FINITE GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM

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CONTENTS

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SUMMARY

The content of this thesis is a proof of the following theorem: Let G be a finite group admitting a fixed-point-free coprime automorphism α of order rst, where r,s and t are distinct primes and rst is a non-Fermat number. Then G is soluble. A non-Fermat number is defined to be one which is not divisible by an integer of the form 2^m+1 $(m \geqslant 1)$; there are infinitely many non-Fermat numbers which are the product of three distinct primes. G is said to admit A, a subgroup of Aut G, the automorphism group of G, fixed-point-freely if and only if $C_G(A) = \{g \in G \mid a(g) = g \text{ for all } a \in A\} = \{1\}$. The result provides a solution to part of this well-known conjecture: let G be a finite group admitting the automorphism group A fixed-point-freely and, if A is non-cyclic; also assume $|A|$ is coprime to $|G|$. Then G is soluble.

INTRODUCTION

Suppose G denotes a finite group and A a subgroup of Aut G, the automorphism group of G. Then A is said to act fixed-point-freely upon G (or, sometimes, G is said to admit A fixed-point-freely) if and only if $C_G(A) = \{g \in G \mid a(g) = g \text{ for all } a \in A\} = \{1\}$.

The following conjecture is well known:-

> Let G be a finite group admitting the automorphism group A fixed-point-freely and, if A is non-cyclic, also assume $|A|$ is coprime to $|G|$. Then G is soluble.

The main result of this work (theorem 10.6) provides a solution to part of this conjecture.

MAIN THEOREM

> Let G be a finite group admitting a fixed-point-free coprime automorphism α of order rst, where r, s and t are distinct primes and rst is a non-Fermat number. Then G is soluble.

A non-Fermat number is defined to be one which is not divisible by an integer of the form 2^m+1 $(m \geq 1)$. Observe that there are infinitely many non-Fermat numbers which are the product of three distinct primes.

The first significant step in establishing the 'fixed-point-free conjecture' stated above was made by J.G. Thompson [17] in 1958 who showed that when A has prime order then G must, in fact, be nilpotent. More recently, E.W. Ralston [15], in 1971, verified the conjecture when $|A|$ is the product of two distinct primes. These two results and the MAIN THEOREM above constitute the present progress on the conjecture when A is of square-free order.

In the case that A is non-cyclic, R.P. Martineau [12 , 13] has successfully resolved the conjecture when A is an elementary abelian r-group, r a prime, (he has actually proved slightly more; see [13]) and M.Pettet [14] has dealt with the case when A is the direct sum of an elementary abelian r-group and an elementary abelian s-group (r and s primes) provided neither r nor s is a Fermat prime.

A brief outline of the proof of the main theorem follows.

After mustering together, in section 1, known results which are required in the proof, section 2 sees the appearance of the 'star group', the concept of 'star-covering' and some important results concerning soluble groups which admit fixed-point-free automorphisms (lemmas 2.10, 2.13, 2.15). Sections 3 and 5 are devoted to pinning down the possible structure of the maximal α-invariant $\{\mu,\eta\}$-subgroups of a minimum counterexample G (to theorem 10.6) where μ and η are subsets of $\pi(G)$. And section 4 consists mostly of criteria for normal complements.

In section 8, it is shown that the α-invariant Sylow subgroups of G of type $\{1,2,3\}$ generate a soluble Hall subgroup of G. Section 6 examines the consequences of various interactions between the sets $\mathfrak{M}_{\mu,\eta}$ as μ and η vary. The purpose of these so called 'linking results' is to aid the 'patching together' of G from information about $\mathcal{M}_{\mu,\eta}$ for different μ and η with the aim of factorizing G as the product of two proper α-invariant subgroups. It is left to sections 9 and 10 to show that these resulting factorizations are inadmissible in a minimal counterexample.

To the best of my knowledge, any work not otherwise attributed is original.

1. NOTATION AND PRELIMINARY RESULTS

The notation employed in this work, except where specified, corresponds to that of Gorenstein [8] and all groups under consideration are assumed to be finite.

First, some well known properties of fixed-point-free automorphisms will be listed. Let G be a group admitting the automorphism α fixed-point-freely.

(1.1) If H is an α-invariant normal subgroup of G, then α induces a fixed-point-free automorphism of G/H.

(1.2) Let $\mathcal{K}$ be a conjugacy class of subsets of G. If $\mathcal{K}$ is invariant under α, then there exists a unique element of $\mathcal{K}$ which is α-invariant.

An important consequence of (1.2) is:

(1.3) For each $p \in \pi(G)$ there exists a unique α-invariant Sylow p-subgroup, P, of G. Furthermore, any α-invariant p-subgroup of G is contained in P.

From (1.3) it follows that if H is an α-invariant subgroup of G and, if P denotes the α-invariant Sylow p-subgroup of G, then $P \cap H$ is the (unique) α-invariant Sylow p-subgroup of H. Also, observe that if G possesses an α-invariant Hall π-subgroup H, then H is the unique such subgroup, since H must be generated by α-invariant Sylow p-subgroups of G, $p \in \pi$; clearly any α-invariant π-subgroups of G will be contained in H.

Proofs for (1.1),(1.2) and (1.3) may be located in Gorenstein [8, Chapter10,Section1]; his theorem 10.1.2 is the same as (1.3) but the proof given also establishes the above stronger statement, (1.2).

Combining (1.3) with a well known result of P.Hall's gives:

(1.4) If G is soluble and π is a subset of $\pi(G)$, then there exists a unique α-invariant Hall π-subgroup of G, which contains every α-invariant π-subgroup of G.

(1.5) (Ralston [15, 2.12]) G is soluble if and only if for each pair of primes $p,q \in \pi(G)$, the corresponding α-invariant Sylow p- and q-subgroups permute.

Now let G be a group with A a subgroup of $\mathrm{Aut}G$.

(1.6) (i) If H is an A-invariant subgroup of G, then so too are $N_G(H)$ and $C_G(H)$.

(ii) $[G,A]$, which is defined to be $\langle g^{-1}a(g) \mid g \in G,\ a \in A \rangle$, is an A-invariant normal subgroup of G.

(iii) If $B \trianglelefteq A$, then $[G,B]$ and $C_G(B)$ are A-invariant.

Further suppose that $(|G|,|A|) = 1$.

(1.7) (i) $G = C_G(A)[G,A]$.

(ii) $[G,A] = [[G,A],A]$.

(iii) If $H \leqslant C_G(A) = C$, then $N_G(H) = N_C(H)C_G(H)$.

(iv) If H is a normal A-invariant subgroup of G, then $C_{G/H}(A) = C_G(A)H/H$.

(v) If H is an A-invariant normal subgroup of G contained in $C_G(A)$, then $[G,A]$ centralizes H.

(vi) If G is soluble and $C_G(A)$ contains a Hall π-subgroup, then $G = C_G(A)O_{\pi'}(G)$.

(vii) For each $p \in \pi(G)$ there exists at least one A-invariant Sylow p-subgroup and any two A-invariant Sylow p-subgroups of G are conjugate by an element of $C_G(A)$. Moreover, every A-invariant p-subgroup of G is contained in some A-invariant Sylow p-subgroup of G.

(viii) If G is abelian, then $G = C_G(A) \times [G,A]$.

(ix) If $G = H \times K$ is abelian with H A-invariant, then there exists an A-invariant direct summand of H in G.

(x) If H is a subgroup of $C_G(A)$ with $C_G(H) \leqslant H$ and G is nilpotent, then $G = C_G(A)$.

(xi) Suppose $[G,A] \leqslant H$ where H is an A-invariant subgroup of G, then $[G,A] = [H,A]$.

(xii) If $G = HK$, where H and K are A-invariant subgroups of G and $K \leqslant C_G(A)$, then $[G,A] \leqslant H$.

(xiii) If H is an A-invariant subgroup of G, then $[H,A] \trianglelefteq HC_{N_G(H)}(A)$.

Parts (i)-(iv) are to be found in Glauberman [3] whilst parts (v) and (vi) are given, respectively, as lemmas 7 and 8 of Glauberman [5] and parts (vii), (viii) and (ix) are respectively 6.2.2, 5.2.3 and 3.3.2 of Gorenstein [8].

Proof (of parts (x)-(xiii))

(x) From (iii), $N_G(C) = N_C(C)C_G(C) = CC_G(C)$ where $C = C_G(A)$ and therefore, as $C_G(H) \leqslant H \leqslant C, N_G(C) = C$. By a well known property of nilpotent groups it follows that $G = C = C_G(A)$.

(xi) As $[G,A] \leqslant H, [[G,A],A] \leqslant [H,A]$ and hence $[G,A] \leqslant [H,A]$ by part (ii). Clearly $[H,A] \leqslant [G,A]$ and so $[G,A] = [H,A]$.

(xii) Direct calculation yields this result.

(xiii) Parts (xi) and (xii) show that $[H,A] = [HC_{N_G(H)}(A),A]$ and then by (1.6)(ii) the result follows.

Some notation and definitions will now be introduced:

Hypothesis A Let G be a non-soluble group admitting a fixed-point-free automorphism α with all proper α-invariant subgroups of G soluble and G possessing no proper non-trivial α-invariant normal subgroups.

Hypothesis B If hypothesis A holds with α being of square-free order.

Definition 1.8 Let d be a positive integer. Then d is said to be a non-Fermat number if and only if d is not divisible by an integer of the form 2^m+1 $(m \geqslant 1)$.

<u>Hypothesis C</u> If hypothesis B holds, $(|\alpha|, |G|) = 1$ and $|\alpha| = r_1 \ldots r_n$ is an odd non-Fermat number.

<u>Hypothesis D</u> If hypothesis C holds with $n = 3$.

If α is a fixed-point-free automorphism of square-free order with $|\alpha| = r_1 \ldots r_n$, then, for $1 \leqslant i \leqslant n$, α_i will denote a fixed element of $\langle\alpha\rangle$ which is of order r_i. When $n = 3$ ρ, σ, τ and r, s, t will also be used in place of $\alpha_1, \alpha_2, \alpha_3$ and r_1, r_2, r_3.

Suppose G admits α fixed-point-freely and H is an α-invariant subgroup of G. Then H_β will also be written for $C_H(\beta)$ where $\beta \in \langle\alpha\rangle$. If hypothesis A holds and $H \neq G$, then H is soluble and so for each subset π of $\pi(H)$ there exists a unique α-invariant Hall π-subgroup of H which will be denoted by H_π. In some circumstances when $\pi = \mu \cup \eta$; H_π will also be written as $H_{\mu,\eta}$. If μ and η are subsets of $\pi(G)$, $\mathcal{M}_{\mu,\eta}$ is defined to be the set of maximal α-invariant $(\mu\cup\eta)$-subgroups of G.

(1.9) (Glauberman[7]) Let G be a p-soluble group (p a prime) admitting a fixed-point-free coprime automorphism group. Then $G = N_G(J(P))C_G(Z(P))O_{p'}(G)$.

Results (1.10)-(1.16) with the exception of (1.13) represent a slight generalization of some work of Martineau ([12],[13]). For the duration of these results it will be assumed that hypothesis A holds and, in addition, for (1.13)-(1.16) that $(|\alpha|, |G|) = 1$. Further, let M and N respectively denote α-invariant nilpotent

Hall μ- and η-subgroups of G. Here μ and η are assumed to be disjoint subsets of $\pi(G)$. Let X (respectively Y) denote the largest α-invariant subgroup of M (respectively N) which is permutable with N (respectively M); note that such X and Y exist. Thus $\{MY, NX\} \subseteq \mathcal{M}_{\mu,\eta}$, and $MY \cap NX = XY$.

(1.10) ('The uniqueness theorem') Let $H, K \in \mathcal{M}_{\mu,\eta}$ with $O_\mu(F(H)) \neq 1 \neq O_\eta(F(H))$ and let L be an α-invariant subgroup of F(H) such that $L \cap O_\mu(F(H)) \neq 1 \neq L \cap O_\eta(F(H))$. If $L \leqslant K$, then $H = K$.

An important corollary of (1.10) is:

(1.11) Let $H \in \mathcal{M}_{\mu,\eta}$ with $O_\mu(F(H)) \neq 1 \neq O_\eta(F(H))$ and let L be a non-trivial α-invariant subgroup of F(H). Then $\{N_G(L)\}_{(\mu\cup\eta)} \leqslant H$.

<u>Remarks</u> (i) The proof given in Martineau [12] for lemma 4 translates into a proof for (1.10) since a suitable analogue of lemma 3 [12] holds.

(ii) A result of the type presented in (1.11) has also been obtained by Ralston [15].

To carry through the programme of extending Martineau's results to nilpotent α-invariant Hall subgroups the following generalization of Glauberman's factorization theorem (1.9) is required:

(1.12) Let G be a soluble group admitting a fixed-point

-free coprime automorphism group A and let N be a nilpotent A-invariant Hall η-subgroup of G. Then $G = \langle N_G(J(P)),\ C_G(Z(P)) \mid p \in \eta \rangle\ O_{\eta'}(G)$.

<u>Proof</u> By induction upon $|G|+|\eta|$. Clearly may assume that $O_{\eta'}(G) = 1$ and, because of (1.9), that $|\eta| > 1$. Let $p \in \eta$ and let P denote the A-invariant Sylow p-subgroup of G and set $\mu = \eta \setminus \{p\}$. Employing (1.9) yields that $G = N_G(J(P))\ C_G(Z(P)) O_{p'}(G)$. By a well known property of soluble groups, $O_{p'}(N) \leqslant O_{p'}(G)$ and so by induction $O_{p'}(G) = \langle N_{O_{p'}(G)}(J(Q)),\ C_{O_{p'}(G)}(Z(Q)) \mid q \in \mu \rangle\ O_{\mu'}(O_{p'}(G))$ $= \langle N_{O_{p'}(G)}(J(Q)),\ C_{O_{p'}(G)}(Z(Q)) \mid q \in \mu \rangle$ (because $O_{\mu'}(O_{p'}(G)) = O_{p'}(G) \cap O_{\mu'}(O_{p'}(G)) \leqslant O_{\eta'}(G) = 1$). Now (1.12) follows.

(1.13) Suppose $Z(N) \leqslant H \in \mathcal{M}_{\mu,\eta}$, then $M \cap H = O_{\mu}(H)(H \cap X)$

<u>Proof</u> Suppose the result is false. Let Q_1 ($= Q \cap H$; Q being the α-invariant Sylow q-subgroup of G) denote the α-invariant Sylow q-subgroup of $(H \cap N) = N_1$. By (1.12) $H = \langle N_H(J(Q_1)),\ C_H(Z(Q_1)) \mid q \in \pi(N_1) \rangle\ O_{\mu}(H)$. If $N_H(J(Q_1))$ and $C_H(Z(Q_1))$ are contained in NX for each $q \in \pi(N_1)$, then $H = O_{\mu}(H)(H \cap NX)$ whence $M \cap H = O_{\mu}(H)(H \cap X)$. Because hypothesis A holds, for each $q \in \pi(N_1)$, $N_G(J(Q_1))$ and $C_G(Z(Q_1))$ are soluble. For each $q \in \pi(N_1)$, since $Z(Q) \leqslant Q \cap N = Q_1$, $C_H(Z(Q_1)) \leqslant \{C_G(Z(Q_1))\}_{\mu,\eta}$ is contained in NX. Therefore, as the result is assumed false there exists a $q \in \pi(N_1)$ such that $N_H(J(Q_1)) \nleqslant NX$.

A contradiction can now be obtained by choosing an α-invariant q-subgroup Q^+ maximal subject to

$Z(Q) \leq Q^+$ and $N_M(J(Q^+)) \not\leq X$ and mimicking the latter part of the proof of lemma 2.1 in Martineau [13].

(1.14) If $MN \neq NM$, then $O_\mu(MY) \neq 1 \neq O_\eta(NX)$.

Let $\mathcal{H}_{\mu,\eta} = \mathcal{M}_{\mu,\eta} \setminus \{MY, NX\}$

(1.15) If $H \in \mathcal{H}_{\mu,\eta}$, then

(i) $O_\mu(F(H)) \neq 1 \neq O_\eta(F(H))$,

(ii) $Z(N), Z(M) \leq H$,

(iii) $H = F(H)(X \cap H)(Y \cap H)$, and

(iv) $X \cap F(H) = 1 = Y \cap F(H)$.

By restricting attention to $\mathcal{H}_{\mu,\eta}$ the following strengthening of (1.10) may be obtained:

(1.16) If $H,K \in \mathcal{H}_{\mu,\eta}$ and L is a non-trivial α-invariant subgroup of $F(H)$ which is also contained in K, then $H = K$.

Proofs for (1.14), (1.15) and (1.16) may be culled from section 2 of Martineau [13] making use of (1.13).

Next some results concerning fixed-point-free automorphisms of square-free order will be reviewed.

(1.17) (Thompson [17]) A finite group admitting a fixed-point-free automorphism of prime order is nilpotent.

(1.18) (Ralston [15]) A finite group admitting a fixed-point-free automorphism of square-free order rs is soluble.(r and s primes).

(1.19) Let G be a group admitting the coprime fixed-point-free automorphism α of square-free order rs, and set $\rho = \alpha^s$ and $\sigma = \alpha^r$. Then

(i) G has Fitting length at most 2.

(ii) If neither r nor s is a Fermat prime, then

$$G/F(G) = (G/F(G))_\rho \times (G/F(G))_\sigma$$

(iii) If $P = P_\rho P_\sigma$ where P is the α-invariant Sylow p-subgroup of G, then G has a normal p-complement.

<u>Remarks</u> (i) By (1.18) G must be soluble and from Berger [1], (i) follows.

(ii) This appears as theorem 3.3(b) of Ralston [15].

(iii) The proof of lemma 3.2 in Ralston [15] furnishes a proof for this result, as the solubility of G removes the necessity for employing the Thompson normal p-complement theorem.

(1.20) (Ralston [15,theorem 4.1]). Let G be a group admitting a fixed-point-free automorphism α of square-free order and let P be the α-invariant Sylow p-subgroup of G. If $P_{\alpha_i} = 1$ for $i = 1,\ldots,n$, then P is a direct summand of G.

The proof given in Ralston [15] is for $n = 2$ but the proof works for any n.

(1.21) Suppose hypothesis D is satisfied and let P and Q be α-invariant Sylow p- and q-subgroups of G. If $P_\rho \neq 1 \neq P_\sigma$, $Q_\rho \neq 1 \neq Q_\sigma$ and $P_{\rho\sigma} = Q_{\rho\sigma} = 1$, then $PQ = QP$.

A proof for (1.21) may be extracted from section 6 of Ralston [15].

(1.22) Suppose G is a p-soluble group admitting α (not necessarily of square-free order) fixed-point-freely. If P^+ is an α-invariant p-subgroup of G such that $P^+_\beta = 1$ for all $\beta \in \langle\alpha\rangle^{\#}$, then $P^+ \leq O_p(G)$.

(1.22) may be established by a proof analogous to the one given in Ralston [15] for lemma 3.5 (and does not require α to be of square-free order).

The remaining results of this section are not directly concerned with fixed-point-free automorphisms.

(1.23) ([10, Satz 17.13]) Let P be an extra-special p-group of order p^{2m+1} admitting a coprime cyclic automorphism group A which centralises Z(P) and acts regularly upon P/Z(P). Let G denote the semi-direct product P with A. Suppose G is faithfully and irreducibly represented on the K-vector space V, where K is an algebraically closed field and (char K, $|G|$) = 1. If χ denotes the character of this representation of G on V and ρ the regular character of A, then

$$\chi|_A = \frac{(p^m - \delta)\rho}{|A|} + \delta\mu \qquad \text{where } \mu \text{ is some}$$

irreducible character of A and $\delta = +1$ or -1.

(1.24) (Wielandt [10,page 680]). If $G = HK$ where H and K are nilpotent Hall subgroups, then G is soluble.

(1.25) (Glauberman [6,theorem 1(iv)]) Let G be a finite group which admits a coprime automorphism group A. If $C_G(A)$ contains $C_G(\tau)$ where τ is an involution of G, then $[G,A]$ is a normal nilpotent subgroup of G.

(1.26) (Janko and Thompson [11]) Let G be a non-abelian simple group, and let $P \in Syl_2G$. Assume that $SCN_3(P) = \emptyset$ and that if x is an involution in P such that $[P:C_P(x)] \leq 2$, then $C_G(x)$ is soluble. Then G is isomorphic to one of the following groups:- $PSL_2(q)$ $(q > 3)$, A_7, M_{11}, $PSL_3(3)$, $PSU_3(3)$ or $PSU_3(4)$.

(1.27) (Gorenstein and Walters [9]) Let G be a finite group with $O_{2'}(G) = 1$ and let $P \in Syl_2G$. Suppose that $SCN_3(P) \neq \emptyset$ and that the centralizer of every involution of G is soluble. Then $O_{2'}(C_G(x)) = 1$ for every involution $x \in G$.

(1.27) is a weaker version of theorem B of [9].

The definition and elementary properties of the Thompson subgroup of a p-group can be found in Gorenstein [8, chapter 8].

2. ON SOLUBLE GROUPS WITH A FIXED-POINT-FREE AUTOMORPHISM, AND THE DEFINITION OF THE 'STAR GROUP'.

Suppose G is a finite group admitting a fixed-point-free automorphism α. Let H be an α-invariant subgroup of G and $\langle\beta\rangle$ a subgroup of $\langle\alpha\rangle$.

Definition 2.1 $\quad H^*_{\langle\beta\rangle} = \langle C_H(\beta^j) \mid 1 \leq j < |\beta| \rangle$.

That is, $H^*_{\langle\beta\rangle}$ is defined to be the subgroup of H which is generated by the fixed point sets (in H) of the non-trivial powers of β.

When $\langle\beta\rangle = \langle\alpha\rangle$ and there is no possibility of confusion $H^*_{\langle\alpha\rangle}$ will be written as H^*.

Remarks (i) (1.20) may be rephrased as: suppose G is a group admitting a fixed-point-free automorphism α of square-free order and assume that $P^* = 1$, where P is the α-invariant Sylow p-subgroup of G. Then P is a direct summand of G.

(ii) If β is of square-free order $r_1 \ldots r_m$, then $H^*_{\langle\beta\rangle} = \langle C_H(\beta_j) \mid 1 \leq j \leq m \rangle$.

Lemma 2.2 (i) $H^*_{\langle\beta\rangle}$ is an α-invariant subgroup of G.

(ii) If K is an α-invariant subgroup of H, then $K^*_{\langle\beta\rangle} \leq H^*_{\langle\beta\rangle}$.

(iii) If $\langle\gamma\rangle \leq \langle\beta\rangle$, then $H^*_{\langle\gamma\rangle} \leq H^*_{\langle\beta\rangle}$.

(iv) If $(|\beta|, |H|) = 1$ and N is an α-invariant subgroup of H, then $\overline{(H^*_{<\beta>})} = (\bar{H})^*_{<\beta>}$ (bars denote quotients by N).

<u>Proof</u> (i) This follows as $H^*_{<\beta>}$ is generated by α-invariant subgroups of G.

(ii) This is clear as $C_K(\beta^j) \leqslant C_H(\beta^j)$ for all j, $1 \leqslant j < |\beta|$.

(iii) Transparent from the definition.

(iv) As $(|\beta|, H) = 1$, by (1.7)(iv) $\overline{(H_{\beta^j})} = (\bar{H})_{\beta^j}$ for all j, $1 \leqslant j < |\beta|$, and so it follows that $(\bar{H})^*_{<\beta>} = \langle (\bar{H})_{\beta^j} | 1 \leqslant j < |\beta| \rangle = \langle \overline{(H_{\beta^j})} | 1 \leqslant j < |\beta| \rangle = \overline{(H^*_{<\beta>})}$.

<u>Lemma 2.3</u> Assume hypothesis A is satisfied and let P denote the α-invariant Sylow p-subgroup of G. Let π be a subset of $\pi(G)$ containing p, such that $\alpha^k = \beta$ acts fixed-point-freely upon all α-invariant π-subgroups of G; or, in other words G_β is a π'-subgroup. If R is a non-trivial α-invariant p-subgroup of G containing $P^*_{<\beta>}$ and $(p, |\beta|) = 1$, then $\{N_G(R)\}_\pi \leqslant \{N_G(P)\}_\pi$.

<u>Proof</u> Choose $\tilde{R}$ maximal with respect to the following:-

(1) $\tilde{R}$ is an α-invariant p-subgroup,

(2) $R \leqslant \tilde{R}$, and

(3) $\{N_G(R)\}_\pi \leqslant \{N_G(\tilde{R})\}_\pi$.

Clearly, there exists at least one such $\tilde{R}$.

Since $\tilde{R}$ is non-trivial, because of hypothesis A, $N_G(\tilde{R})$ is a proper α-invariant subgroup of G and therefore must be soluble. Hence $K/\widetilde{R}$ is soluble, where $K = \{N_G(\widetilde{R})\}_{\pi}$.

As $(p, |\beta|) = 1$, $(N_P(\widetilde{R})/\widetilde{R})^{*}{}_{<\beta>} = 1$ by (1.7)(iv) and so, from (1.21), $N_P(\widetilde{R})/\widetilde{R} \trianglelefteq K/\widetilde{R}$ as K admits β fixed-point-freely. Thus $N_P(\widetilde{R}) \trianglelefteq K$ which together with (3) implies $\{N_G(R)\}_{\pi} \leqslant \{N_G(\widetilde{R})\}_{\pi} = K \leqslant \{N_G(N_P(\widetilde{R}))\}_{\pi}$. Since $N_P(\widetilde{R})$ also satisfies (1) and (2), $N_P(\widetilde{R}) = \widetilde{R}$. Consequently $P = \widetilde{R}$ and therefore $\{N_G(R)\}_{\pi} \leqslant \{N_G(P)\}_{\pi}$.

Corollary 2.4 If $\langle\alpha\rangle = \langle\beta\rangle$ in lemma 2.3, then $N_G(R) \leqslant N_G(P)$.

Proof Immediate from lemma 2.3.

Lemma 2.5 Assume that the hypothesis of lemma 2.3 holds and, in addition, that α is of square-free order. Then, setting $K = \{N_G(R)\}_{\pi\setminus p}$,

(i) $[P,L] = [R,L] \leqslant R$, where L is any α-invariant subgroup of K,

(ii) $C_K(P^{*}{}_{<\beta>}) = C_K(P)$, and

(iii) If $\langle\alpha\rangle = \langle\beta\rangle$, then $\{C_G(P^{*})\}_{p'} = \{C_G(P)\}_{p'}$.

Proof (i) Consider the chain $R = P_0 < P_1 < \ldots < P_m = P$ where $P_0 = R$ and $P_i = N_P(P_{i-1})$ for $i = 1,\ldots,m$. Clearly each P_i is α-invariant. From lemma 2.3, $L \leqslant \{N_G(P)\}_{\pi} \leqslant N_G(P)$ so each P_i is L-invariant. As $(P_i/P_{i-1})^{*}{}_{<\beta>} = 1$, applying (1.20) to $L(P_i/P_{i-1})$ for each i gives that $[P,L] = [[\ldots[P,L],\ldots],L] \leqslant R$ and so $[P,L] = [R,L]$.

therefore must be soluble. Hence $K/\tilde{R}$ is soluble, where $K = \{N_G(\tilde{R})\}_{\pi}$.

As $(p, |\beta|) = 1$, $(N_P(\tilde{R})/\tilde{R})^*_{<\beta>} = 1$ by (1.7)(iv) and so, from (1.21), $N_P(\tilde{R})/\tilde{R} \trianglelefteq K/\tilde{R}$ as K admits β fixed-point-freely. Thus $N_P(\tilde{R}) \trianglelefteq K$ which together with (3) implies $\{N_G(R)\}_{\pi} \leqslant \{N_G(\tilde{R})\}_{\pi} = K \leqslant \{N_G(N_P(\tilde{R}))\}_{\pi}$. Since $N_P(\tilde{R})$ also satisfies (1) and (2), $N_P(\tilde{R}) = \tilde{R}$. Consequently $P = \tilde{R}$ and therefore $\{N_G(R)\}_{\pi} \leqslant \{N_G(P)\}_{\pi}$.

<u>Corollary 2.4</u> If $<\alpha> = <\beta>$ in lemma 2.3, then $N_G(R) \leqslant N_G(P)$.

<u>Proof</u> Immediate from lemma 2.3.

<u>Lemma 2.5</u> Assume that the hypothesis of lemma 2.3 holds and, in addition, that α is of square-free order. Then, setting $K = \{N_G(R)\}_{\pi \setminus p}$,

(i) $[P,L] = [R,L] \leqslant R$, where L is any α-invariant subgroup of K,

(ii) $C_K(P^*_{<\beta>}) = C_K(P)$, and

(iii) If $<\alpha> = <\beta>$,then $\{C_G(P^*)\}_{p'} = \{C_G(P)\}_{p'}$.

<u>Proof</u> (i) Consider the chain $R = P_0 < P_1 < \ldots < P_m = P$ where $P_0 = R$ and $P_i = N_P(P_{i-1})$ for $i = 1,\ldots,m$. Clearly each P_i is α-invariant. From lemma 2.3, $L \leqslant \{N_G(P)\}_{\pi} \leqslant N_G(P)$ so each P_i is L-invariant. As $(P_i/P_{i-1})^*_{<\beta>} = 1$, applying (1.20) to $L(P_i/P_{i-1})$ for each i gives that $[P,L] = [[\ldots[P,L],\ldots],L] \leqslant R$ and so $[P,L] = [R,L]$.

(ii) If $P^*_{<\beta>} = 1$, then the result is obvious, so it may be supposed that $P^*_{<\beta>} \neq 1$. Setting $R = P^*_{<\beta>}$ and $L = C_K(P^*_{<\beta>})$ in part (i) gives $[C_K(P^*_{<\beta>}), P] = 1$ and so $C_K(P^*_{<\beta>}) \leqslant C_K(P)$ which together with $C_K(P) \leqslant C_K(P^*_{<\beta>})$, yields $C_K(P^*_{<\beta>}) = C_K(P)$.

(iii) This follows from (ii) with $\pi = \pi(G)$.

<u>Lemma 2.6</u> Let G be a soluble group admitting the square-free automorphism α fixed-point-freely with P denoting the α-invariant Sylow p-subgroup of G. If $(p, |\alpha|) = 1$ and K is an α-invariant p'-subgroup of G normalized by P, then $K \leqslant O_{p'}(G)$.

<u>Proof</u> Let $\bar{G} = G/O_{p'}(G)$. As $(p, |\alpha|) = 1$, $\bar{P}^* = \overline{P^*}$ by lemma 2.2(iv). Thus $[O_p(\bar{G})^*, \bar{K}] \leqslant O_p(G) \cap K = 1$ and so from lemma 2.5(iii) it follows that $[O_p(\bar{G}), \bar{K}] = 1$. By the Hall, Higman centralizer lemma, $C_{\bar{G}}(O_p(\bar{G})) \leqslant O_p(\bar{G})$ and hence $\bar{K} = 1$. Thus $K \leqslant O_{p'}(G)$.

<u>Lemma 2.7</u> Suppose $G = PA$, where P is a normal p-subgroup of G and $A = \langle\alpha\rangle$ is cyclic with its order a $\{2,p\}'$ non-Fermat number. Let V be a F_qG-module which is faithful for G (F_q denotes the finite field of q elements, q a prime) with $(q, |G|) = 1$. If P has a non-trivial α-invariant section upon which α acts regularly, then $C_V(\alpha) \neq 1$.

<u>Proof</u> Recall that an automorphism group is said to act regularly if and only if each of its non-trivial elements act fixed-point-freely.

Assume the result is false and choose G and V to be a counterexample to the lemma subject to $|G|$ + dim V being minimal. It will be shown for such a pair G and V that $C_V(\alpha)$ must be non-trivial and hence it will follow that no such counterexample exists.

Suppose F is a field which contains a copy of F_q, then, it is well known that, $\dim_{F_q}(C_V(\alpha)) = \dim_F(C_{(F \otimes_{F_q} V)}(\alpha))$. Thus, as the objective is to demonstrate that $C_V(\alpha) \neq 1$, there is no loss in considering V as a vector space over F, which will be done with F chosen to be algebraically closed.

By hypothesis, there exists α-invariant subgroups W and Z of P such that $Z \underset{\neq}{\triangleleft} W$ and the induced action of α upon W/Z is regular. Clearly AW and V satisfy the hypotheses of the lemma and so, if $W \neq P$, it would follow that $C_V(\alpha) \neq 1$ contrary to the choice of G and V. Thus W = P. Further, it may be deduced that W/Z (= P/Z) has no non-trivial proper α-invariant subgroups. For, suppose that W_1 is a subgroup of P containing Z for which W_1/Z is such an α-invariant subgroup of W/Z, then, as the hypotheses of the lemma hold for AW_1 and V (observe that α acts regularly upon W_1/Z ($\neq 1$)), it may be asserted that $C_V(\alpha) \neq 1$. Hence P/Z is an elementary abelian p-group and so, in particular, $\phi(P) \leq Z$. In fact, $\phi(P) = Z$. Suppose otherwise and let bars denote quotients by $\phi(P)$.

Appealing to Maschke's theorem gives $P/\phi(P) = \bar{P} = \bar{Z} \times \bar{Z}_1$ where $\bar{Z}_1$ is an α-invariant subgroup of $\bar{P}$. If Z_1 denotes the inverse image of $\bar{Z}_1$ in P, then Z_1 is a proper α-invariant subgroup of P which has α acting regularly upon Z_1/Z ($= \bar{Z}_1 \neq 1$). Again, as $|Z_1| < |P|$, $C_V(\alpha) \neq 1$ whence it has been shown that $\phi(P) = Z$.

A further reduction of the minimum situation may be obtained in the guise of being able to assume V is irreducible. If V is not an irreducible G-module, then Maschke's theorem gives $V = U_1 \oplus U_2 \oplus \ldots \oplus U_f$ where each U_i is an irreducible G-module and $f > 1$. Since G acts faithfully upon V and P is non-trivial, there exists at least one U_j such that $P \not\leq \ker U_j$. In view of $P/\phi(P)$ being irreducible under the action of α, $P \cap \ker U_j \leq \phi(P)$. Observe that $[(A \cap \ker U_j), P] \leq P \cap \ker U_j \leq \phi(P)$ and so the regularity of α upon $P/\phi(P)$ demands that $A \cap \ker U_j = 1$. Since $\dim U_j < \dim V$ and the pair $G/\ker U_j$ and U_j satisfy the lemma's hypotheses, $C_{U_j}(\alpha) \neq 1$ implying $C_V(\alpha) \neq 1$.

Let D be a non-trivial normal abelian p-subgroup of G. Then, an appeal to Clifford's theorem yields that $V|_D \cong V_1 \oplus \ldots \oplus V_d$ where each V_i is itself the direct sum of irreducible D-modules which are pairwise isomorphic (as D-modules). Usually the V_i are called the Wedderburn components of V (with respect to D). For each $g \in G$, the mapping m_g: $V_i \mapsto V_i g = \{vg \mid v \in V_i\}$ is a permutation of the Wedderburn components of V (with respect to D). Moreover, this permutation representation of G (that is m: $g \mapsto m_g$) upon the set of Wedderburn components of V (with respect to D) is transitive and, because F has been assumed algebraically closed, ker m = $DC_G(D)$. The aim of the ensuing analysis is to show that the number of Wedderburn components of V with respect to D is one. So for a contradiction suppose $d > 1$. Since $d > 1$, V may be regarded as being induced up from a proper subgroup of G. More explicitly: if $1, g_2, \ldots, g_d$ is a set of right coset representatives of the stabilizer of V_1 in G (in the permutation representation on the Wedderburn components), then $V|_D \cong V_1 \oplus \ldots \oplus V_d \cong V_1 \oplus V_1^{g_2} \oplus \ldots \oplus V_1^{g_d}$ and so $V \cong V_1^G$. Since the act of 'inducing a module up' is transitive, it may be taken that $U^G \cong V$ where U is an H-module and H is a maximal subgroup of G.

Suppose that H contains P; clearly $H = P(H \cap A)$ and $[G:H]$ is a prime. Observe that ker U cannot contain P since $P \trianglelefteq G$ and thus $P \cap \ker U \leq \emptyset(P)$. As before, $[(\ker U \cap A), P] \leq \emptyset(P)$ implies that $\ker U \cap A = 1$ and so, as $|H| < |G|$ and the hypotheses of the lemma hold for $H/\ker U$ and U, it may be inferred that $C_U(H \cap A) \neq 1$. Hence $C_V(\alpha) \neq 1$.

Now consider the case that H does not contain P; without loss $H = (P \cap H)(A \cap H)$. If $A \not\leq H$, then $H \lneqq P(A \cap H) \lneqq G$ contradicts the maximality of H. So $A \leq H$ and, by appealing to the fact that $P/\emptyset(P)$ admits A irreducibly, it follows that $P \cap H = \emptyset(P)$. Let $g \in P \setminus \emptyset(P)$ and set $g^A = \{g^a = a^{-1}ga \mid a \in A\}$; clearly $|g^A| = |A|$. It is claimed that g^A is contained in some set of right cosets for $\emptyset(P)$ in P. Clearly it suffices to show that if $g_1, g_2 \in g^A$ are such that $g_1^{-1}g_2 \in \emptyset(P)$ then $g_1 = g_2$. Suppose there is such a pair $g_1, g_2 \in g^A$ for which

$g_1^{-1}g_2 \in \emptyset(P)$. Clearly $g_2 = g_1^{a}$ for some $a \in A$ and so $g_1^{-1}g_1^{a} \in \emptyset(P)$ whence, if $a \neq 1$, $g_1 \in \emptyset(P)$ by the regularity of A on $P/\emptyset(P)$. But this then forces $g \in \emptyset(P)$ which is not so. Thus $a = 1$ and so $g_1 = g_2$. Since a complete set of right coset representatives for $\emptyset(P)$ in P is at the same time a complete set of right coset representatives for H in G, by choosing a complete set of right coset representatives containing g^A for some $g \in P \setminus \emptyset(P)$, it may be seen that, as $V = U^G$, $C_V(\alpha) \neq 1$ in this case also.

Hence the objective, namely showing that $d = 1$, has been attained, and consequently $G = C_G(D)$ and D is cyclic. The first assertion follows from $\ker m = DC_G(D)$ and G acting transitively upon the Wedderburn components of V with respect to D, whilst the second is a consequence of the faithfulness of G on V and $V|_D$ being the direct sum of pairwise isomorphic irreducible D-modules combined with the fact that a non-cyclic abelian group cannot possess a non-trivial faithful irreducible representation over an algebraically closed field. It may thus be deduced that P has class at most 2 (otherwise there would exist a characteristic abelian subgroup of P not contained in $Z(P)$), $Z(P)$ is cyclic and is centralized by A. Let $a \in A^{\#}$ and set $\overline{P} = P/P'$ then, by (1.7)(viii), $\overline{P} = C_{\overline{P}}(a) \times [\overline{P}, a]$. If $C_{\overline{P}}(a) \neq 1$, then the inverse image of $[\overline{P}, a]$, $[P, a]\emptyset(P)$, in P would be a proper α-invariant subgroup of P whence, as the necessary hypotheses hold for $[P, a]\emptyset(P)$ and V, $C_V(\alpha) \neq 1$ could be deduced. Therefore for each $a \in A^{\#}$, $C_{\overline{P}}(a) = 1$ and so $C_P(a) \leq P'$. Evidently P/P' must then be an elementary abelian p-group (if α does not act irreducibly upon P/P', then induction may again be used) and consequently $\emptyset(P) \leq P' \leq Z(P) \leq C_P(A)$. The irreducibility of $P/\emptyset(P)$ as an α-module implies $\emptyset(P) = P' = Z(P) = C_P(A)$. Suppose $x, y \in P$; then $[x, y]^p = [x^p, y]$ (see 8, page 19, lemma 2.2(i)) implies as $x^p \in Z(P)$, that $[x, y]^p = 1$. Thus P' has exponent p as well as being cyclic and so $|P'| = p$.

