q) \\ &\geqslant \phi(q) . \end{aligned}$$

Lemma 6.

In the statement and the note replace q by fq except in the first sentence and where it appears beneath a summation sign (as in $\sum_{q \leqslant Q}$). In the proof replace C_1 by C_1' and replace the calculation by:

$$\sum_{q \leqslant Q} (fq)^{-1}\phi(fq) \geqslant f^{-1} \sum_{q \leqslant Q} q^{-1}\phi(q)$$

$$= f^{-1}\left(\sum_{q \leqslant Q} \Phi(q)(q^{-1} + (q+1)^{-1}) + Q^{-1}\Phi(Q)\right)$$

$$> f^{-1}Q^{-1}\Phi(Q)$$

$$\geqslant f^{-1}C_1'Q = C_1 Q$$

(where $C_1 = f^{-1}C_1'$).

Lemma 7.

Substitute fq for q except where it appears as $\omega(q)$, $\chi(q)$ or beneath a summation sign.

Lemma 8.

No change.

Proof of Theorem I (divergence, $n = 1$) (including Lemma 9).

Substitute fq for q except where it appears as $\psi(q)$, $\tau(q)$, $\omega(q)$, β_q , γ_q or beneath a summation sign.

Lemma 10 and Lemma 11.

Substitute fq for q as above. Also substitute fr for r except where it appears as β_r , γ_r , γ_{qr} or beneath a summation sign. Do not substitute qf for f in γ_{qr}.

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