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A Thesis Submitted for the Degree of PhD at the University of Warwick

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THE MOD 2 COHOMOLOGY OF THE ORTHOGONAL GROUPS OVER A FINITE FIELD.

by Corrado de Concini

Introduction.

The purpose of this paper is to generalize the results of Quillen [8] about the cohomology of (the classifying space of) the general linear groups over a finite field to the orthogonal case.

In the whole paper we will restrict ourselves to the study of the cohomology with mod 2 coefficients of (the classifying space of) the orthogonal groups.

We give a complete computation of $H(BO_n(k),Z_2)$ in the case of split orthogonal groups when k has $q=4m+1$ elements (Theorem5.6).

The computation of the mod p cohomology with p odd and different from the characteristic of k, is basically simpler. It had been announced by Quillen in his Nice talk [6], as a consequence of his study of the etale homotopy types of algebraic varietes. He also announced partial results for the mod 2 case. The details have never appeared. The proof that we give here for the mod 2 case applies, with no essential modifications, essentialy by substituting the Stiefel-Witney classes with the mod p Pontrjagin classes.

The proof that we give follows the general lines of the one given by Quillen for the general linear case. There are in our case some obstacles which did not appear in Quillen's proof, expecially depending by the fact that for a finite group the first KO-theory group

$KO^1(BG)$ is not necessarely zero.

This problem does not arise in mod p computations when p is odd, therefore making the computation in this case considerably simpler.

We now give a summary of parts of the paper. In paragraph 1 we define the space $\widetilde{FO}\,\Psi^q$ which is the real analogue of Quillen's $F\Psi^q$. To construct $\widetilde{FO}\,\Psi^q$ first of all we mimic Quillen and build a space $FO\,\Psi^q$ which turns out to be unsuitable for our computations. Therefore we have to change it with $\widetilde{FO}\,\Psi^q$, essentialy one of its connected components.

In paragraph 2 we give a rough computation of the cohomology of $\widetilde{FO}\,\Psi^q$.

Paragraph 3 deals with a well known technical Lemma.

Paragraph 4 treats the Brauer lifting of orthogonal representation of a finite group over the algebraic closure of k. We show that the Brauer lifting of an orthogonal representation obtained by extension of scalars from an orthogonal representation over k, is left fixed by the action of the Adams operation Ψ^q, allowing us to associate to such a representation an element in $[BG, \widetilde{FO}\,\Psi^q]$. This is applied to the standard representation of $\Theta_n(k)$.

In paragraph 5 we define some elements in $H^*(\widetilde{FO}\,\Psi^q)$ wich will be fundamental in the subsequent computations. Unfortunately their definition depends on the choise of a certain element in $[BO(k), \widetilde{FO}\,\Psi^q]$ where $O(k) = \bigcup_n O_n(k)$.

In paragraph 6 we consider the u_i's relative to a particular choice and we compute a multiplicative formula for them.

In paragraph 7 we give a complete computation of $H^*(\widetilde{FO}\Psi^q, Z_2)$ as an algebra.

In paragraph 8 we give an explicit base for $H_*(\widetilde{FO}\Psi^q, Z_2)$ and for $\bigoplus_{r \geq 0} H_*(BO_r(k), Z_2)$, which allows us to show that $H^*(\widetilde{FO}\Psi^q, Z_2)$ constitutes an upper bound for $H^*(BO_n(k))$ in the sense of the introduction of [8]. This together with the fact that $H^*(\underbrace{BO_2(k) \times \ldots \times BO_2(k)}_{m\text{-times}} \times BZ_2^e, Z_2)$ constitutes a lower bound for $H^*(BO_n(k), Z_2)(n=2m+e(e=0,1))$ gives us the total computation of the mod 2 cohomology algebra of (the classifying space of) $O_n(k)$.

I wish to express my thanks to my supervisor G. Lusztig for his constant help and encouragement during my work on this paper; and my admiration to D.Quillen who first studied the cohomology of the classical groups over finite fields by using this methods.

I finally wish to thank C.N.R. for financially supporting me during the course of this research.

1. The space $\widetilde{FO}\Psi^q$

By the word space we mean a topological space with the homotopy type of a CW-complex.

Let BO be a classifying space, for example the infinite real grasmanians, for the functor $\widetilde{KO}$ defined on compact spaces, i.e. $\widetilde{KO}(X) = [X, BO]$ for X compact.

Let $N((\widetilde{KO})^n, \widetilde{KO})$ denote the set of natural transformations $(\widetilde{KO})^n \longrightarrow \widetilde{KO}$.

We have:

Lemma 1. $$N((\widetilde{KO})^n, \widetilde{KO}) \cong [BO^n, BO]$$

Proof. If we take the Grasmanian model, then $(BO)^n = \underset{m,s}{\underrightarrow{\lim}} (G_{m,s})^n$, where $G_{m,s}$ denotes the real Grasmanian of m-dimensional subspaces of a vector space of dimension m+s.

Then, if we consider the Milnor exact sequence

$$0 \longrightarrow R^1 \underset{m,s}{\underleftarrow{\lim}} KO^1((G_{m,s})^n) \longrightarrow [BO^n, BO] \longrightarrow \underset{m,s}{\underleftarrow{\lim}} \widetilde{KO}((G_{m,s})^n) \longrightarrow 0$$

where R^1 denotes the first derived functor of $\underleftarrow{\lim}$, we must have in order to prove the lemma $R^1 \underset{m,s}{\underleftarrow{\lim}} KO^{-1}((G_{m,s})^n) = 0$.

Now the real completion theorem [2] implies that the inverse system $KO^{-1}((G_{m,s})^n$ is isomorphic as a pro-object to the inverse system

$$RO((O_m)^n) \Big/ \, R((O_m)^n) \Big/ (I((O_m)^n))^s \qquad RO((O_m)^n) \Big/ R((O_m)^n)$$

where, for any group G,$RO(G)$(resp.$R(G)$) denotes the real (resp.complex) representation ring of G and $I(G)$ denotes the real augmentation ideal in $RO(G)$.

It follows that, if we fix m, the inverse system $KO^{-1}((G_{m,s})^n)$ satisfies the Mittag-Leffler condition.

If we make m vary, we notice that it follows from the representation theory of O_m, [1], that, if m is odd, the restriction map $RO((O_h)^n) \longrightarrow RO((O_m)^n)$, for $h \geq m$, is onto and this easily implies that the whole system $KO^{-1}((G_{m,s})^n)$ satisfies the Mittag-Leffler condition, which implies, [2],

$$R^1 \varprojlim_{m,s} KO^{-1}((G_{m,s})^n) = 0$$

thus proving the lemma. q.e.d..

Now let q be an odd integer and let

$$\sigma : BO \longrightarrow BO$$

represent the adams operation ψ^q in $\widetilde{KO}$.

We define the homotopy theoretical fixpoint set of ψ^q as the fibre product

$$\begin{tikzcd} FO\Psi^q \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BO^I \arrow[d, "\Delta"] \\ BO \arrow[r, "(6,id)"'] & BO \times BO \end{tikzcd}$$

where Δ is the map which sends each path to its endpoints.

We want to define a slightly different space from $FO\Psi^q$ which will be more useful for our purposes.

It is well known that $H^*(BO,Z_2) \cong Z_2[w_1,w_2,\ldots\ldots]$ where the w_i's are the universal Stiefel-Witney classes and so, by Kunneth formula we have,

$H^*(BO \times BO,Z_2) \cong Z_2\ [w_1',w_1'',w_2',\ldots\ldots]$ with $w_i'^{(\prime\prime)} = p_{1(2)}^*(w_i)$,

where p_1 (resp. p_2) denote the projection onto the first (resp. the second) factor.

Now let us define B to be the total space of the double covering of $BO \times BO$ associated to the element $w_1' + w_1'' \in H^1(BO \times BO,Z_2)$.

We have:

<u>Proposition 1</u>. $\quad H^*(B,Z_2) \cong H^*(BO \times BO,Z_2) / (w_1' + w_1'')$

<u>Proof</u>. It is clear that the Serre spectral sequence $\{E_r\}$ associated to the fibration

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$$\begin{tikzcd} BSO \times BSO \arrow[d, "g"] \\ B \arrow[d] \\ BZ_2 \end{tikzcd}$$

collapses at the term E_2 because the map $g^*: H^*(B, Z_2) \longrightarrow H^*(BSO \times BSO, Z_2)$ is onto, since the map $(fg)^*: H^*(BO \times BO, Z_2) \longrightarrow H^*(BSO \times BSO, Z_2)$ associated to the fibration

$$\begin{tikzcd} Z_2 \oplus Z_2 \arrow[d] \\ BSO \times BSO \arrow[d, "fg"] \\ BO \times BO \end{tikzcd}$$

where f is the double covering $f: B \longrightarrow BO \times BO$, is known to be onto.

So we have that $E_\infty \cong H^*(BSO \times BSO, Z_2) \otimes Z_2[w]$. and now the proposition follows from the fact that the map

$$d: H^*(BSO \times BSO, Z_2) \longrightarrow H^*(B, Z_2)$$

defined by $d\,((fg)^*(w_i^{\prime(n)})) = f^*(w_i^{\prime(n)})$ for $i \geqslant 2$ provides a right inverse for g^* and from [3] .

q.e.d.

Now consider the map $BO \xrightarrow[(\sigma, id)]{} BO^2$. Since q is odd, we have that σ^* is equal to the identity in mod.2 cohomology, so, we have $(\sigma,\mathrm{id})^*(w_1' + w_1'') = 0$.

This implies that there exists $(\sigma,\mathrm{id})': BO \quad B$ such that the following diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ BO \arrow[ur, "(\sigma,\mathrm{id})'"] \arrow[r, "(\sigma,\mathrm{id})"'] & BO \times BO \end{tikzcd}$$

commutes.

Now let us consider the maps $BO \times BO \xrightarrow[d]{} BO$ representing the difference operation in $\widetilde{KO}$. Fixing a base point $b \in BO$, we can define d, using the homotopy extension theorem, in such a way that $d(x,x) = b$ and $d(x,b) = d(b,x) = x, \forall\ x \in BO$.

If we define $m: BO^I \longrightarrow BO^I \times_{BO} \{b\}$ to be the map wich sends the path p to the path $t \longrightarrow d(p(t),p(1))$ which joins $d\Delta(p) = d(p(0),p(1))$ to the base point, we get a diagram

$$\begin{tikzcd} FO \arrow[r, "\gamma"] \arrow[d, "\varphi"] & BO^I \arrow[r, "m"] \arrow[d, "\Delta"] & BO^I \times_{BO} b \arrow[d, "h"] \\ BO \arrow[r, "(\mathrm{id},\sigma)"'] & BO \times BO \arrow[r, "d"'] & BO \end{tikzcd}$$

which is commutative and in which all the vertical lines are fibrations with the same fiber Ω BO. So BO^I is homotopy equivalent to $(BO^I \times_{BO} \{b\}) \times_{BO} (BO \times BO)$, and we identify BO^I with this space.

Now let us consider the universal double covering of BO

$$\begin{array}{c} BSO \\ \downarrow k \\ BO \end{array}$$

We have that, since the map $d\Delta$ is nullhomotopic, $(d\Delta)^*(w_1)=0$, this means $d^*(w_1) \in \operatorname{Ker} \Delta^* = \{0, w_1' + w_1''\}$ since Δ is homotopic to the diagonal.

So $(d\, f)^*(w_1) = 0$ and so there exists d' such that the following diagram

$$\begin{tikzcd} B \arrow[r, "d'"] \arrow[d] & BSO \arrow[d, "k"] \\ BO \times BO \arrow[r, "d"'] & BO \end{tikzcd}$$

commutes.

Now consider the fibre product

$$\begin{tikzcd} Z \arrow[r, "b"] \arrow[d, "\Delta''"'] & BSO^I \times_{BSO} \{b'\} \arrow[d, "\bar{h}"] \\ B \arrow[r] & BSO \end{tikzcd}$$

where b' is chosen such that $k(b') = b$.

<u>Proposition 2</u>. Z is homotopy equivalent to BO^I

<u>Proof</u>. If we consider the two diagrams

$$\begin{tikzcd} & BSO^I \times_{BSO}\{b'\} \arrow[d, "\bar{h}"] \\ B \arrow[r, "d'"'] & BSO \end{tikzcd}$$

and

$$\begin{tikzcd} & BO^I \times_{BO}\{b\} \arrow[d, "h"] \\ BO \times BO \arrow[r, "d"'] & BO \end{tikzcd}$$

using k and d' we can easily define a map from the first to the second, so a map a: $Z \longrightarrow BO^I$ is defined.

Now since the map $BO^I \xrightarrow{\Delta} BO \times BO$ is homotopic to the diagonal, it clearly lifts to a map $BO^I \xrightarrow{\Delta'} B$.

In order to have a map $f: BO^I \longrightarrow BSO \times_{BSO}\{b'\}$ such that the diagram

$$\begin{tikzcd} BO^I \arrow[r, "f"] \arrow[d, "\Delta'"'] & BSO^I \times_{BSO}\{b'\} \arrow[d, "\bar{h}"] \\ B \arrow[r, "d'"'] & BSO \end{tikzcd}$$

commutes, we have to prove that $d'\Delta'$ is nullhomotopic. But now let us choose an homotopy preserving the base points

$$BO^I \times I \xrightarrow[H]{} BO$$

between $d\Delta$ and the constant map.

The obstruction for lifting such a homotopy to a homotopy between $d'\Delta'$ and the constant map lies in

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$$H^1(BO^I \times I, BO^I \times \{0\} \cup BO^I \times \{1\} \cup \{b\} \times I, Z_2) \cong H^0(BO^I, b, Z_2) = 0$$

So d'' is homotopic to the constant map and we can lift it to $BSO^I \times_{BSO} \{b\}$, thus proving the existence of f and getting a map $\tau : BO^I \longrightarrow Z$.

Now it is clear that $a\tau : BO^I \longrightarrow Z \longrightarrow BO^I$ is equal to the identity of BO^I. Viceversa, for $\tau a: Z \longrightarrow BO^I \longrightarrow Z$, we get $\Delta' \tau a \sim \Delta'$ by a homotopy T because both are liftings of the same map $h^2\Delta'$ and, reasoning as before, we have that the obstruction for these maps to be homotopic lies in

$$H^1(Z \times I, Z \times \{0\} \cup Z \times \{1\} \cup \{b''\} \times I) = 0$$

where $b'' \in Z$ is a base-point and all the maps are chosen to be basepoint preserving.

Again if $\bar{T}$ is the homotopy $d'T$ then $\bar{T}$ is clearly nullhomotopic as a map $\bar{T}: Z \times I \longrightarrow BSO$ so it lifts to a $\bar{T}': Z \times I \longrightarrow BSO \times_{BSO^I} \{b\}$.

It follows that, using the homotopy extension theorem, we can define $\bar{T}'$ in such a way that $\bar{T}'/Z \times \{0\} = b\tau a$ and $\bar{T}'/Z \times \{1\} = b$. This implies that using the universal properties of fibre producd we can define a homotopy $\hat{\bar{T}}: Z \times I \longrightarrow Z$ such that $\hat{\bar{T}}/Z \times \{0\} = a$ and $\hat{\bar{T}}/Z \times \{1\} = id$, thus proving the proposition.

q.e.d.