The position now allows the use of (1.23); so (in the notation of (1.23)) $\chi|_A = \frac{(p^m-\delta)}{A}\rho + \delta\mu$ ($\delta = +1$ or -1). If $\delta = +1$, then $\chi|_A$ will contain the regular character as a constituent which means that $C_V(\alpha)$ would be non-trivial. Thus $\delta = -1$. Moreover, if $p^m + 1/|A| > 1$, $\chi|_A$ would have the regular character as a constituent , so it may be asserted that $p^m + 1 = |A|$. Recalling that $|A|$ is odd, the only possibility available is $p = 2$ and so $|A|$ is not a non-Fermat number. This gives the final contradiction.

<u>Definition 2.8</u> Let G be a group admitting a fixed-point-free automorphism α, H an α-invariant subgroup of G and $\beta \in \langle\alpha\rangle$. Then H is said to be star-covered with respect to β if and only if for each α-invariant subgroup K of H, $K = K^*_{\langle\beta\rangle}$.

<u>Remarks</u> When $\langle\beta\rangle = \langle\alpha\rangle$ and confusion is unlikely, H will just be referred to as being star-covered. Observe that if H is an α-invariant subgroup of G which is star-covered with respect to β, then all α-invariant subgroups of H are likewise star-covered with repect to β. Also note that, if H is an α-invariant subgroup of G such that all its α-invariant Sylow subgroups are star-covered with

respect to β , then so too is H star-covered with respect to β .

Lemma 2.9 Let G be a finite group admitting a fixed-point-free automorphism α, H be an α-invariant subgroup of G and $\beta \in \langle\alpha\rangle$. Assume that $(|H|, |\beta|) = 1$ and let M and N be normal α-invariant subgroups of H.

(i) If $N \leqslant H^*_{\langle\beta\rangle}$ and $H/N = (H/N)^*_{\langle\beta\rangle}$, then $H^*_{\langle\beta\rangle} = H$.

(ii) If H/M is star-covered with respect to β, $H/N = (H/N)^*_{\langle\beta\rangle}$ and $M \cap N = 1$, then $H^*_{\langle\beta\rangle} = H$.

Proof (i) Suppose $H^*_{\langle\beta\rangle} \neq H$ then, as $N \leqslant H^*_{\langle\beta\rangle}$, $(H/N)^*_{\langle\beta\rangle} = (H^*_{\langle\beta\rangle})N/N = H^*_{\langle\beta\rangle}/N \neq H/N$. Thus $H^*_{\langle\beta\rangle} = H$.

(ii) As $N \cap M = 1$, $NM/M \cong N$. Therefore $N = N^*_{\langle\beta\rangle}$ because of lemma 2.2(iv) and the fact that H/M is star-covered with respect to β. By (i), as $N \leqslant H^*_{\langle\beta\rangle}$, $H^*_{\langle\beta\rangle} = H$.

Lemma 2.10 Suppose G is a soluble group admitting the coprime fixed-point-free automorphism α of odd order. Let H and K denote, respectively, the α-invariant Hall π- and π'-subgroups of G and suppose $\langle\beta\rangle$ is a subgroup of $\langle\alpha\rangle$ for which $|\beta|$ is a non-Fermat number and $C_K(\beta) = 1$. Then $H/O_\pi(G)$ is star-covered with respect to β.

Proof By induction upon $|G|$; let G denote a counterexample of minimal order. As $(|\alpha|, |G|) = 1$, clearly it may be assumed that $O_\pi(G) = 1$ and so to derive a contradiction it will suffice to show that H is star-covered with respect to β; also note that, by the Hall, Higman centralizer lemma, $C_G(O_{\pi'}(G)) \leq O_{\pi'}(G)$. Further, H may be assumed to be a p-group (p a prime). For, if $|\pi(H)| > 1$, then applying induction to PK (P denoting the α-invariant Sylow p-subgroup of H) for each $p \in \pi(H) = \pi$ gives that each of the α-invariant Sylow subgroups of H is star-covered with respect to β and hence H must be star-covered with respect to β.

As H is not star-covered with respect to β, there exists an α-invariant subgroup D of H such that $D \neq D^*_{<\beta>}$. Since the hypotheses of the lemma hold for $DO_{\pi'}(G)$ (and $C_G(O_{\pi'}(G)) \leq O_{\pi'}(G)$), it may be supposed that $G = DO_{\pi'}(G)$.

Suppose that $|\pi(O_{\pi'}(G))| > 1$ and let $q \in \pi(O_{\pi'}(G))$; set $\eta = \pi' \setminus \{q\}$. Applying induction to both $D\{O_{\pi'}(G)\}_q$ and $D\{O_{\pi'}(G)\}_\eta$ gives that $D/C_D(\{O_{\pi'}(G)\}_q)$ and $D/C_D(\{O_{\pi'}(G)\}_\eta)$ are both star-covered with respect to β. Since $C_G(O_{\pi'}(G)) \leq O_{\pi'}(G)$, $C_D(\{O_{\pi'}(G)\}_q) \cap C_D(\{O_{\pi'}(G)\}_\eta) = 1$ and consequently, by lemma 2.9(ii), $D = D^*_{<\beta>}$. Thus it may be supposed that $\pi(O_{\pi'}(G)) = \{q\}$ and so $G = DO_q(G)$. Note that, as $D \neq D^*_{<\beta>}$, there exists a non-trivial α-invariant section of D upon which β acts regularly, namely $D/\phi(D)D^*_{<\beta>}$.

An examination of G yields that the necessary hypotheses for the application of lemma 2.7 hold with $<\beta> = A$, $D = P$ and $O_q(G) = V$. Consequently $C_{O_q(G)}(\beta) \neq 1$ which contradicts the hypothesis $C_K(\beta) = 1$. Therefore it follows that $D = D^*_{<\beta>}$ whence H is star-covered with respect to β.

<u>Remark</u> There exist examples showing the hypothesis that $|\beta|$ be an odd non-Fermat number is necessary in lemma 2.10.

A particular case of lemma 2.10 is singled out in:

Lemma 2.11 Assume the hypotheses of lemma 2.10 and, in addition, that β is of prime order. Then

(i) $H = O_\pi(G)C_H(\beta)$, and

(ii) G has π'-length one.

Proof (i) This follows directly from lemma 2.10 as $H = O_\pi(G)H^*_{<\beta>} = O_\pi(G)C_H(\beta)$.

(ii) Consider $G/O_\pi(G) = \bar{G} = \bar{H}\bar{K}$. As $\bar{H} = C_{\bar{H}}(\beta)$ and $C_{\bar{K}}(\beta) = 1$ clearly $\bar{K} \trianglelefteq \bar{G}$ (by (1.7)(xi)), and hence $G = O_{\pi,\pi',\pi}(G)$.

Lemma 2.12 Again suppose the hypotheses of lemma 2.10 hold with β again assumed to have prime order. If $K \trianglelefteq G$ and $[H, C_K(\beta)] = 1$, then $[H,\beta] \leq C_H(K)$.

Proof Induct upon $|G|$; if G is a counterexample of minimal order, by using induction, it may be shown that $C_H(K) = 1$ and that K is a q-group (q a prime). A further use of the minimality of G and a well known property of $\phi(K)$ allow the assumption that $\phi(K) = 1$. Employing Maschke's theorem (and the minimality of G) gives that K is a minimal normal α-invariant subgroup of G. Therefore either $C_K(\beta) = K$ or $C_K(\beta) = 1$. The former implies $[H, K] = 1$ whilst the latter, by lemma 2.11, gives $[H,\beta] = 1$, hence G is not a counterexample to the lemma.

Lemma 2.13 If the assumptions of lemma 2.10 hold and D is an α-invariant subgroup of H, then $D = D^*_{<\beta>}(O_\pi(G) \cap D)$.

(In particular $H = H^{*}{}_{<\beta>}O_{\pi}(G)$.)

Proof This follows from lemma 2.10 and lemma 2.2(iv).

An indication of the type of use to which lemma 2.13 can be put in a 'minimal situation' (to the 'fixed-point-free conjecture') is given in:

Lemma 2.14 Suppose G satisfies hypothesis A and, further that $|\alpha|$ is an odd non-Fermat number coprime to $|G|$. If P denotes an α-invariant Sylow p-subgroup of G which is not star-covered and L_1 and L_2 are two proper α-invariant subgroups of G containing P, then $O_p(PL_1) \cap O_p(PL_2) \neq 1$.

Proof As P is not star-covered, there exists an α-invariant subgroup D of P for which $D^{*} \neq D$. From lemma 2.10 $D/(D \cap O_p(PL_i))$ is star-covered (for $i = 1,2$). Thus, if $D \cap O_p(PL_1) \cap O_p(PL_2) = 1$, then lemma 2.9(ii) would be applicable giving $D = D^{*}$. Thus $D \cap O_p(PL_1) \cap O_p(PL_2) \neq 1$ and consequently $O_p(PL_1) \cap O_p(PL_2) \neq 1$.

Lemma 2.15 Let the hypotheses of lemma 2.10 hold. Moreover, suppose H is nilpotent, D is an α-invariant subgroup of H containing $O_{\pi}(H)$ and $N_H(D)^{*}{}_{<\beta>} \leq D$. Then $D = H$.

Proof From lemma 2.13, $N_H(D) = N_H(D)^{*}{}_{<\beta>}(N_H(D) \cap O_{\pi}(H)) = N_H(D)^{*}{}_{<\beta>}O_{\pi}(H) \leq D \leq N_H(D)$. As H is nilpotent, it follows that $D = H$.

Suppose, for the remainder of this section, G is a group which admits fixed-point-freely the automorphism of square-free order $r_1 \ldots r_n$. Set $\Lambda = \{1, 2, \ldots, n\}$.

Definition 2.16 Let P be an α-invariant Sylow p-subgroup of G. Then P is said to be of type Γ (where Γ is a subset of Λ) if and only if $i \in \Gamma$ implies $P_{\alpha_i} \neq 1$ and $i \notin \Gamma$ implies $P_{\alpha_i} = 1$.

Remark If G also satisfies hypothesis B, then (1.20) shows that G can have at most $2^n - 1$ possible types of α-invariant Sylow subgroups.

Lemma 2.17 Let P be an α-invariant Sylow p-subgroup of G of type Γ and set $\beta = \prod_{i \in \Gamma} \alpha_i$. Then for each $q \in \pi(G) \setminus \{p\}$, $[P, C_Q(\beta)] = 1$ (Q being the α-invariant Sylow q-subgroup of G).

Proof This will be by induction on $|G| + |\alpha|$. Select the pair G and α to be a counterexample to the lemma with $|G| + |\alpha|$ minimal. First, two observations: by Thompson's theorem (1.17), $n > 1$ and, by Ralston's result (1.20), $\Gamma \neq \emptyset$. Let $j \in \Gamma$. As $C_G(\alpha_j)$ admits $\alpha_1 \ldots \alpha_{j-1}\alpha_{j+1} \ldots \alpha_n$ fixed-point-freely, $C_G(\alpha_j)$ satisfies the conclusion of the lemma. Namely $[C_P(\alpha_j), C_{C_G(\alpha_j)}(\gamma)] = 1$ where $\gamma = \prod_{i \in \Gamma \setminus \{j\}} \alpha_i$ (because in $C_G(\alpha_j)$ $C_P(\alpha_j)$ is of type $\Gamma \setminus \{j\}$). However $C_{C_G(\alpha_j)}(\gamma) = C_G(\gamma) \cap C_G(\alpha_j) = C_G(\beta)$. Thus, for each $j \in \Gamma$, $[C_P(\alpha_j), C_Q(\beta)] = 1$ and consequently $[P^*, C_Q(\beta)] = 1$. Choose D to be maximal with respect to being an α-invariant p-subgroup containing P^* which is centralized by $C_Q(\beta)$. If $N_G(D) \neq G$, then using induction on $N_G(D)$ gives $[N_P(D), C_Q(\beta)] = 1$ whence $N_P(D) = P$. Whereas $N_G(D) = G$, as $D \neq 1$, allows induction to be utilized on G/D. Observe that P/D will be of type Δ where $\Delta \subseteq \Gamma$ and hence $[P, C_Q(\delta)] \leq D$ where $\delta = \prod_{i \in \Delta} \alpha_i$. Since $\Delta \subseteq \Gamma$, $C_Q(\beta) \leq C_Q(\delta)$ and so $1 = [[P, C_Q(\beta)], C_Q(\beta)] = [P, C_Q(\beta)]$. Thus either way $[P, C_Q(\beta)] = 1$ and so this supposed counterexample has been found wanting.

<u>Definition 2.18</u> $\hat{L}_i = \langle P \mid P$ is an α-invariant Sylow p-subgroup of G such that $C_P(\alpha_i) = 1 \rangle$ (for $i = 1,\ldots,n$).

<u>Lemma 2.19</u> Suppose G satisfies hypothesis B and assume that any finite group which admits a fixed-point-free automorphism of square free order $r_1 \ldots r_m$ is soluble when $m < n$, Then:

(i) $[\hat{L}_i, \{C_G(\alpha_1 \ldots \alpha_{i-1}\alpha_{i+1} \ldots \alpha_n)\}_{\hat{\pi}_i'}] = 1$ each $i \in \Lambda$ $(\hat{\pi}_i = \pi(\hat{L}_i))$.

(ii) For each $i \in \Lambda$, $\hat{L}_i$ is nilpotent.

(iii) Let H and K be proper α-invariant subgroups of G whose orders are coprime and let $\tilde{H}$ denote the largest α-invariant subgroup of H permutable with K. If π is a set of primes, $\widetilde{H_\pi}$ will denote the largest α-invariant subgroup of H_π which permutes with K. If $H \neq \tilde{H}$, then the α-invariant Hall π-subgroup of $\tilde{H}$ is $\widetilde{H_\pi}$.

<u>Proof</u> (i) Since G is non-soluble, $C_G(\alpha_1 \ldots \alpha_{i-1}\alpha_{i+1} \ldots \alpha_n)$ is a proper α-invariant subgroup of G and therefore soluble.

The result now follows from lemma 2.17.

(ii) Clearly may suppose $n > 1$, and without loss set $i = 1$. Let P and Q be, respectively, (non-trivial) α-invariant Sylow p- and q-subgroups of G such that $P_{\alpha_1} = Q_{\alpha_1} = 1$ $(p \neq q)$. Suppose P and Q are, respectively, of type Δ and Γ. Thus $\Delta, \Gamma \subseteq \{2, \ldots, n\}$ and, as hypothesis B is present, $\Delta \neq \emptyset \neq \Gamma$.

To establish the lemma, it must be shown that $[P, Q] = 1$. This may be achieved by demonstrating the existence of an α-invariant $\{p,q\}$-subgroup H_1 for which $P \cap H_1$ and $Q \cap H_1$ are both non-trivial. Then if H is a maximal α-invariant $\{p,q\}$-subgroup of G containing H_1, H is nilpotent as $H_{\alpha_1} = 1$. By hypothesis B, $N_G(P \cap H)$ is soluble $\{N_G(P \cap H)\}_{p,q} = N_P(P \cap H)N_Q(P \cap H) \geqslant H$ and the maximality of H gives $N_P(P \cap H) \leqslant H$. Hence $N_P(P \cap H) = P \cap H$ and so $P \cap H = P$. Thus $P \leqslant H$ and, similarly, $Q \leqslant H$ implying $[P, Q] = 1$ (as H is nilpotent).

If $\Delta \cap \Gamma \neq \emptyset$, then may take $H_1 = P_{\alpha_j}Q_{\alpha_j}$ where $j \in \Delta \cap \Gamma$.

So it may be supposed that $\Delta \cap \Gamma = \emptyset$. Set $\beta = \prod_{i \in \Delta} \alpha_i$ and $\gamma = \prod_{i \in \Gamma} \alpha_i$. Then, using lemma 2.17, $[P, \{G_\beta\}_{p'}] = 1 = [Q, \{G_\gamma\}_{q'}]$. Since $\Delta \neq \emptyset \neq \Gamma$ and $\Delta \cap \Gamma = \emptyset$, $G_{\beta\gamma}$ is a $\{p,q\}'$-subgroup. Now hypothesis B requires that G be non-soluble whence $G_{\beta\gamma} \neq 1$. Thus $N_G(G_{\beta\gamma})$ is soluble and contains both P and Q in its (unique) α-invariant Hall $\{p,q\}$-subgroup. This completes the proof of (ii).

(iii) Since $\widetilde{H}$ is just the subgroup of H generated by all the α-invariant subgroups of H which permute with K, it follows that $\widetilde{H_\pi} \leqslant \widetilde{H}$. Now, because $H \neq \widetilde{H}$, $\widetilde{H}K$ is a proper α-invariant subgroup of G and hence, by hypothesis B, $\widetilde{H}K$ is soluble. Consequently, there exists an α-invariant Hall $\{\pi \cup \pi(K)\}$-subgroup of $\widetilde{H}K$; or, in other words, $(\widetilde{H})_\pi K = K(\widetilde{H})_\pi$. Therefore, as $\widetilde{H_\pi}$ is just the subgroup of H_π generated by all the α-invariant subgroups of H_π which permute with K, $(\widetilde{H})_\pi \leq \widetilde{H_\pi}$. Thus $\widetilde{H_\pi} = (\widetilde{H})_\pi$ and (iii) is verified.

This section closes with some additional remarks and notation.

<u>Remarks</u> If G satisfies hypothesis C, Then:

(i) For each non-trivial α-invariant subgroup H ($\neq$ G) of G, $N_G(H)$ and $C_G(H)$ must be soluble and hence for any set of primes, π, both $N_G(H)$ and $C_G(H)$ must possess (unique) α-invariant Hall π-subgroups.

(ii) If M, N are nilpotent α-invariant Hall subgroups of G and $H \in \mathcal{M}_{\mu,\eta}$ with $1 \neq M \cap H \trianglelefteq H$, then $M \leq H$.

(iii) Let H_1, H and K be α-invariant subgroups of G with $(|H|,|K|) = 1$ and H_1 a subgroup of H containing $H^*_{<\beta>}$ ($\beta \in <\alpha>$). If $K \leq N_G(H) \cap N_G(H_1)$ and $K_\beta = 1$, then $H = H_1C_H(K)$. (This may be seen by selecting a $K<\alpha>$-invariant Sylow p-subgroup of H, $P \cap H$, (for each $p \in \pi(H)$) and applying (1.20) to each factor in the 'normalizer chain' between $P \cap H_1$ and $P \cap H$).

<u>Notation</u> (with n = 3 and $\{i,j,k\} = \{1,2,3\}$)

For each i and each (distinct) pair j and k define L_i to be the group generated by the α-invariant Sylow subgroups of type $\{j,k\}$ and L_{jk} the group generated by the α-invariant Sylow subgroups of type $\{i\}$. Set $\pi_i = \pi(L_i)$ and $\pi_{jk} = \pi(L_{jk})$. Note that $\hat{L}_i = (L_{ij} L_i L_{ik})$ (when subgroups of G are placed in the same bracket it is to be understood that they permute pairwise). In this situation lemma 2.19(i) gives $[L_i, \{G_{\alpha_{jk}}\}_{\pi_i'}] = 1 =$ $[L_{jk}, \{G_{\alpha_i}\}_{\pi_{jk}'}]$.

Most of the lemmas and theorems to follow are either proved under the assumption of hypothesis D or have the conclusion that hypothesis D cannot hold.

3. ON THE NUMBER OF MAXIMAL α-INVARIANT $\{\mu,\eta\}$-SUBGROUPS

For lemmas 3.1 and 3.2 assume that hypothesis A is satisfied and suppose that M and N are (respectively) nilpotent α-invariant Hall μ- and η-subgroups of G with X (respectively Y) the largest α-invariant subgroup of M (respectively N) which permutes with N (respectively M). Also assume that $\mu \cap \eta = \emptyset$

Lemma 3.1 For (ii), (iii) and (iv) also assume that $(|G|,|\alpha|) = 1$.

(i) $O_\mu(XY) = O_\mu(MY) \cap X$ and $O_\eta(XY) = O_\eta(NX) \cap Y$.

(ii) If $O_\eta(MY) \neq 1 \neq O_\mu(NX)$, then $MN = NM$.

(iii) If $\mathcal{H}_{\mu,\eta} \neq \emptyset$, then $O_\mu(XN) = 1 = O_\eta(MY)$.

(iv) If $H \in \mathcal{H}_{\mu,\eta}$ and α of square-free order, then $O_\mu(H)^* \neq 1 \neq O_\eta(H)^*$.

Proof (i) Clearly $O_\mu(MY) \cap X \leq O_\mu(XY)$ and a well known property of soluble groups gives, as Y normalizes the η'-subgroup $O_\mu(XY)$, the reverse inequality.

(ii) Suppose $O_\eta(MY) \neq 1 \neq O_\mu(NX)$ but $MN \neq NM$. As $MN \neq NM$, $O_\mu(MY) \neq 1 \neq O_\eta(NX)$ by (1.14). Since $MY \neq NX$, the 'uniqueness theorem'(1.10) yields that $O_\mu(MY) \cap X = 1 = O_\eta(NX) \cap Y$. Hence from part (i), $F(XY) = 1$ which is incompatible with XY being a non-trivial soluble group. Consequently, it may be deduced that $O_\eta(MY) \neq 1 \neq O_\mu(NX)$ implies that $MN = NM$.

(iii) Suppose $H \in \mathcal{H}_{\mu,\eta} = \mathcal{H}$ and that $O_\mu(NX) \neq 1$;

observe that $Z(M) \leq \{N_G(O_\mu(NX))\}_{\mu,\eta} = NX$. As $Z(M) \leq H$ by (1.15)(ii), $[Z(M), O_\eta(H)]$ is an η-group and, because $Z(M) < Z(X)$ implies that $Z(M) \leq O_{\eta,\mu}(NX)$ by the Hall, Higman centralizer lemma, $[Z(M), O_\eta(H)]$ is contained in $O_{\eta,\mu}(NX)$. Clearly, $[Z(M), O_\eta(H)]$ must be contained in $O_\eta(NX)$.

As $H \in \mathcal{H}$, $[Z(M), O_\eta(H)]$ must be non-trivial because of (1.15)(iv) and hence application of (1.11) yields that $F(H) \leq \{N_G([Z(M), O_\eta(H)])\}_{\mu,\eta} \leq NX$. Thus $F(H) \leq H \cap NX$ whence $NX = H$ by (1.10). This is against the definition of $\mathcal{H}$ and so it may be concluded that $\mathcal{H} \neq \emptyset$ implies $O_\mu(NX) = 1$ and, similarly, $O_\eta(MY) = 1$.

(iv) If, say, $O_\mu(H)^* = 1$ for some $H \in \mathcal{H}_{\mu,\eta} = \mathcal{H}$, then, from (1.20), $[O_\mu(H), (H \cap N)] = 1$ which implies, as $Z(N) < (H \cap N)$, that $O_\mu(H) \leq X$. However (1.15)(iv) states that $O_\mu(H) \cap X = F(H) \cap X = 1$. Thus $O_\mu(H)^* \neq 1$ and, likewise, $O_\eta(H)^* \neq 1$ where $H \in \mathcal{H}$.

<u>Remarks</u> (i) Clearly parts (i),(ii) and (iii) of lemma 3.1 hold for any fixed-point-free coprime automorphism group in the type of minimal situation typified by hypothesis A.

(ii) Part (ii) of lemma 3.1 has also been obtained by Glauberman and Martineau (unpublished) and by Pettet [14].

<u>Lemma 3.2</u> If $(|G|, |\alpha|) = 1$ and $N_\beta = 1$ where $\beta \in \langle\alpha\rangle$ and $|\langle\beta\rangle|$ is a non-Fermat prime, then $|\mathcal{M}_{\mu,\eta}| \leq 2$.

<u>Proof</u> Since $\mathcal{M}_{\mu,\eta} = \{MY, NX\} \setminus \mathcal{H}_{\mu,\eta}$ the lemma will follow if it can be shown that $\mathcal{H}_{\mu,\eta} = \emptyset$. So suppose $H_1 \in \mathcal{H}_{\mu,\eta} = \mathcal{H}$; note that $H_1 \cap X \neq 1 \neq H_1 \cap Y$. Now $H_1 = M_1N_1$ where $M_1 = M \cap H_1$ and $N_1 = N \cap H_1$ and, from lemma 2.11(i), $M_1 = M_{1_\beta} O_{\mu}(H_1)$, $M = O_{\mu}(M)M_{\beta}$ and $X = O_{\mu}(NX)X_{\beta}$. Hence $[M, \beta] \leq O_{\mu}(MY)$, $[M_1, \beta] \leq O_{\mu}(H_1)$ and, by lemma 3.1(iii), $X = X_{\beta}$.

If $M_{1_\beta} = M_1$ then, as $C_M(M_1) \leq M_1$, (1.7)(x) yields that $M = M_{\beta}$ and so, as $Y_{\beta} = 1$, $Y = [MY, \beta] \trianglelefteq MY$. Hence, as $Y \neq 1$ and $MN \neq NM$, it may be assumed that $M_{1_\beta} \neq M_1$.

Set $\widetilde{M}_1 = O_{\mu}(MY) \cap M_1$; note that $[\widetilde{M}_1, \beta] \neq 1$ as $[M_1, \beta] \leq M_1 \cap O_{\mu}(MY) = \widetilde{M}_1$ implies that $[M_1, \beta] = [\widetilde{M}_1, \beta]$. Claim that $(N_{O_{\mu}(MY)}(\widetilde{M}_1))_{\beta} \leq \widetilde{M}_1$ since $(N_M(\widetilde{M}_1))_{\beta} \leq \{N_G([\widetilde{M}_1, \beta])\}_{\mu,\eta} \leq H$ by, respectively, (1.7)(xiii) and (1.11) and hence $(N_{O_{\mu}(MY)}(\widetilde{M}_1))_{\beta} \leq O_{\mu}(MY) \cap M_1 = \widetilde{M}_1$. Since $Y \cap N_1$ normalises both $\widetilde{M}_1$ and $O_{\mu}(MY)$, an application of (1.17) to $(Y \cap N_1)(N_{O_{\mu}(MY)}(\widetilde{M}_1)/\widetilde{M}_1)$ yields that $N_{O_{\mu}(MY)}(\widetilde{M}_1) = \widetilde{M}_1 C_{N_{O_{\mu}(MY)}(\widetilde{M}_1)}(Y \cap N_1)$.

As $\{C_G(Y \cap N_1)\}_{\mu,\eta} \geq Z(N_1)$, the uniqueness theorem for $\mathcal{H}$,(1.16), yields that $\{C_G(Y \cap N_1)\}_{\mu,\eta}$ is contained in at least one of H_1, NX or MY. The latter possibility implies that $Y \cap F(H_1) = Y \cap O_{\eta}(H_1) \neq 1$ and so cannot occur. If $\{C_G(Y \cap N_1)\}_{\mu,\eta} \leq NX$ then, as $X \leq M_{\beta}$ and $(N_{O_{\mu}(MY)}(\widetilde{M}_1))_{\beta} \leq \widetilde{M}_1$, $N_{O_{\mu}(MY)}(\widetilde{M}_1) = \widetilde{M}_1 C_{N_{O_{\mu}(MY)}(\widetilde{M}_1)}(Y \cap N_1) = \widetilde{M}_1$. A similar conclusion may be drawn when $\{C_G(Y \cap N_1)\}_{\mu,\eta} \leq H_1$ for then $C_{N_{O_{\mu}(MY)}(\widetilde{M}_1)}(Y \cap N_1) \leq M_1 \cap O_{\mu}(MY) = \widetilde{M}_1$ and hence $N_{O_{\mu}(MY)}(\widetilde{M}_1) = \widetilde{M}_1$.

Consequently, $O_{\mu}(MY) \leq M_1$ and so $[M, \beta] \leq \widetilde{M}_1$. Hence $[M, \beta] = [M_1, \beta] \leq O_{\mu}(H_1)$. As $[M_1, \beta]$ is non-trivial and

$[M_1,\beta] \lhd M$, the uniqueness theorem applied to H_1 forces $M \leqslant H_1$ which contravenes the definition of $\mathcal{H}$. This contradiction arose from the assumption that $\mathcal{H} \neq \emptyset$ and so it may be deduced that $|\mathcal{M}_{\mu,\eta}| \leqslant 2$.

When, say, hypothesis C holds lemma 3.2 deals with the interaction between nilpotent α-invariant Hall subgroups at least one of which has some α_1 acting fixed-point-freely upon it. The succeeding lemmas examine the possibilities between Sylow subgroups of type $\{1,2,\ldots,n\}$ (though mostly with $n = 3$).

Let P and Q denote α-invariant Sylow p- and q-subgroups of G both of type $\{1,2,\ldots,n\}$ and let $\mathcal{M}_{p,q} = \{PY,QX\} \dot{\cup} \mathcal{H}_{p,q}$.

<u>Lemma 3.3</u> Suppose hypothesis C holds and $\mathcal{H}_{p,q} \neq \emptyset$. Let $H_1 = P_1Q_1 \in \mathcal{H}_{p,q} = \mathcal{H}$ where $P_1 = P \cap H_1$ and $Q_1 = Q \cap H_1$. Then $(N_p(J(P_1)))^* \not\leqslant P_1$.

<u>Proof</u> Suppose the contrary; that is $N_p(J(P_1))^* \leqslant P_1$. Thus $N_p(J(P_1)) = N_p(J(P_1))^*(N_p(J(P_1)) \cap O_p(PY)) = P_1N_{O_p(PY)}(J(P_1))$ by lemma 2.13. Observe that , for any non-trivial characteristic subgroup R of P_1, $N_Q(R) \leqslant Y$. Thus $N_Q(J(P_1)) \leqslant Y$.

Suppose $N_{Q_1}(J(P_1)) \neq 1$. Evidently, $N_{Q_1}(J(P_1))$ normalizes $J(P_1)$ and, as $N_{Q_1}(J(P_1)) \leqslant Y$, normalizes $O_p(PY)$. Hence $N_{Q_1}(J(P_1))$ normalizes $N_{O_p(PY)}(J(P_1))$.

Also, claim that $N_{Q_1}(J(P_1))$ normalizes $P_1 \cap O_p(PY)$, since $[P_1 \cap O_p(PY), N_{Q_1}(J(P_1))]$ is an α-invariant p-group contained in $N_{H_1}(J(P_1)) = P_1 N_{Q_1}(J(P_1))$, $[P_1 \cap O_p(PY), N_{Q_1}(J(P_1))] \leqslant P_1 \cap O_p(PY)$.

Now, $N_{O_p(PY)}(J(P_1)) = (P_1 \cap O_p(PY)) C_{N_{O_p}(J(P_1))}(N_{Q_1}(J(P_1)))$ because $N_{Q_1}(J(P_1))$ normalizes both $N_{O_p(PY)}(J(P_1))$ and $P_1 \cap O_p(PY)$, and the latter group contains $N_{O_p(PY)}(J(P_1))^*$. As $\{C_G(N_{Q_1}(J(P_1)))\} \geqslant Z(Q_1)$, (1.15)(iv) and (1.16) show that either $C_P(N_{Q_1}(J(P_1))) \leqslant X$ or P_1. However, as $C_P(N_{Q_1}(J(P_1)))$ is not star-covered and, because $O_p(QX) = 1$, X is star-covered by lemma 2.10, $C_P(N_{Q_1}(J(P_1))) \not\leqslant X$. Thus $C_P(N_{Q_1}(J(P_1))) \leqslant P_1$ and hence $N_{O_p(PY)}(J(P_1)) = P_1 \cap O_p(PY)$.

Consequently $N_P(J(P_1)) = P_1 N_{O_p(PY)}(J(P_1)) = P_1$ and thus $P = P_1$ which contradicts the definition of $\mathcal{H}$. Therefore $N_{Q_1}(J(P_1)) = 1$ and so $Y \cap Q_1 = C_{Q_1}(Z(P_1))$ because of Glauberman's factorization theorem (1.9). Thus $[Z(P_1) \cap O_p(H_1), Q_1] = 1$ which, as $Z(Q) \leqslant Q_1$, gives $1 \neq Z(P_1) \cap O_p(H_1) \leqslant X$ which cannot occur because of (1.15)(iv).

Thus, it may be concluded that $N_P(J(P_1))^* \not\leqslant P_1$.

Hypothesis D will be imposed upon the remainder of this section; set $\mathcal{H}_{p,q} = \mathcal{H}$.

<u>Lemma 3.4</u> Let $H_1 \in \mathcal{H}$. If $P_\rho Q_\rho \leqslant H_1$, $Q_\sigma \leqslant Y$ and $P_\tau \leqslant X$, then $PY = \{G_\sigma\}_{p,q}$ and $QX = \{G_\tau\}_{p,q}$.

<u>Proof</u> First, observe that as $\mathcal{H} \neq \emptyset$, $O_q(PY) = 1 = O_p(QX)$

and so, as $[O_p(PY)_\tau, Y \cap O_q(QX)] = 1 = [O_q(QX)_\sigma, X \cap O_p(PY)]$, $Y \cap O_q(QX) \leqslant Q_\tau$ and $X \cap O_p(PY) \leqslant P_\sigma$ by lemma 2.12. Thus $O_q(QX)_\sigma \leqslant O_q(QX)_\tau$ and $O_p(PY)_\tau \leqslant O_p(PY)_\sigma$ and hence $O_q(QX)_{\rho\sigma} = 1 = O_p(PY)_{\rho\tau}$. Since $P_\tau \leqslant X$ and $Q_\sigma \leqslant Y$, by lemma 2.10, P_τ and Q_σ are star-covered with respect to (respectively) $\rho\sigma$ and $\rho\tau$ because $O_q(PY) = O_p(QX) = 1$.

As $Q_\sigma \leqslant Y$ and $Y \cap O_q(H_1) = 1$, $O_q(H_1)_\sigma = 1$ and so, by lemma 2.12, $[P_1, \sigma]$ centralizes $O_q(H_1)$ (where $P_1 = P \cap H_1$). Suppose $[P_1, \sigma] \neq 1$, then, as $C_{P_1}(J(P_1)) \leqslant J(P_1)$, must have $[J(P_1), \sigma] \neq 1$. Since $Z(P_1)$, $O_q(H_1) \leqslant \{N_G([J(P_1), \sigma])\}_{p,q}$ from (1.10) it follows that $(N_P(J(P_1)))_\sigma \leqslant \{N_G([J(P_1), \sigma])\}_{p,q} \leqslant H_1$. Consequently P_ρ, $N_P(J(P_1))_\sigma \leqslant P_1$ and therefore, as $N_P(J(P_1))_\tau = N_P(J(P_1))_\tau{}^{*}{}_{\langle\rho\sigma\rangle}$, $N_P(J(P_1))_\tau \leqslant P_1$ also whence $N_P(J(P_1))^* \leqslant P_1$. However, by lemma 3.3, this cannot occur and so it may be deduced that $[P_1, \sigma] = 1$. As $C_P(P_1) \leqslant P_1$, this leads to $P = P_\sigma$ from whence it follows , as $O_q(PY) = 1$, that $Y = Y_\sigma = Q_\sigma$. Thus $PY = \{G_\sigma\}_{p,q}$.

Similar considerations also yield $QX = \{G_\tau\}_{p,q}$.

<u>Lemma 3.5</u> If $PQ \neq QP$, then either $P_{\alpha_i} \leqslant X$ or $Q_{\alpha_i} \leqslant Y$ for $i = 1,2,3$.

<u>Proof</u> Suppose the result is false. Thus, without loss it may be assumed that $P_\rho \not\leqslant X$ and $Q_\rho \not\leqslant Y$ and so $P_\rho Q_\rho = \{G_\rho\}_{p,q} \leqslant H_1 = P_1Q_1 \in \mathcal{H}$ where $P_1 = P \cap H_1$ and $Q_1 = Q \cap H_1$. First, it will be shown that $|\mathcal{H}| = 1$. Suppose $|\mathcal{H}| > 1$ then, by virtue of the uniqueness theorem (1.16) for

elements of $\mathcal{H}$, it follows that $F(H)_\rho = 1$ for all $H \in \mathcal{H} \setminus \{H_1\}$. Consequently, as $Z(P), Z(Q) \leqslant H$ for all $H \in \mathcal{H}$, $Z(P) \leqslant P_\rho$ and $Z(Q) \leqslant Q_\rho$, because of lemmas 2.12 and 1.15(iv). Further, observe that, as $P_1 = O_p(H_1)(P_1 \cap X)$ and $O_p(H_1) \cap X = 1$, $P_{1_\rho} = O_p(H_1)_\rho (P_1 \cap X)_\rho$ and hence, as $P_\rho = P_{1_\rho} \not\leqslant X$, $O_p(H_1)_\rho \neq 1$; similarly $O_q(H_1)_\rho \neq 1$.

If, say, $Z(P)_\rho \neq Z(P)_\rho{}^*{}_{<\sigma\tau>}$, then, lemma 2.10 forces $O_q(H_1)_\rho$ to lie in Y which is not compatible with (1.15)(iv). So $Z(P)_\rho = Z(P)_\rho{}^*{}_{<\sigma\tau>}$ and, likewise, $Z(Q)_\rho = Z(Q)_\rho{}^*{}_{<\sigma\tau>}$. If both $Z(P)_{\rho\sigma}$ and $Z(P)_{\rho\tau}$ are non-trivial then, as $G_{\rho\sigma}$ and $G_{\rho\tau}$ are nilpotent, $Q_{\rho\sigma}, Q_{\rho\tau} \leqslant Y$ whence $O_q(H_1)_{\rho\sigma} = O_q(H_1)_{\rho\tau} = 1$. From (1.20), $[P_\rho, O_q(H_1)_\rho] = 1$ and again $O_q(H_1)_\rho \leqslant Y$ which is not possible. Similar considerations apply to $Z(Q)_{\rho\sigma}$ and $Z(Q)_{\rho\tau}$.

Thus there are (essentially) two distinct cases:-

$Z(P) = Z(P)_{\rho\sigma}$ and $Z(Q) = Z(Q)_{\rho\sigma}$, or

$Z(P) = Z(P)_{\rho\sigma}$ and $Z(Q) = Z(Q)_{\rho\tau}$.

If the former holds then, because of (1.15)(iv), $O_p(H)_\sigma \neq 1 \neq O_q(H)_\tau$ for all $H \in \mathcal{H}$ and hence, by (1.16) this implies that $|\mathcal{H}| \leqslant 1$. This disposes of the first possibility.

