THE TOPOLOGICAL CLASSIFICATION OF ENDOMORPHISMS OF VECTOR BUNDLES.

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CONTENTS.

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ABSTRACT.

I have made an attempt in this thesis at a systematic topological classification of the endomorphisms of a vector bundle . Chapter one provides the motivation for such a classification and for the notions of 'local' and 'global' topological equivalence of bundle endomorphisms that are given in Chapter 2. Both the local and global classification problems turned out to be a great deal more difficult than anticipated. A complete local and global classification of the generic endomorphisms of real and complex line bundles over a smooth manifold is given in Chapter 2. For real plane bundles, the local classification problem is partly solved, but no attempt is made at the global problem. Chapter 3 gives the complete local classification for generic endomorphisms that have bifurcations of codimension one. Finally in Chapter 4 an attempt is made at the codimensions two and four local classification problems. Only partial results and conjectures are offered. Other notions of local equivalence are suggested.

CHAPTER 1. INTRODUCTION.

1. Some Motivation for the Study of the Endomorphisms of a Vector Bundle.

There seems to have been no systematic attempt at a topological classification of the endomorphisms of a vector bundle. Motivation for such a classification arises in basically two areas of mathematics. The problem may be approached with the view points of these two areas in mind. We shall call one very loosely, the 'homotopy' point of view (embracing the areas of abstract algebraic topology, geometric topology, etc.) and the other the 'differential' point of view (including for example the areas of dynamical systems, differential topology, etc.). In the following, I shall endeavour to explain more precisely what sort of questions arise in these areas that motivate the proposed classification of the endomorphisms of a vector bundle.

Let us consider 'the homotopy point of view'.

It is well known that the image of a vector bundle under a bundle endomorphism is not in general a vector bundle. The classification of these 'bundle - like' objects (i.e. the 'images' of vector bundles) is a difficult unsolved problem. Indeed it is not even clear how the problem should be approached (for example is there a convenient category in which these 'bundle - like' objects occur as objects along side vector bundles?) In this thesis an approach to the problem is suggested, namely via a topological classification of the endomorphisms.

The source of inspiration for the type of classification suggested here, was the paper of Kuiper and Robbin [4] on a classification of the endomorphisms of a vector space. In their paper, they classify the endomorphisms up to topological conjugacy. Trivally one could classify the endomorphisms of a vector bundle. by a 'fibre - by - fibre' approach using their results, that is a 'pointwise' classification that does not take into account the structure or continuity of the bundle Such an approach is clearly unsatisfactory. A definition of topological conjugacy of bundle endomorphisms is suggested in chapter 2, that involves the structure of the bundle. The definitions given, lead to the 'local classification problem' and the 'global classification' problem for bundle-endomorphisms. The problems are very difficult. A complete solution is given (in chapter 2) to both problems only in the case of real and complex line bundles (contrast the 'five lines' in [4] taken to give the classification of endomorphisms of $\mathbb{R}$!). A partial solution to the local classification problem for real plane bundles is given in chapter 3.

So as not to encumber the reader with technicalities and too many formal definitions, in the following discussion, we consider only trivial real or complex finite - dimensional bundles E over some Euclidean space X, and we refer to a bundle endomorphism F of E in the more intuitive way, as a family of matrices (square) over X i.e a continuous (or smooth, etc.) map $\lambda^F : X \to M(n,n)$ where $M(n,n)$ is the space of real or complex matrices and n is the dimension of the fibres of E. The general idea of a global (or local) topological conjugacy between two families of matrices on X (or on some open set $U \subset X$) is a fibre - preserving homeomorphism of E (or E/U) such that its restriction to a fibre E_x of E conjugates the two members of the two families evaluated at x, (a priori, this conjugacy varies continuously with x). The interesting problem from the 'homotopy' point of view is the local classification of families in the neighbourhood of a zero matrix. For example, it is shown in chapter 2 that given two families of 1×1 complex matrices in the neighbourhood of a zero - matrix, if a certain obstruction is zero, then the two families are locally equivalent. In proving the global equivalence of the two families of matrices, a similar obstruction occurs. It is in the local classification of matrices in the neighbourhood of a matrix with at least one eigenvalue of real or complex modulus one, that we are naturally led to impose certain differentiability and transversality restrictions. It seems plausible, that once sufficient differential structure has been introduced to solve the local classification problem no further differential structure is required to solve the global classification problem. Therefore the global classification of locally equivalent families is of particular interest from the homotopy point of view (a complete solution to this problem is given in chapter 2 for real and complex 1×1 matrices over an arbitrary paracompact space X.).

There is a further aspect to the classification of bundle endomorphisms or the classification of 'bundle - like' objects, where a differential structure may simplify the problem. With E and X smooth manifolds, one may define the notion of a generic smooth family, and then a generic 'bundle - like' object. The generic 'bundle - like' objects do not include, unfortunately(?) all vector bundles. For example the Möbius bundle (i.e. the non - trivial line bundle over S^1) is the image of an endomorphism $S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}^2$, which is not generic (see chapter 3). Certainly the notion of genericity here excludes particularly 'nasty' 'bundle - like' objects (e.g. radially shrinking Möbius bundles that disintegrate at the origin). Generic 'bundle - like' objects may be of most interest when the bundle E is T(X), the tangent bundle over X and the endomorphism F is defined as follows : -

given a $\mathcal{C}^\infty$- map $f : X \to X$, F is defined such that the following diagram commutes : -

$$\begin{tikzcd}
T(X) \arrow[r, "F"] \arrow[d, "\pi_f"] \arrow[rr, bend left, "Df"] & f^*(T(X)) \arrow[r, "\tilde{f}"] \arrow[d, "\pi_f"] & T(X) \arrow[d, "\pi"] \\
X \arrow[r, "id_X"] & X \arrow[r, "f"] & X
\end{tikzcd}$$

where $f^*(T(X)) \cong T(X)$ and $\tilde{f} : f^*(T(X)) \to T(X)$ is the standard map covering f.

In the theory of singularities of differentiable maps, a local classification of these tangent bundle endomorphisms in the neighbourhood of a singular point seems an interesting problem. A related local classificatiom problem, is the classification of families of matrices satisfying certain integrability conditions. This brings us to : -

The 'Differential' point of view.

It is not clear just how much information about $\mathcal{C}^\infty$- maps $f : X \to X$ can be deduced from the local classification of the endomorphisms F of T(X) (defined as above). In the neighbourhood of a singular point of f, it may be possible to deduce more about f from the local classification of endomorphisms 'exp F' ('exp F' is defined locally in the obvious way, where exp : $\mathrm{End}(\mathbb{R}^n) \to \mathrm{Aut}(\mathbb{R}^n)$ is the standard exponential map, mapping a neighbourhood of the zero matrix diffeomorphically onto a neighbourhood of the identity). This remark was prompted by the consideration of the following problem.

Consider a family of linear differential equations of the form $\left\{ \frac{dy}{dt} = A_x y \right\}$ where $A : X \to \mathrm{End}(\mathbb{R}^n)$ is continuous (or smooth, etc.). The problem of classifying the families $\{A_x\}$ does not seem of any real interest, but rather the families $\{\exp A_x\}$. The reason here is clear. A solution of the differential equation $\frac{dy}{dt} = A_x y$ with initial condition y_0, or 'orbit' through y_0 is given by the set $\{ \exp(tA_x)y_0,\ t \in \mathbb{R} \}$. The solutions or orbits of the differential equation give a subdivision of $\mathbb{R}^n$, called the orbit system of A_x (we can in fact define the orbit systems of A_x, without reference to differential equations). The natural definition of an equivalence between two orbit systems of $\mathbb{R}^n$ is in terms of a homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$, such that ψ and ψ^{-1} take orbits into orbits. (A complete classification of such orbit systems on $\mathbb{R}^n$ is given in a paper by Kuiper [3].) One may then define in the obvious way a local orbital equivalence between families of differential equations. It would seem from my attempts at classifying up to local topological equivalence, families of 2×2 matrices in the neighbourhood of the identity ($\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$) or a Shear ($\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$) that the related problem of classifying these families up to local orbital equivalence is at least as difficult, if not equivalent. Intuitively the equivalence of the problems seems plausible: broadly - speaking, a topological equivalence between two matrices preserves Z - orbits (i.e. the orbits of the Z - actions on $\mathbb{R}^2$ defined by the matrices), an orbital equivalence between two matrices, preserves $\mathbb{R}$ - orbits (see above) and for matrices arbitrarily close to $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ or $\begin{bmatrix}1&1\\0&1\end{bmatrix}$, the actions become arbitrarily close. ($\bigcup_{0<\lambda<\varepsilon}\{(1+\lambda)^n / n\in Z\}$ is dense in $\mathbb{R}$ for any $\varepsilon > 0$). (Note this does not imply that a given conjugacy $\phi : \mathbb{R}^2\to\mathbb{R}^2$ and a given orbital equivalence $\gamma : \mathbb{R}^2\to\mathbb{R}^2$ are arbitrarily close but rather that by suitable modifications of γ and ϕ, they can be made arbitrarily close: moreover to obtain the result that local topological equivalence and local orbital equivalence are 'the same', the modification of γ and ϕ must be done in a 'continuous manner'.) Restating the local classification problem of families of matrices in terms of the language of bifurcation theory makes clearer why a complete solution is possible or simple for some cases but not for others. In a family of matrices a bifurcation point (relative to topological equivalence, for example) is a point (matrix) such that in any neighbourhood of that point there are matrices of different topological type. More precisely topological equivalence defines a stratification of the space of $n \times n$ matrices (this actually has only been proved for $n = 1,2$), a bifurcation point is a point in the boundary of one of the strata. Its codimension is the codimension of the strata to which it belongs. For families of matrices with no bifurcation points, a local classification can be given relatively simply in any dimension (see chapter 3). The local classification problem at points of codimension one, is essentially the 'differential' problem solved in this thesis. We in fact only give the solution for 1×1

and 2×2 matrices, but these results point to the general solution in higher dimensions (for example, to prove the local equivalence of 'pointwise' equivalent families having just one eigenvalue equal to 1 is quite straightforward) It is in the local classification of families in the neighbourhood of points of codimension 2 or more, that the problem becomes extremely difficult. Chapter 4 gives an attempt at the solution, though what seems to be indicated here, is that methods that solve the codimension one problem do not solve the problem for codimension > 1. Possibly a weaker notion of 'local equivalence' is required. One such, is suggested namely 'deleted local equivalence'. Effectively the problem is reduced to codimension one.

2. Topological Classification of the Endomorphisms of a Vector Space.

In this section, we summarize the results of Kuiper and Robbin [4], for the reader's benefit, and give a simple proof of their unsolved conjecture in the case of the vector space $\mathbb{R}^2$ (a more general, rather deep theorem in [4] does in fact give this result, amongst other things!)

Let V be a real vector space and $f : V \to V$ be a linear endomorphism of V. Then V admits an f - invariant direct sum decomposition :

$$V = W^{\infty}(f) \oplus W^{+}(f) \oplus W^{0}(f) \oplus W^{-}(f)$$

where the eigenvalues λ of

$$f_{\sigma} = f/W^{\sigma}(f) \quad (\sigma = \infty, +, 0, -)$$

satisfy the following conditions : -

f_σ	λ
f_∞	$\lambda = 0$
f_+	$0 < \lvert\lambda\rvert < 1$
f_0	$\lvert\lambda\rvert = 1$
f_-	$\lvert\lambda\rvert > 1$

dim (f_σ) is the dimension of $W^\sigma(f)$ ($= \infty$, $+$, 0, $-$)

or(f_σ) is the sign of the determinant of f_σ

or$(f_\sigma) = 1$ if f_σ preserves orientation and

or$(f_\sigma) = -1$ if f_σ reverses orientation.

Linear endomorphisms $f, g : V \to V$ are said to be topologically conjugate, $f \overset{t}{\sim} g$, if there exists a homeomorphism $h : (V,0) \to (V,0)$ such that the following diagram commutes:-

$$\begin{tikzcd}
(V,0) \arrow[r, "f"] \arrow[d, "h"'] & (V,0) \arrow[d, "h"] \\
(V,0) \arrow[r, "g"] & (V,0)
\end{tikzcd}$$

(one need not require $h(0) = 0$: the weaker form however is easily shown to be equivalent to that given above.)

If h can be chosen linear, then f and g are said to be linearly equivalent, $f \overset{l}{\sim} g$.

We state the result of Kuiper and Robbin as follows :-

<u>Conjecture A.</u> If f and g are periodic linear automorphisms, then $f \overset{t}{\sim} g$ if and only if $f \overset{l}{\sim} g$.

<u>Theorem of Kuiper and Robbin.</u> Let f, g be endomorphisms of a finite dimensional vector space. Assume conjecture A is true. Then $f \overset{t}{\sim} g$ if and only if $\dim(f_+) = \dim(g_+)$, or $(f_+) =$ or (g_+), $\dim(f_-) = \dim(g_-)$, or $(f_-) =$ or (g_-), $f_\infty \overset{l}{\sim} g_\infty$ and $f_0 \overset{l}{\sim} g_0$.

Clearly if $W^o(f) = W^o(g) = \emptyset$, then the assumption concerning conjecture A is not required.

In point of fact, they prove a rather better theorem than that given here. But this will suffice for our purpose of studying endomorphisms of real line and plane bundles.

We prove conjecture A on $\mathbb{R}^2$.

<u>Theorem 2.1.</u> Let f,g be periodic rotations of $\mathbb{R}^2$, then $f \overset{t}{\sim} g$ if and only if $f \overset{\ell}{\sim} g$.

<u>Proof.</u> '<u>if</u>'. Trivial.

'<u>only if</u>' Assume $f \overset{t}{\sim} g$ i.e. there exists a homeomorphism $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that $f \cdot h = h \cdot g$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formula : -

$$f(r, \theta) = (r, \theta + \frac{2\pi p_1}{q_1}), \quad 0 < |p_1| < q_1 \text{ and } (p_1, q_1) \text{ coprime.}$$

Also $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by : -

$$g(r, \theta) = (r, \theta + \frac{2\pi p_2}{q_2}) \quad 0 < |p_2| < q_2 \quad (p_2, q_2) \text{ coprime.}$$

If $f \overset{t}{\sim} g$, then $q_1 = q_2 = q$, $(f^{q_1} = g^{q_1} = f^{q_2} = g^{q_2} = \text{id.})$

Then f and g are linearly equivalent if $p_1 = \pm p_2$.

We may assume $\left|\frac{2\pi p_1}{q}\right| < \pi$ and $\left|\frac{2\pi p_2}{q}\right| < \pi$

if not, for example if $\left|\frac{2\pi p_1}{q}\right| > \pi$, then replace

f by $\bar{f}$ i.e. $\bar{f}(r, \theta) = (r, \theta - \frac{2\pi(q - p_1)}{q})$

and consider the problem for $\bar{f}$ and g, $(\bar{f} \overset{t}{\sim} g)$.

We also assume $h(1,0) = (1,0)$, for if not the linearity of f implies we may replace h by the conjugacy given by the formula : -

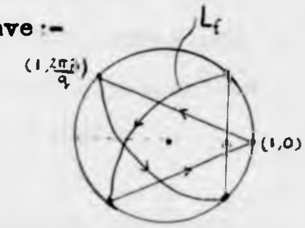
$$(r, \theta) \mapsto \left(\frac{h^1(r, \theta)}{h^1(1,0)}, h^2(r, \theta) - h^2(1,0)\right)$$

where $h/\mathbb{R}^2 - \{0\} = h/\mathbb{R}^+ \times S^1 = (h^1 . h^2)/\mathbb{R}^+ \times S^1$

and $h^1 : \mathbb{R}^+ \times S^1 \to \mathbb{R}^+$ and $h^2 : \mathbb{R}^+ \times S^1 \to S^1$.

Define a loop L_f in $\mathbb{R}^2 - \{0\}$ as follows:-
join $(1,0) \in S^1$ and $(1, \frac{2\pi p_1}{q}) \in S^1$ by a counterclockwise path in $\mathbb{R}^2 - \{0\}$ (if $p_1 > 0$: otherwise, clockwise), then join $(1, \frac{2\pi p_1}{q})$ and $(1, \frac{4\pi p_1}{q})$ by a counterclockwise path in $\mathbb{R}^2 - \{0\}$ (if $p_1 > 0$, etc.) and so on ; finally join $(1, \frac{(q-1)2\pi p_1}{q})$ and $(1,0)$.

Retracting this loop L_f onto S^1, we have:-



retr.$\cdot L_f : (S^1,0) \to (S^1,0)$ is a map of degree p_1 ((p_1,q) are coprime).
Now $h(1,0) = (1,0)$, and $h \cdot f = g \circ h$,
hence $h(1, \frac{2\pi p_1}{q}) = (1, \frac{2\pi p_2}{q})$,
also $h(1, \frac{n(2\pi p_1)}{q}) = (1, \frac{n(2\pi p_2)}{q})$, $n = 0,1,2, \ldots, q-1$, and the path joining $(1,0)$ to $(1, \frac{2\pi p_1}{q})$ maps under h to a path joining $(1,0)$ and $(1, \frac{2\pi p_2}{q})$, and since $h|\mathbb{R}^2 - \{0\}$ is a homeomorphism ,(and therefore homotopic to $\pm \mathrm{id}_{\mathbb{R}^2-\{0\}}$) then the path joining $(1,0)$ and $(1, \frac{2\pi p_2}{q})$ has winding number 0, and the same sense as the path joining $(1,0)$ and $(1, \frac{2\pi p_1}{q})$ if h is orientation preserving (otherwise , the opposite sense). Similarly the path joining $(1, \frac{2\pi p_1}{q})$ and $(1, \frac{4\pi p_1}{q})$ maps under h to a path of winding number 0 joining $(1, \frac{2\pi p_2}{q})$ and $(1, \frac{4\pi p_2}{q})$, ...etc. .

Retracting $h \cdot L_f : S^1 \to \mathbb{R}^2 - \{0\}$ onto S^1, we have,
retr.$\cdot h \cdot L_f : (S^1, 0) \to (S^1, 0)$ is a map of degree p_2 .
But $h|\mathbb{R}^2 - \{0\}$ is a homeomorphism and therefore homotopic to $\pm \mathrm{id}_{\mathbb{R}^2-\{0\}}$. Hence $p_1 = \pm p_2$.

The following result is easily deduced from theorem 2.1..

Corollary 2.1.1. Let $f,g : \mathbb{R}^2 \to \mathbb{R}^2$ be periodic rotations of $\mathbb{R}^2$:-
$f(r,\theta) = (r, \theta + \beta_f)$ and $g(r,\theta) = (r, \theta + \beta_g)$. Then there is an orientation preserving homeomorphism conjugating f and g if and only if $f = g$.

3. Transversality, Jet Bundles, Stratifications and Whitney C^∞-Topology.

By a C^∞-manifold (or smooth manifold), we shall mean a finite-dimensional, C^∞-differentiable manifold without boundary, not neccessarily compact. We have ignored the possibility of obtaining more general results on C^r-manifolds ($r > 1$) so as not to clutter the exposition with details on the'size' of r for particular properties required. Our submanifolds $X_1 \subset X$ are all regular submanifolds.

Transversality. Let X, Y be C^∞- manifolds and $f: X \to Y$, a C^∞-map. Let W be a submanifold of Y and x a point in X. Then f is said to be transversal to W at x ($f \pitchfork W$ at x), if either, (a) $f(x) \notin W$, or

(b) $f(x) \in W$ and $T_{f(x)}(Y) = T_{f(x)}(W) + (Df)_x(T_x(X))$.

If for each $x \in A \subset X$, $f \pitchfork W$ at x, then we say f is transversal to W on A (for A = X we omit 'on X ').

Transversality is a neccessary ingredient in any definition of genericity of maps from one manifold into another. A further ingredient is a topology on the space of maps. Rather than restrict ourselves to compact manifolds, we employ a topology suited to the set of C^∞- maps from a non - compact manifold into another; namely, the Whitney C^∞- topology (or the C^∞- fine topology). The definition is best given in terms of jet bundles. We briefly review the basic definition of : -

Jet Bundles.

Let X,Y be C^∞- manifolds, and f,g, : $X \to Y$, C^∞- maps such that for some $p \in X, q \in Y, f(p) = g(p) = q$. Then

(1). f and g are k - equivalent at p, if there is contact of order k :

$$d_Y(f(x), g(x)) = o(d_X(x,p))^k$$

where d_X and d_Y are metrics on X and Y (respectively).
k - equivalence is independent of the metrics d_X and d_Y, and is an equivalence relation. Also k - equivalence $\Rightarrow$ k - 1 equivalence.

(ii). The k - jet of a smooth map f at a point p in X is the k - equivalence class of maps from X to Y to which f belongs;

$$j^k(f) = \{ g \mid g \overset{k}{\sim} f \text{ at } p \}.$$

$J^k_{(p,q)}(X,Y)$ denotes the set of k - equivalence classes of mappings $f : X \to Y$ where $f(p) = q$.
Finally $J^k(X,Y) = \bigcup_{(p,q) \in X \times Y} J^k_{(p,q)}(X,Y)$, (disjoint union).
$J^k(X,Y)$ has a natural structure as a finite dimensional C^∞- manifold.

<u>The Whitney C^∞- topology on $C^\infty(X,Y)$.</u>

Rather than give a completely formal definition, we give a description of the C^k topology in terms of the neighbourhood basis of a point f in $C^k(X,Y)$ and a metric d on $J^k(X,Y)$, compatible with its topology, (the formal definition makes no such use of a metric d).
Define $\mathcal{N}_\delta(f) = \{ g \in C^\infty(X,Y) \mid \forall\ x \in X,\ d(j^k f(x),\ j^k g(x)) < \delta(x) \}$ where $\delta : X \to \mathbb{R}^+$ is a continuous function.
This is an open set for each such δ. The collection $\{\mathcal{N}_\delta(f)\}$ form a neighbourhood basis of f in the Whitney C^k- topology The Whitney C^∞- topology is given by the union of all the bases for the C^k - topologies, for $k \geqslant 0$. Intuitively, two maps are close together in this 'fine' topology means that these two maps (and any number of their derivatives) approach each other arbitrarily quickly at infinity.
Thus there are neighbourhoods of a C^∞- map f such that 'the perturbations of f' in these neighbourhoods are arbitrarily small 'at infinity'.

We shall mainly be concerned with the space $C^{\infty}(X,Y)$ when Y is a stratified manifold.

<u>Stratifications.</u>

It is not generally agreed which is the 'best' definition of a stratification. We give a fairly weak form.

Let Y be a C^{∞}- manifold . A <u>stratification</u> $\mathcal{S}$ of Y is a partition of Y into a finite number of C^{∞}- submanifolds $\{S_i\}$ called <u>strata</u> (not necessarily connected) such that $Y = \bigcup S_i$ satisfying the <u>boundary condition</u> i.e. for each stratum S_i, $\partial S_i = \bar{S}_i - S_i$ is the union of strata of lower dimension. A further condition is required for the strata to 'fit together nicely'. Let Y be embedded in Euclidean space of some suitable dimension. Then the stratification $\mathcal{S}$ of Y is said to satisfy the '<u>Whitney condition-(a)</u>' if for each pair of strata S_i, S_j and points $y \in Y$, such that $\bar{S}_i \supset S_j \ni y$ the following holds : -
given a sequence $\{x_n\}$ in $\bar{S}_i$ converging to y in S_j, if $\{T_{x_n}(S_i)\}$ converges to some hyperplane τ , then $T_y(S_j) \subseteq \tau$ (the convergence of $\{T_{x_n}(S_i)\}$ is in the Grassman bundle over Y).

The important implication of this condition is that, given a C^{∞}- submanifold W of $\mathbb{R}^n \supset Y$, meeting a stratum S_i transversally at x, then W is transverse to S_j in a neighbourhood of x, where $\partial S_j \supset S_i \ni x$. This is sometimes called condition (t). It has been shown [6] that if the strata are semi-analytic sets, then (t) implies condition (a).

If Y is a stratified C^{∞}- manifold, a C^{∞}- map $f : X \to Y$ is <u>transversal to the stratification</u> $\mathcal{S}$ of Y, if f is transversal to each stratum in the stratification.

The useful theorem motivating these definitions is the following : -

Theorem 2.2. Let X,Y be $\mathcal{C}^\infty$ - manifolds with Y embedded in some Euclidean space. Let $\mathcal{S}$ be a stratification of Y satisfying the Whitney condition-(a). Then the subspace $\mathcal{C}^\infty_T(X,Y)$ of $\mathcal{C}^\infty$- maps transversal to $\mathcal{S}$ is open dense in $\mathcal{C}^\infty(X,Y)$ where $\mathcal{C}^\infty(X,Y)$ has the Whitney $\mathcal{C}^\infty$- topology.

This theorem (quoted but not proved in [8]) appears to be known, though there is no published proof of it . For X compact the result is well documented ([5] , [10]). The 'density' part of theorem 2.2. follows easily from the 'compact version' since $\mathcal{C}^\infty(X,Y)$ is a Baire space [2]. The 'openness' part is false if $\mathcal{C}^\infty(X,Y)$ has some standard 'coarse' topology , defined on compact subspaces of X. However 'openness' holds for $\mathcal{C}^\infty$ (X,Y) with the 'fine' topology. A modification of Theorem 8.1. in [9] gives this result .

CHAPTER 2.

TOPOLOGICAL CLASSIFICATION OF ENDOMORPHISMS OF LINE BUNDLES.

Introduction.

In [4] Kuiper and Robbins classified the endomorphisms of a vector space up to topological conjugacy (the classification was complete modulo a certain well known, unsolved, conjecture, [7]). This classification gives a 'fibre by fibre' or 'pointwise' classification of the endomorphisms of a vector bundle. We suggest in this paper a notion of conjugacy between bundle endomorphisms that involves the continuity of the bundle. A complete classification of the generic endomorphisms of real and complex line bundles over a smooth differentiable manifold is given.

Section 1 contains the main definitions and theorems. In section 2 the classification problem for endomorphisms of real line bundles over an arbitrary paracompact space is considered, the main theorem being a major step towards proving the 'Real Classification Theorem'. Section 3 is similar, but for complex line bundles. Finally in section 4 the main theorems are proved.

I should like to thank Prof. R.L.E. Schwarzenberger of Warwick for his help and advice in preparing this paper and for suggesting the problem in the first place. My thanks are also due to Memorial University of Newfoundland for financial assistance, during the period in which this work was completed.

1. DEFINITIONS AND MAIN THEOREMS.

1.1. Preliminaries. In the following X will denote a paracompact path-connected space; $\pi : E_{\mathbb{R}} \to X$ and $\pi : E_{\mathbb{C}} \to X$ will denote real and complex bundles over X (respectively) (E is used when there is no chance of ambiguity). An endomorphism F of $E_{\mathbb{R}}$ ($E_{\mathbb{C}}$) is a fibre-preserving continuous map that is real linear (complex linear) on each fibre. The vector space of such endomorphisms is denoted by End (E).

Definition 1.1. Let V be a real (complex) vector space, and $h_1, h_2 : V \to V$ be real (complex) endomorphisms of V. Then h_1 and h_2 are said to be topologically equivalent, $h_1 \overset{t}{\sim} h_2$, if there exists a homeomorphism $\psi : (V,0) \to (V,0)$ such that the following diagram commutes:-

$$\begin{tikzcd}
(V,0) \arrow[r, "\psi"] \arrow[d, "h_1"'] & (V,0) \arrow[d, "h_2"] \\
(V,0) \arrow[r, "\psi"] & (V,0)
\end{tikzcd}$$

We say ψ conjugates h_1 and h_2.

Moreover if ψ can be chosen orientation preserving, we say h_1 and h_2 are orientably topologically equivalent, $h_1 \overset{ot}{\sim} h_2$.

Definition 1.2. Let F and G be bundle endomorphisms of E. We say F and G are pointwise topologically equivalent relative to a homeomorphism $\phi : X \to X$, or p.t e.(ϕ) if $F_x \overset{t}{\sim} G_{\phi(x)}$ for all $x \in X$. Similarly we may define pointwise orientably topologically equivalent relative to a homeomorphism $\phi : X \to X$, or p.o.t.e.(ϕ).

Let O_X denote the zero section of E.

Definition 1.3. Let F and G be endomorphisms of E and let $x \in X$. Then F and G are said to be locally topologically equivalent relative to a homeomorphism $\phi : X \to X$ at x or l.t.e.(ϕ) at x if there exists an open set $U \ni x$, and a homeomorphism $\Phi : (E/U, O_x) \rightarrow (\phi^*(E/\phi(U)), O_x)$ such that the following diagram commutes:-

$$\begin{tikzcd}
(E/U,O_x) \arrow[rr, "\Phi"] \arrow[dr, "\pi"] \arrow[dd, "F"'] & & (\phi^*(E/\phi(U)),O_x) \arrow[dl, "\pi_\phi"'] \arrow[dd, "\phi^*(G)"] \\
& U & \\
(E/U,O_x) \arrow[rr, "\Phi"'] \arrow[ur, "\pi"] & & (\phi^*(E/\phi(U)),O_x) \arrow[ul, "\pi_\phi"']
\end{tikzcd}$$

If E is an n-dimensional complex bundle, it has a natural orientation when regarded as a 2n-dimensional real bundle. The natural orientation of E gives a natural orientation on $\phi^*(E)$. Thus if for some $U \ni x, \Phi$ can be chosen orientation preserving in the above diagram, we say F and G are locally orientably topologically equivalent relative to ϕ at x, l.o.t.e. (ϕ) at x. We say Φ conjugates F and G on U or Φ is a local conjugacy between F and G at x.

Definition 1.4. Let F and G be endomorphisms of E. Then F and G are said to be globally topologically equivalent relative to a homeomorphism $\phi : X \rightarrow X$, g.t.e. (ϕ) if there exists a homeomorphism $\Phi : (E, O_x) \rightarrow (\phi^*(E), O_x)$ such that the following diagram commutes:-

$$\begin{tikzcd}
(E,O_x) \arrow[rr, "\Phi"] \arrow[dr, "\pi"] \arrow[dd, "F"'] & & (\phi^*(E),O_x) \arrow[dl, "\pi_\phi"'] \arrow[dd, "\phi^*(G)"] \\
& X & \\
(E,O_x) \arrow[rr, "\Phi"'] \arrow[ur, "\pi"] & & (\phi^*(E),O_x) \arrow[ul, "\pi_\phi"']
\end{tikzcd}$$

If E is an oriented complex bundle and if Φ is orientation preserving, then F and G are said to be globally orientably topologically equivalent relative to ϕ or g.o.t.e.(ϕ).

If F and G are g.t.e.(ϕ), it is easily seen that F and G are l.t.e.(ϕ) at x, for all $x \in X$, and also p.t.e.(ϕ). Most of this paper is concerned with whether or not the converse implications hold: for example if F and G are p.t.e. (ϕ), is it true that F and G are l.t.e.(ϕ) at x, for each $x \in X$?

The definitions p.t.e. (ϕ), etc are flexible in that various restrictions on the homeomorphism ϕ can be imposed. When X is a differentiable manifold, we require $\phi : X \to X$ to be a diffeomorphism.