Note. By abuse of language let us identify, from now on, BO^I with Z and, since clearly $\Delta' a \sim \Delta''$ we also identify Δ' with the fibration Δ''.

<u>Definition</u>. $\widetilde{FO}\Psi^q$ is the fibre product

$$\begin{tikzcd}
\widetilde{FO}\Psi^q \arrow[r, "\gamma'"] \arrow[d, "\varphi'"'] & BO^I \arrow[d, "\Delta'"] \\
BO \arrow[r, "(\sigma,\mathrm{id})'"'] & B
\end{tikzcd}$$

By Lemma 1 it is clear that we can extend the definition of Ψ^q to the groups $[Y,BO]$, where Y denotes any space.

Now let Y be a connected space and let $y \in [Y,BO]$ an element such that $\Psi^q(y)=y$ and let $s:Y \longrightarrow BO$ be a map representing y. Choose a basepoint $z \in Y$ such that $s(z)=b$, then have that the map

$$Y \xrightarrow[s]{} BO \xrightarrow[(\sigma,\mathrm{id})']{} B \xrightarrow[d']{} BSO$$

is nullhomotopic by reasoning as in the proof of Proposition 2. So $d'(\sigma,\mathrm{id})'s$ lifts to $BSO^I \times_{BSO} \{b\}$ thus defining a map $Y \xrightarrow{\ '\ } \widetilde{FO}\Psi^q$.

This proves the following:

<u>Lemma 2</u>. If Y is a connected space and $y \in [Y,BO]$ is such that $\Psi^q(y) = y$, then if $s:Y \longrightarrow BO$ represents y, there exists $s':Y \longrightarrow \widetilde{FO}\Psi^q$ such that the diagram

$$\begin{tikzcd}
 & \widetilde{FO}\Psi^q \arrow[d] \\
Y \arrow[ur, "s'"] \arrow[r, "s"'] & BO
\end{tikzcd}$$

commutes.

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2. A first computation of $H^*(\widetilde{FO}\Psi^q, Z_2)$.

From now on, given any space X, $H^*(X)$ will denote the mod 2 cohomology of X.

Lemma 3. For a suitable filtration of the ring $H^*(\widetilde{FO}\Psi^q)$ we have,

$$\mathrm{gr}\, H^*(\widetilde{FO}\Psi^q) = [Z_2\ w_1, w_2 \ldots\ldots] \otimes \Lambda [u_2, u_3, \ldots\ldots]$$

with $\deg(w_i) = i$ and $\deg(u_i) = i-1$.

In particular the Poincaré series of $H^*(\widetilde{FO}\Psi^q)$ is

$$\prod_{i=1}^{\infty} \frac{1+t^i}{1-t^i}$$

Proof. We consider the square

(a)

$$\begin{array}{ccc} \widetilde{FO}\Psi^q & \xrightarrow{\gamma'} & BO^I \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \Delta'} \\ BO & \xrightarrow[(\sigma, id)]{} & B \end{array}$$

of the preceeding paragraph.

In order to apply the result in [9] asseting that, given a fibre square

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

were the vertical lines are fibrations and Y is simply connected, there exists a spectral sequence $\{E_r\} \Longrightarrow H^*(X \times_Y Z)$

such that $E_2 \cong \mathrm{Tor}^{H^*(Y)}(H^*(Z),H^*(X))$, we should have B simply connected; but it is easy to see that the proof in [9] goes over verbatim in the weaker ipothesis that the fibration $X \longrightarrow Y$ is orientable, i.e. if the action of $\pi_1(X)$ over the homology of the fiber is trivial.

The fibration $BO^I \xrightarrow{\Delta} B$ is clearly orientable since it is induced by the fibration $BSO^I \times_{BSO} \{b'\} \longrightarrow BSO$ which has a simply connected base space.

The above discussion implies that we have an Eilenberg-Moore spectral sequence $\{E_r\} \Longrightarrow H^*(\widetilde{FO}\Psi^q)$ with

$$E_2^{s,*} \cong \mathrm{Tor}_{-s}^{H^*(B)}(H^*(BO),H^*(BO^I)).$$

From lemma 1 we have $H^*(B) \cong Z_2[w_1,w_2',w_2'',\dots]$ with $w_1 = f^*(w_1') = f^*(w_1'')$ and $w_i'^{(\prime)} = f^*(w_i'^{(\prime)})$ for each $i \; 2$.

Since q is odd we have already noted that σ^* acts as the identity in cohomology and since Δ' (resp.$(\sigma,id)'$) is a lifting to B of Δ (resp.(id,σ)), we have $(\sigma,id)'^*(w_i') = (\sigma,id)'^*(w_i'') = \Delta'(w_i') = \Delta'(w_i'') = w_i$ for $i \geq 2$ and $(\sigma,id)'^*(w_1) = \Delta'^*(w_1) = w_1$.

This means that $(\sigma,id)'^*$ and Δ'^* define the same $H^*(B)$ module structure on the two isomorphic groups $H^*(BO)$ and $H^*(BO^I)$, and that they are both equal, as $H^*(B)$ modules, to the module $H^*(B)/I$; where I is the ideal generated by $w_i' + w_i''$ for $i \geqslant 2$.

Now let A_1 and A_2 be the two subrings of $H^*(B)$ generated respectively by $w_i' + w_i''$ for $i \geqslant 2$

w_1 and w_i' for $i \geqslant 2$.

We have

$$H^*(B) = A_1 \otimes A_2$$

$$H^*(B)/I = A_2$$

Then, by the Kunneth formula 5, we have :

$$E_2 = \mathrm{Tor}^{A_1 \otimes A_2}(A_2, A_2) = \mathrm{Tor}^{A_1}(Z_2, Z_2) \otimes A_2.$$

Since A_1 is a polinomial algebra with generators in degrees 2,3,......, we have [5]

$$\mathrm{Tor}^{A_1}(Z_2, Z_2) = \bigwedge [u_2, u_3, \ldots\ldots]$$

with $\deg(u_i) = i-1$.

This implies

$$E_2 = Z_2[w_1, w_2, \ldots\ldots] \otimes \bigwedge [u_2, u_3, \ldots\ldots]$$

with $w_i \in E_2^{0,i}$ and $u_i \in E_2^{-1,i-1}$. Since E_2 is generated by elements in $E_2^{0,*}$ and $E_2^{-1,*}$, and on these the differentials are all zero, we get $E_2 = E_\infty$ and hence the result.

q.e.d.

3. A tec^h^nical fact.

All the results in this paragraph are reproduced from [8].

Let X and Y be two spaces, and $f: X \longrightarrow Y$ be a map, and let Cyl f be the mapping cylinder of f. Then Cyl f is homotopy equivalent to Y and, if we put $H^*(\mathrm{Cyl}\ f, X \times \{0\}, G) = H^*(f,G)$ we get an exact cohomology sequence

$$\cdots \longrightarrow H^{i-1}(X,G) \longrightarrow H^i(f,G) \longrightarrow H^i(Y,G) \longrightarrow H^i(X,G) \longrightarrow \cdots$$

with G any group of coefficients.

Let us consider now two maps $f:X \longrightarrow Y$ and $f':X' \longrightarrow Y'$ and a morphism $g:f \longrightarrow f'$, i.e. a pair (g_1,g_2) of maps, $g_1:X \longrightarrow X'$, $g_2:Y \longrightarrow Y'$, such that the following diagram

$$\begin{tikzcd} X \arrow[r, "g_1"'] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g_2"'] & Y' \end{tikzcd}$$

commutes.

We get a morphism of exact sequences

(*)
$$\begin{tikzcd} \cdots \arrow[r] & H^{i-1}(X',G) \arrow[r, "\delta"] \arrow[d, "g_1^*"] & H^i(f',G) \arrow[r, "j'"] \arrow[d, "g^*"] & H^i(Y',G) \arrow[r, "f'^*"] \arrow[d, "g_2^*"] & H^i(X',G) \arrow[r] \arrow[d, "g_1^*"] & \cdots \\ \cdots \arrow[r] & H^{i-1}(X,G) \arrow[r, "\delta"] & H^i(f,G) \arrow[r, "j"] & H^i(Y,G) \arrow[r, "f^*"] & H^i(X,G) \arrow[r] & \cdots \end{tikzcd}$$

This morphism gives rise to a homomorphism

$$D_g : \left\{ \mathrm{Ker}: H^i(Y',G) \xrightarrow{(f'^*,\, g_2^*)} H^i(X',G) \oplus H^i(Y,G) \right\} \longrightarrow \frac{H^{i-1}(X,G)}{f^* H^{i-1}(Y,G) + g_1^* H^{i-1}(X',G)}$$

for each i, obtained by δ (Dg u) = g^* v with j' v = u. Now take a ring as a coefficient group for cohomology so that cup products are defined.

<u>Lemma 4</u>.

(i) Dg is an homomorphism of $H^*(Y')$ - modules, i. e. if $v \in H^i(Y')$ and $u \in \mathrm{Ker}(f'^*, g_1^*)$ we have $Dg(vu) = (-1)^i f^* g_2^* \, v \, Dg\, u$

(ii) If u Ker f'^* and $v \in \mathrm{Ker}\, g_1^*$ then Dg(uv) = 0.

<u>Proof</u>.

(i) is obvious since all the maps in the diagram (*) are $H^*(Y')$ - modules homomorphisms.

(ii) Let x be such that j'x=u. Then j'(x v)=u v and $g^*(x\, v) = g^* x\, g_2^* \, v = 0$, so (ii) is clear.

q.e.d..

4. The Brauer lifting

Let $\bar{k}$ be an algebraic closure of the field with q elements (q odd) k.

Since $\bar{k}^*$ is the union of an expanding sequence of finite cyclic groups, we can define an embedding $\rho : \bar{k}^* \longrightarrow C^*$ where C is the field of complex numbers.

Let G be a finite group and let us consider a finite dimensional representation π of G over $\bar{k}$.

The modular character of π is defined ad the complex valued function

$$\chi_\pi(g) = \sum \rho(\alpha_i)$$

where the set $\{\alpha_i\}$ is the set of eigenvalues counted with multiplocity, of $\pi(g)$.

It is known [4] that the function χ_π is the character of a complex virtual representation, i.e. $\chi_\pi \in R(G)$, the complex representation ring. χ_π is called the Brauer lifting of π.

Now let $R_{\bar{k}}(G)$ be the Grothendiek group of the representations of G over $\bar{k}$. Since the map χ which associates to each representation over $\bar{k}$ its Brauer lifting is clearly additive we get a homomorphism

$$\chi : R_{\bar{k}}(G) \longrightarrow R(G)$$

Now consider an orthogonal representation τ of G. By [7] we have that in this case $\chi\tau \in RO(G)$, the real representation ring of G. Thus, by reasoning as above, if we denote by $RO_{\bar{k}}(G)$ the Grothendiek group of orthogonal representations of G over $\bar{k}$, we get a homomorphism

$$\chi : RO_{\bar{k}}(G) \longrightarrow RO(G).$$

Now it is easy to prove that, if ψ^r denotes the r-th Adams operation in R(G), i.e. the operation which associates to an element $a \in R(G)$ the element $Q_r(\lambda^1(a),\ldots,\lambda^r(a))$ where the λ^is denote the exterior powers of a and Q_k is the Neuton polinomial expressing $t_1^k+\ldots+t_k^k$ in terms of the elementary symmetric functions, we have

$$\psi^r \chi_\pi(g) = \chi_\pi(g^r)$$

for any g G and any representation π of G over $\bar{k}$.

If we consider a representation π of G over k then, by extension of scalars we get a representation $\bar{\pi}$ of G over $\bar{k}$. Since $\bar{\pi}$ comes from π it is clear that the set of eigenvalues $\{\alpha_i\}$ of $\bar{\pi}(g)$ is stable under the action of the Frobenius homomorphism $x \longrightarrow x^q$, for each g G. So, by the above relation we get

$$\psi^q \chi_\pi = \chi_\pi$$

This clearly implies that we get a homomorphism

$$R_k(G) \longrightarrow R(G)^{\psi q}$$

where by $R(G)^{\psi^q}$ we denote the subgroup of $R(G)$ which is fixed under the action of ψ^q , and by $R_k(G)$ the Grothendiek group of representations of G over k.

The same is evidently true for the orthogonal case,thus giving a homomorphism

$$\tilde{\chi} : RO_k(G) \longrightarrow RO(G)^{\psi^q}$$

From now on we shall consider only the orthogonal case.

It is well known that by associating to a real representation π of a finite group G, the corresponding vector bundle over BG we get a map

$$RO(G) \longrightarrow [BG,BO]$$

This map is clearly a homomorphism and is compatible with the action of Adams operations; so it takes $RO(G)^{\psi^q}$ into $[BG,BO]^{\psi^q}$. Composing with the homomorphism $\tilde{\chi} : RO_k(G) \longrightarrow RO(G)^{\psi^q}$, we get a homomorphism

$$RO_k(G) \longrightarrow [BG,BO]^{\psi^q}.$$

Applying lemma 2 we see that we can associate to an element in $RO_k(G)$ a map $BG \longrightarrow \widetilde{FO}\psi^q$.

Remark. The above map $BG \longrightarrow \widetilde{FO}\psi^q$ is not uniquely defined up to homotophy as we will show later, and this is the main problem when one tries to extend to the orthogonal case the proof in [8].

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Now let $k_{(s)}$ be the finite subfield of $\bar{k}$ with q^s elements. We recall $\bar{k}=\bigcup_s k_{(s)}$.

For each s and n let us consider the vector space $k^n_{(s)}$ over $k_{(s)}$ together with the bilinear form $\sum_{i=1}^{m} x_i y_i$ and let $O_n(k_{(s)})$ be the group of isometries of $k^n_{(s)}$ with respect to this bilinear form. Then, since if $s \subseteq s'$ we have that $k_{(s)}$ is a subfield of $k_{(s')}$ and since for $n \leq n'$ we can consider $k^n_{(s)}$ as the subspace of $k^{n'}_{(s)}$ with the last n'-n coordinates equal to zero, we get inclusions $O_n(k_{(s)})$ $O_{n'}(k_{(s')})$ for $n \leq n'$ and $s \leq s'$, which are clearly compatible with one another.

Using this inclusions we define

$$O(\bar{k}) = \bigcup_{(n,s)} O_n(k_{(s)}) .$$

We have,

<u>Proposition 3</u> $\quad [BO(\bar{k}),BO] \cong \varprojlim_{s,n} [BO_n(k_{(s)}),BO]$

<u>Proof</u>. We consider the Milnor construction for the classifying space of a topological group G, we have

$$BG=\bigcup BG^{(m)}, \text{ with } BG^{(m)}= \underbrace{G*G*G* \ldots *G}_{m \text{ times}}/G$$

and $*$ denotes the join operation.

Now by the definition of $BO(\bar{k})$ we have $BO(\bar{k}) = \bigcup_{m,n,s} B^{(m)}_{n,s}$ with $B^{(m)}_{n,s}=(BO_n(k_{(s)}))^{(m)}$.