Consider the case $Z(P) = Z(P)_{\rho\sigma}$ and $Z(Q) = Z(Q)_{\rho\tau}$. Clearly $O_q(H)_\tau \neq 1 \neq O_p(H)_\sigma$ for all $H \in \mathcal{H}$. Therefore, because of (1.16) and the assumption that $|\mathcal{H}| > 1$, neither $P_\tau Q_\tau$ nor $P_\sigma Q_\sigma$ can be contained in an element of $\mathcal{H}$. Consequently, in view of $O_p(H)_\sigma \neq 1$ and $O_q(H)_\tau \neq 1$ for any $H \in \mathcal{H}$, it follows that $P_\tau \leqslant X$ and $Q_\sigma \leqslant Y$. Lemma 3.5 may now be applied to give $PY = \{G_\sigma\}_{p,q}$ and $QX = \{G_\tau\}_{p,q}$.

In particular, $Z(P) = Z(P)_{\rho\sigma} = [Z(P),\tau] \leq [P \cap H, \tau] \leq O_p(H)$ for any $H \in \mathcal{H}$ and so, by (1.11), $P \leq H$ which shows that the second possibility cannot occur.

Thus $\mathcal{H} = \{H_1\}$ and therefore there are (essentially) four different possibilities:-

(i) $P_\rho Q_\rho \leq H_1$, $P_\sigma Q_\sigma \leq PY$ and $P_\tau Q_\tau \leq QX$;

(ii) $P_\rho Q_\rho \leq H_1$ and $P_\sigma Q_\sigma$, $P_\tau Q_\tau \leq PY$;

(iii) $P_\rho Q_\rho$, $P_\sigma Q_\sigma \leq H_1$ and $P_\tau Q_\tau \leq PY$; or

(iv) $P_\rho Q_\rho$, $P_\sigma Q_\sigma$, $P_\tau Q_\tau \leq H_1$.

(i) For this case lemma 3.4 is available to give that $PY = \{G_\sigma\}_{p,q}$ and $QX = \{G_\tau\}_{p,q}$. Thus, as $O_q(PY) = 1$ and PY has Fitting length at most 2, $P \trianglelefteq PY$ and, so since $Y = Y_{\sigma\tau}$ and $P_1 \geq P_\rho$, $P = P_1 C_P(Q_1 \cap Y)$. If $\{C_G(Q_1 \cap Y)\}_{p,q} \leq PY$ then, by (1.7)(x), $Q = Q_{\sigma\tau}$ whence $QX = \{G_{\sigma\tau}\}_{p,q}$, which implies, as $X \neq 1$, that $PQ = QP$. Thus either $\{C_G(Y \cap Q_1)\}_{p,q} \leq H_1$ or XQ; the first possibility leads to $P = P_1$ which is against the definition of $\mathcal{H}$. Hence $P = P_1 X = O_p(H_1)X$. As $X = P_\tau$ and $X \neq P$, $1 \neq [P,\tau] \leq O_p(H_1)$ which implies, by virtue of (1.11) that $P \leq H_1$. This shows that (i) cannot occur.

(ii) As $Q_\sigma, Q_\tau \leq Y$, $O_q(H_1)_\sigma = O_q(H_1)_\tau = 1$ and so both $[P_1,\sigma]$ and $[P_1,\tau]$ centralize $O_q(H_1)$ from lemma 2.12. Hence, if both $[P_1,\sigma]$ and $[P_1,\tau]$ are non-trivial then $N_P(J(P_1))^* \leq P_1$ which cannot occur by lemma 3.3. Therefore, without loss, may take $P_1 = P_{1\sigma}$ which then gives, as $C_P(P_1) \leq P_1$, $P = P_\sigma$. By (1.7)(vi), $[Y,\sigma] \leq O_q(PY)$ and so $Y = Y_\sigma$. Consequently, $P_\tau Q_\tau \leq P_\sigma Q_\sigma$ which implies that $\{G_{\rho\tau}\}_{p,q} = 1$ whence $PQ = QP$ by (1.21).

(iii) As $Q_\tau \leqslant Y$, $O_q(H_1)_\tau = 1$ and so it may be asserted that $[P_1, \tau] \leqslant C_{P_1}(O_q(H_1))$. If $[P_1, \tau]$ is non-trivial, then $N_P(J(P_1))^* \leqslant P_1$ which is untenable by lemma 3.3. Therefore $P_{1_\tau} = P_1$ and hence $P = P_\tau$. Further, as $O_q(PY) = 1$, $Y = Y_\tau = Q_\tau$. Without loss it may be supposed that $[X \cap P_1, \sigma] \neq 1$.

Since $X \cap P_1$ normalizes both $O_q(QX)$ and $O_q(QX) \cap Q_1$, $O_q(QX) = (O_q(QX) \cap Q_1)C_{O_q(QX)}([X \cap P_1, \sigma])$ as $O_q(QX) \cap Q_1 \geqslant O_q(QX)_\sigma$. The uniqueness theorem, (1.10), dictates that either $C_Q([X \cap H_1, \sigma]) \leqslant Y$ or Q_1. Thus (respectively) either $O_q(QX) = (O_q(QX) \cap Q_1)(O_q(QX) \cap Y)$ or $O_q(QX) = (O_q(QX) \cap Q_1)$. Since $X = X_\tau$, $Q = O_q(QX)Q_\tau = O_q(QX)Y$ and so (whichever possibility for $O_q(QX)$ occurs) $= YQ_1 = YO_q(H_1)$. As $Y = Q_\tau(\neq Q)$, this gives $1 \neq [Q, \tau] \leqslant O_q(H_1)$ and thence $Q \leqslant H_1$ which contradicts the definition of $\mathcal{H}$. Consequently (iii) cannot arise.

(iv) This possibility is easily vanquished by lemma 3.3.

The proof of lemma 3.5 is now complete.

Lemma 3.6 Suppose $PQ \neq QP$.

(i) If P_ρ, P_σ, $P_\tau \leqslant X$, then $\mathcal{H} = \emptyset$.

(ii) If $P_\rho \leqslant X$ and Q_σ, $Q_\tau \leqslant Y$, then $O_p(PY) \cap X \neq 1 \neq O_q(QX) \cap Y$ and $\mathcal{H} = \emptyset$.

Proof (i) Suppose $\mathcal{H} \neq \emptyset$ and let $H \in \mathcal{H}$. As $P^* \leqslant X$ and $X \cap O_p(H) = 1$, $O_p(H)^* = 1$ which, by lemma 3.1(iv) is impossible. Thus $\mathcal{H} = \emptyset$.

(ii) First note that $X \neq 1 \neq Y$. Initially it will be shown that $O_p(PY) \cap X \neq 1 \neq O_q(QX) \cap Y$; this will be done by considering each case in turn.

Suppose $O_p(PY) \cap X = 1$. As $P_\rho \leq X$, clearly $O_p(PY)_\rho = 1$ and so by lemma 2.12 $[O_p(PY), [Y,\rho]] = 1$.

If $[Y,\rho] \neq 1$, then by the Hall-Higman centralizer lemma $O_q(PY) \neq 1$. Moreover, lemma 3.1(ii) shows that $\mathcal{H}$ must be empty. Because $P_\rho \leq X$, P_ρ normalizes both $O_q(QX)$ and $O_q(QX) \cap Y$ and, as $O_q(QX)^*_{<\sigma\tau>} \leq O_q(QX) \cap Y$, $O_q(QX) = C_{O_q(QX)}(P_\rho)(O_q(QX) \cap Y)$. It is claimed that $C_P(P_\rho) \leq X$. For suppose $C_P(P_\rho) \not\leq X$ then $C_{O_q(QX)}(P_\rho) \leq Y$ giving $O_q(QX) \leq Y$. Recalling that $[Y,\rho] \neq 1$ the uniqueness theorem (1.10) yields that $N_Q(Y)^* \leq Y$. Consequently, from lemma 2.15, it follows that $Y = Q$. Thus $C_P(P_\rho) \leq X$ and, in particular, $Z(P) \leq X$. However $X \cap O_p(PY) = 1$ and so $Z(P) \cap O_p(PY) = 1$ which contradicts the conclusion of (1.14).

Therefore it may be taken that $[Y,\rho] = 1$ whence $P = O_p(PY)P_\rho$. Hence $[X,\rho] \leq [P,\rho] \leq O_p(PY)$ which gives, as $X \cap O_p(PY) = 1$, $[X,\rho] = 1$ and so $X = P_\rho$. Further, $[Y,\rho] = 1$ also yields that $Q^* = Q_\rho$. The preceding observations imply that $O_q(QX) = C_{O_q(QX)}(X)\ O_q(QX)_\rho$. As X is non-trivial either $C_{O_q(QX)}(X) \leq Y$ or Q_1 where $Q_1 = Q \cap H_1$ and $H_1 \in \mathcal{H}$. If $C_{O_q(QX)}(X) \leq Y \leq Q_\rho$ then, together with the fact that $Q = Q_\rho O_q(QX)$, it follows that $Q = Q_\rho$. The other possibility yields that $Q = Q_\rho O_q(QX) = Q_\rho Q_1 = Q_\rho O_q(H_1)$, (because $Y \leq Q_\rho$) and so if $Q \neq Q_\rho$ then an applicationof (1.11) would force $Q \leq H_1$. Therefore $Q = Q_\rho$ and so $XQ = \{G_\rho\}_{p,q}$. If $O_p(XQ) \neq 1$ then $P = P_\rho = X$ and so $O_p(XQ) = 1$ which, in turn, gives $Q \trianglelefteq QX$ by 1.19(i). ThusQ $= YC_Q(X)$ as $Q^*_{<\sigma\tau>} \leq Y$. If $\mathcal{H} \neq \emptyset$ then, as $X \neq 1$, $C_Q(X) \leq Y$ implying $Q = Y$. Whereas, if $\mathcal{H} \neq \emptyset$ then, because of the uniqueness theorem (1.10), $Z(P) \leq P_\rho \leq X$ which also yields $Q = Y$.

Hence $O_p(PY) \cap X \neq 1$.

Suppose $O_q(XQ) \cap Y = 1$. Clearly $O_q(QX)_\sigma = O_q(QX)_\tau = 1$ and so $[X,\sigma]$ and $[X,\tau]$ both centralize $O_q(QX)$. By hypothesis, $P_\rho \leq X$ and hence at least one of $[X,\sigma]$ and $[X,\tau]$ must be non-trivial. Thus $O_p(XQ) \neq 1$. Further, from lemma 3.1(ii) and (iii) it follows that $\mathcal{H} = \emptyset$ and $O_q(PY) = 1$. If $[X,\sigma] = 1$ then, as $C_p(X) \leq X$, $P = P_\sigma$ which leads to $Y = Y_\sigma$. Thus $\{G_\sigma\}_{p,q} = PY \geq P_\tau Q_\tau$ and so $\{G_{\rho\tau}\}_{p,q} = 1$. Employing (1.21) gives $PQ = QP$. A similar conclusion follows if $[X,\tau] = 1$. Hence $[X,\sigma] \neq 1 \neq [X,\tau]$. Consequently $N_p(X)^* \leq X$.

Let $D = O_p(PY) \cap X$ ($= O_p(XY)$, by lemma 3.1(i)). If $[D,\sigma] = 1$ then $O_p(XY) \leq X_\sigma$ and hence $Y = Y_\sigma$ because, by lemma 3.1(i), $O_q(XY) = O_q(QX) \cap Y = 1$. From 1.7(vi), $X = X_\sigma O_p(XY) = X_\sigma$. However $X = X_\sigma$ has already been excluded. Hence $[D,\sigma] \neq 1$ and likewise $[D,\tau] \neq 1$. Applying the uniqueness theorem to XQ gives that $N_p(D)_\tau$, $N_p(D)_\sigma \leq X$ (this is possible since $[D,\sigma] \neq 1 \neq [D,\tau]$ and $[D,\sigma]$ and $[D,\tau]$ centralize $O_q(XQ)$). Clearly $N_p(D)_\rho \leq P_\rho \leq X$ and hence $N_{O_p(PY)}(D)_\rho$, $N_{O_p(PY)}(D)_\sigma$, $N_{O_p(PY)}(D)_\tau \leq X \cap O_p(PY) = D$ implying that $N_{O_p(PY)}(D)^* \leq D$.

As Y normalizes $O_p(PY)$ and D,Y normalizes $N_{O_p(PY)}(D)$ and so $N_{O_p(PY)}(D) = DC_{N_{O_p(PY)}(D)}(Y)$. Because $Y \neq 1$ and $\mathcal{M}_{p,q} = \{PY, XQ\}$, $C_p(Y) \leq {}^P X$ which yields that $C_{N_{O_p(PY)}(D)}(Y) \leq X \cap O_p(PY) = D$ whence $N_{O_p(PY)}(D) = D$. Thus $O_p(PY) \leq X$ which, when combined with $N_p(X)^* \leq X$, and lemma 2.15 yields $P = X$.

Therefore $O_q(XQ) \cap Y \neq 1$.

Next it will be demonstrated that $\mathcal{H} = \emptyset$. Since PY $\neq$ XQ, $X \cap O_p(PY) \neq 1 \neq Y \cap O_q(QX)$ and both $(X \cap O_p(PY))O_q(PY)$ and $(Y \cap O_q(QX))O_p(QX)$ are contained in $PY \cap QX$, by (1.10), it may be concluded that $O_q(PY) = 1 = O_p(QX)$. As $[X \cap O_p(PY), Y \cap O_q(QX)] = 1$, it follows that $X \cap O_p(PY)$ centralizes $O_q(QX)_\sigma$ and $O_q(QX)_\tau$ and that $Y \cap O_q(QX)$ centralizes $O_p(PY)_\rho$. Thus $X \cap O_p(PY) \leqslant P_{\sigma\tau}$ and $Y \cap O_q(QX) \leqslant Q_\rho$, by lemma 2.12, because $O_q(PY) = 1 = O_p(XQ)$.

By hypothesis $P_\rho \leqslant X$ whence, as α is fixed-point-free, $O_p(PY)_\rho = 1$ from which it may be asserted, as $O_q(PY) = 1$, that $Y = Y_\rho$. Hence $Q^* = Q_\rho$. Claim that $Q_\rho \neq Q$ for if $Q_\rho = Q$ then $X = X_\rho$ because $O_p(XQ) = 1$ hence giving $1 \neq X \cap O_p(PY) \leqslant P_{\sigma\tau} \cap P_\rho$ which cannot occur.

Now suppose there exists $H_1 = P_1Q_1 \in \mathcal{H}$, where $P_1 = P \cap H_1$ and $Q_1 = Q \cap H_1$. As $Y \leqslant Q_\rho$, $[Q_1, \rho] \leqslant O_q(H_1)$. Since $Q_1 = Q_{1\rho}$ would force $Q = Q_\rho$, $Q_1 \neq Q_{1\rho}$. Consequently $[J(Q_1), \rho] \neq 1$ and as $[J(Q_1), \rho] \leqslant O_q(H_1)$ employing (1.11) it may be deduced that $N_Q(J(Q_1))^* = N_Q(J(Q_1))_\rho \leqslant Q_1$. By lemma 3.3 this cannot happen and so a contradiction has been obtained.

Thus $\mathcal{H} = \emptyset$.

Lemmas 3.5 and 3.6 taken together yield:

<u>Lemma 3.7</u> If PQ $\neq$ QP, then $|\mathcal{M}_{p,q}| = 2$.

4. NORMAL P-COMPLEMENTS

This section contains, with the exception of lemma 4.6 and corollary 4.7, results whose conclusion is the existence of a normal p-complement.

Lemma 4.1 Let G be a group admitting a coprime automorphism group A. If $C_G(A)$ contains a Sylow p-subgroup of G and $C_G(A) = O_{p',p}(C_G(A))$, then $G = O_{p',p}(G)$.

Proof Let R be a non-trivial subgroup of P where $P \in Syl_pG$ and $P \leqslant C_G(A) = C$. From (1.7)(iii), $N_G(R) = N_C(R)C_G(R)$ and so $N_G(R)/C_G(R) \cong N_C(R)/C_C(R)$. Because $C_G(A)$ has a normal p-complement, $N_C(R)/C_C(R)$ must be a p-group. A well known result of Frobenius (see [8,7.4.5(a)]) yields the desired conclusion.

Lemma 4.2 Suppose G is a soluble group admitting the coprime automorphism γ of order rs, where r and s are distinct primes. Let $\rho = \gamma^s$ and $\sigma = \gamma^r$ and let P be a γ-invariant Sylow p-subgroup of G. If $P_\rho = P_\sigma = G_\gamma$, then G has a normal p-complement.

Proof Suppose the result is false and choose G to be a counterexample of minimal order. Observe that, as P is the only γ-invariant Sylow p-subgroup of G by(1.7)(vii), the hypotheses of the lemma carry over to γ-invariant subgroups of G and γ-invariant quotients of G. If $O_{p'}(G) \neq 1$, then applying induction to $G/O_{p'}(G)$ it may be deduced that G has a normal p-complement. Thus familiar properties of soluble groups give that $O_p(G) \neq 1$ and that $C_G(O_p(G)) \leqslant O_p(G)$. From the latter property, $G = O_p(G)Q$ where Q is a (non-trivial) γ-invariant Sylow q-subgroup of G, $p \neq q$ and, moreover, Q possesses no non-trivial proper γ-invariant subgroups. If $\phi(O_p(G)) \neq 1$, then, by induction, $[O_p(G),Q] \leqslant \phi(O_p(G))$ whence, from a well known property of the Frattini subgroup of a p-group, $[O_p(G),Q] = 1$. Thus, it may be assumed that $\phi(O_p(G)) = 1$ and further, because of Maschke's theorem, $O_p(G)$ must be a minimal normal γ-invariant subgroup of G. Clearly, $C_{O_p(G)}(Q) = 1$.

If $G_\gamma = 1$, then (1.20) is applicable to give that $G = O_p(G) \times Q$. Thus $G_\gamma = P_\gamma \neq 1$. Note that as $G_\gamma = P_\rho = P_\sigma$, from (1.7)(vi), both G_ρ and G_σ have normal p-complements. Therefore, if either $Q = Q_\rho$ or $Q = Q_\sigma$, then $1 \neq P_\gamma = P_\rho = P_\sigma \leqslant C_{O_p(G)}(Q) = 1$. Thus $Q_\rho = Q_\sigma = 1$ and so $Q\langle\gamma\rangle$ is a Frobenius group which is faithfully and irreducibly represented on $O_p(G)$. By a well known result this representation, when restricted to $\langle\gamma\rangle$ contains the regular representation of $\langle\gamma\rangle$ and, under these circumstances, $P_\gamma = P_\rho = P_\sigma$ cannot hold. Thus a contradiction has been reached and the verification of the lemma is finished.

<u>Lemma 4.3</u> Suppose G is a soluble group admitting the coprime fixed-point-free automorphism α of square-free order rst. Assume that $P_\rho = P_{\rho\sigma}P_{\rho\tau}$, $P_\sigma = P_{\rho\sigma}P_{\sigma\tau}$ and $P_\tau = P_{\rho\tau}P_{\sigma\tau}$ where P is the α-invariant Sylow p-subgroup of G. Then G has a normal p-complement.

<u>Proof</u> Deny the result and let G be a counterexample of minimal order. As the hypotheses of the lemma hold for α-invariant subgroups of G and α-invariant quotients of G, using a reduction of the type given in lemma 4.2 yields that $G = O_p(G)Q$ where Q is an α-invariant Sylow q-subgroup with $q \neq p$ and Q possessing no non-trivial proper α-invariant subgroups. Again, $O_p(G)$ may be taken to be a minimal normal α-invariant subgroup of G and $C_{O_p(G)}(Q) = 1$.

If one of Q^* or P^* is trivial, then (1.20) gives $G = O_p(G) \times Q$ and consequently it may be supposed that $P^* \neq 1 \neq Q^*$. Without loss it may be assumed that $Q_\tau \neq 1$ and so, as Q_τ is α-invariant, $Q = Q_\tau$. By (1.19)(iii), G_ρ, G_σ and G_τ must have normal p-complements and so $P_\tau \leq C_P(Q) = 1$. Hence $P_\rho = P_\sigma = P^*$. If either Q_ρ or Q_σ is non-trivial, then similarly, it would follow that $P^* \leq C_P(Q)$ and so $P_\rho = P_\sigma = G_{\rho\sigma}$. Lemma 4.2 shows that G has a normal p-complement. Therefore there is no counterexample and so the lemma is established.

Lemma 4.3 when combined with work in section 2 gives rise to:

<u>Lemma 4.4</u> Suppose hypothesis D is satisfied and let P be an α-invariant Sylow p-subgroup of P, $p \in \pi(G)$, which is not star-covered. If $Z(J(P))_{\rho} = Z(J(P))_{\rho\sigma} Z(J(P))_{\rho\tau}$, $Z(J(P))_{\sigma} = Z(J(P))_{\rho\sigma} Z(J(P))_{\sigma\tau}$, and $Z(J(P))_{\tau} = Z(J(P))_{\rho\tau} Z(J(P))_{\sigma\tau}$, then P is contained in a unique maximal α-invariant subgroup, K, of G. Moreover, $K = C_G(R)$ where R is a non-trivial α-invariant subgroup of Z(P).

<u>Proof</u> Let K_1 and K_2 be two maximal α-invariant subgroups of G which contain P. Since K_1 and K_2 are soluble, by lemma 2.14, as P is not star-covered, $O_p(K_1) \cap O_p(K_2) \neq 1$ and, from (1.9), $K_i = N_{K_i}(J(P))C_{K_i}(Z(P))O_{p'}(K_i)$ for $i = 1,2$. As $N_G(Z(J(P)))$ is soluble, by lemma 4.3, $N_G(Z(J(P)))/C_G(Z(J(P)))$ is a p-group. Set $R = O_p(K_1) \cap O_p(K_2) \cap Z(P)$; as $O_p(K_1) \cap O_p(K_2)$ is non-trivial, R is non-trivial. Thus $K_1, K_2 \leqslant C_G(R) \neq G$. The maximality of K_1 and K_2 forces $K_1 = C_G(R) = K_2$ and the lemma is proven.

Suppose G satisfies hypothesis D and P is an α-invariant Sylow p-subgroup of G where $p \in \pi(G)$. If, p is odd and say, $P_{\rho} = 1$, then, because of the Thompson normal p-complement theorem [19] and lemma 4.3, $P_{\sigma} = P_{\tau}$ cannot occur. The purpose of the next result is to give some information which will be of relevance to this type of situation when $p = 2$.

<u>Lemma 4.5</u> Suppose G admits a coprime automorphism γ of order rs, where r and s are distinct primes; set $\rho = \gamma^s$ and $\sigma = \gamma^r$. Assume further that G = PH, where P is a

γ-invariant Sylow 2-subgroup of G and H is a γ-invariant soluble Hall 2'-subgroup of G. If $P_\rho = P_\sigma = G_\gamma \leq N_P(H)$, then G has a normal 2-complement.

Proof Let G be a counterexample of minimal order. Note that for each $q \in \pi(G)$ there is a unique γ-invariant Sylow q-subgroup of G by (1.7)(vii), from which it follows that all γ-invariant subgroups of G satisfy the hypothesis of the lemma. By lemma 4.2 it may be assumed that G is not soluble. If S(G), the largest normal soluble subgroup of G is non-trivial then, as the hypotheses of the lemma carry over to G/S(G), and H is soluble, G/S(G) is soluble. Hence G is soluble and so S(G) must be trivial. Observe that $Z(P) \cap G_\gamma = 1$ for, if $\tilde{P} = Z(P) \cap G_\gamma \neq 1$ then $\tilde{P}^H (\leq N_P(H)H \neq G)$ is a non-trivial normal soluble subgroup of G. A further consequence of $S(G) = 1$ is that P is a maximal γ-invariant subgroup of G; for if $P \lneqq K \lneqq G$ where K is a γ-invariant subgroup of G, then $(K \cap H)^H$ is a non-trivial normal soluble subgroup of G.

From (1.18), it may be assumed that $G_\gamma \neq 1$. Let $x \in G_\gamma$ be an involution. Now, $C_G(x)$ is a proper γ-invariant subgroup of G and so, by induction, $C_G(x)$ has a normal 2-complement (namely $C_H(x)$). Since $Z(P)C_H(x)$ admits γ fixed-point-freely with $Z(P)_\rho = Z(P)_\sigma = 1$, $[Z(P), C_H(x)] = 1$ by (1.20), which implies that $C_H(x) \leq C_G(Z(P))$. Consequently $C_H(x) = 1$ as $C_G(Z(P))$ is a proper γ-invariant subgroup of G containing P. By (1.17)

H must be nilpotent and so by Wielandt's result (1.24), G is soluble.

This contradiction completes the proof of lemma 4.5.

The next result is of interest in its own right.

Lemma 4.6 Suppose G is a finite group admitting a coprime automorphism group A with $C_G(A)$ soluble. If $C_G(A)$ contains a Sylow 2-subgroup of G, then G is soluble.

Proof By induction on $|G|$. Choose G to be a counter-example of minimal order to the lemma and let P denote a Sylow 2-subgroup of G which is contained in $C_G(A)$. From the Feit, Thompson theorem [2], it may be supposed that $P \neq 1$. If H is an A-invariant subgroup of G and let $\tilde{P}$ denote an A-invariant Sylow p-subgroup of H. By (1.7)(vii) there exists $y \in C_G(A)$ such that $\tilde{P}^y \leq P \leq C_G(A)$ and so $\tilde{P} \leq C_G(A)$. Since A-invariant quotients of G satisfy the hypotheses of the lemma, G cannot have any non-trivial proper normal A-invariant subgroups. Clearly G must be characteristic simple and so $G = G_1 \times \ldots \times G_n$ where the G_i, which are non-abelian simple groups, are pairwise isomorphic and comprise the set of minimal normal subgroups of G. Evidently $P = P_1 \times \ldots \times P_n$ where $P_i = P \cap G_i$. For each $a \in C_G(A)$, $a(G_1)$ must be one of the G_i but as $1 \neq P_1 \leq G_1 \cap a(G_1)$ it follows that $a(G_1) = G_1$ for all $a \in A$. Thus G is a non-abelian simple group.

Claim that $SCN_3(P) \neq \emptyset$; for if $SCN_3(P) = \emptyset$, then by (1.26) since for each $x \in P^{\#}$ $C_G(x)$ is soluble and G is a non-abelian simple group, G is one of the following groups:- PSL(2,q) ($q > 3$), A_7, M_{11}, PSL(3,3) or PSU(3,4). However, none of these groups can be counterexamples to the lemma. Now a result of Gorenstein and Walters (1.27), may be applied to give that $O_{2'}(C_G(x)) = 1$ for each involution x of G. Thus if x is an involution contained in P, then, as $C_G(x)$ is soluble, by (1.7)(vi) $C_G(x) = O_{2'}(C_G(x))\, C_{C_G(x)}(A) = C_{C_G(x)}(A)$ $\leq C_G(A)$.

This gives the desired contradiction since under these circumstances, Glauberman has shown (see(1.25)) that $[G,A]$ is nilpotent and so $G = C_G(A)[G,A]$ must be soluble.

The preceding lemma may be used to give a partial solution to a conjecture of Thompson's (see [4,problem 4])

<u>Corollary 4.7</u> If G, a finite group, admits the coprime automorphism group A and for each $a \in A^{\#}$, $C_G(a)$ is soluble, then G contains an A-invariant soluble subgroup which admits A faithfully.

<u>Proof</u> The proof is immediate from the Feit, Thompson theorem [2] and lemma 4.6.

5. THE STRUCTURE OF MAXIMAL α-INVARIANT $\{\mu,\eta\}$-SUBGROUPS

Hypothesis D will be assumed to hold for the remainder of this work (though the next two lemmas hold in a much wider context). Let M and N be α-invariant nilpotent Hall Subgroups of G with X and Y having the same meaning as at the beginning of section 3.

Lemma 5.1 Suppose $MN \neq NM$ and $N_\rho = 1$. If $O_\mu(XN) \neq 1$, then $Y = 1$.

Proof Suppose $Y \neq 1$. If $X = X_\rho$ then by (1.7)(x), as $C_M(X) \leq X$, $M = M_\rho$ whence, as $N_\rho = 1$ and $Y \neq 1$, $MN = NM$. Thus it may be assumed that $1 \neq [X,\rho] \leq O_\mu(XN)$. By lemma 2.11(i), $[X,\rho] \leq [M,\rho] \leq O_\mu(MY)$ and, with the help of (1.10), it may be shown that $(N_{O_\mu(MY)}(X \cap O_\mu(MY)))_\rho \leq X \cap O_\mu(MY)$. Clearly, as Y normalizes both $O_\mu(MY)$ and $X \cap O_\mu(MY)$, $N_{O_\mu(MY)}(X \cap O_\mu(MY)) = (X \cap O_\mu(MY)) \cdot C_{N_{O_\mu(MY)}(X \cap O_\mu(MY))}(Y)$. Now, as $Y \neq 1$, $Y \neq N$ and $\mathcal{M}_{\mu,\eta} = \{MY, NX\}$, $N_M(Y) \leq X$ which gives that $X \geq O_\mu(MY)$. In particular, $[M,\rho] \leq X$ and so $[M,\rho] = [X,\rho] \leq O_\mu(XN)$. Hence $MN = NM$ by (1.11) as $[M,\rho]$ is a non-trivial normal subgroup of M and from this contradiction it may be inferred that $Y = 1$.

Lemma 5.2 Suppose $MN \neq NM$ and $N_\rho = 1$, then $X = N_M(N)$.

Proof By lemma 2.11(ii), XN has η-length one so $X = O_{\mu}(XN)N_X(N)$. If $O_{\mu}(XN) = 1$ then $X \leqslant N_M(N)$ and so $X = N_M(N)$. On the other hand, $O_{\mu}(XN) \neq 1$ yields, from lemma 5.1, that $Y = 1$. Then, as $Z(M) \leqslant X$, (1.13) implies that $N = O_{\eta}(XN)Y = O_{\eta}(XN)$ whence $X = N_M(N)$.

Thus $X = N_M(N)$.

Let $\{i,j,k\} = \{1,2,3\}$.

Lemma 5.3 If $L_{ij}P \neq PL_{ij}$ where P is an α-invariant Sylow p-subgroup of G of type $\{1,2,3\}$, then $\mathcal{M}_{\pi_{ij},p} = \{L_{ij}N_P(L_{ij}),P\}$.

Proof From lemma 3.2, $|\mathcal{M}_{p,\pi_{12}}| = 2$. Since $(L_{ij})^{*}_{\langle\alpha_i\alpha_j\rangle} = 1$, it follows that $[(L_{ij})_{\alpha_k},\{G_{\alpha_k}\}_{\pi_{ij}'}] = 1$ and that $(L_{ij})^{*} = (L_{ij})_{\alpha_k}$. Hence, from lemma 2.5(iii), $[L_{ij},P_{\alpha_k}] = 1$, and so $C_P(L_{ij}) \neq 1$. By employing lemmas 5.1 and 5.2 the result follows.

Lemma 5.4 If $L_iL_j \neq L_jL_i$, then $\mathcal{M}_{\pi_i,\pi_j} = \{L_iN_{L_j}(L_i), L_jN_{L_i}(L_j)\}$.

Proof From lemma 3.2, $|\mathcal{M}_{\pi_i,\pi_j}| = 2$; then a double application of lemma 5.2 gives the result.

Lemma 5.5 If $L_iP \neq PL_i$ where P is an α-invariant Sylow p-subgroup of type $\{1,2,3\}$, then one of the following holds (without loss may set $i = 1$ and, because of lemma 3.2, may take $\mathcal{M}_{p,\pi_1} = \{PY, XL_1\}$):-

(i) $L_1^* \leq Y$, and furthermore

(a) $X = 1$;

(b) either $L_{1_\sigma} = L_{1_\tau}$ or $Z(L_1) \leq Y$;

(c) $Y = N_{L_1}(P)$ and so $\mathcal{M}_{p,\pi_1} = \{PN_{L_1}(P), L_1\}$;

(d) if $Z(L_1) \leq Y$, then $Z(L_1) = Z(L_1)_{\sigma\tau}$;

(e) $P_{\sigma\tau} = 1$;

(f) P is not equal to P_ρ, P_σ or P_τ;

(g) $P_{\rho\sigma} \neq 1 \neq P_{\rho\tau}$

(ii) $P_\sigma, P_\tau \leq X$ with $O_p(XL_1) \neq 1$, and furthermore

(a) $X = N_P(L_1) = N_P(L_1)_\rho C_P(L_1)$;

(b) $Y = 1$, so $\mathcal{M}_{p,\pi_1} = \{N_P(L_1)L_1, P\}$;

(c) $Z(P) = Z(P)_\rho \leq N_P(L_1)$;

(d) either $N_P(N_P(L_1))^* \leq N_P(L_1)$ or $P = P_\rho$ (that is either P is not star-covered or $P = P_\rho$); or

(iii) $P_\sigma, P_\tau \leq X$ with $O_p(XL_1) = 1$, and furthermore

(a) $P^* = P_\rho \geq X = N_P(L_1)$;

(b) $P_{\sigma\tau} = 1$;

(c) $Y = N_{L_1}(P) \leq L_{1_{\sigma\tau}}$;

(d) if $L_1 = L_1^*$, then $P = P_\rho$.

<u>Proof</u> It will first be demonstrated that either P_σ, $P_\tau \leq X$ or L_{1_σ}, $L_{1_\tau} \leq Y$. This will be achieved by showing that either of $P_\sigma \leq X$ and $L_{1_\tau} \leq Y$ or $P_\tau \leq X$ and $L_{1_\sigma} \leq Y$ imply that $PQ = QP$. Because of the symmetry of the arguments it will suffice to examine the case when $P_\sigma \leq X$ and $L_{1_\tau} \leq Y$.

As $PL_1 \neq L_1P$ and $L_{1_\rho} = 1$, lemma 5.2 implies that $X = N_P(L_1)$. Observe that, if $O_p(XL_1) \neq 1$, then lemma 5.1 may be applied with the result that $X = 1$ which is not so.

Hence $O_p(XL_1) = 1$ and so $P_\sigma \leqslant X \leqslant P_\rho$ by lemma 2.11(i).

As X normalizes both L_1 and Y with Y containing L_{1_τ}, $L_1 = C_{L_1}([X,\tau])Y$ by lemma 2.12. Clearly $[X,\tau] \neq 1$ since $P_\sigma \leqslant X \leqslant P_\rho$. As $C_{L_1}([X,\tau]) \leqslant Y$ forces $L_1 = Y$, whereas $PL_1 \neq L_1P$, therefore $C_{L_1}([X,\tau]) \not\leqslant Y$. Hence $C_P([X,\tau]) \leqslant X \leqslant P_\rho$ which, from (1.7)(x) implies that $P = P_\rho$ and so $Y \trianglelefteq PY$. Since $Y \neq 1$, this yields that $PL_1 = L_1P$.

Thus either $L_1^* \leqslant Y$ or $P^*_{\langle\sigma\tau\rangle} \leqslant X$. Shall now proceed to verify the remainder of the lemma.

<u>(i) $L_1^* \leqslant Y$</u>

(a) From lemma 5.2, $X = N_P(L_1)$ and as X also normalizes $Y \geqslant L_1^*$, $L_1 = YC_{L_1}(X)$. If $X \neq 1$ then, necessarily, $C_{L_1}(X) \leqslant Y$ giving $L_1 = Y$. Therefore $X = 1$.

(b) If $L_{1_\sigma} \neq L_{1_\tau}$, then may suppose, without loss that $L_{1_\sigma} \not\leqslant L_{1_\tau}$. Hence $O_{\pi_1}(P_\sigma L_{1_\sigma}) \neq 1$ and so $\{N_G(O_{\pi_1}(P_\sigma L_{1_\sigma}))\}_{p,\pi_1} \geqslant P_\sigma, L_{1_\sigma}, Z(L_1)$. As $X = 1$ and $P_\sigma \neq 1$, the only possibility is that $Z(L_1) \leqslant Y$.

(c) If $L_{1_\sigma} = L_{1_\tau}$ then applying lemma 4.3 to PY gives that $P \trianglelefteq PY$ whilst, if $Z(L_1) \leqslant Y$, then, from (1.13), $P = O_p(PY)X = O_p(PY)$. Hence $Y = N_{L_1}(P)$.

(d) If $Z(L_1) \leqslant Y$ then by lemma 2.13 and (1.11), as $PL_1 \neq L_1P$, $Z(L_1)^* = Z(L_1)$. If $Z(L_1)_\sigma \not\leqslant Z(L_1)_\tau$, then $Z(L_1) \cap O_{\pi_1}(P_\sigma L_{1_\sigma}) \neq 1$ by lemma 2.13 hence $\{N_G(Z(L_1) \cap O_{\pi_1}(P_\sigma L_{1_\sigma}))\}_{p,\pi_1} \geqslant O_p(P_\sigma L_{1_\sigma}), L_1$. As $P_\sigma L_{1_\sigma}$ is p-closed this contradicts (a). Hence $Z(L_1)_\sigma \leqslant Z(L_1)_\tau$. Similar considerations also show that $Z(L_1)_\tau \leqslant Z(L_1)_\sigma$. Thus $Z(L_1)_\sigma = Z(L_1)_\tau = Z(L_1)^* = Z(L_1)$.

(e) As $[P_{\sigma\tau}, L_1] = 1$, clearly $P_{\sigma\tau} \leqslant X = 1$.

(f) If $P = P_\sigma$ or P_τ, then, as $P_{\sigma\tau} = 1$, P could not be of type $\{1,2,3\}$.

Suppose $P = P_\rho$. As $1 \neq L_1^* \leqslant Y$ and $L_{1_\rho} = 1$ this clearly forces $PL_1 = L_1P$ which contradicts one of the hypotheses of the lemma. Thus $P = P_\rho$ cannot occur either.

(g) Suppose (say) that $P_{\rho\sigma} = 1$ then $P_\sigma{}^*{}_{\langle\rho\tau\rangle} = 1$ and so $[P_\sigma, L_{1_\sigma}] = 1$. If $Z(L_1) \leqslant Y$ then $Z(L_1) = Z(L_1)_{\sigma\tau}$ and so $P_\sigma \leqslant X$ whereas, if $Z(L_1) \not\leqslant Y$ then $L_{1_\sigma} = L_{1_\tau}$ and again $P_\sigma \leqslant X$. Thus $P_{\rho\sigma} = 1$ is impossible; so too, by similar arguing, is $P_{\rho\tau} = 1$.

<u>(ii) $P_\sigma, P_\tau \leqslant X$ with $O_p(XL_1) \neq 1$</u>

That $Y = N_P(L_1)$ and $X = 1$ follow (respectively) from lemmas 5.2 and 5.1. Furthermore $N_P(L_1) = N_P(L_1)_\rho C_P(L_1)$ follows from lemma 2.11(i) and hence $Z(P)_\rho = Z(P) \leqslant N_P(L_1)$.

If $[N_P(L_1), \rho] \neq 1$, then, by (1.10) and lemma 2.12, $(N_P(N_P(L_1)))_\rho \leqslant N_P(L_1)$ and so $(N_P(N_P(L_1)))^* \leqslant N_P(L_1)$. On the other hand $[N_P(L_1), \rho] = 1$ implies, as $C_P(N_P(L_1)) \leqslant N_P(L_1)$, that $P = P_\rho$ so establishing part (d).