1.2. Line Bundles. Let $\pi : E \to X$ denote a real or complex line bundle over X. Every endomorphism of $E_{\mathbb{R}}$ ($E_{\mathbb{C}}$) has the form:- $e_x \longmapsto \lambda^F(x)e_x$, where $\lambda^F : X \to \mathbb{R}$ ($\lambda^F : X \to \mathbb{C}$) is a continuous function. If E is a C^∞-differentiable bundle over a C^∞-differentiable manifold X, then $\lambda^F : X \to \mathbb{R}$ ($\lambda^F : X \to \mathbb{C}$) is a C^∞-map.

Recall [3] that if $F_x \sim G_x$ for some $x \in X$ where (a) F and G are real endomorphisms of $E_{\mathbb{R}}$, (b) F and G are complex endomorphisms of $E_{\mathbb{C}}$, then $\lambda^F(x)$ and $\lambda^G(x)$ belong to the same element of the partition of (a) $\mathbb{R}$. (b) $\mathbb{C}$, determined by the following sets:-

(a) (i) $\{y \in \mathbb{R} / -\infty < y < -1\}$ (ii) $\{y = -1\}$, (iii) $\{y \in \mathbb{R} / -1 < y < 0\}$, (iv) $\{y \in \mathbb{R} / y = 0\}$, (v) $\{y \in \mathbb{R} / 0 < y < 1\}$, (vi) $\{y \in \mathbb{R} / y = 1\}$ (vii) $\{y \in \mathbb{R} / y > 1\}$:

(b) (i) $\{z \in \mathbb{C} / z = 0\}$, (ii) $\{z \in \mathbb{C} / 0 < |z| < 1\}$, (iii) the one-parameter family of equivalence classes determined by θ, $0 \leq \theta \leq \pi$, where each element of the family is of the form, $\{z \in \mathbb{C} / z = e^{\pm i\theta}$, for some $\theta\}$, (iv) $\{z \in \mathbb{C} / |z| > 1\}$.

Note. If $\lambda^F(x) = e^{i\theta}$, $0<\theta<\pi$, and $F_x \overset{ot}{\sim} G_x$, then $\lambda^G(x) = e^{i\theta}$.

1.3. Classification Theorems. Let X be a connected $\mathcal{C}^\infty$-differentiable manifold without boundary and E be a $\mathcal{C}^\infty$-line bundle over X. Denote by $End^\infty(E)$, the space of $\mathcal{C}^\infty$-endomorphisms of E. Let $\mathcal{C}^\infty(X, \mathbb{R})$ denote the space of $\mathcal{C}^\infty$-maps of X into $\mathbb{R}$, with the Whitney $\mathcal{C}^\infty$-topology [2]. It is well known [2] that the subspace $\mathcal{C}^\infty_T(X, \mathbb{R}) \subset \mathcal{C}^\infty(X, \mathbb{R})$ of maps that are transversal to a closed submanifold of $\mathbb{R}$ is open dense in $\mathcal{C}^\infty(X, \mathbb{R})$.

Definition 1.5. Let $F \in End^\infty(E)$. Then F is said to be generic if $\lambda^F : X \to \mathbb{R}$ is transversal to the submanifold $S^0 = \{1, -1\} \subset \mathbb{R}$.

Theorem 1.1. (Real Classification Theorem.)

Let F and G be generic $\mathcal{C}^\infty$-endomorphisms of a $\mathcal{C}^\infty$-real line bundle $E_{\mathbb{R}}$ over a $\mathcal{C}^\infty$-manifold X. Let $\phi : X \to X$ be a $\mathcal{C}^\infty$-diffeomorphism such that $\phi^*(E_{\mathbb{R}}) \cong E_{\mathbb{R}}$. Then F and G are g.t.e.(ϕ) if and only if F and G are p.t.e.(ϕ).

Before stating the complex classification theorem, more notation is required.

Consider the finite stratification $\mathcal{S}$ of $\mathbb{R}^2$, defined as follows:- $\mathcal{S}$ consists of the following submanifolds:-
(i) $\{(r,\theta) \in \mathbb{R}^2 / r>1\}$, (ii) $\{(r,\theta) \in \mathbb{R}^2 / r = 1, \theta = 0\}$,
(iii) $\{(r,\theta) \in \mathbb{R}^2 / r = 1, 0<\theta<\pi\}$, (iv) $\{(r,\theta) \in \mathbb{R}^2 / r = 1, \pi<\theta<2\pi\}$,
(v) $\{(r,\theta) \in \mathbb{R}^2 / r = 1, \theta = \pi\}$ (vi) $\{(r,\theta) \in \mathbb{R}^2 / 0 < r < 1\}$.

$\mathcal{S}$ satisfies the Whitney regularity condition -(a). It is known [5] that the subspace $\mathcal{C}^\infty_T(X, \mathbb{R}^2) \subset \mathcal{C}^\infty(X, \mathbb{R}^2)$ of $\mathcal{C}^\infty$-maps transversal to such a stratification $\mathcal{S}$ is open dense in $\mathcal{C}^\infty(X, \mathbb{R}^2)$, where $\mathcal{C}^\infty(X, \mathbb{R}^2)$ has the Whitney $\mathcal{C}^\infty$-topology. (There seems to be no published proof of this.: the proof follows that of theorem 8.1 in [9]).

<u>Definition I.6.</u> Let $F \in \mathrm{End}^{\infty}(E_{\mathbb{C}})$. Then F is said to be generic if $\lambda^F : X \to \mathbb{R}^2$ is transversal to the stratification $\mathcal{S}$.

Consider $F \in \mathrm{End}(E_{\mathbb{C}})$, define $\Sigma_F(0) = \{ x \in X \,/\, \lambda^F(x) = 0 \}$. Write $\lambda^F : X - \Sigma_F(0) \to \mathbb{C} - \{0\}$, such that $\lambda^F = (|\lambda|^F, \beta^F)$ where $|\lambda|^F : X - \Sigma_F(0) \to \mathbb{R}^+$ is given by $|\lambda|^F(x) = |\lambda^F(x)|$, and $\beta^F : X - \Sigma_F(0) \to S^1$ is given by $\beta^F(x) = \exp(i\arg(\lambda^F(x)))$.

Assume F and $G \in \mathrm{End}(E)$ are p.o.t.e. (ϕ).

Define $\Sigma_F(1) = \{ x \in X \,/\, |\lambda|^F(x) = 1 \}$.

Then $\Sigma_F(1) = \phi^{-1}(\Sigma_G(1)) = \Sigma(1)$ (say).

Also $\Sigma_F(0) = \phi^{-1}(\Sigma_G(0)) = \Sigma(0)$ (say).

Define $j(F,G) : (X - \Sigma(0), \Sigma(1)) \to (S^1, 0)$

such that :- $: x \mapsto (\beta^G(\phi(x)) - \beta^F(x))$,

where F and G are p.o.t.e. (ϕ).

Intuitively j measures the amount of twisting of the fibres of E by F relative to G, as they 'move around' $X - \Sigma(0)$. Notice that F and G agree on $\Sigma(1)$, since they are p.o.t.e. (ϕ).

To simplify the statement of the following theorem, we shall assume $X - \Sigma(0)$ is path - connected. A more general theorem can be obtained by applying the weaker theorem to the path - components of $X - \Sigma(0)$.

<u>Theorem 1.2.</u> (<u>Complex Classification Theorem.</u>)

Let X be a C^{∞}- manifold and let $F, G \in \mathrm{End}^{\infty}(E_{\mathbb{C}})$ be generic. If F and G are p.t.e. (ϕ) where $\phi : X \to X$ is a C^{∞}- diffeomorphism, and $x_0 \in X - \Sigma(0)$, then F and G are l.o.t.e. (ϕ) at x_0, or F and $\bar{G}$ are l.o.t.e. (ϕ) at x_0, (where $\lambda^{\bar{G}}(x) = \overline{\lambda^G(x)}$).

Let F and G be p.o.t.e (ϕ), and $\phi^*(E_{\mathbb{C}}) \cong E_{\mathbb{C}}$. Assume $X - \Sigma(0)$

is path - connected. Then F and G are g.o.t.e.(ϕ) if and only if (a) when $\Sigma(1) \neq \emptyset$, $(j$ [illegible]$)_*(\pi_1(X - \Sigma(0), \Sigma(1))) = 0$, (b) when $\Sigma(1) = \emptyset$, $j_*(\pi_1(X - \Sigma(0))) = 0$.

2. REAL LINE BUNDLES.

2.1. Main results . In this section $\pi : E \to X$ will denote a real line bundle over a paracompact space X . The main theorem given here, will be used to prove the ' Real Classification Theorem'. It gives the relationship of local to global equivalence.

Theorem 2.1. Let $F, G \in End(E)$ be p.t.e.(ϕ) where $\phi : X \to X$ is a homeomorphism such that $\phi^*(E) \cong E$. Then F and G are g.t e. (ϕ) if and only if for all $x \in \Sigma(1)$, F and G are l.t.e.(ϕ) at x.

The proof is given later.

The following theorems are of a more technical nature, involving the use of a function c .

Let $F, G \in End(E)$ be p.t.e.(ϕ)

Define $\Sigma(0) = \{ x \in X / \lambda^F(x) = \lambda^G(\phi(x)) = 0 \}$

$\Sigma(1) = \{ x \in X / |\lambda^F(x)| = |\lambda^G(\phi(x))| = 1 \}$ where $\lambda^F : X \to R$ and $\lambda^G : X \to R$ are the previously defined functions associated with F and G.

Define $c : X - (\Sigma(0) \cup \Sigma(1)) \to R^+$:-

$$c(x) = \frac{\log |\lambda^G(\phi(x))|}{\log |\lambda^F(x)|}$$. c is clearly continuous.

The following theorem illustrates the relevance of the function c.

Theorem 2.2. Let $F, G \in End(E)$ be p.t.e.(ϕ) where $\phi : X \to X$ is a homeomorphism such that $\phi^*(E) \cong E$. If $c : X - (\Sigma(0) \cup \Sigma(1)) \to R^+$ can be extended over X to a continuous function $\bar{c} : X \to R^+$, then F and G are g.t.e.(ϕ).

<u>Proof.</u> It is required to find a homeomorphism $\Phi : (E, O_X) \to (\phi^*(E), O_X)$ such that the following diagram commutes :-

$$\begin{tikzcd}
(E,O_X) \arrow[rr, "\Phi"] \arrow[dd, "F"'] \arrow[dr, "\pi"] & & (\phi^*(E),O_X) \arrow[dl, "\pi_\phi"'] \arrow[dd, "\phi^*(G)"] \\
& X & \\
(E,O_X) \arrow[rr, "\Phi"'] \arrow[ur, "\pi"] & & (\phi^*(E),O_X) \arrow[ul, "\pi_\phi"']
\end{tikzcd}$$

Let $\{(U_i, \tau_i)\}$ be a trivialisation of E with transition functions $\{g_{ij}\}$ with values in $O(1) = \{1,-1\}$ (such a trivialisation exists, since X is paracompact). Let $\chi : \phi^*(E) \to E$ be an isomorphism. Then (U_i, τ_i') is a trivialisation of $\phi^*(E)$ with transition functions $\{g_{ij}'\}$ where $\tau_i' = \tau_i \cdot \chi$ and $g_{ij} = g_{ij}'$ for all i,j.

On U_i, F is represented by F_i that is $F_i = \tau_i F \tau_i^{-1}$

On U_i, $\phi^*(G)$ is represented by $\phi^*(G)_i$.

$$F_i : U_i \times \mathbb{R} \to U_i \times \mathbb{R}$$

$$F_i(x,y) = (x.\ \lambda^F(x)y)$$

$$\phi^*(G)_i : U_i \times \mathbb{R} \to U_i \times \mathbb{R}$$

$$\phi^*(G)_i(x,y) = (x, \lambda^G(\phi(x))y)$$

Define $\Phi : E \to \phi^*(E)$ on U_i, such that

$\Phi_i = \tau_i' \Phi \tau_i^{-1} : U_i \times \mathbb{R} \to U_i \times \mathbb{R}$ is given by :-

$$: (x,y) \mapsto (x, y|y|^{\alpha(x)-1})$$

Φ_i is easily seen to be a conjugacy between F_i and G_i on U_i .

To show Φ is well defined on X , it is sufficient to show that the following diagram commutes :-

$$\begin{tikzcd}
(U_i \cap U_j)\times\mathbb{R} \arrow[r, "\Phi_i"] \arrow[d, "\tau_j\tau_i^{-1}"'] & (U_i \cap U_j)\times\mathbb{R} \arrow[d, "\tau_j'\tau_i'^{-1} = \tau_j\tau_i^{-1}"] \\
(U_i \cap U_j)\times\mathbb{R} \arrow[r, "\Phi_i"] & (U_i \cap U_j)\times\mathbb{R}
\end{tikzcd}$$

We have $\tau_j\tau_i^{-1}\cdot(U_i \cap U_j)\times\mathbb{R} \to (U_i \cap U_j)\times\mathbb{R}$ is given by :-

$$(x,y) \to (x, g_{ij}(x)y) \quad \text{where } g_{ij}(x) = 1 \text{ or } -1 .$$

Also $\tau_j\tau_i^{-1} = \tau_j'(\tau_i')^{-1}$

Thus Φ is well defined on X .

Definition 2.1. An endomorphism $F \in End(E)$ is said to be hyperbolic if $\Sigma_F(0) = \Sigma_F(1) = \emptyset$

Corollary 2.2.1. Let $F, G \in End(E)$ be hyperbolic, and $F_{x_0} \overset{t}{\sim} G_{x_0}$ for some $x_0 \in X$, where X is a connected space . Then F and G are g.t.e.(id_X) .

Theorem 2.2 . pointed to the relevance of the function c to the problem of constructing conjugacies. The next two theorems make the relationship more precise.

Theorem 2.3. (Local Equivalence on $\Sigma(1)$). Let $F, G \in End(E)$ be p.t.e.(ϕ) in some neighbourhood V of $q \in \Sigma(1)$. Assume $V - \Sigma(1) \neq \emptyset$ Then F and G are l.t.e.(ϕ) at q if and only if for some neighbourhood $U \subset V$ of q , the function $c : U - \Sigma(1) \to \mathbb{R}^+$ extends over U to a function $\tilde{c} : U \to \mathbb{R}^+$

In contrast to theorem 2.3., theorem 2.4. shows that the behavior of c near $\Sigma(0)$ does not determine local equivalence on $\Sigma(0)$.

Theorem 2.4. (Local Equivalence on $\Sigma(0)$.)

Let $F, G \in End(E)$ be p.t.e.(ϕ) in some neighbourhood V of a point $s \in \Sigma(0)$. Then F and G are l.t.e.(ϕ) at s.

From theorem 2.3. we can construct a counterexample to the conjecture :- F and G are p.t e.(ϕ) in a neighbourhood U of $x \in X$ implies F and G are l.t.e.(ρ) at x. Moreover in the counterexample there is no homeomorphism $\rho : X \to X$, such that F and G are l.t.e.(ρ).

Counter-example 2.1. Let $X = \mathbb{R}$, $E = \mathbb{R} \times \mathbb{R}$ and $F, G \in End(\mathbb{R} \times \mathbb{R})$ be such that $\lambda^F : \mathbb{R} \to \mathbb{R}$ and $\lambda^G : \mathbb{R} \to \mathbb{R}$ are defined as follows:-

$$\lambda^G(x) = \exp\left(|2x + x \sin 1/x|\right) , \quad x \neq 0$$
$$= 1 \qquad , \quad x = 0$$
$$\lambda^F(x) = \exp(|x|)$$

F and G are p.t.e.(ϕ) for any homeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. We show F and G are not l t.e.(ϕ) at x = 0 for any homeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. It is sufficient, in view of theorem 2.3. to show that

$$\lim_{x \to 0} \frac{\log(\exp|2\phi(x) + \phi(x) \sin 1/\phi(x)|)}{\log(\exp|x|)} , \quad \text{does not exist}$$

for anyhomeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.

If for some $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $\lim_{x \to 0} \dfrac{2\phi(x) + \phi(x) \sin 1/\phi(x)}{x}$ exists then $h(x) = \phi(x) \; (2 + \sin 1/\phi(x))$ is differentiable at x = 0. It is easily checked that this is falsefor any choice of ϕ. (Intuitively, no change of variable on $\mathbb{R}$, effectively dampens the oscill ations of h.)

2.2. Proof of Theorems.

To prove theorems 2.2 and 2.3., lemmas 2.1. and 2.2. are needed. They give the general form for a local conjugacy between two endomorphisms that are locally equivalent at any point $x \in (X - (\Sigma(0) \cup \Sigma(1)))$

Lemma 2.1. Let $h : \mathbb{R} \to \mathbb{R}$ be an endomorphism of $\mathbb{R}$ given by, $h : y \mapsto \lambda y$, $\lambda \in \mathbb{R} - \{-1, 0, 1\}$. Then any homeomorphism $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, such that $\psi(\mathbb{R}^+) = \mathbb{R}^+$, conjugating h with itself is of the form:-

$$\psi : y \longmapsto y \exp(\mu(\log y)), \ y>0$$

where $\mu : \mathbb{R} \to \mathbb{R}$ is some periodic function of period $\log|\lambda|$, that is $\mu(y + \log|\lambda|) = \mu(y)$, $y \in \mathbb{R}$. Furthermore μ satisfies the inequality :-

$$|\mu(y) - \mu(0)| < |\log|\lambda|| \quad , \quad \text{for all } y \in \mathbb{R}.$$

Proof. Assume $\lambda > 0$.

If $\psi : (\mathbb{R},0) \to (\mathbb{R},0)$ is a conjugacy between h and itself, (and $\psi(\mathbb{R}^+) = \mathbb{R}^+$), then $\psi \cdot h = h \cdot \psi$ implies

$$\psi(\lambda y) = \lambda \psi(y) \ .$$

Let $\psi'(\log y) = \log \psi(y)$, for $y>0$

then $\psi' : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and

$$\log(\psi(\lambda y)) = \log(\lambda \psi(y)) \ , \text{for } y>0 ,$$

implies $\psi'(\log\lambda + \log y) = \log\lambda + \log\psi'(y)$.

Let $\mu(y) = \psi'(y) - y$, $y \in \mathbb{R}$

then $\mu : \mathbb{R} \to \mathbb{R}$ is continuous and $\mu(y + \log\lambda) = \mu(y)$

Clearly $\psi(y) = y\exp(\mu(\log y))$, for $y > 0$.

To prove the inequality for μ, it is sufficient to show it is satisfied on the interval $[0, |\log(\lambda)|]$

The inequality on this interval follows easily from the fact

$$\psi'(y) = y + \mu(y)$$

is an increasing function.

Lemma 2.2. Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be conjugate endomorphisms of $\mathbb{R}$ given by :-

$$h_1 : y \longmapsto \lambda_1 y \ , \ |\lambda_1| \neq 1 \text{ or } 0$$
$$h_2 : y \longmapsto \lambda_2 y \ , \ |\lambda_2| \neq 1 \text{ or } 0$$

then any conjugacy $\chi : (\mathbb{R},0) \to (\mathbb{R},0)$ such that $\chi(\mathbb{R}^+) = \mathbb{R}^+$ between h_1 and h_2 i.e. $h_2 = \chi h_1 \chi^{-1}$ is of the form :-

$$\chi : y \longmapsto y^c \exp(c\mu(\log y)) \quad , \quad y>0$$

where μ is some periodic function of period $\log|\lambda_1|$ and

$$c = \frac{\log|\lambda_2|}{\log|\lambda_1|}$$

Proof. Let $\chi_1 : (\mathbb{R},0) \to (\mathbb{R},0)$ be the homeomorphism :

$$\chi_1(y) = y|y|^{c-1}$$

then $h_2 = \chi_1 h_1 \chi_1^{-1}$

But $h_2 = \chi \cdot h_1 \chi^{-1}$, hence $h_1 = (\chi_1^{-1}\chi)h_1(\chi_1^{-1}\chi)^{-1}$

that is, $(\chi_1^{-1}\chi)$ conjugates h_1 with itself, thus from lemma 2.1. , $(\chi_1^{-1}\chi)$ is of the form :- $y \mapsto y\exp(\mu(\log y))$, for $y>0$, for some periodic function $\mu : \mathbb{R} \to \mathbb{R}$ of period $\log|\lambda_1|$

Hence for $y>0$ $\chi(y) = y^c\exp(c\mu(\log y))$.

We now prove theorem 2.3. .

Proof of theorem 2.3. . (Local equivalence on $\Sigma(1)$.)

Since the problem is local we may assume E is trivial .

'if' . The result follows from theorem 2.2.

'only if' . Assume F and G are l.t.e. (ϕ) at $q \in \Sigma(1)$. Let Φ be a conjugacy between F and G on some neighbourhood $U \subset \bar{U} \subset V$ of q . Without loss of generality assume $\Phi_x(\mathbb{R}^+) = \mathbb{R}^+$, then from lemma 2.2. , for $x \in U - \Sigma(1)$, Φ_x is of the form :-

$$y \longmapsto y^{c(x)}\exp(c(x)\mu_x(\log y)) , \quad \text{for } y > 0$$

where $\mu_x : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $\log|\lambda^F(x)|$, that varies continuously with x , for $x \in U - \Sigma(1)$.

We assume $\mu_x(0) = 0$ for all $x \in U - \Sigma(1)$, then lemma 2.1. implie

$$c(x)|\mu_x(y)| < c(x)|\log|\lambda^F(x)|| = |\log|\lambda^G\phi(x)||$$

But $\log|\lambda^G\phi(x)| = \log|\lambda^F(x)| = 0$, for all $x \in \Sigma(1)$.

Hence if $\{x_n\}$ is a sequence of points in $U - \Sigma(1)$, that converges to $q \in U \cap \Sigma(1)$, then

$\lim_{n\to\infty} \exp(c(x_n)\mu_{x_n}(\log y)) = 1$, for all $y > 0$.

Thus $\lim_{n\to\infty} \Phi_{x_n}$ exists and is a homeomorphism if and only if $\lim_{n\to\infty} c(x_n)$ exists and is non-zero . Therefore $c : U - \Sigma(1) \to \mathbb{R}^+$ extends over $\overline{U - \Sigma(1)} \cap \bar{U}$, which is a closed subspace of a normal subspace $\bar{U}$ of X . Hence by the 'Tietze Extension Theorem' , c extends over U to some function $\tilde{c} : U \to \mathbb{R}^+$.

Proof of theorem 2.4. (Local Equivalence on $\Sigma(0)$.)

We assume E is trivial since the problem is local .

Let $U \subset V$ be a neighbourhood of $s \in \Sigma(0)$ such that $|\lambda^F(x)| < 1$ for all $x \in U$. Assume $U - \Sigma(0) \neq \emptyset$.

Let $a(x) = -\log|\lambda^F(x)|$ and $b(x) = -\log|\lambda^F(\phi(x))|$, then $c(x) = \dfrac{b(x)}{a(x)}$

Define $\Phi : (U \times \mathbb{R}, 0_x) \longrightarrow (U \times \mathbb{R}, 0_x)$ as follows :-

for $x \in \Sigma(0)$, $\Phi_x(y) = y$;

for $x \in U - \Sigma(0)$, $\Phi_x(y) = y^{c(x)} \exp(c(x)\mu_x(\log y))$, for $y > 0$,

$\Phi_x(-y) = -\Phi_x(y)$;

where $\mu_x : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $\log|\lambda^F(x)|$ defined as follows :-

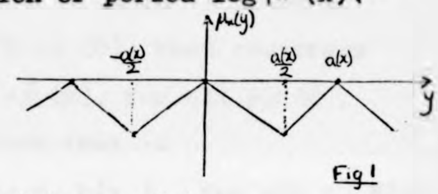


Fig 1

if $b(x) > a(x)$;

$$\mu_x(y) = \frac{y}{c(x)} - y \quad , \quad 0 < y < \frac{a(x)}{2}$$

$$= \left(1 - \frac{a(x)}{b(x)}\right)(y - a(x)) \quad , \quad \frac{a(x)}{2} \leq y < a(x)$$

Elsewhere $\mu_x(y)$ is given by $\mu_x(y) = \mu_x(y + a(x))$.

If $b(x) = a(x)$,

$\mu_x(y) = 0$, for all $y \in \mathbb{R}$.

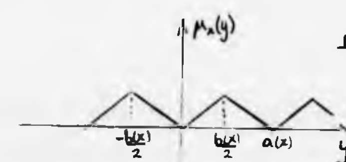


Fig 2

If $b(x) < a(x)$,

$$\mu_x(y) = \frac{y}{c(x)} - y \quad , \quad 0 \leq y < \frac{b(x)}{2}$$

$$= \frac{(a(x) - b(x))}{(b(x) - 2a(x))}(y - a(x)) \quad , \quad \frac{b(x)}{2} \leq y < a(x)$$

Elsewhere $\mu_x(y)$ is given by $\mu(y) = \mu_x(y + a(x))$

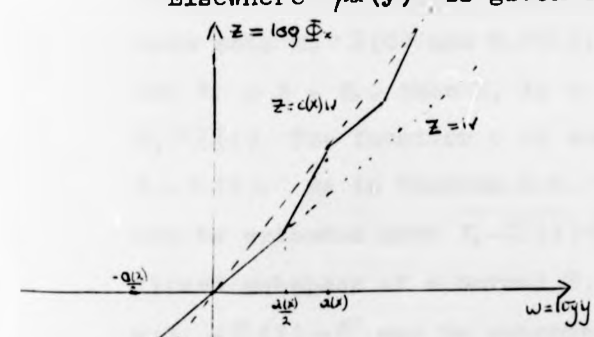


Fig 3 $(b(x) > a(x))$

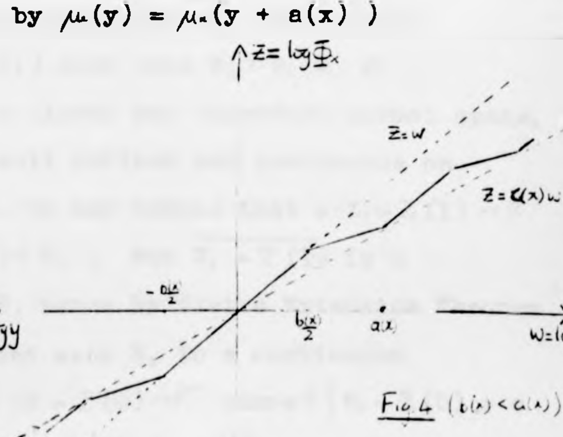


Fig 4 $(b(x) < a(x))$

Proof of theorem 2.4. (Local Equivalence on $\Sigma(0)$.)

We assume E is trivial since the problem is local .

Let $U \subset V$ be a neighbourhood of $s \in \Sigma(0)$ such that $|\lambda^F(x)| < 1$ for all $x \in U$. Assume $U - \Sigma(0) \neq \emptyset$.

Let $a(x) = -\log|\lambda^F(x)|$ and $b(x) = -\log|\lambda^s(\phi(x))|$, then $c(x) = \dfrac{b(x)}{a(x)}$

Define $\Phi : (U \times \mathbb{R}, 0_x) \longrightarrow (U \times \mathbb{R}, 0_x)$ as follows :-

for $x \in \Sigma(0)$, $\Phi_x(y) = y$;

for $x \in U - \Sigma(0)$, $\Phi_x(y) = y^{c(x)} \exp(c(x)\mu_x(\log y))$, for $y > 0$,

$\Phi_x(-y) = -\Phi_x(y)$;

where $\mu_x : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $\log|\lambda^F(x)|$ defined as follows :-

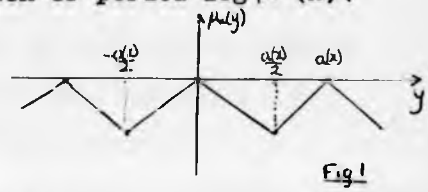


Fig 1

if $b(x) > a(x)$:

$$\mu_x(y) = \frac{y}{c(x)} - y \quad , \quad 0 < y < \frac{a(x)}{2}$$

$$= \left(1 - \frac{a(x)}{b(x)}\right)(y - a(x)) \quad , \quad \frac{a(x)}{2} \leq y < a(x)$$

Elsewhere $\mu_x(y)$ is given by $\mu_x(y) = \mu_x(y + a(x))$.

If $b(x) = a(x)$,

$\mu_x(y) = 0$, for all $y \in \mathbb{R}$.

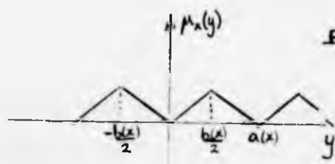


Fig 2

If $b(x) < a(x)$,

$$\mu_x(y) = \frac{y}{c(x)} - y \quad , \quad 0 \leq y < \frac{b(x)}{2}$$

$$= \frac{(a(x) - b(x))}{(b(x) - 2a(x))}(y - a(x)) \quad , \quad \frac{b(x)}{2} \leq y < a(x)$$

Elsewhere $\mu_x(y)$ is given by $\mu_x(y) = \mu_x(y + a(x))$

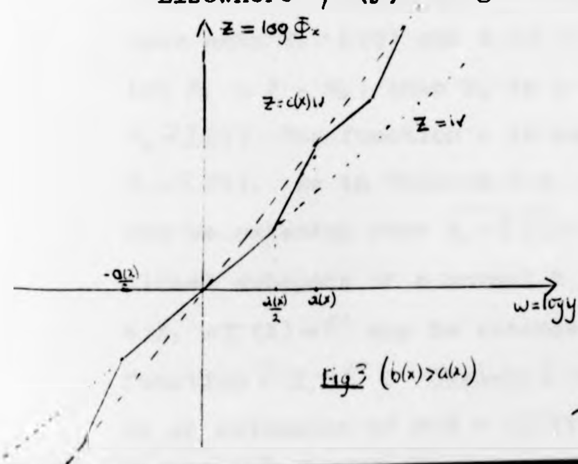


Fig 3 $(b(x) > a(x))$

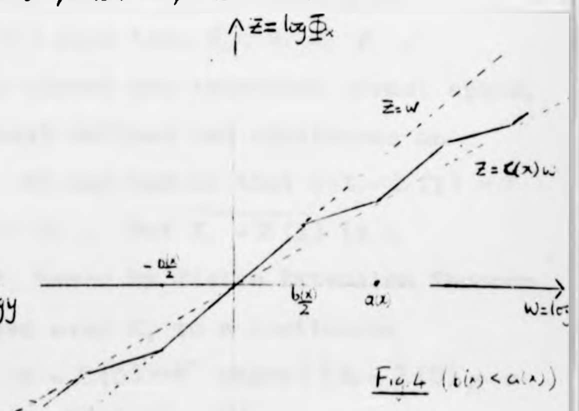


Fig 4 $(b(x) < a(x))$

Since a,b,and c are continuous non-zero functions on $U - \Sigma(0)$, μ_x is continuous with respect to x. To show Φ_x is a homeomorphism for $x \in U - \Sigma(0)$ it is sufficient to show that the function, $\gamma_x : \mathbb{R} \to \mathbb{R}$ given by:-

$$\gamma_x(y) = y + \mu_x(y)$$

is an increasing function on $[0, a(x)]$. Since $\gamma_x : \mathbb{R} \to \mathbb{R}$ is differentiable except at the obvious isolated points, this is easily checked.