The Milnor exact sequence in this case gives us the following exact sequence:

$$0 \longrightarrow R^1 \varprojlim_{m,n,s} \left[B^{(m)}_{n,s}, \Omega BO\right] \longrightarrow \left[BO(\bar{k}), BO\right] \longrightarrow \varprojlim_{m,n,s} \left[B^{(m)}_{n,s}, BO\right] \longrightarrow$$

$$\longrightarrow 0.$$

So, in order to prove the proposition we have to show

(1) $R^1 \varprojlim_{m,n,s} \left[B^{(m)}_{n,s}, \Omega BO\right] = 0$

(2) $\varprojlim_{m} \left[B^{(m)}_{n,s}, BO\right] = \left[BO_n(k_{(s)}), BO\right]$.

But (1) follows because, if we fix a couple (n,s) we have [2] that the inverse system $\left\{\left[B^{(m)}_{n,s}, \Omega BO\right]\right\}_m$ with only m varying, is isomorphic as a pro-object to the inverse system

$$(*) \quad \left\{ \begin{matrix} RO(O_n(k_{(s)})) \,/\, R(O_n(k_{(s)})) & & \\ & (I(O_n(k_{(s)})))^m & \\ & & RO(O_n(k_{(s)})) \,/\, R(O_n(k_{(s)})) \end{matrix} \right\}_m$$

(we use the notations in [2]), and we have that this inverse system consists of finite groups. So the entire inverse system $\left\{\left[B^{(m)}_{n,s}, BO\right]\right\}$ is isomorphic to an inverse system of finite groups.

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(2) follows from the Milnor exact sequence

$$0 \longrightarrow R^1 \varprojlim_m \left[B^{(m)}_{n,s}, \Omega BO\right] \longrightarrow \left[BO_n(k_{(s)}), BO\right] \longrightarrow$$

$$\longrightarrow \varprojlim_m \left[B^{(m)}_{n,s}, BO\right] \longrightarrow 0$$

for each couple (n,s), using the isomorphism between the system $\left\{\left[B^{(m)}_{n,s}, \Omega BO\right]\right\}_m$ and the system (∗) .

q.e.d.

If we put $O(k_{(s)}) = \bigcup_n O_n(k_{(s)})$ we get,

<u>Corollary</u>. $\left[BO(k_{(s)}), BO\right] = \varprojlim_n \left[BO_n(k_{(s)}), BO\right]$

for each s.

<u>Proof</u>. It follows immediately by repeating the proof of Proposition 3, considering the system $\left\{\left[B^{(m)}_{n,s}, BO\right]\right\}_{(m,n)}$ instead of the system $\left\{\left[B^{(m)}_{n,s}, BO\right]\right\}_{(m,n,s)}$.

q.e.d.

If we consider the canonical n-dimensional orthogonal representation over $k_{(s)}$ of $O_n(k_{(s)})$ we have already shawed how to associate to such a representation an element of $\left[BO_n(k_{(s)}), BO\right]$, let us call it $\pi^{(s)}_n$.

Further, if we consider the inclusion $O_n(k_s) \subset O_{n'}(k_{(s')})$ for $n \leq n'$, $s \leq s'$, we can associate to this inclusion an element in $\left[BO_n(k_{(s)}), BO_{n'}(k_{(s')})\right]$, let

(2) follows from the Milnor exact sequence

$$0 \longrightarrow R^1 \varprojlim_m \left[B_{n,s}^{(m)}, \Omega BO\right] \longrightarrow \left[BO_n(k_{(s)}), BO\right] \longrightarrow$$

$$\longrightarrow \varprojlim_m \left[B_{n,s}^{(m)}, BO\right] \longrightarrow 0$$

for each couple (n,s), using the isomorphism between the system $\left\{\left[B_{n,s}^{(m)}, \Omega BO\right]\right\}_m$ and the system (✱) .

q.e.d.

If we put $O(k_{(s)}) = \bigcup_n O_n(k_{(s)})$ we get,

<u>Corollary</u>. $\left[BO(k_{(s)}), BO\right] = \varprojlim_n \left[BO_n(k_{(s)}), BO\right]$

for each s.

<u>Proof</u>. It follows immediately by repeating the proof of Proposition 3, considering the system $\left\{\left[B_{n,s}^{(m)}, BO\right]\right\}_{(m,n)}$ instead of the system $\left\{\left[B_{n,s}^{(m)}, BO\right]\right\}_{(m,n,s)}$.

q.e.d.

If we consider the canonical n-dimensional orthogonal representation over $k_{(s)}$ of $O_n(k_{(s)})$ we have already shawed how to associate to such a representation an element of $\left[BO_n(k_{(s)}), BO\right]$, let us call it $\pi_n^{(s)}$.

Further, if we consider the inclusion $O_n(k_s) \subset O_{n'}(k_{(s')})$ for $n \leq n'$, $s \leq s'$, we can associate to this inclusion an element in $\left[BO_n(k_{(s)}), BO_{n'}(k_{(s')})\right]$, let us call it $\pi^{(s,s')}_{(n,n')}$. It follows immediately, by computing the modular chracters, that we have:

$$\pi^{(s)}_{(n)} = \pi^{(s,s')}_{(n,n')}\ \pi^{(s')}_{(n')}$$

as elements of $[BO_n(k_{(s)}),BO]$.

<u>Lemma 5</u> (i) The sequence $\left\{\pi^{(s)}_{(n)}\right\}_{(n,s)}$ defines a unique element $\pi \in [BO(\bar{k}),BO]$.

(ii) The sequence $\left\{\pi^{(s)}_{(n)}\right\}_{(n)}$ defines a unique element $\pi_{(s)} \in [BO(k_{(s)}),\ BO]^{\psi^{q^s}}$ for each s.

<u>Proof</u>. (i) is clear by Proposition 3

(ii)follows from the Corollary and the fact that $\pi^{(s)}_{(n)} \in [BO_n(k_{(s)}),\ BO]^{\psi^{q^s}}$ for each n.

q.e.d.

<u>Note</u>. It is clear by unicity that if $\pi^{(s)} \in [BO(k_{(s)}),BO(\bar{k})]$ denotes the element associated to the inclusion $O(k_{(s)}) \subset O(\bar{k})$, we have $\pi_{(s)} = \pi^{(s)}\pi$.

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5. The elements u_i

In this paragraph k is again a field with q elements.

It is clear that our definition of $O_n(k)$ allows us to identify $O_n(k)$ with the group consisting of n x n invertible matrices with entries in k, with the property $T^{-1} = \tilde{T}$ where $\tilde{T}$ indicates the transpose of a matrix T.

Under this identification let Q(n) be the sugroup of diagonal matrices in $O_n(k)$. Thus Q(n) is the subgroup consisting of matrices with entries 1 or -1 on the diagonal, and 0 elsewhere. Thus Q(n) is a 2 elementary abelian group of rank n.

If we consider the canonical inclusion $i_n: Q(n) \hookrightarrow O_n(k)$ as a representation of Q(n) and we compute its modular character (i.e. the modular character of the representation of Q(n) over $\bar{k}$ we can define by extension of scalars starting from i_n) it is easy to see that such a modular character is equal to the character of the corresponding inclusion $\bar{i}_n$ of Q(n) in O_n as the subgroup of diagonal matrices.

Thus it is clear that the map

$$\chi : RO_k(Q(n)) \longrightarrow [BQ(n), BO]$$

carries i_n into the element j_n of $[BQ(n), BO]$ which corresponds to the n-dimensional vector bundle associated to $\bar{i}_n$.

Now by Lemma 5(ii) we can choose an element

$\pi \in [BO(k), \widetilde{FO}\psi^q]$ such that

$$[\varphi']\pi = \pi_{(1)}$$

as elements of $[BO(k), BO]$. It is clear from the above that, if $\bar{j}_n$ is the element in $[BQ(n), BO(k)]$ associated to the composition of inclusions $Q(n) \subset O_n(k) \subset O(k)$ we have

$$[\varphi']\pi\, \bar{j}_n = j_n$$

In consideration of these facts we get:

<u>Lemma 6</u>

(i) The homomorphism $\varphi'^* : H^*(BO) \longrightarrow H^*(FO\quad)$ is into.

(ii) Let the symmetric group on n letters Σ_n act on the subgroup of diagonal matrices $Q(n)$ by permuting the entries. Then, if $H^*(BQ(n))^{\Sigma_n}$ denotes the subring of $H^*(BQ(n))$ of invariants under the induced action of Σ_n on cohomology, the homomorphism

$(\pi j_n)^* : H^*(\widetilde{FO}\psi^q) \longrightarrow H^*(BQ(n))$ maps $H^*(\widetilde{FO}\psi^q)$ onto $H^*(BQ(n))^{\Sigma_n}$, for each n.

<u>Proof</u>. (i) Since j_n comes from the representation $j \times \bar{i}_n$ it is well know that $j_n^* : H^t(BO) \longrightarrow H^t(BQ(n))$ is injective for $t \leq n$ for each n. Since for each n we have $[\varphi']\pi\, j_n = j_n$, (i) follows.

(ii) It is known that for each n j_n^* maps $H^*(BO)$ onto $H^*(BQ(n))^{\Sigma_n}$ so it will be sufficient to prove

$$\mathrm{Im}(\bar{\pi}\,\bar{j}_n)^* \subset H\,(BQ(n))^{\Sigma_m}$$

But this follows at once because, if $N(Q(n))$ denotes the normalizer of $Q(n)$ in $O_n(k)$, we have $N(Q(n))/Q(n) \cong \Sigma_m$ and Σ_m acts on $Q(n)$ exactly by permuting the entries in the diagonal.

q.e.d.

Let us consider now, for each $t \geq 2$ the elements $\{w'_t + w''_t\}$ in $H^*(B)$ (B has the same meaning as in paragraph 1). We have $(w'_t + w''_t) \in \mathrm{Ker}((\sigma,\mathrm{id})'^*, \Delta'^*)$ so by paragraph 3, we can define, by considering the couple of maps $(\gamma', (\sigma,\mathrm{id})')$ in the square

$$\begin{tikzcd} FO\psi^q \arrow[r, "\gamma'"] \arrow[d, "\varphi"'] & BO^I \arrow[d, "\Delta'"] \\ BO \arrow[r, "(\sigma{,}\mathrm{id})'"'] & B \end{tikzcd}$$

as a map $\Gamma : \varphi' \longrightarrow \Delta'$, the element

Ø $\tilde{u}_t = D_\Gamma(w'_t + w''_t) \quad H^{t-1}(\widetilde{FO}\psi^q) / \varphi'^* H^{t-1}(BO)$,

since it follows from the fact that Δ'^* and $(\sigma,\mathrm{id})'^*$ are onto that $\mathrm{Im}\,\varphi'^* = \mathrm{Im}\,\gamma'^*$.

But it is clear by Lemma 6 that there is only one element in the lateral class u_t which is in the kernel of $(\bar{\pi}\, j_t)^*$.

So we can give the following,

<u>Definition</u> For each t the elements $\bar{\pi}\, u_t \in H^{t-1}(\widetilde{FO}\psi^q)$, $t \geq 2$, are defined as the unique elements in the lateral classes $\tilde{u}_t$ such that $(\bar{\pi}\, j_t)^*(u_t) = 0$.

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Remarks

(1) It is obvious to verify that, for each $t' \geq t$, $_{\bar{\pi}}u_t \in Ker(\bar{\pi} j_{t'})^*$ and that $_{\bar{\pi}}u_t$ is the unique element in the class u_t with this property.

(2) By putting a subscript $\bar{\pi}$ under $\tilde{u}_t$ we want to enfasize the fact that the construction of the $\{_{\bar{\pi}}u_t\}$ depends on the choice of $\bar{\pi}$.

(3) We have defined the $\{_{\bar{\pi}}u_t\}$ in $H^*(\widetilde{FO}\psi^q)$ only when q is the order of a finite field of. odd characte ristic (i.e. $q=p^a$ for some odd prime p).

The case of any odd integer can we treated in the same way since the role of O(k) in the above discussion is irrelevant, because we could have studied directly the elements in $[BQ(n), \widetilde{FO}\psi^q]$,which again are not uniquely defined, that arise in any case from $j_n \in [BQ(n), BO]$, j_n depending only by the diagonal representation of Q(n) in O(n).

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6. Multiplicative formulas.

Again let k be the field with q elements and let $\bar{k}$ be its algebraic closure.

Let $O_n(\bar{k})$ be the n-th orthogonal group of the vector space $\bar{k}^n$ with bilinear form $\sum_{i=1}^{n} x_i y_i$

If $k_{(s)}$ are defined, for each s, as in paragraph 4 we clearly have $O_n(\bar{k}) = \bigcup_{s=1}^{\infty} O_n(k_{(s)})$.

Let $x \longrightarrow x^q$ be the q-th Frobenius automorphism in $\bar{k}$, and let $\bar{F}_n : O_n(\bar{k}) \circlearrowleft$ be the automorphism of $O_n(\bar{k})$ defined by

$$\bar{F}_n(a_{ij}) = (a_{ij}^q)$$

where $(a_{ij}) = A$ denotes an $n \times n$ matrix in $O_n(\bar{k})$.

If $G_n \subset O_n(\bar{k}) \times O_n(\bar{k})$ is the kernel of the homomorphism $d : O_n(\bar{k}) \times O_n(\bar{k}) \longrightarrow \{-1,1\}$ defined as $d(A,B) = \det A \det B$, let $\bar{\Delta}_n : O_n(\bar{k}) \longrightarrow G_n$ be the homomorphism defined as $\bar{\Delta}_n(A) = (A,A)$, and let $\bar{F}'_n : O_n(\bar{k}) \longrightarrow G_n$ be the homomorphism defined as $\bar{F}'_n(A) = (\bar{F}_n(A), A)$.

Now let us consider a map $\Delta_n : BO_n(\bar{k}) \longrightarrow BG_n$ (resp. $F_n : BO_n(\bar{k}) \longrightarrow BG_n$) representing the element in $[BO_n(\bar{k}), BG_n]$ associated to $\bar{\Delta}_n$ (resp. $\bar{F}'_n$). Further, since Δ_n is an inclusion let us choose Δ_n to be a fibration with fiber $G_n / O_n(\bar{k})$.

We define X_n to be the fibre product

$$\begin{tikzcd} X_n \arrow[r, "\gamma_n"] \arrow[d, "\varphi_n"'] & BO_n(\bar{k}) \arrow[d, "\Delta_n"] \\ BO_n(\bar{k}) \arrow[r, "F_n"'] & BG_n \end{tikzcd} .$$

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<u>Proposition 4.</u> $\pi_i(X_n)=0$ if $i\neq 1$ $\pi_1(X_n)=O_n(k)$. Thas X_n is a classifying space for $O_n(k)$.

<u>Proof.</u> Let us take basepoints in $BO_n(\bar{k})$ and BG_n so that F_n and Δ_n are based maps (this is possible since we can vary F_n up to homotopy).

It follows that also φ_n and γ_n can be considered as basepoint preserving maps.

Now since Δ_n is a fibration we have that the map $\partial:\pi_1(BG_n)\longrightarrow\pi_0(G_n/O_n(\bar{k}))$ is just the map which assigns to an element $(A,B)\in G_n=\pi_1(BG_n)$ its left lateral class modulo $O_n(\bar{k})$.