<u>(iii) $P_\sigma, P_\tau \leqslant X$ with $O_p(XL_1) = 1$</u>

(a) By lemma 5.2, $X = N_P(L_1)$. Since it is assumed that $O_p(XL_1) = 1$, it follows that $[X, \rho] = 1$ and so $P^* = P_\rho \geqslant X$.

(b) This is immediate from (a).

(c) As P_σ, $P_\tau \leqslant X = N_P(L_1)$, $[Y, O_p(PY)_\sigma] = [Y, O_p(PY)_\tau] = 1$ and hence, by lemma 2.12, $[Y, \sigma]$ and $[Y, \tau]$ centralize $O_p(PY)$. Suppose $O_{\pi_1}(PY) \neq 1$. As $L_1 = L_{1_\sigma}$ or $L_1 = L_{1_\tau}$ would imply that $O_p(XL_1)$ is non-trivial this means that both $[Y, \sigma]$ and $[Y, \tau]$ must be non-trivial. By employing the uniqueness theorem 1.10 may obtain that

$N_{L_1}(Y)_\sigma, N_{L_1}(Y)_\tau \leqslant Y$; that is $N_{L_1}(Y)^* \leqslant Y$. Now, $X = N_P(L_1)$ normalizes both Y and $N_{L_1}(Y)$ whence $N_{L_1} = C_{N_{L_1}(Y)}(X)Y$. Because $X \neq 1$ it may be deduced that $C_{L_1}(X) \leqslant Y$ giving $N_{L_1}(Y) = Y$ which, in view of L_1 being nilpotent, implies $Y = L_1$.

Thus it may be concluded that $O_{\overline{\pi}_1}(PY) = 1$ and so $Y \leqslant L_{1_{\sigma\tau}}$. Moreover, as $P_{\sigma\tau} = 1$, it follows that $P = [P, \sigma\tau] \trianglelefteq PY$ and therefore $Y = N_{L_1}(P)$.

(d) Suppose $P \neq P_\rho$ and let $\bar{L}_1 = L_1/\emptyset(L_1)$. By lemma 2.2(iv), $\bar{L}_1 = \bar{L}_1^* = \bar{L}_{1_\sigma}\bar{L}_{1_\tau}$ (because $\bar{L}_1$ is abelian). As $P_\sigma \leqslant X = N_P(L_1)$ and $[P_\sigma, \tau] = P_\sigma$, P_σ acts by conjugation upon $\bar{L}_1$ with the result that $\bar{L}_1 = \bar{L}_{1_\sigma} C_{\bar{L}_1}(P_\sigma)$. If $C_P(P_\sigma) \leqslant X$, then clearly $P = P_\rho$ thus $C_{L_1}(P_\sigma) \leqslant Y \leqslant L_{1_{\sigma\tau}}$ from (c). Hence, as $C_{\bar{L}_1}(P_\sigma) = \overline{C_{L_1}(P_\sigma)}$, $C_{\bar{L}_1}(P_\sigma) \leqslant \bar{L}_{1_{\sigma\tau}}$ and therefore $\bar{L}_1 = \bar{L}_{1_\sigma}$. By similar arguments, $\bar{L}_{1_\tau} = \bar{L}_1$. Using a well known result about the Frattini subgroup of a nilpotent group, it may be deduced that $L_1 = L_{1_{\sigma\tau}}$. As $G_{\sigma\tau}$ is nilpotent, lemma 4.1 shows this situation to be at variance with the supposed non-permuting of P and L_1. Thus $L_1 = L_1^*$ implies that $P = P_\rho$.

<u>Remark</u> If $PL_1 \neq L_1P$ (with P as in lemma 5.5 and $\mathcal{M}_{p,\pi_1} = \{PY, XL_1\}$),then the proof of lemma 5.5 also shows that if one of P_σ or P_τ is contained in X then this implies that both must be contained in X. Similarly, if one of L_{1_σ} or L_{1_τ} is contained in Y, then both must be contained in Y.

<u>Lemma 5.6</u> Let P and Q be α-invariant Sylow p- and q-subgroups of G of type $\{1,2,3\}$ which do not permute (and let $\mathcal{M}_{p,q} = \{PY, QX\}$). Then, with possible interchanging of p and q and rearrangement of ρ, σ and τ, one of the following occurs :

(i) $P^* \leq X$, and furthermore

(a) $Y = 1$, so $\mathcal{M}_{p,q} = \{P, XQ\}$;

(b) $Z(P) \leq X$ and $X = N_P(Q)$;

(c) $Z(P)$ is contained in one of $P_{\sigma\tau}$, $P_{\rho\sigma}$ or $P_{\rho\tau}$;

(d) (Suppose in (c), $Z(P) \leq P_{\sigma\tau}$) $Q_{\sigma\tau} = 1$ and $Q_{\rho\sigma} \neq 1 \neq Q_{\rho\tau}$;

(e) Q cannot be equal to either Q_ρ, Q_σ or Q_τ.

<u>or</u> (ii) $P_\rho \leq X$ and $Q_\sigma, Q_\tau \leq Y$, and furthermore

(a) $X \cap O_p(PY) \neq 1 \neq Y \cap O_q(QX)$;

(b) $O_q(PY) = O_p(XQ) = 1$ and so $X \cap O_p(PY) \leq P_{\sigma\tau}$ and $Y \cap O_q(QX) \leq Q_\rho$;

(c) $Y \leq Q_\rho$ and so $Q^* = Q_\rho$ (also $Q_{\sigma\tau} = 1$);

(d) $Q \neq Q_\rho$ (so Q is not star-covered);

(e) for all non-trivial α-invariant subgroup R of P_ρ, $N_P(R) \leq X$;

(f) $X = X_{\sigma\tau} \times P_\rho$;

(g) $Z(P) \leq X_{\sigma\tau}$;

(h) $[P, \rho] = O_p(PY)$;

(i) $P_\rho = P_{\rho\sigma} P_{\rho\tau}$ (and hence G_ρ has a normal p-complement);

(j) $XQ \trianglerighteq Q$;

(k) $X \neq X_\sigma$ or X_τ (so $P_{\rho\sigma} \neq 1 \neq P_{\rho\tau}$);

(l) $N_P(X)^* \leq X$ so P is not star-covered either;

(m) either P is contained in a unique maximal α-invariant subgroup of G or $J(P) \trianglelefteq PY$ and $J(P)_\rho = 1$.

<u>Proof</u> From lemma 3.5, without loss, one of the following cases occur: $P^* \leq X$ or $P_\rho \leq X$ and $Q^*_{<\sigma\tau>} \leq Y$.

First, the additional assertions made when $P^* \leq X$ holds will be proven.

<u>(i) $P^* \leq X$</u>

(a) Now $P = O_p(PY)\ X$ since, by lemma 2.10, $P = O_p(PY)P^*$. Clearly Y acts upon $O_p(PY)$ and $O_p(PY) \cap X$, the latter group containing $O_p(PY)^*$. Thus $O_p(PY) = (O_p(PY) \cap X)\ C_{O_p(PY)}(Y)$ and so $P = XC_P(Y)$. If $Y \neq 1$, then $C_P(Y) \leq X$ which would give $P = X$ and this establishes (a)

(b) If, say, $P_\rho \neq P_\rho{}^*{}_{<\sigma\tau>}$ then $O_p(G_\rho) \neq 1$ and so $\{N_G(O_p(G_\rho))\}_{p,q} \geq Z(P), Q_\rho$ which then implies, as $Y = 1$ and $Q_\rho \neq 1$, that $Z(P) \leq X$. Therefore may suppose that $P_\rho = P_\rho{}^*{}_{<\sigma\tau>}$, $P_\sigma = P_\sigma{}^*{}_{<\rho\tau>}$ and $P_\tau = P_\tau{}^*{}_{<\rho\sigma>}$. It is clear, as P is of type $\{1,2,3\}$, that at least two of $P_{\sigma\tau}$, $P_{\rho\sigma}$, $P_{\rho\tau}$ are non-trivial. Suppose $P_{\rho\sigma}$ and $P_{\rho\tau}$ are non-trivial. If either of $Q_{\rho\sigma}$ or $Q_{\rho\tau}$ is non-trivial then as $G_{\rho\sigma}$ and $G_{\rho\tau}$ are nilpotent it would follow that $Z(P) \leq X$. Thus it may be assumed $Q_{\rho\sigma} = Q_{\rho\tau} = 1$ whence $[P_\rho, Q_\rho] = 1$. Since $\{N_G(P_\rho)\}_{p,q} \geq Z(P), Q_\rho$ and $Y = 1$, $Z(P) \leq X$. Thus $Z(P) \leq X$ and an application of (1.13) yields that $X = N_P(Q)$.

(c) The proof of (c) depends upon the two following observations:-

(c)' $Z(P)_\rho = Z(P)_\rho{}^*{}_{\langle\sigma\tau\rangle}$, $Z(P)_\sigma = Z(P)_\sigma{}^*{}_{\langle\rho\tau\rangle}$ and $Z(P)_\tau = Z(P)_\tau{}^*{}_{\langle\rho\sigma\rangle}$.

(c)'' at least two of $Z(P)_{\rho\tau}$, $Z(P)_{\rho\sigma}$ and $Z(P)_{\sigma\tau}$ are trivial.

(c)' If, say, $Z(P)_\rho \neq Z(P)_\rho{}^*{}_{\langle\sigma\tau\rangle}$ then $Z(P) \cap O_p(G_\rho) \neq 1$ and this gives $P, Q_\rho \leq \{N_G(Z(P) \cap O_p(G_\rho))\}_{p,q}$ (because $Q_\rho = O_q(\{G_\rho\}_{p,q})$) which contravenes (a). Similar considerations apply to $Z(P)_\sigma$ and $Z(P)_\tau$.

(c)'' If this assertion were false then could choose $Z(P)_{\rho\sigma}$ and $Z(P)_{\rho\tau}$ (say) to be non-trivial. Because $Y = 1$, $Q_{\rho\tau} = Q_{\rho\sigma} = 1$ and hence $[P_\rho, Q_\rho] = 1$ which produces yet another compromising situation with $Q_\rho \leq Y$.

Returning to the verification of (c), because of (c)'' it may be supposed, without loss, that $Z(P)_{\rho\tau} = Z(P)_{\rho\sigma} = 1$. Combining this with (c)' gives $Z(P)_\sigma = Z(P)_\tau = Z(P)_{\sigma\tau} = Z(P)^*$. As $Z(P) \leq X$ and $PQ \neq QP$, $Z(P) = Z(P)^*$ and so (c) is established.

(d) As $Z(P) \leq P_{\sigma\tau}$ clearly $Q_{\sigma\tau} \leq Y = 1$ and if, say, $Q_{\rho\sigma} = 1$ then $[P_\sigma, Q_\sigma] = 1$ implying $1 \neq Q_\sigma \leq Y = 1$. Consequently $Q_{\sigma\tau} = 1$ and $Q_{\rho\sigma} \neq 1 \neq Q_{\rho\tau}$.

(e) If Q is equal to either Q_σ or Q_τ then, as $Q_{\sigma\tau} = 1$, Q could not then be of type $\{1,2,3\}$. $Q = Q_\rho$ would force $Z(P) = [Z(P),\rho] \leq [X,\rho] \leq O_p(XQ) (= C_P(Q))$ which contradicts the assumption that $PQ \neq QP$.

<u>(ii) $P_\rho \leqslant X$ and $Q^*_{<\sigma\tau>} \leqslant Y$.</u>

Parts (a) and (b) have already been dealt with in lemma 3.6.

(c) This is clear as $O_p(PY)_\rho = 1$ and $O_q(PY) = 1$.

(d) If $Q = Q_\rho$ then, as $O_p(XQ) = 1$, it follows that $X = P_\rho$ whereas, by (b), $X_{\sigma\tau} \neq 1$ so contradicting the fixed-point-freeness of α.

(e) Let R be a non-trivial α-invariant subgroup of $P_\rho \leqslant X$ and suppose $N_P(R) \not\leqslant X$. As $Q^* = Q_\rho$, $O_q(QX) = O_q(QX)_\rho$ $C_{O_q(QX)}(R) = O_q(QX)_\rho$ because $C_Q(R) \leqslant Y \leqslant Q_\rho$. Hence $Q = Q^*O_q(QX) = Q_\rho$ which is prohibited by (d) and therefore it may be inferred that $N_P(R) \leqslant X$.

(f) Since $Y = Y_\rho$, $X = O_p(XY)X_\rho = (O_p(PY) \cap X)X_\rho = X_{\sigma\tau}P_\rho = X_{\sigma\tau} \times P_\rho$ (that it is a direct sum follows from the fact that $X_{\sigma\tau} = [X,\rho]$, $P_\rho = [X,\sigma\tau]$ and $P_\rho \cap X_{\sigma\tau} = 1$).

(g) From (e), $Z(P) \leqslant N_P(P_\rho) \leqslant X$ and, as $X \neq P$, $Z(P)_\rho = 1$ giving $Z(P) \leqslant X_{\sigma\tau}$.

(h) Since $Y = Y_\rho$, clearly $[P,\rho] \leqslant O_p(PY)$ so $[P,\rho] = [O_p(PY),\rho]$. Further, $O_p(PY)_\rho \leqslant O_p(PY) \cap X \leqslant P_{\sigma\tau}$ giving $O_p(PY)_\rho = 1$ whence $[O_p(PY),\rho] = O_p(PY)$. Consequently $[P,\rho] = O_p(PY)$.

(i) It will suffice to show, because of Ralston's result (1.19)(ii), that $O_p(P_\rho Q_\rho) = 1$. As $\overline{XY} = XY/O_p(XY) = (\overline{XY})_\rho$, it follows that $\overline{O_p(P_\rho Q_\rho)} \leqslant O_p(\overline{XY}) = 1$ so giving $O_p(P_\rho Q_\rho) \leqslant O_p(XY) = X_{\sigma\tau}$. Hence $O_p(P_\rho Q_\rho) = 1$ as required.

(j) Observe that $X_{\sigma\tau} = O_p(XY)$ implies that $[Y,\sigma\tau] = Y$ centralizes $O_p(XY)$ and also that, by (i), $Y = Y_\rho \lhd Y_\rho P_\rho$.

Hence $Y \trianglelefteq XY$ and so, as $Q = O_q(QX)Y$, it follows that $Q = O_q(QX)$ so proving (j).

(k) Suppose it were tha case that $X = X_\sigma$ then (e) forces $P = P_\sigma$ whence, as $Q_{\sigma\tau} = 1$ and $Q_\tau \leqslant Y$, $O_q(PY) \neq 1$ contradicting (b). Likewise for X_τ.

(l) From (k) $[X,\sigma]$ and $[X,\tau]$ are non-trivial and also are clearly subgroups of P_ρ. Hence using (e), $N_P(X)_\sigma \leqslant N_P([X,\sigma]) \leqslant X$ and $N_P(X)_\tau \leqslant N_P([X,\tau]) \leqslant X$. Combined with the fact that $P_\rho \leqslant X$, (l) follows.

(m) Suppose $J(P)_\rho = 1$. Then $J(P) \leqslant [P,\rho] = O_p(PY)$ and so by a well known property of the Thompson subgroup, $J(P) = J(O_p(PY))$ implying that $J(P) \trianglelefteq PY$. Thus to establish (m) it will be sufficient to show that $J(P)_\rho \neq 1$ implies that P is contained in a unique maximal α-invariant subgroup of G. Applying (e) gives $Z(J(P)) \leqslant X$. Because $X_\rho = X_{\rho\sigma}X_{\rho\tau}$, $X_\sigma = X_{\sigma\tau}X_{\sigma\rho}$ and $X_\tau = X_{\rho\tau}X_{\sigma\tau}$ it follows that $Z(J(P))_\rho = Z(J(P))_{\rho\sigma}Z(J(P))_{\rho\tau}$, $Z(J(P))_\sigma = Z(J(P))_{\sigma\tau}Z(J(P))_{\sigma\rho}$ and $Z(J(P))_\tau = Z(J(P)_{\rho\tau}$ $Z(J(P))_{\sigma\tau}$. Since, by (l), P is not star-covered, lemma 4.4 shows that P is contained in a unique maximal α-invariant subgroup of G.

<u>Definition 5.7</u> Set $\{i,j,k\} = \{1,2,3\}$ and let P be an α-invariant Sylow p-subgroup of G of type $\{1,2,3\}$; P is said to be of:

<u>Type I</u> if there exists an α-invariant Sylow q-subgroup Q of G of type $\{1,2,3\}$ such that $PQ \neq QP$ and $P^* \leqslant N_P(Q)$;

<u>Type II</u> if there exists an α-invariant Sylow q-subgroup Q of G of type $\{1,2,3\}$ such that $PQ \neq QP$ and $Q^* \leqslant N_Q(P)$;

<u>Type III(i)</u> if there exists an α-invariant Sylow q-subgroup Q of G of type $\{1,2,3\}$ such that $PQ \neq QP$ and $P_{\alpha_i} \leqslant X$ and $Q_{\alpha_j}, Q_{\alpha_k} \leqslant Y$;

<u>Type IV(j,k)</u> if there exists an α-invariant Sylow q-subgroup Q of G of type $\{1,2,3\}$ such that $PQ \neq QP$ and $Q_{\alpha_i} \leqslant X$ and $P_{\alpha_j}, P_{\alpha_k} \leqslant Y$;

<u>Type V</u> if P permutes with all α-invariant Sylow subgroups of G of type $\{1,2,3\}$.

<u>Remark</u> By lemma 5.6, every α-invariant Sylow subgroup of G of type $\{1,2,3\}$ is of type I,II,III( ),IV(,) or V though at the moment, excepting type V, there is no reason why these types should be 'well defined'. This and related matters will be examined in section 7.

Sometimes, types III(i) and IV(j,k) will just be written as (respectively) types III and IV and by the phrase P is of type I with respect to Q it is meant that Q is of type $\{1,2,3\}$, $PQ \neq QP$ and $P^* \leqslant N_P(Q)$. Analogous interpretations hold for the phrases P is of type $\begin{cases} \text{II} \\ \text{III} \\ \text{IV} \end{cases}$ with respect to Q.

6. LINKING THEOREMS.

This section is devoted to the examination of possible relations between certain $\mathfrak{M}_{\mu,\eta}$ as μ and η vary. Again set $\{1,2,3\} = \{i,j,k\}$.

Lemma 6.1 Let L, M, and N be nilpotent α-invariant Hall subgroups of G. Suppose the following hold:

(i) $NL = LN$ and $ML = LM$;

(ii) $MN \neq NM$; and

(iii) $N^*_{\langle\beta\rangle} \leq Y$ (Y being the largest α-invariant subgroup of N permutable with M) where β ($\in \langle\alpha\rangle$) acts fixed-point-freely upon L.

Then $O_\pi(LM) = 1$ where $\pi = \pi(L)$ (and hence L is star-covered with respect to γ, if γ ($\in\langle\alpha\rangle$) is acting fixed-point-freely upon M).

Proof By lemma 2.10, $N = O_\eta(NL)N^*_{\langle\beta\rangle}$ and so $N = O_\eta(NL)Y$. The largest α-invariant subgroup of NL permutable with M is YL and hence L normalizes both $O_\eta(NL)$ and $O_\eta(NL)\cap Y \geq O_\eta(NL)^*_{\langle\beta\rangle}$. Hence, as $L_\beta = 1$, $O_\eta(NL) = (O_\eta(NL)\cap Y)C_{O_\eta(NL)}(L)$ and consequently $N = YC_N(L)$. Thus it now follows that if $O_\pi(LM)$ is non-trivial then $C_N(L) \leq Y$ giving $N = Y$ which contradicts assumption (ii).

An example of the use to which lemma 6.1 will be put is given in the next result (which is of interest in connection with lemma 5.5(i)).

Lemma 6.2 Let P and Q denote α-invariant Sylow p- and q-subgroups of G of type $\{1,2,3\}$, and suppose $PQ = QP$, $PL_i = L_iP$ and $QL_i \neq L_iQ$. If, further, P is not star-covered, then $L_i^* \nleq N_{L_i}(Q)$.

Proof The proof follows immediately from the preceding lemma.

Lemma 6.3 If P denotes an α-invariant Sylow p-subgroup of G of type $\{1,2,3\}$, then at least two of P,L_i and L_j permute.

Proof Suppose the lemma is false and (without loss) that $i = 1$ and $j = 2$. Thus $L_1L_2 \neq L_2L_1$, $PL_1 \neq L_1P$ and $PL_2 \neq L_2P$ is assumed to hold; the proof is broken up into cases which depend upon the form of $\mathcal{M}_{p,\pi_1}$ and $\mathcal{M}_{p,\pi_2}$. Let $\mathcal{M}_{p,\pi_1} = \{PY_1, L_1X_1\}$ and $\mathcal{M}_{p,\pi_2} = \{PY_2, L_2X_2\}$. (Note that the arguments are symmetric with respect to L_1 and L_2).

Case 1 P_σ, $P_\tau \leq X_1$ and P_ρ, $P_\tau \leq X_2$.

If, furthermore , $O_p(L_1X_1) \neq 1$ and $O_p(L_2X_2) \neq 1$, then by appealing to lemma 5.4(ii), $Y_1 = Y_2 = 1$, $Z(P) = Z(P)_{\rho\sigma} \leq X_1 \cap X_2$, $X_1 = N_P(L_1)$ and $X_2 = N_P(L_2)$. As $L_1L_2 \neq L_2L_1$, without loss of generality , it may be taken that $L_{1_\tau} \leq N_{L_1}(L_2)$ ($\neq L_1$). Now, $Z(P) \leq N_P(L_1) \cap N_P(L_2)$ so normalizing $N_{L_1}(L_2)$ and hence, as $Z(P)_\tau = 1$, $C_{L_1}(Z(P)) \neq 1$. This, however, is against $Y_1 = 1$.

Now suppose $O_p(L_1X_1) \neq 1$ and $O_p(L_2X_2) = 1$. By lemma 5.4(ii) and (iii), $Z(P) = Z(P)_\rho \leq X_1 = N_P(L_1)$ and

P_ρ, $P_\tau \leqslant X_2 = N_P(L_2) \leqslant P_\sigma$. Clearly, $Z(P) = Z(P)_{\rho\sigma} \leqslant N_P(L_1) \cap N_P(L_2)$ and, as either $L_{1_\tau} \leqslant N_{L_1}(L_2)$ or $L_{2_\tau} \leqslant N_{L_2}(L_1)$, it follows that at least one of $C_{L_1}(Z(P))$ and $C_{L_2}(Z(P))$ is non-trivial. The former cannot occur as $Y_1 = 1$ and the latter is dealt with by using lemma 5.5(iii)(c) which gives $C_{L_2}(Z(P)) \leqslant L_{2_{\rho\tau}}$, whence $L_2 = N_{L_2}(L_1)\ C_{L_1}(Z(P)) = N_{L_2}(L_1)\ L_{2_{\rho\tau}} = N_{L_2}(L_1)$ and this contradicts the assumption that $L_1L_2 \neq L_2L_1$.

Finally, consider the possibility that both $O_p(L_1X_1)$ and $O_p(L_2X_2)$ are trivial. Then from lemma 5.5(iii)(a), $P_\rho = P^* = P_\sigma$ which implies that P is not of type 1,2,3 σ.

<u>Case 2</u> P_σ, $P_\tau \leqslant X_1$ and L_{2_ρ}, $L_{2_\tau} \leqslant Y_2$.

From lemma 5.5, $X_1 = N_P(L_1)$, $Y_2 = N_{L_2}(P)$ and either $L_{2_\rho} = L_{2_\tau}$ or $Z(L_2) = Z(L_2)_{\rho\tau} \leqslant Y_2$. Suppose $L_{2_\tau} \leqslant N_{L_2}(L_1)$, then clearly $L_{2_\tau} \leqslant N_{L_2}(L_1) \cap N_{L_2}(P)$. Hence L_{2_τ} normalizes both P and $N_P(L_1)$, the latter subgroup containing P_σ and therefore, as $N_P(L_1) \neq P$, $C_P([L_{2_\tau}, \sigma])$ is non-trivial. If it is the case that $Z(L_2) = Z(L_2)_{\rho\tau} \leqslant Y_2$, then $C_P(Z(L_2))$ is non-trivial which is against the shape of $\mathcal{M}_{p,\pi_2}$. Whereas, if $L_{2_\rho} = L_{2_\tau}$ then (as $[L_{2_\tau}, \sigma] = L_{2_\tau}$) $C_P(L_2) = C_P(L_2^*) = C_P(L_{2_\tau}) \neq 1$ which again does not agree with the supposed form of $\mathcal{M}_{p,\pi_2}$.

Thus it may be concluded that $L_{1_\tau} \leqslant N_{L_1}(L_2)$ and so $N_{L_1}(L_2) \neq 1$. If $Z(L_2) = Z(L_2)_{\rho\tau} \leqslant N_{L_2}(P)$, then $[N_{L_1}(L_2), Z(L_2)] = 1$ and consequently $Z(L_2) \leqslant N_{L_2}(L_1)$, again giving $Z(L_2) \leqslant N_{L_2}(P) \cap N_{L_2}(L_1)$. A contradiction to this configuration may be deduced as before. Therefore $L_{2_\rho} = L_{2_\tau}$ and so by lemma 4.3 $N_{L_1}(L_2) \trianglelefteq N_{L_1}(L_2)L_2$ yielding, as $N_{L_1}(L_2) \neq 1$ and L_1 is nilpotent, that $L_1L_2 = L_2L_1$. This disposes of case 2.

Case 3 $L_1^* \leqslant Y_1$ and $L_2^* \leqslant Y_2$.

Applying lemma 5.5(i) to this situation gives :

$P_{\sigma\tau} = 1$, $P_{\rho\sigma} \neq 1 \neq P_{\rho\tau}$ and $P_{\rho\tau} = 1$, $P_{\sigma\tau} \neq 1 \neq P_{\rho\sigma}$.
Clearly this configuration is untenable.

The proof of the lemma is complete.

The next two results will be required in the proof of theorem 6.6.

Lemma 6.4 Suppose $L_iL_j \neq L_jL_i$ and $L_jL_k \neq L_kL_j$ and let L be a non-trivial α-invariant subgroup of $N_{L_k}(L_i) \cap N_{L_k}(L_j)$. If $L_{j_{\alpha_k}} \leqslant N_{L_j}(L_i)$, then $C_{L_j}(L) \nleqslant N_{L_j}(L_k)$.

Proof As $L \leqslant N_{L_k}(L_i) \cap N_{L_k}(L_j)$, L normalizes $N_{L_j}(L_i)$ $(\geqslant L_{j_{\alpha_k}})$ and hence, since $L_{\alpha_k} = 1$, $L_j = C_{L_j}(L)N_{L_j}(L_i)$. Now suppose $C_{L_j}(L) \leqslant N_{L_j}(L_k)$. Observe that $[N_{L_j}(L_k), \alpha_k]$ $\neq 1$ because otherwise $C_{L_j}(L) \leqslant L_{j_{\alpha_k}} \leqslant N_{L_j}(L_i)$ would give $L_j = N_{L_j}(L_i)$ so contradicting $L_iL_j \neq L_jL_i$. Thus $O_{\pi_j}(L_kN_{L_j}(L_k))$ $(= C_{L_j}(L_k)) \neq 1$ and applying lemma 5.1 it follows that $N_{L_k}(L_j) = 1$. However, by hypothesis, $1 \neq L \leqslant N_{L_k}(L_j)$ so giving a contradiction. Thus $C_{L_j}(L) \nleqslant N_{L_j}(L_k)$.

Corollary 6.5 Assume the hypotheses of lemma 6.4 hold. Then $L \cap Z(L_k) = 1$.

Proof This comes directly from lemma 6.4.

Theorem 6.6 Suppose that no two of L_1, L_2 and L_3 permute. Then one of the following occurs:

(i) $L_{1_\sigma} = L_1, L_{2_\tau} = L_2, L_{3_\rho} = L_3$; or

(ii) $L_{1_\tau} = L_1$, $L_{2_\rho} = L_2$, $L_{3_\sigma} = L_3$.

Proof By lemma 5.4, as no pair of L_1, L_2 and L_3 permute, $\mathcal{M}_{\pi_i,\pi_j} = \{L_iN_{L_j}(L_i),\ L_jN_{L_i}(L_j)\}$ for $i,j \in \{1,2,3\}$, $i \neq j$.

First, it will be established that it is not possible for both L_{2_τ} and L_{3_σ} to be contained in $N_G(L_1)$. Suppose the contrary , and suppose also, without loss of generality, that $\{N_G(L_1)\}_{\pi_2,\pi_3} \leqslant L_2N_{L_3}(L_2)$. Consequently $L_{3_\sigma} \leqslant N_{L_3}(L_1) \cap N_{L_3}(L_2)$ which, together with $L_{2_\tau} \leqslant N_{L_2}(L_1)$, implies, by lemma 6.4, that $C_{L_2}(L_{3_\sigma}) \not\leqslant N_{L_2}(L_3)$. Therefore $Z(L_3) \leqslant N_{L_3}(L_2)$ and $Z(L_3)_\sigma = 1$. However, as $L_{2_\sigma} = 1$, $Z(L_3)L_2$ admits σ fixed-point-freely and hence $[Z(L_3),L_2] = 1$ which is against $L_2L_3 \neq L_3L_2$. This shows that L_{2_τ} and L_{3_σ} cannot both be contained in $N_G(L_1)$. Similarly, L_{1_τ} and L_{3_ρ} (respectively L_{1_σ} and L_{2_ρ}) cannot both be contained in $N_G(L_2)$ (respectively $N_G(L_3)$).

Because of the form of $\mathcal{M}_{\pi_i,\pi_j}$ for $i,j \in \{1,2,3\}$, $i \neq j$, together with the conclusions of the preceding paragraph one of the following must hold:

(a) $L_{1_\tau} \leqslant N_{L_1}(L_2)$, $L_{2_\rho} \leqslant N_{L_2}(L_3)$ and $L_{3_\sigma} \leqslant N_{L_3}(L_1)$; or

(b) $L_{1_\sigma} \leqslant N_{L_1}(L_3)$, $L_{2_\tau} \leqslant N_{L_2}(L_1)$ and $L_{3_\rho} \leqslant N_{L_3}(L_2)$.

The succeeding arguments are applicable to both cases (a) and (b) so, without loss of generality, it will be supposed that case (a) holds.

The next part of the proof is concerned with showing that $L_{1_\sigma} \geq L_{1_\tau}$. Suppose $L_{1_\sigma} \not\leq L_{1_\tau}$. Now, as $L_{2_\sigma} = 1$ and $L_{1_\tau} \leq N_{L_1}(L_2)$, it may be deduced that $O_{\pi_1}(L_2 N_{L_1}(L_2)) \neq 1$ and hence that $Z(L_1) \leq N_{L_1}(L_2)$. Moreover, as $L_1L_2 \neq L_2L_1$, $Z(L_1) \leq L_{1_\sigma}$. If, additionally, $Z(L_1) \leq N_{L_1}(L_3)$ then $Z(L_1) \leq N_{L_1}(L_3) \cap N_{L_1}(L_2)$ and so, as $L_{2_\rho} \leq N_{L_2}(L_3)$ already, corollary 6.5 shows this situation cannot occur. Hence $Z(L_1) \not\leq N_{L_1}(L_3)$.

Suppose, for the moment, that $L_{3_\sigma} \neq N_{L_3}(L_1)$. Because $Z(L_1) \leq L_{1_\sigma}$, $[[N_{L_3}(L_1),\sigma], Z(L_1)] = 1$ and, since $Z(L_1) \not\leq N_{L_1}(L_3)$ and $[N_{L_3}(L_1),\sigma] \neq 1$, $Z(L_3) \leq N_{L_3}(L_1)$. Furthermore, $[Z(L_3),\sigma] \neq 1$ would imply that $Z(L_1) \leq N_{L_1}(L_3)$ and so $Z(L_3) \leq L_{3_\sigma}$. Consider the group $N_{L_2}(L_3)Z(L_3)$. Since $Z(L_3) \leq L_{3_\sigma}$ and $L_{2_\sigma} = 1$, $[N_{L_2}(L_3), Z(L_3)] = 1$. Now $1 \neq L_{2_\rho} \leq N_{L_2}(L_3)$ and so $Z(L_3) \leq N_{L_3}(L_2)$. Hence a contradiction in the form of the situation occurring in corollary 6.5 with $Z(L_3) \leq N_{L_3}(L_2) \cap N_{L_3}(L_1)$ and $L_{1_\tau} \leq N_{L_1}(L_2)$ has been reached. Therefore $L_{3_\sigma} = N_{L_3}(L_1)$.

If $O_{\pi_3}(L_1 N_{L_3}(L_1)) \neq 1$ then, as $L_{3_\sigma} \geq C_{L_3}(L_{3_\sigma})$, it follows from (1.7)(x) that $L_3 = L_{3_\sigma} = N_{L_3}(L_1)$ which is against $L_3L_1 \neq L_1L_3$. Consequently $L_{3_\sigma} = N_{L_3}(L_1) \leq L_{3_\rho}$.

Next, it is claimed that $L_3 = L_{3_\rho}$. As L_{2_ρ} normalizes both L_3 and L_{3_ρ} ($= L_3^*$), $L_3 = L_3\, C_{L_3}(L_{2_\rho})$. Suppose $L_3 \neq L_{3_\rho}$. Because of lemma 5.1 and the fact that $N_{L_2}(L_3)$ is non-trivial, clearly $C_{L_3}(L_{2_\rho}) \not\leq N_{L_3}(L_2)$. Thus $Z(L_2) \leq N_{L_2}(L_3)$ and, obviously, $Z(L_2)_\rho = 1$. Hence $Z(L_2)N_{L_1}(L_2)$ admits ρ fixed-point-freely which implies, as $1 \neq L_{1_\tau} \leqslant N_{L_1}(L_2)$, that $Z(L_2) \leqslant N_{L_2}(L_1)$. However, this means that $Z(L_2) \leqslant N_{L_2}(L_1) \cap N_{L_2}(L_3)$ which as corollary 5.5 shows, is untenable and thus the above claim, that $L_3 = L_{3_\rho}$, is substantiated.

If $C_{L_2}(L_3)$ ($= O_{\pi_2}(N_{L_2}(L_3)L_3)$) is non-trivial then, because $L_2L_3 \neq L_3L_2$, $Z(L_2) \leqslant L_{2_\rho}$. On the other hand, $C_{L_2}(L_3) = 1$ gives $L_{2_\rho} \leqslant L_{2_\tau}$. If the former case occurs then $[N_{L_1}(L_2), Z(L_2)] = 1$ because $L_{1_\rho} = 1$. Since $N_{L_1}(L_2) \neq 1$, $Z(L_2) \leqslant N_{L_2}(L_1)$ in addition to $Z(L_2) \leqslant N_{L_2}(L_3)$ and another application of corollary 6.5 shows this cannot happen. Therefore, must have $L_{2_\rho} \leqslant L_{2_\tau}$. The same line of argument as used in the previous paragraph to show $L_3 = L_{3_\rho}$ will also yield that $L_2 = L_{2_\tau}$. In brief: if $L_{2_\tau} \neq L_2$ then $C_{L_2}(L_{1_\tau}) \neq N_{L_2}(L_1)$, by lemma 6.4, hence $Z(L_1) \leqslant N_{L_1}(L_2)$ and $Z(L_1)_\tau = 1$; as $N_{L_3}(L_1) \neq 1$ and $N_{L_3}(L_1)Z(L_1)$ admits τ fixed-point-freely, $Z(L_1) \leqslant N_{L_1}(L_3)$ and the desired contradiction again comes from corollary 6.5. Recall that $Z(L_1) \leqslant N_{L_1}(L_2)$ and so, as $L_1L_2 \neq L_2L_1$, $Z(L_1) \leqslant L_{1_\tau}$. Now, $L_{3_\tau} = 1$ implies $[N_{L_3}(L_1), Z(L_1)] = 1$ which in turn implies $Z(L_1) \leqslant N_{L_1}(L_3)$ and so corollary 6.5 is applicable. Therefore $L_{2_\rho} \leqslant L_{2_\tau}$ cannot occur either and this contradiction establishes that $L_{1_\sigma} \geqslant L_{1_\tau}$.

Since $L_{3_\sigma} \leqslant N_{L_3}(L_1)$, invoking the same kind of argument as used above in establishing that $L_3 = L_{3_\rho}$ and $L_2 = L_{2_\tau}$, it may be demonstrated that $L_1 = L_{1_\sigma}$.

By similar arguments, $L_{2_\tau} = L_2$ and $L_{3_\rho} = L_3$ may be obtained so giving part (i) in the statement of the theorem. Of course (b) gives rise to part (ii), so the proof is finished.

Theorem 6.7 Let P and Q be α-invariant Sylow subgroups of type $\{1,2,3\}$. If $PQ = QP$, $PL_j = L_jP$ and $QL_i = L_iQ$, then at least one of $PL_i = L_iP$, and $QL_j = L_jQ$ holds.

Proof Suppose the lemma is false and set $i = 1$ and $j = 2$. Thus the following is assumed to hold:

$$\left.\begin{array}{l} PQ = QP,\ PL_2 = L_2P,\ QL_1 = L_1Q, \\ PL_1 \neq L_1P \text{ and } QL_2 \neq L_2Q. \end{array}\right\} \quad (+)$$

(1) The inequalities $L_2^* \leqslant N_{L_2}(Q)$ and $L_1^* \leqslant N_{L_1}(P)$ are mutually exclusive. For suppose that both $L_2^* \leqslant N_{L_2}(Q)$ and $L_1^* \leqslant N_{L_1}(P)$ then $\mathcal{M}_{\pi_2,q} = \{QN_{L_2}(Q),\ L_2\}$ and $\mathcal{M}_{\pi_1,p} = \{PN_{L_1}(P),\ L_1\}$. Furthermore, $P_{\sigma\tau} = 1 = Q_{\rho\tau}$ and hence PL_2 and QL_1 admit, respectively, $\sigma\tau$ and $\rho\tau$ acting fixed-point-freely. Consequently, as $L_2{}^*{}_{<\sigma\tau>} = L_{2\tau}$ and $L_1{}^*{}_{<\rho\tau>} = L_{1\tau}$, $[P_\tau,\ L_2] = 1 = [Q_\tau,\ L_1]$. Hence $O_q(Q_\tau P_\tau) = 1$ for if not then $1 \neq P_\tau \leqslant N_P(L_1) = 1$. Similarly $O_p(Q_\tau P_\tau) = 1$ and so $F(Q_\tau P_\tau) = 1$ which contradicts a well known property of soluble groups.

(2) It is asserted that if $L_1L_2 = L_2L_1$ then the configuration (+) is impossible. So suppose $L_1L_2 = L_2L_1$. Because of (1) it may be assumed that (say) $Q_\rho,\ Q_\tau \leq N_Q(L_2)$. Employing lemma 6.1 to Q, L_1 and L_2 yields that $O_{\pi_1}(L_1L_2) = 1$ and hence $L_1 = L_{1\sigma}$. Consequently $P_\sigma,\ P_\tau \leqslant N_P(L_1)$. Repeating the preceding argument using P in place of Q gives $O_{\pi_2}(L_1L_2) = 1$ whence $F(L_1L_2) = 1$.