Lemma 2.2. implies Φ_x conjugates F_x and $(\phi^*(G))_x$ for all $x \in U$. Finally it is only necessary to check Φ is continuous with respect to x on U., in particular at each $s \in U \cap \Sigma(0)$. Let $\{x_n\}$ be a sequence of points in $U - \Sigma(0)$, that converges to $s \in U \cap \Sigma(0)$, we prove $\lim_{n\to\infty} \Phi_{x_n}(y) = \Phi_s(y)$, for all $y \in \mathbb{R}$. Given $y \in \mathbb{R}$, there exists $N(y) > 0$ such that :-

$$-\frac{a(x_n)}{2} < y < \frac{a(x_n)}{2} \quad \text{and} \quad -\frac{b(x_n)}{2} < y < \frac{b(x_n)}{2}, \text{ for all } n > N(y).$$

Hence for $n > N(y)$, $\Phi_{x_n}(y) = y$, that is,

$$\lim_{n\to\infty} \Phi_{x_n}(y) = \Phi_s(y)$$

We now prove theorem 2.1. .

<u>Proof of theorem 2.1.</u> .

<u>'only if'</u> : It is trivial that if F and G are g.t.e. (ϕ) then F and G are l.t.e. (ϕ) at x, for all $x \in X$

<u>'if '</u>. Assume F and G are l.t.e. (ϕ) at each $x \in \Sigma(1)$. Since X is paracompact and therefore normal, there exist open sets $W_0 \supset \Sigma(0)$ and $W_1 \supset \Sigma(1)$ such that $W_0 \cap W_1 = \emptyset$.

Let $X_1 = X - W_0$, then X_1 is a closed and therefore normal space; $X_1 \supset \Sigma(1)$. The function c is well defined and continuous on $X_1 - \Sigma(1)$. As in Theorem 2.3. we may deduce that $c : X_1 - \Sigma(1) \to \mathbb{R}^+$ can be extended over $\overline{X_1 - \Sigma(1)} \subset X_1$. But $\overline{X_1 - \Sigma(1)}$ is a closed subspace of a normal X_1 hence by Tietze Extension Theorem $c : X_1 - \Sigma(1) \to \mathbb{R}^+$ may be extended over X_1 to a continuous function $\tilde{c} : X_1 \to \mathbb{R}^+$. Clearly $\tilde{c} : X - \Sigma(0) \to \mathbb{R}^+$ where $\tilde{c} \mid W_0 - \Sigma(0) = c$ is an extension of $c : X - (\Sigma(1) \cup \Sigma(0)) \to \mathbb{R}^+$.

Let $\{(U_i, \tau_i)\}$ be a trivialisation of E, with transition functions $\{g_{ij}\}$ with values in O (1). Define a conjugacy Φ between F and G on X, such that Φ is represented by Φ_i on U_i where Φ_i is defined as follows : -

for $x \in U_i - (\Sigma(0) \cup \Sigma(1))$,

$$\Phi_{ix}(y) = y^{\tilde{c}(x)} \exp(\tilde{c}(x)\mu_x(\log y)) \quad , \text{ for } y > 0 ,$$

$$\Phi_{ix}(y) = -\Phi_{ix}(y) \quad ;$$

where $\mu_x : \mathbb{R} \to \mathbb{R}$ is the periodic function of period $\log |\lambda^F(x)|$ as defined in theorem 2.4. ;

for $x \in U_i \cap \Sigma(0)$, Φ_x is the identity on $\mathbb{R}$;

for $x \in U_i \cap \Sigma(1)$, $\Phi_x(y) = y\,|y|^{\tilde{c}(x)-1}$, for all $y \in \mathbb{R}$.

Clearly Φ_i conjugates F_i and $(\phi^*(G)_i)$ on U_i .

As in theorem 2.2. we may show that Φ is well defined and continuous on X.

3. COMPLEX LINE BUNDLES .

3.1. Preliminaries . Throughout this section $\pi : E \to X$ or E for short , will denote an oriented complex line bundle over X (with the natural orientation) , where X is a paracompact path-connected space. The interrelationships of p.t.e.(ϕ), l.t.e.(ϕ), etc. , for complex endomorphisms F and G are considerably more complicated than those for F and G real . The local classification problems however , have alot in common . Instead of the local extendability of the function c determining local equivalence, it is determined by the local extendability of two functions. One of these we shall again denote by c , owing to the obvious resemblance to the function previously denoted by c. The other function is d . In the global classification problem d plays a central role. Intuitively c measures the rate of expanding or contracting of the fibres of E by F relative to G as they 'move around' X

Also d measures the amount of twisting of the fibres of E by F relative to G as they 'move around' X.

Recall that, if $F \in \mathrm{End}(E)$, F_x is of the form : -

$$F_x \quad : v \mapsto \lambda^F(x)\, v \quad , \quad v \in E_x \quad ,$$

where $\lambda^F : X \to \mathbb{C}$ is continuous.

Also $\Sigma_F(0) = \{x \in X \,/\, \lambda^F(x) = 0\}$

and $\Sigma_F(1) = \{x \in X \,/\, |\lambda^F(x)| = 1\}$

Then write $\lambda^F = (|\lambda|^F, \beta^F) : X - \Sigma_F(0) \to \mathbb{C} - 0$

where $|\lambda|^F : X - \Sigma_F(0) \to \mathbb{R}^+$ is given by $|\lambda|^F(x) = |\lambda^F(x)|$

and $\beta^F : X - \Sigma_F(0) \to S^1$ is given by $\beta^F(x) = \exp(i \arg \lambda^F(x))$

$0 \leq \arg \lambda^F(x) < 2\pi$.

Let F and $G \in \mathrm{End}(E)$ be p.t.e.(ϕ), then

$$\Sigma_F(0) = \phi^{-1}(\Sigma_G(0)) = \Sigma(0) \quad \text{(say)} \quad \text{and}$$

$$\Sigma_F(1) = \phi^{-1}(\Sigma_G(1)) = \Sigma(1) \quad \text{(say)} .$$

Define $c : X - (\Sigma(0) \cup \Sigma(1)) \to \mathbb{R}^+$ such that,

$$c(x) = \frac{\log|\lambda^G(\phi(x))|}{\log|\lambda^F(x)|} .$$

c is well defined and continuous.

Define $\quad j(F,G) : (X - \Sigma(0), \Sigma(1)) \to (S^1, 0)$

$$: x \mapsto (\beta^G(\phi(x)) - \beta^F(x)) ,$$

where F and G are p.o.t.e.(ϕ).

Let $Y \subset X$. We say a map $k : (X,Y) \to (S^1, 0)$ lifts , if there exists a map $\hat{k} : (X,Y) \to (\mathbb{R}, 0)$ such that the following diagram commutes : -

$$\begin{tikzcd} & (\mathbb{R}, 0) \arrow[d, "p"] \\ (X,Y) \arrow[ur, "\hat{k}"] \arrow[r, "k"'] & (S^1, 0) \end{tikzcd}$$

where p is the standard covering map $\theta \mapsto e^{i\theta}$.

In general j does not lift. However j lifts locally in $X - \Sigma(0)$. That is to say , if j lifts locally, then for each $x_0 \in X - \Sigma(0)$, there exists a neighbourhood U of x_0, such that $j/U - \Sigma(0)$ lifts. For ease of exposition we shall assume from now on that $X - \Sigma(0)$ is path-connected and locally path-connected. Otherwise the various path components need to be considered.

For each $x_0 \in X - \Sigma(0)$ define $d_{x_0} : U - \Sigma(1) \to \mathbb{R}$ for some neighbourhood U of x_0 such that,

$$d_{x_0}(x) = \frac{\widehat{j|U}(x)}{\log|\lambda^F(x)|}, \quad \text{for all } x \in U - \Sigma(1)$$

Then d_{x_0} is well defined and continuous for each $x_0 \in X - \Sigma(0)$.

If j lifts globally. we define :-

$d : X - \Sigma(1) \to \mathbb{R}$,

$$d(x) = \frac{\hat{j}(x)}{\log|\lambda^F(x)|}, \quad x \in X - (\Sigma(1) \cup \Sigma(0))$$

$$= 0 \quad , \quad x \in \Sigma(0) .$$

For each $x_0 \in X$, there is a neighbourhood U of x_0 such that $d|U = d_{x_0}$.

Recall that if $F_q \overset{t}{\sim} G_q$ where $\lambda^F(q) \in S' - ((1,0) \cup (1,\pi))$ then $\lambda^F(q) = \lambda^G(q)$ or $\lambda^F(q) = \overline{\lambda^G}(q)$. If the former holds, any homeomorphism conjugating F and G is orientation preserving, and if the latter, orientation reversing.

$\bar{G}$ is the endomorphism of E such that $\lambda^{\bar{G}}(x) = \overline{\lambda^G}(x)$.
Clearly G and $\bar{G}$ are g.t.e.(id_X) .

<u>Lemma 3.1.</u> Let F, $G \in$ End (E) .

(a) If F and G are l.t e.(ϕ) at $q \in \Sigma(1)$, then at least one of the following hold , (i) F and G are l.o.t.e.(ϕ) at q, or (ii) F and $\bar{G}$ are l.o.t.e.(ϕ) at q .

(b) If F and G are g.t.e.(ϕ) , then at least one of the following hold, (i) F and G are g.o.t.e.(ϕ) , or (ii) F and $\bar{G}$ are g.o.t.e.(ϕ) .

<u>Proof (a)</u> Let Φ be a conjugacy between F and G on some neighbourhood V of q . Then if Φ_q is orientation preserving , there exists a neighbourhood $U \subset V$ of q , such that Φ_x is orientation preserving for all x in U . Hence F and G are l.o.t.e.(ϕ) at q . Similarly, if Φ_q is orientation reversing , F and $\bar{G}$ are l.o.t.e.(ϕ) at q .

<u>(b)</u> Let Φ be a conjugacy between F and G on X . Assume at $x_1 \in X$, Φ_{x_1} is orientation preserving and at x_2 ($\neq x_1$) Φ_{x_2} is orientation reversing . But X is path-connected . Thus Φ restricted to the path joining x_1 and x_2 gives a homotopy between an orientation preserving and an orientation reversing map , which is impossible . Therefore if F and G are g.t.e.(ϕ) , then F and G or F and $\bar{G}$ are g.o.t.e.(ϕ) .

It is clear from lemma 3.1. that we may construct a counter-example in which endomorphisms F and G are p.t.e.(ϕ) in some neighbourhood V of a point $x \in X$, but are not l.t.e.(ϕ) at x .

<u>Counter-example 3.1.</u>

Let $X = \mathbb{R}^2 - \{0\}$, and $E = X \times \mathbb{C}$.

Define $F, G \in \mathrm{End}(E)$ such that $\lambda^F : X \to \mathbb{C}$ and $\lambda^G : X \to \mathbb{C}$ are given as follows :-

$$\lambda^F(r, \theta) = re^{i\theta} \quad , \quad 0 \leq |\theta| \leq \pi \, .$$

$$\lambda^G(r, \theta) = re^{i|\theta|} \quad , \quad 0 \leq |\theta| \leq \pi \, .$$

F and G are p.t.e.(id_X) on X, but F and G are not l.t.e.(id_X) at $x = (1,0)$ or $(1,\pi)$. Moreover there is no homeomorphism $\phi: X \to X$ such that F and G are l.t.e.(ϕ) at $x = (1,0)$ or $(1,\pi)$ Further, taking $X = \mathbb{R}^2 - \{(0,0) \cup (1,0) \cup (1,\pi)\}$, also F and G as above, then it is easily seen that F and G are l.t.e.(id_X) at x, for all $x \in X$, but F and G are not g.t.e.(id_X). In fact there is no homeomorphism $\phi: X \to X$, such that F and G are g.t.e.(ϕ

In the above counterexample, F and G are p.t.e.(id_X) on X but neither F and G are p.o t.e.(id_X) nor F and $\bar{G}$ are p.o.t.e.(id_X). In view of the above counterexample and lemma, we shall consider in this section the concepts of p.o.t.e.(ϕ), l.o.t.e.(ϕ) and g.o.t.e.(ϕ).

3.2. Main Theorems.

Theorem 3.1. is the main result of this section in that the 'Complex Classification Theorem', proved in section 4, follows easily from it. It is proved later.

Theorem 3.1. Let $F, G \in End(E)$ be l.o.t.e.(ϕ) at each $x \in X - \Sigma(0)$, where $\phi : X \to X$ is a homeomorphism such that $\phi^*(E) \cong E$. Assume $X - \Sigma(0)$ is path-connected. Then F and G are g.o.t.e.(ϕ) if and only if (i) if $\Sigma(1) \neq \emptyset$,

$$(j_{})_*(\pi_1(X - \Sigma(0), \Sigma(1))) = 0 ,$$

(ii) if $\Sigma(1) = \emptyset$,

$$j_*(\pi_1(X - \Sigma(0)) = 0$$

Definition 3.1. An endomorphism $F \in End(E)$ is said to be hyperbolic if $\Sigma_F(0) = \Sigma_F(1) = \emptyset$.

Theorem 3.2. (Local Equivalence of Hyperbolic Endomorphisms.) Let $F, G \in End(E)$ be hyperbolic such that for some $x_0 \in X$ $F_{x_0} \sim G_{\phi(x_0)}$. Then F and G are l.t.e.(ϕ) at x_0.

Corollary 3.2.1. (Global Equivalence of Hyperbolic Endomorphisms)
Let F , $G \in End(E)$ be hyperbolic endomorphisms that are p.t.e.(ϕ) for some homeomorphism $\phi : X \to X$ such that $\phi^*(E) \equiv E$. Then F and G are g.t.e.(ϕ) if and only if $j_*(\pi_1(X)) = 0$.

The proof of Corollary 3.2.1. follows easily from theorems 3.1. and 3.2. .

Theorem 3.3. (Local Equivalence on $\Sigma(0)$) .

Let $F, G \in End(E)$ be p.t.e.(ϕ) in some neighbourhood V of $s \in \Sigma(0)$. Assume $X - \Sigma(0)$ is locally path-connected Then F and G are l.t.e.(ϕ) at s if and only if for some neighbourhood $U \subset V$ of s ,

$$j_*(\pi_1(U - \Sigma(0))) = 0 .$$

Theorem 3.4. (Local Equivalence on $\Sigma(1)$)

Let $F, G \in End(E)$ be p.o.t.e.(ϕ) in some neighbourhood V of $q \in \Sigma(1)$. Then F and G are l o.t.e.(ϕ) at q if and only if for some neighbourhood $U \subset V$ of q , the functions $c : U - \Sigma(1) \to \mathbb{R}^+$ and $d_q : U - \Sigma(1) \to \mathbb{R}$ can be extended over U to continuous functions $\tilde{c} : U \to \mathbb{R}^+$ and $\tilde{d}_q : U \to \mathbb{R}$.

3.3. Proofs of Theorems

Theorem 3.1. is proved last , as it depends on the proofs of the other theorems .

Proof of Theorem 3.2. (Local Equivalence of Hyperbolic Endomorphisms) .

We assume E is trivial since the problem is local.

If $F_{x_0} \overset{t}{\sim} G_{\phi(x_0)}$ for some $x_0 \in X$, then F and G are p.t.e.(ϕ) in some neighbourhood U of x_0 .Choose U so that d_{x_0} is well defined and continuous .

A conjugacy Φ between F and G is given by the following formula :-

$$\Phi_x(r,\theta) = (r^{c(x)}, \theta + p(d_u(x)\log r)) \quad , \quad r \neq 0$$
$$\Phi_x(0) = 0$$

where $p : \mathbb{R} \to S^1$ is the standard covering map .

It is easily checked that Φ is the required conjugacy .

Proof of Theorem 3.3. (Local Equivalence on $\Sigma(0)$).

The problem is local , so we take E to be trivial .

'if'. Assume $j_*(\pi_1(U - \Sigma(0)) = 0$

Then the map $j : (U - \Sigma(0), x_0) \longrightarrow (S^1, j(x_0))$ lifts for each $x_0 \in U - \Sigma(0)$ to a map $\hat{j} : (U - \Sigma(0), x_0) \to (\mathbb{R}, \hat{j}(x_0))$.

Therefore $d_s(x) = \dfrac{\hat{j}(x)}{\log|\lambda^F(x)|}$, for $x \in U - \Sigma(0)$

is well defined and continuous . ($d_s(x) = 0$, for $x \in U \cap \Sigma(0)$).

Define $\Phi : (U \times \mathbb{C}, 0_x) \longrightarrow (U \times \mathbb{C}, 0_x)$, such that :-

for $x \in U - \Sigma(0)$,

$$\Phi_x(r,\theta) = (r^{c(x)}\exp(c(x)\mu_x(\log r)), \theta + p(d_s(x)\log r)) :$$

for $x \in \Sigma(0)$, Φ_x is the identity on $\mathbb{C}$;

where $\mu_x : \mathbb{R} \to \mathbb{R}$ is the periodic function of period $\log|\lambda^F(x)|$ as defined in theorem 2.4. . It may checked that Φ is the required conjugacy between F and G on U .

'only if' . Let F ,G be l.t.e. (ϕ) at $s \in \Sigma(0)$.

Let $\Phi : (U \times \mathbb{C}, U \times \{0\}) \to (U \times \mathbb{C}, U \times \{0\})$ be aconjugacy between F and G on a neighbourhood U of s .

If $j_*(\pi_1(U - \Sigma(0)) \neq 0$, then there exists a loop $L_0 : (S^1, 0) \to (U - \Sigma(0), x_0)$, for some $x_0 \in U - \Sigma(0)$, that maps S^1 homeomorphically onto $L_0(S^1)$, and $j \cdot L_0 : (S^1, 0) \to (S^1, j(x_0))$ is a map of non-zero degree .

Since Φ maps $U \times \{0\}$ homeomorphically onto $U \times \{0\}$, we identify $\mathbb{C} - \{0\}$ with $\mathbb{R}^+ \times S^1$ and write

$$\Phi | U \times \mathbb{C} - \{0\} = (\Phi^1, \Phi^2)$$

where $\Phi^1_x : \mathbb{R}^+ \times S' \to \mathbb{R}^+$ and $\Phi^2_x : \mathbb{R}^+ \times S' \to S'$.

We assume Φ is orientation preserving . Also we may assume $\Phi | (L_0(S') \times \mathbb{C} - \{0\})$ is homotopic to the identity on $L_0(S') \times \mathbb{C} - \{0\}$. For if not , Φ may be replaced by the conjugacy given by the formula :-

$$(x,r,\theta) \longmapsto \left(x, \ \frac{\Phi^1_x(r,\theta)}{\Phi^1_x(1,0)} , \ \Phi^2_x(r,\theta) - \Phi^2_x(1,0) \right) .$$

Clearly $(x,1,0) \longmapsto (x,1,0)$, for $x \in U$, and Φ is the identity on the loop $L_0(S')$. Also fixing $(r_0, \theta_0) \in \mathbb{C} - \{0\}$, Φ maps the path $L_0(S') \times \{(r_0, \theta_0)\}$ to a path homotopic to $L_0(S') \times \{(1,0)\}$. Thus Φ is homotopic to the identity on $L_0(S') \times (\mathbb{C} - \{0\})$.

Furthermore

$$\Phi_x \cdot F_x = (\psi^*(G))_x \cdot \Phi_x , \quad \text{for } x \in U ,$$

hence $\quad \Phi_x(|\lambda^F(x)|, \beta^F(x)) = (|\lambda^G(\phi(x))|, \beta^G(\phi(x)))$

Define $f : (S',0) \to (L_0(S') \times (\mathbb{C} - \{0\}), (x_0, \lambda^F(x_0)))$,

$\qquad f(w) = (L_0(w), \lambda^F(L_0(w)))$:

and $\tilde{f} : (S',0) \to (L_0(S') \times S', (x_0, \beta^F(x_0)))$,

$\qquad \tilde{f}(w) = (L_0(w), \beta^F(L_0(w)))$

Also define $g : (S',0) \to (L_0(S') \times (\mathbb{C} - \{0\}), (x_0, \lambda^G(\phi(x_0))))$,

$\qquad g(w) = (L_0(w), \lambda^G(\phi(L_0(w))))$:

and $\tilde{g} : (S',0) \to (L_0(S') \times (S'), (x_0, \beta^G(\phi(x_0))))$,

$\qquad \tilde{g}(w) = (L_0(w), \beta^G(\phi(L_0(w))))$.

We have, $\Phi \cdot f = g$.

Hence f and g are homotopic maps: also, then $\tilde{f}$ and $\tilde{g}$ are homotopic.

But if $\tilde{f}$ and $\tilde{g}$ are homotopic,

$j \cdot L_0 = (\beta^G \cdot \phi \cdot L_0 - \beta^F \cdot L_0) : (S',0) \to (S', j(x_0))$

is a map of degree 0, - contradiction.

Hence $j_*(\pi_1(U - \Sigma(0))) = 0$

The following lemmas are required to prove Theorems 3.1. and 3.4.

Lemma 3.2. Let $F \in End(E)$ and $\Psi : (E/U, O_X) \longrightarrow (E/U, O_X)$ be a local conjugacy between F and itself at $q \in \Sigma(1)$, on some neighbourhood U of q. Assume $U - \Sigma(1) \neq \emptyset$. Let $\{(U_i, \tau_i)\}$ be a trivialisation of E. Assume $q \in U_i$ and let Ψ_i represent Ψ on U_i; also write

$$\Psi_{iq} | C - \{0\} = \Psi_{iq} | (\mathbb{R}^+ \times S') = (\Psi^1_{iq}, \Psi^2_{iq})$$

where $\Psi^1_{iq} : \mathbb{R}^+ \times S' \to \mathbb{R}^+$ and $\Psi^2_{iq} : \mathbb{R}^+ \times S' \to S'$. Then $\Psi^1_{iq}(r, \theta)$ is a linear function of $r \in \mathbb{R}^+$ and constant in $\theta \in S'$, and $\Psi^2_{iq}(r, -) : S' \to S'$ is a rotation, for each $r \in \mathbb{R}^+$.

Proof. Clearly if Ψ^1_{iq} and Ψ^2_{iq} have the required properties for the trivialisation $\{(U_i, \tau_i)\}$, they have the required properties on any other trivialisation. Therefore we may assume E is trivial and consider Ψ^1_q and Ψ^2_q.

Let U be a neighbourhood of $q \in \Sigma(1)$ such that $U \cap \Sigma(0) = \emptyset$.

$$F_x \cdot \Psi_x = \Psi_x \cdot F_x \ , \text{ for } x \in U.$$

Then (1) $|\lambda^F(x)| \, \Psi^1_x(r, \theta) = \Psi^1_x(\, |\lambda^F(x)| r, \ \theta + \beta^F(x) \,)$,

and (2) $\Psi^2_x(r, \theta) + \beta^F(x) = \Psi^2_x(\, |\lambda^F(x)| \, r, \theta + \beta^F(x) \,)$,

for $x \in U$, and $(r, \theta) \in \mathbb{R}^2 - \{0\}$.

Let $\{x_n\}$ be a sequence of points in $U - \Sigma(1)$ that converges to q, such that $\arg(\lambda^F(x_n))$ is ~~a~~ an irrational multiple of 2π, for all $n > 0$. Given $\gamma \in S'$ and $\varepsilon_1 > 0$, there exists an integer $m(\varepsilon_1, \gamma) > 0$ and an integer $N_1(m) > 0$ such that :-

$$| \gamma - m\beta^F(x_n) | < \varepsilon_1, \text{ for all } n > N_1(m).$$

From (2),

$$\Psi^2_{x_n}(\, |\lambda^F(x_n)|^m \, r, m\beta^F(x_n) \,) = \Psi^2_{x_n}(r, 0) + m\beta^F(x_n),$$

for all $n > 0$, and $r \in \mathbb{R}^+$.

Since $\lim_{n\to\infty} |\lambda^F(x_n)|^m = 1$ and $\bar{\Psi}^2$ is continuous with respect to $x \in U$, then for each $r \in \mathbb{R}^+$, given $\varepsilon_2 > 0$, there exists $N_2(\varepsilon_2, m, r) > N_1(m)$ such that, for all $n > N_2(\varepsilon_2, m, r)$:-

$$\left| \bar{\Psi}^2_{x_n}(|\lambda^F(x_n)|^m r, m\beta^F(x_n)) - \bar{\Psi}^2_q(r,\gamma) \right| < \varepsilon_2$$

Thus $\bar{\Psi}^2_q(r,\gamma) = \bar{\Psi}^2_q(r,0) + \gamma$, for each $r \in \mathbb{R}^+$, and all $\gamma \in S'$,

i.e. $\bar{\Psi}^2_q(r,-)$ is a rotation of S'.

We now show, $\bar{\Psi}^1_q(r,\theta) = \bar{\Psi}^1_q(r,\theta+\gamma)$, for all $\gamma \in S'$ and $(r,\theta) \in \mathbb{R}^2 - \{0\}$

From (1), taking $m(\varepsilon_1, \gamma)$ as above :-

$$|\lambda^F(x_n)|^m \bar{\Psi}^1_{x_n}(r,\theta) = \bar{\Psi}^1_{x_n}(|\lambda^F(x_n)|^m r, \theta + m\beta^F(x_n))$$

Given $\varepsilon_3 > 0$, there exists $N_3(\varepsilon_3, m, r) > N_1(m)$ such that : -

$$\left| |\lambda^F(x_n)|^m \bar{\Psi}^1_{x_n}(r,\theta) - \bar{\Psi}^1_q(r,\theta) \right| < \varepsilon_3 \text{, for } n > N_3.$$

Hence $\bar{\Psi}^1_q(r,\theta+\gamma) = \bar{\Psi}^1_q(r,\theta)$, for all $\gamma \in S'$.

Lastly to show $\bar{\Psi}^1_q(r,\theta)$ is a linear function of r, and constant in θ, it is possible to prove (as in lemma 2.1.), that $\bar{\Psi}^1_x$, for $x \in U - \Sigma(1)$, is of the form : -

$$\bar{\Psi}^1_x(r,\theta) = r \exp(\mu_x(\log r, \theta)),$$

where $\mu_x : \mathbb{R}^2 \to \mathbb{R}$ is a periodic function of period $(\log|\lambda^F(x)|, \beta^F(x))$, that is :-

$$\mu_x(r + \log|\lambda^F(x)|, \theta + \beta^F(x)) = \mu_x(r, \theta).$$

Let $\{x_n\}$ be a sequence of points in $U - \Sigma(1)$ converging to q, then $\lim_{n\to\infty} \log|\lambda^F(x_n)| = 0$,

and since μ_{x_n} is clearly bounded, for $n > 0$, $\lim_{n\to\infty} \mu_{x_n}$ is a periodic function of period $(0, \beta^F(q))$.

But $\bar{\Psi}^1_q(r,\theta) = \bar{\Psi}^1_q(r,\theta+\gamma)$, for all $\gamma \in S'$.

Hence $\bar{\Psi}^1_q(r,\theta)$ is a linear function in r and constant in $\theta \in S'$.

<u>Lemma 3.3.</u> Let $F, G \in End(E)$ be l.o.t.e.(ϕ) at $q \in \Sigma(1)$. Let $\Phi : (E/V, 0_x) \to (\phi^*(E/\phi(V)), 0_x)$ be a local conjugacy at q, between F and G , where $V - \Sigma(1) \neq \emptyset$ and $V \cap \Sigma(0) = \emptyset$. Let $\{(U_i, \tau_i)\}$ be a trivialisation of E . Then for some neighbourhood $U \subset V$, and $U \subset U_i$, of q , Φ is represented by Φ_i on U_i and $\Phi_{ix}|\mathbb{C} - \{0\}$, for $x \in U - \Sigma(1)$, is of the form :-

$$\Phi_{ix}(r,\theta) = ((\Psi^1_{ix}(r,\theta))^{c(x)}, \Psi^2_{ix}(r,\theta) + p(d_q(x)\log(\Psi^1_{ix}(r,\theta)))),$$

where $\Psi_i = (\Psi^1_i, \Psi^2_i)$ is a conjugacy on U_i , between F_i and itself (as in lemma 3.2.) and $p : \mathbb{R} \to S^1$ is the standard covering map .

<u>Proof</u> . The proof follows that of lemma 2.2. , and so is ommitted .

We now prove theorem 3.4. .

<u>Proof of Theorem 3.4. (Local Equivalence on $\Sigma(1)$)</u>.

The problem is local , so we take E to be trivial .

<u>'if'</u>. Assume $c : U - \Sigma(1) \to \mathbb{R}^+$ and $d_q : U - \Sigma(1) \to \mathbb{R}$ extend over U , to $\tilde{c}$ and $\tilde{d}_q$, respectively . Then the following formula defines a local conjugacy between F and G at q :-

$$\Phi_x(r,\theta) = (r^{\tilde{c}(x)}, \theta + p(\tilde{d}_q(x)\log r)) , \quad r \neq 0 ,$$

$$\Phi_x(0) = 0$$

<u>'only if'</u>. Let Φ be a conjugacy between F and G on some neighbourhood U of q , where $U - \Sigma(1) \neq \emptyset$ and $U \cap \Sigma(0) = \emptyset$.

Then by lemma 3.3. , $\Phi_x|\mathbb{C} - \{0\}$, for $x \in U - \Sigma(1)$ is of the form :-

$$\Phi_x(r,\theta) = ((\Psi^1_x(r,\theta))^{c(x)}, \Psi^2_x(r,\theta) + p(d_q(x)\log\Psi^1_x(r,\theta))) .$$

By lemma 3.2. $\Psi^1_x(r,\theta)$ is linear in r and constant in θ for all $x \in U \cap \Sigma(1)$ and $(r,\theta) \in \mathbb{C} - \{0\}$, and

$\Psi^1_x(r,-) : S^1 \to S^1$ is a rotation of S^1, for each $r \in \mathbb{R}^+$, and $x \in U \cap \Sigma(1)$.

The proof now follows that of theorem 2.3. ;that is, it can be shown that $c : U - \Sigma(1) \to \mathbb{R}^+$ and $d_j : U - \Sigma(1) \to \mathbb{R}$ extend over U if and only if $\Phi | U - \Sigma(1)$ extends over U (which of course it does).

It is now possible to construct as in section 2, a counterexample in which endomorphisms F and G of a complex line bundle are p.o.t.e.(ϕ) in some neighbourhood U of $q \in \Sigma(1)$, but are not l.o.t.e.(ϕ) at q. Moreover F and G are not l.o.t.e.(ϕ) at q for any choice of homeomorphism $\phi : X \to X$. A minor modification of that given in section 2, will give an appropiate counterexample for the complex case.

<u>Proof of Main Theorem 3.1.</u>

<u>'if'</u> Assume F and G are l.o.t.e.(ϕ) at x, for all $x \in X - \Sigma(0)$ and

$$(j\quad)_*(\pi_1(X - \Sigma(0), \Sigma(1)) = 0 \quad , \quad (\Sigma(1) \neq \emptyset).$$

Then $j\quad : (X - \Sigma(0), \Sigma(1)) \to (S^1, 0)$ lifts. ~~for each $q \in \Sigma(1)$~~.

Hence $d : X - \Sigma(1) \to \mathbb{R}$ where

$$d(x) = \frac{\hat{j}\quad(x)}{\log|\lambda^F(x)|}$$

is well defined and continuous.

The function $c \quad X - (\Sigma(0) \cup \Sigma(1)) \to \mathbb{R}^+$ is also well defined and continuous.

