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Site percolation and isoperimetric inequalities for plane graphs

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Abstract

We use isoperimetric inequalities combined with a new technique to prove upper bounds for the site percolation threshold of plane graphs with given minimum degree conditions. In the process we prove tight new isoperimetric bounds for certain classes of hyperbolic graphs. This establishes the vertex isoperimetric constant for all triangular and square hyperbolic lattices, answering a question of Lyons and Peres. We prove that plane graphs of minimum degree at least 7 have site percolation threshold bounded away from $1/2$, which was conjectured by Benjamini and Schramm, and make progress on a conjecture of Angel, Benjamini, and Horesh that the critical probability is at most $1/2$ for plane triangulations of minimum degree 6. We prove additional bounds for stronger minimum degree conditions, and for graphs without triangular faces.

KEYWORDS

isoperimetric inequality, percolation, planar graphs

1 | INTRODUCTION

In their highly influential paper [5], Benjamini and Schramm made several conjectures that generated a lot of interest among mathematicians and led to many beautiful mathematical results [2, 4, 6, 7, 12, 13], just to name a few. Despite the substantial amount of work, most of these conjectures are still open, while for a few of them, hardly anything is known. One of their conjectures states that $\dot{p}_c(G) < 1/2$ on any planar graph G of minimal degree at least 7; they additionally conjecture that there are infinitely many infinite open clusters on the interval $(\dot{p}_c(G), 1 - \dot{p}_c(G))$. As Benjamini and Schramm observe in

their paper, every planar graph of minimal degree at least 7 is nonamenable. The conjecture has been verified for the d-regular triangulations of the hyperbolic plane in [6].

The connection between percolation thresholds and isoperimetric (or Cheeger) constants is well known, and in [5] it is proved that the site percolation threshold for a graph G is bounded above by $(1 + \dot{h}(G))^{-1}$, where $\dot{h}(G)$ is the vertex isoperimetric constant. In their book [14], Lyons and Peres give the edge isoperimetric constants for the regular hyperbolic tessellations $H_{d,d'}$, where $(d-2)(d'-2) > 4$ (which were established by Häggström, Jonasson, and Lyons [11]), and ask [14, Question 6.20] for the corresponding vertex isoperimetric constants.

Angel, Benjamini, and Horesh considered isoperimetric inequalities for plane triangulations of minimum degree 6 in [1], and proved a discrete analogue of Weil's theorem, showing that any such triangulation satisfies the same isoperimetric inequality as the Euclidean triangular lattice $\mathbb{T}_6$. They conjectured that $\mathbb{T}_6$ is extremal in other ways which might be expected to have connections with isoperimetric properties. First, they conjecture that the connective constant $\mu(T)$—that is, the exponential growth rate of the number of self-avoiding walks of length n on T—is minimized by $\mathbb{T}_6$ among triangulations of minimum degree at least 6. Second, they conjecture that percolation is hardest to achieve on $\mathbb{T}_6$ in the sense that both the critical probability for bond percolation $p_c(T)$ and the critical probability for site percolation $\dot{p}_c(T)$ are maximized by $\mathbb{T}_6$. Intuitively these conjectures are connected, in that if fewer long self-avoiding paths exist then long connections might be expected to be less robust, making percolation less likely to occur. See also [3] for several other conjectures regarding planar triangulations.

Georgakopoulos and Panagiotis [8] proved that $p_c(T) \leq 1/2$ for any planar triangulation T with minimum degree at least 6, and a well-known result of Grimmett and Stacey [10] shows that $\dot{p}_c(T) \leq 1-(1/2)^d$ when the degrees in T are bounded by d. The latter bound can be improved to $\dot{p}_c(T) \leq 1-\frac{1}{d-1}$ [8] but in both cases, the bounds converge to 1 as the maximal degree converges to infinity. We remark that Benjamini and Schramm [5] made an even stronger conjecture than that one of Angel, Benjamini, and Horesh mentioned above, namely that $\dot{p}_c(T) \leq 1/2$ for any planar triangulation without logarithmic cut sets.

In Section 3 we consider the conjecture of Angel, Benjamini, and Horesh for site percolation, and we prove the following theorem.

Theorem 1. *For any plane graph G of minimum degree at least 6,*

$$\dot{p}_c(G) \leq 2/3 \,.$$

In Section 4 we study plane graphs of minimal degree at least 7 and we prove the aforementioned conjecture of Benjamini and Schramm.

Theorem 2. *Let G be a plane graph with minimum degree at least $d \geq 7$. Then*

$$\dot{p}_c(G) \leq \frac{2 + \alpha_d}{(d-3)(1+\alpha_d)},$$

where $\alpha_d = \frac{d-6+\sqrt{(d-2)(d-6)}}{2}$. In particular, for plane graphs with minimum degree at least 7

$$\dot{p}_c(G) \leq \frac{2 + \alpha_7}{4(1 + \alpha_7)} \approx 0.3455.$$

For plane graphs without faces of degree 3, a minimum vertex degree of 5 is sufficient to ensure nonamenability; in Section 5 we give bounds on the site percolation threshold in this case.

Theorem 3. *Let G be a plane graph with minimum degree at least $d \geq 5$ and minimum face degree at least 4. Then*

$$\dot{p}_c(G) \leq \frac{(2 + \alpha_{d+2})(d - 2)}{(1 + \alpha_{d+2})(d^2 - 3d + 1)}.$$

As far as we know, these results are new even for the d-regular triangulations and quadrangulations of the hyperbolic plane. In the process, we obtain best-possible bounds on the vertex Cheeger constants $\dot{h}(G)$ of such graphs.

Theorem 4. *Let G be a plane graph with minimum degree at least $d \geq 7$. Then*

$$\dot{h}(G) \geq \alpha_d.$$

If G is the d-regular triangulation of the hyperbolic plane, then we have equality.