But, given an element (A,B) G_n, we have that

$$\partial(A,B)=\partial(AB^{-1},1)\ .$$

So ∂ factors through the map $\underline{\partial}:G_n\longrightarrow SO_n(\bar{k})$, which assigns to each $(A,B)\in G_n$ the element $AB^{-1}\in SO_n(\bar{k})$ and the map $\underline{\underline{\partial}}:SO_n(\bar{k})\longrightarrow G_n/O_n(\bar{k})$ which assigns to each $A\in SO_n(\bar{k})$ the lateral class $[(A,1)]\in G_n/O_n(\bar{k})$.

The map $\underline{\underline{\partial}}$ is clearly bijective.

Now let us consider the map of homotopy exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_1(X_n) \arrow[r] \arrow[d, "\gamma_{n\#}"] & \pi_1(BO_n(\bar{k})) \arrow[r] \arrow[d, "F_{n\#}"] & \pi_0(G_n/O_n(\bar{k})) \arrow[r] \arrow[d, "id"] & \pi_0(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(BO_n(\bar{k})) \arrow[r] & \pi_1(BG_n) \arrow[r] & \pi_0(G_n/O_n(\bar{k})) \arrow[r] & 0 &
\end{tikzcd}$$

We have $F_{n\#}=\bar{F}_n:\pi_1(BO_n(\bar{k}))\cong O_n(\bar{k})\longrightarrow\pi_1(BG_n)=G_n$.

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Since $\varphi_{n\#}$ is injective, we have that $\pi_1(X)$ is isomorphic to the subgroup of $O_n(\bar{k})$ which is mapped by $\bar{F}_n$ into the kernel of ∂. But this is exactly the subgroup of matrices $A \in O_n(\bar{k})$ such that $F_n(A)=A$ i.e. the subgroup of the matrices with entries in k, $O_n(k)$.

So we have proved $\pi_1(X_n) \cong O_n(k)$.

Since we have $\partial' = \partial F_{n\#}$ and since $\partial = \underline{\underline{\partial}}\,\underline{\partial}$ and $\underline{\underline{\partial}}$ is bijective it is sufficient to prove that $\underline{\partial} F_{n\#} : O_n(\bar{k}) \longrightarrow SO_n(\bar{k})$ is onto.

We have

$$\underline{\partial} F_{n\#}(A) = A^F A^{-1}$$

and $\partial_n F_n$ is onto since, by the Lang isomorphism [6], the restriction of $\underline{\partial} F_{n\#}$ to $SO_n(\bar{k})$ is onto.

So we have proved $\pi_0(X_n)=0$.

Now $\pi_i(X_n)=0$ for $i \geqslant 2$ follows from the homotopy exact sequence of the fibration

$$\begin{array}{c} G_n/O_n(\bar{k}) \\ \downarrow \\ X_n \\ \downarrow \\ BO_n(\bar{k}) \end{array}$$

since $\pi_i(G_n\ O_n(\bar{k})) = 0$ for $i \geqslant 1$ and $\pi_i(BO_n(\bar{k}))=0$ for $i \geqslant 2$.

q.e.d.

Since X_n is a classifying space for $O_n(k)$ we shall

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denote it by $BO_n(k)$.

Now let us consider the groups $O(\bar{k})$, $O(k)$ which have already been defined, and $G = \bigcup_{n} G_n$. Clearly $O(\bar{k}) = \bigcup_{n} O_n(\bar{k})$. Since the F_n's are compatible we can define an homomorphism $\tilde{F}: O(\bar{k}) \longrightarrow O(\bar{k})$ by taking $\tilde{F} = \bigcup_{n} \bar{F}_n$ and also an homomorphism $\bar{F}': O(\bar{k}) \longrightarrow G$ which is the union of the $\left\{ \bar{F}'_n \right\}$.
Similarly we can define the homomorphism $\bar{\Delta} = \bigcup_{n} \bar{\Delta}_n : O(\bar{k}) \longrightarrow G$. Now let us denote by $\Delta : BO(\bar{k}) \longrightarrow BG$ the fibration induced by $\bar{\Delta}$ with fiber $G/O(\bar{k})$, and let $F: BO(\bar{k}) \longrightarrow BG$ a map in the homotopy class of $[BO(\bar{k}), BG]$ induced by $\bar{F}'$.

We define X to be the fiber product

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BO(\bar{k}) \arrow[d, "\Delta"] \\ BO(\bar{k}) \arrow[r, "F"'] & BG \end{tikzcd}$$

It follows immediately from Proposition 4, by passing to the limit that X is a classifying space for $O(k)$. In view of this we shall denote X by $BO(k)$.

Now let us considered the element $\pi \in [BO(k), BO]$ defined in paragraph 5. We have

Theorem (Quillen)7.
$$H^*(BO(\bar{k})) \cong Z_2[\bar{w}_1, \bar{w}_2, \ldots] \quad \text{where } \bar{w}_i = \pi^*(w_i).$$

Using this theorem we get:

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Proposition 5.

$$H^*(BG) \cong H^*(BO(\bar{k}) \times BO(\bar{k})) \Big/ (\bar{w}_1' + \bar{w}_1'')$$

whith $\bar{w}_i'^{(\prime\prime)} = pr_{1(2)}(\bar{w}_i)$, where pr_i is the projection of $BO(\bar{k}) \times BO(\bar{k})$ on the i-th factor ($i = 1,2$).

Proof. It follows verbatim from the proof of Proposition 1.

q.e.d.

Now let us consider the element $\eta \in [BG, BO \times BO]$ defined as $\bar{\eta} = (\pi \times \pi)\alpha$ where $\alpha \in [BG, BO(\bar{k}) \times BO(\bar{k})]$ denotes the element associated to the inclusion of G into $O(\bar{k}) \times O(\bar{k})$.

If we take a map $\bar{e}: BG \longrightarrow BO \times BO$ in the homotopy class $\bar{\eta}$ the proposition gives us the existence of $e: BG \longrightarrow B$ such that the diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ BG \arrow[ur, "e"] \arrow[r, "\bar{e}"'] & BO \times BO \end{tikzcd}$$

where f denotes as in paragraph 1 the double covering, commutes.

It also follows, since the Brauer lifting is additive, that, if we consider the composite map $e\Delta: BO(\bar{k}) \longrightarrow B$, then there exists a map $a: BO(\bar{k}) \longrightarrow BO^I$ such that the following diagram

$$\begin{tikzcd} & BO^I \arrow[d, "\Delta'"] \\ BO(\bar{k}) \arrow[ur, "a"] \arrow[r, "e\Delta"'] & B \end{tikzcd}$$

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Δ' being the fibration with fiber Ω BSO, commutes. Thus we can define the following commutative diagram

$$(*)\qquad \begin{array}{ccccc} BO(k) & \xrightarrow{\ \gamma\ } & BO(\bar{k}) & \xrightarrow{\ a\ } & BO^I \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta'} \\ BO(\bar{k}) & \xrightarrow[F]{\quad} & BG & \xrightarrow[e]{\quad} & B \end{array} .$$

Now let K be a finite group and let ε be an n-dimensional orthogonal representation of K over $\bar{k}$. By using the same notations of paragraph 4, it is easy to see, by direct computation,

$$\psi^q \chi_\varepsilon(g) = \chi_{F_n \varepsilon}(g)$$

for any $g \in K$. This and Proposition 3 clearly imply that

$$[eF] = [(\sigma,id)'_{\pi}]$$

as elements of $[BO(\bar{k}),B]$. Thus we can choose a homotopy $H_t : BO(\bar{k}) \times I \longrightarrow B$ such that $H_0 = eF$ and $H_1 = (\sigma,id)'$ where $\tilde{\pi}$ is a representative for the class $\pi \in [BO(\bar{k}),BO]$.

If we apply H_t it follows, by the covering homotopy theorem that there exists a homotopy $H'_t : BO(k) \times I \longrightarrow BO^I$ such that $H'_0 = a\gamma$ and H'_t covers H_t for each $t \in I$. At the end of these homotopies the diagram $(*)$ will be transformed into the diagram

$$\begin{array}{ccccc} BO(k) & \xrightarrow{\ \tilde{\pi}\ } & FO & \xrightarrow{\ \gamma'\ } & BO^I \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\Delta} \\ BO(\bar{k}) & \xrightarrow[\tilde{\pi}]{\quad} & BO & \xrightarrow[(\sigma,id)]{\quad} & B \end{array}$$

in fact it follows immediately by the universal property

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of fiber product that H_1' factors through γ'.

It follows from lemma 5 and the note under it, that using the notation of paragraph 4, $[\varphi'\tilde{\pi}] = [\tilde{\pi}\varphi] = \pi_{(1)}$

So we have that we can define the elements ${}_{[\tilde{\pi}]}u_i$ $H^{i-1}(\widetilde{FO}\psi^q)$, for each $i \geq 2$.

Note. Since from now on we shall consider only the elements ${}_{\tau}u_i$ with $\tau = [\tilde{\pi}]$ we shall put ${}_{[\tilde{\pi}]}u_i = u_i$.

Now let us take up the notations of paragraph 5, we have:

Lemma 7.

(i) the homomorphism $\varphi^*: H^*(BO(\bar{k})) \longrightarrow H^*(BO(k))$ is into.

(ii) The homomorphism $j_n^*: H^*(BO(k)) \longrightarrow H^*(BQ(n))$ maps $H^*(BO(k))$ onto $H^*(BQ(n))^{\varepsilon\sim}$.

Proof. By the theorem, the proof proceeds exactly as the proof of Lemma 6.

q.e.d.

Now, by reasoning as in paragraph 5 we can define, for each $t \geq 2$, the elements $\bar{u}t \in H^{t-1}(BO(k))$ as the unique elements in the lateral class $D_\Gamma(\bar{w}'_{t+1} + \bar{w}''_{t+1})$ such that $j_t^*(\bar{u}_t) = 0$, where we put $\Gamma : \varphi \longrightarrow \Delta$ equal to the couple of maps (γ, F).

Since from the construction of $\tilde{\pi}$ and from the fact that Dg clearly depends on tje homotopy class of g, it follows that $D_\Gamma e^* = \tilde{\pi}^* D_{\Gamma'}$ as maps from $\ker((\sigma, id)'^*, \Delta'^*)$ to Coker (φ^*).

Lemma 8: $\tilde{\pi}^*(u_t) = \bar{u}_t$.

Proof. The lemma is an immediate consequence of the definition of the u_t's and $\bar{u}_t$'s and of the relation $D_{\Gamma}\, e^* = \tilde{\bar{\pi}}^*\, D_{\Gamma'}$.

q.e.d.

Now let us consider the homomorphism $m : O(\bar{k}) \times O(\bar{k}) \longrightarrow$ $\longrightarrow O(\bar{k})$ defined as the union of the direct sum homomorphism $m_{(n,t)} : O_n(\bar{k}) \times O_t(\bar{k}) \longrightarrow O_{n+t}(\bar{k})$. By the definition of G_n we have that, if we consider the restriction $v_{(n,t)}$ of the homomorphism $m_{(n,t)} \times m_{(n,t)}$: : $O_n(\bar{k}) \times O_t(\bar{k}))^2 \longrightarrow (O_{n+t}(\bar{k}))^2$ to the subgroup $G_n \times G_t$ we get $\mathrm{Im}\, v_{(n,t)} \subset G_{n+t}$.

Further it is immediate to verify that the following diagram

$$\begin{tikzcd}
O_n(\bar{k}) \times O_t(\bar{k}) \arrow[r, "m_{(n,t)}"] \arrow[d, "F'_n \times F'_t"] & O_{n+t}(\bar{k}) \arrow[d, "F'_{n+t}"] \\
G_n \times G_t \arrow[r, "v_{(n,t)}"'] & G_{n+t}
\end{tikzcd}$$

commutes.

So this implies that the diagram

$$\begin{tikzcd}
O(\bar{k}) \times O(\bar{k}) \arrow[r, "m"] \arrow[d, "\bar{F}'"] & O(\bar{k}) \arrow[d, "\bar{F}'"] \\
G \times G \arrow[r, "v"'] & G
\end{tikzcd}$$

where $v : G \times G \qquad G$ is defined as the union of the $v_{(n,t)}$'s, commutes.

By taking representatives for the homotopy classes of maps induced by the homomorphisms in the above diagram we get a diagram

(τ_1)

$$\begin{tikzcd} BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "\tilde{m}"] \arrow[d, "F \times F"] & BO(\bar{k}) \arrow[d, "F"] \\ BG \times BG \arrow[r, "\tilde{v}"] & BG \end{tikzcd}$$

which is commutative up to homotopy.

Similarly we get the homotopy commutative diagram

(τ_2)

$$\begin{tikzcd} BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "\tilde{m}"] \arrow[d, "\Delta \times \Delta"] & BO(\bar{k}) \arrow[d, "\Delta"] \\ BG \times BG \arrow[r, "\tilde{v}"] & BG \end{tikzcd}$$

Since in this case we have chosen Δ to be a fibration we can make (τ_2) into a commutative diagram by the covering homotopy theorem. So, from now on we fix $\tilde{m}$ and $\tilde{v}$ in such a way that (τ_2) is commutative.

Now let us consider the diagram

$(\Omega$

$$\begin{tikzcd} BO(k) \times BO(k) \arrow[r, "\gamma \times \gamma"] \arrow[d, "\varphi \times \varphi"] & BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "\tilde{m}"] \arrow[d, "\Delta \times \Delta"] & BO(\bar{k}) \arrow[d, "\Delta"] \\ BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "F \times F"'] & BG \times BG \arrow[r, "\tilde{v}"] & BG \end{tikzcd}$$

which is commutative by the above discussion; and let us choose a homotopy $H_t : BO(\bar{k}) \times BO(\bar{k}) \times I \longrightarrow BG$ such that $H_0 = \tilde{v}(F \times F)$ and $H_1 = F\,\tilde{m}$. By the covering homotopy theorem there exists a homotopy $L_t : BO(k) \times BO(k) \times I \longrightarrow BO(\bar{k})$ covering H_t. So, at the end of these homotopies, the above diagram will be transformed in the commutative diagram

$$\begin{tikzcd} BO(k) \times BO(k) \arrow[r, "\mu"] \arrow[d, "\varphi \times \varphi"] & BO(k) \arrow[r, "\gamma"] \arrow[d, "\varphi"] & BO(\bar{k}) \arrow[d, "\Delta"] \\ BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "m"'] & BO(\bar{k}) \arrow[r, "F"'] & BG \end{tikzcd}$$

where $L_1 = \gamma\mu$ by the universal property of fibre product.

Lemma 9. $\mu : BO(k) \times BO(k) \longrightarrow BO(k)$ represents the homomorphism defined as the union of the direct sum homomorphism $\mu_{(n,t)} ; O_n(k) \times O_t(k) \longrightarrow O_{n+t}(k)$.

Proof. Since $\varphi\mu = \tilde{m}(\varphi \times \varphi)$ and we have seen that φ represents the inclusion $O(k) \subset O(\bar{k})$ we must have that μ must represent the restriction to $O(k) \times O(k)$ of the homomorphism m, thus proving the lemma.

q.e.d.