As in the proof of theorem 2.1.; it can be shown, using theorem 3.4., that c and d extend over $X - \Sigma(0)$ and X to function $\tilde{c}$ and $\tilde{d}$ (respectively). (Note that if $(j\quad)_*(\pi_1(X - \Sigma(0), \Sigma(1)) = 0$, then for each $s \in \Sigma(0)$, there exists a neighbourhood U of s such that $j_*(\pi_1(U - \Sigma(0))) = 0$, thus F and G are l.t.e.(ϕ) at s).

Let $\{(U_i, \tau_i)\}$ be a trivialisation of E with transition maps $\{g_{ij}\}$ that take values in the one dimensional complex unitary group ($\cong S'$). This is possible since X is paracompact.

Let Φ be the conjugacy between F and G on X such that Φ is represented on Ui, by Φ_i where Φ_{ix} is given by the formula : -

for $x \in U_i - (\Sigma(0) \cup \Sigma(1))$,

$$\Phi_{ix}(r,\theta) = (r^{\tilde{c}(x)} \exp(\tilde{c}(x)\mu_x(\log r)), \theta + p(\tilde{d}(x)\log r$$

for $x \in U_i \cap \Sigma(1)$,

$$\Phi_{ix}(r,\theta) = (r^{\tilde{c}(x)}, \theta + p(\hat{d}(x)\log r))$$

for $x \in U_i \cap \Sigma(0)$,

Φ_{ix} is the identity,

where μ_x is the periodic function of period $\log|\lambda^F(x)|$ as described in theorem 2.4.

It may be checked that Φ_i is the required conjugacy between F_i and G_i on U_i . The proof that Φ is well defined on X , follows that of theorem 2.2. .

<u>'only if</u> ' . Assume F and G are g.o.t.e.(ϕ) . Let $\Phi : (E, O_x) \to (\phi^*(E), O_x)$ be a conjugacy between F and G . The proof is by contradiction.

Assume $(j_{\_})_*(\pi_1(X - \Sigma(0), \Sigma(1))) \neq 0$, $(\Sigma(1) \neq \emptyset)$ then <u>(a)</u> there exists a path P_{01} in $X - \Sigma(0)$, not in $\Sigma(1)$, joining q_0 and q_1 in $\Sigma(1)$ ($q_0 \neq q_1$), such that $j\,) : (P_{01}, q_0) \to (S', 0)$ has winding number $k \neq 0$; and/or <u>(b)</u> there exists a loop L_0 at q_0 in $X - \Sigma(0)$, not in $\Sigma(1)$ such that $j \circ L_0 : (S', 0) \to (S', 0)$ has non-zero degree .

<u>Consider case (a)</u> . $E|P_{01}$ is trivial and since we shall only be considering Φ over P_{01} , we take E to be trivial .

As in theorem 3.3., we assume

$$\Phi_x(1,0) = (\Phi^1_x(1,0),\ \Phi^2_x(1,0)) = (1,0)$$

for $x \in P_{01}$.

Since $\Phi_x \cdot F_x = (\phi^*(G))_x \cdot \Phi_x$,

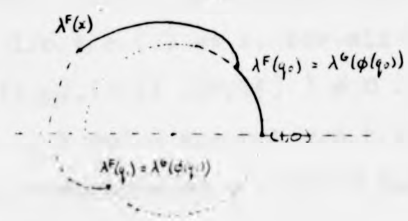
$$\Phi_x(|\lambda^F(x)|,\ \beta^F(x)) = (|\lambda^G(\phi(x))|,\ \beta^G(\phi(x)))$$

for $x \in P_{01}$.

Assume $\arg(\lambda^F(q_0)) < \arg(\lambda^F(q_1))$.

Define $i_F : P_{01} \times I \to P_{01} \times \mathbb{R}^2 - \{0\}$ such that $i_F(q_0, -) : I \to \{q_0\} \times \mathbb{R}^2 - \{0\}$ is a counterclockwise path in S^1 joining $(1,0)$ and $\lambda^F(q_0)$; and $i_F(x, -) : I \to \{x\} \times \mathbb{R}^2 - \{0\}$ is a path joining $(1,0)$ and $\lambda^F(x)$ in $\{x\} \times \mathbb{R}^2 - \{0\}$, which for convenience we take to be the union of copies of the paths $i_F(q_0, -)$ and $\lambda^F(P_{0x})$, where $P_{0x} \subset P_{01}$, is the path joining q_0 and x.

Define $i_G = \Phi \cdot i_F : P_{01} \times I \to P_{01} \times \mathbb{R}^2 - \{0\}$. Then $i_G(x,-)$ is a path in $\{x\} \times \mathbb{R}^2 - \{0\}$ joining $(1,0)$ and $\lambda^G(\phi(x))$.



We show that $i_F(q_1, -)$ and $i_G(q_1, -)$ have the same winding number. Note that, since Φ is orientation preserving $\Phi | P_{01} \times \mathbb{R}^2 - \{0\}$ is homotopic to the identity. Thus if $i_F(x, -)$ is a counterclockwise path in $\{x\} \times \mathbb{R}^2 - \{0\}$, then so is $i_G(x, -)$.

Theorem 3.4. implies there exist neighbourhoods U_0 and U_1 of q_0 and q_1 (respectively) such that $c : U_i - \Sigma(1) \to \mathbb{R}^+$ and $d_{q_i} : U_i - \Sigma(1) \to \mathbb{R}$ extend over U_i to functions $\tilde{c} : U_i \to \mathbb{R}^+$ and $\tilde{d}_{q_i} : U_i \to \mathbb{R}$, $(i = 0,1)$. Then from lemma 3.3. :-

$$\Phi_{q_i}(r,\theta) = (r^{\tilde{c}(q_i)}, \theta + p(\tilde{d}_{q_i}(q_i)\log r)) , \quad (i = 0,1) .$$

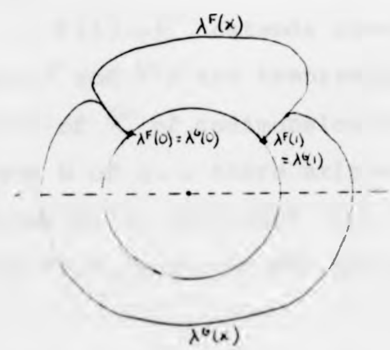
Clearly Φ_{q_i} is the identity on S', and a rotation in θ for $r \neq 1$. Hence it is easily seen that $i_F(q_1,-)$ and $i_G(q_1,-)$ have the same winding number k_1.

But $j = (\beta^G\phi - \beta^F) : (P_{01}, q_0) \longrightarrow (S', 0)$ is of winding number $k \neq 0$, which implies $i_G(q_1,-)$ is a path of winding number $k + k_1 \neq k_1$ - contradiction.

Case (b). The proof here follows that of theorem 3.3. and so is ommitted.

Counterexample 3.2.

Let $X = I$ and $E = I \times \mathbb{C}$. Let F, $G \in End(E)$ be such that $\lambda^F : I \to \mathbb{C}$ and $\lambda^G : I \to \mathbb{C}$ are as shown in the figure below. It is easily seen from the figure that F and G are l.o.t.e.(id) at x, for all $x \in I$. It is also clear that $(j)_*(\pi_1(I, \{0,1\})) \neq 0$. Hence from theorem 3.1., F and G are not g.o.t.e.(id). Moreover there is no homeomorphism $\phi : X \to X$ such that F and G are g.o.t.e.(ϕ).



4. PROOF OF CLASSIFICATION THEOREMS .

Most of the work needed to prove the classification theorems has been done . In this section we consider conditions under which the functions c and d may be extended. The most natural, arise when X is a finite dimensional differentiable manifold without boundary. The line bundles E , in this section are all $\mathcal{C}^\infty$- bundles over a $\mathcal{C}^\infty$- manifold X .The endomorphisms of E are $\mathcal{C}^\infty$- maps.

Lemma 4.1. (Complex case) . Let F, G be generic complex $\mathcal{C}^\infty$- endomorphisms of the bundle $E_{\mathbb{C}}$ that are p.o.t.e.(ϕ) in some neighbourhood V of $q_0 \in \Sigma(1)$ where $\phi : X \to X$ is a $\mathcal{C}^\infty$- diffeomorphism . Then for some neighbourhood $U \subset V$ of q_0 , $c : U - \Sigma(1) \to \mathbb{R}^+$ and $d_{\gamma} : U - \Sigma(1) \to \mathbb{R}$ extend over U to functions $\hat{c} : U \to \mathbb{R}^+$ and $\tilde{d}_{\gamma} : U \longrightarrow \mathbb{R}$.

Proof. The problem is local , so we may take E to be trivial and $X = \mathbb{R}^n$, $n \geqslant 1$.

If $\lambda^G : \mathbb{R}^n \to \mathbb{R}^2$ is transversal to the stratification $\mathcal{S}$, then so is $\lambda^G \circ \phi : \mathbb{R}^n \to \mathbb{R}^2$ ($\mathcal{S}$ is defined in section 1.3.) We shall in fact only require that λ^F and $\lambda^G \cdot \phi$ are transversal to $S^1 \subset \mathbb{R}^2$.

We prove firstly that for some neighbourhood $U \subset V$ of q_0, $c : U - \Sigma(1) \to \mathbb{R}^+$ extends over U .

Since λ^F and $\lambda^G \cdot \phi$ are transversal to S^1 , $\Sigma(1)$ is a submanifold of $\mathbb{R}^n$ of codimension one [2] .For some neighbourhood U of q_0 , there exists a local coordinate system centred at q_0 such that $x = (x_1, \ldots , x_n) \in U \cap \Sigma(1)$ if and only if $x_n = 0$. We may use polar coordinates to write :-

$$\lambda^F(x_1,\ldots,x_n) = (|\lambda|^F(x_1,\ldots,x_n),\ \beta^F(x_1,\ldots,x_n))$$

$$\lambda^G \cdot \phi(x_1,\ldots,x_n) = (|\lambda|^G \cdot \phi(x_1,\ldots,x_n),\ \beta^G \cdot \phi(x_1,\ldots,x_n)) .$$

Write $b(x) = \log|\lambda|^G \cdot \phi(x)$ and $a(x) = \log|\lambda|^F(x)$, then

$c(x) = \frac{b(x)}{a(x)}$, for all $x \in U - \Sigma(1)$. $b(x) = a(x) = 0$ for all $x \in U \cap \Sigma(1)$.

To prove $c : U - \Sigma(1) \to \mathbb{R}^{+}$ extends over U , it is sufficient to show that , for each $q = (x_1, \ldots x_{n-1}, 0) \in U \cap \Sigma(1)$
$\lim_{x_n \to 0} \frac{b(x_1, \ldots, x_n)}{a(x_1, \ldots, x_n)}$ exists, is non-zero , and varies continuously with q . Then define $\tilde{c}(q) = \lim_{x_n \to 0} \frac{b(x_1, \ldots, x_n)}{a(x_1, \ldots, x_n)}$, for each $q \in U \cap \Sigma(1)$.

Since λ^F and $\lambda^G \cdot \phi$ are transversal to S' ,

$$\left(\frac{\partial b(x)}{\partial x_n}\right)_q \neq 0 \qquad \text{and} \qquad \left(\frac{\partial a(x)}{\partial x_n}\right)_q \neq 0 \text{ , for } q \in U \cap \Sigma(1).$$

Thus $\lim_{x_n \to 0} \frac{b(x_1, \ldots, x_n)}{a(x_1, \ldots, x_n)} = \lim_{x_n \to 0} \left(\frac{\frac{\partial b}{\partial x_n}(x_1, \ldots, x_n)}{\frac{\partial a}{\partial x_n}(x_1, \ldots, x_n)} \right)$ exists and is non-zero . Furthermore the continuity of $\frac{\partial b}{\partial x_n}$ and $\frac{\partial a}{\partial x_n}$ on U , ensure the limit exists and depends continuously on $q \in U$.

To prove $d_q \cdot U - \Sigma(1) \to \mathbb{R}$ extends for some neighbourhood U of q , over U. it is sufficient to note that if F and G are p.o.t e.(ϕ) on $V \supset U$, then $\beta^F(q) = \beta^G(\phi(q))$ for all $q \in U \cap \Sigma(1)$. Then the proof follows that of the first half of the lemma.

The 'real' version of lemma 4.1. is stated without proof.

<u>Lemma 4.1. (Real Case)</u> Let F.G be real generic $\mathcal{C}$-endomorphisms of $E_{\mathbb{R}}$ that are p.t.e.(ϕ) in some neighbourhood V of $q \in \Sigma(1)$ where $\phi : X \to X$ is a $\mathcal{C}'$ - diffeomorphism, then for some neighbourhood $U \subset V$ of q, $c : U - \Sigma(1) \to \mathbb{R}^{+}$ extends over U to a continuous function $\tilde{c} : U \to \mathbb{R}^{+}$.

Proof of Theorem 1.1. (Real Classification Theorem)
Lemma 4.1. (real case) and Theorem 2.3. imply that F and G are l.t.e.(ϕ) at x for all $x \in X$. Hence the result follows immediately from Theorem 2.1..

Lemma 4.2. Let F,G be complex generic $\mathcal{C}^\infty$- endomorphisms of the bundle $E_{\mathbb{C}}$ that are p.t.e.(ϕ) in some neighbourhood V of $q_0 \in \Sigma(1)$ where $\phi : X \to X$ is a $\mathcal{C}^\infty$ - diffeomorphism. Then in some neighbourhood $U \subset V$ of q_0, (i) F and G are p.o.t.e.(ϕ) or (ii) F and $\bar{G}$ are p.o.t.e.(ϕ) on U.

Proof The problem is local, so we assume E is trivial and $X = \mathbb{R}^n$,$n \geq 1$.

For q_0 not in the counterimage of (1,0) or (1,π) the result is immediate.

Let $q_0 \in (\lambda^F)^{-1}(1,0)$.

We need to show that there exists a neighbourhood $U \subset V$ of q_0 such that (i) for each $q \in U \cap \Sigma(1)$, $\lambda^F(q) = \lambda^G(\phi(q))$ (and hence F and G are p.o.t.e.(ϕ) on U) or (ii) for each $q \in U \cap \Sigma(1)$, $\lambda^F(q) = \overline{\lambda^G(\phi(q))}$ (and hence F and $\bar{G}$ are p.o.t.e.(ϕ) on U).

If dimX = 1 , $\lambda^F : X \to \mathbb{R}^2$ is not transversal to $\{(1,0)\}$, and therefore F is not generic . Hence we assume $X = \mathbb{R}^n$ and $n > 1$.

Since λ^F and $\lambda^G \cdot \phi$ are transversal to the stratification $\mathcal{S}$, λ^F and $\lambda^G \cdot \phi$ are locally surjective at q_0 .

Furthermore $\Sigma_F^+(1) = \{ x \in \Sigma(1) | \ 0 < \arg(\lambda^F(x)) < \pi \}$

and $\Sigma_F^-(1) = \{ x \in \Sigma(1) | \ \pi < \arg(\lambda^F(x)) < 2\pi \}$ are submanifolds of $\mathbb{R}^n$ of codimension one and

$\Sigma_F^0(1) = \{ x \in \Sigma(1) | \ \lambda^F(x) = (1.0) \}$ is a submanifold of codimension two. Clearly $\Sigma_F^+(1) \cap \Sigma_F^-(1) = \emptyset$. Similarly $\Sigma_G^+(1) \cap \Sigma_G^-(1) = \emptyset$.

Note $\Sigma_F^0(1) = \Sigma_G^0(1) = \Sigma^0(1)$. Hence for some neighbourhood $U \subset V$ of q_0,

$$U \cap \Sigma(1) = U \cap (\Sigma_F^+(1) \cup \Sigma_F^-(1) \cup \Sigma^0(1))$$
$$= U \cap (\Sigma_G^+(1) \cup \Sigma_G^-(1) \cup \Sigma^0(1))$$

Trivially (i) $U \cap \Sigma_F^+(1) = U \cap \Sigma_G^+(1)$ and $U \cap \Sigma_F^-(1) = U \cap \Sigma_G^-(1)$

or (ii) $U \cap \Sigma_F^+(1) = U \cap \Sigma_G^-(1)$ and $U \cap \Sigma_F^-(1) = U \cap \Sigma_G^+(1)$

Proof of Theorem 1.2. (Complex Classification Theorem.)

Let F and G be p.t.e.(ϕ) where $\phi : X \to X$ is a C^∞ - diffeomorphism. From lemma 4.2. if $x \in \Sigma(1)$, F and G are p.o.t.e.(ϕ) in a neighbourhood of x, or F and $\bar{G}$ are p.o.t.e.(ϕ) in a neighbourhood of x. Applying lemma 4.1. (complex case) and theorem 3.4. we have, F and G or F and $\bar{G}$ are l.o.t.e.(ϕ) at x. For $x \in X - (\Sigma(0) \cup \Sigma(1))$, the result follows from theorem 3.2..

Similarly if F and G are p.o.t.e.(ϕ), then F and G are l.o.t.e.(ϕ) at x, for all $x \in X - \Sigma(0)$. The result then follows from theorem 3.1..

Chapter 3.

THE LOCAL CLASSIFICATION OF CERTAIN ENDOMORPHISMS OF REAL PLANE BUNDLES.

Introduction. In this chapter we give a partial solution to the local classification problem for generic endomorphisms of a plane bundle $E = X \times \mathbb{R}^2$, where X is a manifold. The solution is partial in that local equivalences are given only at points $x \in X$, such that the endomorphism on the fibre over x are not of the type $\pm$'Id', $\pm$'Shears', $\pm$'0×0' and 'Null' (these are all defined in section 1).

We shall identify the space of real 2×2 matrices, M(2,2) with End($\mathbb{R}^2$). Also M(2,2) is identified with $\mathbb{R}^4$.

In section 1, we list the topological conjugacy classes of M(2,2): indicate their structure as submanifolds of $\mathbb{R}^4$, their interrelationships (bifurcation diagrams) and the stratifications of $\mathbb{R}^4$ that they define (a more detailed treatment of neighbourhoods of $\pm$'Id', $\pm$'Shears', $\pm$'0×0', and 'Null', is given in chapter 4). The main theorem of the chapter is then stated at the end of section one without proof. The proof is given in section 3. Section 2 deals with the local equivalence problem for endomorphisms of $X \times \mathbb{R}^2$ where X is an arbitrary paracompact space.

1. CLASSIFICATION OF ENDOMORPHISMS OF $\mathbb{R}^2$.

1.1. The Topological Conjugacy Classes. The work of Kuiper and Robbins [4] leads to a complete classification of the endomorphisms of $\mathbb{R}^2$. The results in this section extend those of de Oliveira in [6] who considered the stratification of $GL^r(\mathbb{R}^2) \subset M(2,2)$ defined by the relation of topological conjugacy, (a number of the results were found independently by the author).

To simplify later discussions, we give a table of the topological conjugacy classes of M(2,2). Some of the classes split naturally into two components. Two endomorphisms belong to the same component if and only if they are orientably topologically equivalent. Roughly speaking, there are three groups; (i) those determined by ranges of eigenvalues, with determinant non - zero. (ii) those determined by linear type. (iii) those with at least one eigenvalue zero and type determined by range of the other eigenvalue.

Table 1 is given overleaf.

$GL^+(\mathbb{R}^2)$ has fourteen classes and two 1 - parameter families of classes determined by orientable topological conjugacy. It has twelve classes and one 1 - parameter family of classes determined by topological equivalence.

End($\mathbb{R}^2$) has thirty- two classes and two 1 - parameter families of classes determined by orientable equivalence. It has twenty-nine classes and one 1 - parameter family of classes determined by topological equivalence.

TABLE 1. (Table of conjugacy classes of M(2,2)

Range of Eigenvalues of M	Name of class if det $M>0$	Name of class if det $M<0$
$\lvert\lambda_1\rvert>1$, $\lvert\lambda_2\rvert>1$	expansions	- expansions
$0<\lvert\lambda_1\rvert<1$, $0<\lvert\lambda_2\rvert<1$	contractions	- contractions
$\lvert\lambda_1\rvert>1$, $0<\lambda_2<1$	saddles	- saddles
$\lvert\lambda_1\rvert>1$, $-1<\lambda_2<0$	twisted saddles	- twisted saddles
$\lambda_1=1$, $\lvert\lambda_2\rvert>1$	1 × expansion	1 × - expansion
$\lambda_1=1$, $0<\lvert\lambda_2\rvert<1$	1 × contraction	1 × - contraction
$\lambda_1=-1$, $\lvert\lambda_2\rvert>1$	- 1 × - expansion	- 1 × expansion
$\lambda_1=-1$, $0<\lvert\lambda_2\rvert<1$	- 1 × - contraction	- 1 × contraction
$\lambda_1=1$, $\lambda_2=-1$		1 × - 1

Linear type of M, det $M>0$	Conjugacy Class. Name of 1st component	Name of 2nd component
$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ $0<\lvert\theta\rvert<\pi$	**Rotations** , the one-parameter family of classes determined by θ , $0<\theta<\pi$.	-**Rotations**. the one-parameter family of classes determined by θ , $\pi<\theta<2\pi$.
$\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ $\lambda_1=\lambda_2$ $\lvert\lambda_1\rvert=1$	$Id=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; $-Id=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$	
	Shears	-**Shears**
$\begin{bmatrix}\lambda_1 & \pm 1\\ 0 & \lambda_2\end{bmatrix}$ $\lambda_1=\lambda_2$ $\lvert\lambda_1\rvert=1$	$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$; $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$	$\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$; $\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix}$
$\begin{bmatrix}0 & \pm 1\\ 0 & 0\end{bmatrix}$	$0\times 0=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$	$-0\times 0=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}$
$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$	Null.	

$\lambda_1=0$, Range of λ_2			Names of Classes		
$\lambda_2>1$	$\lambda_2=1$	$0<\lambda_2<1$	0×expansion	0 × 1	0×contraction
$-1<\lambda_2<0$	$\lambda_2=-1$	$\lambda_2<-1$	0× -contraction	0× -1	0×-expansion

1.2. Differential Structure and Bifurcation of the Conjugacy Classes.

Much essential information concerning the topological and differential structure of the conjugacy classes and their interrelationships can be found by considering the smooth map:-

$$f : M(2,2) \longrightarrow \mathbb{R}^2$$
$$f(M) = (\det M, \operatorname{tr} M)$$

We indicate on the following graph the subsets representing the various classes of M(2,2) i.e. f^{-1}(subset indicated) is a certain conjugacy class.

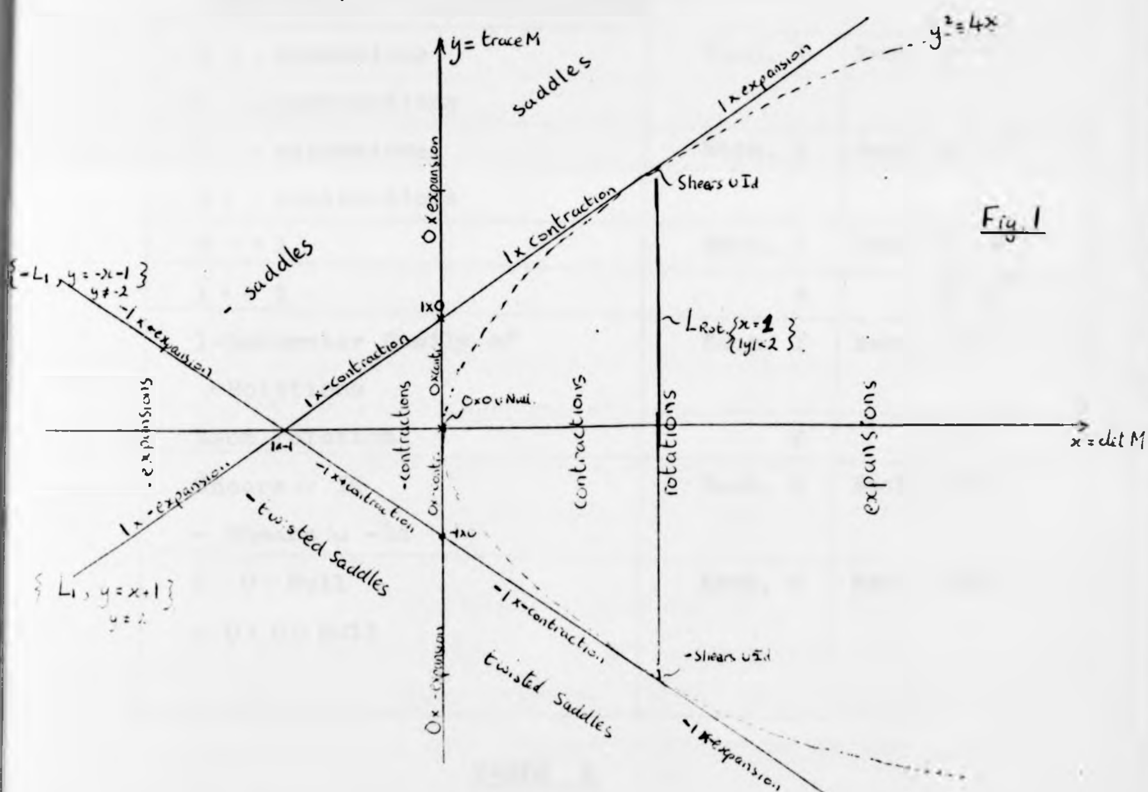


Fig. 1

Inside the parabola $y^2 = 4x$, the points represent matrices with complex eigenvalues. On it, they represent matrices with equal eigenvalues: and outside it, matrices with distinct eigenvalues.

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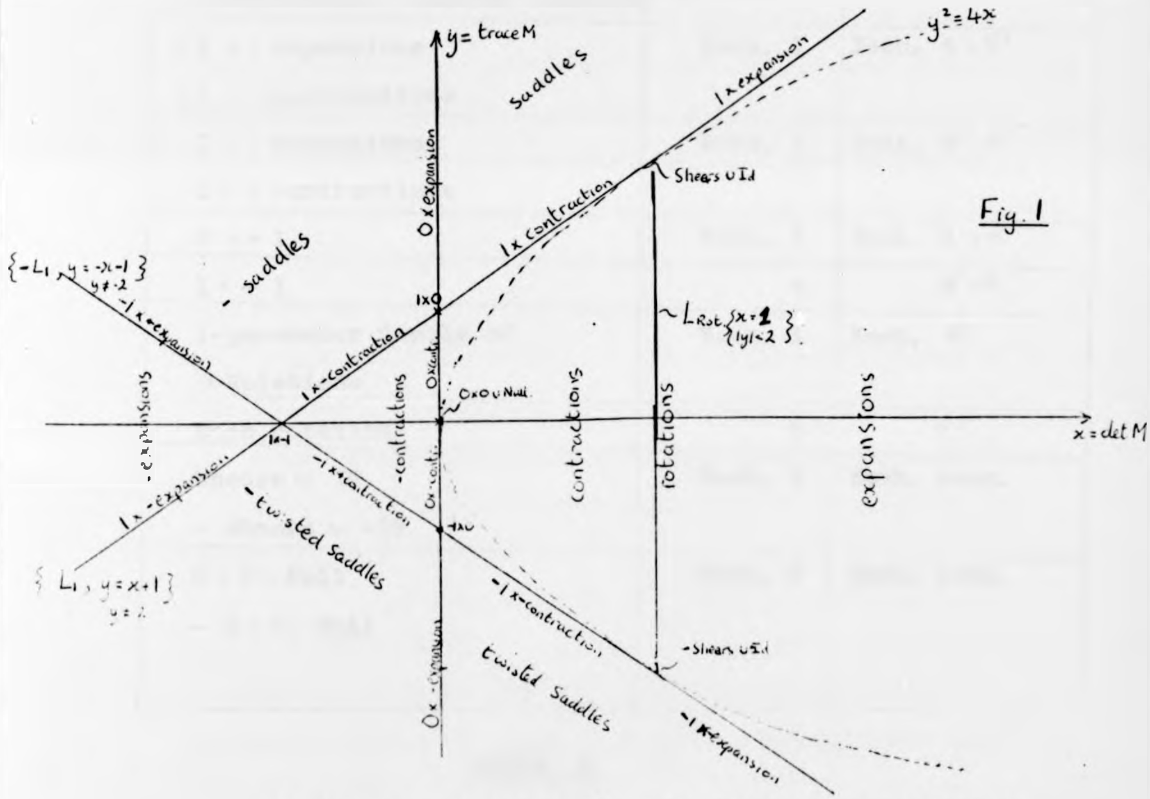

Fig. 1

Inside the parabola $y^2 = 4x$, the points represent matrices with complex eigenvalues. On it, they represent matrices with equal eigenvalues: and outside it, matrices with distinct eigenvalues.

If $M \in M(2,2)$ is not a diagonal matrix with equal eigenvalues, then f is regular (or a submersion) at M. Thus $f^{-1}(L_1)$ and $f^{-1}(-L_1)$ and $f^{-1}(L_{Rot})$ are C^∞-submanifolds of $M(2,2)$ of codimension one. Furthermore for each point P on these lines $f^{-1}(P) - \{\pm Id\}$ is a submanifold of codimension two in $M(2.2)$.

We give the following table of codimension and topological type of the classes.

Conjugacy Class.	Codimension.	Topological Type
$\pm$expansions, $\pm$saddles, $\pm$contraction, $\pm$twisted saddles	Each. 0	Each, $S^1 \times \mathbb{R}^3$
$\pm 1 \times \pm$expansions $\pm 1 \times \pm$contractions	Each, 1	Each, $S^1 \times \mathbb{R}^2$
$0 \times \pm$ expansions $0 \times \pm$ contractions	Each, 1	Each, $S^1 \times \mathbb{R}^2$
$0 \times \pm 1$	Each, 2	Each, $S^1 \times \mathbb{R}$
1×-1	2	$S^1 \times \mathbb{R}$
1-parameter family of $\pm$Rotations	Each, 1	Each, $\mathbb{R}^3$
Each rotation	2	$\mathbb{R}^2$
Shears $\cup$ Id $-$ Shears $\cup$ $-$Id	Each, 2	Each, cone.
$0 \times 0 \cup$ Null $- 0 \times 0 \cup$ Null	Each, 2	Each, cone.

TABLE 2.

The bifurcation diagrams at any point $M \in M(2,2)$ are easily deduced from Fig. 1 and table 2. For example if $M \in$ 'Rotations', we have the following figure:-

$(\mathbb{R}^3)$

$(\mathbb{R}^4)$ contractions $(\mathbb{R}^4)$ | M expansions $(\mathbb{R}^4)$

Rotations

1.3. Stratification of M(2, 2) and Main Theorem.

The twenty-nine classes and one 1-parameter family of classes of M(2,2), determined by topological equivalence are all regular submanifolds of M(2,2) and stratify M(2,2) (to obtain a connected stratification, take the thirty - two classes and two 1 - parameter families determined by orientable equivalence). Moreover it is easily checked that the stratification satisfies the Whitney condition-(a) (See de Oliveira for a similar result for $GL^+(\mathbb{R}^2)$, [6]). Call this stratification $\mathcal{S}$. As previously, giving $\mathcal{C}^\infty(X, \mathbb{R}^4)$ the Whitney $\mathcal{C}^\infty$-topology, the subspace of $\mathcal{C}^\infty(X, \mathbb{R}^4)$ of $\mathcal{C}^\infty$-maps transversal to $\mathcal{S}$, is open dense in $\mathcal{C}^\infty(X, \mathbb{R}^4)$

If $F \in End^\infty(X \times \mathbb{R}^2)$, then $F_x : \{x\} \times \mathbb{R}^2 \longrightarrow \{x\} \times \mathbb{R}^2$ is given by :-

$$F_x(x, y) = (x, \lambda^F(x)y) \quad \text{where}$$

$\lambda^F : X \to End(\mathbb{R}^2) \cong \mathbb{R}^4$ is a $\mathcal{C}^\infty$-map (see section 2). We shall in future identify $\lambda^F(x)$ with its matrix representation with respect to the standard basis on $\mathbb{R}^2$ for all $x \in X$.