Theorem 5. *Let G be a plane graph with minimum degree at least $d \geq 5$ and minimum face degree at least 4. Then*

$$\dot{h}(G) \geq \alpha_{d+2}.$$

If G is the d-regular quadrangulation of the hyperbolic plane, then we have equality.

The second halves of the last two theorems answer the aforementioned question of Lyons and Peres for all cases with $d' = 3$ and $d' = 4$. In particular, the vertex isoperimetric constant for both the 7-regular triangulation and the 5-regular quadrangulation, α_7, is the golden ratio.

Our results on percolation thresholds follow from bounding the "surface to volume ratio" of *outer interfaces*. This notion was introduced in [8] to prove that in 2-dimensional Bernoulli percolation, the percolation density is an analytic function on the interval $(p_c, 1]$. In the case of triangulations, outer interfaces coincide with the standard notion of inner vertex boundary appearing in the literature. See also [9] for some further interesting properties and open problems regarding them.

2 | DEFINITIONS AND MAIN TECHNIQUE

Let G be an infinite connected locally finite plane graph, and fix a root vertex o. Throughout, we assume that G is embedded in the plane without accumulation points, that is, only finitely many vertices lie in the region enclosed by any cycle of the graph. Site percolation with intensity p on G is a random function $\omega : V(G) \mapsto \{0, 1\}$ for which $\{\omega(v) : v \in V(G)\}$ are independent Bernoulli variables with parameter p. We say that a vertex v is *occupied* in an instance ω if $\omega(v) = 1$ and *unoccupied* otherwise. If o is occupied, the *occupied cluster* of o is the component of o in the subgraph induced by all occupied vertices. The critical probability $\dot{p}_c(G)$ is the infimum of all intensities for which there is a positive probability of this cluster being infinite, which does not depend on o.

Let C_o be any finite connected induced subgraph containing o. The *outer interface* of C_o consists of all vertices of C_o meeting the unbounded face of C_o. Deleting all vertices of C_o from G divides the remaining graph into components, at least one of which lies in the unbounded face of C_o. Write C_∞
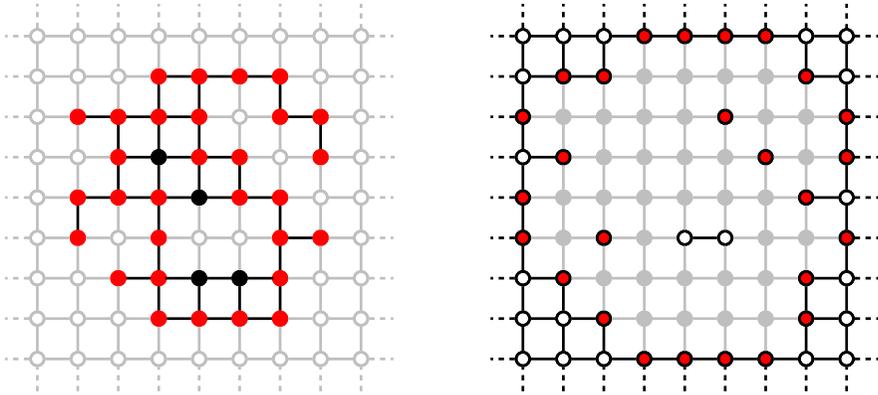
FIGURE 1 A connected induced subgraph C_o, with its outer interface highlighted (left) and the remaining graph with the corresponding outer boundary highlighted (right) [Colour figure can be viewed at wileyonlinelibrary.com]

for the union of all components lying in the unbounded face of C_o. The *outer boundary* of C_o is the set of vertices in C_∞ adjacent to C_o. Denote the outer interface by M and the outer boundary by B. By definition, M induces a connected subgraph of G and B forms a vertex cut separating o from infinity. See Figure 1 for an example; in this case C_∞ is not connected but there is also another component not in C_∞.

Note that, while (in general) neither M nor B uniquely determine C_o, each uniquely determines C_∞. In fact, it is the union of components of $G\backslash M$ lying in the outer face of M, and it is also the union of B and the set of vertices not connected to o in $G\backslash B$. Since M is also the outer interface of $G\backslash C_\infty$ and B coincides with the set of vertices in C_∞ adjacent to M, each of M and B uniquely determines the other. Let the set of feasible pairs (M, B) be $\mathfrak{D}$, and for each n let $\mathfrak{D}_n = \{(M, B) \in \mathfrak{D} : |B| = n\}$.

We say that a pair $(M, B) \in \mathfrak{D}$ *occurs* in a site percolation instance ω if $\omega(m) = 1$ for each $m \in M$ and $\omega(b) = 0$ for each $b \in B$. Note that, in a site percolation instance ω with $\omega(o) = 1$, the occupied cluster of o is infinite if and only if no pair $(M, B) \in \mathfrak{D}$ occurs. Since each outer boundary forms a vertex cut separating o from infinity, the occurrence of a pair certainly precludes o being in an infinite cluster, whereas if o is in a finite cluster C_o then the outer interface and outer boundary of C_o form an occurring pair.

Our main technique is to upper bound the ratio $|M|/|B|$ for $(M, B) \in \mathfrak{D}$ and consequently to show that the probability of occurrence of any given pair is decreasing for p above a certain value. Provided G satisfies an isoperimetric inequality of moderate strength, we then deduce that with positive probability no pair occurs. This latter condition essentially requires nonpositive curvature, and so the smallest minimum degree to guarantee this is 6 for the general case and 5 for the triangle-free case.

Theorem 6. *Suppose that there exist a real number α and a function $f(n)$ of subexponential growth with the following properties.*

(i) *For each pair $(M, B) \in \mathfrak{D}$, we have $|M| \leq \alpha|B|$.*
(ii) *For each outer boundary B, the component of o in $G\backslash B$ contains at most $f(|B|)$ vertices.*

Then $\dot{p}_c(G) \leq \frac{\alpha}{1+\alpha}$.