This establishes the assertion.

Thus may assume for the remainder of the proof that $L_1L_2 \neq L_2L_1$.

First, suppose P and Q are not star-covered. As both P and Q are not star-covered, a double application of lemma 6.2 shows that P_σ, $P_\tau \leqslant N_P(L_1)$ and Q_ρ, $Q_\tau \leqslant N_Q(L_2)$. Furthermore it is the case that both $[N_P(L_1),\rho]$ and $[N_Q(L_2),\sigma]$ are non-trivial. For, suppose $[N_P(L_1),\rho] = 1$. Then $P^* = P_\rho$. As Q is not star-covered, by lemma 2.14, $O_q(PQ) \cap O_q(QL_1) \neq 1$ and hence either $O_p(PQ) \leqslant N_P(L_1)$ or $O_{\pi_1}(QL_1) \leqslant N_{L_1}(P)$. The former possibility implies that $P = P^*O_p(PQ) = P_\rho$ whereas P has been assumed to not be star-covered. Thus $O_{\pi_1}(QL_1) \leqslant N_{L_1}(P) \leqslant L_{1_{\sigma\tau}}$ and so $L_1 = L_1^*$. However lemma 5.5 (iii)(d) also yields that $P = P_\rho$. Thus it may be deduced that $[N_P(L_1),\rho] \neq 1$ and, likewise, that $[N_Q(L_1),\sigma] \neq 1$. Clearly $\mathcal{M}_{p,\pi_1} = \{P, N_P(L_1)L_1\}$ and $\mathcal{M}_{q,\ 2} = \{Q, N_Q(L_2)L_2\}$.

As $L_1L_2 \neq L_2L_1$ it may be taken that $1 \neq L_{1_\tau} \leqslant N_{L_1}(L_2)$. Since $N_Q(L_2) \geqslant Q_\rho$ and $Q = O_q(QL_1)Q_\rho$, $Q = N_Q(L_2)C$ where $C = C_Q(N_{L_1}(L_2))$. Claim that C must be star-covered for if it were not then, by lemma 2.13, $C \cap O_q(PQ) \neq 1$ hence giving $O_p(PQ) \leqslant N_P(L_1)$ which, by lemma 2.15, forces $P = N_P(L_1)$ as $(N_P(N_P(L_1)))^* \leqslant N_P(L_1)$.

Now $N_Q(N_Q(L_2)) = N_Q(L_2)N_C(N_Q(L_2))$ and as $N_Q(N_Q(L_2))^* \leqslant N_Q(L_2)$, it follows that $N_C(N_Q(L_2)) = N_C(N_Q(L_2))^* \leqslant N_Q(N_Q(L_2))^* \leqslant N_Q(L_2)$. Therefore $N_Q(N_Q(L_2)) = N_Q(L_2)$ which implies that $QL_2 = L_2Q$. This settles the lemma when both P and Q are not star-covered.

Next consider the situation: P is not star-covered and Q is star-covered.

Because P is not star-covered, lemma 6.2 is available to give that Q_ρ, $Q_\tau \leqslant N_Q(L_2)$. As Q is star-covered, $[N_Q(L_2), \sigma] \neq 1$ is untenable and so $Q_\sigma = Q^* = Q$ whence $\mathcal{M}_{q,\pi_2} = \{Q, N_Q(L_2)L_2\}$. Furthermore may deduce that $O_{\pi_2}(PL_2) = 1$ and hence that L_2 is star-covered. For $[P,\sigma] \trianglelefteq PQ$ and $[P,\sigma] \leqslant O_p(PL_2)$ which, as $[P,\sigma] \neq 1$ because P is not star-covered, implies that $O_{\pi_2}(PL_2) \leqslant N_{L_2}(Q) = 1$.

As $L_1L_2 \neq L_2L_1$, either $L_{1_\tau} \leqslant N_{L_1}(L_2)$ or $L_{2_\tau} \leqslant N_{L_2}(L_1)$. If the latter alternative occurs then, as L_2 is star-covered, $[N_{L_2}(L_1), \rho] \neq 1$ is impossible and so $L_{2_\rho} = L_2^* = L_2$. Consequently $[N_Q(L_2), \rho]$ centralizes L_2 and, in particular, $[Q_\tau, L_2] = 1$. Since $Q_{\rho\tau} = 1$, QL_1 admits $\rho\tau$ fixed-point-freely and therefore $[Q_\tau, L_1] = 1$ because $L_1{}^*_{\langle\rho\tau\rangle} = L_{1_\tau}$. Now $C_G(Q_\tau) \geqslant L_2, L_1$ so forcing $L_1L_2 = L_2L_1$. Therefore $L_{1_\tau} \leqslant N_{L_1}(L_2)$ must hold.

Since Q_ρ, $Q_\tau \leqslant N(L_2)$ also and $L_{1_\rho} = 1$, it follows that $Q = N_Q(L_2)C_Q(L_{1_\tau})$. Let $\bar{L}_2 = L_2/\phi(L_2)$. Then $\bar{L}_2 = \bar{L}_{2_\rho}\bar{L}_{2_\tau}$, by lemma 2.2(iv), as L_2 is star-covered. Since $L_{1_\tau} \leqslant N_{L_1}(L_2)$ and $L_{1_\rho} = 1$, $\bar{L}_2 = \bar{L}_{2_\tau}C_{\bar{L}_2}(L_{1_\tau})$. Hence, if $\bar{L}_{2_\tau} \neq \bar{L}_2$, then $C_{L_2}(L_{1_\tau}) \neq 1$ from which it follows that $C_Q(L_{1_\tau}) \leqslant N_Q(L_2)$ (because of the shape of $\mathcal{M}_{q,\pi_2}$). Therefore as $L_2Q \neq QL_2$, $\bar{L}_{2_\tau} = \bar{L}_2$ whence $L_{2_\tau} = L_2$.

It is asserted that $Z(Q) \leqslant Z(QL_1)$. Since $[N_Q(L_2), \tau]$ is non-trivial, $Z(Q) \leqslant N_Q(L_2)$ and hence $Z(Q) \leqslant Q_\tau$. Combined with the fact that QL_1 admits $\rho\tau$ fixed-point-freely so giving $[Q_\tau, L_1] = 1$ the assertion is now clear.

Suppose $O_q(PQ) \neq 1$. Hence $1 \neq Z(Q) \cap O_q(PQ)$ and consequently $C_G(Z(Q) \cap O_q(PQ)) \geqslant O_p(PQ), L_1$. Recall that P is not star-covered so $O_p(PQ) \neq 1$. Clearly, in this situation, $L_1^* \leqslant N_{L_1}(P)$ is not possible. Thus P_σ, $P_\tau \leqslant N_P(L_1)$. However, if $N_P(L_1) \leqslant P_\rho$ then $P = O_p(PQ)P^* = P_\rho$ which is against P being not star-covered. Whereas $[N_P(L_1), \rho] \neq 1$ also produces a contradiction because $N_P(N_P(L_1))^* \leqslant N_P(L_1)$ together with $O_p(PQ) \leqslant N_P(L_1)$ implies, by lemma 2.15, that $P = N_P(L_1)$. Therefore $O_q(PQ) = 1$ must hold and so $Q = N_Q(J(P))C_Q(Z(P))$, from Glauberman's factorization theorem.

Observe that if $Z(J(P)) \leqslant P_\rho$, then $L_2Q = QL_2$ because, as P is not star-covered, $L_2 = C_{L_2}(D)L_{2\rho}$ and $Q = C_Q(D)Q_\rho$ with $D = O_p(PL_2) \cap O_p(PQ) \cap Z(P)$ $(\neq 1)$; since $Q_\rho \leqslant N_Q(L_2)$ and $L_2Q \neq QL_2$, $C_Q(D) \not\leqslant N_Q(L_2)$ and so $C_{L_2}(D)$ would have to be trivial hence $L_2 = L_{2\rho}$. Since it has already been shown that $L_2 = L_{2\tau}$, lemma 4.1 demonstrates that the assumption $L_2Q \neq QL_2$ was incorrect.

It will now be shown that $P_{\sigma\tau} = 1$. For when this has been done it will then be the case that PL_2 admits $\sigma\tau$ fixed-point-freely and so will have Fitting length $\leqslant 2$. Recalling that $O_{\pi_2}(PL_2) = 1$, so $P \trianglelefteq PL_2$ whence $N_Q(J(P))$, $C_Q(Z(P)) \leqslant N_Q(L_2)$ so giving a contradiction in the form of $L_2Q = QL_2$.

Towards the above end it will be assumed that $P_{\sigma\tau} \neq 1$ from which a contradiction will be derived. Hence, by lemma 5.5(ii), $\mathcal{M}_{p,\pi_1} = \{N_P(L_1)L_1, P\}$ (i.e. $Y = 1$). Suppose $O_{\pi_1}(N_{L_1}(L_2)L_2) \neq 1$ then $Z(L_1) \leqslant N_{L_1}(L_2)$ hence, as $L_2 = L_{2_\tau}$, $L_{2_\sigma} = 1$ and $L_1L_2 \neq L_2L_1$, $Z(L_1) \leqslant L_{1_{\sigma\tau}}$. However, this forces $Z(L_1) \leqslant Y = 1$ as $[Z(L_1), N_P(L_1)] = 1$. Thus $O_{\pi_1}(N_{L_1}(L_2)L_2) = 1$ and therefore $L_1^* = L_{1_\sigma}$.

Claim that if $L_1 = L_{1_\sigma}$, then L_2 and Q permute. For $L_1 = L_{1_\sigma}$ implies that $N_P(L_1) = C_P(L_1)(N_P(L_1))_{\tau\rho}$ thence $[L_1, P_\tau] = 1$. As $[P_{\rho\tau}, L_2] = 1$, were it the case that $P_{\rho\tau} \neq 1$ then $\{C_G(P_{\rho\tau})\}_{\pi_1,\pi_2} = L_1L_2$ so giving $L_1L_2 = L_2L_1$. Thus $P_{\rho\tau} = 1$ and so $\rho\tau$ acts fixed-point-freely upon PQ. Since $O_q(PQ) = 1 = O_{\pi_2}(PL_2)$, $P \trianglelefteq PQ$ and $L_2 = N_{L_2}(J(P))C_{L_2}(Z(P))$ from which it follows that $QL_2 = L_2Q$.

Therefore it may be inferred that $L_1 \neq L_{1_\sigma} = L_1^*$. As $P_\sigma \leqslant N_P(L_1)$, $L_1 = C_{L_1}(P_\sigma)L_{1_\sigma}$ and clearly $C_{L_1}(P_\sigma) \neq 1$. Moreover, as $\mathcal{M}_{p,\pi_1} = \{N_P(L_1)L_1, P\}$, $Z(P)_\sigma = 1$. As $Z(P) \leqslant N_P(L_1)$ and $[Z(P), P_\sigma] = 1$, by (1.6)(iii), $Z(P)$ normalizes $[L_1, P_\sigma]$. Now $[P_\sigma, L_1] \leqslant L_{1_\sigma}$ together with $Z(P)_\sigma = 1$ gives that $[[P_\sigma, L_1], Z(P)] = 1$. The shape of $\mathcal{M}_{p,\pi_1}$ dictates that $[P_\sigma, L_1] = 1$ so $P_\sigma \leqslant C_P(L_1)$.

If $J(P)_\sigma \neq 1$ then, clearly, $Z(J(P)) \leqslant N_P(L_1)$ and, as $Z(J(P)) \leqslant P_\rho$ gives $L_2Q = QL_2$, it follows that $[Z(J(P)), \rho] \neq 1$ which in turn implies that $P_\rho \leqslant N_P(L_1)$. Thus $P \stackrel{*}{\leqslant} N_P(L_1)$ and consequently, from lemma 6.1, $O_q(QL_1) = 1$. However $[Z(Q), L_1] = 1$, so $O_q(QL_1) \neq 1$. Therefore $J(P)_\sigma = 1$ in which case $J(P) \leqslant [P, \sigma] \leqslant O_p(PL_2)$ giving $J(P) \trianglelefteq PL_2$. Moreover, as $J(P)L_2$ admits σ fixed-point-freely, $[L_2, J(P)] = 1$. Hence $Q = N_Q(J(P))C_Q(Z(P)) \leqslant N_Q(L_2)$ which is the final contradiction to the assumption $P_{\sigma\tau} \neq 1$.

This settles the case: P is not star-covered and Q is star-covered.

To finish the proof it only remains to examine the configuration (+) when both P and Q are star-covered.

If P_σ, $P_\tau \leq N_P(L_1)$ and Q_ρ, $Q_\tau \leq N_Q(L_2)$ then as both P and Q are assumed star-covered, $P = P_\rho$ and $Q = Q_\sigma$. Clearly $[Q, \rho]$ is non-trivial and so, as $L_{1_\rho} = 1$, $O_{\pi_1}(QL_1) \leq N_{L_1}(P) = 1$ whence $L_1 = L_{1_\tau}$ because $(QL_1)_{\rho\tau} = 1$ and $L_1^*{}_{<\rho\tau>} = L_1$. Further, as $Q = Q_\sigma$, must also have $L_1 = L_{1_\sigma}$ and so $L_1 = L_{1_{\sigma\tau}}$ which yields, from lemma 4.1, that G has a normal π_1-complement.

Therefore, without loss, it may be assumed that $N_P(L_1) \geq P_\sigma$, P_τ and $L_2^* \leq N_{L_2}(Q)$. A contradiction will now be deduced in the guise of showing that $L_1L_2 = L_2L_1$. Since P is star-covered, $P = P_\rho$ and so $\mathcal{M}_{p,\pi_1} = \{N_P(L_1)L_1, P\}$. Because $Q_{\rho\tau} = 1$, $Q \neq Q_\rho$ and so, as has been deduced before, $O_{\pi_1}(QL_1) \leq Y$ (the largest α-invariant subgroup of L_1 permutable with P) = 1. Therefore, as QL_1 admits $\rho\tau$ fixed-point-freely $L_1 = L_{1_\tau}$ and so $L_{2_\tau} \leq N_{L_2}(L_1)$. If $[N_{L_2}(L_1), \rho]$ $\neq 1$ then as $[L_2, \rho] \leq O_{\pi_1}(PL_2)$ it follows that $\{N_G([N_{L_2}(L_1), \rho]\ )\}_{p,\pi_1} \geq O_p(PL_2), L_1$ and hence that $P = O_p(PL_2)P_\sigma \leq N_P(L_1)$. Thus $L_{2_\tau} \leq N_{L_2}(L_1) \leq L_{2_\rho} = L_2^*$ and, in view of this fact, $C_{L_2}(L_1) = 1$, otherwise $L_2^* = L_{2_\rho} = L_2$ which would contradict $QL_2 \neq L_2Q$. Therefore $L_{2_\tau} \ntrianglelefteq N_{L_2}(L_1)$ and, as $Q \trianglelefteq QL_1$, $L_2^* \leq N_{L_2}(L_1)$ which forces $L_{2_\tau} = L_{2_\rho}$. Applying lemma 4.3 to PL_2 gives $P \trianglelefteq PL_2$, hence $[P, O_{\pi_2}(PL_2)] = 1$. From $L_1 = L_{1_\tau}$, it follows that $[P_\sigma, L_1] = 1$ whence $C_G(P_\sigma) \geq L_1, O_{\pi_2}(PL_2)$ and so because $(PL_2)_{\sigma\tau} = 1$, $L_2 = L_{2_\tau}O_{\pi_2}(PL_2) \leq N_{L_2}(L_1)$.

This completes the verification of theorem 6.7.

The next linking result is of a very similar nature to theorem 6.7.

Lemma 6.8 Let P and Q be α-invariant Sylow p- and q-subgroups of G of type $\{1,2,3\}$ which permute. If $PL_{jk} = L_{jk}P$ and $QL_i = L_iQ$, then at least one of $PL_i = L_iP$ and $QL_{jk} = L_{jk}Q$ must hold.

Proof Suppose the lemma is false and set $i = 1$, $j = 2$, $k = 3$; that is:

$$\left.\begin{array}{l} PQ = QP,\ PL_{23} = L_{23}P,\ QL_1 = L_1Q, \\ PL_1 \neq L_1P \text{ and } QL_{23} \neq L_{23}Q. \end{array}\right\} \quad (+)$$

From the assumption $QL_{23} \neq L_{23}Q$, $Z(Q) \leq Q_{\sigma\tau}$ (note that $[L_{23}, Q_\rho] = 1$) and so, as $[Q_{\sigma\tau}, L_1] = 1$, $[Z(Q), L_1] = 1$. Further, observe that $L_{23}^* = L_{23_\rho} \neq L_{23}$ for $L_{23_\rho} = L_{23}$ when combined with lemma 4.1 would violate hypothesis D.

Now suppose Q is not star-covered, then, by lemma 6.1 $L_1^* \not\leq N_{L_1}(P)$ and so $P_\sigma, P_\tau \leq N_P(L_1)$. Moreover, as Q is not star-covered, $O_q(PQ) \neq 1$ so $Z(Q) \cap O_q(PQ) \neq 1$ which gives $O_p(PQ) \leq N_P(L_1)$. Because of lemma 2.15, there is now no alternative other than $P = P_\rho$ and so $\mathcal{M}_{p,\pi_1} = \{P, N_P(L_1)L_1\}$. Also as $[Q,\rho] \neq 1$, because Q is not star-covered, it follows that $O_{\pi_1}(L_1Q)$, $P \leq N_G([Q,\rho])$ and hence that $O_{\pi_1}(L_1Q) = 1$; evidently L_1 is star-covered.

As PL_{23} admits $\sigma\tau$ fixed-point-freely with

${L_{23}}^{*}_{\langle\sigma\tau\rangle} = 1$, $[P, L_{23}] = 1$. Clearly $L_{23}L_1 = L_1L_{23}$ is untenable because, if $L_{23}L_1 = L_1L_{23}$, then $L_{23} \neq {L_{23}}^{*}$ would produce $P, L_1 \leq N_G(O_{\pi_{23}}(L_{23}L_1))$. Thus $L_{23}L_1 \neq L_1L_{23}$. As $P_{\sigma\tau} = 1$, $P_{\sigma}, P_{\tau} \leq N_P(L_1)$ and $L_1 = {L_1}^{*}$ it follows that for at least one of P_{σ}, and P_{τ}, (say P_{σ}) $C_{L_1}(P_{\sigma}) \neq 1$ and $C_{L_1}(P_{\sigma})$ is not contained in $L_{1_{\sigma\tau}}$. Clearly $\{C_G(P_{\sigma})\}_{\pi_{23},\pi_1} \geq L_{23}, C_{L_1}(P_{\sigma})$ and because $C_{L_1}(P_{\sigma}) \neq L_{1_{\sigma\tau}}$ it follows that $O_{\pi_1}(L_{23}\,N_{L_1}(L_{23})) \neq 1$ which in turn implies that $Z(L_1) \leq L_{1_{\sigma\tau}}$. From the latter consequence, it may be inferred that $[Z(L_1), N_P(L_1)] = 1$ whence $Z(L_1) \leq Y$ (the largest α-invariant subgroup of L_1 permuting with P) = 1; a contradiction. Thus it has been shown that the configuration (+) cannot occur when Q is not star-covered.

So now suppose Q is star-covered. Then (without loss) $[N_Q(L_{23}), \sigma] = 1$ and hence, as $C_Q(L_{23}) \neq 1$, $Q = Q_{\sigma}$. If $[P, \sigma]$ is non-trivial, then $1 \neq O_{\pi_{23}}(PL_{23})$ is contained in the largest α-invariant subgroup of L_{23} permutable with Q, which does not agree with lemma 5.1. Therefore $P = P_{\sigma}$ and, clearly, must also have $P_{\sigma\tau} = 1$ (if $P_{\sigma\tau} \neq 1$ then $P_{\sigma}, P_{\tau} \leq N_P(L_1)$ so giving $P = P_{\sigma} \leq N_P(L_1)$). But then $P_{\tau} = P_{\sigma\tau} = 1$ which is at variance with the supposed type of P.

The lemma is now proven.

<u>Lemma 6.9</u> Suppose L_i permutes with P but not with Q, where P and Q are α-invariant Sylow subgroups of type $\{1,2,3\}$. Moreover, if $PQ \neq QP$, then $L_i^* \leqslant N_{L_i}(Q)$ and $P^* \leqslant N_P(Q)$ cannot both hold.

<u>Proof</u> Assume the hypotheses of the lemma with (say) $i = 1$ and also that both $L_1^* \leqslant N_{L_1}(Q)$ and $P^* \leqslant N_P(Q)$ hold. Recall from lemmas 5.5 and 5.6 that $\mathcal{M}_{q,\pi_1} = \{L_1, N_{L_1}(Q)Q\}$ and $\mathcal{M}_{p,q} = \{P, N_P(Q)Q\}$. Thus, if $O_{\pi_1}(PL_1)_\sigma \not\leqslant O_{\pi_1}(PL_1)_\tau$ then $O_{\pi_1}(PL_1) \cap O_{\pi_1}(L_{1\sigma}Q_\sigma) \neq 1$ whence $\{N_G(O_{\pi_1}(PL_1) \cap O_{\pi_1}(L_{1\sigma}Q_\sigma))\}_{p,q} \geqslant O_p(PL_1)$, Q_σ (as $Q_\sigma \trianglelefteq L_{1\sigma}Q_\sigma$) and therefore $O_p(PL_1) \leqslant N_P(Q)$. As this leads to $P = O_p(PL_1)P^* \leqslant N_P(Q)$, it may be concluded that $O_{\pi_1}(PL_1)_\sigma \leqslant O_{\pi_1}(PL_1)_\tau$ and, similarly, that $O_{\pi_1}(PL_1)_\tau \leqslant O_{\pi_1}(PL_1)_\sigma$. Using lemma 4.3 gives $[O_{\pi_1}(PL_1), P] = 1$. If $O_p(P_\rho Q_\rho) \neq 1$ then $\{N_G(O_p(P_\rho Q_\rho))\}_{q,\pi_1} \geqslant O_{\pi_1}(PL_1)$, Q_ρ and so $O_{\pi_1}(PL_1) \leqslant N_{L_1}(Q)$ which, as $L_1^* \leqslant N_{L_1}(Q)$ already forces $L_1 \leqslant N_{L_1}(Q)$. Thus $O_p(P_\rho Q_\rho) = 1$ and so $P_\rho = P_{\rho\sigma}P_{\rho\tau}$. From lemma 5.6(i)(d), $Q_{\rho\sigma} \neq 1 \neq Q_{\rho\tau}$ and, as at least one of $P_{\rho\sigma}$ and $P_{\rho\tau}$ must be non-trivial, say, $P_{\rho\sigma}$, $\{N_G(P_{\rho\sigma})\}_{q,\pi_1} \geqslant Q_{\rho\sigma}$, $O_{\pi_1}(PL_1)$ again implying that $L_1Q = QL_1$.

Thus, in the given circumstances, $L_1^* \leqslant N_{L_1}(Q)$ and $P^* \leqslant N_P(Q)$ cannot hold simultaneously.

<u>Lemma 6.10</u> Let P be an α-invariant Sylow p-subgroup of G of type $\{1,2,3\}$ which is not star-covered. If P permutes with L_i and L_j but not L_k, then either $L_iL_j = L_jL_i$ or $Z(J(P)) \leqslant N_P(L_k)$.

<u>Proof</u> Without loss, may take $i = 1$, $j = 2$ and $k = 3$. Assuming the hypotheses of the lemma hold, together with $L_1L_2 \neq L_2L_1$ and $Z(J(P)) \not\leq N_P(L_3)$, a contradiction will be deduced.

Evidently, as $[P_{\rho\sigma}, L_3] = 1$, $J(P)_{\rho\sigma} = 1$. It is claimed that either $L_1 = L_{1_\sigma}$ or $L_2 = L_{2_\rho}$. Observe that, as $J(P)$, L_1 and L_2 admit $\rho\sigma$ fixed-point-freely and P is not star-covered, may obtain $L_1 = C_{L_1}(D)L_{1_\sigma}$ and $L_2 = C_{L_2}(D)L_{2_\rho}$ with $(1 \neq)\ D = O_p(PL_1) \cap O_p(PL_2) \cap Z(P)$ by use of Glauberman's factorization theorem (1.9). Since $L_1L_2 \neq L_2L_1$, it may be deduced that either $C_{L_1}(D) \leq N_{L_1}(L_2)$ or $C_{L_2}(D) \leq N_{L_2}(L_1)$ holds. Suppose the former occurs; it will now be shown that $N_{L_1}(L_2) \leq L_{1_\sigma}$. If it were the case that $[N_{L_1}(L_2), \sigma] \neq 1$, then $C_{L_1}(L_2) \neq 1$ whence, by lemma 5.1, $N_{L_2}(L_1) = 1$. Thus $L_{1_\tau} \leq N_{L_1}(L_2)$ which, combined with $[N_{L_1}(L_2), \sigma] \neq 1$, yields that $N_{L_1}(N_{L_1}(L_2))^* \leq N_{L_1}(L_2)$. Now lemma 2.15 contradicts the supposition that $L_1L_2 \neq L_2L_1$ because $O_{\pi_1}(PL_1) \leq C_{L_1}(D) \leq N_{L_1}(L_2)$. Therefore $N_{L_1}(L_2) \leq L_{1_\sigma}$ and hence $L_1 = C_{L_1}(D)L_{1_\sigma} = L_{1_\sigma}$. If $C_{L_2}(D) \leq N_{L_2}(L_1)$ were to hold, then it would give $L_2 = L_{2_\rho}$. Consequently, the above claim has been substantiated. Without loss, it will be assumed that $L_1 = L_{1_\sigma}$ holds. One consequence of $L_1 = L_{1_\sigma}$ is that $\mathcal{M}_{\pi_1,\pi_2} = \{L_1, L_2N_{L_1}(L_2)\}$. Further, because P is not star-covered and $[P, \sigma] \trianglelefteq PL_1$, it may be deduced that $O_{\pi_2}(PL_2) \leq N_{L_2}(L_1) = 1$. From $L_1 = L_{1_\sigma}$ and $L_{1_\rho} = 1$ it follows that $P = O_p(PL_1)P_{\rho\sigma}$ and so $J(P) \leq [P, \rho\sigma] \leq O_p(PL_1)$. Hence $J(P) = J(O_p(PL_1))$ and hence $J(P) \trianglelefteq PL_1$. Also, $L_2 = N_{L_2}(J(P))C_{L_2}(Z(P))O_{\pi_2}(PL_2) = C_{L_2}(Z(P))$.

Observe that $P_{\rho\sigma} = 1$ is untenable for then PL_2 would admit $\rho\sigma$ fixed-point-freely whence, as $O_{\pi_2}(PL_2) = 1$, $P \trianglelefteq PL_2$ which implies that $L_1, L_2 \leq N_G(J(P))$. Consequently, $C_P(L_3) \neq 1$ and thus $\mathcal{M}_{p,\pi_3} = \{P, L_3N_P(L_3)\}$ with $P_\rho, P_\sigma \leq N_P(L_3)$.

It is clear from lemma 6.1 that if $L_1L_3 = L_3L_1$ then $O_{\pi_1}(L_1L_3)$ must be trivial and so $L_1 = L_{1_\rho}$. Lemma 4.1 shows this situation to be at variance with hypothesis D. Hence $L_1L_3 \neq L_3L_1$. Clearly $L_{3_\sigma} \leq N_{L_3}(L_1)$. If $C_{L_3}(L_1)$ were non-trivial then $Z(L_3) \leq L_{3_{\rho\sigma}}$ which gives $[N_P(L_3), Z(L_3)] = 1$ so contradicting the form of $\mathcal{M}_{p,\pi_3}$. Thus $L_{3_\sigma} \leq N_{L_3}(L_1) \leq L_{3_\rho}$. Suppose $L_{3_\rho} = L_3^* \neq L_3$. Clearly $C_{L_3}(P_\rho) \neq 1$. If $J(P)_\rho \neq 1$, then the shape of $\mathcal{M}_{p,\pi_3}$ would dictate that $Z(J(P)) \leq N_P(L_3)$. Therefore $J(P)_\rho = 1$ and, as $J(P) \trianglelefteq PL_1$ and $L_{1_\rho} = 1$ it follows that $[J(P), L_1] = 1$. Hence $L_1, L_2 \leq C_G(Z(P))$ implying $L_1L_2 = L_2L_1$. Thus it may be inferred that $L_3 = L_{3_\rho}$. Therefore it follows that $P_\sigma \leq C_P(L_3)$. If $P_{\sigma\tau} \neq 1$ then, as $[P_{\sigma\tau}, L_1] = 1$, $L_1L_3 = L_3L_1$ which cannot occur. Hence $P_{\sigma\tau} = 1$. But then PL_2 admits $\sigma\tau$ fixed-point-freely with the result that $P \trianglelefteq PL_2$ (recall $O_{\pi_2}(PL_2) = 1$) thence $L_1, L_2 \leq N_G(J(P))$. With this contradiction the proof is complete.

7. LINKING THEOREMS BETWEEN α-INVARIANT SYLOW SUBGROUPS OF TYPE $\{1,2,3\}$.

This section describes some of the possible interactions between α-invariant Sylow subgroups of G of type I, II, III, and IV.

Lemma 7.1 Let Q be an α-invariant Sylow q-subgroup of G of type II. Then Q cannot, additionally, be of type I, III or IV.

Proof Let P be an α-invariant Sylow p-subgroup of type I with respect to Q. Without loss may suppose $Z(P) = Z(P)_{\sigma\tau} \leqslant N_P(Q)$ and (so) $Q_{\sigma\tau} = 1$. If Q were also either of type I or type IV then, from lemma 5.6(i)(b) and (ii)(g), $Z(Q) \leqslant Q_{\alpha_i\alpha_j}$ where $i, j \in \{1,2,3\}$, $i \neq j$. But then $[Z(Q), N_P(Q)] = 1$ which contravenes the form of $\mathcal{M}_{p,q}$ and hence it may be inferred that Q cannot be of type I or IV. Were Q to be of type III then $Q^* = Q_{\alpha_i}$, some $i \in \{1,2,3\}$. In view of the prevailing situation in Q, $\alpha_i = \rho$ but then $[Q_\sigma, Z(P)] = [Q_\tau, Z(P)] = 1$ which contradicts the supposed form of $\mathcal{M}_{p,q}$ and finishes the proof of the lemma.

Lemma 7.2 There does not exist an α-invariant Sylow subgroup of G which is both of type III and IV.

<u>Proof</u> Let P be an α-invariant Sylow p-subgroup of G. If P is of type III then, from lemma 5.6(ii)(g, k), $P_{\rho\tau}$, $P_{\rho\tau}$ and $P_{\sigma\tau}$ are all non-trivial. Whereas if P is of type IV then again from lemma 5.6(ii)(c) one of P ,P and P must be trivial. Clearly these two possibilities are incompatible.

<u>Lemma 7.3</u> Let P be an α-invariant Sylow p-subgroup of G. Then P cannot both be of type I and type III.

<u>Proof</u> Suppose the contrary; that is, there exists Sylow q- and w-subgroups Q and W which are (respectively) of type II and type IV with respect to P. Without loss it may be taken that $Z(P) = Z(P)_{\sigma\tau} \leq N_P(Q)$ and (so) $Q_{\sigma\tau} = 1$. Set $\mathcal{M}_{p,w} = \{PY, WX\}$. As $Z(P) \leq P_{\sigma\tau}$ has already been fixed, the only possibility, because of lemma 5.6(ii), concerning P and W is:- $P_\rho \leq X$ and W_σ, $W_\tau \leq Y$. Note $X < P^* \leq N_P(Q)$.

Since $Q_{\sigma\tau}$ and $W_{\sigma\tau}$ are trivial from (1.21), $QW = WQ$. Now $W = W^* O_w(WQ) = W_\rho O_w(WQ) = W_\rho C_W(Q_\rho)$. Consider $\{N_G(Q_\rho)\}_{q,p} \geq C_P(Q), C_W(Q_\rho)$. If $C_W(Q_\rho) \leq Y \leq W_\rho$ then $W = W_\rho$ which is forbidden by lemma 5.6(ii)(d). Therefore $C_P(Q) \leq X$. As $N_P(X)^* \leq X$ and $N_P(Q)/C_P(Q)$ is star-covered it follows that $X = N_P(Q)$. In particular, $P^* \leq X$. However lemma 5.6(i) shows that in this situation Y must be trivial, which is not so. Thus it may be concluded that P cannot be simultaneously of types I and III.

Lemma 7.4 Let P_1, P_2, Q_1 and Q_2 be α-invariant Sylow subgroups of G.

(i) Further suppose P_1 is of type I with respect to Q_1 and $Z(P_1) \leqslant P_{1_{\sigma\tau}}$ (and so $Q_{1_{\sigma\tau}} = 1$).

(a) If P_2 is of type I with respect to Q_1, then $Z(P_2) \leqslant P_{2_{\sigma\tau}}$.

(b) If P_1 is of type I with respect to Q_2, then $Q_{2_{\sigma\tau}} = 1$.

(ii) Further suppose P_1 is of type III(i) with respect to Q_1, and set $\{i,j,k\} = \{1,2,3\}$.

(a) If P_2 is of type III(h) ($h \in \{1,2,3\}$) with respect to Q_1, then $h = i$.

(b) If P_1 is of type III(h) ($h \in \{1,2,3\}$) with respect to Q_2, then $h = i$.

Proof (i)(a) Because Q_1 is of type II with respect to P_1 (and $Z(P_1) \leqslant P_{1_{\sigma\tau}}$), $Q_{1_{\sigma\tau}} = 1$, $Q_{1_{\rho\sigma}} \neq 1 \neq Q_{1_{\rho\tau}}$ from lemma 5.6(i). Clearly $Z(P_2) \leqslant P_{2_{\sigma\tau}}$ is the only possible candidate, for $Z(P_2) \leqslant P_{2_{\rho\sigma}}$ (say) dictates, by lemma 5.6(i), that $Q_{1_{\rho\sigma}} = 1$.

(b) Since $Z(P_1) \leqslant P_{1_{\sigma\tau}}$, the shape of $\mathcal{M}_{P_1,Q_2}$ and $[P_{1_{\sigma\tau}}, Q_{2_{\sigma\tau}}] = 1$ give that $Q_{2_{\sigma\tau}} = 1$.

(ii)(a) As P_1 is of type III(i) with respect to Q_1, by lemma 5.6(ii), $Q_1^* = Q_{1_{\alpha_i}}$. If P_2 is of type III(h) with respect to Q_1 and $h \neq i$, then $Q_{1_{\alpha_h}} = Q_1^* = Q_{1_{\alpha_i}}$ whence Q_1 cannot be of type $\{1,2,3\}$. Thus $h = i$.

(b) Since P_1 is of type III(i) with respect to Q_1, $Z(P_1) \leqslant P_{1_{\alpha_j\alpha_k}}$, by lemma 5.6(ii) and, if $h \neq i$ again from lemma 5.6(ii) this would give $Z(P_1) \leqslant P_{1_{\alpha_i}}$ contradicting the fixed-point-freeness of α. Hence $h = i$.

Lemma 7.5 Let P and Q be α-invariant Sylow p- and q-subgroups of G which do not permute. Additionally, let N be an α-invariant nilpotent Hall η-subgroup of G which permutes with both P and Q. Then:

(i) If P is of type I with respect to Q, then $C_Q(N_1) = 1$ for all non-trivial α-invariant subgroups N_1 of N.

(ii) If P is of type III with respect to Q, then N is star-covered.

Proof (i) As $P^* \leq N_P(Q)$, the arguments given in lemma 6.1 may be used to produce $P = N_P(Q)C_P(N)$. If $C_Q(N_1) \neq 1$ for some non-trivial α-invariant subgroup N_1 of N, then the shape of $\mathcal{M}_{p,q}$ would force $C_P(N) \leq N_P(Q)$ so giving $PQ = QP$. Therefore (i) holds.

(ii) Suppose $\mathcal{M}_{p,q} = \{PY, QX\}$ (with $P_\rho \leq X$, $Q_\sigma, Q_\tau \leq Y$) and that N is not star-covered. Then either $O_p(PN) \leq X$ or $O_q(QN) \leq Y$. The first possibility, as $N_P(X)^* \leq X$, produces $P = X$ (using lemma 2.15) whereas the latter gives $Q = O_q(QN)Q^* = Q_\rho$ (as $Y \leq Q_\rho = Q^*$). Both conclusions are inadmissible because P is of type III with respect to Q. Hence N must be star-covered.

Lemma 7.6 Let P be of type IV with respect to W. If, further, P is of type I (say) with respect to Q then W is contained in a unique maximal α-invariant subgroup of G.

<u>Proof</u> Deny the lemma. Clearly, by lemma 7.1, $Q \neq W$.

If $WQ \neq QW$ then, by lemma 7.1, W could only be of type I (with respect to Q). Then $\{N_G(Q)\}_{p,w} \geq P^*, W^*$ and hence the III, IV configuration between P and W would be impossible, by lemma 5.6(i). Thus $WQ = QW$.

Before continuing fix the following notation:- $Z(P) = Z(P)_{\sigma\tau} \leq N_P(Q)$ (and hence $Q_{\sigma\tau} = 1$); because P is of type IV, $P^* = P_{\alpha_i}$, some $i \in \{1,2,3\}$. Clearly $P^* = P_\tau$ or P_σ so, without loss, it may be taken that $P^* = P_\tau$ and, if $\mathcal{M}_{p,w} = \{PX, WY\}$, having assumed that $P^* = P_\tau$ necessarily: P_ρ, $P_\sigma \leq Y$ and $W_\tau \leq X$. From lemma 5.6(ii), $X = N_W(P) = X_\tau \times X_{\rho\sigma}$ and $Z(W) \leq X_{\rho\sigma}$.

By lemma 5.6(ii)(m), as W is not assumed to be contained in a unique maximal α-invariant subgroup of G, $J(W)_\tau = 1$.

If S is a non-trivial characteristic subgroup of Q then $N_W(S) \leq X = N_W(P)$ because $\{N_G(S)\}_{p,w} = N_P(Q)N_W(S)$ and $P^* \not\leq Y$, by lemma 5.6(i), hence $N_W(S) \leq X$. In particular $N_W(J(Q))$, $C_W(Z(Q)) \leq X$.

If $C_W(Z(Q)) \neq 1$ then, as $C_W(Z(Q))$ normalizes both P and $N_P(Q) \geq P^*$, $P = C_P(C_W(Z(Q)))N_P(Q)$ which then yields, as $\{C_G(C_W(Z(Q)))\}_{p,w} \geq C_P(C_W(Z(Q)))$, $Z(Q)$, that $C_P(C_W(Z(Q))) \leq N_P(Q)$, because of the shape of $\mathcal{M}_{p,q}$, and hence $P = N_P(Q)$. Consequently, must have $C_W(Z(Q)) = 1$ and so $W = N_W(J(Q))O_w(WQ)$.