<u>Definition 1.1.</u> Let $F \in End^\infty(X \times \mathbb{R}^2)$, where X is a $\mathcal{C}^\infty$-manifold. Then F is said to be <u>generic</u> if $\lambda^F : X \to \mathbb{R}^4$ is transversal to the stratification $\mathcal{S}$.

Let $H \subset M(2,2)$ be the subspace consisting of real 2×2 matrices , none of which are linearly equivalent to (i) $\pm$ Shears , (ii) $\pm$ Id, (iii) $\pm\, 0 \times 0$, or (iv) Null. The main theorem of the chapter is :-

Theorem 1.1. (Local Classification , relative to H).
Let F ,G be generic C^∞-endomorphisms of a C^∞ -bundle $X \times \mathbb{R}^2$, where X is a C^∞ -manifold . Let F and G be p.t.e.(ϕ) where $\phi : X \to X$ is a diffeomorphism and $\lambda^F(x)$ and $\lambda^G\circ\phi(x)$ are in H for all $x \in X$. Then F and G are l.t.e.(ϕ) at x for all $x \in X$.

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2. REAL PLANE BUNDLES

2.1. Canonical Model for H . In this section, we take X to be a paracompact space and E to be the bundle $X \times \mathbb{R}^2$. It is in fact not neccessary to take E to be trivial in the local classification problem. However since there is no continuous function $\lambda^F : X \to \mathrm{End}(\mathbb{R}^2)$ associated with $F \in \mathrm{End}(E)$,when E is non-trivial, it is not clear how to define 'genericity'. There are also the technical problems associated with choice of trivialisation .

A canonical model $\mathcal{M}'$ in $\mathbb{R}^4$ is constructed for the endomorphisms $H < M(2,2)$ of $\mathbb{R}^2$ .That is, points in the model space $\mathcal{M}'$ represent matrices other than those of the type :- $\pm$ Shears , $\pm\, 0 \times 0$, in canonical form relative to the standard basis on $\mathbb{R}^2$.

$$\text{Let } \mathcal{M}' = \left\{ (w_1,w_2,w_3,w_4) \in \mathbb{R}^4 / \begin{array}{l} w_1 \neq w_2 \Rightarrow w_3 = w_4 = 0 \\ w_1 = w_2 \Rightarrow w_3 = -w_4 \end{array} \right\}$$

We take the following identification of $M(2,2)$ with $\mathbb{R}^4$:-

$$\begin{bmatrix} w_1 & w_3 \\ w_4 & w_2 \end{bmatrix} \longleftrightarrow (w_1,w_2,w_3,w_4)$$

Thus $w \in \mathcal{M}'$ is identified with the matrix

$$\begin{bmatrix} w_1 & w_3 \\ -w_3 & w_2 \end{bmatrix} \quad \text{where } w_3 = 0 \text{ if } w_1 \neq w_2 .$$

For $F \in \mathrm{End}(E)$, $F_x : \{x\} \times \mathbb{R}^2 \to \{x\} \times \mathbb{R}^2$ is of the form :-

$$F_x(x,y) = (x, \lambda^F(x)\, y)$$

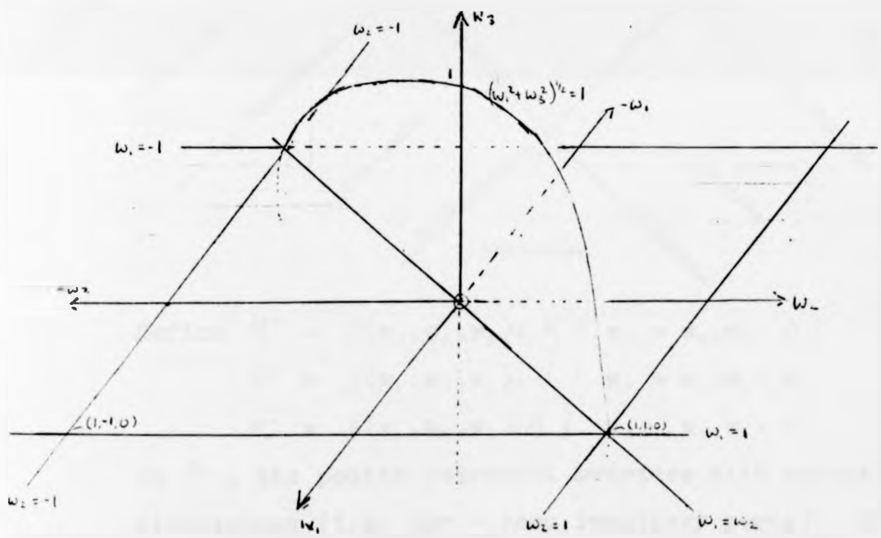
where $\lambda^F : X \to \mathrm{End}(\mathbb{R}^2) \leftrightarrow M(2,2) \leftrightarrow \mathbb{R}^4$ is continuous .

$$\begin{tikzcd} X \times \mathbb{R}^2 \arrow[r, "id \times id \times \lambda^F"] \arrow[rr, bend left, "F"] & X \times \mathbb{R}^2 \times \mathrm{End}(\mathbb{R}^2) \arrow[r, "id \times ev"] & X \times \mathbb{R}^2 \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rrrr, bend right, "(\lambda^F)_{ij}"'] & X \times \{e_j\} \arrow[r, hook] & X \times \mathbb{R}^2 \arrow[r, "F"] & X \times \mathbb{R}^2 \arrow[r, "p_i"] & \mathbb{R} \end{tikzcd} \qquad (i,j = 1,2)$$

Thus F is a C^r-map $\Leftrightarrow$ λ^F is a C^r-map $\Leftrightarrow$ $(\lambda^F)_{ij}$ is C^r-map.

Write $(\lambda^F)_{11} = \lambda^F_1$, $(\lambda^F)_{22} = \lambda^F_2$, $(\lambda^F)_{12} = \lambda^F_3$, $(\lambda^F)_{21} = \lambda^F_4 = -\lambda^F_3$



$\mathcal{M}'$

$\mathcal{M}'$ lies in the hyperplane of $\mathbb{R}^4$ given by $w_3 = -w_4$.
We shall in future omit the w_4 - co-ordinate when speaking of points in $\mathcal{M}'$.
Because of symmetry about the line $w_1 = w_2$ and the (w_1, w_2) plane, we restrict our attention to the subset $\mathcal{M}$ of $\mathcal{M}'$ given by :-

Thus $w \in \mathcal{M}'$ is identified with the matrix

$$\begin{bmatrix} w_1 & w_3 \\ -w_3 & w_2 \end{bmatrix} \quad \text{where } w_3 = 0 \text{ if } w_1 \neq w_2 .$$

For $F \in \mathrm{End}(E)$, $F_x : \{x\} \times \mathbb{R}^2 \longrightarrow \{x\} \times \mathbb{R}^2$ is of the form :-

$$F_x(x,y) = (x,\ \lambda^F(x)\,y)$$

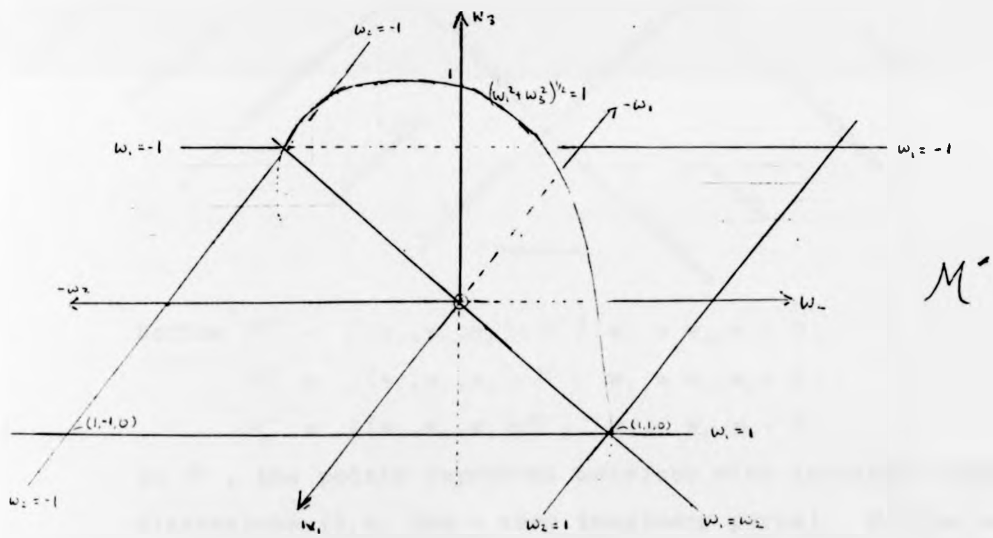
where $\lambda^F : X \to \mathrm{End}(\mathbb{R}^2) \leftrightarrow M(2,2) \leftrightarrow \mathbb{R}^4$ is continuous.

$$\begin{tikzcd} X \times \mathbb{R}^2 \arrow[r, "id \times id \times \lambda^F"] \arrow[rr, bend left, "F"] & X \times \mathbb{R}^2 \times \mathrm{End}(\mathbb{R}^2) \arrow[r, "id \times ev"] & X \times \mathbb{R}^2 \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rrrr, bend right, "(\lambda^F)_{ij}"'] & X \times \{e_j\} \arrow[r, hook] & X \times \mathbb{R}^2 \arrow[r, "F"] & X \times \mathbb{R}^2 \arrow[r, "p_i"] & \mathbb{R} \end{tikzcd} \qquad (i,j = 1,2)$$

Thus F is a $\mathcal{C}^r$-map $\Leftrightarrow$ λ^F is a $\mathcal{C}^r$-map $\Leftrightarrow$ $(\lambda^F)_{ij}$ is $\mathcal{C}^r$-map.

Write $(\lambda^F)_{11} = \lambda^F_1$, $(\lambda^F)_{22} = \lambda^F_2$, $(\lambda^F)_{12} = \lambda^F_3$, $(\lambda^F)_{21} = \lambda^F_4 = -\lambda^F_3$



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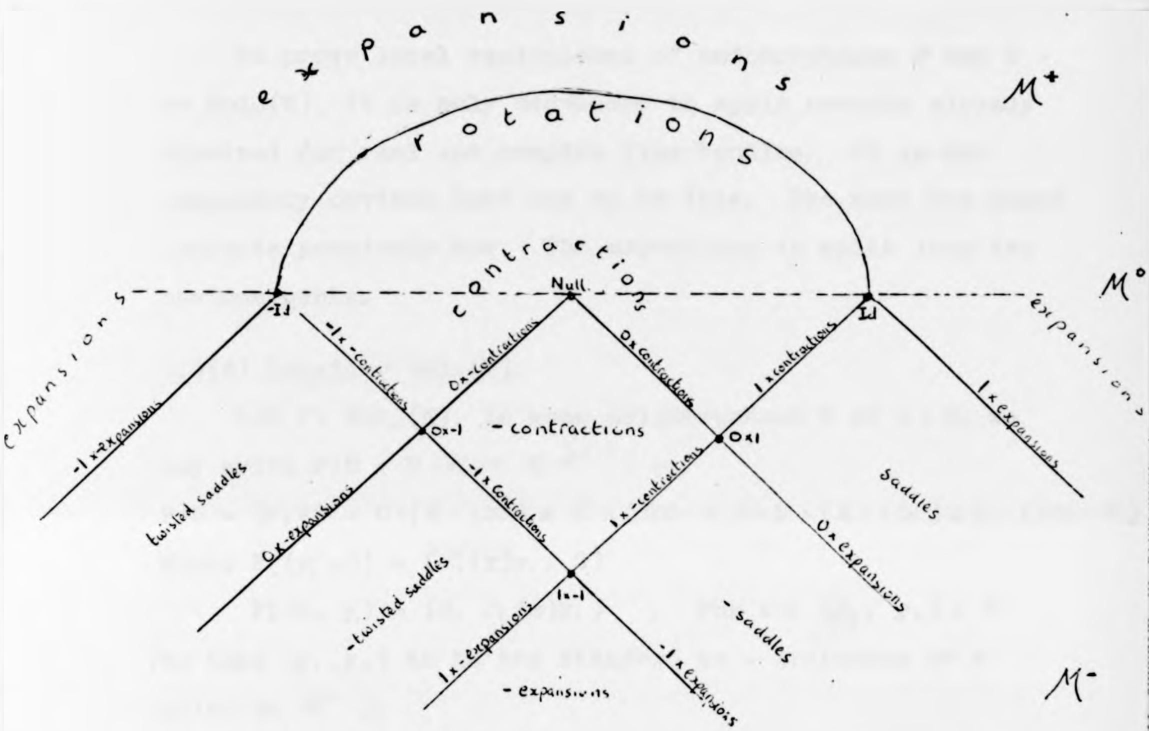
Because of symmetry about the line $w_1 = w_2$ and the (w_1, w_2) plane, we restrict our attention to the subset $\mathcal{M}$ of $\mathcal{M}'$ given by :-

$\mathcal{M} = \{(w_1, w_2, w_3) \in \mathcal{M}' \mid w_1 \geq w_2 \text{ if } w_3 = 0, \text{and } w_3 \geq 0 \text{ if } w_1 = w_2\}$
(Roughly speaking, any family of matrices in $\mathcal{M}' - \{\text{line } w_1 = w_2\}$ is locally linearly equivalent to a family in $\mathcal{M} - \{\text{line } w_1 = w_2\}$



Define $\mathcal{M}^+ = \{(w_1, w_2, w_3) \in \mathcal{M} \mid w_1 = w_2, w_3 > 0\}$

$\mathcal{M}^0 = \{(w_1, w_2, w_3) \in \mathcal{M} \mid w_1 = w_2, w_3 = 0\}$

$\mathcal{M}^- = \{(w_1, w_2, w_3) \in \mathcal{M} \mid w_1 > w_2, w_3 = 0\}$

In $\mathcal{M}^+$, the points represent matrices with strictly complex eigenvalues (i.e. non - zero imaginary parts). Points in $\mathcal{M}^0$ represent diagonal matrices with equal eigenvalues: and those in $\mathcal{M}^-$, matrices with real distinct eigenvalues.

Define $\mathrm{End}_{\mathcal{M}'}(E) \subset \mathrm{End}(E)$ to be the subspace consisting of endomorphisms F such that $\lambda^F(x) \in \mathcal{M}'$ for all $x \in X$. Similarly define $\mathrm{End}_{\mathcal{M}}(E)$, etc..

2.2. Local Classification of $\mathrm{End}_{\mathcal{M}^2}(E)$.

To prove local equivalence of endomorphisms F and G in $\mathrm{End}_{\mathcal{M}^2}(E)$, it is only necessary to apply results already obtained for real and complex line bundles. It is not completely obvious just how to do this. The next few pages indicate precisely how. The exposition is split into two obvious parts.

2.2(A) Consider $\mathrm{End}_{\mathcal{M}^-}(E)$.

Let $F \in \mathrm{End}_{\mathcal{M}^-}(E)$. In some neighbourhood U of $x \in X$, we may write $F|U : U \times \mathbb{R}^2 \to U \times \mathbb{R}^2$:-

$F|U = (F^1, F^2) : U \times (\mathbb{R} \times \{0\}) \oplus U \times (\{0\} \times \mathbb{R}) \to U \times (\mathbb{R} \times \{0\}) \oplus U \times (\{0\} \times \mathbb{R})$

Where $F^1_x(y_1, 0) = (\lambda^F_1(x)y_1, 0)$

$F^2_x(0, y_2) = (0, \lambda^F_2(x)y_2)$, for all $(y_1, y_2) \in \mathbb{R}^2$

(We take (y_1, y_2) to be the standard co - ordinates of a point in $\mathbb{R}^2$).

We identify F^1 and F^2 with obvious endomorphisms of the line bundle $U \times \mathbb{R}$.

Define, $\Sigma^F_1(1) = \{x \in X \mid |\lambda^F_1(x)| = 1\}$

$\Sigma^F_2(1) = \{x \in X \mid |\lambda^F_2(x)| = 1\}$

where $F \in \mathrm{End}_{\mathcal{M}^-}(E)$

Let $F, G \in \mathrm{End}_{\mathcal{M}^-}(E)$ be p.t.e.(ϕ)

Then $\Sigma^F_1(1) = \phi^{-1}(\Sigma^G_1(1)) = \Sigma_1(1)$, (say),

and $\Sigma^F_2(1) = \phi^{-1}(\Sigma^G_2(1)) = \Sigma_2(1)$, (say).

Similarly define $\Sigma_1(0)$ and $\Sigma_2(0)$.

Define $c_1 : X - (\Sigma_1(1) \cup \Sigma_1(0)) \to \mathbb{R}^+$:-

$$c_1(x) = \frac{\log |\lambda^G_1 \cdot \phi(x)|}{\log |\lambda^F_1(x)|}$$

And similarly define $c_2 : X - (\Sigma_2(1) \cup \Sigma_2(0)) \to \mathbb{R}^+$.

Lemma 2.1. Let $F, G \in End_{M}(E)$ be p.t.e.(ϕ) in a neighbourhood V of $x_0 \in X$. Then F and G are l.t.e.(ϕ) at x_0 if andonly if F^1 and G^1 are l.t.e.(ϕ) at x_0, as endomorphisms of $V \times \mathbb{R}$; also F^2 and G^2 are l.t.e.(ϕ) at x_0 as endomorphisms of $V \times \mathbb{R}$.

Proof.

'if'. Assume F^1 and G^1 are l.t.e.(ϕ) at x_0, and let $\Phi^1 : (U_1 \times \mathbb{R}, 0_x) \to (U \times \mathbb{R}, 0_x)$ be a conjugacy between F^1 and G^1 on some neighbourhood $U_1 \subset V$ of x_0 .

Similarly let $\Phi^2 : (U_2 \times \mathbb{R}, 0_x) \to (U_2 \times \mathbb{R}, 0_x)$ be a conjugacy between F^2 and G^2 on $U_2 \subset V$ of x_0.

In the obvious sense,

$$\Phi = (\Phi^1, \Phi^2) : (U_1 \cap U_2 \times \mathbb{R}^2, 0_x) \to (U_1 \cap U_2 \times \mathbb{R}^2, 0_x)$$

is a conjugacy between F and G at x_0.

'only if'. Assume F^2 and G^2 are not l.t.e.(ϕ) at x_0 . Then it is clear from chapter 2.2. that

$$\lambda_2^F(x_0) = \lambda_2^G \cdot \phi(x_0) = 1 \quad (\text{or } -1) .$$

Let $\Phi : (U \times \mathbb{R}^2, 0_x) \to (U \times \mathbb{R}^2, 0_x)$ be a conjugacy between F and G on some neighbourhood $U \subset V$ of x_0.

It is easy to see that Φ_{x_0} maps the y_2-axis to the y_2-axis. For U 'small enough' , we may assume Φ_x maps the y_2-axis to the y_2-axis in $\{x\} \times \mathbb{R}^2$, for all $x \in U$.

But $\Phi_x | y_2$-axis , for $x \in U - \Sigma_2(1)$, is of the form :-

$$y_2 \mapsto y_2 \exp(c_2(x)\mu_x(\log y)) , \quad \text{for } y > 0$$

where $\mu_x : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $\log|\lambda_2^F(x)|$.

Hence $\Phi | U - \Sigma_2(1)$ extends over U if and only if $c_2 : U - \Sigma_2(1) \to \mathbb{R}^+$ extends over U .

But if c_2 extends over U , F^2 and G^2 are l.t.e.(ϕ) at x_0, (chapter 2, theorem2.3.) - cotradiction .

The following theorem is immediate from lemma 2.1. and theorem 2.3. in chapter 2..

Theorem 2.1. (Local Classification on $\Sigma_1(1) \cup \Sigma_2(1)$).
Let $F, G \in End_{M}(E)$ be p.t.e.(ϕ) in a neighbourhood V of $x_0 \in X$, where (i) $x_0 \in \Sigma_1(1)$ and/or (ii) $x_0 \in \Sigma_2(1)$. Then F and G are l.t.e.(ϕ) at x_0, if and only if for some neighbourhood $U \subset V$ of x_0 (i) $c_1 : U - \Sigma_1(1) \to \mathbb{R}^+$ and/or (ii) $c_2 : U - \Sigma_2(1) \to \mathbb{R}^+$ extend over U .
(Note , $x_0 \in \Sigma_1(1) \cap \Sigma_2(1)$ if $\lambda^F(x_0) = (1,-1,0)$.)

A more general form of this theorem can easily be obtained. Let $F \in End(E)$ and $\lambda^F : X \to End(\mathbb{R}^3)$ be such that for some $x_0 \in X$, $\lambda^F(x_0)$ is a matrix with real distinct eigenvalues. Then for some neighbourhood U of x_0 , there exists a unique $F_M \in End_M(E/U)$ such that F/U and F_M are locally linearly equivalent at x_0 , (the eigenvectors of $\lambda^F(x)$ can be chosen so as to depend smoothly on $\lambda^F(x)$ and therfore continuously on $x \in U$) Thus if F and $G \in End(E)$ are p.t.e.(ϕ) , then for $x \in X$, such that $\lambda^F(x)$ and $\lambda^G \cdot \phi(x)$ have real distinct eigenvalues , there are neighbourhoods U of x such that $\Sigma_1(1)$, $\Sigma_2(1)$, c_1 and c_2 are well defined . Hence theorem 2.1. is extended in the obvious way .

Similarly a more general form of the following theorem is easily given . $\Sigma_1(0)$ and $\Sigma_2(0)$ are defined in the obvious way .

Theorem 2.2. (Local Classification on $\Sigma_1(0) \cup \Sigma_2(0)$).
Let $F, G \in End_M(E)$ be p.t.e.(ϕ) in a neighbourhood V of $x_0 \in \Sigma_1(0) \cup \Sigma_2(0)$. Then F and G are l.t.e.(ϕ) at x_0 .
Proof. The proof follows from lemma 2.1. and theorem 2.4. in chapter 2 .

2.2.(B) Local Classification on $\mathcal{M}^+$

The local classification of endomorphisms $F \in End_{\mathcal{M}^+}(E)$ follows easily from our results on endomorphisms of complex line bundles.

Let $F \in End_{\mathcal{M}^+}(E)$, then $F_x : \mathbb{R}^2 \to \mathbb{R}^2$ is given by :-

$F_x(y_1, y_2) = (\lambda_1^F(x)y_1 + \lambda_3^F(x)y_2 , - \lambda_3^F(x)y_1 + \lambda_1^F(x)y_2)$

(recall $\lambda_1^F(x) = \lambda_2^F(x)$).

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we have :-

$F_x : z \longrightarrow (\lambda_1^F(x) + i\, \lambda_3^F(x)) z$

Let $F, G \in End_{\mathcal{M}^+}(E)$ be p.t.e.(ϕ).

Define $\Sigma(1) = \{ x \in X \mid (\lambda_1^F(x))^2 + (\lambda_3^F(x))^2 = (\lambda_1^G \cdot \phi(x))^2 + (\lambda_3^G \cdot \phi(x))^2 = 1 \}$

and $\Sigma(0) = \{ x \in X \mid \lambda^F(x) = \lambda^G \cdot \phi(x) = 0 \}$

Also define $c : X - (\Sigma(1) \cup \Sigma(0)) \longrightarrow \mathbb{R}^+$ such that :

$$c(x) = \frac{\log((\lambda_1^G \cdot \phi(x))^2 + (\lambda_3^G \cdot \phi(x))^2)}{\log((\lambda_1^F(x))^2 + (\lambda_3^F(x))^2)}$$

Let $\beta^F(x) = \arg(\lambda_1^F(x) + i\, \lambda_3^F(x))$, $\beta^G(x) = \arg(\lambda_1^G(x) + i\, \lambda_3^G(x))$

We assume $X - \Sigma(0)$ is locally path-connected. Then d_{x_0} is defined in a neighbourhood U of $x_0 \in X - \Sigma(0)$ as in chapter 2.

The following theorem is just the real plane bundle version of Theorem 3.2. of chapter 2 (restated for the readers benefit).

Theorem 2.3. (Local Classification on $\Sigma(1)$).

Let $F, G \in End_{\mathcal{M}^+}(E)$ be p.o.t.e.(ϕ) in a neighbourhood V of $x_0 \in \Sigma(1)$. Then F and G are l.o.t.e.(ϕ) at x_0 if and only if for some neighbourhood $U \subset V$ of x_0, $c : U - \Sigma(1) \to \mathbb{R}^+$ and $d_{x_0} : U - \Sigma(1) \to \mathbb{R}$ can be extended over U.

Let $F \in End(E)$ and $x_0 \in X$, such that $\lambda^F(x_0)$ has strictly complex eigenvalues. Then for some neighbourhood U of x_0, there exists a unique endomorphism $F_{\mathcal{M}} \in End_{\mathcal{M}^+}(E/U)$ such that F/U and $F_{\mathcal{M}}$ are locally linearly equivalent at x_0.

Thus theorem 2.3. extends to the obvious more general form.

2.3. Hyperbolic Endomorphisms.

We have not yet considered local equivalence on $\mathcal{M}^\circ$. It is not difficult to give a local classification with respect to $\mathcal{M}^\circ$ (i.e. a classification of $\mathrm{End}_{\mathcal{M}^\circ}(E)$) at points other than 'Null, and $\pm$Id', by a direct method, but it is lengthy. Moreover the results do not extend easily to more general endomorphisms $F \in \mathrm{End}(E)$ at points $x \in X$ such that $\lambda^F(x)$ has real, equal eigenvalues (not $= \pm 1, 0$) (the construction of local conjugacies around matrices of the form $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\lambda \neq \pm 1, 0$ is not trivial). Therefore we give the following theorem, which provides a local classification of hyperbolic endomorphisms of not only $X \times \mathbb{R}^2$ but also $X \times \mathbb{R}^n$.

<u>Definition 2.1.</u> An endomorphism $F \in \mathrm{End}(X \times \mathbb{R}^n)$ is <u>hyperbolic</u> if it is an automophism on $X \times \mathbb{R}^n$ and for each $x \in X$, $\lambda^F(x) \in GL(\mathbb{R}^n)$ has no real eigenvalue of modulus 1 nor complex eigenvalue of modulus 1.

The following theorem is a modification of a result due to Robbin [7].

<u>Theorem 2.4. (Local Classification of Hyperbolic Endomorphisms)</u>
Let E be the bundle $X \times \mathbb{R}^n$, $n > 0$. Let $F, G \in \mathrm{End}(E)$ be hyperbolic endomorphisms that are p.t.e.(ϕ) at $x_0 \in X$. Then F and G are l.t.e.(ϕ) at x_0.

<u>Proof</u>. We shall in fact prove that F and F_0 are l.t.e.(idx) at x_0, where $F_0 \in \mathrm{End}(E)$ is the 'constant' endomorphism, such that $\lambda^{F_0}(x) = \lambda^F(x_0)$, for all $x \in X$. Similarly it can be shown that $\phi^*(G_0)$ and $\phi^*(G\ )$ are l.t.e.(idx) at x . The result follows.

To simplify the proof we assume F is 'pure contraction' (i.e. all the eigenvalues of $\lambda^F(x)$ are of modulus < 1) at each x in some neighbourhood V of x . If not, we split

$F|V$ into (F^+, F^-) and $V \times \mathbb{R}^n$ into $(V \times W_F^+) \oplus (V \times W_F^-)$ where $W_F^+(x)$ and $W_F^-(x)$ are the subspaces of $\{x\} \times \mathbb{R}^n$, invariant under F_x^+ and F_x^- (respectively). Similarly $F_o = (F_o^+, F_o^-)$. Since F and F_o are p.t.e.(idx) in V, the dimensions of $W_F^+(x)$ and $W_{F_o}^+(x)$ agree for $x \in V$ (and of course those of $W_F^-(x)$ and $W_{F_o}^-(x)$). We may assume that the subspaces $W_F^+(x)$ and $W_{F_o}^+(x)$ are equal for all $x \in V$, if not, we may replace F by an endomorphism locally linearly conjugate to F. A conjugacy Φ (if it exists) between F and F_o may be defined by (Φ^+, Φ^-) where Φ^+ conjugates F^+ and F_o^+ and Φ^- conjugates F^- and F_o^-. Hence it is sufficient to consider F, pure contraction, or pure expansion.

Given $\xi : \mathbb{R}^n \to \mathbb{R}^n$, the $\mathcal{C}^o$-norm of ξ is given by :-

$$\|\xi\|_o = \sup_{x \in \mathbb{R}^n} \|\xi(x)\|$$

If ξ is $\mathcal{C}^1$, define the $\mathcal{C}^1$-norm :-

$$\|\xi\|_1 = \|\xi\|_o + \sup_{x \in \mathbb{R}^n} \|D\xi(x)\|$$

Robbin in [7] proves that a hyperbolic endomorphism f of $\mathbb{R}^n$ is structurally stable (this is a restatement of Hartman's theorem [7]) that is any diffeomorphism g sufficiently $\mathcal{C}^1$-close to f is conjugate to f. It is clear from the proof that the conjugacy set up between f and g varies continuously with g in a neighbourhood of f in Diff $(\mathbb{R}^n)$. He in fact goes on to prove f is absolutely structurally stable. To apply this result to obtain our theorem, we would require F_x and F_{ox} to be $\mathcal{C}^1$-close for x near x_o. But $\|F_x\|_1$ is unbounded. Hence we first approximate F_x, for $x \in V$ by a $\mathcal{C}^1$- function which is $\mathcal{C}^1$- close to F_{ox} for V sufficiently small.

Define $\theta : V \times \mathbb{R}^n \to \mathbb{R}$, such that for each $x \in V$, θ_x is C^∞, non - negative, with compact support and equal to 1 on a neighbourhood W of $0 \in \mathbb{R}^n$.

Now define $f_x : \mathbb{R}^n \to \mathbb{R}^n$ such that :-

$f_x(y) = \theta_x(y)F_x(y) + (1 - \theta_x(y))F_{ox}(y)$,

for all $x \in V$, and $y \in \mathbb{R}^n$.

Then f_x is C' and $f_x|W = F_x|W$, for all $x \in V$. Furthermore f_x can be made arbitrarily C'-close to F_{ox} by suitable choice of neighbourhood $U \subset V$ of x_o. Thus by the absolute structural stability of F_{ox} , $x \in V$ (Robbins' result [7]), there is a conjugacy $\psi_x : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ between f_x and F_{ox} , that varies continuously with x, for x in some neighbourhood U of x .

Now define $\Phi_x : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ by :-

$\Phi_x(y) = f_x^{-n} \cdot \psi_x \cdot F_{ox}^n(y)$, for all $n \geq k$.