We will use a Peierls-type argument (see e.g., [15, Theorem 4.1]) to show that with positive probability o is in an infinite component for site percolation with any intensity $p > \frac{\alpha}{1+\alpha}$. It is sufficient

to show that with positive probability no pair $(M, B) \in \mathfrak{D}$ occurs; in fact we shall find it more convenient to work with a slightly weaker notion than occurrence. The key observations are that not too many pairs of any given size can occur for $p = \frac{\alpha}{1+\alpha}$, and that the expected number of occurring pairs at any given higher intensity is exponentially smaller.

In G, pick a geodesic R from o, going to infinity. For any $(M, B) \in \mathfrak{D}$, let R_B be the longest initial segment of R which does not intersect B (so the next vertex on R is the first intersection with B, which must exist since B is a vertex cut), and define $M' = M \backslash R_B$. We say that the pair (M, B) *almost occurs* in a percolation instance if the vertices of M' are occupied and the vertices of B are unoccupied.

We first need to show that (ii) gives a bound on the number of almost-occurring pairs.

Lemma 7. *At most $f(n)$ elements of $\mathfrak{D}_n$ almost occur in any instance ω.*

Proof. Fix an instance ω. Suppose $(M, B) \in \mathfrak{D}_n$ almost occurs in ω. Now define a new instance ω' by setting each vertex in R_B to be occupied, leaving the states of other vertices unchanged. Note that (M, B) occurs in ω'; in fact, since $M \cup R_B$ induces a connected subgraph, (M, B) is the outer interface and boundary of the occupied cluster of o in ω'. Thus ω' uniquely determines (M, B).

Since $o \in R_B$ and R_B lies entirely within the component of o in $G\backslash B$, by (ii) we have $1 \leq |R_B| \leq f(n)$. Thus, given ω, there are at most $f(n)$ possibilities for ω' and hence at most this many pairs $(M, B) \in \mathfrak{D}_n$ almost occur. ∎

Proof of Theorem 6. We can always assume that f is a strictly increasing function as otherwise we can work with $g(n) := n + \max_{k \leq n} f(k)$ which is a strictly increasing function and also satisfies the properties of f in the statement of the theorem.

Let $b_{n,m}$ be the number of pairs $(M, B) \in \mathfrak{D}_n$ for which $|M'| = m$. By (i), whenever $(M, B) \in \mathfrak{D}_n$ we have $|M| \leq \alpha n$, and so $b_{n,m} = 0$ for $m > \alpha n$.

Let X_n be the number of pairs $(M, B) \in \mathfrak{D}_n$ which almost occur, and write $q = \frac{\alpha}{1+\alpha}$. By Lemma 7, $X_n \leq f(n)$, and hence $\mathbb{E}_q(X_n) \leq f(n)$. Thus

$$f(n) \geq \sum_{(M,B) \in \mathfrak{D}_n} \mathbb{P}_q((M, B) \text{ almost occurs})$$

$$= \sum_{m \leq \alpha n} b_{n,m} q^m (1 - q)^n.$$

Now for any $p > q$ we have

$$\mathbb{E}_p(X_n) = \sum_{m \leq \alpha n} b_{n,m} p^m (1 - p)^n$$

$$= \sum_{m \leq \alpha n} b_{n,m} q^m (1 - q)^n (p/q)^m \left(\frac{1 - p}{1 - q} \right)^n$$

$$\leq \sum_{m \leq \alpha n} b_{n,m} q^m (1 - q)^n \cdot (p/q)^{\alpha n} \left(\frac{1 - p}{1 - q} \right)^n$$

$$\leq \left(\frac{(1 + \alpha)^{1+\alpha}}{\alpha^\alpha} p^\alpha (1 - p) \right)^n f(n).$$

The arithmetic-geometric mean inequality implies $p^\alpha (1 - p) < \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$. Consequently, since $f(n)$ is subexponential, $\sum_{n>0} \mathbb{E}_p(X_n)$ is finite, and in particular there is some n_0 such that $\mathbb{E}_p\left(\sum_{n \geq n_0} X_n \right) < 1$.

Let ω be a site percolation instance with intensity p. Notice that any pair (M, B) with $|R_B| \geq f(n_0)$ satisfies $|B| \geq n_0$ by the monotonicity of f. We can now deduce from the union bound that with

positive probability no pair (M, B) with $|R_B| \geq f(n_0)$ almost occurs in $\boldsymbol{\omega}$; call this event E_1. Let E_2 be the event that the first $f(n_0)$ vertices along R are all occupied in $\boldsymbol{\omega}$; clearly $\mathbb{P}_p(E_2) > 0$. As E_1 and E_2 are determined by the states of disjoint sets of vertices, they are independent, and so their intersection also has positive probability. In the event $E_1 \cap E_2$, no pair (M, B) occurs in $\boldsymbol{\omega}$. Thus o is in an infinite cluster with positive probability, so $p \geq \dot{p}_c(G)$. ∎

Following [5], we use the notation ∂S, where S is a set of vertices, to denote the set of vertices which are not in S but are adjacent to some vertex in S, and we define the (site) Cheeger constant

$$\dot{h}(G) = \inf_{|S| < \infty} \frac{|\partial S|}{|S|} .$$

3 | GRAPHS OF MINIMUM DEGREE AT LEAST 6

In this section we will consider the following problem of Angel, Benjamini, and Horesh [1].

Problem 8 (from [1, Problem 4.2]). If T is a plane triangulation with all vertex degrees at least 6, is it necessarily true that $\dot{p}_c(T) \leq 1/2$?

The authors ask also similar questions for bond percolation and self-avoiding walks.

While [1] does not give a precise definition of "plane triangulation," it is clear from context that accumulation points are not permitted. Without this assumption, the answer to the question would be negative. Indeed, consider the graph of an infinite cylinder formed by stacking congruent antiprisms, which can be embedded in the plane with a single accumulation point. Notice that all vertices have degree 6, and furthermore this graph contains infinitely many disjoint cut sets of fixed size separating o from infinity, hence $\dot{p}_c = 1$ by the Borel-Cantelli lemma. We remark in passing that the same graph demonstrates that [1, Theorem 2.4] also assumes accumulation points are not permitted. With this assumption, we firmly believe that the answer to Problem 8 is affirmative.