Let us return to the diagrams (Ω_1) and (Ω_2). Since, as we have already noticed, the homomorphism D_g depends only by the homotopy class of g, we have

$$(*) \qquad \mu^* D_\Gamma = D_{\Gamma \times \Gamma} \tilde{v}^* .$$

Now let us consider the canonical projections of the square

$$\begin{tikzcd}
BO(k) \times BO(k) \arrow[r, "\gamma \times \gamma"] \arrow[d, "\varphi \times \varphi"] & BO(\bar{k}) \times BO(\bar{k}) \arrow[d, "\Delta \times \Delta"] \\
BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "F \times F"'] & BG \times BG
\end{tikzcd}$$

onto the square

$$\begin{tikzcd}
BO(k) \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BO(\bar{k}) \arrow[d, "\Delta"] \\
BO(\bar{k}) \arrow[r, "F"'] & BG
\end{tikzcd}$$

If we denote by $x \otimes 1$ (resp. $1 \otimes x$) the image of an element of $H^*(X)$, X is any space in the above square, in $H^*(X \times X)$ under the cohomology homomorphism induced by the first (resp. the second) projection, we get, by the functoriality of D_g, that $D_{\Gamma^2}(y\ 1) = D_\Gamma(y) \otimes 1$ for $y \in H^*(BG)$, and similarly for $1 \otimes y$.

<u>Lemma 10</u>.

$$D_{\Gamma^2}((\bar{w}_i \otimes \bar{w}_j)' + (\bar{w}_i \otimes \bar{w}_j)'') =$$

$$= (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes (\varphi^* \bar{w}_j) + (\varphi^* \bar{w}_i) \otimes (D_\Gamma(\bar{w}_j' + \bar{w}_j'')) \qquad \text{for } i,j > 2$$

$$= (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes (\varphi^* \bar{w}_1) \qquad \text{for } j=1, i > 2$$

$$= 0 \qquad \text{for } j=i=1$$

<u>Proof</u>. From what we have noticed above, it follows

$$D_{\Gamma^2}((\bar{w}_i' + \bar{w}_i'') \otimes 1) = (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes 1$$

and similarly for $1 \otimes (\bar{w}_i' + \bar{w}_i'')$.

Since

$$(\bar{w}_i \otimes \bar{w}_j)' + (\bar{w}_i \otimes \bar{w}_j)'' = (\bar{w}_i' + \bar{w}_i'') \otimes \bar{w}_j' + \bar{w}_i'' \otimes (\bar{w}_j' + \bar{w}_j'')$$

for $i,j \geqslant 2$,

we have by Lemma 4:

$$D_{\Gamma}^{2}((\bar{w}_i \otimes \bar{w}_j)' + (\bar{w}_i \otimes \bar{w}_j)'') = (D_{\Gamma}(\bar{w}_i' + \bar{w}_i'')) \otimes \bar{w}_j' +$$

$$+ \bar{w}_i'' \otimes (D \quad (\bar{w}_j' + \bar{w}_j'')) = (D_{\Gamma}(\bar{w}_i' + \bar{w}_i'')) \otimes (\varphi^* \bar{w}_j) +$$

$$+ (\varphi^* \bar{w}_i) \otimes (D_{\Gamma}(\bar{w}_j' + \bar{w}_j'')).$$

Now suppose $j=1$ $i \geqslant 2$.

Then by Proposition 5, $\bar{w}_1' = \bar{w}_1'' = \bar{w}_1$. So,

$(\bar{w}_i \otimes \bar{w}_1)' + (\bar{w}_i \otimes \bar{w}_1)'' = (\bar{w}_i' + \bar{w}_i'') \otimes \bar{w}_1$, then by Lemma 4:

$$D_{\Gamma}^{2}((\bar{w}_i' + \bar{w}_i'') \otimes \bar{w}_1) = (D_{\Gamma}(\bar{w}_i' + \bar{w}_i'')) \otimes \varphi^*(\bar{w}_1).$$

Finally if $i=j=1$,

$$(\bar{w}_1 \otimes \bar{w}_1)' = (\bar{w}_1 \otimes \bar{w}_1)''$$

and so the proof of the Lemma is complete.

q.e.d.

Let us recall that the Brauer lifting defines a map $RO_{\bar{k}}(G) \longrightarrow [BG, BO]$, for each finite group G. This, together with the definition of $m: O(\bar{k}) \times O(\bar{k}) \longrightarrow O(\bar{k})$ and Proposition 3 implies that the square

$$\begin{tikzcd} BO(\bar{k}) \times BO(\bar{k}) \arrow[r, "\tilde{m}"] \arrow[d, "\pi \times \pi"'] & BO(\bar{k}) \arrow[d, "\pi"] \\ BO \times BO \arrow[r, "s"'] & BO \end{tikzcd}$$

where s is a map representing addition in $\widetilde{KO}$, is homotopy commutative.

This implies:

Proposition 6. $\widetilde{m}^*(\bar{w}_i) = \sum_{k+j=i} \bar{w}'_k \otimes \bar{w}''_j$.

Proof. By the know multiplicative formulas for Stiefel-Witney classes we have

$$s^*(w_i) = \sum_{k+j=i} w_k \otimes w_j ,$$

and by the above diagram

$$\widetilde{m}^*(\bar{w}_i) = \widetilde{m}^*(\pi^*(w_i)) = (\pi \times \pi)^*(s^*(w_i)) .$$

So, we have

$$\widetilde{m}^*(\bar{w}_i) = (\pi \times \pi)^*\Big(\sum_{k+j=i} w_k \otimes w_j\Big) = \sum_{k+j=i} \bar{w}_k \otimes \bar{w}_j .$$

q.e.d.

We are now ready to prove:

Proposition 7

$$\mu^*(\bar{u}_i) = \sum_{a+b=i} \bar{u}_a \otimes (\varphi^* \bar{w}_b) + (\varphi^* \bar{w}_a) \otimes \bar{u}_b$$

for each $i \geq 2$, where we put $u_1 = u = 0$.

Proof. If we consider the image of $\mu^*(\bar{u}_i)$ modulo $\mathrm{Im}(\varphi \times \varphi)^*$ we get:

$$\mu^*(\bar{u}_i) = \mu^*(D_\Gamma(\bar{w}'_i + \bar{w}''_i)) = D_{\Gamma^2}(\tilde{v}^*(\bar{w}'_i + \bar{w}''_i))$$

by $(*)$.

But,

$$D_{\Gamma^2}(\tilde{v}^*(\bar{w}'_i + \bar{w}''_i)) = D_{\Gamma^2}\Big(\Big(\sum_{a+b=i} (\bar{w}_a \otimes \bar{w}_b)'\Big) + \Big(\sum_{a+b=i} (\bar{w}_a \otimes \bar{w}_b)''\Big) =$$

$= D_{\Gamma^2}(\sum_{a+b=i}((\bar{w}_a \otimes \bar{w}_b)' + (\bar{w}_a \otimes \bar{w}_b)'') =$

$= \sum_{a+b=i}((D\,(\bar{w}_a' + \bar{w}_a''))\otimes(\varphi^* \bar{w}_b) + (\varphi^* \bar{w}_a)\otimes(D_\Gamma(\bar{w}_b' + \bar{w}_b''))) ,$

by Lemma 9, whith $D_\Gamma(\bar{w}_a' + \bar{w}_a'') = 0$ when $a=0,1$.

Now by the definition of $\bar{u}_i$ it follows that $\mu^*(\bar{u}_i)$ must be the only element in the lateral class $D_{\Gamma^2}(\hat{v}^*(\bar{w}_i' + \bar{w}_i''))$ which lies in $\mathrm{Ker}(j_i \times j_i)^*$; in fact it is clear that $v(j_i \times j_i)$ is homotopic to j_{2i}.

But this element is clearly just

$\sum_{a+b=i}(u_a \otimes (\varphi^* \bar{w}_b) + u_b \otimes (\varphi^* \bar{w}_a))$ with $u = u_1 = 0$,

thus proving the proposition

q.e.d.

<u>Corollary</u>.

$$(\widetilde{\pi}\, v)^*(u_i) = \sum_{a+b=i}(\bar{u}_a \otimes (\varphi^* \bar{w}_b) + u_b \otimes (\varphi^* \bar{w}_a)) .$$

7. The algebra $H^*(\widetilde{FO\psi^3})$

From now on we put $\bar{w}_i = \varphi^*(\bar{w}_i)$ and $w_i = \varphi'^*(w_i)$, for each i.

So we can write the multiplicative formulas of the preceeding paragraph as

$$\mu^* (\bar{w}_i) = \sum_{a+b=i} w_a \otimes w_b$$

$$\mu^* (\bar{u}_i) = \sum_{a+b=1} (\bar{u}_a \otimes \bar{w}_b + \bar{w}_a \otimes \bar{u}_b)$$

with $u_1 = u = 0$.

Now let us introduce indeterminates t,s with $s^2=0$. If we put

$$\bar{w}_{ts} = 1 + \sum_{i \geq 1} \bar{w}_i\, t^i + \bar{u}_i\, t^{i+1} s \qquad (u_1 = 0)$$

we can rewrite our multiplicative formulas as

$$\mu^* (\bar{w}_{ts}) = \bar{w}_{ts} \otimes \bar{w}_{ts} \text{ .}$$

Now let k be the field with q elements with the restriction q = 4m+1, and let us consider the group $O_2(k)$.

It is easy to see that this group is a diedral group with 2(q-1) elements and it is known [7] that

$$H^*(O_2(k)) \cong Z_2 \left[x_1, x_2, l\right] / (l^2 + lx_1)$$

with deg x_1=deg l = 1 and deg x_2 = 2, and with $\varphi_2(\bar{w}_i) = x_i$, i=1,2.

<u>Proposition 8</u>. If $f\in[BO_2(k),BO(k)]$ is the homotopy class associated to the canonical inclusion of $O_2(k)$ in $O(k)$ then:

(i) If A is the subalgebra of $H^*(O_2(k))$ generated by $x_1,x_2,f^*(\bar{u}_2)$,we have $A=H^*(O_2(k))$. In particular $f^*(u_2)\neq 0$.

(ii) $f^*(\bar{w}_i)=f^*(\bar{u}_i)=0$,for $i\geqslant 3$.

<u>Proof.</u>

£ (i) Let us consider the two squares

$$\begin{tikzcd} BO(k) \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BO(\bar{k}) \arrow[d, "\Delta"] \\ BO(\bar{k}) \arrow[r, "F"'] & BG \end{tikzcd}$$

and

$$\begin{tikzcd} BSO_2(k) \arrow[r, "\bar{\gamma}_2"] \arrow[d, "\bar{\varphi}_2"'] & BSO(\bar{k}) \arrow[d, "\bar{\Delta}_2"] \\ BSO_2(\bar{k}) \arrow[r, "(\bar{F},id)"'] & BSO_2(\bar{k}) \times BSO_2(\bar{k}) \end{tikzcd}$$

where the second is defined in exactly the same way as the corresponding square for $O_2(k)$, $\bar{F}$ denotes a map induced by the homomorphism $F:SO_2(\bar{k})\longrightarrow SO_2(\bar{k})$ defined using the Frobenius homomorphism.

By using the same methods of the preceeding paragraph, it is easy to see that, if $f\in[BSO_2(k),BO(k)]$ denotes the homotopy class corresponding to the canonical inclusion of $SO_2(k)$ in $O(k)$ and $\tilde{\tilde{f}} \in [BSO_2(\bar{k}) \times BSO_2(\bar{k}), BG]$ denotes the homotopy class corresponding to the canonical inclusion of $SO_2(\bar{k}) \times SO_2(\bar{k})$ in G, we have:

$$\tilde{f}^* D_{\Gamma} = D_{\Gamma_2} \tilde{\tilde{f}}$$

where $\Gamma_2 = (\chi_2, (\bar{F}, id))$ and Γ has its usual meaning.

Now it is known $SO_2(\bar{k}) \cong \bar{k}^*$ and, since $\bar{k}^*$ is a union of an expanding sequence of finite cyclic groups of order prime to char $\bar{k}$ and since the relevant Bocksteins are all zero $H^*(SO_2(\bar{k}), C) \cong C[x]$ with deg $x = 2$, where C is any finite cyclic group of order prime to char $\bar{k}$. In particular if $C = Z_2$, it follows immediately from the theorem in paragraph 6 that, if $\bar{f} \in [BSO_2(\bar{k}), BO(\bar{k})]$ is the homotopy class induced by the canonical inclusion of $SO_2(\bar{k})$ in $O(\bar{k})$, then $x = \bar{f}^*(\bar{w}_2)$.

Now let us take coefficients in $Z/h(q-1)$, where $h \geqslant 1$ is an integer prime to $q-1$ and to char $\bar{k}$, and let us consider the following map of exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(BSO_2(\bar{k}), Z/h(q-1)) \arrow[r, "\delta"] \arrow[d, "\bar{\gamma}_2^*"] & H^2(\ _2, Z/h(q-1)) \arrow[r, "\tau"] \arrow[d, "\Gamma_2^*"] & H^2((BSO_2(\bar{k}))^2, Z/h(q-1)) \arrow[d, "(F, id)^*"] \\
\cdots \arrow[r] & H^1(BSO_2(k), Z/h(q-1)) \arrow[r, "\delta"] & H^2(\ _2', Z/h(q-1)) \arrow[r, "\tau'"] & H^2(BSO_2(\bar{k}), Z/h(q-1))
\end{tikzcd}$$

We have that, if we put $x'\,'' = pr_{1(2)}(x)$ where pr_i $(i=1,2)$ denotes the i-th canonical projection $BSO_2(\bar{k}) \times BSO_2(\bar{k}) \longrightarrow BSO_2(\bar{k})$, $\Delta_2(x' + x'') = 0$. This implies that there is an element $z \in H^2(\Delta_2, Z/h(q-1))$ such that $\tau(z) = x' - x''$.

Now let us consider $\Gamma_2^*(z)=z'$.
Since, if we consider the homomorphism $Z/h(q-1) \qquad Z_2$ which sends 1 to 1 and the corresponding homomorphism

$$\mathcal{J} : H^2(BSO_2(\bar{k}), Z/h(q-1)) \longrightarrow H^2(BSO_2(\bar{k}), Z_2),$$

we get that $\mathcal{J}(x)=\bar{f}(w_2)$, so by the definition of Dg and the fact that $\tilde{f}^* D_{\Gamma} = D_{\Gamma_2} \tilde{f}^*$ we get that, in order to prove that $\tilde{f}^*(u_2) \neq 0$, it is sufficient to prove that there is no element $\bar{z} \in H^2(\varphi_2, Z/h(q-1))$ such that $2\bar{z}=z'$.
Now, since $4/q-1$ it is easily seen that $SO_2(k) \cong k^* = Z/(q-1)$

Since h is prime to q-1 it follows from the universal coeffients exact sequence that $H^2(SO_2(k), Z/h(q-1)) \cong Z/(q-1)$ and we can choose $\varphi_2^*(x)$ as a generator.
Since $(F,id)^*(x'-x")=(q-1)x$ we have $\tau'(z')=(q-1)x$ so if we suppose that there exists $\bar{z}$ such that $2\bar{z}=z'$ we have $\tau'(\bar{z}) = \frac{q-1}{2} x$.

But, by exactness $\varphi_2^* \tau'(\bar{z}) = 0 = \frac{q-1}{2} \varphi_2^*(x)$ which is absurd since $\varphi_2^*(x)$ is a generator of $H^2(BSO(k), Z/h(q-1)) \cong$ $= Z/q-1$.