Where $F_{ox}^n(y) \in \psi_x^{-1}(W)$, for all $n \geq k$ and $x \in U$. Φ_x is well-defined, independent of n, and the required conjugacy between F_{ox} and F_x, for all $x \in U$. Moreover Φ_x varies continuously with x, $x \in U$: i.e. F and F_o are l.t.e. (id_x).

3. PROOF OF LOCAL CLASSIFICATION THEOREM RELATIVE TO H.

In this section, we take X to be a C^∞ manifold and E to be the bundle $X \times \mathbb{R}^2$. The space of C^∞-endomorphisms of E is denoted by $End^\infty(E)$.

We reduce the general local classification theorem 1.1. to a local classification theorem for endomorphisms in $End_{M^-}(E) \cup End_{M^+}(E)$, that are generic in the following sense.

Consider the following stratification of $\mathcal{M}^-$:-

(i) $\{(w_1, w_2, 0) \in \mathcal{M}^- \mid w_1 = 1, \quad w_2 \neq -1\}$

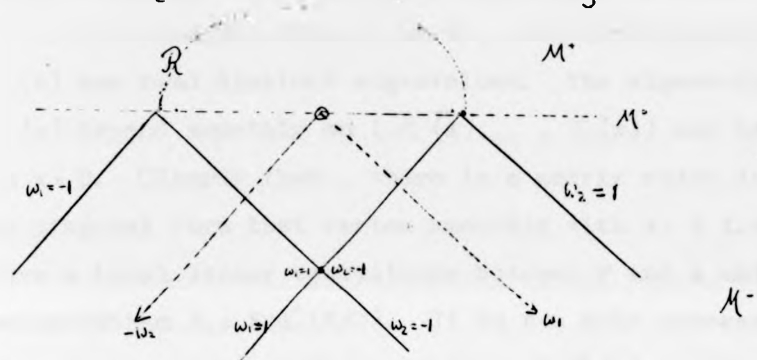
(ii) $\{(w_1, w_2, 0) \in \mathcal{M}^- \mid w_2 = -1, \; w_1 \neq 1\}$

(iii) $\{(w_1, w_2, 0) \in \mathcal{M}^- \mid w_2 = 1\}$

(iv) $\{(w_1, w_2, 0) \in \mathcal{M}^- \mid w_1 = -1\}$

(v) $\{(w_1, w_2, 0) \in \mathcal{M}^- \mid w_1 = 1, \quad w_2 = -1\}$

(vi) $\mathcal{M}^- - \{(i) \cup (ii) \cup (iii) \cup (iv) \cup (v)\}$



$\mathcal{S}(\mathcal{M}^-)$

Trivially it is a stratification satisfying the Whitney condition-(a).

<u>Definition 3.1.</u> Let $F \in \mathrm{End}^{\infty}_{\mathcal{M}^-}(E)$. Then F is <u>generic</u> if $\lambda^F : X \to \mathcal{M}^-$ is transversal to the stratification $\mathcal{S}(\mathcal{M}^-)$.

It is clear from the previous sections that given F and G in End(E) that are p.t.e.(ϕ) then for any $x \in X$, such that $\lambda^F(x)$ and $\lambda^G \cdot \phi(x)$ have real distinct or 'hyperbolic' eigenvalues, F and G are l.t e.(ϕ) at x. This holds true, irrespective of the differentiability conditions on E, X etc. The differential criteria are needed to prove local equivalence in the neighbourhood of matrices with at least one eigenvalue of modulus (real or complex) one ($\neq \pm$ Id, $\pm$ Shears).

The following lemmas reduce this general problem to the equivalent problem on $\mathcal{M}^-$ and $\mathcal{M}^+$.

Lemma 3.1.($\mathcal{M}^-$) Let F be a generic C^∞ - endomorphism on $E = X \times \mathbb{R}^2$, where X is a C^∞-manifold. Then for each $x_0 \in X$, such that $\lambda^F(x_0)$ has real distinct eigenvalues, there exists a neighbourhood U of x_0 and a unique endomorphism $F_{\mathcal{M}} \in \mathrm{End}^\infty_{\mathcal{M}^-}(E/U)$ such that $F_{\mathcal{M}}$ is generic and such that F and $F_{\mathcal{M}}$ are locally linearly equivalent at x_0.

Proof. Assume $\lambda^F(x_0)$ has real distinct eigenvalues, then there is a neighbourhood U of x_0, such that for all $x \in U$, $\lambda^F(x)$ has real distinct eigenvalues. The eigenvalues of $\lambda^F(x)$ depend smoothly on $(\lambda^F_1(x), \ldots, \lambda^F_4(x))$ and therefore on $x \in U$. Clearly then, there is a matrix reducing $\lambda^F(x)$ to diagonal form that varies smoothly with $x \in U$ i.e. we have a local linear equivalence between F and a unique endomorphism $F_{\mathcal{M}} \in \mathrm{End}^\infty_{\mathcal{M}^-}(E/U)$. It is now only necessary to show $F_{\mathcal{M}}$ is generic. The reduction of $\lambda^F(x)$ to diagonal form determines a smooth map L in a neighbourhood of $\lambda^F(x_0) \in \mathbb{R}^4$ such that the following diagram commutes :-

$$\begin{tikzcd} & \mathbb{R}^4 \arrow[d, "L"] \\ U \arrow[ur, "\lambda^F"] \arrow[r, "\lambda^{F_{\mathcal{M}}}"'] & \mathcal{M}^- \end{tikzcd}$$

It is easily checked that L is regular in a neighbourhood of $\lambda^F(x_0)$.

Thus if λ^F is transversal to the stratification $\mathcal{S}(M(2,2))$ at x_0, $\lambda^{F_{\mathcal{M}}}$ is transversal to the stratification $\mathcal{S}(\mathcal{M}-)$ i.e. $F_{\mathcal{M}}$ is generic.

Lemma 3.2. Let $F, G \in \mathrm{End}^\infty_{\mathcal{M}}(E)$ be generic and p.t.e.(ϕ) in a neighbourhood V of x_0 in (i)$\Sigma_1(1)$ and / or (ii)$\Sigma_2(1)$, where $\phi : X \to X$ is a C^∞-diffeomorphism. Then for some neighbourhood $U \subset V$ of x_0, (i) $c_1 : U - \Sigma_1(1) \to \mathbb{R}^r$ and / or (ii) $c_2 : U - \Sigma_2(1) \to \mathbb{R}^r$ extend over U.

<u>Proof.</u> The proof follows that of lemma 4.1. in chapter 2, and so is ommitted.

And now a lemma , similar to lemma 3.1., reducing the general local classification problem in the neighbourhood of a matrix with eigenvalues of complex modulus one , to the simpler corresponding problem on $\mathcal{M}^+$.

Let $\mathcal{R} = \{ (w_1, w_2, w_3) \in \mathcal{M}^+ \mid w_1 = w_2 \,,\ (w_1^2 + w_3^2) = 1 \}$,

i.e. $\mathcal{R}$ represents the matrices in $\mathcal{M}^+$ with eigenvalues of complex modulus one .

<u>Lemma 3.3.</u> <u>($\mathcal{M}^+$)</u> Let $F \in \mathrm{End}^\infty(E)$ be a generic $\mathcal{C}^\infty$-endomorphism. Then for each $x_0 \in X$ such that $\lambda^F(x_0)$ has strictly complex eigenvalues, there is a neighbourhood U of x_0 and a unique endomorphism $F_{\mathcal{M}} \in \mathrm{End}_{\mathcal{M}^+}(E/U)$ such that F and $F_{\mathcal{M}}$ are locally linearly equivalent at x_0, and furthermore $\lambda^{F_{\mathcal{M}}} : U \to \mathcal{M}^+$ is transversal to $\mathcal{R}$ at x_0.

<u>Proof.</u> Assume $\lambda^F(x_0)$ has strictly complex eigenvalues. Then there is a neighbourhood U of x_0, such that $\lambda^F(x)$ has strictly complex eigenvalues for all $x \in U$. Hence $\lambda^F(x)$ reduces to the unique form :-

$$\begin{bmatrix} a(x) & b(x) \\ -b(x) & a(x) \end{bmatrix}$$

where $a(x) + ib(x)$ are the eigenvalues of $\lambda^F(x)$.

($b(x) > 0$ for $x \in U$). Clearly $a(x)$ and $b(x)$ vary smoothly with $x \in U$. Thus F determines a unique endomorphism $F_{\mathcal{M}} \in \mathrm{End}_{\mathcal{M}^+}(E/U)$. It is easily checked that F and $F_{\mathcal{M}}$ are locally linearly equivalent at x_0 . Further the local equivalence determines a smooth map $L : \mathbb{R}^4 \to \mathcal{M}^+$ such that the following diagram commutes :-

$$\begin{tikzcd} & \mathbb{R}^4 \arrow[d, "L"] \\ U \arrow[ur, "\lambda^F"] \arrow[r, "\lambda^{F_{\mathcal{M}}}"'] & \mathcal{M}^+ \end{tikzcd}$$

It is easily seen that L is regular at $\lambda^F(x_0)$.

Thus if λ^F is transversal to the stratification $\mathcal{S}$ of $M(2,2)$, λ^{F_M} is transversal to $\mathcal{R}$ at x_0.

Lemma 3.4. Let $F, G \in End^{\infty}_{M+}(E)$ be such that λ^F and λ^G are transversal to $\mathcal{R}$. Also let F and G be p.t.e. (ϕ) in a neighbourhood V of $x_0 \in \Sigma(1)$ where $\phi : X \to X$ is a C^{∞}- diffeomorphism. Then for some neighbourhood $U \subset V$ of x_0, $c : U - \Sigma(1) \to \mathbb{R}^+$ and $d_{x_0} : U - \Sigma(1) \to \mathbb{R}^+$, can be extended over U.

Proof. The proof follows that of lemma 4.1. in chapter 2.

The proof of the 'main local classification theorem' is now complete.

To summarize:

local classification of generic endomorphisms $F \in End^{\infty}_{H}(E)$ at points x_0 such that $\lambda^F(x_0) \in H$ and :-

(i) $\lambda^F(x_0)$ has one eigenvalue 0, follows from theorem 2.2. and lemma 4.1. ;

(ii) $\lambda^F(x_0)$ has at least one real eigenvalue of modulus one, follows from theorem 2.1. and lemmas 3.1., 3.2.;

(iii) $\lambda^F(x_0)$ has an eigenvalue of complex modulus one, follows from lemmas 3.3. 3.4. and theorem 2.3.

(iv) $\lambda^F(x_0)$ is hyperbolic. follows from theorem 2.4..

CHAPTER 4.

THE LOCAL CLASSIFICATION PROBLEM AT BIFURCATION POINTS OF CODIMENSION GREATER THAN ONE.

1. Introduction.

As previously indicated, the local classification problem so far considered is effectively 'the codimension one' problem (see chapter 1.). In this chapter, the codimension n, (n = 2,4) problem is considered; that is the local classification problem in a neighbourhood of '± Shears' (codim. 2), '± O × O' (codim. 2), '±Id.' (codim. 4) and 'Null' (codim. 4). At the time of writing no complete solution to these local classification problems has been obtained. However there are some partial solutions. These are included in this chapter, together with sub - stantiated conjectures and some plausible but unsubstantiated ones.

The general aim of the chapter is to point to the difficulties of the problem, to present some techniques I have used to obtain my partial solutions, indicate their shortcomings, and suggest modifications of the problem itself. For example in the neighbourhood of ± Id. or ± Shear, a more natural (but not easier) problem is the local classification with respect to 'local orbital equivalence' (see section 2). I conjecture that the problems are equivalent.

A further notion of equivalence is suggested, 'deleted local equivalence'. Basically a local deleted conjugacy at x_0 (say) is defined only on an open - dense subset of a neighbourhood of x_0. It reduces the problem, effectively, to a codimension one problem, and as such may be easier to handle.

The difficulty of writing this chapter, has been in the sifting through, of the mass of partial results, long messy proofs etc., and putting them in a form convincing enough for the reader to understand the difficulties and believe the conjectures. Unfortunately the techniques are inelegant, unsophisticated and generally lengthy. This is not totally or simply a reflection on the state of my mind or my thesis, but of the subject of bifurcations.

The contents of the chapter are then as follows. Section 2 deals with the local equivalence problem in the neighbourhood of 'Id' (although this may be the most difficult problem, the simple situation we consider (on $\mathcal{M}$) best illustrates the techniques employed in this chapter). Section 3 deals with the local classification in the neighbourhood of a 'Shear', and the notion of 'deleted local equivalence' is considered informally. Section 4 considers briefly the $O \times O$ and Null bifurcations; only conjectures are offered, (possibly naive).

Lastly throughout this chapter we take X to be a $\mathcal{C}^\infty$-manifold (unless otherwise stated) and E to be the $\mathcal{C}^\infty$- bundle $X \times \mathbb{R}^2$. We reserve the right to call upon whatever class of endomorphisms and whatever assumptions best illustrate or simplify the exposition.

2. The Local Classification Problem in a Neighbourhood of 'Id'.

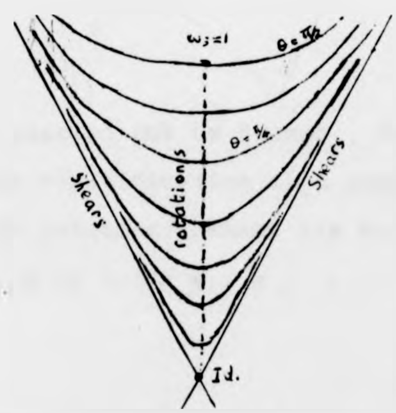
2.1. The Stratification $\mathcal{S}$ of $\mathbb{R}^4$ near 'Id'.

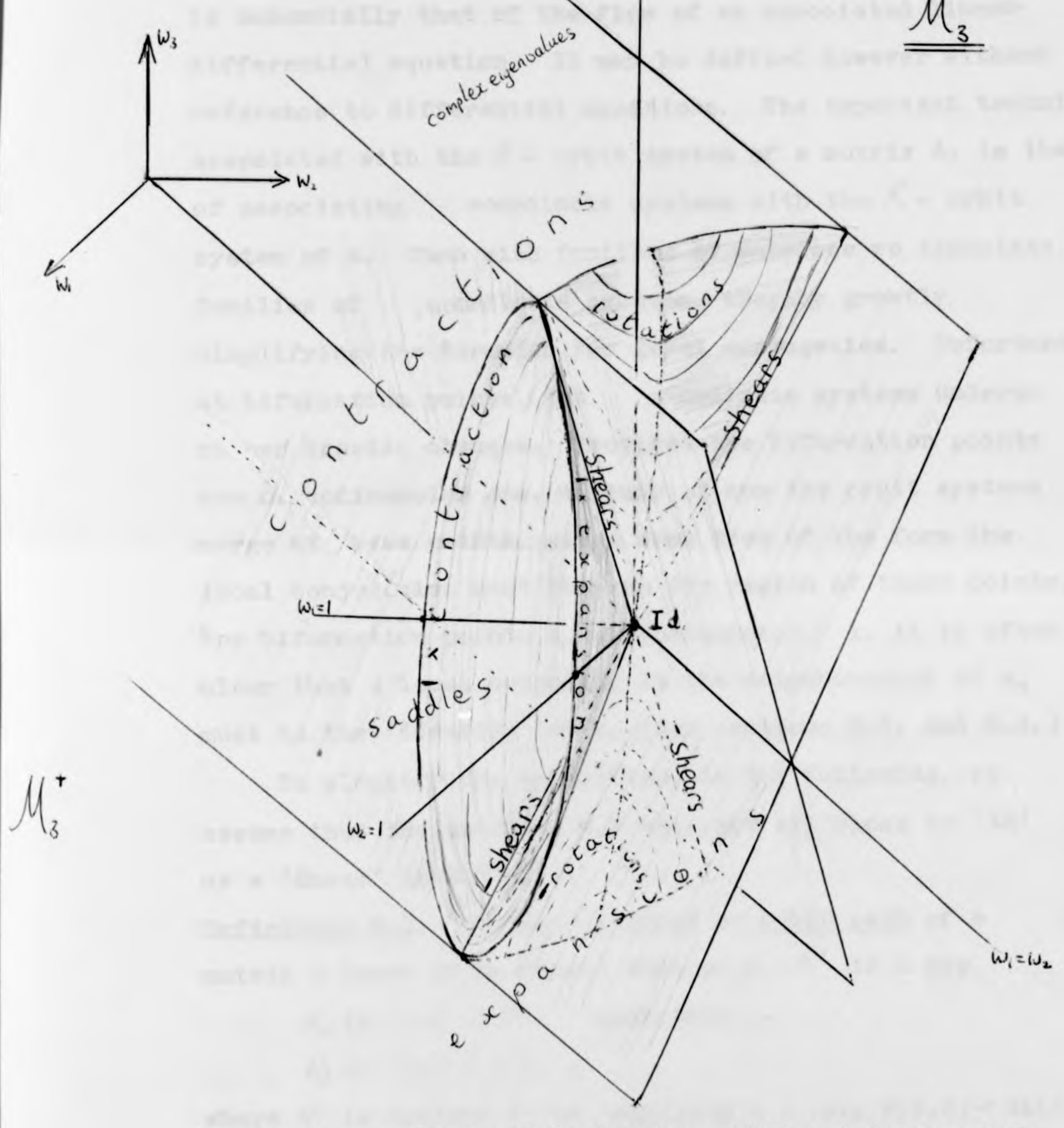
Recall $\mathcal{M}' = \{(w_1, w_2, w_3, w_4) \in \mathbb{R}^4 / w_1 \neq w_2 \Rightarrow w_3 = 0,\ w_3 = -w_4\}$,

and $\mathcal{M} = \{(w_1, w_2, w_3, w_4) \in \mathcal{M}' / w_1 > w_2$ if $w_3 = 0$, $w_3 > 0$ if $w_1 = w_2\}$.

A neighbourhood of 'Id' in $\mathcal{M}'$ (or $\mathcal{M}$) is the simplest (non-trivial) on which to consider the local equivalence problem. We can even give a complete local classification of $\mathrm{End}_{\mathcal{M}}(E)$. The model $\mathcal{M}'$ (or $\mathcal{M}$) is far from being generic. A three-dimensional neighbourhood of 'Id' gives a more precise picture of the stratification at 'Id'. The easiest way to obtain this picture is to 'fill out' $\mathcal{M}'$ i.e. take the whole hyperplane $\{w_3 = -w_4\}$ of $\mathbb{R}^4$. Call it $\mathcal{M}_3$. The picture is given overleaf.

Notice that 'Rotations' is part of an elliptic hyperboloid caught between the planes $|w_1 - w_2| = 2w_3$ (these planes represent matrices with equal eigenvalues). Further this surface intersects these planes in the lines given by its intersection with the plane $w_1 + w_2 = 2$, (these lines represent $\pm$ Shears). To see how the individual rotations foliate this surface, we give a 2-dimensional view of the surface looking along the line $w_1 = w_2$, $w_3 = 0$ towards (0,0,0) on the positive side of (1,1,0).





It has been pointed out by Zeeman , that a 4-dimensional picture of the stratification in a neighbourhood of Id can be obtained by rotating $\mathcal{M}_3^+$ about its boundary $\partial\mathcal{M}_3^+$, where $\mathcal{M}_3^+ = \{(w_1, w_2, w_3, w_4) \in \mathcal{M}_3 \,/\, w_1 > w_2\}$.

2.2. Z- and $\mathbb{R}$ - orbits.

The notion of an $\mathbb{R}$ - orbit system of a matrix $A \in GL^+(\mathbb{R}^2)$ is essentially that of the flow of an associated linear differential equation. It may be defined however without reference to differential equations. The important technique associated with the $\mathbb{R}$ - orbit system of a matrix A, is that of associating coordinate systems with the $\mathbb{R}$ - orbit system of A. Then with families of matrices we associate families of coordinate systems, thereby greatly simplifying the formulas for local conjugacies. Unfortunately at bifurcation points, the coordinate systems undergo rather drastic changes. Provided the bifurcation points are of codimension one, a study of how the orbit systems merge at these points, gives some idea of the form the local conjugacies must take in the region of these points. For bifurcation points x_0 of codimension > 1, it is often clear that a local conjugacy in the neighbourhood of x_0 must be the 'identity' at x_0 (see sections 2.3. and 2.4.)

To simplify the exposition, in the following, we assume that the matrices A,B etc. are all close to 'Id' or a 'Shear' in $GL^+(\mathbb{R}^2)$.

Definition 2.1. The $\mathbb{R}$ - orbit or orbit path of a matrix A (near Id or Shear) through $y_0 \in \mathbb{R}^2$ is a map

$$\mathbb{R}_A(y_0) : \mathbb{R} \rightarrow \mathbb{R}^2 \quad , \quad \text{such that :-}$$

$$\mathbb{R}_A(y\ )(t) = A^t y_0 \ ,$$

where A^t is defined to be $\exp(t \log A)$; $\exp : M(2,2) \rightarrow GL(\mathbb{R}^2)$ is the standard exponential map and 'log' is its local inverse in the neighbourhood of Id or a Shear; (e.g. $\log \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$).

The $\mathbb{R}$ -orbit system of A in $\mathbb{R}^2$ is the map

$$\mathbb{R}_A : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ ,$$ such that $\mathbb{R}_A(y_0)$ is the $\mathbb{R}$ - orbit through y_0 of A , $y_0 \in \mathbb{R}^2$.

Alternatively we may define it as a subdivision of $\mathbb{R}^2$ determined by the solutions or orbits of the differential equation , $\frac{dy}{dt} = (\log A)y$.

<u>Definition 2.2.</u> The <u>Z-orbit</u> of a matrix A through y_0 is a map $Z_A(y_0) : Z \to \mathbb{R}^2$ such that ,

$$Z_A(y_0)(n) = A^n y_0 \quad , \quad y_0 \in \mathbb{R}^2 \quad .$$

We shall refer ambiguously , to $\mathbb{R}_A(y_0)$ and $Z_A(y_0)$ as sets in $\mathbb{R}^2$,('directed' in the sense of t or n increasing). Clearly $Z_A(y_0) \subset \mathbb{R}_A(y_0)$.

<u>Definition 2.3.</u> If A and $B \in GL^+(\mathbb{R}^2)$ are matrices near 'Id' or a 'Shear', A and B are said to be 'orbitally equivalent' (orb. e.), if there is a homeomorphism $\gamma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that γ takes $\mathbb{R}_A$-orbits into $\mathbb{R}_B$-orbits and γ^{-1} takes $\mathbb{R}_B$ -orbits into $\mathbb{R}_A$ -orbits .

We may also define the notion of <u>pointwise orbitally equivalent</u> (ϕ), p.orb.e.(ϕ). Further, l.orb.e.(ϕ) at x_{Id} or x_{shear}.

We conjecture that:if F and G $\in$ End(E) are p.o.t.e.(ϕ) in a neighbourhood V of $x_0 \in \Sigma$(Shear)(or Σ(Id)), then F and G are l.o.t.e.$(\phi)^{\text{at } x_0}_{\lambda}$, if and only if F and G are l.orb.e.(ϕ) at x_0.

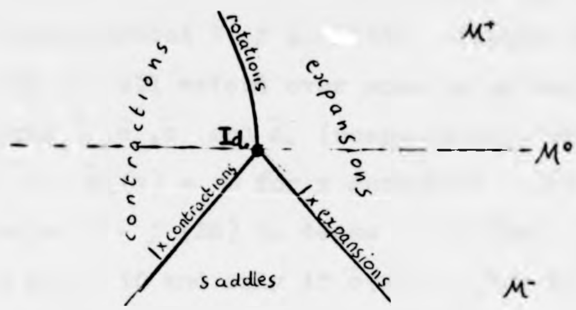
We make the following informal observation in support of the conjecture : l.o.t.e.(ϕ) $\Rightarrow$ l.orb.e.(ϕ) .

Let $\{A_n\}$ be a sequence of matrices in $GL^+(\mathbb{R}^2)$ converging to Id . $Z_{A_n}(y_0) \subset \mathbb{R}_{A_n}(y_0)$, for $y_0 \in \mathbb{R}^2$, and all $n > 0$. Taking the 'limit' of these sets $Z_{A_n}(y_0)$ and $\mathbb{R}_{A_n}(y_0)$ (for example the 'inverse limit' can easily be defined), we have :- $\lim\limits_{n\to\infty} Z_{A_n}(y_0) = \lim\limits_{n\to\infty} \mathbb{R}_{A_n}(y_0)$. Thus any local conjugacy in a neighbourhood of Id, is arbitrarily close to any local orbital equivalence .

The converse i.e. l.orb.e.(ϕ) $\Rightarrow$ l.o.t.e.(ϕ) is more difficult. To obtain a local conjugacy from a local orbital equivalence , one has to 'reparametrize' all the orbit paths so that 'Z-orbits ' go to 'Z-orbits'.

2.3. Local Classification at 'Id' in $\mathcal{M}$.

A neighbourhood of 'Id' in $\mathcal{M}$ is shown below.



The main result of this section is the theorem 2.1. The important part of the theorem is the 'only if' (or'neccessity') part . It indicates a new type of condition for local equivalence (new in the sense that it was not needed in the codimension one classification problem). The 'if' (or'sufficiency') part does not have any real importance since the choice of neighbourhood of 'Id' (i.e. in $\mathcal{M}$) is very artificial. Nevertheless a proof is given in section 2.4. in order to demonstrate the technique of using families of coordinate systems.

Let F , $G \in End_{\mathcal{M}}(E)$ be p.t.e. (ϕ).

Define $\sum$ (Id) $= \{ x \in X \; / \; \lambda^F(x) = \lambda^G \phi(x) = \text{Id} \}$.

If in some neighbourhood U of $x_o \in \Sigma(\text{Id})$, F and G have the property that $\lambda^F(x) \in \mathcal{M}^o$ if and only if $\lambda^G \phi(x) \in \mathcal{M}^o$, then F and G are said to satisfy the '$\mathcal{M}^o$ - condition'. In the following theorem the endomorphisms satisfy the '$\mathcal{M}^o$-condition', but it is only used in proving the 'if' part ,(it is not needed to prove 'deleted local equivalence' - see section 3.).

Theorem 2.1. (Local Classification on Σ (Id)).
Let $F, G \in End_{\mathcal{M}}(E)$ be p t.e. (ϕ) and satisfy the '$\mathcal{M}^{\circ}$-condition' in some neighbourhood V of $x_0 \in \Sigma(Id)$. Assume the functions c_1, c_2, c and d_{λ}, all extend over some neighbourhood $U \subset V$ of x_0 to functions $\tilde{c}_1, \tilde{c}_2, \tilde{c}$ and $\tilde{d}_{\lambda}$ (respectively) where $\tilde{c}_1(x) = \tilde{c}(x)$, for $x \in U$, and $\tilde{d}_{\lambda}(x) = 0$ for x such that $\lambda^F(x)$ and $\lambda^G\phi(x) \in \mathcal{M}^-$, $x \in U$. Assume $U - \Sigma(Id)$ is dense in U. Then F and G are l.t.e.(ϕ) at x_0 if and only if $\tilde{c}(x) = 1$, for all $x \in U \cap \Sigma(Id)$.
Proof. 'if'. This is proved in section 2.4.

'only if'. Let $\Phi : (U \times \mathbb{R}^2, O_x) \to (U \times \mathbb{R}^2, O_x)$ be a conjugacy between F and G on $U \ni x_0$.
Let U(sd), denote the subset of U such that $\lambda^F(x)$ and $\lambda^G\phi(x)$ are in 'saddles' for all $x \in U(sd)$. Similarly define U(rt) U(ex), U(ct), U(lex),U(lct), U(Id).

As in lemma2.2., in chapter 2. , the general form for a conjugacy defined on U(sd) can be found. Firstly it is clear that for each $x \in U(sd)$, any homeomorphism conjugating F_x and $\phi^*(G)_x$ must preserve axes . Then the general form for a conjugacy between F_x and $\phi^*(G)_x$, $x \in U(sd)$, is :-
for $y_1, y_2 > 0$,

$$(y_1, y_2) \mapsto (y_1^{c_1(x)}\exp(c_1(x)\mu_{1x}(\log y_1, \log y_2)),\ y_2^{c_2(x)}\exp(c_2(x)\mu_{2x}(\log y_1, \log y_2)))$$

where $\mu_{1x} : \mathbb{R}^2 \to \mathbb{R}$ is a periodic function of period $(\log \lambda_1^F(x)\ ,\ \log \lambda_2^F(x))$, and μ_{2x} is similarly defined.
As in previous arguments of this type, it can be seen, if $x \in U(Id)$, that Φ_x is given by :-

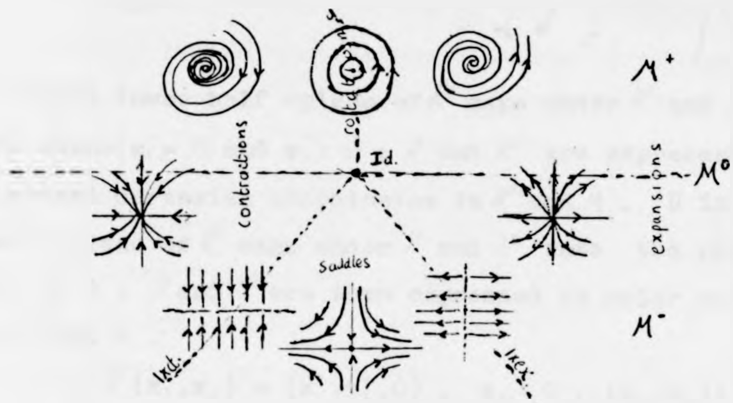
$$(y_1, y_2) \mapsto (y_1^{\tilde{c}_1(x)}, y_2^{\tilde{c}_2(x)}) = (y_1^{\tilde{c}(x)}, y_2^{\tilde{c}(x)}) \text{ , for } y_1, y_2 \geq 0 .$$

But it has been shown (lemma 3 3. chapter 2.), that Φ_x for $x \in U(rt)$ is of the form -

$$(r, \theta) \mapsto (r^{\tilde{c}(x)}, \theta + \tilde{d}_{\lambda}(x)\log r)$$

Trivially , then Φ_x is the identity . for $x \in U \cap \Sigma(\mathrm{Id})$. Therefore $\hat{c}(x) = 1$, for all $x \in U \cap \Sigma(\mathrm{Id})$.

The orbit systems of F_x and G_x in a neighbourhood of $x_0 \in \Sigma(\mathrm{Id})$ are as shown below.



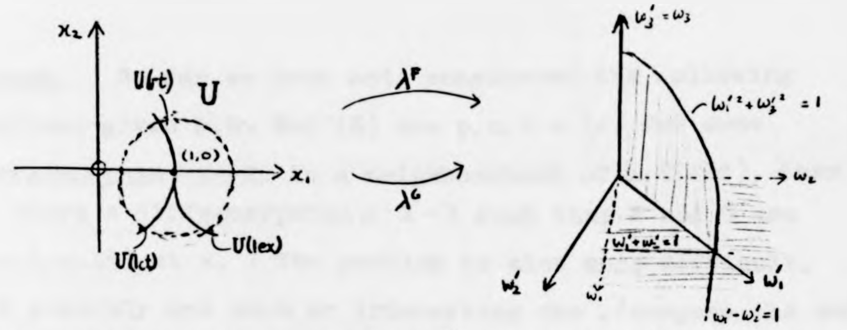
It is evident that if F and G are p.o.t.e.(ϕ), then any local orbital equivalence between F and G (if it exists) must be the identity at $x_0 \in \Sigma(\mathrm{Id})$. It is not so clear however that this implies $\hat{c}(x_0) = 1$. I conjecture (and can probably prove)that this is so.