The aim of this section is to give upper bounds for $\dot{p}_c$ on general plane graphs of minimum degree at least 6 that are not necessarily triangulations. As we will see, the general case can be easily reduced to the case of triangulations.

The following two results of Angel, Benjamini, and Horesh [1] about plane triangulations T with finitely many vertices will be used in our proofs. A *plane triangulation* is a plane graph without accumulation points in which every bounded face has degree 3. The *boundary* of T is the set of vertices incident with the unbounded face. The remaining vertices of T are called *internal*. The *total boundary length* of T counts all edges induced by the boundary vertices of T exactly once, except for those not incident with a triangular face of T which are counted twice. When the boundary vertices of T span a cycle, we will say that its boundary is *simple* and T is a *disc triangulation*.

Lemma 9. *Let T be a disc triangulation with a simple boundary of length n and at least one internal vertex. Let T' be the triangulation induced by the internal vertices of T and let m be the total boundary length of T'. Suppose all internal vertices of T have degree at least 6. Then $m \leq n - 6$.*

Lemma 10. *Any disc triangulation with k vertices and n boundary vertices, and with all internal vertices having degree at least 6, satisfies $k \leq \lfloor n^2/12 + n/2 + 1 \rfloor$.*

For the remainder of this section, we fix some plane graph G of minimum degree at least 6. For any pair $(M, B) \in \mathfrak{D}_n$, consider the subgraph of G induced by B and the finite components of $G \backslash B$.

Write $B°$ for the set of vertices in B which were adjacent to the infinite component of $G\backslash B$; note that $|B°| \leq |B| = n$. By adding edges, if necessary, to the subgraph we may obtain a plane triangulation $T = T(M, B)$ with finitely many vertices, say k, and boundary $B°$. We do this by first adding edges joining vertices of $B°$ cyclically so that no other vertices meet the unbounded face, then adding internal edges to triangulate bounded faces of degree greater than 3. Each time that we triangulate a face between B and M, we do so by adding all diagonals meeting some vertex of B; this will ensure that in the final triangulation the subgraph induced by B is connected, and every vertex of M is adjacent to B. Let us fix such a triangulation T. Since M is connected, T must be a disc triangulation, and the choice of $B°$ ensures that all internal vertices have degree at least 6. The next corollary follows now from Lemma 10, noting that the number of internal vertices is at most $k - n$.

Corollary 11. *For each pair $(M, B) \in \mathfrak{D}_n$, the component of o in $G\backslash B$ consists of at most $\lfloor n^2/12 - n/2 + 1 \rfloor$ vertices.*

We wish to apply Lemma 9 to bound $|M|$ in terms of $|B|$, for $(M, B) \in \mathfrak{D}$. However, the boundary of $T(M, B)$ does not necessarily coincide with B; to deal with this we show that the triangulation can be modified to give a disc triangulation with boundary which is not too much larger than $|B|$. First we need a simple application of Euler's formula.

Lemma 12. *Consider a pair $(M, B) \in \mathfrak{D}$ and the corresponding triangulation $T = T(M, B)$. Let H be the subgraph of T induced by B, and f a face of H. Write ∂f for the boundary of f, and write $\partial°f \subseteq \partial f$ for that part of the boundary which forms a cycle separating f from infinity. Follow ∂f clockwise, writing down a list of vertices visited (so the same vertex can appear in the list multiple times). The length of the list so obtained is at most $2|\partial f| - |\partial°f|$, which is at most $2|\partial f| - 3$.*

Proof. We may assume that f is an internal face. Add a new vertex inside f, and join it to each vertex in the list in turn. This gives a plane multigraph H' with $|\partial f| + 1$ vertices in which each face incident with the new vertex has degree 3 and, since H was simple, all other faces have degree at least 3. The unbounded face has degree $|\partial°f|$, but there may be other faces inherited from H.

Suppose there are k internal faces. Then Euler's formula gives $e(H') - (k+1) = |\partial f| - 1$. Since each edge is incident with two faces, the sum of face degrees coincides with $2e(H')$. Hence $3k + |\partial°f| \leq 2e(H')$, so $k \leq 2|\partial f| - |\partial°f|$. Since the length of the list is the number of faces incident with the new vertex, which is at most k, the result follows. ∎

We are now ready to bound $|M|$ in terms of $|B|$ and $|B°|$.

Lemma 13. *For each pair $(M, B) \in \mathfrak{D}_n$ we have $|M| \leq 2n - |B°|$.*

Proof. Fix such a pair, and let $T = T(M, B)$ be the plane triangulation fixed above. Recall that T is formed in such a way that B is internally connected, and every vertex of M is adjacent to B. Let C_o be the component containing o of $T\backslash B$. Removing the vertices of C_o would leave a face f with boundary vertices B.

We define an "unzipping" operation on B as follows. We imagine that each edge of B has positive width so that each such edge has two edge-sides, where each of them is incident with exactly one face. Moreover, each edge-side has two ends reaching the endvertices of the corresponding edge. Proceed clockwise around the boundary of f along the edge-sides incident with f, recording the ends of edges-sides of f which are crossed in a cyclic ordering. Group these edge-ends by the vertex in B which they reach; since f is a face of $T\backslash C_o$, no edge of T inside f connects two vertices of B.
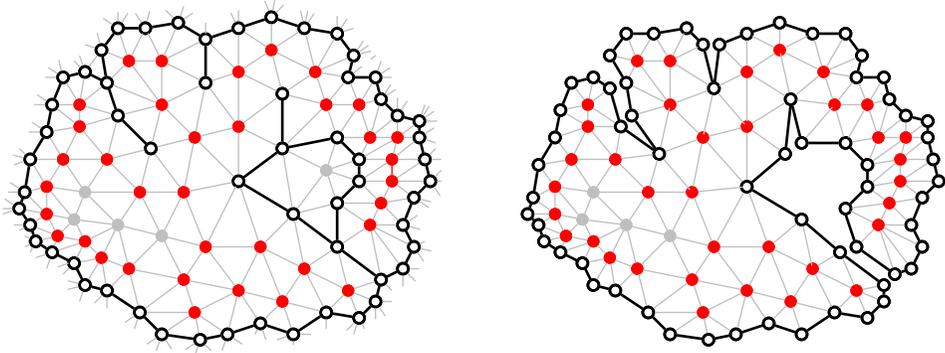
FIGURE 2 Unzipping an outer boundary B (bold vertices and edges); M is shown in red [Colour figure can be viewed at wileyonlinelibrary.com]