Now, since if we consider the homotopy class $h \in [BSO_2(k), BO_2(k)]$, induced by the canonical inclusion, we clearly get

$$\tilde{f} = fh$$

and since we have proved $\tilde{f}^*(u_2) \neq 0$ while it is known $\tilde{f}^*(\bar{w}_1) = 0$, we have that $f^*(u_2) \neq f^*(w_1)$. So, by the structure of $H^*(BO_2(k))$ we have that $f^*(u_2)=1$ or $f^*(u_2) = 1+f^*(w_1)$. In either case it is immediate to see that $f^*(w_1), f^*(w_2)$ and $f^*(u_2)$ generate the whole

$H^*(BO_2(k))$.

So (i) is proved

(ii)follows immediately from the relation

$$\widetilde{f}^* D = D_{f(2)} \,\widetilde{\widetilde{f}}^*$$

and the theorem in paragraph 6.

q.e.d.

Remark. Given a finite group G and an orthogonal representation $\varkappa$ of G over k, if $\widetilde{\varkappa} \in [BG, BO(k)]$ corresponds to $\varkappa$, we can consider the elements $\widetilde{\varkappa}^*(w_i)$, $\widetilde{\varkappa}^*(u_i)$ as characteristic classes for the representation $\varkappa$ and the class

$$w_{ts}(\varkappa) = 1 + \sum_{i \geq 1} \widetilde{\varkappa}^*(w_i)\, t^i + \widetilde{\varkappa}^*(u_i)\, t^{i-1} s$$

as a total cohomology characteristic class for $\varkappa$.

With these notations, Proposition 7 asserts that if $\varkappa$ is the canonical two dimensional representation of $O_2(k)$, then:

$$w_{ts}(\varkappa) = 1 + f^*(\bar{w}_1)t + f^*(\bar{w}_2)t^2 + f^*(u_2)ts,$$

and the coefficients of the non constant terms of the above polinomial in t and s, generate $H^*(BO_2(k))$.

We have the following (we suppose q=4m+1):

Theorem 1. The monomials

$$w_1^{\alpha_1} \; w_2^{\alpha_2} \; \ldots\ldots\ldots \; u_2^{\beta_1} \; u_3^{\beta_2} \; \ldots\ldots\ldots$$

where $\alpha_i \geq 0$, $0 \leq \beta_i \leq 1, \forall i$ and $\alpha_i = \beta_i = 0$ for all but a finite number of i's, form a basis for the algebra $H^*(\widetilde{FO}(q))$.

Proof. let us consider the map

$$\tilde{h}_n : \underbrace{BO_2(k) \times \dots \times BO_2(k)}_{n \text{ times}} \longrightarrow BO(k)$$

defined by induction in the following way:

$$\tilde{h}_1 = f \qquad \tilde{h}_n = \mu(\tilde{h}_{n-1} \times \tilde{h}_1).$$

We put $\tilde{\tilde{\pi}}\,\tilde{h}_n = h_n$, where $\tilde{\tilde{\pi}} : BO(k) \longrightarrow \widetilde{FO}\,\psi^q$ is the map defined in paragraph 6.

Now let us define an homomorphism F from

$$H^*(O_2(k) \cong Z_2\left[h_1^*(\bar{w}_1), h_1^*(\bar{w}_2), h_1^*(u_2)\right] \Big/ (h_1^*(u_2)^2 + h_1^*(u_2) \times h_1^*(w_1)) =$$

$= A$ to the algebra $Z_2\left[x', x'', y\right] \Big/ (y^2 + y) = B$

by $F(h_1^*(w_1)) = x' + x''$, $F(h_1^*(w_2)) = x'x''$, $F(h_1^*(u_2)) = (x'+x'')\,y$.

It is clear that F is injective.

Consider the homomorphism:

$$F^{\otimes n} : \underbrace{A \otimes \dots \otimes A}_{n \text{ times}} \qquad \underbrace{B \otimes \dots \otimes B}_{n \text{ times}} = Z_2\left[x_1', x_1'', \dots x_n', x_n'', y_1, \dots y_n\right] \Big/ (y_1^2 + y_1, \dots y_n^2 + y_n)$$

Since $H^*(\underbrace{O_2(k) \times \dots \dots \times O_2(k)}_{n \text{ times}}) \cong \underbrace{A \otimes \dots \otimes A}_{n \text{ times}}$,

by Kunneth formula, we immediately get from the Corollary to Proposition 7, Proposition 8 and the definition of $F^{\otimes n}$ that

$$F^{\otimes n} h_n^*(w_i) = \sigma_i$$

where σ_i denotes the i-th elementary symmetric function in $(x_1',x_1'', \dots x_n',x_n'')$ for $i \leq 2n$;

$$F^{\otimes n} h_n^*(w_i) = 0 \qquad \text{for } i \geq 2n ;$$

and also

$$F^{\otimes n} h_n^*(u_i) = \sum_{k=1}^{n} \sigma_{i-2}(x_1',x_1'',\dots,\hat{x}_k',\hat{x}_k'',\dots,x_n',x_n'')(x_k'+x_k'')y_k \qquad \text{for } i \leq 2n$$

$$F^{\otimes n} h_n^*(u_i) = 0, \qquad \text{for } i > 2n.$$

Now we want to prove that the elements

$$F^{\otimes n} h_n^*(w_1^{\alpha_1} \dots w_{2n}^{\alpha_{2n}} u_2^{\beta_1} \dots u_{2n}^{\beta_{2n}}) \text{ with}$$

$\alpha_1,\dots,\alpha_{2n}$ 0, $0 \leq \beta_1,\dots,\beta_n \leq 1$ are independent in $\underbrace{B\otimes \dots\dots \otimes B}_{n \text{ times}} = B^{\otimes n}$

It is readily seen that we can consider $B^{\otimes n}$ as a quotient of the algebra

$$Z_2\left[x_1',x_1'',\dots,x_n',x_n'',y_1',y_1'',\dots,y_n',y_n''\right] \Big/ (y_1'^2+y_1',y_1''^2+y_1'',\dots,y_n''^2+y_n'')$$

over the ideal generated by the elements $y_1'+y_1'',\dots,y_n'+y_n''$. Let us call q the quotient homomorphism.

Lemma 11. The following identity holds

$$F^{\otimes n} h_n^*(u_i) = q\Big(\sum_{s=1}^{n} (\sigma_{i-1}(x_1',x_1'',\dots,\hat{x}_s',x_s'',\dots,x_n'')y_s' +$$

$$+\sigma_{i-1}(x_1',x_1'',\dots,\hat{x}_s'',\dots,x_n'')y_s'')\Big).$$

Proof. We can write:

(α) $\sigma_{i-1}(x_1',x_1'',\ldots\hat{x}_s',\ldots,x_n'')= x_s''\,\sigma_{i-2}(x_1',x_1'',\ldots\hat{x}_s',\hat{x}_s'',\ldots,x_n'')+$
$+\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s',\hat{x}_s'',\ldots,x_n'')$.

We have:

$$q(\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s',\ldots,x_n'')y_s'+\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s'',\ldots,x_n'')y_s'') =$$
$$= \left[\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s',\ldots,x_n'')+\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s'',\ldots,x_n'')\right] y_s.$$

Introducing the relations we get, for each $s \leq n$:

$$\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s',\ldots,x_n'')+\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s'',\ldots,x_n'')=$$
$=(x_s'+x_s'')\sigma_{i-2}(x_1',x_1'',\ldots,\hat{x}_s',\hat{x}_s'',\ldots,x_n'')$ which proves the lemma.

q.e.d.

Now let us put for $2 \leq i \leq 2n$,

$$v_i = \sum_{s=1}^{n}(\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s',\ldots,x_n'')y_s'+\sigma_{i-1}(x_1',x_1'',\ldots,\hat{x}_s'',\ldots,x_n'')y_s''),$$

and $v_1 = y_1'+y_1'' + \ldots + y_n' + y_n''$.

Lemma 12. The monomials

$$v_1^{\beta}\ldots v_{2n}^{\beta} \qquad 0 \leq \beta_1, \ldots, \beta_{2n} \leq 1$$

are linearly independent over $Z_2(x_1',x_1'',\ldots x_n',x_n'')$, the field of fractions of $Z_2[x_1',x_1'',\ldots,x_n',x_n'']$.

Proof. Suppose we have an expression

$\sum a_I v_I = 0$ where $a_I \in Z_2(x_1',x_1'',\ldots,x_n',x_n'')$ and

$v_I = v_{i_1} \dots v_{i_k}$ for some subset $I=(i_1,\dots,i_k)\subset(1,\dots,2n)$.

Suppose that for some of the I's, $a_I \neq 0$ and let $\bar{I}$ be a set of maximal order among those. We can suppose $a_{\bar{I}} = 1$.

Let J be the complement of $\bar{I}$ in $(1,\dots,2n)$.

We have

$$\Big(\sum_I a_I v_I\Big) v_J = 0$$

But now, by maximality, only the term $a_{\bar{I}} v_{\bar{I}} v_J$ can contain a monomial of type $b\, y_1' y_1'' \dots y_n''$. So we must have $b\, y_1' y_1'' \dots y_n'' = 0$.

Since $a_{\bar{I}} = 1$ we have that b is equal to the coefficient of $y_1' y_1'' \dots y_n''$ in $v_1 \dots v_{2n}$. So b comes to be equal to the determinant of the Jacobian matrix:

$$\begin{vmatrix} 1 & \dots & 1 \\ \sigma_1(\hat{x}_1',\dots,x_n'') & \dots & \sigma_1(x_1',\dots,\hat{x}_n'') \\ \vdots & & \vdots \\ \sigma_{2n-1}(\hat{x}_1',\dots,x_n'') & \dots & \sigma_{2n-1}(x_1',\dots,\hat{x}_n'') \end{vmatrix}$$

which is different from zero by the algebraic independence of the elementary symmetric functions. So, also $b\, y_1' y_1'' \dots y_n'' \neq 0$ and this implies that $a_{\bar{I}} = 0$ thus giving a contradiction.

q.e.d.

Now for any two by two partition $\mathfrak{p}$ of the set

$(y_1',\ldots,y_n'')$, let us consider the corresponding algebra Q_p given by taking the quotient of the algebra

$$Z_2(x_1',\ldots,x_n'')\left[y_1',\ldots,y_n''\right] \Big/ ({y_1'}^2+y_1',\ldots,{y_n''}^2+y_n'') = \tilde{R}$$

obtained by identifying, two by two, the elements coupled in the partition p.

Let us take the vector space over $\tilde{K}=Z_2(x_1',\ldots,x_n'')$ given by $\bigoplus_{p\in T} Q_p$ where T is the set of two by two partitions of $(1,\ldots,2n)$, and $G:R\longrightarrow \bigoplus_{p\in T} Q_p$ the vector space homomorphism which is the quotient defined above on each factor.

We want to prove $\dim(\operatorname{Ker} G)=2^{n-1}$.

In order to do so let us prove the following,

<u>Lemma 13</u>. Let K be any field and

$$R=K\left[y_1,\ldots,y_{2n}\right] \Big/ (y_1^2+y_1,\ldots,y_{2n}^2+y_{2n}).$$

Let us consider, for each element p of the set T of two by two partitions of the set $(1,\ldots,2n)$, the quotient Q_p defined as above. And let $G:R\longrightarrow \bigoplus_{p\in T} Q_p$ also be defined as above. Then, $\dim(\operatorname{Im} G)\geqslant 2^{n-1}$.

<u>Proof</u>. Let R' be the subalgebra of R generated by $y_1,\ldots,y_{2n-1}$. It will be sufficient to prove $R'\cap \operatorname{Ker}(G)=0$.

Now suppose $G(\sum a_I y_I)=0$, where $a_I \in K$ and $y_I = y_{i_1} \cdots y_{i_k}$ with $I=(i_1,\dots,i_k) \subset (1,\dots,2n-1)$.
Clearly $a_\emptyset = 0$; so we can make induction on the order of I and suppose $a_I=0$ for $|I| < m$.

Consider any element $a_I y_{i_1} \cdots y_{i_m}$ and suppose m to be even.
Now take any partition p containing the couples $(i_1,i_2),\dots,(i_{m-1},i_m)$ and consider the image of $y_{i_1}\cdots y_{i_m}$ in Q_p.
It is clear that there is no set J with $|J| \geq |I|$ such that y_J and y_I are mapped to the same element in Q_p, so this implies $a_I=0$.

If I is odd, consider any p containing $(i_2,i_3),\dots,(i_{m-1},i_m),(i_1,2n)$ and also in this case one proves readily that $a_I=0$.

q.e.d.

If we go back to $\tilde{K}$, then Lemma 12 and Lemma 13 imply that a basis for Ker(G) is given by the elements

$$v_1 v_2^{\beta_2} \cdots v_{2n}^{\beta_{2n}} \qquad 0 \leq \beta_2,\dots,\beta_{2n} \leq 1 .$$

Now let us restrict to the subring $\bar{R}$ R generated by the elementary symmetric functions $\sigma_i(x_1',\dots,x_n'')$ and by the v_i's.

It is clear that an element $x \in \bar{R} \cap \mathrm{Ker}(G)$ if and only if $x \in \mathrm{Ker}(G_p) \cap \bar{R}$ where G_p denotes the quotient $\tilde{R} \longrightarrow Q_p$ relative to any partition $p \in T$. If we consider the partition $\bar{p}:(y_1',y_1''),\dots,(y_n',y_n'')$ the above implies that the elements $G_{\bar{p}}(v_2^{\beta_2},\dots,v_{2n}^{\beta_{2n}})$, $0 \leq \beta_2,\dots,\beta_{2n} \leq 1$ are linearly independent over $Z_2(\sigma_1,\dots,\sigma_{2n})$ with $\sigma_i=\sigma_i(x_1',x_1'',\dots,x_n'')$.

In particular the elements

$$\sigma_1^{\alpha_1}\dots\sigma_{2n}^{\alpha_{2n}}\ G_{\bar p}(v_2^{\beta_2})\dots G_{\bar p}(v_{2n}^{\beta_{2n}})$$

$\alpha_1,\dots,\alpha_{2n}\geq 0$, $0\leq\beta_2,\dots,\beta_{2n}\leq 1$, are linearly independent over Z_2. Since we know that $\sigma_i=F^{\otimes n}\ h_n^*(w_i)$ and, by Lemma 11, $G_{\bar p}(v_j)=F^{\otimes n}\ h_n^*(u_j)$, for $j\geq 2$, we have that the monomials:

$$w_1^{\alpha_1}\cdots\cdots w_{2n}^{\alpha_{2n}}u_2^{\beta_2}\cdots\cdots u_{2n}^{\beta_{2n}}\qquad \alpha_1,\dots,\alpha_{2n}\geq 0,\ 0\leq\beta_2,\dots,\beta_{2n}\leq 1$$

are linearly independent in $H^*(\widetilde{FO}\,\Psi^q)$.

Applying this for larger and larger n we get that the w_i's and u_i's generate a subalgebra of $H^*(\widetilde{FO}\,\Psi^q)$ with Poincaré series

$$\frac{(1+t)(1+t^2)\dots\dots\dots\dots}{(1-t)(1-t^2)\dots\dots\dots\dots}$$

but this, by Lemma 3, is just the Poincaré series of $H^*(\widetilde{FO}\,\Psi^q)$ and the Theorem follows.

q.e.d.