We show by counterexample , using theorem 2.1. , that if F ,$G \in \mathrm{End}(E)$ are p.o.t.e.(ϕ) in a neighbourhood V of x_0 in $\Sigma(\mathrm{Id})$, then it is not in general true that F and G are l.o.t.e.(ϕ) at x_0 , even if the functions c_1, c_2 ,c and d_{x_0} extend (as in theorem 2 1.) over V .

<u>Counterexample 2.1.</u>

Let $X = \mathbb{R}^2$, $E = X \times \mathbb{R}^2$, and $F, G \in \mathrm{End}_{\mathcal{M}}(E)$ are given in a neighbourhood U of (1,0) as follows. Before defining $\lambda^F : U \to \mathcal{M}$ and $\lambda^G : U \to \mathcal{M}$, we simplify their forms by defining new coordinates on $\mathcal{M}'$, (w_1', w_2', w_3'), where :-

$$w_1' = \frac{w_1 + w_2}{2} \quad , \qquad w_2' = \frac{w_1 - w_2}{2} \; , \qquad w_3' = w_3 \; .$$



U in the lower half -plane of $\mathbb{R}^2$ maps under λ^F and λ^G into the plane$\{w_3' = 0\}$ and $w_2' \geq 0$; λ^F and λ^G are expressed in standard cartesian coordinates in $\mathbb{R}^2$ and $\mathcal{M}$. U in the upper half -plane of $\mathbb{R}^2$ maps under λ^F and λ^G into the plane$\{w_2' = 0\}$ and $w_3' \geq 0$; λ^F and λ^G are then expressed in polar coordinates in $\mathbb{R}^2$ and $\mathcal{M}$.

$$\begin{cases} \lambda^F(x_1,x_2) = (x_1^2, x_2^2, 0) , & x_2 \leq 0 , \ (x_1,x_2) \in U , \\ \lambda^F(r,\theta) = (r, 0, \theta^2) , & \theta \geq 0 , \ (r,\theta) \in U . \end{cases}$$

$$\begin{cases} \lambda^G(x_1,x_2) = (2x_1^2 - 1, 2x_2^2, 0) , & x_2 \geq 0 , \ (x_1,x_2) \in U , \\ \lambda^G(r,\theta) = (2r^2 - 1, 0, \theta^2) , & \theta \geq 0 , \ (r,\theta) \in U . \end{cases}$$

F and G are p.t.e.(id_X) , c_1, c_2 ,c and d extend over U , but $\tilde{c}(1,0) = 2$. Hence F and G are not l.t.e. (id_X) at (1,0).

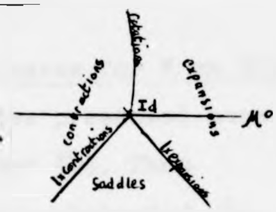
Similarly a $\mathcal{C}^\omega$- counterexample can be constructed.

Theorem 2.1. is stated in terms of endomorphisms in $End_{\mathcal{M}}(E)$. However a similar theorem concerning generic endomorphisms can be given , i.e. we may show that, given $F,G \in End^\infty(E)$ are generic and p.o.t.e.(ϕ) in a neighbourhood of $x_0 \in \Sigma(Id)$ where $\phi : X \to X$ is a diffeomorphism , then if F and G are l.o.t.e.(ϕ) at x_0 , $\tilde{c}(x) = 1$ for all $x \in U \cap \Sigma(Id)$, (the converse is false). To construct a counterexample similar to 2.1. but for generic endomorphisms is very complicated since the dimension of X must be at least four. It is reasonable to assume such an example exists.

<u>Remark.</u> So far we have not considered the following problem; given $F, G \in End^{\omega}(E)$ are p.o.t.e (ϕ) for some diffeomorphism $\phi: X \to X$ in a neighbourhood of $x_0 \in \Sigma(Id)$, then is there a diffeomorphism $\phi_1: X \to X$ such that F and G are l.o.t.e. (ϕ_1) at x_0 ? The problem is alot more difficult, and possibly not such an interesting one ,(compare the work of Arnold on <u>Versal families</u> of matrices [1]).

<u>2.4. Proof of 'Sufficiency ' in Theorem 2.1.</u>

Recall a neighbourhood of 'Id' in $\mathcal{M}$.



We shall construct a local conjugacy at $x_0 \in \Sigma(Id)$ between F and G on $U \subset X$ ($x_0 \in U$). The construction is non-trivial; and so ,rather than write down the local conjugacy without explanation, the details of the construction are presented(informally). The techniques employed will be used again in a more difficult construction in section 3 .

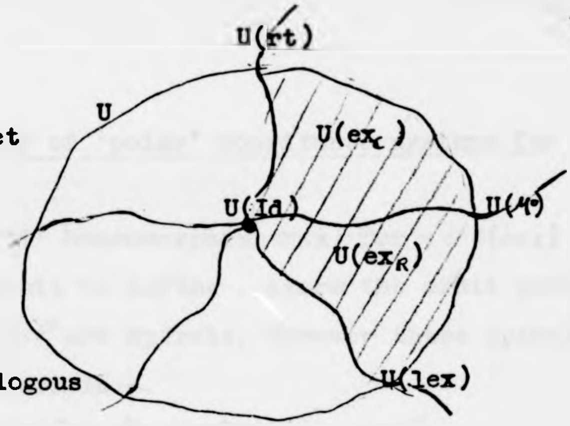
The difficulty for the reader , will be in the mass of notation needed . It is a good idea for the reader to think of $U \subset X$ as a 'copy' of a neighbourhood of 'Id' in $\mathcal{M}$ (taking $X = \mathbb{R}^2$) .

Recall $U(ex) = \{ x \in U / \lambda^F(x)$ and $\lambda^G\phi(x)$ are 'expansions'$\}$. Since F and G satisfy the '$\mathcal{M}^{\circ}$-condition ' , some further simplifying notation can be given :-

Define $U(ex_{\mathbb{C}}) = \{ x \in U(ex) / \lambda^F(x)$ and $\lambda^G\phi(x)$ have complex or real equal eigenvalues $\}$

$U(ex_{\mathbb{R}}) = \{ x \in U(ex) / \lambda^F(x)$ and $\lambda^G\phi(x)$ have real eigenvalues$\}$.

The sets $U(ex\ )$, $U(rt)$, etc. are shown below in a neighbourhood U of x_0 in $X = \mathbb{R}^2$. (<u>Note</u>. $\mathcal{M}$ is a subspace of $\mathbb{R}^4$)



We shall only construct the conjugacy on the shaded region of U. The conjugacy elsewhere in U , is constructed in an analogous or obvious fashion .

The construction is simplified by associating with $F \in \mathrm{End}_{\mathcal{M}}(E)$ a family of 'polar-coordinates' on U(ex) .

<u>Construction of a family of 'polar' coordinates for F on U(ex)</u> .

The idea of the change of coordinates presented here is to 'straighten out', the orbit paths for $\lambda^F(x)$. Then relative to the new coordinate system the orbits for $\lambda^F(x)$ will be rays through the origin , giving a 'natural system of polar coordinates' relative to F_x .

<u>For $x \in U(ex_R)$</u>, the change of coordinates is given by the homeomorphism $\sigma_F(x) : \mathbb{R}^2 \to \mathbb{R}^2$, defined by the formula:-

$$\sigma_F(x)(y_1, y_2) = (y_1 , y_2 |y_2|^{h_F(x)-1})$$

where $h_F : U(ex_R) \to \mathbb{R}^+$ is given by :-

$$h_F(x) = \frac{\log \lambda_1^F(x)}{\log \lambda_1(x)}$$, which is well defined and continuous.

We then take polar coordinates (r_{F_x}, v_{F_x}) ,where

$$r_{F_x}(y_1, y_2) = (y_1^2 + y_2^{2h_F(x)})^{1/2}$$, and

$$v_{F_x}(y_1, y_2) = \tan^{-1}\left(\frac{y_2|y_2|^{h_F(x)-1}}{y_1}\right)$$.

With respect to these polar coordinates, the endomorphism F_x , now written as F_x^{σ} , i.e. $F_x^{\sigma} = \sigma_F(x) \cdot F_x \sigma_F^{-1}(x)$, is given by:-

$$F_x^{\sigma}(r_{F_x}, v_{F_x}) = (\lambda_1^F(x) r_{F_x} , v_{F_x})$$.

In fact $\sigma_F : U(ex_R) \times \mathbb{R}^2 \to U(ex_R) \times \mathbb{R}^2$,is a homeomorphism.

(<u>Note</u>. $\tan(v_{F_x}(y_1, y_2))$ is the intersection of the orbit path of $\lambda^F(x)$ through (y_1, y_2) with the line $y_1 = 1$ (for $y_1 > 0$).)

Construction of a family of 'polar' coordinate systems for F on $U(ex_c)$.

The 'straightening out' homeomorphism $\sigma_F(x)$ for $x \in U(ex_c)$ is a little more difficult to define, since the orbit paths for $\lambda^F(x)$ through $(r,\theta) \in \mathbb{R}^2$ are spirals. However these spirals are given by a simple formula :-

$$\mathbb{R}_{\lambda^F(x)}(r,\theta)(t) = (|\lambda^F(x)|^t r \ , \ \theta + t\beta^F(x)\ ) \ , \quad t \in \mathbb{R} .$$

The point of intersection of each spiral with the unit circle (and any other circle centered at 0) is unique. Thus we express the points of $\mathbb{R}^2$ in terms of these intersections and r . We have then,

$$\sigma_F \ : U(ex_c) \times \mathbb{R}^2 \longrightarrow U(ex_c) \times \mathbb{R}^2 \qquad , \text{ is a}$$

homeomorphism given by :-

$$\sigma_F(x)(r,\theta) = (r_{F_x}, v_{F_x}) \ , \text{ where } \ r_{F_x}(r,\theta) = r$$

$$\text{and} \quad v_{F_x}(r,\theta) = \theta - \frac{\beta^F(x)\log r}{\log|\lambda^F(x)|} \quad .$$

As previously, define $F^\sigma : U(ex_c) \times \mathbb{R}^2 \to U(ex_c) \times \mathbb{R}^2$, such that $F^\sigma = \sigma_F \cdot F|U(ex_c) \cdot \sigma_F^{-1}$, and then

$$F_x^\sigma(r_{F_x}, v_{F_x}) = (|\lambda^F(x)| r_{F_x} \ , v_{F_x})$$

Note that $\sigma_F|(U(ex_c) \cap U(ex_R) \times \mathbb{R}^2)$, is well defined, i.e. $\sigma_F : \ U(ex) \times \mathbb{R}^2 \longrightarrow U(ex) \times \mathbb{R}^2$ is a homeomorphism.

$$\text{Write } \tilde{\lambda}^F(x) = \lambda_1^F(x) \ , \ x \in U(ex_R),$$
$$= |\lambda^F(x)| \ , \ x \in U(ex_c) .$$

Then for each $x \in U(ex)$, $F_x^\sigma(r_{F_x}, v_{F_x}) = (\tilde{\lambda}^F(x) r_{F_x}, v_{F_x})$.

For ease of exposition, take $\phi : X \to X$ to be id_X.

As above, define a homeomorphism $\sigma_G : U(ex) \times \mathbb{R}^2 \to U(ex) \times \mathbb{R}^2$ and G^σ, such that $G_x^\sigma(r_{G_x}, v_{G_x}) = (\tilde{\lambda}^G(x) r_{G_x}, v_{G_x})$.

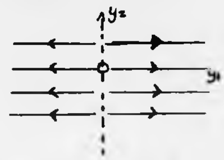
Note that $\tilde{c}(x) = \dfrac{\log \tilde{\lambda}^G(x)}{\log \tilde{\lambda}^F(x)}$, for $x \in U(ex)$.

The construction of a conjugacy Φ^σ on U(ex).

To define a local conjugacy Φ between F and G on U, we first define it on U(ex) . This is done by first constructing a conjugacy Φ^σ on U(ex) between F^σ and G^σ . The following observations motivate our choice for Φ^σ .

(i). The conjugacy to be constructed is to preserve orbits.

(ii). The orbit system for $\lambda^F(x)$, $x \in U(lex)$ is as shown.



Any local conjugacy on a neighbourhood of $x \in U(lex)$ that preserves orbits must be of the form :-

$(y_1, y_2) \mapsto (- , y_2^{\tilde{c}_2(x)})$ on U(lex).

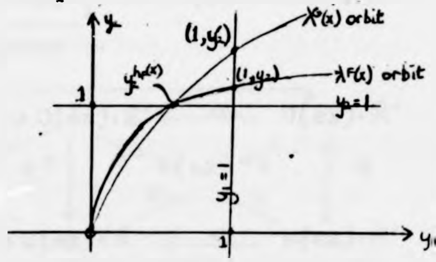
(iii). Fix $(1, y_2) \in \mathbb{R}^2$ $(y_2 > 0)$, then $\tan(v_{F_\lambda}(1, y_2)) = y_2^{h_F(x)}$.

Let $(1, y_2') \in \mathbb{R}^2$ be such that $\tan(v_{G_\lambda}(1, y_2')) = \tan(v_{F_\lambda}(1, y_2))$, then $y_2' = y_2^{\frac{h_F(x)}{h_G(x)}}$.

Finally note that ,

$$c_2(x) = \frac{c_1(x) h_F(x)}{h_G(x)} ,$$

for all $x \in U(ex_R)$.



(iv). A local conjugacy in the neighbourhood of a point $x_1 \in U(rt)$ is of the form :-

$$(r, \theta) \mapsto (r^{\tilde{c}(x_1)}, \theta + \tilde{d}_{x_0}(x_1) \log r) \text{ at } x_1$$

where $\tilde{d}_{x_0}(x_1) \to 0$ as $x_1 \to U(Id)$.

(v). Fix $(1, \theta) \in S'$, then $v_{F_\lambda}(1, \theta) = \theta - \dfrac{\beta^F(x)}{\log|\lambda^F(x)|}$.

Let $(1, \theta') \in S'$ be such that $v_{G_\lambda}(1, \theta') = v_{F_\lambda}(1, \theta)$,

then $\theta' = \theta + \dfrac{\beta^G(x)}{\log|\lambda^G(x)|} - \dfrac{\beta^F(x)}{\log|\lambda^F(x)|}$.

Now for $x \in U(ex)$, define $\Phi^\sigma : U(ex)\times\mathbb{R}^2 \to U(ex)\times\mathbb{R}^2$,
such that $\Phi^\sigma = (\Phi^{\sigma 1}, \Phi^{\sigma 2})$, where $r_{G_x}(\Phi^\sigma_x(r_{F_x}, v_{F_x})) = \Phi^{\sigma 1}_x(r_{F_x}, v_{F_x})$,
and $v_{G_x}(\Phi^\sigma_x(r_{F_x}, v_{F_x})) = \Phi^{\sigma 2}_x(r_{F_x}, v_{F_x})$,
and Φ^σ_x is given by the formula :-

$$\boxed{\Phi^\sigma_x(r_{F_x}, v_{F_x}) = (r_{F_x}^{\tau(x)},\ \tan^{-1}((\tan v_{F_x})|\tan v_{F_x}|^{e(x)-1}))}$$

where $e(x) = \tilde{c}(x) + \dfrac{(1-\tilde{c}(x))}{(h_F(x))^2}$, $x \in U(ex)$.

Recall $\tilde{c}(x) = c(x)$, for $x \in U(ex_L)$, (and $h_F(x) = 1$),
$= c_1(x)$, for $x \in U(ex_R)$.

Then $e(x) = 1$, for $x \in U(ex_C) \cup U(Id)$, and $e(x) \to c_1(x)$ as $x \to U(lex)$, ($h_F(x) = {}'\infty{}'$ on $U(lex)$).

The construction of the conjugacy Φ.

Φ^σ conjugates F^σ and G^σ on U(ex). The local conjugacy Φ on U between F and G is defined such that :-

$$\underline{\Phi|U(ex) = \sigma_G^{-1} \circ \Phi^\sigma \circ \sigma_F}, \qquad \cdots\cdots (1)$$

then the following diagram commutes :-

$$\begin{tikzcd}
U(ex)\times\mathbb{R}^2 \arrow[rr, "\sigma_F"] \arrow[dr, "\pi"] \arrow[dd, "F"'] \arrow[rrrrrr, bend left, "\Phi|U(ex)"] & & U(ex)\times\mathbb{R}^2 \arrow[rr, "\Phi^\sigma"] \arrow[dl, "\pi"'] \arrow[dr, "\pi"] \arrow[dd, "F^\sigma"] & & U(ex)\times\mathbb{R}^2 \arrow[dl, "\pi"'] \arrow[dr, "\pi"] \arrow[dd, "G^\sigma"'] & & U(ex)\times\mathbb{R}^2 \arrow[ll, "\sigma_G"'] \arrow[dl, "\pi"'] \arrow[dd, "G"] \\
& U(ex) & & U(ex) & & U(ex) & \\
U(ex)\times\mathbb{R}^2 \arrow[ur, "\pi"] \arrow[rr, "\sigma_F"'] \arrow[rrrrrr, bend right, "\Phi|U(ex)"'] & & U(ex)\times\mathbb{R}^2 \arrow[ul, "\pi"'] \arrow[ur, "\pi"] \arrow[rr, "\Phi^\sigma"] & & U(ex)\times\mathbb{R}^2 \arrow[ul, "\pi"'] \arrow[ur, "\pi"] & & U(ex)\times\mathbb{R}^2 \arrow[ul, "\pi"'] \arrow[ll, "\sigma_G"]
\end{tikzcd}$$

It is neccessary to check that $\Phi|U(ex)$ extends naturally to a conjugacy on U(lex) and U(rt).

For $x \in U(ex_C)$;

from (1), we have that,

$$\Phi_x(r,\theta) = (r^{\tau(x)},\ \theta + \tilde{d}_{x_0}(x)\log r)$$

which clearly extends naturally to a conjugacy on U(rt), (see ch.2), and further extends to the identity on U(Id) provided $\tilde{c}(x) = 1$ on U(Id).

<u>For $x \in U(ex_R)$,</u>

$$\Phi_x(y_1, y_2) = \left(y_1^{\tilde{c}_1(x)} \frac{\left(1 + (y_2^{h_0(x)}/y_1)^2\right)^{\frac{\tilde{c}_1(x)}{2}}}{\left(1 + (y_2^{h_0(x)}/y_1)^{2e(x)}\right)^{1/2}},\ y_2^{\tilde{c}_2(x)} \frac{\left(1 + (y_1/y_2^{h_0(x)})^2\right)^{\frac{\tilde{c}_1(x)}{2h_0(x)}}}{\left(1 + (y_1/y_2^{h_0(x)})^{2e(x)}\right)^{1/2h_0(x)}} \right),$$

for $y_1, y_2 \geqslant 0$, and then symmetrically about the axes, (the function is well defined at $(0,0)$ and on the axes).

It is not difficult to check that $\Phi | U(ex)$ extends naturally to a conjugacy on $U(lex)$ given by :-

$$\Phi_x(y_1, y_2) = (y_1 |y_1|^{\tilde{c}_1(x)-1}, y_2 |y_2|^{\tilde{c}_2(x)-1}) .$$

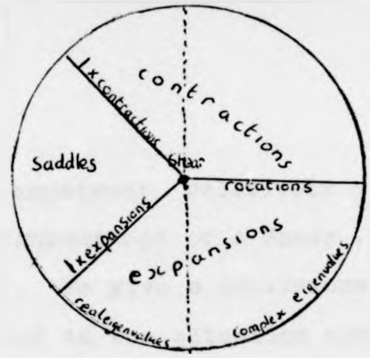
The conjugacy Φ on $U(sd)$ is defined by a similar formula. On $U(ct)$, the conjugacy is constructed as on $U(ex)$ and thence on $U(lct)$. Finally Φ_x is the identity on $U(Id)$.

<u>Remark.</u> We could have defined a conjugacy Φ on a deleted neighbourhood of $U(Id)$ i.e. on $U - U(Id)$ (a'deleted' conjugacy) irrespective of the behavior of $\tilde{c}$ on $U(Id)$. It is also apparent from the construction of Φ^{σ} that the $\mathcal{M}^{0}$-condition was not needed to set up a deleted local conjugacy. The $\mathcal{M}^{0}$-condition should be regarded as a condition determining the extendability of a 'deleted local conjugacy ' to a local conjugacy . (It is actually difficult to prove this assertion rigorously) . We expand on the theme of deleted local conjugacy in section 3.

<u>3. The Local Classification problem for Shears.</u>

<u>3.1 Deleted Local Equivalence</u> .

From the picture in section 2. showing the stratification $\mathcal{S}$ of $\mathbb{R}^4$ in a neighbourhood of Id in the hyperplane $\{w_3 = -w_4\}$ we deduce that any two-dimensional disc containing a shear transversal to $\mathcal{S}$ bifurcates as shown below , (take, for example, a disc in in a plane parallel to the plane $\{w_3 = 0\}$.



In order to obtain some insight into the local classification problem for families in the neighbourhood of a Shear, we need to consider some restricted class of endomorphisms of E (such as $\mathrm{End}_{\mathcal{M}}(E)$) . The 'restricted' class is chosen so as to make explicit formulas for local conjugacies as simple as possible . Unfortunately no such class of endomorphisms exists that are generic in a neighbourhood of a Shear. Since our solution to the problem is incomplete , we choose a class of endomorphisms that best illustrates the technical difficulties of constructing local conjugacies in the neighbourhood of a Shear, and further best illustrates the technical problem of the construction of 'deleted local conjugacies'. The endomorphisms in the class chosen are not generic.

Let $\mathcal{M}_5 = \{ (w_1, w_2, w_3, w_4) \in \mathbb{R}^4 / \ w_3 = 1, \ w_4 = 0 \}$.

A point in $\mathcal{M}_5$ represents a matrix in the form :-

$$\begin{bmatrix} w_1 & 1 \\ 0 & w_2 \end{bmatrix}$$

The formulas for explicit conjugacies between matrices in the simple form of those in $\mathcal{M}_5$, are fairly complicated ; this makes the problem of constructing local conjugacies in the neighbourhood of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ technically very difficult. I have been unable to construct a single local conjugacy , even after assuming the 'nicest' possible behavior of the endomorphisms near $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

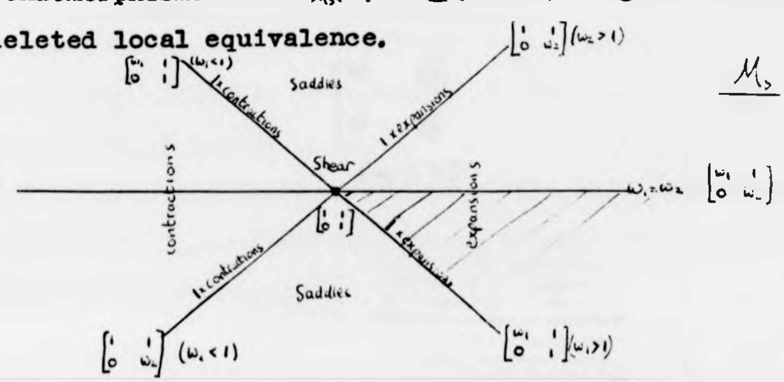
However one may construct relatively simply, local conjugacies in a deleted neighbourhood of a Shear , that is a 'deleted local conjugacy'. We give a provisional definition of the notion as applied to the situation considered in this section.

Definition 3.1. Let F, $G \in \mathrm{End}(E)$ be p.t.e.(ϕ) where $\phi: X \to X$ is a homeomorphism . Then F and G are deleted l.t.e.(ϕ) at $x_0 \in \Sigma(\text{Shear})$, if for some neighbourhood U of x_0 ,there exists a homeomorphism $\Phi : U - \Sigma(\text{Shear}) \times \mathbb{R}^2 \to U - \Sigma(\text{Shear}) \times \mathbb{R}^2$, such that the following diagram commutes :-

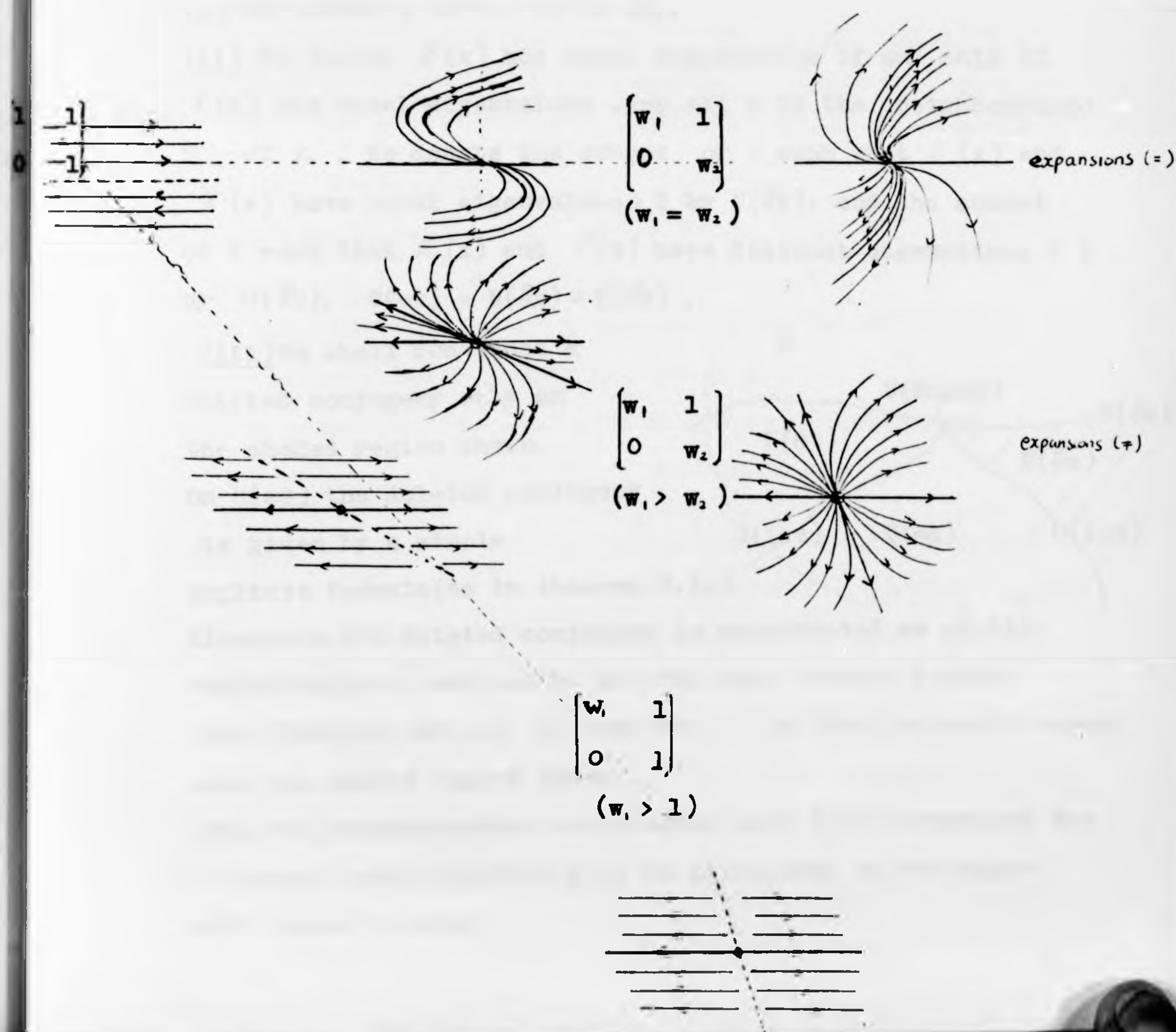
$$\begin{tikzcd}
(U - \Sigma(\text{Shear}) \times \mathbb{R}^2, 0_x) \arrow[rr, "F"] \arrow[dr, "\pi"] \arrow[dd, "\Phi"'] & & (U - \Sigma(\text{Shear}) \times \mathbb{R}^2, 0_x) \arrow[dl, "\pi"'] \arrow[dd, "\Phi"] \\
& U - \Sigma(\text{Shear}) & \\
(U - \Sigma(\text{Shear}) \times \mathbb{R}^2, 0_x) \arrow[rr, "\phi^*(G)"'] \arrow[ur, "\pi"] & & (U - \Sigma(\text{Shear}) \times \mathbb{R}^2, 0_x) \arrow[ul, "\pi"']
\end{tikzcd}$$

The definition may be reformulated so as to apply to similar situations , such as deleted equivalence at $x_0 \in \Sigma(\mathrm{Id})$.
The general idea is to delete bifurcation points of codimension> 1 . I conjecture : if F , $G \in \mathrm{End}^{\infty}(E)$ are generic and are p.o.t.e.(ϕ) where $\phi: X \to X$ is a diffeomorphism , then F and G are deleted l.o.t.e.(ϕ) at x , for each $x \in X - \Sigma(\text{Null})$.

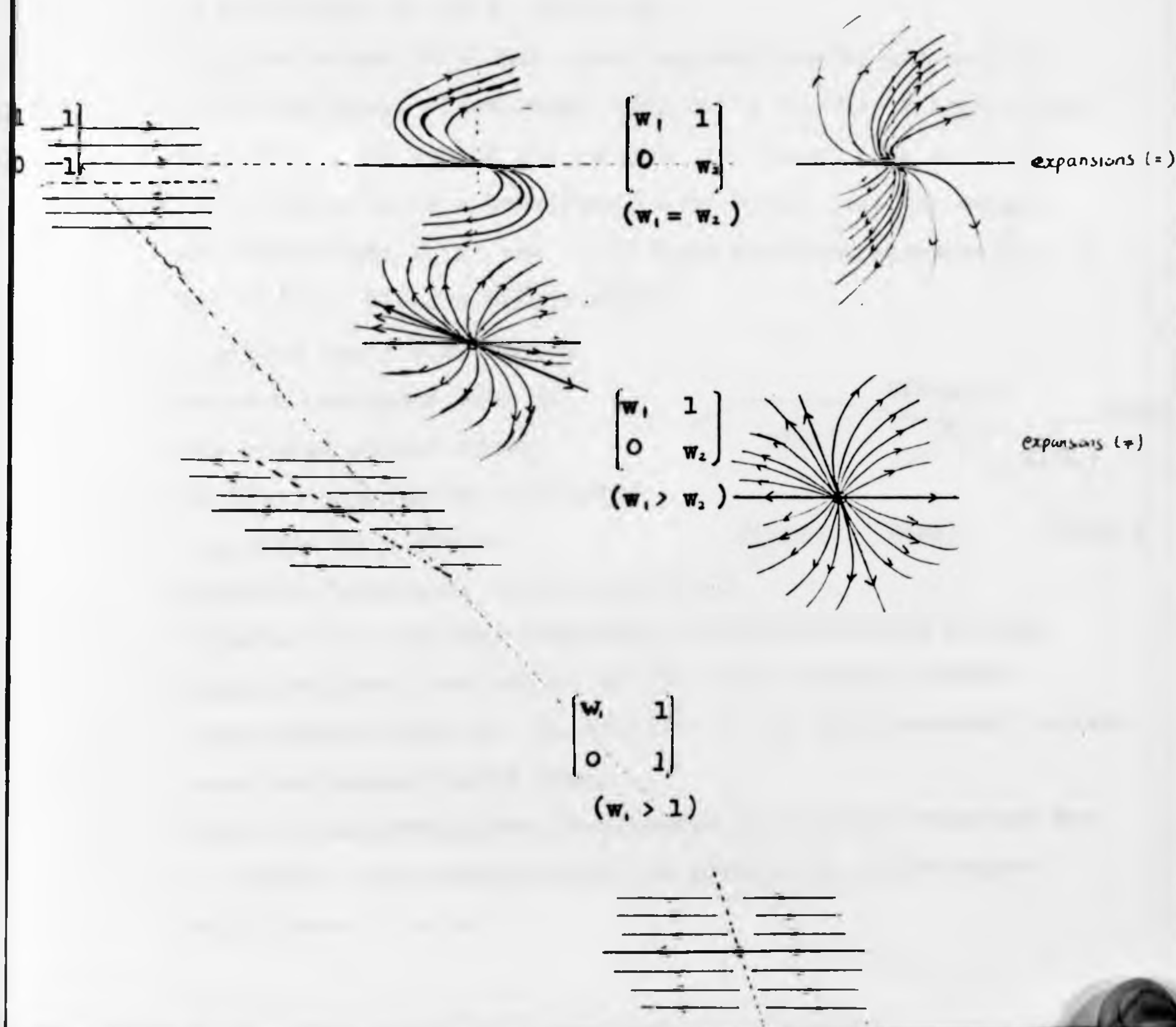
The construction of deleted local conjugacies is non-trivial and is illustrated by Theorem 3.1. , where a local classification for endomorphisms in $\mathrm{End}_{\mathcal{M}_S}(E)$ on $\Sigma(\text{Shear})$ is given relative to deleted local equivalence.