Note that, since T is a triangulation, every time a vertex of B is encountered when proceeding around ∂f clockwise, at least one edge-end of T incident with that vertex is crossed. Thus the number of groups in the cyclic ordering of edge-ends is precisely the number of entries in the list constructed in Lemma 12; since $|B| = n$, this list has at most $2n - |B°|$ entries.

We now "unzip" B by replacing vertices in B by the entries of the list, so that each vertex which appears more than once in the list is split into multiple vertices distinguished by list position. We also replace edges in ∂f by edges between consecutive entries in the list; this means that any edges of ∂f which were surrounded by f will also be split into two.

There is a one-to-one correspondence between groups of edge-ends of T inside f and entries in the list; we use this correspondence to replace every edge between C_o and B by an edge between C_o and a specific list entry. This will ensure that the graph obtained is still plane, and every vertex of T inside f retains its original degree. Finally, remove all vertices and edges which lie completely outside f. Figure 2 illustrates this unzipping operation.

This produces a disc triangulation with all internal vertices having degree 6. The simple boundary has at most $2n - |B°|$ vertices, and M is precisely the boundary of the internal vertices. The required bound now follows from Lemma 9. ∎

We now have all the ingredients required for Theorem 1.

Proof of Theorem 1. We apply Theorem 6; by Lemma 13 we have (i) with $\alpha = 2$, and by Corollary 11 we have (ii) with $f(n) = \lfloor n^2/12 - n/2 + 1 \rfloor$. ∎

If we further assume that G is nonamenable, we obtain better bounds. We say G is *nonamenable* if $\dot{h}(G) > 0$.

Theorem 14. *Let G be a nonamenable plane graph of minimum degree at least 6, and set $\beta = \dot{h}(G) > 0$. Then $\dot{p}_c(G) \leq \frac{2+\beta}{3+3\beta}$.*

Remark. This result is only interesting for $0 < \beta < 1$, since for $\beta \geq 1$ it is weaker than the bound of $\frac{1}{1+\beta}$, due to Benjamini and Schramm [5], which applies to general graphs. This is as we would expect, since for $\beta \geq 1$ a stronger form of Lemma 9 may be deduced directly from the Cheeger constant.

Proof. Consider some $(M, B) \in \mathfrak{D}_n$ and the corresponding triangulation T. Let $B° \subseteq B$ be the minimum vertex cut separating o from infinity. Note that $B°$ is the boundary of some set A consisting

of those vertices of T separated from infinity by $B°$. In particular, $A \supseteq M \cup (B \backslash B°)$. Consequently $|A| \geq |M| + n - |B°|$, giving

$$|B°| \geq \beta(|M| + n - |B°|),$$

that is,

$$|B°| \geq \frac{\beta}{1+\beta}(|M| + n). \tag{1}$$

Unzipping B as in the proof of Lemma 13 gives a disc triangulation with boundary of length at most $2n - |B°|$ by Lemma 12, and so applying Lemma 9 and (1) gives $|M| \leq 2n - \frac{\beta}{1+\beta}(|M| + n)$, that is, $|M| \leq \left(\frac{2+\beta}{1+2\beta}\right)n$. Applying Theorem 6 with $\alpha = \frac{2+\beta}{1+2\beta}$ gives the required result. ∎

We remark that Angel, Benjamini, and Horesh [1] proved that for any $r > 0$, if a plane triangulation T has minimum degree at least 6 and every ball of radius r contains a vertex of degree greater than 6, then T is nonamenable.

4 | HYPERBOLIC GRAPHS

In this section we consider more stringent degree conditions, motivated by regular hyperbolic graphs. Benjamini and Schramm [5] conjecture that a plane graph G of minimum degree at least 7 has $\dot{p}_c < 1/2$, and furthermore that there are infinitely many infinite open clusters for every $p \in (\dot{p}_c, 1 - \dot{p}_c)$. We show the first half of this conjecture. In the process we give tight bounds on the vertex Cheeger constant.

Let G be a plane graph with minimum degree at least $d \geq 7$, fixed throughout this section.

Lemma 15. *Fix a minimal vertex cut X separating o from infinity. Let Y be the vertices of the finite component of $G \backslash X$ which are adjacent to X, and Z be the other vertices of this component. Then $|X| \geq (d-5)|Y| + (d-6)|Z| + 5$.*

Proof. Consider the induced graph on $X \cup Y \cup Z$. We may add edges if necessary so that this is a disc triangulation with boundary X, and we may do this in such a way that the set of vertices in $G \backslash X$ which are adjacent to X is precisely Y. Notice that the vertices of X induce a cycle, and the graph induced by $Y \cup Z$ is a connected plane triangulation because X is a minimal vertex cut.

Now add a new vertex u in the unbounded face of this graph adjacent to all vertices in X to obtain a new graph H. We now have a triangulation of a sphere with $|X| + |Y| + |Z| + 1$ vertices, that is, all faces, including the unbounded one, have degree 3. Since each edge is incident with two faces, we can easily deduce that $3F = 2E$, where F and E denote the number of faces and edges of H. Using Euler's formula we now obtain that $E = 3(|X| + |Y| + |Z|) - 3$.