It is claimed that $J(W) \trianglelefteq WQ$ is untenable. Suppose that $J(W) \trianglelefteq WQ$ then, as $J(W)_\tau = 1$, $[J(W), [Q, \tau]] = 1$. Lemma 5.6(i)(e) states that $[Q,\tau] \neq 1$ and so $\{N([Q,\tau])\}_{p,w} \geq P_\tau$ $(= P^*)$, $J(W)$. So either $P^* \leq Y$ or

$J(W) \leq X$. The first alternative is ruled out by lemma 5.6(i) and the second gives $J(W) \leq W_{\rho\tau}$ which implies, as $C_W(J(W)) \leq J(W)$, that $W = W_{\rho\tau}$ and contradicts lemma 5.6(ii)(1). Hence the claim is verified.

Suppose that it has been shown that $N_P(Q) = P^* = P_\tau$. As $N_W(J(Q))$ normalizes both P and $N_P(Q)$ $(\geq P^*)$, $[N_W(J(Q)), P] \leq N_P(Q) = P_\tau$. Clearly $[N_W(J(Q))_{\rho\sigma}, P] = [N_W(J(Q))_{\rho\sigma}, [N_W(J(Q))_{\rho\sigma}, P]] = 1$ because $[N_W(J(Q))_{\rho\sigma}, P] \leq P_\tau$. Since, from lemma 5.6(ii)(b), $O_w(PX) = 1$ it follows that $N_W(J(Q))_{\rho\sigma} = 1$ and so $N_W(J(Q)) = [N_W(J(Q)), \rho\sigma] \leq [X, \rho\sigma] \leq W_\tau$. Hence $W = N_W(J(Q))O_w(WQ) = W_\tau O_w(WQ)$ and so $J(W) \leq [W, \tau] \leq O_w(WQ)$ which implies that $J(W) \trianglelefteq WQ$ and this possibility has already been excluded. Thus the desired contradiction follows once $N_P(Q) = P^* = P_\tau$ has been established.

Suppose that $N_P(Q) \neq P^*$. If $O_q(QW) \neq 1$ then $\{N_G(O_q(QW))\}_{p,w} \geq W$, $C_P(Q)$ and hence $C_P(Q) \leq P_\tau$. Therefore $N_P(Q) = C_P(Q)N_P(Q)^* = C_P(Q)P_\tau = P_\tau$. Hence it may be assumed that $O_q(QW) = 1$ and so Q is star-covered.

As $Z(P) = Z(P)_{\sigma\tau} \leq N_P(Q)$ and $\mathcal{M}_{p,q} = \{P, N_P(Q)Q\}$, it follows that $S_\rho \neq 1$ for all non-trivial characteristic subgroups, S, of G. If, furthermore, S is abelian then $S = S_\sigma S_\tau$. For, as Q is star-covered and $Z(P)$ normalizes both S and $S_\sigma S_\tau = S^*_{<\sigma\tau>}$, $S = S^*_{<\sigma\tau>}C_S(Z(Q)) = S^*_{<\sigma\tau>} = S_\sigma S_\tau$. Moreover, when S is abelian, $S_\rho = S_{\rho\sigma}S_{\rho\tau}$.

Now, if $1 \neq N_W(J(Q))_\rho$ $(\leq N_W(P))$ then

$\{C_G(N_W(J(Q))_\rho)\}_{p,q} \geqslant C_P(N_W(J(Q))\ ),\ Z(J(Q))_\rho$ because, from the previous paragraph, $Z(J(Q))_\rho = Z(J(Q))_{\rho\sigma} Z(J(Q))_{\rho\tau}$. As $Z(J(Q))_\rho \neq 1$, this gives $C_P(N_W(J(Q))_\rho) \leqslant N_P(Q)$ and hence $P = C_P(N_W(J(Q))_\rho) N_P(Q) \leqslant N_P(Q)$. Thus it may be deduced that $N_W(J(Q))_\rho = 1$ and also, as $N_W(J(Q)) \leqslant X = W_\tau \times X_{\rho\sigma}$, that $N_W(J(Q)) \leqslant W_\tau$. Hence $W = N_W(J(Q)) O_w(WQ) = W_\tau O_w(WQ)$ and so a contradiction has been obtained in the form of $J(W) \trianglelefteq WQ$. This establishes that $N_P(Q) = P^*$ and so completes the proof of the lemma.

8. ON THE SUBGROUP GENERATED BY α-INVARIANT SYLOW SUBGROUPS OF G OF TYPE $\{1,2,3\}$.

The purpose of this section is to show that all α-invariant Sylow subgroups of G of type $\{1,2,3\}$ are of type V or, in other words, that the α-invariant Sylow subgroups of G of type $\{1,2,3\}$ generate a soluble Hall subgroup of G.

Theorem 8.1 If P is an α-invariant Sylow p-subgroup of G of type $\{1,2,3\}$ then P is of type V.

Proof Assuming that the theorem is false, a suitable contradiction will be derived.

The proof will be presented in a series of lemmas and broken down into three main cases:

Case(1) Only types I, II and V can occur;

Case(2) Only types III, IV and V can occur;

Case(3) Types I, II, III and IV all can occur.

Case (1) For lemmas 8.2 to 8.6, P will denote an α-invariant Sylow p-subgroup of G which is of type I with respect to Q and, additionally, it will be supposed that $Z(P) \leq P_{\sigma\tau}$ (so $Q_{\sigma\tau} = 1$).

Lemma 8.2 (i) Q permutes with L_{23};

(ii) $L_{12} = L_{13} = 1$;

(iii) Q permutes with at least one of L_2 and L_3;

and (iv) Q permutes with L_1.

<u>Proof</u> (i) Suppose $L_{23}Q \neq QL_{23}$ then, as $L_{23}N_Q(L_{23})$ admits $\sigma\tau$ fixed-point-freely with $L_{23}{}^{*}_{<\sigma\tau>} = 1$, $N_Q(L_{23}) \trianglelefteq L_{23}N_Q(L_{23})$ and therefore, since $N_Q(L_{23}) \neq 1$, $L_{23}Q = QL_{23}$.

(ii) Since $[L_{12}, Q_\tau] = [L_{13}, Q_\sigma] = 1$, from lemma 7.5(i) both of L_{12} and L_{13} must be trivial.

(iii) As $\{G_{\sigma\tau}\}_{q,\pi_2,\pi_3} = 1$, from lemma 2.5(ii), $[L_2, Q_\tau] = [L_3, Q_\sigma] = 1$. If $QL_2 \neq L_2Q$ and $QL_3 \neq L_3Q$ then as $L_{2_\sigma} = L_{3_\tau} = 1$, $Z(Q) \leqslant Q_{\sigma\tau}$ which is at variance with the fact that $Q_{\sigma\tau} = 1$.

(iv) Suppose that $L_1Q \neq QL_1$. As $Z(P) \leqslant P_{\sigma\tau}$, $L_1P = PL_1$ and hence, by lemma 6.9, $L_1^* \not\leqslant N_{L_1}(Q)$. Hence Q_σ, $Q_\tau \leqslant N_Q(L_1)$. If $[N_{L_1}(Q), \rho] \neq 1$ then $Z(Q) \leqslant Q_\rho$ which yields that $[Z(P), Z(Q)] = 1$ contradicting the form of $\mathcal{M}_{p,q}$. Thus $Q^* = Q_\rho$ but then, from (1.19)(iii), $[P_\sigma, Q_\sigma] = [P_\tau, Q_\tau] = 1$ again contradicting the form of $\mathcal{M}_{p,q}$.

<u>Lemma 8.3</u> Suppose $PL_i = L_iP$ for some $i \in \{1,2,3\}$ and let W be an α-invariant Sylow w-subgroup of type $\{1,2,3\}$ which permutes with P. Then W permutes with L_i.

<u>Proof</u> Suppose $L_iW \neq WL_i$; clearly $L_i \neq 1 \neq W$. From lemma 6.2, as P is not star-covered, $L_i^* \not\leqslant N_{L_i}(W)$.

First consider the case $i = 1$. Then $L_{1_\sigma} \neq 1 \neq L_{1_\tau}$

and hence, as $[P_{\sigma\tau}, L_1] = 1$, $[Z(P), L_1] = 1$. Because P is not star-covered, $O_p(PW) \neq 1$ whence $O_p(PW) \cap Z(P) \neq 1$ and consequently $O_w(PW) \leq N_W(L_1)$. Since $N_W(L_1) \neq W$ and $N_W(L_1) \geq W_\sigma$, W_τ the only conclusion that may be drawn, in view of lemma 2.15, is that $W = W_\rho$. Clearly W must permute with Q as W is not a suitable candidate to be of type I (if $WQ \neq QW$, then by lemma 6.1 W would have to be of type I). So lemma 6.1 becomes available and yields that $O_w(WQ) = 1$. Therefore, as WQ admits $\sigma\tau$ fixed-point-freely, $W = W_\sigma W_\tau$, from (1.19)(ii). Consequently, by (1.19)(iii), G_ρ has a normal w-complement and so, as $W = W_\rho (\neq 1)$, lemma 4.1 implies that G does not satisfy hypothesis D. Thus for $i = 1$, $WL_i = L_1W$.

Now examine the case $i = 2$; as for the case $i = 1$, it will first be shown that $WL_2 \neq L_2W$ implies that $WQ = QW$. Since $WL_2 \neq L_2W$, L_2 must be non-trivial and so if $L_2Q = QL_2$ then, as $[L_2, Q_\tau] = 1$, lemma 7.5(i) shows this cannot occur. Thus $L_2Q \neq QL_2$ and, as $Q_{\rho\sigma} \neq 1$ by lemma 5.6(i)(d), Q_ρ, $Q_\tau \leq N_Q(L_2)$ with $N_Q(N_Q(L_2))^* \leq N_Q(L_2)$.

Now, suppose that $QW \neq WQ$. The prevailing hypotheses demand that, by lemma 7.1 W be of type I. Hence, from lemma 7.4(i)(a), $Z(W) \leq W_{\sigma\tau}$ and so $Z(W) \leq N_W(L_2)$. As $Q_\rho \leq N_Q(L_2)$, $Z(W)$ normalizes Q and $N_Q(L_2)(\neq Q)$, it follows that $C_Q(Z(W)) \neq 1$. But this is not possible by lemma 5.6(i)(a) and so it may be concluded that $QW = WQ$.

At least one of $[N_W(L_2), \rho]$ and $[N_W(L_2), \tau]$ must be non-trivial; suppose $[N_W(L_2),\rho]$ is non-trivial. Because $O_q(QW) \cap N_Q(L_2) \geqslant O_q(QW)_\rho$, $O_q(QW)_\tau$ and $N_W(L_2)$ normalizes $O_q(QW)$ and $O_q(QW) \cap N_Q(L_2)$, $O_q(QW) = (O_q(QW) \cap N_Q(L_2))C_{O_q(QW)}([N_W(L_2),\rho])$. By lemma 7.5(i), $C_{O_q(QW)}([N_W(L_2),\rho])$ must be trivial. Therefore $O_q(QW) \leqslant N_Q(L_2)$ which, when combined with $N_Q(N_Q(L_2))^* \leqslant N_Q(L_2)$ and lemma 2.15, forces $Q = N_Q(L_2)$. Thus the lemma holds for $i = 2$. When $i = 3$, a similar argument to that presented for $i = 2$ will suffice.

<u>Lemma 8.4</u> Suppose L_i permutes with P, where $i = 2$ or 3. Then $L_1L_i = L_iL_1$.

<u>Proof</u> Suppose the lemma is false and, without loss, set $i = 2$; note that $L_1P = PL_1$. As $[L_2, Q_\tau] = 1$, if $O_{\pi_2}(L_2P)$ is non-trivial then $\{N_G(O_{\pi_2}(L_2P))\}_{p,q} \geqslant P, Q_\tau$ which forces $PQ = QP$. Hence $O_{\pi_2}(L_2P) = 1$ and so L_2 is star-covered. From lemma 8.2(iii), $L_1Q = QL_1$ and hence, by lemma 6.1, $O_{\pi_1}(L_1Q) = 1$ and so L_1 is also star-covered.

Thus either $L_1 = L_{1_\sigma}$ or $L_2 = L_{2_\rho}$ (because either $L_{1_\tau} \leqslant N_{L_1}(L_2)$ or $L_{2_\tau} \leqslant N_{L_2}(L_1)$; if say $L_{1_\tau} \leqslant N_{L_1}(L_2)$ then $[N_{L_1}(L_2),\rho] \neq 1$ would mean L_1 could not be star-covered and so $L_{2_\rho} = L_2^* = L_2$). The permutability of L_2 and Q is excluded by lemma 7.5(i) as $[L_2, Q_\tau] = 1$. Since $C_Q(L_2) \neq 1$ and $QL_2 \neq L_2Q$, if $L_2 = L_{2_\rho}$, then $Z(Q) \leqslant Q_\rho$ giving $[Z(Q), Z(P)] = 1$ which cannot occur.

Thus $L_1 = L_{1_\sigma}$ and furthermore, as $L_{2_\sigma} = 1$, $1 \neq L_{1_\tau} \leqslant N_{L_1}(L_2)$. As $O_q(L_1Q) \cap N_Q(L_2) \geqslant O_q(L_1Q)_\rho$ and $L_{1_\rho} = 1$, either $1 \neq C_{O_q(L_1Q)}(N_{L_1}(L_2)) \leq C_Q(N_{L_1}(L_2))$ or $O_q(L_1Q) \leq N_Q(L_2)$. The first possibility contradicts the conclusion of lemma 7.5(i), whereas the second possibility, since $N_Q(N_Q(L_2))^* \leq N_Q(L_2)$, forces $QL_2 = L_2Q$.

Hence the lemma is established.

<u>Lemma 8.5</u> Suppose U and V are α-invariant Sylow u- and v-subgroups of type $\{1,2,3\}$ both of which permute with P. Then UV = VU.

<u>Proof</u> As only Sylow subgroups of types I, II and V are assumed to occur, if $UV \neq VU$, without loss, it may be supposed that V is of type I with respect to U. Hence $V^* \leq N_V(U)$ and so, from lemma 6.1, $O_p(PV) = 1$. However, as P is of type I, P is not star-covered. Therefore UV = VU.

<u>Lemma 8.6</u> If $PL_i \neq L_iP$ (where i = 2 or 3), then $QL_i = L_iQ$.

<u>Proof</u> Deny the result, and suppose i = 2. Since $Z(P) \leq P_{\sigma\tau}$, $[Z(P), N_{L_2}(P)] = 1$ and hence if $L_2^* \not\leq N_{L_2}(P)$, then this implies, as $N_{L_2}(P) \neq 1$, that $\mathcal{M}_{p,\pi_2}$ has a form contrary to that given in lemma 5.6(i). Therefore $L_2^* \not\leq N_{L_2}(P)$ hence P_ρ, $P_\tau \leq N_P(L_2)$ and, in particular, $Z(P) = Z(P)_{\sigma\tau} \leq N_P(L_2)$.

Now $[L_2, Q_\tau] = 1$ implies that Q_ρ, $Q_\tau \leq N_Q(L_2)$. Since Z(P) normalizes Q and $N_Q(L_2)$ $(\neq Q) \geq Q_\rho$, clearly $C_Q(Z(P))$ must be non-trivial. This violates the conclusion of lemma 5.6(i)(a) and hence lemma 8.6 is proven.

<u>Lemma 8.7</u> If Q_1 and Q_2 are α-invariant Sylow subgroups of type $\{1,2,3\}$ which do not permute with P. Then $Q_1Q_2 = Q_2Q_1$.

<u>Proof</u> As case (1) holds P must be of type I with respect to Q_1 and Q_2 because of lemma 7.1. From lemma 7.4(i)(b), as $Q_{\sigma\tau} = 1$, it follows that $Q_{1_{\sigma\tau}} = Q_{2_{\sigma\tau}} = 1$ and hence, by (1.21), $Q_1Q_2 = Q_2Q_1$.

It will now be shown that case (1) is untenable. To this end, for the three ensuing lemmas, P will denote some fixed α-invariant Sylow ~~p~~-subgroup of G of type I (as theorem 8.1, is assumed to be false together with the assumption of case (1), there must exist at least one such P). Also assume $Z(P) \leq P_{\sigma\tau}$.

Define $H = \langle W, L_{\cdot}, L_{\cdot\cdot} \mid L_{\cdot}P = PL_{\cdot}, L_{\cdot\cdot}P = PL_{\cdot\cdot}$ and $WP = PW$, W α-invariant Sylow subgroup of type $\{1,2,3\}\rangle$ and $K = \langle W, L_{\cdot} L_{\cdot\cdot} \mid L_{\cdot}P \neq PL_{\cdot}, L_{\cdot\cdot}P \neq PL_{\cdot\cdot}$ and $WP \neq PW$, W α-invariant Sylow subgroup of type $\{1,2,3\}\rangle$. Further, set $H^+ = \langle W \mid W$ is an α-invariant Sylow subgroup of H of type $\{1,2,3\}\rangle$ and $K^+ = \langle W \mid W$ is an α-invariant Sylow subgroup of K of type $\{1,2,3\}\rangle$.

<u>Lemma 8.8</u> $G = HK$

<u>Proof</u> First recall from lemma 8.2(ii) that $L_{12} = L_{13} = 1$;

also, as $L_{23}Q = QL_{23}$ (where Q is an α-invariant Sylow subgroup of type II with respect to P) and $L_{23} \neq L_{23}^{*}$ (if $L_{23} \neq 1$); if L_{23} is non-trivial, then $L_{23}P \neq PL_{23}$ because of lemma 6.1. Clearly $PL_1 = L_1P$ and from lemmas 8.5 and 8.7 both H^+ and K^+ are soluble Hall subgroups of G.

Suppose both L_2 and L_3 do not permute with P. Thus lemma 6.3 yields that $L_2L_3 = L_3L_2$. Also lemma 8.6 shows that both L_2 and L_3 permute with K^+. Applying lemma 8.3 gives $L_1H^+ = H^+L_1$ and so $G = (H^+L_1)(L_2L_3L_{23}K^+) = HK$ since L_{23} clearly permutes with $L_2L_3K^+$.

Suppose L_2 permutes with P but L_3 does not permute with P. Using lemmas 8.6 and 8.3 gives respectively that $L_3K^+ = K^+L_3$ and that both L_1 and L_2 permute with H^+. Thus $G = (H^+L_1L_2)(K^+L_3L_{23}) = HK$ by virtue of lemma 8.4.

Suppose L_2 and L_3 permute with P. By lemma 8.2(iii) it may be supposed that $QL_3 = L_3Q$ and hence, as $[L_3, Q_{\mathcal{G}}] = 1$, lemma 7.5(i) demands that L_3 be trivial. Again, use of lemmas 8.3, 8.4 and 8.6 gives $G = (H^+L_1L_2)(K^+L_{23}) = HK$.

This exhausts all the possibilities and therefore lemma 8.8 follows.

The next result will be used in showing that the factorization obtained in the previous lemma is incompatible with hypothesis D.

<u>Lemma 8.9</u> Let P be of type I with respect to Q and assume $PL_i \neq L_iP$ ($i = 2$ or 3). If $Z(J(P)) \leq N_P(Q)$, then $Z(J(P)) \leq N_P(L_i)$.

<u>Proof</u> Suppose the lemma is false (and assume $i = 3$). Recall that, from lemma 8.6, $QL_3 = L_3Q$ and observe that $L_3^* \not\leq N_{L_3}(P)$ because $Z(P) = Z(P)_{\sigma\tau}$ renders $L_3^* \leq N_{L_3}(P)$ impossible.

If (say) $Z(J(P))_\sigma \neq Z(J(P))_\sigma{}^*_{<\rho\tau>}$, then $O_p(P_\sigma L_{3_\sigma}) \cap Z(J(P)) \neq 1$ and $\{N_G(O_p(P_\sigma L_{3_\sigma}) \cap Z(J(P)))\}_{p,\pi_3} \geq Z(J(P))$, L_{3_σ}. As P_σ, $P_\rho \leq N_P(L_3)$, the remark succeeding lemma 5.5 demands that $Z(J(P)) \leq N_P(L_3)$. Therefore $Z(J(P))_\sigma = Z(J(P))_\sigma{}^*_{<\rho\tau>}$ and, for similar reasons, $Z(J(P))_\rho = Z(J(P))_\rho{}^*_{<\sigma\tau>}$. As $P_{\rho\sigma} \leq C_P(L_3)$, clearly $Z(J(P))_{\rho\sigma} = 1$ may be assumed hence $Z(J(P))_\rho$, $Z(J(P))_\sigma \leq Z(J(P))_\tau$. Thus $Z(J(P))^* = Z(J(P))_\tau$ and so $[Z(J(P)),\tau]^* = 1$. Since $L_{3_\tau} = 1$, $O_q(L_3Q) = 1$ would dictate that $Q = Q_\tau$ which would contravene the type of Q (see lemma 5.6(ii)). Hence, as $[Z(J(P)),\tau]^* = 1$, $[Z(J(P)),\tau]$, $L_3 \leq \{N_G(O_q(L_3Q))\}_{p,\pi_3}$ and, therefore, $[Z(J(P)),\tau] \leq C_P(L_3)$. If $[Z(J(P)),\tau] \neq 1$ then $Z(J(P)) \leq N_P(L_3)$. Hence $[Z(J(P)),\tau]$ must be trivial.

Claim that $N_{L_3}(P) = 1$ (from lemma 5.5 $N_{L_3}(P)$ is the largest α-invariant subgroup of L_3 permutable with P). For if not then, as $Z(J(P)) \leq P_\tau$ and $L_{3_\tau} = 1$, $[Z(J(P)), N_{L_3}(P)] = 1$ whence $Z(J(P)) \leq N_P(L_3)$. Therefore $\mathcal{M}_{p,\pi_3} = \{L_3N_P(L_3), P\}$ and so, if one of $C_{L_3}(Z(Q))$ or $N_{L_3}(J(Q))$ is non-trivial, then $Z(J(P))$ is forced to be contained in $N_P(L_3)$. Thus, applying Glauberman's factorization theorem, $L_3 \trianglelefteq L_3Q$ and, as $L_{3_\tau} = 1$, $[L_3, [Q,\tau]] = 1$. Since $[Q,\tau] \neq 1$, $\{N_G([Q,\tau])\}_{p,\pi_3} \geqslant P_\tau$, L_3 and so $Z(J(P)) \leqslant N_P(L_3)$ which has been assumed not to hold. Hence the lemma is established.

<u>Lemma 8.10</u> G does not satisfy hypothesis D.

<u>Proof</u> Observe that one of the following two possibilities must hold: <u>either</u> $Z(J(P)) \leqslant N_P(Q)$ for each α-invariant Sylow subgroup Q which is of type II with respect to P <u>or</u> $Z(J(P))_\rho = Z(J(P))_{\rho\sigma} . Z(J(P))_{\rho\tau}$, $Z(J(P))_\sigma = Z(J(P))_{\rho\sigma} Z(J(P))_{\sigma\tau}$ and $Z(J(P))_\tau = Z(J(P))_{\rho\tau} Z(J(P))_{\sigma\tau}$.

The latter possibility, combined with lemma 4.3 and the fact that P is not star-covered, gives that $H \leqslant C_G(D)$ where $1 \neq D = O_p(H) \cap Z(P)$. Clearly $Z(P) = Z(P)_{\sigma\tau} \leqslant N_P(Q)$ for each Q of type II with respect to P, so $Z(P) \leqslant N_G(K^+)$. If L_{23} is non-trivial then $L_{23}P \neq PL_{23}$ and hence $Z(P) \leqslant N_P(L_{23})$. If $L_2(L_3)$ does not permute with P then, as $P_\rho, P_\tau \leqslant N_P(L_2)$ $(P_\rho, P_\sigma \leqslant N_P(L_3))$, it follows that $Z(P) \leqslant N_P(L_2)$ $(Z(P) \leqslant N_P(L_3))$ also. These remarks taken together give that $G = C_G(D)K$ with $D \leqslant Z(P) \leqslant N_G(K)$ $(\neq G)$ and clearly G has a non-trivial proper normal α-invariant subgroup, namely D^K.

When the first possibility holds, $Z(J(P)) \leqslant N_P(K^+)$ and if $L_3P \neq PL_3$ then, by recourse to lemma 8.9, $Z(J(P)) \leqslant N_P(L_3)$ also. Similarly, $L_2P \neq PL_2$ leads to $Z(J(P)) \leqslant N_P(L_2)$. If $L_{23} \neq 1$ then $L_{23}P \neq PL_{23}$ and, as $[L_{23}, K^+] = 1$, this produces $Z(J(P)) \leqslant N_P(L_{23})$ as well.

Thus $Z(J(P)) \leqslant N_P(K)$. Let $D = O_p(H) \cap Z(P)$ $(\neq 1)$. Then (from Glauberman's factorization theorem) $D^H \leqslant Z(J(P)) \leqslant N_G(K)$ and so D^G $(\leqslant N_G(K))$ is a non-trivial proper normal α-invariant subgroup of G.

Consequently G does not satisfy hypothesis D.

Case(2) The first few lemmas under the assumption of case(2) parallel some of those of the preceding case.

Until stated otherwise P will denote an α-invariant Sylow p-subgroup of type III with respect to Q. Further the following notation will also be fixed:- $\mathcal{M}_{p,q} = \{PY, QX\}$ with $P_\rho \leqslant X$ and Q_σ, $Q_\tau \leqslant Y$. Thus referring to lemma 5.6(ii): $Z(P) = Z(P)_{\sigma\tau} \leqslant X = N_P(Q)$, $Y \leqslant Q_\rho$ so $Q_{\sigma\tau} = 1$ and $Q^* = Q_\rho$.

Lemma 8.11 (i) Q permutes with L_{12}, L_{13} and L_{23}.

(ii) $L_{12} = L_{13} = 1$.

Proof (i) As $Q_{\sigma\tau} = 1$, $L_{23}{}^*{}_{<\sigma\tau>} = 1$ and $C_Q(L_{23}) \neq 1$, clearly $QL_{23} = L_{23}Q$.

Consider L_{12}. Since $Z(P) \leqslant P_{\sigma\tau}$, $\hat{L}_1 P = P\hat{L}_1$ so, in particular, $PL_{12} = L_{12}P$. Suppose $L_{12}Q \neq QL_{12}$ then $\mathcal{M}_{\pi_{12},q} = \{Q, L_{12}N_Q(L_{12})\}$ with $Q_\tau \leqslant C_Q(L_{12})$. Observe that $N_Q(L_{12}) = C_Q(L_{12})(N_Q(L_{12}))_{\rho\sigma}$. Since $L_{12} \neq 1$ and G satisfies hypothesis D, $L_{12} \neq L_{12}{}^*$ hence $O_{\pi_{12}}(PL_{12}) \neq 1$, and so $\{N_G(O_{\pi_{12}}(PL_{12}))\}_{p,q} \geqslant C_Q(L_{12})$, P. Therefore $C_Q(L_{12}) \leqslant Y \leqslant Q_\rho$ and hence $N_Q(L_{12}) \leqslant Q_\rho$ leading, because of (1.7)(x), to $Q = Q_\rho$ which is contrary to lemma 5.6(ii)(d). Thus $L_{12}Q = QL_{12}$. That $L_{13}Q = QL_{13}$

follows by similar reasoning.

(ii) Suppose L_{12} is non-trivial. Then, because hypothesis D holds, $L_{12}^{*} \neq L_{12}$. As L_{12} permutes with both P and Q this situation is impossible in the light of lemma 7.5(ii). Again, similar arguing gives $L_{13} = 1$.

Lemma 8.12 If Q_1 and Q_2 are α-invariant Sylow subgroups of type $\{1,2,3\}$ which are of type IV with respect to P, then $Q_1Q_2 = Q_2Q_1$.

Proof Let $\mathcal{M}_{p,q_1} = \{PY_1, X_1Q_1\}$ and $\mathcal{M}_{p,q_2} = \{PY_2, X_2Q_2\}$. Then from lemma 7.4(ii)(b), the shape of $\mathcal{M}_{p,q}$ forces $P_\rho \leq X_1$, Q_{1_σ}, $Q_{1_\tau} \leq Y_1$ and $P_\rho \leq X_2$, Q_{2_σ}, $Q_{2_\tau} \leq Y_2$. Thus $Q_{1_{\sigma\tau}} = Q_{2_{\sigma\tau}} = 1$ and hence $Q_1Q_2 = Q_2Q_1$ by (1.21).

Lemma 8.13 If $L_iP \neq PL_i$, where $i = 2$ or 3, then $L_iQ = QL_i$.

Proof Suppose $L_iQ \neq QL_i$ then (taking $i = 2$), as $[Q_\tau, L_2] = 1$, Q_ρ, $Q_\tau \leq N_Q(L_2)$. Since $Z(P) \leq P_{\sigma\tau}$, it may also be deduced that P_ρ, $P_\tau \leq N_P(L_2)$. Consequently $Z(P) = Z(P)_{\sigma\tau} \leq N_P(L_2) \cap N_P(Q)$ and therefore, as $N_Q(L_2) \geq Q_\rho$, $Q = C_Q(Z(P))N_Q(L_2)$. However $C_Q(Z(P)) \leq Y \leq Q_\rho \leq N_Q(L_2)$ giving $Q = N_Q(L_2)$ which contradicts the original supposition.

Thus $L_iQ = QL_i$.

Lemma 8.14 Let W be an α-invariant Sylow subgroup of type $\{1,2,3\}$ which permutes with P. If, further, it is assumed that $J(P)_\rho = 1$, then $L_1W = WL_1$.

Proof Suppose the contrary. Since $PL_1 = L_1P$, and P is not star-covered, by lemma 6.2, $L_1^* \nleq N_{L_1}(W)$. Thus W_σ, $W_\tau \leq N_W(L_1)$. Furthermore, as $Z(P) \leq P_{\sigma\tau}$, $[Z(P), L_1] = 1$ which forces $O_w(PW) \leq N_W(L_1)$ (because $Z(P) \cap O_p(PW) \neq 1$). Lemma 2.15 demands that $W = W_\rho$ and so $[P, \rho] \trianglelefteq PW$; hence $[P, \rho] \leq O_p(PW) \cap O_p(PL_1)$. By assumption, $J(P) \leq [P, \rho]$ and so, by a well known property of the Thompson subgroup, $J(P)$ is a characteristic subgroup of both $O_p(PL_1)$ and $O_p(PW)$. Therefore $\{N_G(J(P))\}_{w,\pi_1} \geq L_1, W$ from which it follows that $L_1W = WL_1$.

Lemma 8.15 If L_i ($i = 2$ or 3) permutes with P and $J(P)_\rho = 1$, then $L_1L_i = L_iL_1$.

Proof Without loss, examine the lemma for $i = 2$; note $PL_1 = L_1P$. Suppose $L_1L_2 \neq L_2L_1$. As $[Z(P), L_1] = 1$ and P is not star-covered, $O_p(PL_2) \cap Z(P) \neq 1$, hence $O_{\pi_2}(PL_2) \leq N_{L_2}(L_1)$.

If $O_p(PL_2)_\rho \neq 1$ then by lemma 5.6(ii)(e) $Z(O_p(PL_2)) \leq X = X_{\sigma\tau} \times P_\rho$. If $Z(O_p(PL_2))_\rho \neq 1$ then a furhter application of lemma 5.6(ii)(e) yields that $O_p(BL_2) \leq X$. Since $N_P(X)^* \leq X$ it follows, from lemma 2.15, that $P = X$ which contradicts the type of P. Thus $Z(O_p(PL_2))_\rho = 1$ and therefore $Z(O_p(PL_2)) \leq X_{\sigma\tau} \leq P_{\sigma\tau}$. Hence $N_G(Z(O_p(PL_2))) \geq L_1, L_2$ because $[L_1, P_{\sigma\tau}] = 1$ so giving $L_1L_2 = L_2L_1$.

Thus it may be deduced that $O_p(PL_2)_\rho = 1$ from which it follows that $O_p(PL_2)O_{\pi_1}(PL_1)$ admits fixed-point-freely. Hence $O_{\pi_1}(PL_1) \leq N_{L_1}(L_2)$.

Therefore, depending on whether $L_{1\tau} \leqslant N_{L_1}(L_2)$ or $L_{2\tau} \leqslant N_{L_2}(L_1)$, either $L_1 = L_{1\sigma}$ or $L_2 = L_{2\rho}$ (because $O_{\pi_1}(PL_1) \leqslant N_{L_1}(L_2)$ and $O_{\pi_2}(PL_2) \leqslant N_{L_2}(L_1)$).

If $L_2 = L_{2\rho}$ then $[P,\rho] \leqslant O_p(PL_2) \cap O_p(PL_1)$ and, as $J(P)_\rho = 1$, this gives $J(P)$ normal in both L_1P and L_2P. Hence $L_1L_2 = L_2L_1$.

Therefore $L_1 = L_{1\sigma}$ and, since $L_{2\sigma} = 1$ and $L_1L_2 \neq L_2L_1$, $\mathcal{M}_{\pi_1,\pi_2} = \{L_1, N_{L_1}(L_2)L_2\}$. As $1 \neq [P,\sigma] \leqslant PL_1$ and $[P,\sigma] \leqslant O_p(PL_2)$ it may be deduced that $O_{\pi_2}(PL_2) \leqslant N_{L_2}(L_1) = 1$.

As $[Z(P), L_1] = 1$ and $J(P) \leqslant [P, \rho] \leqslant O_p(PL_1)$, $C_{L_1}(Z(P)) = L_1 = N_{L_1}(J(P))$. Taken together with the fact that $L_2 = C_{L_2}(Z(P))N_{L_2}(J(P))$, it follows that $L_1L_2 = L_2L_1$ which gives the required contradiction.

<u>Lemma 8.16</u> Suppose U and V are α-invariant Sylow subgroups of type $\{1,2,3\}$ both of which permute with P. Then $UV = VU$.

<u>Proof</u> Because case(2) holds, if $UV \neq VU$, without loss, V may be taken to be of type III and U of type IV. Since P is not star-covered, lemma 7.5(ii) shows this configuration to be impossible.

<u>Lemma 8.17</u> If L_2 and L_3 permute with P, then $L_2L_3 = L_3L_2$.

<u>Proof</u> It will first be demonstrated that $O_p(PL_2)_\rho \neq 1 \neq$

$O_p(PL_3)_\rho$. This follows from the fact that $[X,\sigma] \leqslant [P,\sigma] \leqslant O_p(PL_2)$ and by lemma 5.6(ii)(k) that $1 \neq [X,\sigma] \leqslant P_\rho$. Similarly for $O_p(PL_3)_\rho$.

Consequently by lemma 5.6(ii)(e), $Z(O_p(PL_2))$ and $Z(O_p(PL_3))$ are contained in $N_P(Q) = X = X_{\sigma\tau} \times P_\rho$. If (say) $Z(O_p(PL_2))_\rho \neq 1$ then, using lemma 5.6(ii)(e), $O_p(PL_2) \leqslant X$ and so, as $N_P(X)^* \leqslant X$, lemma 2.15 gives $PQ = QP$. Hence $Z(O_p(PL_2))$, $Z(O_p(PL_2)) \leqslant X_{\sigma\tau}$ and therefore $[Z(O_p(PL_2)), L_2] = [Z(O_p(PL_3)), L_3] = 1$. As P is not star-covered, $O_p(PL_2) \cap O_p(PL_3) \neq 1$. From $1 \neq O_p(PL_2) \cap O_p(PL_3) \trianglelefteq O_p(PL_2)$ it follows that $O_p(PL_2) \cap O_p(PL_3) \cap Z(O_p(PL_2)) \neq 1$. However $O_p(PL_2) \cap O_p(PL_3) \cap Z(O_p(PL_2))$ is normal in P so $O_p(PL_2) \cap O_p(PL_3) \cap Z(O_p(PL_2)) \trianglelefteq O_p(PL_3)$ hence $O_p(PL_2) \cap O_p(PL_3) \cap Z(O_p(PL_2)) \cap Z(O_p(PL_3)) \neq 1$. In particular, $Z(O_p(PL_2)) \cap Z(O_p(PL_3)) \neq 1$ and thus $L_3L_2 = L_2L_3$.

<u>Lemma 8.18</u> If $PL_i = L_iP$ (where $i = 2$ or 3) then $L_iQ = QL_i$.

<u>Proof</u> As in the proof of lemma 8.17 it may be asserted that $Z(O_p(PL_2)) \leqslant X_{\sigma\tau}$ and hence $[Z(O_p(PL_2)), L_2] = 1$. If $L_2Q \neq QL_2$ (say), then as $Q_\rho, Q_\tau \leqslant N_Q(L_2)$ (with $Q^* = Q_\rho$) and $Z(O_p(PL_2)) \leqslant N_P(L_2) \cap N_P(Q)$, $Q = N_Q(L_2)C_Q(Z(O_p(PL_2)))$ $= N_Q(L_2)Q_\rho$ (because $Z(O_p(PL_2)) \cap Z(P) \neq 1) = N_Q(L_2)$ is obtained.

Hence $QL_2 = L_2Q$ must hold; the result for $i = 3$ may be established in a similar manner.

<u>Remark</u> Lemma 8.18 shows that if Q_h is any α-invariant Sylow subgroup of type IV with respect to P and $PL_i = L_iP$ ($i = 2$ or 3), then $Q_hL_i = L_iQ_h$.

<u>Lemma 8.19</u> Suppose $J(P)_\rho = 1$ and that $PL_i \neq L_iP$ for $i = 2$ or 3. If $i = 3$, assume that $Z(J(P))_\sigma \not\leq Z(J(P))_\tau$ and, if $i = 2$, assume that $Z(J(P))_\tau \not\leq Z(J(P))_\sigma$, then:

(i) L_i does not permute with any α-invariant Sylow subgroup of type $\{1,2,3\}$ which permutes with P; and

(ii) For any Q_h which is of type IV with respect to P and any α-invariant Sylow subgroup W of type $\{1,2,3\}$ which permutes with P, $W_\rho \leq N_W(Q_h)$.

<u>Proof</u> Assume the hypotheses of lemma 8.19 and that $i = 3$. Observe that $L_3P \neq PL_3$ and $Z(P) \leq P_{\sigma\tau}$ dictates that P_ρ, $P_\sigma \leq N_P(L_3)$. As $Z(J(P))_\sigma \not\leq Z(J(P))_\tau$, $Z(J(P)) \cap O_p(P_\sigma L_{3\sigma}) \neq 1$ and so, as $P_\sigma L_{3\sigma}$ is π_3-closed, $\{N_G(Z(J(P)) \cap O_p(P_\sigma L_{3\sigma}))\}_{p,\pi_3} \geq J(P), L_{3\sigma}$. The remark following lemma 5.5 is relevant here and hence $J(P) \leq N_P(L_3)$. Clearly $J(P) = J(P)_\tau C_{J(P)}(L_3)$ and, since P is not star-covered, obviously $J(P) \neq J(P)_\tau$.

If $O_{\pi_3}(QL_3) \neq 1$ (by lemma 8.13, L_3 and Q permute), then $\{N_G(O_{\pi_3}(L_3Q))\}_{p,q} \geq C_P(L_3), Q$. In particular, $C_{J(P)}(L_3) \leq X = X_{\sigma\tau} \times P_\rho$ and so, as $J(P)_\rho = 1$, $C_{J(P)}(L_3) \leq X_{\sigma\tau}$ giving $J(P) \leq P_\tau$ which, as has been mentioned, cannot occur. Therefore $O_{\pi_3}(QL_3) = 1$ and consequently $L_3 = L_{3\sigma}$ as QL_3 admits $\sigma\tau$ fixed-point-freely.