Given a group G, we say that a family $\{N_i\}_{i\in I}$ of subgroups of G detects the (mod.2) cohomology of G when, if we consider the elements $j_{N_i}\in[BN_i,BG]$ for each $i\in I$, associated to the inclusions of the N_i's in G, the homomorphism $\prod_{i\in I}j_N^*: H^*(BG)\longrightarrow\prod_{i\in I}H^*(BN_i)$ is injective.

It is known [7] that the cohomology of $O_2(k)$ is detected by its family of maximal elementary abelian 2-subgroups.

Since there are just two conjugacy classes of maximal elementary abelian 2-subgroups, one of which containing the subgroup of diagonal matrices $Q(2)$, by taking a representative V for the class not containing $Q(2)$, we have that the cohomology of $O_2(k)$ is detected by $Q(2)$ and V (both $Q(2)$ and V have rank 2).

By the definition of u_2 we have $j^*_{Q(2)}(h^*_1(u_2))=0$, so we must have $j^*_V(h^*_1(u_2)) \neq 0$. Since the center C of $O_2(k)$ has order 2, by maximality C is contained in both $Q(2)$ and V. Let us take polinomial generators x,y (resp. $\bar{x},\bar{y}$), for $H^*(BQ(2))$ (resp. $H^*(BV)$) with the property that the kernel of the homomorphism $H^*(BQ(2)) \longrightarrow H^*(BC)$ (resp. $H^*(BV) \longrightarrow H^*(BC)$) induced by inclusion, is the ideal $(x+y)$ (resp. $(\bar{x}+\bar{y})$).

We get:

$$(*)\qquad \begin{aligned} j^*_{Q(2)}(h^*_1(w_1)) &= x+y \\ j^*_{Q(2)}(h^*_1(w_2)) &= xy \\ j^*_V(h^*_1(w_2)) &= \bar{x}\,\bar{y} \\ j^*_V(h^*_1(w_1)) = j^*_V(h^*_1(u_2)) &= \bar{x}+\bar{y}. \end{aligned}$$

This follows for the w_i's because the two subgroups $Q(2)$ and V are conjugate in $O_2(\bar{k})$ and for u_2 by the definition of $\bar{x}$ and $\bar{y}$ and by the fact that $C=Q(2)\cap V$.

It follows from the above properties that the cohomology of $\underbrace{O_2(k) \times \dots\dots \times O_2(k)}_{n\text{-times}}$ is detected by the subgroups of type $E_1 \times \dots\dots \times E_n$ where each E_i can be equal to $Q(2)$ or V.

Since the proof of Theorem 1 implies that the homomorphism

$$h_n^*: H^i(\widetilde{FO}\Psi^q) \longrightarrow H^i(\underbrace{BQ_2(k) \times \dots\dots \times BQ_2(k)}_{n\text{-times}})$$

is injective for $i \leq 2n-1$,we have that the homomorphism

$$(\oplus(j_{E_1} \times \dots \times j_{E_n})^*)\, h_n^*: H^i(\widetilde{FO}\Psi^q) \longrightarrow \oplus H^i(BE_1 \times \dots \times BE_n),$$

where the sum is taken over the number of different subgroups of type $E_1 \times \dots \times E_n$, is injective for $i \quad 2n-1$.

By definition

$$h_n = \widetilde{\widetilde{\pi}}\,\widetilde{h}_n,$$

$\widetilde{h}_n$ being induced by the canonical inclusion of $\underbrace{O_2(k) \times \dots\dots \times O_2(k)}_{n\text{-times}}$ in $O(k)$. Since in $O(k)$ any two subgroups $E_1 \times \dots \times E_n$ and $E_1' \times \dots \times E_n'$ with the same number of E_i's and E_j''s equal to $Q(2)$, are conjugate, we get that the homomorphism

$$\lambda_n^m : \Big(\bigoplus_{m=0}^{n}(j_{Q(2)_1} \times \dots \times j_{Q(2)_m} \times j_{V_1} \times \dots \times j_{V_{n-m}})^*\Big)\, h_n^* :$$

$$: H^i(\widetilde{FO}\Psi^q) \longrightarrow \bigoplus_{m=0}^{n} H^i(BQ(2)_1 \times \dots \times BQ(2)_m \times BV_1 \times \dots \times BV_{n-m})$$

is injective for $i \leq 2n-1$.

<u>Theorem 2</u>. In $H^*(\widetilde{FO}\,\Psi^q)$

$$u_k^2 = \sum_{\substack{a+b=2k-1\\ b\geq 2}} w_a\, u_b \qquad \text{for each } k\geq 2.$$

<u>Proof</u>. It follows from the above discussion that it is sufficient to prove, for any fixed $n\geq k$

$$\lambda_n^m(u_k^2)=\lambda_n^m\Big(\sum_{\substack{a+b=2k-1\\ b\geq 2}} w_a\, u_b\Big)$$

for each $0\leq m\leq n$.

Let us fix such an n and let us put for simplicity $\lambda_n^m(w_i)=w_i$ and $\lambda_n^m(u_i)=u_i$.

First of all suppose m=n. Then, by the definition of the u_i's, we have

$$0=\lambda_n^n(u_k^2)=\lambda_n^n\Big(\sum_{\substack{a+b=2k-1\\ b\geq 2}} w_a\, u_b\Big).$$

Now suppose m=0. We have the following relations:
if g is odd

$$\lambda_n^0(u_g)=0$$

if g is even

$$\lambda_n^0(u_g)=w_{g-1} \qquad \text{for } g\leq 2n$$

$$\lambda_n^0(u_g)=0 \qquad \text{for } g>2n.$$

To prove this, let us make induction on n, for n=1 the above relations follow from proposition 7 and the relations (*). Suppose they are true for $n-1$ and let us put $\lambda^0_{n-1}(w_i)=w'_i$, $\lambda^0_{n-1}(u_i)=u'_i$, $\lambda^0_1(w_i)=w''_i$, $\lambda^0_1(u_i)=u''_i$.

Using the multiplicative relations and the induction ipothesis we have:

$$\lambda^0_n(w_{t,s})=\Big(1+\sum_{i=0}^{2(n-1)} w'_i t^i+\sum_{\substack{j=1\\ j=odd}}^{2(n-1)-1} w'_j t^j s\Big)(1+w''_1 t+w''_2 t^2+w''_1 ts).$$

This implies if g is odd and $g \leq 2n-1$,

$$u_g=w'_{g-2}w''_1+w'_{g-2}w''_1=0$$

if $g \geq 2n+1$

$$u_g = 0.$$

If g is even and $g \leq 2n$

$$u_g=w'_{g-2}w''_1+w'_{g-3}w''_2+w'_{g-1}= w_{g-1}$$

if $g > 2n$

$$u_g = 0,$$

so the above relations are proved.

They implie, if k is odd,

$$\lambda^0_n\Big(\sum_{a+b=2k-1} w_a u_b\Big)= \sum_{e+f=2k-2}^{f=odd} w_e w_f=0=\lambda^0_n(u_k^2) ,$$

if k is even

$$\lambda^0_n\Big(\sum_{a+b=2k-1} w_a u_b\Big)= \sum_{e+f=2k-2}^{f=odd} w_e w_f=w^2_{k-1}=\lambda^0_n(u_k^2) .$$

Finally suppose $0 < m < n$.

Let us put $w'_j=\lambda^m_n(w_j)$ and $w''_j=\lambda^0_{n-m}(w_j)$.

The above relations and the multiplicative formulas implie:

$$\lambda_n^m\Big(\sum_{a+b=2k-1} w_a u_b\Big)=\sum_{a+b=2k-1} w_a\Big(\sum_{e+f=b-1}^{f=odd} w'_e w''_f\Big) =$$

$$=\sum_{a+b=2k-1}\Big(\Big(\sum_{u+v=a} w'_u w''_v\Big)\Big(\sum_{e+f=b-1}^{f=odd} w'_e w''_f\Big)\Big).$$

Take any 4-ple (e,f,u,v) with f odd, $(e,f)\neq(u,v)$, $e+f+u+v=2k-2$. For this 4-ple we get the element

$$w'_e w''_f w'_u w''_v$$

in the above sum.

We separate two casos:

1) If v is odd we get four 4-ple

$$(e,f,u,v),(u,v,e,f),(e,v,u,f),(u,f,e,v)$$

which give the same element in the above sum (clearly if $e=u$ or $f=v$ the four 4-ple reduce to two).

2) if u is even we get two 4-ple

$$(e,f,u,v),(u,f,e,v)$$

which give the same element in the above sum.

Now it is clear that in either cases the elements associated to those 4-ple cancel two by two.

So, we are left with the case $e=u$, $f=v$.

This implies

$$\lambda_n^m\Big(\sum_{a+b=2k-1} w_a u_b\Big)=\Big(\sum_{h+s=2k-2}^{s=odd} w'_h w''_s\Big)^2 = \lambda_n^m(u_k^2)$$

where the second equality follows from the multiplicative relations.

Thus

$$\lambda_n^m\Big(\sum_{a+b=2k-1} w_a u_b\Big) = \lambda_n^m(u_k^2) \qquad \text{for each } 0 \leq m \leq n$$

and the Theorem is proved.

q.e.d.

Remark. Just by using diedral groups and a multiplicative relation which can be easily defined for $H^*(\widetilde{FO}\,\Psi^q)$ one could prove similar results to Theorems 1 and 2 without the restrictions $q=|k|$, k a finite field with $4m+1$ elements.

8. The algebras $H^*(O_n(k))$.

In this paragraph we suppose that $q=4m+1$ and that k is a field with q elements.

Let Q' and V' two proper subgroups of the groups $Q(2)$ and V considered in the preceeding paragraph, which are both different from C. Since both Q' and V' are elementary abelian 2-subgroups of rank 1, $H_i(Q') \cong H_i(V') \cong Z_2$ for each $i \geq 0$, where by H_i we denote the i-th homology group with coefficients in Z_2.

Let $\bar{\xi}_i$ (resp. $\bar{\eta}_i$) the unique non zero element in $H_i(Q')$ (resp. $H_i(V')$) for $i \geq 1$.

Let $R = M_1 \times \ldots \times M_n$ any subgroup of $\underbrace{O_2(k) \times \ldots \times O_2(k)}_{n\text{-times}}$ which is the product of copies of Q' and V'.

For each R we get the homomorphism

$$(h_n j_R)_* : H_*(BR) \longrightarrow H_*(\widetilde{FO}\Psi^q).$$

We put $\xi_i = (h_n j_{Q'})_* \bar{\xi}_i$ and $\eta_i = (h_n j_{V'})_* \bar{\eta}_i$.

Now let $x, y \in H_*(\widetilde{FO}\Psi^q)$ be such that $\tau = (h_{n'} j_{R'})(\bar{\tau})$ and $\kappa = (h_{n''} j_{R''})_*(\bar{\kappa})$ for two subgroups R' and R'' of the type described above. We can define $\cancel{=(h_{n'+n''} j_{R' \times R''})}$

$\tau\kappa = (h_{n'+n''} j_{R' \times R''})_* (\tau \otimes \kappa)$ by using the Kunneth formula.

Theorem 3. $H_*(\widetilde{FO}\Psi^q)$ has a basis formed by the monomials

$$\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \cdots \eta_1^{\beta_1} \eta_2^{\beta_2} \cdots \cdots$$

with $\alpha_i \geq 0$, $0 \leq \beta_i \leq 1$ and all but a finite number of

α_i's and β_i's equal to zero.

Further $(\eta_i+\xi_i)^2=0$

<u>Proof</u>. Let $t_1,\dots,t_N;\ s_1,\dots,s_N$ be indeterminates with $s_j^2=0$ for each $1\leq j\leq N$. We define the homomorphism

$$T_N:H_*(\widetilde{FO}\psi^q)\longrightarrow Z_2[t_1,\dots,t_N]\otimes\Lambda[s_1,\dots,s_N]$$

by

$$T_N(z)=\langle z,\prod_{j=1}^{N}w_{t_js_j}\rangle$$

where by $\langle\ \ \rangle$ we mean the canonical pairing between homology and cohomology.

Now let $\tilde{\xi}_i=I_N(\xi_i)$ and $\tilde{\eta}_i=T_N(\eta_i)$.

The multiplicative relations and the definition of Q' and V' clearly implie that,if $x\in H^1(BQ')$(resp.$y\in H^1(BV')$) is the one dimensional polinomial generator of $H^*(BQ')$ (resp.$H^*(BV')$),

$$\tilde{\xi}_i=\langle\bar{\xi}_i,\prod_{j=1}^{N}(1+xt_j)\rangle$$

and

$$\tilde{\eta}_i=\langle\bar{\eta}_i,\prod_{j=1}^{N}(1+y(t_j+s_j))\rangle,$$

and that,given two elements $\tau,\varkappa\in H_*(\widetilde{FO}\psi^q)$ for which is defined

$$T_N(\tau\varkappa)=T_N(\tau)\,T_N(\varkappa).$$

The above relations give:

$$\tilde{\xi}_i=\sigma_i(t_1,\dots,t_N)$$

$$\tilde{\eta}_i = {}_i((t_1+s_1),\dots,(t_N+s_N)))$$

where by σ_i we mean the elementary symmetric function of the variables in brackets.

We also have

$$\tilde{\xi}_i+\tilde{\eta}_i = \sigma_i(s_1,\dots,s_N) + \sum_{h=1}^{N} \sigma_{i-1}(s_1,\dots,\hat{s}_h,\dots,s_n)t_h +$$

$$\dots + \sum_{h=1}^{N} s_h \; \sigma_{i-1}(t_1,\dots,\hat{t}_h,\dots,t_N) .$$

So

$$(*) \quad T_N(\xi_i+\eta_i)^2=(\tilde{\xi}_i+\tilde{\eta}_i)^2=0.$$

Finally we can filter $Z_2[t_1,\dots,t_N]\otimes\Lambda[s_1,\dots,s_N]$ by powers of the ideal $(s_1,\dots,s_N)$; then under this filtration, the leading term of $\tilde{\xi}_i+\tilde{\xi}_i$ is $\sum_{h=1}^{N} s_h(\sigma_{i-1}(t_1,\dots,\hat{t}_h,\dots,t_N))$.

If we consider $Z_2[t_1,\dots,t_N]\otimes\Lambda[s_1,\dots,s_N]$ as a De Rham complex with $dt_i=s_i$ we get that

$$\sum_{h=1}^{N} s_h(\sigma_{i-1}(t_1,\dots,\hat{t}_h,\dots,t_N)) = d\sigma_i(t_1,\dots,t_N).$$

We apply the following:

<u>Lemma 14</u> [8]. The ring homomorphism

$$Z_2[\sigma_1,\dots,\sigma_N]\otimes\Lambda[d\sigma_1,\dots,d\sigma_N] \longrightarrow Z_2[x_1,\dots,x_N]\otimes$$

$$\otimes\ \Lambda[dx_1,\dots,dx_N]$$

defined in the obvious way is injective.

We clearly get from the above Lemma that the monomials

$\tilde{\xi}_1^{\alpha_1} \cdots \tilde{\xi}_N^{\alpha_N} \tilde{\eta}_1^{\beta_1} \cdots \tilde{\eta}_N^{\beta_N}$ with $\alpha_i \geqslant 0$, $0 \leq \beta_i \leq 1$ are linearly independent. Thus by applying this result for larger and larger N to-gether with Lemma 3, we get the first part of the Theorem.