Look at faces between X and Y in H, that is, faces having two vertices from X and one from Y or vice versa. There are $|X|$ of the first type (one for each edge between vertices of X), because the vertices of X span a cycle, and at least $|Y| - 1$ of the second, because there is at least one face for each edge between vertices of Y, and Y spans a connected graph (in fact we only need the -1 in the case $|Y| = 1$). When walking along X the faces between X and Y appear in a cyclic order because X spans a cycle, hence the number of edges between X and Y coincides with the number of faces between X and Y. It follows that there are at least $|X| + |Y| - 1$ edges between X and Y.

Thus we can calculate the sum of the degrees $d(v)$ of H to be

$$\sum_{v \in V(H)} d(v) \geq \sum_{v \in Y \cup Z} d(v) + |X| + |Y| - 1 + 4|X| \geq 5|X| + (d+1)|Y| + d|Z| - 1.$$

To see that the first inequality holds notice that the sum $\sum_{v \in X \cup \{u\}} d(v)$ counts the edges between X and Y only once and the remaining edges with at least one endvertex in X exactly twice. Now the number of edges connecting vertices of X and the number of edges between X and u are both equal to $|X|$. In the second inequality we used that $d(v) \geq d$. Since $\sum_{v \in V(H)} d(v)$ is equal to $2E = 6(|X| + |Y| + |Z|) - 6$ by the handshake lemma, the result follows. ∎

In particular, we have $(d - 5)|Y| \leq |X| - 5$ and $(d - 6)|Y \cup Z| \leq |X| - 5$. In fact, the second inequality can be improved.

Lemma 16. *For X, Y, Z as in Lemma 15, we have $\alpha_d |Y \cup Z| < |X| - 5$, where*

$$\alpha_d = \frac{d - 6 + \sqrt{(d - 2)(d - 6)}}{2}.$$

In particular, $\alpha_7 \approx 1.618$ is the golden ratio.

Proof. We prove this by induction on $|Z|$. If $Z = \varnothing$ then the required inequality holds since $\alpha_d < d - 5$. Otherwise, Y contains $k \geq 1$ minimum vertex cuts which separate clusters of vertices in Z from infinity. Write $(Y_i)_{i=1}^k$ for the cuts (which may overlap) and $(Z_i)_{i=1}^k$ for the clusters; by Lemma 15 $|Y_i| \geq d$ for each i.

We may add edges, where necessary, between vertices of Y such that each of the cuts forms a cycle. The auxiliary graph H consisting only of these cycles is outerplanar, that is, all vertices of H belong to the unbounded face of H, by definition of Y. Notice that H has k faces bounded by the vertex cuts, one unbounded face, and possibly a few more faces that are formed when some vertex cuts overlap in a cyclic way. Let us denote by ℓ the number of the latter faces. Moreover, H has $|Y|$ vertices and its unbounded face has degree at least $|Y|$, because H is a connected graph and has at least one cycle. Since each edge of H is incident with two faces, the sum of face degrees coincides with $2e(H)$. It follows that $\sum_{i=1}^k |Y_i| \leq 2e(H) - |Y| - 3\ell$, and by Euler's formula the right-hand side of the inequality is equal to $|Y| + 2k - \ell - 2$, implying that $\sum_{i=1}^k (|Y_i| - 2) \leq |Y| - 2$.

By the induction hypothesis, we have $\alpha_d |Z_i| \leq |Y_i| - 5$, and thus $\alpha_d |Z| = \alpha_d \sum_{i=1}^k |Z_i| \leq \sum_{i=1}^k (|Y_i| - 5) \leq |Y| - 2 - 3k \leq |Y| - 5$.

It follows that $|Y| > \frac{\alpha_d}{1+\alpha_d}(|Y \cup Z|)$, so applying Lemma 15 we get

$$\left(d - 6 + \frac{\alpha_d}{1 + \alpha_d}\right)|Y \cup Z| < |X| - 5,$$

and since $\alpha_d = d - 6 + \frac{\alpha_d}{1+\alpha_d}$ the result follows. ∎

Corollary 17. *Let W be any finite subset of $V(G)$. Then $|\partial W| \geq \alpha_d |W|$.*

Proof. We may assume every vertex in ∂W meets the infinite component of $G \backslash \partial W$, since otherwise we may find a larger set with smaller boundary. Now splitting ∂W into minimum vertex cuts surrounding clusters of W as above, and applying Lemma 16 to each cut, gives the required result. ∎

The following result, which shows that Lemma 16 is tight, may be of independent interest.

Theorem 18. *For each $d \geq 7$ the d-regular hyperbolic triangulation $H_{d,3}$ has vertex Cheeger constant $\dot{h}(H_{d,3}) = \alpha_d$.*

Proof. Corollary 17 immediately gives $\dot{h}(H_{d,3}) \geq \alpha_d$, and so it suffices to exhibit a sequence of sets S_n satisfying $|\partial S_n| = (\alpha_d + o(1))|S_n|$. In fact the balls B_n have this property. Note that, for $n \geq 1$, ∂B_n forms a minimum vertex cut, the set of vertices on the unbounded face of B_n is precisely $B_n \backslash B_{n-1}$, and the induced subgraph on this set is a cycle.

Thus, following the proof of Lemma 15, we obtain $|\partial B_n| = (d-5)|B_n \backslash B_{n-1}| + (d-6)|B_{n-1}| + 6$, or equivalently, noting that $B_n \cup \partial B_n = B_{n+1}$,

$$|B_{n+1}| - (d-4)|B_n| + |B_{n-1}| = 6. \tag{2}$$

Standard techniques on recurrence relations imply that the solution of (2) is given by $|B_n| = a(1 + \alpha_d)^n + b(1 + \alpha_d)^{-n} + c$ for suitable constants a, b, c (where clearly $a > 0$). In particular, $|\partial B_n| = |B_{n+1}| - |B_n| = (\alpha_d + o(1))|B_n|$, as required. ∎

Lemma 16 therefore shows that among plane graphs with minimum degree $d \geq 7$, $H_{d,3}$ minimizes the vertex Cheeger constant. This fact already implies an upper bound on the critical probability, using a result of Benjamini and Schramm [5] that $\dot{p}_c(G) \leq (1 + \dot{h}(G))^{-1}$. However, combining these facts with our method of interfaces yields better bounds.