Now (i) will be established. Let W be an α-invariant Sylow subgroup of type $\{1,2,3\}$ such that $PW = WP$ and suppose $WL_3 = L_3W$.

As $J(P) \leq N_P(L_3)$ and $J(P) \not\leq P_\tau$ by considering $J(P)P_\tau$ it follows, as $[J(P)P_\tau, \tau] = [J(P), \tau] \leq C_P(L_3)$, that $P_\tau \leq N_P(L_3)$. Hence $P^* \leq N_P(L_3)$. By appealing to lemma 6.1, it may be seen that $O_w(WL_3) = 1$. Since $L_{3_\tau} = 1$ and $L_3 = L_{3_\sigma}$, it may be inferred that $W = W_{\sigma\tau}$. Consequently $W_\rho = 1$ which is against the type of W. Thus it may be concluded that $WL_3 \neq L_3W$ and so (i) follows.

Now consider part (ii); use the notation given in the statement of (ii).

Suppose $WQ_h \neq Q_hW$. The prevailing circumstances taken together with lemma 7.2, dictate that W must be of type III with respect to Q_h. As $P_\rho \leq N_P(Q)$, by lemma 7.4(ii)(a) and (b), it may be deduced that $W_\rho \leq N_W(Q_h)$.

Consider (ii) when $WQ_h = Q_hW$. Observe that lemma 7.5(ii) may now be employed to deduce that W is star-covered. From (i), $L_3W \neq WL_3$ and moreover, as $L_3 = L_{3_\sigma} = L_3^*$, it is evident that W_σ, $W_\rho \leq N_W(L_3)$. Since W is star-covered, $W = W_\tau$ and so, from (1.19(iii)) $Q_{h_\rho} \trianglelefteq Q_{h_\rho}W_\rho$. As $P_\rho \leq N_P(Q)$, it follows from lemma 7.4(ii)(b) that $P_\rho \leq N_P(Q_h)$ also. Hence $Q_h^* = Q_{h_\rho}$ and so, by corollary 2.4, $W_\rho \leq N_G(Q_h)$. Therefore (ii) is proven.

<u>Lemma 8.20</u> Suppose $PL_i = L_iP$ (for i = 2 or 3) and W is an α-invariant Sylow w-subgroup of type $\{1,2,3\}$ such

that $PW = WP$. Then $WL_2 \neq L_2W$ (respectively $WL_3 \neq L_3W$) implies that $W = W_\sigma$ (respectively $W = W_\tau$).

<u>Proof</u> Let W be as in the statement of the lemma with $WL_2 \neq L_2W$.

Observe that either $W = W_\sigma$ or $[N_W(L_2), \sigma] \neq 1$. For, as P is not star-covered, $L_2^* \not\leq N_{L_2}(W)$, by lemma 6.2, and consequently W_ρ, $W_\tau \leq N_W(L_2)$. If $[N_W(L_2), \sigma] = 1$ then, as P is not star-covered, either $O_{\pi_2}(PL_2) \leq N_{L_2}(W)$ or $O_w(PW) \leq N_W(L_2)$. The latter immediately gives $W = W_\sigma$ (because of lemma 2.15) whilst the former, as $N_{L_2}(W) \leq L_{2\rho\tau}$ (see lemma 5.5(iii)(c)), implies that $L_2 = L_2^*$. This situation (plus the fact that G satisfies hypothesis D forbids $L_2 = L_{2\rho\tau}$) gives (by considering $L_2/\emptyset(L_2)$) that either $C_W(W_\rho)$ or $C_W(W_\tau) \leq N_W(L_2)$ ($\leq W_\sigma$) whence $W = W_\sigma$. Thus to complete the proof of the lemma, it will suffice to show that $[N_W(L_2), \sigma] \neq 1$ cannot hold.

Suppose then that $[N_W(L_2), \sigma] \neq 1$; clearly W is not star-covered. Furthermore W cannot permute with Q because of lemma 7.5(ii). From lemma 7.2 (as case(2) holds) W must be of type III with respect to Q; thus $\mathcal{M}_{w,q} = \{WZ, N_W(Q)Q\}$ with (by courtesy of lemma 7.4(ii)(a)) $W_\rho \leq N_W(Q)$ and Q_σ, $Q_\tau \leq Z$.

It will now be shown that $N_W(L_2) \leq N_W(Q)$. As $PL_2 = L_2P$, by lemma 8.18 $L_2Q = QL_2$. Observe that the largest α-invariant subgroup of L_2Q permutable with P is L_2Y. As $Y \geq Q_\sigma$, Q_τ and L_2Q admits $\sigma\tau$ fixed-point-freely, $Q = C_Q(L_2)Y$.

Consider $\{C_G(L_2)\}_{q,w}$. If $C_Q(L_2) \leq Z \leq Q_\rho$, then $Q = Q_\rho Y = Q_\rho$, which is against lemma 5.6(ii)(d). Thus $C_W(L_2)$ ($\geqslant [N_W(L_2),\sigma]$) is contained in $N_W(Q)$ and so $[N_W(L_2),\sigma] \leq [N_W(Q),\sigma]$. By lemma 5.6(ii)(e and f), $1 \neq [N_W(L_2),\sigma] \leqslant W_\rho$ hence $N_W([N_W(L_2),\sigma]) \leqslant N_W(Q)$ and so $N_W(L_2) \leqslant N_W(Q)$.

A contradiction may now be obtained as follows: $N_W(L_2) \leqslant N_W(Q)$ implies that $W_\tau \leqslant N_W(Q)$ and so $[O_w(WZ)_\tau,\ Z] = 1$. However, lemma 5.6(ii)(b) demands that $O_q(WZ) = 1$ and so $Z \leqslant Q_\tau$. Thus $Q_\sigma \leqslant Q_{\rho\tau}$ which contravenes the fact that Q is of type $\{1,2,3\}$ and therefore the possibility $[N_W(L_2),\sigma] \neq 1$ is excluded.

Hence lemma 8.20 is verified.

<u>Lemma 8.21</u> If $J(P)_\rho \neq 1$, then hypothesis D does not hold for G.

<u>Proof</u> From lemma 5.6(ii)(m), P is contained in a unique maximal α-invariant subgroup H of G. Moreover $H = C_G(D)$ where D is a non-trivial α-invariant subgroup of Z(P). The combined effect of lemmas 8.11, 8.12 and 8.13 is to show that the group generated by the α-invariant Sylow subgroups of type $\{1,2,3\}$, $L_.$ and $L_{..}$ which do not permute with P, is a soluble Hall subgroup, say K, of G. Clearly HK = G. For any α-invariant Sylow subgroup Q_h of K of type $\{1,2,3\}$, $Z(P) = Z(P)_{\sigma\tau} \leqslant N_P(Q_h)$; further, if say $PL_2 \neq L_2P$, then, because $Z(P) = Z(P)_{\sigma\tau}$, P_ρ, $P_\tau \leqslant N_P(L_2)$ and so $Z(P) \leqslant N_P(L_2)$. Should $L_{23}P \neq PL_{23}$ occur then, as $[L_{23},\ Q] = 1$,

evidently $Z(P) \leqslant N_P(L_{23})$ also. Hence $Z(P) \leqslant N_H(K)$ and so D^G is a non-trivial proper normal α-invariant subgroup of G.

Therefore, when $J(P)_\rho \neq 1$, G cannot satisfy hypothesis D.

In view of lemma 8.21, for the remainder of case(2) it will be assumed that P is of type III with respect to Q (with $P_\rho \leqslant N_P(Q)$), and that $J(P)_\rho = 1$. Observe that lemmas 8.14 and 8.15 now become available.

<u>Lemma 8.22</u> P permutes with L_1, L_2 and L_3.

<u>Proof</u> Deny the lemma; it is already known that $PL_1 = L_1P$, thus it is being assumed that P does not permute with at least one of L_2 and L_3. Most of the proof is directed, under these assumptions, to showing that at least one of $Z(J(P))_\sigma \leqslant Z(J(P))_\tau$ and $Z(J(P))_\tau \leqslant Z(J(P))_\sigma$ must hold; to this end it will be supposed that neither possibility occurs, and from this a contradiction will be deduced.

Suppose, further that, $PL_2 \neq L_2P$ and $PL_3 \neq L_3P$. Then L_2, L_3, L_{23} and those α-invariant Sylow subgroups of type IV with respect to P generate a soluble Hall subgroup, K, of G by lemmas 8.11, 8.12 and 8.13. As in lemma 8.21, it may be deduced that $Z(P) \leqslant N_P(K)$. By lemma 8.19(i) applied to both L_2 and L_3, it follows that no α-invariant Sylow subgroup of type $\{1,2,3\}$ which permutes with P permutes with either of L_2 or L_3. Furthermore $L_2 = L_{2_\tau}$ and $L_3 = L_{3_\sigma}$ (a demonstration of this may be extracted from the beginning of the proof of lemma 8.19) and so it follows for each α-invariant Sylow subgroup W of type $\{1,2,3\}$ that permutes with P that W_ρ, $W_\tau \leqslant N_W(L_2)$ and W_ρ, $W_\sigma \leqslant N_W(L_3)$. In particular, for each such W, $W_\rho \leqslant N_W(L_2L_3)$.

If H denotes the subgroup of G generated by L_1 and the α-invariant Sylow subgroups of type $\{1,2,3\}$ which permute with P, then H is a soluble Hall subgroup by lemmas 8.14 and 8.16. Appealing to lemma 8.19(ii) and recalling that $[L_{23}, \{G_\rho\}_{\pi_{23}'}] = 1$, it follows that $H_\rho \leqslant N_H(K)$. Since $J(P)_\rho = 1$, from lemma 2.12 $\{N_G(J(P))\}_{p'} = \{C_G(J(P))\}_{p'}(\{N_G(J(P))\}_{p'})_\rho$, and so, using (1.9), it may be seen that $H = C_H(D)H_\rho$ where $D = O_p(H) \cap Z(P)$ $(\neq 1)$. It is now clear that $G = HK = (C_H(D)H_\rho)K = C_H(D)N_G(K)$ cannot satisfy hypothesis D. So when $PL_2 \neq L_2P$ and $PL_3 \neq L_3P$ hold, it may be inferred that at least one of $Z(J(P))_\sigma \leqslant Z(J(P))_\tau$ and $Z(J(P))_\tau \leqslant Z(J(P))_\sigma$ occurs.

Now consider the case when (say) $PL_2 = L_2P$ and $PL_3 \neq L_3P$. Since $Z(J(P))_\sigma \not\leqslant Z(J(P))_\tau$ (by assumption), as before, from lemma 8.19(ii) $W_\rho \leqslant N_W(Q_h)$ where Q_h is any α-invariant Sylow subgroup of type IV with respect to P and W is any α-invariant Sylow subgroup of type $\{1,2,3\}$ such that $PW = WP$; also $L_3 = L_{3_\sigma}$.

It is now aimed to show that $L_2 = 1$; suppose otherwise. As $P_\sigma \leqslant N_P(L_3)$, $L_2L_3 = L_3L_2$ would compel, because of lemma 6.1 and $L_3 = L_{3_\sigma}$ implying $L_2 \trianglelefteq L_2L_3$, P and L_3 to permute. Hence $L_2L_3 \neq L_3L_2$ and, as $L_3 = L_{3_\sigma}$, it follows that $\mathcal{M}_{\pi_2,\pi_3} = \{L_3, N_{L_3}(L_2)L_2\}$. Appealing to lemmas 8.13 and 8.18 gives that $QL_2 = L_2Q$ and $QL_3 = L_3Q$. Since $Q \neq Q_\sigma$, $L_3 = L_{3_\sigma}$ and $L_{2_\sigma} = 1$, it follows that $O_{\pi_2}(QL_2) \leq N_{L_2}(L_3) = 1$. Thus, as QL_2 admits $\sigma\tau$ fixed-point-freely, $Q \trianglelefteq QL_2$. Moreover, $L_3 = L_{3_\sigma}$ implies, by (1.19)(iii), that $Q \trianglelefteq QL_3$ whence $L_2L_3 = L_3L_2$. Consequently L_2 must be trivial.

Proceeding as in the previous case when both L_2 and L_3 did not permute with P, it is possible (because $L_2 = 1$) to obtain $G = HK$ with H_ρ, $Z(P) \leq N_H(K)$ (here H is the subgroup of G generated by L_1 and those α-invariant Sylow subgroups of type $\{1,2,3\}$ which permute with P and , K is the subgroup of G generated by the remaining α-invariant Sylow subgroups); again, G cannot satisfy hypothesis D.

Thus, without loss, it may be taken that $Z(J(P))_\sigma \leq Z(J(P))_\tau$. Consequently, as $J(P)_\rho = 1$, $Z(J(P))^* = Z(J(P))_\tau$. By lemma 5.6(ii)(m), as $J(P)_\rho = 1$, $Z(J(P)) \trianglelefteq PY$ and so $Z(J(P)) = Z(J(P))_\tau C_{Z(J(P))}(Q_\tau)$ (since $Q_\tau \leq Y$). If $C_Q(Q_\tau) \leq Y$ $(\leq Q_\rho)$ then (1.7)(x) forces $Q = Q_\rho$ which is contrary to lemma 5.6(ii)(d). Therefore $C_{Z(J(P))}(Q_\tau) \leq X = X_{\sigma\tau} \times P_\rho$ giving, because $J(P)_\rho = 1$, $C_{Z(J(P))}(Q_\tau) \leq P_\tau$. Hence $Z(J(P)) = Z(J(P))_\tau$. Now $[Z(J(P)), [Y,\tau]] = 1$ and $1 \neq Q_\sigma \leq [Y,\tau]$. As before, $C_Q([Y,\tau]) \leq Y$ is untenable so $Z(J(P)) \leq X$ and hence $Z(J(P)) \leq P_{\sigma\tau}$.

Referring to lemma 4.4 shows that P is contained in a unique maximal α-invariant subgroup of G. Arguing as in lemma 8.21, it may be shown that hypothesis D does not hold for G. This completes the proof of lemma 8.22.

The next result sounds the death knell for case(2). Let H^+ and K^+ denote (respectively) the subgroups generated by the α-invariant Sylow subgroups of type $\{1,2,3\}$ which do permute and not permute with P.

Lemma 8.23 G does not satisfy hypothesis D.

Proof Suppose G satisfies hypothesis D. As P permutes with L_2 and L_3 by lemma 8.22, it may be inferred, using lemma 8.17, that $L_2L_3 = L_3L_2$. Also, as it is assumed that $J(P)_\rho = 1$; lemma 8.15 shows that $L_1L_2 = L_2L_1$ and $L_1L_3 = L_3L_1$. Invoking lemmas 6.3, 8.12, 8.14 and 8.18 yields that $G = (H^+L_1)(K^+L_2L_3L_{23})$ = (say) HK (clearly H and K are soluble Hall subgroups). Observe that K admits $\sigma\tau$ fixed-point-freely.

It is claimed that $L_2 = L_{2\tau}$ and $L_3 = L_{3\sigma}$ (and hence $K^+ \trianglelefteq K$). If, say, $H^+L_2 = L_2H^+$ then, as $L_1L_2 = L_2L_1$, $G = (H^+L_1L_2)(K^+L_3L_{23})$ also and so $L_2 = L_2{}^*_{<\sigma\tau>} = L_{2\tau}$. On the other hand, if $H^+L_2 \neq L_2H^+$ let W be an α-invariant Sylow subgroup of H^+ such that $WL_2 \neq L_2W$. By lemma 8.20, $W = W_\sigma$. Hence, as W clearly cannot be of type III, WQ = QW. Since $Q \neq Q_\sigma$, $L_{2\sigma} = 1$, $W = W_\sigma$ and $\mathcal{M}_{W,\pi_2} = \{L_2N_W(L_2), W\}$, it follows that $O_{\pi_2}(QL_2) = 1$, whence $L_2 = L_2{}^*_{<\sigma\tau>} = L_{2\tau}$. For analogous reasons, $L_3 = L_{3\sigma}$.

Next, it will be shown that $C_{L_2L_3}(K^+) = 1$; suppose otherwise. Then, without loss, $C_{L_2}(K^+) \neq 1$. If $L_2H^+ \neq H^+L_2$ then the argument presented in the preceding paragraph may be used to show that $O_{\pi_2}(QL_2) = 1$

and clearly this is incompatible with $C_{L_2}(K^+) \neq 1$. Thus $L_2H^+ = H^+L_2$ and so $G = (H^+L_1L_2)(K^+L_3L_{23})$. Consequently, as L_2 commutes with L_3L_{23} so giving $C_{L_2}(K^+) \trianglelefteq K$, $(C_{L_2}(K)^+)^G$ is a non-trivial proper normal α-invariant subgroup of G. Therefore $C_{L_2L_3}(K^+) = 1$.

Let L be the subgroup of H generated by those α-invariant Sylow subgroups of type$\{1,2,3\}$ which permute with K. Since $K^+ \trianglelefteq K^+L_2L_3$, the largest normal $(\pi(K^+)\cup\pi_{23})'$-subgroup of KL must be a $\pi(L)$ group. In view of the present circumstances, the largest normal $(\pi(K^+)\cup\pi_{23})'$-subgroup of KL must be trivial. Hence as $[L_{23}, \{G_\rho\}_{\pi_{23}'}] = 1$ and for each α-invariant Sylow subgroup Q_h of K^+, $Q_h^* = Q_{h_\rho}$, it follows from lemma 2.6 that $O_{\pi(L)}(K^+_\rho L_\rho) = 1$. Thus for each α-invariant Sylow subgroup W of L, $W_\rho = W_{\rho\sigma}W_{\rho\tau}$ and consequently, because of (1.19)(iii), $K^+_\rho \trianglelefteq K^+_\rho L_\rho$. Employing corollary 2.4, yields that $L_\rho \leq N_G(K^+)$.

Now let W denote an α-invariant Sylow w-subgroup of H^+ which is not contained in L. If $WK^+ \neq K^+W$ then, by lemma 5.6(ii)(i), G_ρ has a normal w-complement and so a further application of corollary 2.4 gives that $W_\rho \leq N_G(K^+)$. Suppose W is such that $WK^+ = K^+W$. Then W does not permute with $L_2L_3L_{23}$. Since $[Q, L_{23}] = 1$ and $Q^* = Q_\rho \neq Q$, it is clear that L_{23} and W must permute. Thus either $WL_2 \neq L_2W$ or $WL_3 \neq L_3W$ holds and so, from lemma 8.20, either $W = W_\sigma$ or $W = W_\tau$. Either way G_ρ will have a normal w-complement and, so, again enlisting the aid of corollary 2.4, this gives $W_\rho \leq N_G(K^+)$.

Therefore $H_\rho \leq N_G(K^+)$. Since $Z(P) \leq N_G(K^+)$, $J(P)_\rho = 1$ and $K^+ \trianglelefteq K$, mimicking the arguments given in the third paragraph of the proof of lemma 8.22, it may be shown that $G = N_G(K^+)C_G(D)$ where D is a non-trivial α-invariant subgroup of $Z(P)(\leq N_G(K^+))$.

It is now clear that G cannot satisfy hypothesis D and so lemma 8.23 is established.

<u>Case(3)</u>

<u>Lemma 8.24</u> G does not satisfy hypothesis D.

<u>Proof</u> In view of cases(1) and (2) having been settled, it may be assumed that G possesses α-invariant Sylow subgroups of types I, II, III and IV.

Let U be an α-invariant Sylow subgroup of G of type III with respect to V. It is claimed that if P is an α-invariant Sylow subgroup of type I, then P must be of type IV with respect to U. Suppose P is not of type IV with respect to U; clearly $P \neq V$ and also, by lemma 7.3, $P \neq U$. If $PU \neq UP$, then by lemma 7.1, as P is of type I and U is of type III, neither P nor U can be of type II and so either U is of type IV or P is of type IV. As the latter alternative is excluded for the moment, it follows that U must be of type III and IV which is contrary to lemma 7.2. Hence $PU = UP$. Further, it will also be shown that P permutes with V. Suppose $PV \neq VP$; again, by lemma 7.1, neither P nor V can be of type II. As V is already of type IV, from lemma 7.2, V cannot be of type III and therefore P must be of type III. By lemma 7.3, this situation cannot occur and so $PV = VP$.

Since P is of type I, P is not star-covered and now lemma 7.5(ii) shows that this configuration is untenable. Thus the claim is established.

Now lemma 7.6 is applicable and yields that U is contained in a unique maximal α-invariant subgroup H of G; moreover, $H = C_G(D)$ where D is a non-trivial α-invariant subgroup of $Z(U)$.

Those α-invariant Sylow subgroups of type $\{1,2,3\}$ which do not permute with U must be either of type I or type IV (U cannot be of type II or IV as U is already of type III; see lemmas 7.1 and 7.2). Hence the α-invariant Sylow subgroups of type $\{1,2,3\}$ which do not permute with U must all be of type IV (with respect to U).

By appealing to lemmas 6.3, 8.10(i), 8.12 and 8.13 (the latter three results clearly apply to this situation), it may be deduced that the group K, generated by the α-invariant Sylow subgroups of type IV with respect to V, the L_i and L_{jk} which do not permute with V forms a soluble Hall subgroup. Clearly $G = HK$ and it is possible, arguing along the same lines as in lemma 8.21, to obtain that $Z(U) \leq N_G(K)$ whence lemma 8.24 follows.

Since it has been shown that cases(1), (2) and (3) are incompatible with G satisfying hypothesis D, theorem 8.1 is now proven.

9. FACTORIZATION THEOREMS

The two main results of this section are concerned with showing that if G factorizes in a particular way as the product of two proper α-invariant subgroups, then G cannot satisfy hypothesis D. The configurations that are studied here will make an encore in section 10.

Theorem 9.1 Suppose $G = K\hat{L}_i$ where K is an α-invariant soluble subgroup of G, $i \in \{1,2,3\}$ and $(|K|, |\hat{L}_i|) = 1$. Then G does not satisfy hypothesis D.

Proof Deny the theorem and suppose $i = 1$; let $\tilde{K}$ denote the largest α-invariant subgroup of K permutable with $\hat{L}_1$. As $L_{1\rho} = 1$ and hypothesis D holds for G, $O_{\pi(K)}(KL_1) = 1$ and hence $\tilde{K} = N_K(L_1)$ $(\leq K_\rho)$, by corollary 2.11(i) and (ii). Observe that $\hat{L}_1 \neq 1$.

As $[\hat{L}_1, K_{\sigma\tau}] = 1$, clearly K admits $\sigma\tau$ fixed-point-freely. Let W be a non-trivial α-invariant Sylow w-subgroup of K which permutes with $\hat{L}_1$. The fact that G possesses no non-trivial proper normal α-invariant subgroups forces $O_w(K)$ and $O_w(WL_1)$ to both be trivial an d so, as $L_{1\rho} = K_{\sigma\tau} = 1$, $W = W_\rho = W_\sigma W_\tau$ which implies, by lemma 4.1, that G has a normal w-complement. Therefore, no (non-trivial) α-invariant Sylow subgroup of K is contained in $\tilde{K}$.

First, it is aimed to show that K_σ, $K_\tau \leq N_K(\hat{L}_1) = \tilde{K}$.

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Proof Deny the theorem and suppose $i = 1$; let $\tilde{K}$ denote the largest α-invariant subgroup of K permutable with $\hat{L}_1$. As $L_{1\rho} = 1$ and hypothesis D holds for G, $O_{\pi(K)}(KL_1) = 1$ and hence $\tilde{K} = N_K(L_1)$ $(\leq K_\rho)$, by corollary 2.11(i) and (ii). Observe that $\hat{L}_1 \neq 1$.

As $[\hat{L}_1, K_{\sigma\tau}] = 1$, clearly K admits $\sigma\tau$ fixed-point-freely. Let W be a non-trivial α-invariant Sylow w-subgroup of K which permutes with $\hat{L}_1$. The fact that G possesses no non-trivial proper normal α-invariant subgroups forces $O_w(K)$ and $O_w(WL_1)$ to both be trivial and so, as $L_{1\rho} = K_{\sigma\tau} = 1$, $W = W_\rho = W_\sigma W_\tau$ which implies, by lemma 4.1, that G has a normal w-complement. Therefore, no (non-trivial) α-invariant Sylow subgroup of K is contained in $\tilde{K}$.

First, it is aimed to show that $K_\sigma, K_\tau \leq N_K(\hat{L}_1) = \tilde{K}$.

If L_{12} and L_{13} are both non-trivial, this is easily accomplished as $[K_\sigma, L_{13}] = [K_\tau, L_{12}] = 1$ and $\hat{L}_1$ is nilpotent. So it may be supposed that at least one of L_{12} and L_{13} is trivial. If $L_1 = 1$, then $\hat{L}_1 = L_{13}$ (say) and hence $[\hat{L}_1, K_\sigma] = 1$ which, as G is supposed to satisfy hypothesis D, implies that K admits σ fixed-point-freely and so is nilpotent by Thompson's result. Consequently, by a theorem of Wielandt's (1.24), G is soluble. Therefore it may be supposed that $\hat{L}_1 = L_1L_{13}$ with $L_1 \neq 1$.

Additionally, it will be assumed that K^+ (the Hall subgroup of K generated by the α-invariant Sylow subgroups of K of type $\{1,2,3\}$) is non-trivial. It is asserted that L_1 does not permute with any (non-trivial) α-invariant Sylow subgroup of K^+ nor with L_i ($i = 2$ or 3) provided $L_i \neq 1$. Of course, if $L_1 = \hat{L}_1$ this has already been shown to be the case. So suppose $L_{13} \neq 1$. Let W be a (non-trivial) α-invariant Sylow w-subgroup of K^+. If $WL_{13} = L_{13}W$, then, as $L_{13} \neq L_{13_\sigma}$, $O_{\pi_{13}}(WL_{13}) \neq 1$ so giving $W \leq \tilde{K}$. Thus $WL_{13} \neq L_{13}W$ and so $\mathcal{M}_{w,\pi_{13}} = \{W, L_{13}N_W(L_{13})\}$. Clearly $N_W(L_{13}) \leq \tilde{K}$ and so $N_W(L_{13}) \leq W_\rho$. Hence, as $C_W(L_{13}) \neq 1$, $W = W_\rho$. If it were the case that $L_1W = WL_1$, then $L_1 \trianglelefteq L_1W$ so forcing $W \leq \tilde{K}$; therefore $L_1W \neq WL_1$. It only remains to show (assuming $L_2 \neq 1$) that $L_1L_2 \neq L_2L_1$ (because when $L_{13} \neq 1$ it follows that $L_3 \leq \tilde{K}$ and so $L_3 = 1$). If $L_1L_2 = L_2L_1$ then, in order to ensure that $L_2 \leq \tilde{K}$ does not occur, $O_{\pi_1}(L_1L_2)$ must be trivial and so $L_1 = L_{1_\sigma}$. From the fact that $L_1 = L_{1_\sigma}$, as $L_1W \neq WL_1$ where W is an α-invariant

Sylow subgroup of K^+, it may be deduced that $Z(W) \leq W_\sigma$. However recall that $WL_{13} \neq L_{13}W$ and so, since $[L_{13}, W_\sigma] = 1$, a contradiction has arisen. Hence $L_1L_2 \neq L_2L_1$ and so the assertion has been verified.

It will now be demonstrated that the following is impossible: that there is a Sylow p-subgroup P of K^+ such that $Z(L_1) = Z(L_1)_{\sigma\tau} \leq N_{L_1}(P)$. Supposing there is such a P it will be shown that $Z(L_1)$ $(\leq Z(\hat{L}_1))$ is contained in a proper α-invariant subgroup of G together with K, whence it would follow that G could not satisfy hypothesis D. Let Q be any (non-trivial) α-invariant Sylow q-subgroup of K^+. As $QL_1 \neq L_1Q$, by lemma 5.5, either $L_{1\sigma}, L_{1\tau} \leq N_{L_1}(Q)$ or $Q_\sigma, Q_\tau \leq N_Q(L_1)$. For the first possiblity, $Z(L_1) = Z(L_1)_{\sigma\tau} \leq N_{L_1}(Q)$; for the second $[N_Q(L_1), Z(L_1)] = 1$ implies, as $N_Q(L_1) \neq 1$, that $Z(L_1) \leq N_{L_1}(Q)$. Thus $Z(L_1) \leq N_{L_1}(K^+)$. If L_2 is non-trivial then $L_1L_2 \neq L_2L_1$ and so either $L_{1\tau} \leq N_{L_1}(L_2)$ or $L_{2\tau} \leq N_{L_2}(L_1)$. If $L_{1\tau} \leq N_{L_1}(L_2)$ then clearly $Z(L_1) \leq N_{L_1}(L_2)$ and if $L_{2\tau} \leq N_{L_2}(L_1)$ then $N_{L_2}(L_1) \neq 1$ which together with $[N_{L_2}(L_1), Z(L_1)] = 1$ gives $Z(L_1) \leq N_{L_1}(L_2)$. Similarly, if $L_3 \neq 1$, it may also be deduced that $Z(L_1) \leq N_{L_1}(L_3)$. Thus $Z(L_1) \leq N_{L_1}(K^+L_2L_3)$. As the α-invariant Sylow subgroups of K for which σ and τ both act fixed-point-free are direct summands of K, $N_G(K^+L_2L_3)$ also contains K.

If P is an α-invariant Sylow subgroup of K^+ for which $P_\sigma, P_\tau \not\leq N_P(L_1)$, then from lemma 5.5(i)(b and d) either $Z(L_1) = Z(L_1)_{\sigma\tau} \leq N_{L_1}(P)$ or $L_{1\sigma} = L_{1\tau}$. The first possibility is precluded by the conclusion of the previous paragraph; for the second possibilty, the supposed properties of G combined with lemma 4.3 and the Thompson normal p-complement theorem dictate that $\pi(L_1) = \{2\}$. Observe that (by lemma 4.3) all proper α-invariant subgroups of G have normal 2-complements and so, clearly, $\widetilde{K}$ must be trivial. In particular, $L_{13} = 1$ and so $L_1 = \hat{L}_1$. As G_σ and G_τ both have normal 2-complements, by exploiting the 'normalizer chain' of an α-invariant Sylow subgroup (and the fact that all proper α-invariant subgroups of G have normal 2-complements) it may be demonstrated that $L_{1_\sigma} = L_{1_\tau} \leqslant N_G(P)$ for each α-invariant Sylow subgroup P of G for which at least one of P_σ and P_τ is non-trivial. Since $L_1^* \leqslant N_{L_1}(K^+L_2L_3)$, it follows that $L_{1_\sigma} = L_{1_\tau} \leqslant N_{L_1}(K)$. Thus G fulfills the conditions of lemma 4.5 and so G must itself have a normal 2-complement which is against hypothesis D. Hence for each α-invariant Sylow subgroup P, of K^+, $P_\sigma, P_\tau \leqslant N_P(L_1)$.

Suppose (say) that $L_{2_\tau} \not\leqslant N_{L_2}(L_1)$ then, as $L_2L_1 \neq L_1L_2$ (note $L_2 \neq 1$), $L_{1_\tau} \leqslant N_{L_1}(L_2)$. If $C_{L_1}(L_2) \neq 1$ then $Z(L_1) \leqslant N_{L_2}(L_1)$ and $Z(L_1) \leqslant L_{1_\sigma}$. Let P be some (non-trivial) α-invariant Sylow p-subgroup of K^+. Then $[[N_P(L_1), \sigma], Z(L_1)] = 1$ because $P_\sigma, P_\tau \leqslant N_P(L_1) \leqslant \widetilde{K} \leqslant K_\rho$ and hence either $Z(L_1) \leqslant N_{L_1}(P) \leqslant L_{1_{\sigma\tau}}$ (by lemma 5.5) or $C_P([N_P(L_1), \sigma]) \leqslant N_P(L_1) \leqslant P_\rho$. The first possibility has been shown not to occur and the second forces $P = P_\rho$. As $N_{L_1}(P) \leqslant L_{1_{\sigma\tau}}$ it is clear that, since $PL_1 \neq L_1P$, $N_{L_1}(P)$ must be trivial and so $\mathcal{M}_{p,\pi_1} = \{P, N_P(L_1)L_1\}$.

to be untenable. Thus $Z(P)_\tau = 1$ and consequently $Z(P)^*_{<\sigma\tau>} = 1$ which then gives $[Z(P), N_K(P)] = 1$ and $Z(P) \leq O_p(K)$. Hence, since K has Fitting length at most two, $Z(P) \leq Z(K)$ and, as $Z(P) \leq N_P(L_1) \leq \widetilde{K}$, this shows that G cannot satisfy hypothesis D. Thus it may be inferred that $L_{2\tau} \leq N_{L_2}(L_1)$. A similar argument applies to give that $L_{3\sigma} \leq N_{L_3}(L_1)$ (note that when $L_3 \neq 1$, then $L_{13} = 1$).

Therefore, when $K^+ \neq 1$, it has been shown that $K_\sigma, K_\tau \leq N_K(L_1) \leq \widetilde{K}$.

Now the case $K^+ = 1$ will be considered; because $L_{13} \neq 1$ gives $L_3 = 1$ so making it possible to apply Wielandt's result to deduce that G is soluble, it may be taken that $\hat{L}_1 = L_1$. For this case, in order to establish that $K_\sigma, K_\tau \leq N_K(L_1)$, it will suffice, due to the symmetry of the arguments, to demonstrate that the two possibilities $L_{1\sigma} \leq N_{L_1}(L_3)$, $L_{1\tau} \leq N_{L_1}(L_2)$ and $L_{1\sigma} \leq N_{L_1}(L_3)$, $L_{2\tau} \leq N_{L_2}(L_1)$ cannot occur.

First consider the possibility $L_{1\sigma} \leq N_{L_1}(L_3)$, $L_{1\tau} \leq N_{L_1}(L_2)$. Since $L_{1\sigma} = L_{1\tau}$ implies, by lemma 4.3, and the Thompson normal p-complement theorem,that $\pi(L_1) = \{2\}$ which in turn, using similar arguments for the analogous situation when $K^+ \neq 1$, gives that G must have a normal 2-complement, it may be assumed that (say) $L_{1\tau} \not\leq L_{1\sigma}$. Thus $C_{L_1}(L_2) \neq 1$ and so $Z(L_1) \leq N_{L_1}(L_2)$ and $Z(L_1) \leq L_{1\sigma} (\leq N_{L_1}(L_3))$. Consequently $Z(L_1) \leq N_{L_1}(L_2L_3)$. As $K = L_2L_{23}L_3$, $N_G(L_2L_3) \geq K$, $Z(L_1)$ which implies G cannot satisfy hypothesis D and so this deals with the situation when $L_{1\sigma} \leq N_{L_1}(L_3)$ and $L_{1\tau} \leq N_{L_1}(L_2)$.

If $O_{\pi_2}(PL_2) \neq 1$ then $\{N_G(O_{\pi_2}(PL_2))\}_{p,\pi_1} \geqslant P,\ C_{L_1}(L_2)$ which is incompatible with the assumption that $C_{L_1}(L_2) \neq 1$. Hence $O_{\pi_2}(PL_2) = 1$ and consequently, as PL_2 admits $\sigma\tau$ fixed-point-freely and $P = P_\rho$, $L_2 = L_{2_\tau} = L_{2_\rho}$ which implies that G has a normal π_2-complement.

Thus it may be supposed that $C_{L_1}(L_2) = 1$ and hence $L_{1_\tau} \leqslant L_{1_\sigma}$. Therefore $L_1{}^* = L_{1_\sigma}$ and $L_{1_\sigma} \neq L_1$. For suppose $L_1 = L_{1_\sigma}$ occurs; one consequence of this is that L_{13} must be trivial because $PL_1 \neq L_1P$ together with $L_1 = L_{1_\sigma}$ gives $Z(P) \leqslant P_\sigma$ whence $PL_{13} = L_{13}P$ which implies that $L_{13} = 1$; thus $\hat{L}_1 = \hat{L}_{1_\sigma}$ and hence, by (1.7)(xii), $[G, \sigma] \leqslant K \neq G$ so giving rise to a non-trivial proper normal α-invariant subgroup of G.

As $P_\sigma \leqslant N_P(L_1)$ (P still denotes some fixed (non-trivial) α-invariant Sylow subgroup of K^+), $L_1 = L_{1_\sigma} C_{L_1}(P_\sigma)$. If $C_{L_1}(P_\sigma) \leqslant N_{L_1}(P)$ then, as $N_{L_1}(P) \leqslant L_{1_{\sigma\tau}}$, this gives $L_1 = L_{1_\sigma}$ which cannot hold. Thus $C_{L_1}(P_\sigma) \not\leqslant N_{L_1}(P)$ and so $Z(P)_\sigma = 1$ and $Z(P) \leqslant N_P(L_1) \leqslant K$. If $C_{L_{13}}(L_2) \neq 1$, then $C_G(C_{L_{13}}(L_2)) \geqslant L_1, L_2$ which is against $L_1L_2 \neq L_2L_1$. So $C_{L_{13}}(L_2) = 1$ which, as $C_{L_1}(L_2) = 1$ already, gives $C_{\hat{L}_1}(L_2) = 1$ and so $C_G(L_2) \leqslant K$.

As $L_2{}^*{}_{\langle\sigma\tau\rangle} = L_{2_\tau}$, $[P_\tau, L_2] = 1$ and so $P_\tau \leqslant C_G(L_2)$ whence $[N_{L_1}(L_2), P_\tau] \leqslant C_G(L_2) \leqslant K$. Moreover, as $P_\tau \leqslant N_P(L_1)$, $[N_{L_1}(L_2), P_\tau] \leqslant L_1$ and therefore $[N_{L_1}(L_2), P_\tau] \leqslant K \cap L_1 = 1$. Recalling that L_1 $N_{L_1}(L_2)$ and hence L_1 , P = 1. If $Z(P)_\tau \neq 1$ then $L_{1_\tau} \leqslant N_{L_1}(P)$. However, it is already known that P_σ, $P_\tau \leqslant N_P(L_1)$ and the remark succeeding lemma 5.5 shows such a situation

Now suppose that $L_{1_\sigma} \leq N_{L_1}(L_3)$, $L_{2_\tau} \leq N_{L_2}(L_1)$ occurs. If $L_{1_\sigma} \leq L_{1_\tau}$ then $L_{1_\tau} = L_1^* \neq L_1$ and $L_1 = L_{1_\tau} C_{L_1}(L_{2_\tau})$ with $C_{L_1}(L_{2_\tau}) \not\leq L_{1_\sigma}$ (since $L_1 = L_{1_\tau}$ implies that $[G, \tau] \neq G$). If $C_{L_1}(L_{2_\tau}) \leq N_{L_1}(L_2)$ then, as $C_{L_1}(L_{2_\tau}) \not\leq L_{1_\sigma}$, $C_{L_1}(L_2) \neq 1$ and hence, by lemma 5.1, $N_{L_2}(L_1) = 1$ whereas L_{2_τ} is non-trivial and contained in $N_{L_2}(L_1)$. Thus $C_{L_1}(L_{2_\tau}) \not\leq N_{L_1}(L_2)$ whence $Z(L_2)_\tau = 1$ and $Z(L_2) \leq N_{L_2}(L_1)$ $(\leq \tilde{K})$. As K admits $\sigma\tau$ fixed-point-freely and $Z(L_2)^*_{\langle\sigma\tau\rangle} = Z(L_2)_\tau = 1$, $Z(L_2) \leq O_{\pi_2}(K)$ giving $Z(L_2) \trianglelefteq K$ which then implies that G possesses a non-trivial proper α-invariant subgroup.