Note that there is symmetry about the line $\{w_1 = w_2\}$. Any family of matrices in the lower half plane is locally linearly equivalent to some family in the upper half plane. Thus it is only neccessary to construct a deleted conjugacy in the lower half plane. In fact, we consider only the shaded region shown, elsewhere, as in Theorem 2.1., it will be clear how the deleted conjugacy should be constructed. The construction is in terms of $\mathcal{R}$ -orbits as in Theorem 2.1.. These are shown for matrices in the shaded region of M_5.



Note that there is symmetry about the line $\{w_1 = w_2\}$. Any family of matrices in the lower half plane is locally linearly equivalent to some family in the upper half plane. Thus it is only neccessary to construct a deleted conjugacy in the lower half plane. In fact, we consider only the shaded region shown, elsewhere, as in Theorem 2.1., it will be clear how the deleted conjugacy should be constructed. The construction is in terms of $\mathcal{R}$-orbits as in Theorem 2.1.. These are shown for matrices in the 'shaded region' of M_s.



The functions c_1, c_2 are well defined for $F,G \in \mathrm{End}_{M_5}(E)$ that are p.t.e.(ϕ) .

<u>Theorem 3.1. (Deleted Local Equivalence on Σ (Shear) for $\mathrm{End}_{M_5}(E)$)</u>

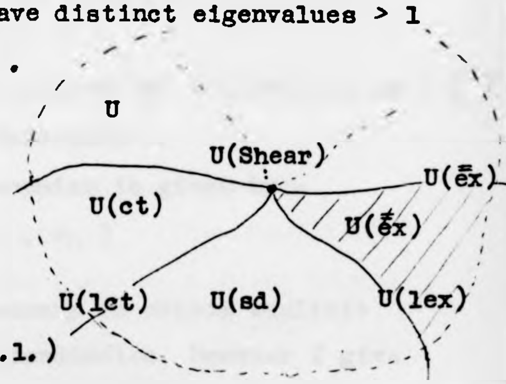
Let $F,G \in \mathrm{End}_{M_5}(E)$ be p.t.e.(ϕ) in a neighbourhood V of $x_0 \in \Sigma$ (Shear). Assume for some neighbourhood $U \subset V$ of x_0 , that the functions c_1, c_2 extend over U to functions $\tilde{c}_1$ and $\tilde{c}_2$. Then F and G are deleted l.t.e.(ϕ) at x_0 .

<u>Proof.</u>

To shorten and simplify the proof we make the following assumptions without loss of generality:-

<u>(i)</u>. We assume $\phi : X \to X$ to be id_X.

<u>(ii)</u> We assume $\lambda^F(x)$ has equal eigenvalues if and only if $\lambda^G(x)$ has equal eigenvalues ,for all x in the neighbourhood U of x_0 . We denote the subset of U such that $\lambda^F(x)$ and $\lambda^G(x)$ have equal eigenvalues > 1 by $U(\overset{=}{e}x)$, and the subset of U such that $\lambda^F(x)$ and $\lambda^G(x)$ have distinct eigenvalues > 1 by $U(\overset{\neq}{e}x)$. $U(ex) = U(\overset{=}{e}x) \cup U(\overset{\neq}{e}x)$.



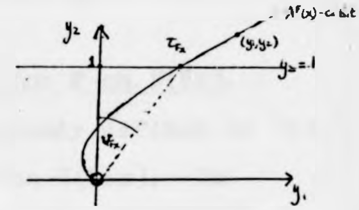
<u>(iii)</u>We shall construct a deleted conjugacy only on the shaded region shown. On U(sd) the deleted conjugacy is given by a simple explicit formula(as in theorem 2.1.) Elsewhere the deleted conjugacy is constructed as on the shaded region , and can be written down without further justification (and so is ommitted) By U(ex) we shall always mean the shaded region shown .

<u>(iv)</u> The homeomorphisms constructed will leave invariant the y_1- axis , and therefore will be given only on the upper half-plane $\cup \{y_1\text{-axis}\}$.

In order to obtain simple formulas for the deleted conjugacy, families of 'polar'coordinates for F and G are constructed as in theorem 2.1..

<u>A family of 'polar' coordinates for F on U(ex).</u>

The orbit paths through $(y_1, y_2) \in \mathbb{R}^2$ $(y_2 > 0)$ for $\lambda^F(x)$, $x \in U(ex)$, intersect the line $y_2 = 1$ (and any line parallel to it) just once. Let $\tau_{Fx}(y_1, y_2)$ denote this point of intersection. $\tau_{Fx} : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function of (y_1, y_2) and $x \in U(ex)$.



$$\text{Let } v_{Fx}(y_1, y_2) = \tan^{-1}(\tau_{Fx}(y_1, y_2)) \quad , \; y_2 > 0$$
$$= \pi/2, (-\tfrac{\pi}{2}) \quad , \; y_2 = 0 \; , \; y_1 > 0, (y_1 < 0).$$

$v_{Fx} : \mathbb{R}^2_+ \to S'$, is a continuous function of (y_1, y_2) and $x \in U(ex)$.

$$\text{Let } r_{Fx}(y_1, y_2) = (y_2^{2h_F(x)} + (y_2^{h_F(x)} \tan v_{Fx}(y_1, y_2))^2)^{1/2}, \; y_2 > 0$$
$$= y_1^2 \; , \; y_2 = 0 \; .$$

(a more natural choice would have been $(y_2^2 + y_2^2(\tan v_{Fx}(y_1, y_2)))^{1/2}$ however the former simplifies later work).

The'straightening out' homeomorphism is given by:-

$$\sigma_F(x) : (y_1, y_2) \mapsto (r_{Fx}, v_{Fx})$$

<u>Remark.</u> It is in fact not neccessary to obtain explicit formulas for $\sigma_F(x)$ in standard coordinates. However I give them below to indicate the technical problems of working in standard coordinates .

<u>For $x \in U(\bar{\bar{e}}x)$</u> ,

$$\mathbb{R}_{\lambda^F(x)}(y_1, y_2)(t) = \left((\lambda_0^F(x))^t y_1 + t(\lambda_0^F(x))^{t-1} y_2 , (\lambda_0^F(x))^t y_2 \right) , t \in \mathbb{R} ;$$

where $\lambda_0^F(x) = \lambda_1^F(x) = \lambda_2^F(x)$.

$$v_{Fx}(y_1, y_2) = \tan^{-1}\left(\frac{y_1}{y_2} - \frac{\log y_2}{\lambda_0^F(x) \log \lambda_0^F(x)} \right), \; y_2 \geq 0$$

$$r_{Fx}(y_1, y_2) = \left(y_2^2 + \left(y_1 - \frac{y_2 \log y_2}{\lambda_0^F(x) \log \lambda_0^F(x)} \right)^2 \right)^{1/2} ; \; y_2 \geq 0.$$

For $x \in U(\vec{\bar{\bar{e}}}x)$:-

$$\mathcal{R}_{N_F(x)}(y_1, y_2)(t) = \left((\lambda_1^F(x))^t y_1 + \frac{(\lambda_1^F(x)^t - \lambda_2^F(x)^t) y_2}{k_F(x)}, \lambda_2^F(x)^t y_2 \right), t \in \mathbb{R},$$

where $k_F(x) = \lambda_1^F(x) - \lambda_2^F(x)$.

$$v_{F_x}(y_1, y_2) = \tan^{-1}\left(\frac{y_1}{y_2^{h_F(x)}} + \frac{(y_2 - y_2^{h_F(x)})}{k_F(x) y_2^{h_F(x)}} \right), \quad y_2 \geq 0$$

$$r_{F_x}(y_1, y_2) = \left(y_2^{2h_F(x)} + \left(y_1 + \frac{(y_2 - y_2^{h_F(x)})}{k_F(x)} \right)^2 \right)^{\frac{1}{2}}, \quad y_2 \geq 0$$

<u>Note.</u> $\dfrac{y_2^{h(x)} - y_2}{k_F(x)} \rightarrow \dfrac{\log y_2}{\lambda_2^F(x) \log \lambda_2^F(x)}$, as $x \longrightarrow U(\bar{\bar{e}}x)$.

Another family of 'polar' coordinates for F on $U(\bar{\bar{e}}x)$.

The family of 'polar' coordinates already defined on U(ex) are clearly well defined near but not on U(lex). In setting up a local conjugacy on U(ex), the question must be asked as to what determines whether a local conjugacy on U(ex) extends to a local conjugacy on U(lex). This question has already been considered in theorem 2.1. . A new consideration arises in the present problem. Namely, any local conjugacy near U(lex) must send eigenvectors to eigenvectors (they are no longer fixed as in theorem 2.1.) Any simple conjugacy defined in terms of the (r_{F_x}, v_{F_x})-coordinates (above), does not send eigenvectors to eigenvectors. We therefore construct another family of 'polars' on $U(\bar{\bar{e}}x)$. They will be based on a local linear equivalence between F and $F_N \in End_N(E)$ (see chapter 3.)

Define a linear change of coordinates :-

$$A_x^F : \mathbb{R}^2 \to \mathbb{R}^2 \text{ , such that,}$$

$$A_x^F(y_1, y_2) = \left(y_1 + \frac{y_2}{k_F(x)}, \ y_2 \right), \quad x \in U(\bar{\bar{e}}x) .$$

(A_x^F gives the linear equivalence between $\begin{bmatrix} \lambda_1^F(x) & 1 \\ 0 & \lambda_2^F(x) \end{bmatrix}$ and $\begin{bmatrix} \lambda_1^F(x) & 0 \\ 0 & \lambda_2^F(x) \end{bmatrix}$, $\lambda_1^F \neq \lambda_2^F$)

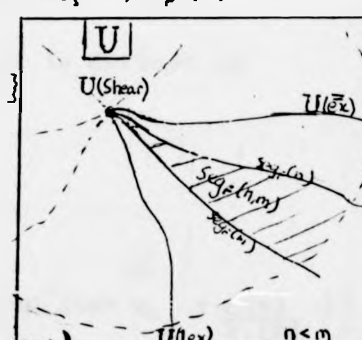
Then we'straighten out' with respect to this new basis as in theorem 2.1. ,i.e. define $\sigma_F(x) : \mathbb{R}^2 \to \mathbb{R}^2$, $x \in U(\vec{e}x)$ as the composite map:-

$$(y_1, y_2) \longmapsto \left(y_1 + \frac{y_2}{k_F(x)}, \; y_2 \right) \longmapsto \Big(\overbrace{\Big(\Big(y_1 + \frac{y_2}{k_F(x)}\Big)^2 + y_2^{2h_F(x)}\Big)^{1/2}}^{r'_{F_x}}, \; \overbrace{\tan^{-1}\Big(\frac{y_1 + y_2/k_F(x)}{y_2^{h_F(x)}}\Big)}^{v'_{F_x}} \Big)$$

$$\sigma_F(x)(y_1, y_2) = (r'_{F_x}, v'_{F_x}) .$$

<u>The transition from one family of 'polars' to another.</u>

We have a family of 'polars' for F with the required properties near U(lex) defined on $U(\vec{e}x)$, and a family defined on U(ex). The families are not equivalent . Hence we require an intermediate stage 'between' $U(\bar{\bar{e}}x)$ and U(lex). Fortunately there is a natural 'segmentation' of U(ex) associated with F that enables us to effect a continuous transition from one family of 'polars' to the other . The segmentation is defined by the sets $\{ x \in U(ex) \,/\, h_F(x) = \text{const.}\}$; $h_F(x) = 1$ on $U(\bar{e}x)$ and $h_F(x) = {'\infty'}$ on U(lex) .



Define $seg_F(n,m) = \{ x \in U(ex) \,/\, n \leq h_F(x) \leq m \}$ and $seg_F(n,\infty) = \{ x \in U(ex) / \, n \leq h_F(x) < \infty \}$ (<u>Note</u>. A similar segmentation is defined by the function $d_{x_0}(x) = \dfrac{\beta^s(x) - \beta^r(x)}{\log|\lambda^r(x)|}$, in a neighbourhood of $x_0 \in \Sigma(Id)$ for F and G in $End_M(E)$).

<u>The required family of 'polars' for F on U(ex)</u> .

Before defining the final choice for a family of 'polars ' for F ,((s_F, ω_{F_x})), we make the following observations motivating our choice:-

<u>(i)</u>. If $s_{F_x}(y_1, y_2) = r_{F_x}(y_1, y_2) . \left(\dfrac{r'_{F_x}(y_1, y_2)}{r_{F_x}(y_1, y_2)}\right)^{\varepsilon(x)}$, $0 \leq \varepsilon(x) \leq 1$, $x \in U(\bar{e}x)$

$$s_{F_x}(\lambda_1^F(x) y_1 + y_2, \lambda_2^F(x) y_2) = \lambda_1^F(x) r_{F_x}(y_1, y_2) \left(\frac{\lambda_1^F(x) r'_{F_x}(y_1, y_2)}{\lambda_1^F(x) r_{F_x}(y_1, y_2)} \right)^{\varepsilon(x)}$$

$$= \underline{\lambda_1^F(x) s_{F_x}(y_1, y_2)}$$

(ii). Fix $(y_1, y_2) \in \mathbb{R}^2$. $\tan v_{F_x}(y_1, y_2)$, $(y_2 > 0)$, is the point of intersection of the orbit of $\lambda^F(x)$ through (y_1, y_2) with the line $y_2 = 1$, whereas $\tan v'_{F_x}(y_1, y_2)$ is the point of intersection of the orbit path of $\lambda^F(x)$ through $(y_1 + \frac{y_2}{k_F(x)}, y_2)$ with the line $y_2 = 1$.

Clearly $\tan v'_{F_x}(y_1, y_2) = \tan v_{F_x}(y_1, y_2) + \frac{1}{k_F(x)}$, $x \in U(ex)$,

$$\text{or, } v'_{F_x}(y_1, y_2) = \tan^{-1}\left(\tan v_{F_x}(y_1, y_2) + \frac{1}{k_F(x)}\right).$$

We have, $v_{F_x}(y_1, y_2) \mapsto v'_{F_x}(y_1, y_2)$ defines a homeomorphism of S^1, as (y_1, y_2) ranges over the orbit paths of $\lambda^F(x)$, (for example fix y_2 and let y_1 range over $\mathbb{R}$). Note also that $\omega_{F_x}(y_1, y_2) = \tan^{-1}\left(\tan v_{F_x}(y_1, y_2) + \frac{\varepsilon(x)}{k_F(x)}\right)$, $0 \leq \varepsilon(x) \leq 1$ defines a homeomorphism of S^1 as (y_1, y_2) ranges over the orbit paths of $\lambda^F(x)$, for each $\varepsilon(x)$, $x \in U(ex)$.

The 'straightening out' function σ_F is defined as follows :-

On $seg_F(1,2)$,

$$\sigma_F(x)(y_1, y_2) = (\overset{s_{F_x}}{r_{F_x}}, \overset{\omega_{F_x}}{v_{F_x}}).$$

On $seg_F(2,3)$

$$\sigma_F(x)(y_1, y_2) = \left(\overset{s_{F_x}}{r_{F_x}\left(\frac{r'_{F_x}}{r_{F_x}}\right)^{\varepsilon(x)}}, \overset{\omega_{F_x}}{\tan^{-1}\left(\tan v_{F_x} + \frac{\varepsilon(x)}{k_F(x)}\right)} \right)$$

where $\varepsilon : seg_F(2,3) \to [0,1]$ is the function such that $\varepsilon(x) = (h_F(x) - 2)$.

On $seg_F(3,\infty)$,

$$\sigma_F(x)(y_1, y_2) = (\overset{s_{F_x}}{r'_{F_x}}, \overset{\omega_{F_x}}{v'_{F_x}}).$$

$$\sigma_F : U(ex) \times \mathbb{R}^2 \longrightarrow \overline{U(ex) \times \mathbb{R}^2}, \text{ is a homeomorphism.}$$

Define $F^\sigma = \sigma_F \cdot F|U(ex) \circ \sigma_F^{-1}$, then F^σ_x maps (s_{F_x}, ω_{F_x}) to

$$(\lambda^F_1(x)s_{F_x}, \omega_{F_x}).$$

Similarly define $\sigma_G : U(ex)\times\mathbb{R}^2 \to U(ex)\times\mathbb{R}^2$, and G^σ .
G^σ_x maps (s_{Gx}, ω_{Gx}) to $(\lambda^G_1(x)s_{Gx}, \omega_{Gx})$.

A deleted local conjugacy on U(ex).

Define $\Phi : (U - \Sigma(\text{Shear})\times\mathbb{R}^2 , 0_x) \to (U - \Sigma(\text{Shear})\times\mathbb{R}^2 , 0_x)$,
such that $\Phi | U(ex) = \sigma_G^{-1} \circ \Phi^\sigma \circ \sigma_F$,
where Φ^σ is defined as follows :-

On $seg_F(1,2)$,

$$\Phi^\sigma_x(s_{Fx}, \omega_{Fx}) = (s_{Fx}^{\tau_1(x)}, \omega_{Fx}) .$$

On $seg_F(2,3)$,

$$\Phi^\sigma_x(s_{Fx}, \omega_{Fx}) = (s_{Fx}^{\tau_1(x)}, \tan^{-1}(\tan\omega_{Fx} |\tan\omega_{Fx}|^{\varepsilon(x)\tau_1(x)-1})).$$

On $seg_F(3,\infty)$,

$$\Phi^\sigma_x(s_{Fx}, \omega_{Fx}) = (s_{Fx}^{\tau_1(x)}, \tan^{-1}(\tan\omega_{Fx} |\tan\omega_{Fx}|^{\tau_1(x)-1})).$$

Note that in $seg_F(3,\infty) \cap seg_G(3,\infty)$, the homeomorphism Φ_x is similar to that defined in theorem 2.1. near U(lex), with a linear change in coordinates . Thus it is easily checked that Φ extends naturally to U(lex) , such that:-

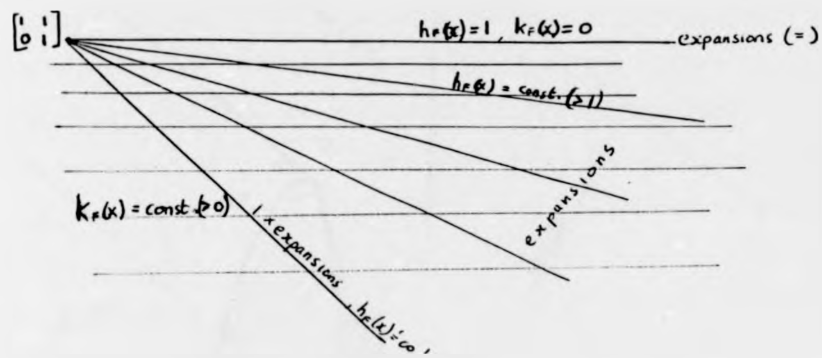
$$\Phi_x(y_1, y_2) = \left(\left(y_1 + \frac{y_2}{k_F(x)}\right)\left|y_1 + \frac{y_2}{k_F(x)}\right|^{\tau_1(x)-1} - \frac{y_2^{\tau_1(x)}}{k_F(x)} , \; y_2^{\tau_1(x)} \right) , y_2 \geq 0.$$

On U(sd), Φ_x is defined by a similar explicit formula .
On U(ct), Φ_x is defined as on U(ex) , and thence on U(lct).
Thus Φ is the required deleted local conjugacy at $x_0 \in \Sigma(\text{Shear})$

3.2. Remarks on the Extension of a Deleted Local Conjugacy to a Local Conjugacy .

It is apparent from the above that any deleted local conjugacy in the neighbourhood of $x_0 \in \Sigma(\text{Shear})$,when expressed in standard coordinates is extremely complicated; and deciding when, how or if, a deleted conjugacy extends to a local conjugacy is very difficult. It seems reasonable to conjecture that a neccessary condition for a deleted conjugacy to extend is that $\tilde{c}(x) = 1$ for $x \in U(\text{Shear})$. It is very likely that the condition is not sufficient . In section 2. ,a deleted local conjugacy was constructed (effectively) in a neighbourhood of $x_0 \in \Sigma(\text{Id})$, that extended to a local conjugacy. It is reasonably certain (though difficult to prove) that this extension was possible for two reasons. Firstly we assumed that $\tilde{c}(x) = 1$ on U(Id), and secondly we assumed F and G satisfied the $\mathcal{M}^0$-condition i.e. that $\lambda^F(x) \in \mathcal{M}^0$ if and only if $\lambda^G(x) \in \mathcal{M}^0$, $x \in U$. The $\mathcal{M}^0$-condition ensured that any conjugacy constructed sent spirals to spirals and non-spirals to non-spirals. The analogous situation in the neighbourhood of a Shear is greatly complicated by the behavior of the eigenvectors. The eigenvectors of $\lambda^F(x)$, $x \in U(\mathcal{E}x)$, coalesce as $\lambda^F(x)$ approaches not only $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, but also $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ (the rate at which this happens is determined by the function $k_F(x) = \lambda_1^F(x) - \lambda_2^F(x)$). A condition that effectively 'matches up' orbit systems would need to be firstly in terms of the segmentations of U determined by F and G (e.g. $\text{seg}_F(n,m) \subset \text{seg}_G(n,m)$ for all $n,m \in Z$) and secondly of the form $\frac{k_F(x)}{k_G(x)} \to 1$ as $x \to U(\text{Shear})$ Even under these severe restrictions I have been unable to construct a deleted conjugacy on any segment that extended. It is clear that a detailed analysis of local conjugacy in a neighbourhood of a Shear is best attempted in terms of the natural segmentations. One may alternately view such an analysis as a study of local conjugacy on the 'blown up' neighbourhood of a singular point (that is a point in U(Shear)

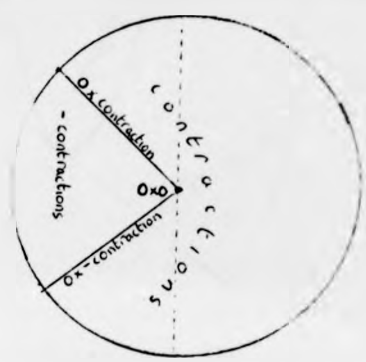
As a final comment on the ' matching up' problem for orbit systems (cf. $\mathcal{M}^\circ$- condition), we give the following picture showing the images of the sets $\{k_F(x) = \text{const.}\}$ and $\{h_F(x) = \text{const.}\}$ under the map χ^r in $\mathcal{M}_5$, $x \in U(ex)$.



4. The Local Classification Problem for 0 × 0 and Null.

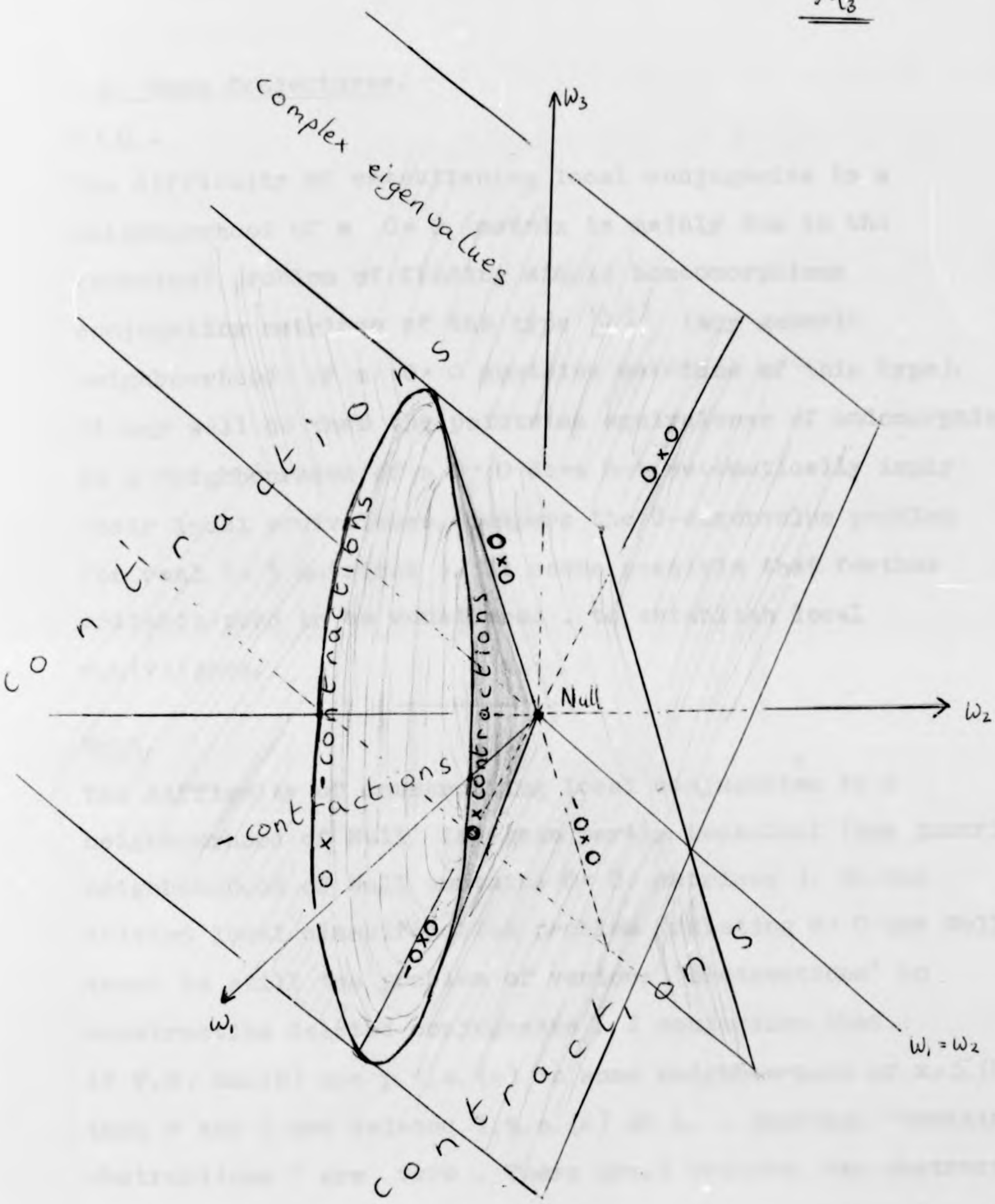
4.1. Bifurcation diagrams and stratification near Null and 0 × 0

From the picture overleaf showing the stratification $\mathcal{S}$ of $\mathbb{R}^4$ in a neighbourhood of Null in the hyperplane $\{w_3 = -w_4\}, (\mathcal{M}_1)$, we deduce that any two-dimensional disc transversal to $\mathcal{S}$ containing a 0× 0 , bifurcates as shown below.



The four dimensional bifurcation diagram for Null is obtained by rotating $\mathcal{M}_3^+$ about its boundary plane $\partial\mathcal{M}_3^+$. (see section 2)

$\mathcal{M}_3$



ω_3
ω_2
ω_1
$\omega_1 = \omega_2$
complex eigenvalues
contradictions
contractions
0 x contractions
-contractions
0 x -contractions
-0x0
0x0
Null

4.2. Some Conjectures.

0×0.

The difficulty of establishing local conjugacies in a neighbourhood of a 0×0 matrix is mainly due to the technical problem of finding simple homeomorphisms conjugating matrices of the type $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ (any generic neighbourhood of a 0×0 contains matrices of this type). It may well be that the pointwise equivalence of endomorphisms in a neighbourhood of a 0×0 does not automatically imply their local equivalence, (compare the 0-eigenvalue problem for real 1×1 matrices). It seems possible that further criteria need to be considered , to establish local equivalence.

Null.

The difficulty of constructing local conjugacies in a neighbourhood of Null is again partly technical (any generic neighbourhood of Null contains 0×0 matrices). In the deleted local classification problem (deleting 0×0 and Null) there is still the problem of various 'obstructions' to constructing deleted conjugacies . I conjecture that : if $F, G \in \mathrm{End}(E)$ are p.t.e.(ϕ) in some neighbourhood of $x_0 \in \Sigma(\mathrm{Null})$ then F and G are deleted l.t.e.(ϕ) at x_0 , provided 'certain obstructions ' are zero . There are, I beleive, two obstruction conditions . The first of these is analogous to that for the local classification on $\Sigma(0)$ for complex line bundles (see chapter 2.3.) , suitably rephrased for the present situation . The second is a 'mod. 2' obstruction described below .

Let $F \in End(E)$.

Define $\Sigma_F(\lambda,0) = \{ x \in X - \Sigma_F(Null) / \lambda^F(x)$ has at least one eigenvalu zero

Let $x_0 \in \Sigma_F(\lambda,0)$. Define a map $\gamma^F_{x_0} : \Sigma_F(\lambda,0) \to (S' , \gamma^F_{x_0}(x_0))$

such that $\gamma^F_{x_0}(x) = \tan^{-1}\left(\frac{\lambda^F_1(x)}{\lambda^F_4(x)}\right)$,

(Note that if $\lambda^F(x)$ is singular $\lambda^F_1(x).\lambda^F_3(x) = \lambda^F_2(x).\lambda^F_4(x)$).

Conjecture. Assume $\Sigma_F(\lambda,\dot{0})$ is locally path connected. Let $F,G \in End(E)$ be p.t.e.(ϕ) in a neighbourhood V of $x_0 \in \Sigma(Null)$. Then F and G are not l.t.e.(ϕ) at x_0 , if for each neighbourhood $U \subset V$ of x_0 ,

$$2(\gamma^F + \gamma^G)_*(\pi_1(U \cap \Sigma(\lambda,0))) \neq 0 \quad (\text{mod.}2) .$$

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