Theorem 19. *Let G be a plane graph with minimum degree $d \geq 7$. Then $\dot{p}_c(G) \leq \frac{2+\alpha_d}{(d-3)(1+\alpha_d)}$.*

Remark. For $d = 7$, $(1 + \alpha_d)^{-1} \approx 0.3820$, whereas $\frac{2+\alpha_d}{(d-3)(1+\alpha_d)} \approx 0.3455$.

Proof. Let M, B be an outer interface and its boundary, let B° be the minimal vertex cut part of B, and fix a triangulation $T(M, B)$ as described in Section 3. We can unzip B as in Lemma 13 and then apply Lemma 15 to obtain

$$(d-5)|M| \leq 2|B| - |B^\circ|. \tag{3}$$

Also, applying Lemma 16 to the vertex cut B°, we have $\alpha_d(|M| + |B \backslash B^\circ|) \leq |B^\circ|$, that is,

$$\alpha_d |M| \leq (1 + \alpha_d)|B^\circ| - \alpha_d |B|. \tag{4}$$

Taking a linear combination of (3) and (4) we can cancel $|B^\circ|$ to obtain $(d - 5 + (d-4)\alpha_d)|M| \leq (2 + \alpha_d)|B|$. Now Theorem 6 gives $\dot{p}_c(G) \leq \frac{2+\alpha_d}{(d-3)(1+\alpha_d)}$. ∎

5 | HYPERBOLIC QUADRANGULATIONS

Let G be a plane graph with no triangular faces, and minimum degree $d \geq 5$, fixed throughout this section. While we will primarily be interested in the case where G is a quadrangulation, our results in this section apply more generally to any such graph, even though it is not necessarily possible to create a quadrangulation from such a graph by adding edges. Our first step is an analogue of Lemma 15.

Lemma 20. *Fix a minimal vertex cut X separating o from infinity. Let Y be the vertices of the finite component of $G \backslash X$ which are adjacent to X, and Z be the other vertices of this component. Then $|X| \geq (d-3)|Y| + (d-4)|Z| + 3$.*

Remark. In fact the proof gives $|X| \geq (d-3)|Y| + (d-4)|Z| + 4$ unless $|Y| = 1$.

Proof. We may assume $|Y| > 1$ since otherwise the result is trivial. Take the induced subgraph on $X \cup Y \cup Z$ and add edges as necessary so that X is a cycle, giving a finite graph H. Note that H may have faces of degree 3. Furthermore, the unbounded face has degree $|X|$ and, by minimality of X, each other face meets X at one vertex, at two vertices with an edge between them, or not at all. Write A for the set of internal faces meeting X along an edge; note that $|A| = |X|$. Write A' for the set of faces meeting X at a single vertex. Each face in A has degree at least 3; let q be the number of faces in A of degree exactly 4, and p be the number of faces in A of degree at least 5.

Once again the sum of face degrees equals $2E$, hence we have $2E \geq |X| + 4(F - |X| - 1) + 3|A| + q + 2p = 4F - 4 + q + 2p$, where F and E are the number of faces and edges of H. By Euler's formula, $F = E - |X| - |Y| - |Z| + 2$. Thus $2E \leq 4(|X| + |Y| + |Z|) - q - 2p - 4$. Also, by the handshake lemma, $2E \geq d|Y| + d|Z| + \sum_{x \in X} d(x)$. The $|X|$ edges on the unbounded face are double-counted by this sum and there are as many edges between X and Y as faces because X induces a cycle. We can now deduce that $\sum_{x \in X} d(x) = 3|X| + |A'|$.

We now claim that $|A'| \geq |Y| - q - p$. Indeed, write $d'(y)$, $y \in Y$ for the number of edges between y and X. Since $|A| + |A'|$ coincides with number of edges between X and Y, we have $|A| + |A'| = \sum_{y \in Y} d'(y)$. We can rewrite the latter sum as $|Y| + \sum_{y \in Y}(d'(y) - 1)$. Notice that $d'(y) > 1$ only when y is incident with a triangular face, and moreover $d'(y) - 1$ is not less than the number of triangular faces incident with y, because $|Y| > 1$ and so not all faces incident with y are triangular. Thus $\sum_{y \in Y}(d'(y) - 1) \geq |A| - p - q$, which implies that $|A'| \geq |Y| - p - q$.

Consequently we have $3|X| + (d + 1)|Y| + d|Z| - q - p \leq 4(|X| + |Y| + |Z|) - q - 2p - 4$; since $p \geq 0$ the result follows. ∎

We next give an analogue of Lemma 16 for this setting; perhaps surprisingly, the same sequence of constants arises.

Lemma 21. *For X, Y, Z as in Lemma 20, we have $\alpha_{d+2}|Y \cup Z| < |X| - 3$.*

Proof. We prove this by induction on $|Z|$. If $Z = \varnothing$ then the required inequality holds since $\alpha_{d+2} < d - 3$. Otherwise, as in the proof of Lemma 16, Y contains minimum vertex cuts $(Y_i)_{i=1}^k$ separating clusters $(Z_i)_{i=1}^k$, where $\sum_{i=1}^k |Z_i| = |Z|$ and $\sum_{i=1}^k (|Y_i| - 2) \leq |Y| - 2$.

By the induction hypothesis, we have $\alpha_{d+2}|Z_i| \leq |Y_i| - 3$, hence $\alpha_{d+2}|Z| = \sum_{i=1}^k \alpha_{d+2}|Z_i| \leq \sum_{i=1}^k (|Y_i| - 3) \leq |Y| - 2 - k \leq |Y| - 3$.