The second follows from $(*)$ and the fact that $T_N/H_i(\widetilde{FO}\,\Psi^q)$ is injective for $i \leq N$.

q.e.d.

Remarks

1) The same remark of the end of Paragraph 7 is valid in the case of this theorem.

2) Lemma 14 is essentialy Lemma 12.

3) It comes out from the proof of Theorem 3 that we can define a ring structure on $H^*(\widetilde{FO}\,\Psi^q)$. With this ring structure $H_*(\widetilde{FO}\,\Psi^q) \cong Z_2\,[\xi_1, \xi_2, \ldots] \otimes \otimes \Lambda[\xi_1+\eta_1, \xi_2+\eta_2, \ldots]$.

Now let us consider the group $\bigoplus_{r \geqslant 1} H_*(BO_r(k))$. The direct sum homomorphism $O_n(k) \times O_m(k) \longrightarrow O_{n+m}(k)$ clearly induces a multiplication in $\bigoplus_{r \geqslant 1} H(BO_r(k))$ which is associative and commutative.

Let ε be the generator of $H_0(BO_1(k))$, then ε^r will be the generator of $H_0(BO_r(k))$.

By its definition we can choose Q' to be $O_1(k)$ under the canonical inclusion in $O_2(k)$. Thus let us consider the elements

$$\xi_i \in H_i(BO_1(k)), \qquad \forall i \geqslant 1,$$

$$\eta_i = j_{V'_*}(\bar{\eta}_i) \in H_i(BO_2(k)), \forall i \geqslant 1.$$

We have:

Theorem 4. If, for each n $\vartheta_n \in [BO_n(k), BO(k)]$ is the homotopy class associated to the canonical inclusion of $O_n(k)$ in $O(k)$, then the homomorphism:

$$(\tilde{\tilde{\pi}}\, \vartheta_n)_* : H_*(BO_n(k)) \longrightarrow H_*(\widetilde{FO}\,\Psi^q)$$

is injective.

Proof. It is clear that $\tilde{\tilde{\pi}}\, \vartheta_2 = h_1$, so we have that $\tilde{\tilde{\pi}}\, \vartheta_2$ takes the η_i's into the elements denoted by the same name in $H_*(\widetilde{FO}\,\Psi^q)$.

It also follows from the multiplicative relations that each monomial in the ξ_i's and η_i's in $\bigoplus_{r \geqslant 1} H_*(BO_r(k))$ goes into the corresponding monomial in the ξ_i's and η_i's, in $H_*(\widetilde{FO}\,\Psi^q)$.

In order to prove the Theorem we need some Lemmas.

Lemma 15 (Quillen) [6]. The cohomology of $O_n(k)$ is detected by its elementary abelian 2-subgroups.

Lemma 16. If $n=2m+e$ $(e=0,1)$, then the cohomology of $O_n(k)$ is detected by the subgroup which is the image of $\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}} \times Z_2^e$ under the canonical inclusion

<u>Proof</u>. By Lemma 15, it is sufficient to prove that each elementary abelian 2-subgroup of $O_n(k)$ is conjugate to a subgroup of $\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}} \times Z_2^e$.

Since given such a subgroup $A \subset O_n(k)$, we can consider k^n as an orthogonal n-dimensional representation of A, it is sufficient to prove that any orthogonal representation of A can be decomposed as a sum of 1 and 2-dimensional representations.

Since for 1-dimensional representations this is trivial we suppose, by induction, that any m-dimensional representation of A can be written as a sum of 1 and 2-dimensional representations for $m \angle n$.

Let us consider an -dimensional orthogonal representation W of A, and let L an irreducible invariant subspace for this representation. Since the exponent of A divides q-1, L is of dimension 1. We divide two cases:

1) if $L^{\perp}$ is not an isotropic subspace, then $W \cong L \oplus L^{\perp}$ where L is the space orthogonal to L, and by applying induction for $L^{\perp}$ W can be written as a sum of 1 and 2-dimensional representations.

2) If L is an isotropic subspace, then, by choosing an invariant subspace which is complementary to $L^{\perp}$ (this exists because the order of A is prime to the

characteristic of k), we write W as a direct sum of an iperbolic orthogonal representation and an n-2 dimensional representation. Thus also in this case the induction ipothesis implies that W can be written as a sum of 1 and 2-dimensional representations, and the Lemma is proved.

q.e.d.

We are now ready to prove Theorem 4.

Let us consider the group $V \subset O_2(k)$ of the preceeding paragraph and let V' and V" the two proper subgroups of $O_2(k)$ which are different from the center of $O_2(k)$. We have $H_*(BV) = H_*(BV') \otimes H_*(BV'')$ by the Kunneth formula.

Since V' and V" are clearly conjugate in $O_2(k)$ it follows that if $\bar{\bar{\eta}}_i \in H_i(BV'')$ denotes the generator of $H_i(BV'')$, for each $i \geq 1$, $j_V(\eta_i \otimes \varepsilon'') = j_V(\varepsilon' \otimes \eta_i) = \eta_i$, where ε' (resp. ε'') is the generator of $H_0(BV')$ (resp. $H_0(BV'')$).

Now, if we consider the two subgroups of $O_4(k)$ obtained one by composing the inclusion of V in $O_2(k)$ with the canonical inclusion of $O_2(k)$ in $O_4(k)$, the other by composing the product inclusion of V' x V' in $O_2(k) \times O_2(k)$ with the direct sum homomorphism $O_2(k) \times O_2(k) \longrightarrow O_4(k)$, it is easy to see, by direct computation that the two subgroups are conjugate by a conjugation which is the identity on their intersection and takes the subgroup which is the image of V" under the first inclusion into the subgroup which is the image of $\{1\} \times V'$ under the second.

This clearly implies that in $H_*(\widetilde{FO}\,Y^q)$

$$(\tilde{\pi}\vartheta_2)_*(j_{V*}(\bar{\eta}_j \otimes \bar{\bar{\eta}}_k)) = \eta_j \eta_k \quad \forall i,k \geq 1$$

Since $\tilde{\pi}\vartheta_2 = h_1$ it follows from Proposition 8 that $(\tilde{\pi}\vartheta_2)$ is injective so, by (τ) and Theorem 3 we get:

$$j_{V_*}(\bar{\eta}_i \otimes \bar{\eta}_i) = \xi_i^2 \qquad \forall i \geq 1.$$

(τ) also implies that the elements $\eta_{j,k} = j_{V_*}(\bar{\eta}_j \otimes \bar{\bar{\eta}}_k)$, $0 \leq j < k$, where we put $\eta_{0,k} = j_{V_*}(\varepsilon' \otimes \bar{\bar{\eta}}_k) = j_{V_*}(\bar{\eta}_k \otimes \varepsilon")$ and the elements $\xi_j \xi_h$, $0 \leq j \leq h$, where we put $\xi_0 = \varepsilon$, are linearly independent, thus they generate a submodule of $H_*(BO_2(k))$ with Poincaré series $\frac{1+t}{(1-t)(1-t^2)}$. But, by the known structure of $H^*(BO_2(k))$, this is just the Poincaré series of $H_*(BO_2(k))$.

Thus the above elements for $H_*(BO_2(k))$.

Now Lemma 16 implies that, if $n=2m+e$ $(e=0,1)$, the homomorphism

$$\partial_{n*} : H_*(\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}} \times Z_2^e) \longrightarrow H_*(O_n(k))$$

induced by inclusion, is onto. Thus we get that the elements

$$\text{(H)} \qquad \varepsilon^{\alpha} \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m} \cdots\cdots \eta_{0,1}^{\beta_{0,1}} \cdots\cdots \eta_{i,k}^{\beta_{i,k}} \qquad \begin{matrix} \alpha \geq 0 \\ \alpha_i \geq 0 \\ \beta_{ik} \geq 0 \\ i < k \end{matrix}$$

where only a finite number of α_i's and β_{ik}'s are different from zero, form a set of generators over Z_2 for $\bigoplus_{r \geq 1} H_*(O_r(k))$.

If, for each $m \geq 2$ we consider the subgroup K of $O_{2m}(k)$ obtained by composing the inclusion of $\underbrace{V \times \ldots\ldots \times V}_{m\text{-times}}$ in $\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}}$ with the inclusion of $\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}}$ in $O_{2m}(k)$ it is known that if $N(K)$ denotes the normalizer of K in $O_{2m}(k)$ then $N(K)/K = \Sigma_{2m}$, the symmetric group on 2 m letters, and an element $s \in \Sigma_{2m}$ acts on H (BK) by sending the element $\bar{\bar{\eta}}_{i_1} \otimes \bar{\bar{\eta}}_{i_2} \otimes \cdots \otimes \bar{\bar{\eta}}_{i_{2m}}$ to the element $\bar{\bar{\eta}}_{s(i_1)} \otimes \cdots \otimes \bar{\bar{\eta}}_{s(i_{2m})}$, where $(i_1, \ldots, i_{2m})$ is any set of 2 m integers with $i_j \geq 0$ (we put $\bar{\bar{\eta}}_0 = \varepsilon'$ $\bar{\bar{\eta}} = \varepsilon''$) This clearly implies that

$$(\Delta) \qquad \eta_{(t_1 t_2)} \cdots \eta_{t_{2m-1} t_{2m}} = \eta_{s(t_1) s(t_2)} \cdots \eta_{s(t_{2m-1}) s(t_{2m})}$$

where $(t_1, \ldots, t_{2m})$ is any set of integers with $t_j \geq 0$, $t_{2j-1} < t_{2j}$ and $s \in \Sigma_{2m}$, in the ring $\bigoplus_{r \geq 1} H_*(O_r(k))$.

Thus (Δ) together with (H) and the fact that $\eta_{jj} = \xi_j^2$ implies that the elements

$$(\tau) \qquad \varepsilon^{\alpha} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdots \eta_{01}^{\beta_{01}} \cdots \eta_{i,k}^{\beta_{ik}}$$

with $\alpha, \alpha_1 \geq 0$, $0 \leq \beta_{i,k} \leq 1$, if $\eta_{i,k}$ compares on the left of $\eta_{i',k'}$ and $\beta_{i,k} = \beta_{i',k'} = 1$, then $i < k < i' < k'$ and only a finite number of α_1's and $\beta_{i,k}$'s are different from zero, form a set of generators over Z_2 for $\bigoplus_{r \geq 1} H_*(BO_r(k))$.

Now, if for such a monomial A we define

$$\deg(A) = \alpha + \sum_l \alpha_l + \sum_{i,k} 2\beta_{i,k}$$

we have that $A \in H_*(O_r(k))$ if and only if $\deg(A)=r$; so, in order to prove our theorem, it is sufficient to prove that the monomials of a fixed degree are mapped by $(\tilde{\tilde{\pi}}\vartheta_n)$ to independent monomials.

We have from the above,

$$(\tilde{\tilde{\pi}}\vartheta_n)_* \left(\varepsilon^{\alpha} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdots \eta_{0,1}^{\beta_{0,1}} \cdots \eta_{1k}^{\beta_{1,k}}\right) =$$
$$= \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdots \eta_1^{\beta_{0,2}} \cdots \eta_i^{\beta_{i,k}} \eta_k^{\beta_{i,k}}$$

which by theorem 3 clearly implies that the monomials satisfying (τ') of the same degree are mapped to independent monomials by $(\tilde{\pi}\vartheta_n)_*$.

q.e.d.

<u>Theorem 5.</u> $H^*(BO_n(k))$ is generated as an algebra by elements $\bar{w}_1, \dots, \bar{w}_n$; $\bar{u}_2, \dots, \bar{u}_n$, with $\deg(\bar{w}_i)=i$, $\deg(\bar{u}_i)=i-1$, subject to the following relations

$$\bar{u}_i^2 = \sum_{\substack{a+b=2i-1 \\ b \geq 2}} \bar{w}_a \bar{u}_b, \qquad \text{where } \bar{w}_0 = 1.$$

<u>Proof.</u> It follows by theorem 4 that the homomorphism

$(\tilde{\pi}\vartheta_n)^* : H^*(\widetilde{FO}\,\psi^q) \longrightarrow H^*(BO_n(k))$

is onto for each $n \geq 1$; and we known, by Lemma 16 that, if $n=2m+e$ $(e=0,1)$, the homomorphism

$$\delta^*_n : H^*(BO_n(k)) \longrightarrow H^*(\underbrace{BO_2(k) \times \cdots\cdots \times BO_2(k)}_{m\text{-times}} \times BZ_2^e)$$

induced by inclusion, is into.

If $n=2m$, then $(\tilde{\pi}\vartheta_n\delta_n)=h_n$, so the Theorem follows from the proof of Theorem 1 and Theorem 2 by taking $\bar{w}_i=(\tilde{\pi}\vartheta_n)_*(w_i)$ and $\bar{u}_i=(\tilde{\pi}\vartheta_n)_*(u_i)$.

If $n=2m+1$, we have that, by definition the inclusion of $\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}} \times Z_2$ in $O_n(k)$ is obtained by composing the inclusion of $\underbrace{O_2(k) \times \ldots\ldots \times O_2(k)}_{m\text{-times}} \times Z_2$ in $O_{n-1}(k) \times (Z_2 \cong O_1(k))$ with the direct sum homomorphism $O_{n-1}(k) \times O_1(k) \longrightarrow O_n(k)$; thus by the multiplicative relations and the result for $O_{n-1}(k)$, we get

$$(\tilde{\pi}\vartheta_n S_n)^*(w_{ts})=(1+\sum_{i=1}^{n-1}\bar{w}_i^! t^i+\sum_{j=2}^{n-1}\bar{u}_j^! t^{j-1}s)\otimes(1+xt)$$

where $\bar{w}_i^!=h_{n-1}^*(w_i)$, $\bar{u}_j^!=h_{n-1}^*(u_j)$ and $x\in H^1(BZ_2)$ is the one dimensional polinomial generator of $H^*(BZ_2)$.

Thus we get $(\tilde{\pi}\vartheta_n S_n)^*(w_i)=0$ and $=(\tilde{\pi}\vartheta_n S_n)(u_i)=$ $=0$ for $i>n$.

This means that the ideal generated by $w_{n+1}, w_{n+2}, \ldots\ldots, u_{n+1}, u_{n+2}, \ldots\ldots$ lies in the kernel of $(\tilde{\pi}\vartheta_n)^*$.

Since by the proof of Theorem 4 the Poincaré series of $H^*(BO_n(k))$ is

$$\frac{(1+t)\ldots\ldots(1+t^{n-1})}{(1-t)\ldots\ldots(1-t^n)}$$

and since also the algebra $H^*(\widetilde{FO}\Psi^q)/(w_{n+1}, w_{n+2}, \ldots, u_{n+1}, u_{n+2}, \ldots)$ has this Poincaré series, the Theorem follows also for $n=2m+1$, because of $(\tilde{\pi}\vartheta_n)^*$ being onto by putting $\bar{w}_i=(\tilde{\pi}\vartheta_n)^*(w_i)$ and $\bar{u}_i=(\tilde{\pi}\vartheta_n)^*u_i$.

q.e.d.

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