Consequently $|Y| > \frac{\alpha_{d+2}}{1 + \alpha_{d+2}}(|Y \cup Z|)$, and applying Lemma 20 gives

$$\left(d - 4 + \frac{\alpha_{d+2}}{1 + \alpha_{d+2}} \right)|Y \cup Z| < |X| - 3 ,$$

whence the result follows since $\alpha_{d+2} = d - 4 + \frac{\alpha_d}{1 + \alpha_d}$. ∎

Arguing as in the proof of Corollary 17 we obtain

Corollary 22. *Let W be any finite subset of $V(G)$. Then $|\partial W| \geq \alpha_{d+2}|W|$.*

Again, this result is best possible.

Theorem 23. *For each $d \geq 5$ the d-regular hyperbolic quadrangulation $H_{d,4}$ has vertex Cheeger constant $\dot{h}(H_{d,4}) = \alpha_{d+2}$.*

Proof. Again, it suffices to show that $|\partial B_n| = (\alpha_{d+2} + o(1))|B_n|$, where B_n is the ball of radius n.

Note that, for $n \geq 1$, ∂B_n forms a minimum vertex cut, the set of vertices in B_n adjacent to ∂B_n is precisely $B_n \backslash B_{n-1}$, and in the graph obtained by adding a cycle through ∂B_n to the induced subgraph on B_{n+1}, every vertex in B_n has degree d, every face meeting ∂B_n along an edge has degree 3, and every other face has degree 4. Thus, following the proof of Lemma 20, we have $q = p = 0$ and equality at every step, giving $|\partial B_n| = (d - 3)|B_n \backslash B_{n-1}| + (d - 4)|B_{n-1}| + 4$, or equivalently

$$|B_{n+1}| - (d - 2)|B_n| + |B_{n-1}| = 4. \tag{5}$$

Again, it follows that $|B_n| = a(1 + \alpha_{d+2})^n + b(1 + \alpha_{d+2})^{-n} + c$ for suitable constants a, b, c with $a > 0$, and so $|\partial B_n| = |B_{n+1}| - |B_n| = (\alpha_{d+2} + o(1))|B_n|$, as required. ∎

Lemma 21 implies that $\dot{p}_c(G) \leq (1 + \alpha_{d+2})^{-1}$ if G has all vertex degrees at least $d \geq 4$ and all face degrees at least 4. Our method yields again better bounds.

Theorem 24. *Let G be a plane graph with minimum degree $d \geq 5$ and no faces of degree 3. Then $\dot{p}_c(G) \leq \frac{(2+\alpha_{d+2})(d-2)}{(d^2-3d+1)(1+\alpha_{d+2})}$.*

Remark. For $d = 5$, $(1 + \alpha_{d+2})^{-1} \approx 0.3820$, whereas $\frac{(2+\alpha_{d+2})(d-2)}{(1+\alpha_{d+2})(d^2-3d+1)} \approx 0.3769$.

Proof. Let M, B be an outer interface and its boundary, and let B° be the minimal vertex cut part of B. Delete all vertices of the infinite component of $G \backslash B$. Notice that any vertex of M belongs to a common face with a vertex of B. By adding edges if necessary, we can achieve that all faces between M and B have degree 4 or 5 and also preserve this property. Then every vertex of M has distance at most 2 from B.

We can argue as in the proof of Lemma 13 to unzip B and obtain a new graph H with at most $2|B| - |B^\circ|$ "boundary" vertices in its unbounded face. To be more precise, we first add a set of edges, which we denote by S, to obtain the triangulation $T = T(M, B)$, and then we unzip B as in the proof of Lemma 13. This process gives rise to a correspondence between the edges of the graph obtained after unzipping B and the edges of T. The desired graph H is obtained after removing the preimages of S. Write X for the "boundary" vertices of H. Let Y be the set of vertices of $H \backslash X$ that are adjacent to B', and Z the remaining vertices of H. Lemma 20 implies that

$$2|B| - |B^\circ| \geq |X| \geq (d - 3)|Y| + (d - 4)|M \backslash Y| = |Y| + (d - 4)|M|.$$

We now claim that $|Y| \geq (d - 3)(|M| - |Y|) + 3$. Indeed, if the graph induced by Z is connected we can apply Lemma 20 to $Y \cup Z$, noting that all vertices of $M \backslash Y$ are adjacent to some vertex of Y because M has distance at most 2 from B. If it is not connected we can split Y into minimal vertex cuts $(Y_i)_{i=1}^k$ and Z into its components $(Z_i)_{i=1}^k$, and define $M_i := (Y_i \cup Z_i) \cap M$. Then we have as above that $|Y_i| \geq (d - 3)(|M_i| - |Y_i|) + 3$. Arguing as in the proof of Lemma 16 we obtain that $\sum_{i=1}^k (|Y_i| - 2) \leq |Y| - 2$. The desired claim follows now from the fact that the sets $M_i \backslash Y_i$ partition $M \backslash Y$.

We can now deduce that $|Y| \geq \frac{d-3}{d-2}|M|$, which implies that

$$\left(d - 4 + \frac{d-3}{d-2}\right)|M| \leq 2|B| - |B^\circ|. \tag{6}$$

Also, applying Lemma 21 to the vertex cut B°, we have $\alpha_{d+2}(|M| + |B \backslash B^\circ|) \leq |B^\circ|$, that is,

$$\alpha_{d+2}|M| \leq (1 + \alpha_{d+2})|B^\circ| - \alpha_{d+2}|B|. \tag{7}$$

Taking a linear combination of (6) and (7) we obtain

$$\left(\alpha_{d+2} + (1+\alpha_{d+2})\left(d-4+\frac{d-3}{d-2}\right)\right)|M| \leq (2+\alpha_{d+2})|B|.$$

Using Theorem 6 we deduce that $\dot{p}_c(G) \leq \frac{(2+\alpha_{d+2})(d-2)}{(d^2-3d+1)(1+\alpha_{d+2})}$. ∎

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