In the light of the preceding arguments it may be supposed that $L_{1_\sigma} \not\leq L_{1_\tau}$; hence $C_{L_1}(L_3) \neq 1$ and therefore $Z(L_1) \leq N_{L_1}(L_3)$ with $Z(L_1) \leq L_{1_\tau}$. Now it is aimed to show that $L_3 = L_{3_\sigma}$, so suppose $L_3 \neq L_{3_\sigma}$. Then, as $[L_3, \sigma] \leq O_{\pi_3}(L_2L_3)$ and $L_{1_\sigma} \leq N_{L_1}(L_3)$, $\{N_G([L_3, \sigma])\}_{\pi_2,\pi_1}$ $\geq O_{\pi_2}(L_2L_3)$, L_{1_σ}. If $O_{\pi_2}(L_2L_3) \leq N_{L_2}(L_1)$ then $L_2 = L_{2_\tau} O_{\pi_2}(L_2L_3) \leq N_{L_2}(L_1)$. While if $L_{1_\sigma} \leq N_{L_1}(L_2)$ then $L_{1_\sigma} \leq N_{L_1}(L_2L_3)$ and so $N_G(L_2L_3) \geq K$, L_{1_σ}. The fact that G is supposed to satisfy hypothesis D forces $Z(L_1)_\sigma = 1$. But then σ acts fixed-point-freely on $Z(L_1)N_{L_2}(L_1)$ and so $[Z(L_1), N_{L_2}(L_1)] = 1$. Since $N_{L_2}(L_1)$ is non-trivial, $Z(L_1) \leq N_{L_1}(L_2)$ whence $Z(L_1) \leq N_{L_1}(L_2L_3)$ which, as before, contradicts the assumption that hypothesis D holds. Hence $L_3 = L_{3_\sigma}$ and so, as $Z(L_1) \leq N_{L_1}(L_3)$, $Z(L_1) \leq L_{1_\sigma}$ from which it follows, since $L_{2_\sigma} = 1$, that $[N_{L_2}(L_1), Z(L_1)] = 1$. Because $N_{L_2}(L_1) \neq 1$, again $Z(L_1) \leq N_{L_1}(L_2)$ and therefore $Z(L_1) \leq N_{L_1}(L_2L_3)$. Again this leads to the conclusion that G must contain a non-trivial proper normal α-invariant subgroup.

Hence, when $K^+ = 1$, it may also be deduced that $K_\sigma, K_\tau \leqslant N_K(L_1)$.

Let P be an α-invariant Sylow p-subgroup of K. By lemma 2.10, $\{K\}_{p'} = O_{p'}(K)(\{K\}_{p'} {}^*_{<\sigma\tau>})$ and therefore $G = K\hat{L}_1 = N_G(\hat{L}_1)(O_{p'}(K)P)$. As $P = O_p(K)P^*_{<\sigma\tau>}$, $K^*_{<\sigma\tau>} \leqslant N_K(\hat{L}_1)$ and $[O_p(K), O_{p'}(K)] = 1$, it follows that $[P, O_{p'}(K)] \leqslant N_K(\hat{L}_1)$. Since $[P, O_{p'}(K)] \trianglelefteq O_{p'}(K)P$, were $[P, O_{p'}(K)]$ non-trivial then $[P, O_{p'}(K)]^G$ would be a non-trivial proper α-invariant normal subgroup of G. Thus $[P, O_{p'}(K)] = 1$ and, as K has Fitting length at most 2, $K = N_K(P)O_{p'}(K)$ which implies that $P \trianglelefteq K$

Hence K is p-closed for each $p \in \pi(K)$ and so K is nilpotent whence, by a result of Wielandt's, G is soluble. Therefore G cannot satisfy hypothesis D and the proof is complete.

<u>lemma 9.2</u> If P is a star-covered α-invariant Sylow p-subgroup of G of type $\{1,2,3\}$, then P permutes with at least two of L_1, L_2, L_3.

<u>Proof</u> Suppose $PL_1 \neq L_1P$ and $PL_2 \neq L_2P$. Additionally, assume that $L_1^* \leqslant N_{L_1}(P)$. By lemma 5.5(i)(e and g), $P_{\sigma\tau} = 1$, $P_{\rho\sigma} \neq 1 \neq P_{\rho\tau}$ and so $L_2^* \leqslant N_{L_2}(P)$ is excluded. Therefore $P_\rho, P_\tau \leqslant N_P(L_2)$ and so, as P is star-covered, $P = P_\sigma$ which is, from lemma 5.5(i)(f), at variance with $L_1^* \leqslant N_{L_1}(P)$. Consequently $P_\sigma, P_\tau \leqslant N_P(L_1)$ and $P_\rho, P_\tau \leqslant$

$N_P(L_2)$ is the only possibility. However, P being star-covered demands that $P = P_\rho = P_\sigma$ which is contrary to the type of P. Thus P permutes with at least two of L_1, L_2 and L_3.

Let L denote the subgroup of G generated by the α-invariant Sylow subgroups of type $\{1,2,3\}$; from theorem 8.1, L is a soluble Hall subgroup of G.

<u>Theorem 9.3</u> If $G = (LL_1)(L_2L_3L_{23})$ and all α-invariant Sylow subgroups of G of type $\{1,2,3\}$ are star-covered, then G does not satisfy hypothesis D.

<u>Proof</u> Suppose the result is false. Let $\tilde{L}$ be the subgroup of L generated by the α-invariant Sylow subgroups of L which permute with L_2L_3 and let L^+ be the subgroup of L generated by the α-invariant Sylow subgroups of L which do not permute with L_2L_3. Clearly $L = L^+\tilde{L}$ and $L^+ \cap \tilde{L} = 1$.

<u>(1)</u> Because of theorem 9.1, it may be assumed that $L_2 \neq 1 \neq L_3$. A further restriction may be placed upon G in the form of : L^+ permutes with one of L_2 and L_3 and, further, if (say) $L^+L_2 = L_2L^+$ then, for each α-invariant Sylow subgroup P of L^+, $PL_3 \neq L_3P$. This may be seen as follows: for each α-invariant Sylow subgroup P of L^+, from lemma 9.2, P must permute with at least one of (and only one) L_2 and L_3. If the above assertion were false, then it would be possible to find α-invariant Sylow subgroups P and Q of L^+ such that

$PL_2 = L_2P$, $PL_3 \neq L_3P$, $QL_3 = L_3Q$ and $QL_2 \neq L_2Q$ and this type of configuration has already been ruled out of contention by theorem.6.7. Without loss, it will be assumed that $L^+L_2 = L_2L^+$. Clearly then L permutes with L_2.

(2) If P is a non-trivial α-invariant Sylow subgroup of L^+ for which P_ρ, $P_\sigma \leq N_P(L_3)$, then L_3 is not equal to either L_{3_ρ} or L_{3_τ}. First observe that, as P is star-covered and P_ρ, $P_\sigma \leq N_P(L_3)$, $P = P_\tau$ and therefore $\mathcal{M}_{p,\pi_3} = \{P, N_P(L_3)L_3\}$.

Suppose that $L_3 = L_{3_\sigma}$; this immediately yields $[P_\rho , L_3] = 1$. Since $P_{\rho\sigma} = 1$, PL_2 admits $\rho\sigma$ fixed-point-freely and so $[P_\rho , L_2] = 1$. As $[P_\rho , L_{23}] = 1$, $[(L_2L_3L_{23}), P_\rho] = 1$ and hence $P_\rho{}^G(\leq LL_1)$ is a non-trivial proper normal α-invariant subgroup of G. Thus $L_3 \neq L_{3_\sigma}$.

Next consider $L_3 = L_{3_\rho}$; of course this gives $[P_\sigma, L_3] = 1$. It will be shown that under the prevailing conditions $L_{23} = 1$. First, observe that if $L_{23}P \neq PL_{23}$ then $Z(P) \leq P_{\sigma\tau}$ which contradicts $L_3P \neq PL_3$. Whence $L_{23}P = PL_{23}$ and, if $L_{23} \neq 1$, then $L_{23} \neq L_{23}^*$ which yields that $\{N_G(O_{\pi_{23}}(L_{23}P))\}_{p,\pi_3} \geq L_3, P$. Hence $L_{23} = 1$. Now, as $(PL_1)_{\rho\sigma} = 1$, $[P_\sigma, L_1] = 1$ and therefore $L_1L_3 = L_3L_1$ which leads to $G = (LL_2)(L_1L_3)$ with $[(L_1L_3), P_\sigma] = 1$. Consequently hypothesis D cannot hold and so $L_3 \neq L_{3_\rho}$.

(3) If $L_{23} \neq 1$, then (i) for each α-invariant Sylow subgroup P of L^+, $PL_{23} \neq L_{23}P$ and (ii) for each α-invariant Sylow subgroup P of $\widetilde{L}$, $PL_{23} = L_{23}P$.

(i) Suppose $PL_{23} = L_{23}P$ where P is an α-invariant Sylow subgroup of L^+, then, as $L_{23} \neq 1$ and hence $L_{23}^* \neq L_{23}$,

$\{N_G(O_{\pi_{23}}(PL_{23}))\}_{p,\pi_3} \geq P$, L_3 whereas $P \leq L^+$. Therefore $PL_{23} \neq L_{23}P$.

(ii) Let P be an α-invariant Sylow subgroup of $\widetilde{L}$ and suppose $PL_{23} \neq L_{23}P$. As P is star-covered, either $P = P_\tau$ or P_σ. These possibilities give rise respectively to $L_3 \trianglelefteq L_3P$ or $L_2 \trianglelefteq L_2P$ and hence $PL_{23} = L_{23}P$. So (ii) is verified.

<u>(4)</u> $L^+ \neq 1$. For $L^+ = 1$ implies that $L = \widetilde{L}$ and so, appealing to (3)(ii), it follows that L permutes with $(L_2L_3L_{23})$. Hence $G = (LL_2L_3L_{23})L_1$ which is dealt with in theorem 9.1. Thus $L^+ \neq 1$.

<u>(5)</u> It is claimed that for each non-trivial α-invariant Sylow subgroup P of L^+, P_ρ, $P_\sigma \leq N_P(L_3)$. Suppose otherwise and let P be an α-invariant Sylow subgroup of L^+ such that $L_3^* \leq N_{L_3}(P)$. Since $Z(P) \leq P_{\sigma\tau}$ is incompatible with the supposed shape of $\mathcal{M}_{p,\pi_3}$, $PL_{23} \neq L_{23}P$ is impossible. Thus, with regard to 3(i), L_{23} must be trivial.

Suppose Q is a non-trivial α-invariant Sylow subgroup of L^+ such that Q_ρ, $Q_\sigma \leq N_Q(L_3)$. By assumption Q is star-covered whence $Q = Q_\tau$ and thus $\mathcal{M}_{q,\pi_3} = \{Q, N_Q(L_3)L_3\}$. From lemma 5.5(i)(b and d) either $Z(L_3) = Z(L_3)_{\rho\sigma} \leq N_{L_3}(P)$ or $L_{3_\rho} = L_{3_\sigma}$. Since $N_Q(L_3)$ is non-trivial, if $Z(L_3) = Z(L_3)_{\rho\tau}$, then $[Z(L_3), N_Q(L_3)] = 1$ which is against the shape of $\mathcal{M}_{q,\pi_3}$ while $L_{3_\rho} = L_{3_\sigma}$ yields that $N_Q(L_3) \trianglelefteq L_3N_Q(L_3)$ by lemma 4.3. Thus Q_ρ, $Q_\sigma \leq N_Q(L_3)$ cannot hold. Consequently $L_3^* \leq N_{L_3}(L^+)$.

As $L_3^* \leq N_{L_3}(P)$, $P_{\rho\sigma} = 1$ and so $[P_\rho, L_2] = 1$. Thus, as $\mathcal{M}_{p,\pi_3} = \{L_3, N_{L_3}(P)P\}$, $O_{\pi_2}(L_3L_2)$ must be trivial hence $L_2 = L_{2_\tau}$ and $L_3 \trianglelefteq L_2L_3$. A further consequence of $O_{\pi_2}(L_3L_2) = 1$ is that $L_{3_\rho} \neq L_{3_\sigma}$ and so, from lemma 5.5(i)(b and d), it follows that $Z(L_3) = Z(L_3)_{\rho\sigma} \leq N_{L_3}(L^+)$. It is asserted that L_3 is a normal subgroup of $L_2L_3\tilde{L}$. By lemma 2.11(ii), this would follow if it could be shown that $O_{\pi_3'}(L_2L_3\tilde{L}) = 1$. If $O_{\pi_3'}(L_2L_3\tilde{L})$ were non-trivial then, as $O_{\pi_2}(L_2L_3) = 1$, it would have to be a $\pi(\tilde{L})$-group and so be contained in $\tilde{L}$. Hence $(O_{\pi_3'}(L_2L_3\tilde{L}))^G$ $(\leq (HL_1))$ would be a non-trivial proper normal α-invariant subgroup of G. Thus $O_{\pi_3'}(L_2L_3\tilde{L}) = 1$ and so $L_3 \trianglelefteq L_2L_3\tilde{L}$.

Next, it will be demonstrated that $Z(L_3) \leq N_{L_3}(L_1)$. First consider the possibility $L_1L_3 = L_3L_1$; since $[P_\sigma, L_1] = 1$ and $\mathcal{M}_{p,\pi_3} = \{L_3, N_{L_3}(P)P\}$, $O_{\pi_1}(L_1L_3) = 1$ whence $L_1 = L_{1_\tau}$ and so, as $L_1, L_2 \leq G_\tau$, $L_1L_2 = L_2L_1$. Consequently $G = (L_1L_2L)L_3$ which, by theorem 9.1, may be discounted. Thus it may be taken that $L_1L_3 \neq L_3L_1$. If $L_{3_\sigma} \leq N_{L_3}(L_1)$ then the desired conclusion follows as $Z(L_3) = Z(L_3)_{\rho\sigma}$. The other alternative $L_{1_\sigma} \leq N_{L_1}(L_3)$ gives, in particular, $N_{L_1}(L_3) \neq 1$ which, when combined with $[Z(L_3), N_{L_1}(L_3)] = 1$, also yields $Z(L_3) \leq N_{L_3}(L_1)$.

Therefore $Z(L_3) \leq N_{L_3}(L^+L_1)$ which, together with $L_3 \trianglelefteq L_2L_3\tilde{L}$ and $G = (L_2L_3)(LL_1) = (L_2L_3\tilde{L})(L^+L_1)$, gives that hypothesis D is violated yet again and so (5) is verified.

<u>(6)</u> Here it will be shown that having $L^+ \neq 1$ and each α-invariant Sylow subgroup P of L^+ such that P_ρ, $P_\sigma \leq N_P(L_3)$ is also incompatible with hypothesis D. In view

of all α-invariant Sylow subgroups of type $\{1,2,3\}$ being star-covered, $L^+ = L^+_\tau$.

If P is an α-invariant Sylow subgroup of L^+ then $P = P_\tau$, $PL_2 = L_2P$ and $PL_3 \neq L_3P$. Thus, if $[L_2,\tau] \neq 1$, then $\{N_G([L_2,\tau])\}_{p,\pi_3} \geqslant P$, $O_{\pi_3}(L_2L_3)$ implying, as $P = P_\tau$, that $O_{\pi_3}(L_2L_3) = 1$. Consequently $L_3 = L_{3_\sigma}$ which is forbidden by (2). Hence $L_2 = L_{2_\tau}$. Since $(PL_2)_{\rho\sigma} = 1$ and P_ρ, $P_\sigma \leqslant N_P(L_3)$, by lemma 6.1, $O_{\pi_2}(L_2L_3) = 1$.

In the same manner as in (5), it may now be established (using the fact $O_{\pi_2}(L_2L_3) = 1$ and (3)(ii)) that $(L_3L_{23}) \trianglelefteq (L_2L_3L_{23}\tilde{L})$; clearly $L_3 \trianglelefteq (L_2L_3L_{23}\tilde{L})$.

If $L_3L_1 = L_1L_3$ then, as $L_3 \neq L_{3_\rho}$ (by (2)) and so $O_{\pi_3}(L_1L_3) \neq 1$, $L_1L_{23} = L_{23}L_1$ also thence $G = (L_1L_3L_{23}\tilde{L})(L^+L_2)$ which, as $L^+L_2 \leqslant G_\tau$, implies that $[G,\tau] \neq G$. So it may be supposed that $L_3L_1 \neq L_1L_3$.

If $L_{1_\sigma} \leqslant N_{L_1}(L_3)$, then $(L^+L_1)_\rho$, $(L^+L_1)_\sigma \leqslant N_{(L^+L_1)}(L_3)$. As L^+L_1 admits $\rho\sigma$ fixed-ppint-freely and (because $L_3 \trianglelefteq (L_2L_3L_{23}L)$) $G = N_G(L_3)(L^+L_1)$, by mimicking the last part of the proof of theorem 9.1 it may be proven that L^+L_1 is nilpotent. Thus, if P is an α-invariant Sylow subgroup of L^+, then $[P,L_1] = 1$ yielding, since $[P_\rho, L_2] = 1$, that $L_1L_2 = L_2L_1$. Clearly $G = (L_1L_2L)(L_3L_{23})$ and so theorem 9.1 may again be applied.

Therefore it may be taken that $L_{3_\sigma} \leqslant N_{L_3}(L_1)$. If $C_{L_3}(L_1) \neq 1$ then $\mathcal{M}_{p,\pi_3} = \{P, L_3N_P(L_3)\}$ dictates that $O_{\pi_1}(PL_1) = 1$ and so, as $(PL_1)_{\rho\sigma} = 1$, $L_1 = L_1{}^*_{\langle\rho\sigma\rangle} = L_{1_\sigma}$. Consequently $Z(L_3) \leqslant L_{3_{\rho\sigma}}$ so giving $[Z(L_3), N_P(L_3)] = 1$ which is contrary to the form of $\mathcal{M}_{p,\pi_3}$. Hence $C_{L_3}(L_1) = 1$

and so $L_{3_\sigma} \leq L_{3_\rho}$. So $L_3^* = L_{3_\rho} \neq L_3$ (by (2)). Thus, as $P_\rho \leq N_P(L_3)$, $L_3 = L_{3_\rho} C_{L_3}(P_\rho)$. Since $C_{L_3}(P_\rho) \neq 1$, $Z(P)_\rho = 1$ and $Z(P) \leq N_P(L_3)$. As PL_1 admits $\rho\sigma$ fixed-point-freely, $L_1 = N_{L_1}(Z(P))O_{\pi_1}(PL_1)$ and so, because $Z(P)_\rho = 1$, $[Z(P), L_1] = 1$. Therefore $Z(P) \leq N_P(L_1) \cap N_P(L_3)$ and so, additionally, $Z(P)$ also normalizes $N_{L_3}(L_1)$ $(\geq L_{3_\sigma})$. The shape of $\mathcal{M}_{p,\pi_3}$ and $L_3 \neq N_{L_3}(L_1)$ demand that $Z(P) \leq P_\sigma$.

Now (L^+L_1) admits $\rho\sigma$ fixed-point-freely and so $(L^+L_1) = N_{(L^+L_1)}(P)O_{p'}(L^+L_1)$. From the previous paragraph, $Z(P) \leq P_\sigma$ whence $[Z(P), N_{(L^+L_1)}(P)] = 1$. Since $P^*_{\langle\rho\sigma\rangle} \leq N_P(L_3) \neq P$, clearly $O_p(L^+L_1) \neq 1$ from which it follows that $1 \neq D = Z(P) \cap O_p(L^+L_1)$ $(\leq N_G(L_3))$. Thus $G = (L_2L_3L_{23})(LL_1) = (L_2L_3L_{23}\tilde{L})(L^+L_1) = N_G(L_3)C_G(D)$ and once again hypothesis D is seen to be out of the question.

This completes the proof of theorem 9.3.

10. CONCLUSIONS

Lemma 10.1 At least two of L_1, L_2 and L_3 permute.

Proof Deny the lemma; clearly L_{12}, L_{13} and L_{23} may be assumed to be trivial. Clearly theorem 6.6 is available and thus it may be taken that $L_1 = L_{1_\tau}$, $L_2 = L_{2_\rho}$ and $L_3 = L_{3_\sigma}$. Hence:

$$\left.\begin{aligned} \mathcal{M}_{\pi_1,\pi_2} &= \{L_1 N_{L_2}(L_1), L_2\} \\ \mathcal{M}_{\pi_1,\pi_3} &= \{L_1, L_3 N_{L_1}(L_3)\} \\ \mathcal{M}_{\pi_2,\pi_3} &= \{L_2 N_{L_3}(L_2), L_3\} \end{aligned}\right\} (+)$$

Should $2 \in \pi_1 \cup \pi_2 \cup \pi_3$ ($2 \in \pi(G)$ by the Feit, Thompson result [2]) then, by lemma 4.6, G must be soluble. Thus, if P denotes the α-invariant Sylow 2-subgroup of G, P must be of type $\{1,2,3\}$. If two of L_1, L_2 and L_3 do not permute with P then, by virtue of lemma 6.3, they would permute with each other. Therefore there are (essentially) two cases requiring examination: (i) P permutes with L_2 and L_3 but does not permute with L_1; and (ii) P permutes with L_1, L_2 and L_3. Because of lemma 4.6 it will be supposed that P is not equal to P_ρ, P_σ or P_τ.

(i) As $L_1 = L_{1_\tau}$ and $PL_1 \neq L_1P$, it follows that P_σ, $P_\tau \leqslant N_P(L_1)$ and, furthermore, $[P_\sigma, L_1] = 1$ because $N_P(L_1) = O_P(L_1)(N_P(L_1))_{\rho\tau}$. Since $[P_{\rho\sigma}, L_3] = 1$ and $L_1L_3 \neq L_3L_1$, it may be deduced that $P_{\rho\sigma} = 1$ and so PL_2 admits $\rho\sigma$ fixed-point-freely. Now $[P,\sigma] \leqslant O_p(PL_2)$

and $[P,\sigma] \trianglelefteq PL_3$ so $O_{\pi_2}(PL_2) \leq N_{L_2}(L_3) = 1$. Since $C_P(L_1) \neq 1$, $[N_P(L_1),\tau] \neq 1$ and, because $[P,\tau] \leq O_p(PL_3)$, it may be inferred that $O_{\pi_3}(PL_3) \leq N_{L_3}(L_1) = 1$. Consequently $O_{\pi_3}(PL_3) = 1 = O_{\pi_2}(PL_2)$ so giving $L_2 = N_{L_2}(P)$ and $L_3 = N_{L_3}(J(P))C_{L_3}(Z(P))$. Clearly L_3 is forced to be contained in $N_{L_3}(L_2)$ and so this disposes of (i).

<u>(ii)</u> As $1 \neq [P,\rho] \leq O_p(PL_1)$ and $[P,\rho] \trianglelefteq PL_2$, it may be concluded that $O_{\pi_1}(PL_1) \leq N_{L_1}(L_2) = 1$. Likewise it may be shown that $O_{\pi_i}(PL_i) = 1$ for $i = 1,2,3$ and hence $L_i = N_{L_i}(J(P))C_{L_i}(Z(P))$ for $i = 1,2,3$. It is claimed that for each i (= 1,2,or 3) $N_{L_i}(J(P)) \neq 1 \neq C_{L_i}(Z(P))$; for suppose (say) that $L_1 = N_{L_1}(J(P))$. Then, because of the shape of $\mathcal{M}_{\pi_1,\pi_3}$, $N_{L_3}(J(P)) = 1$ and so $L_3 = C_{L_3}(Z(P))$. Moreover, since $N_{L_2}(L_3) = 1$, it follows that $C_{L_2}(Z(P)) = 1$ and so $L_2 = N_{L_2}(J(P))$ whence $L_1L_2 = L_2L_1$. If $L_1 = C_{L_1}(Z(P))$, an analogous sequence of arguments will also produce $L_1L_2 = L_2L_1$. Since the same reasoning works for i = 2 or 3, the claim is verified.

So $N_{L_1}(J(P)) \neq 1 \neq C_{L_1}(Z(P))$ and $N_{L_2}(J(P) \neq 1 \neq C_{L_2}(Z(P))$ and so the fact that $N_{L_1}(L_2) = 1$ forces $L_2 = N_{L_2}(J(P))C_{L_2}(Z(P)) \leq N_{L_2}(L_1)$. This finishes (ii) and the proof of the lemma.

<u>Lemma 10.2</u> If P is an α-invariant Sylow subgroup of G of type $\{1,2,3\}$ which is not star-covered, then P permutes with at least two of L_1, L_2 and L_3.

<u>Proof</u> Suppose it is the case that P does not permute with both L_2 and L_3. An appeal to lemma 6.3 gives that $L_2L_3 = L_3L_2$.

It will first be shown that P_ρ, $P_\tau \leq N_P(L_2)$ and P_ρ, $P_\sigma \leq N_P(L_3)$ with $Z(P) = Z(P)_{\sigma\tau} \leq N_P(L_2) \cap N_P(L_3)$. Towards this end, suppose that (say) $L_3^* \leq N_{L_3}(P)$; because of lemma 5.5(i)(e and g), $L_2^* \leq N_{L_2}(P)$ is not possible so P_ρ, $P_\tau \leq N_P(L_2)$. Further $N_P(L_2) \leq P_\sigma$ cannot occur since $P_{\rho\sigma} = 1$. Consequently $C_P(L_2) \neq 1$ and so, by considering the form of $\mathcal{M}_{p,\pi_3}$, it may be inferred that $O_{\pi_2}(L_2L_3) = 1$, hence $L_2 = L_{2\tau}$. This leads to $Z(P) \leq P_\tau$ which violates the supposition $L_3^* \leq N_{L_3}(P)$. Thus P_ρ, $P_\tau \leq N_P(L_2)$ and P_ρ, $P_\sigma \leq N_P(L_3)$; note that $P_\sigma = P^* = P_\tau$ is not possible. If both $C_P(L_2)$ and $C_P(L_3)$ are non-trivial, then $Z(P) = Z(P)_{\sigma\tau} \leq N_P(L_2) \cap N_P(L_3)$; whereas if $C_P(L_2) \neq 1 = C_P(L_3)$ then $Z(P) = Z(P)_\sigma \leq N_P(L_2)$ whence $Z(P) = Z(P)_\sigma \leq P_\sigma \leq N_P(L_3) \leq P_\tau$.

As $Z(P) \leq P_{\sigma\tau}$; $\hat{L}_1P = P\hat{L}_1$. Taking into consideration the fact that $[L_3, L_{13}] = 1 = [L_2, L_{12}]$ and P does not permute with L_2 and L_3, it may be concluded that $L_{12} = L_{13} = 1$.

Let W be an α-invariant Sylow subgroup of L (here L has the same meaning as in section 9) and suppose $L_1W \neq WL_1$. With the complicity of lemma 6.2, it may be taken that W_σ, $W_\tau \leq N_W(L_1)$. Further, $[L_1, Z(P)] = 1$ and $O_p(PW) \neq 1$ coerce $O_w(PW)$ into being contained in $N_W(L_1)$ whence, with the aid of lemma 2.15, it may be asserted that $W = W_\rho$. (Note this line of argument has also appeared in lemmas 8.3 and 8.14). Consulting lemma 5.5

yields that W permutes with L_2L_3. Since $W_\rho = W \neq W_\sigma W_\tau$ (because of hypothesis D), $O_w(WL_2L_3) \neq 1$ and, without loss, $[O_w(WL_2L_3), \sigma] \neq 1$. As $W = W_\rho$ and $P_\rho \leq N_P(L_2) \cap N_P(L_3)$, $P = P_\rho O_p(PW) = N_P(L_i) O_p(PW)$, $(i = 2,3)$. Now, from $O_p(PW) = N_{O_p(PW)}(L_3) C_{O_p(PW)}([W, \sigma])$. So $\{N_G([O_w(WL_2L_3), \sigma])\}_{p,\pi_3} \geqslant C_{O_p(PW)}([W, \sigma]), L_{3_\sigma}$. If $C_{O_p(PW)}([W, \sigma]) \leq N_P(L_3)$ then $P \leqslant N_P(L_3)$ whereas $L_{3_\sigma} \leq N_{L_3}(P)$ is ruled out by the remark following lemma 5.5. Thus there is no such W and hence $G = (LL_1)(L_2L_3L_{23})$.

If $L_{23} \neq 1$, then $PL_{23} \neq L_{23}P$ (otherwise P would be required to permute with L_2 and L_3). Consequently $Z(P) \leqslant N_P(L_2L_3L_{23})$. If $Z(J(P))_\rho = Z(J(P))_{\rho\sigma} Z(J(P))_{\rho\tau}$, $Z(J(P))_\sigma = Z(J(P))_{\rho\sigma} Z(J(P))_{\sigma\tau}$ and $Z(J(P))_\tau = Z(J(P))_{\rho\tau} Z(J(P))_{\sigma\tau}$, then $(1 \neq)$ $D = Z(P) \cap O_p(LL_1) \leqslant Z(LL_1) \cap N_P(L_2L_3L_{23})$ and so D^G is a non-trivial proper normal α-invariant subgroup of G. If $Z(J(P))_\rho = Z(J(P))_{\rho\sigma} Z(J(P))_{\rho\tau}$, $Z(J(P))_\sigma = Z(J(P))_{\rho\sigma} Z(J(P))_{\sigma\tau}$ and $Z(J(P))_\tau = Z(J(P))_{\rho\tau} Z(J(P))_{\sigma\tau}$ does not hold then it may be deduced that either $J(P) \leqslant N_P(L_2) \cap N_P(L_3)$, $J(P) \leqslant N_P(L_2)$ or $J(P) \leqslant N_P(L_3)$.

If $J(P) \leqslant N_P(L_2) \cap N_P(L_3)$, then $J(P) \leq N_P(L_2L_3L_{23})$ whence $(O_p(LL_1) \cap Z(J(P)))^G$ $(\leqslant (L_2L_3L_{23}) N_P(L_2L_3L_{23}))$ is a non-trivial proper normal α-invariant subgroup of G. Thus, the proof will be completed when (without loss) the situation $J(P) \leq N_P(L_2)$ and $J(P) \not\leq N_P(L_3)$ has been successfully analysed.

As it is supposed that $J(P) \leq N_P(L_2)$, $J(P) = C_{J(P)}(L_2) J(P)_\sigma$ and, since $P_\sigma \leq N_P(L_3)$, $O_{\pi_2}(L_2L_3) \neq 1$ implies that $J(P) \leq N_P(L_3)$. Thus $O_{\pi_2}(L_2L_3) = 1$ so $L_3 \trianglelefteq$

L_2L_3 and $L_2 = L_{2_\tau}$. Note that if $L_{23} \neq 1$, then $N_P(L_2) = N_P(L_{23}) = N_P(L_3)$ and so $L_{23} = 1$. Clearly $N_P(L_3) \leq N_P(L_2)$ and so $N_P(L_3) \leq N_P(L_2L_3)$. Now $[N_P(L_3),\tau] \leq C_P(L_3)$ and also, since $L_2 = L_{2_\tau}$, $[N_P(L_3),\tau] \leq C_P(L_2)$. As G does not possess any non-trivial proper normal -invariant subgroups $N_P(L_3) \leq P_\tau$ and so $P^* = P_\tau$ $(\leq N_P(L_2))$.

Observe that $P \neq L$ since $P = L$ would give $Z(P) \leq N_P(L_2L_3) \cap Z(LL_1)$ so contravening hypothesis D; let Q be an α-invariant Sylow q-subgroup of L, $p \neq q$. If $QL_2 = L_2Q$, then lemma 6.1 is available and yields that $O_q(QL_2) = 1$. However, as $L_{2_\sigma} = 1$ and $L_{2_\tau} = L_2$, this would require $Q = Q_{\sigma\tau}$ contrary to Q being of type $\{1,2,3\}$. Thus $QL_2 \neq L_2Q$ and, as $L_2 = L_{2_\tau}$, Q_ρ, $Q_\tau \leq N_Q(L_2)$. Since $J(P) \leq N_P(L_2)$ and P is not star-covered, it follows that $[N_P(L_2),\tau]$, which is contained in $C_P(L_2)$, is non-trivial. Moreover, as $P^* = P_\tau$, $[N_P(L_2),\tau] \leq O_p(LL_1)$. Clearly, then $O_q(LL_1) \leq N_Q(L_2)$ whence (by lemmas 2.10 and 2.15) $Q = Q_\sigma$. Hence, by employing lemma 5.5, it may be seen that $QL_3 = L_3Q$. Two consequences of $Q = Q_\sigma$ are that $Q_\rho \leq O_q(QL_3) \cap C_Q(L_2)$. Clearly $\{N_G(Q_\rho)\}_{\pi_2,\pi_3} \geq O_{\pi_3}(QL_3)$, L_2. Should it be the case that $N_{L_3}(Q_\rho) = L_3$, then Q_ρ^G would be a non-trivial proper normal α-invariant subgroup of G. Therefore it may be taken that $N_{L_3}(Q_\rho) \neq L_3$. Further, since $Q = Q_\sigma$, $[L_3,\sigma] \leq O_{\pi_3}(QL_3) \leq N_{L_3}(Q_\rho)$ hence $L_3 = N_{L_3}(Q_\rho)L_{3_\sigma}$. Hence, as $L_{2_\sigma} = 1$, L_2 normalizes L_3 and $N_{L_3}(Q_\rho)$, it may be inferred that $C_{L_3}(L_2) \neq 1$. Now consider $\{N_G(L_2)\}_{p,\pi_3} \geq C_{L_3}(L_2)$, $N_P(L_2)$. As $N_P(L_2) \leq N_P(L_3)$ is excluded since $J(P) \not\leq N_P(L_3)$, it may be inferred that $C_{L_3}(L_2) \leq N_{L_3}(P)$ with $C_{L_3}(L_2)$

also normalizing $N_P(L_2) \geqslant P^*$. Therefore $P = N_P(L_2)C_P(C_{L_3}(L_2))$ and clearly $C_P(C_{L_3}(L_2))$ must be contained in $N_P(L_2)$ which contradicts $PL_2 \neq L_2P$.

This resolves the situation: $J(P) \leqslant N_P(L_2)$ but $J(P) \not\leqslant N_P(L_3)$ and so it may be concluded that P permutes with at least two of L_1, L_2 and L_3.

<u>Lemma 10.3</u> Each α-invariant Sylow subgroup of G of type $\{1,2,3\}$ permutes with at least two of L_1, L_2 and L_3.

<u>Proof</u> This result follows from lemmas 9.2 and 10.2.

<u>Lemma 10.4</u> With a possible re-ordering of 1,2,3, either $G = (LL_2L_3L_{23})L_1$ or $G = (LL_1)(L_2L_3L_{23})$.

<u>Proof</u> The proof will be broken into two parts depending on whether or not all of L_{12}, L_{13} and L_{23} are trivial.

So, first suppose that (say) $L_{23} \neq 1$. Clearly $\hat{L}_2\hat{L}_3 = \hat{L}_3\hat{L}_2$. Suppose P is an α-invariant Sylow subgroup of L which permutes with L_{23}. Since $[L_{23}, (L_2L_3L_{21}L_{31})] = 1$ and $L_{23} \neq L_{23}^*$, $N_G(O_{\pi_{23}}(PL_{23})) \geqslant P$, $\hat{L}_2$, $\hat{L}_3$ so giving that P permutes with $\hat{L}_2\hat{L}_3$. Let L^+ denote the group generated by those α-invariant Sylow subgroups of L which permute with L_{23} and let $\tilde{L}$ denote the α-invariant Hall $\pi(L^+)'$-subgroup of L. If Q is an α-invariant Sylow subgroup of $\tilde{L}$ then, clearly, $Z(Q) \leqslant Q_{\sigma\tau}$ and hence $Q\hat{L}_1 = \hat{L}_1Q$. Consequently $G = (\hat{L}_2\hat{L}_3L^+)(\tilde{L}\hat{L}_1)$. In order that hypothesis D be preserved, L_{12} and L_{13} must be trivial, and thus $G = (L_2L_3L_{23}L^+)(\tilde{L}L_1)$. If the conclusion of the lemma were false then there would exist an α-invariant Sylow subgroup, P, of L^+ such that $PL_1 \neq L_1P$ and, clearly, $\tilde{L}$ would have to be non-trivial. Thus there exists an α-invariant Sylow subgroup of G, Q such that $QL_{23} \neq L_{23}Q$. In addition, $PL_{23} = L_{23}P$ and $QL_1 = L_1Q$, but lemma 6.8 asserts that a configuration of this kind is untenable. Consequently, either $G = (LL_2L_3L_{23})L_1$ or $G = (LL_1)(L_2L_3L_{23})$.

Now suppose that $L_{12} = L_{13} = L_{23} = 1$ and furthermore, because of lemma 10.1, it may be assumed that $L_2L_3 = L_3L_2$. If P is an α-invariant Sylow subgroup of L, then, by lemma 10.3, P permutes with at least two of L_1, L_2 and L_3. Hence P permutes with at least one of L_1 and (L_2L_3). Therefore $G = (L_2L_3L^+)(\tilde{L}L_1)$ where L^+ and $\tilde{L}$ are subgroups of L which are generated (respectively) by those α-invariant Sylow subgroups of L which permute with (L_2L_3) and L_1. Again, if the lemma does not hold then it is possible to select α-invariant Sylow subgroups P and Q of (respectively) L^+ and $\tilde{L}$ such that $PL_1 \neq L_1P$ and $Q(L_2L_3) \neq (L_2L_3)Q$ (so may suppose that $QL_2 \neq L_2Q$). Since $PL_2 = L_2P$ and $QL_1 = L_1Q$, theorem 6.7 denies the credibility of this situation. Therefore, in this case also, either $G = (LL_2L_3)L_1$ or $G = (L_2L_3)(LL_1)$.

<u>Lemma 10.5</u> If $G = (LL_1)(L_2L_3L_{23})$, then G does not satisfy hypothesis D.

<u>Proof</u> Suppose the result is false; let $\tilde{L}$ be the subgroup

of L generated by the α-invariant Sylow subgroups of L which permute with (L_2L_3) and let L^+ be the subgroup of L generated by the α-invariant Sylow subgroups of L which do not permute with L_2L_3. Hence $L = L^+\tilde{L}$ with $L^+ \cap \tilde{L} = 1$.

Theorem 9.1 allows the assumptions: $L \neq 1$, $L_2 \neq 1 \neq L_3$. Also, as in the proof of theorem 9.3, it may be supposed that (say) $L^+L_2 = L_2L^+$ (so $LL_2 = L_2L$) and that for each α-invariant Sylow subgroup Q of L^+, $QL_3 \neq L_3Q$. Because of theorem 9.3, L must contain an α-invariant Sylow subgroup P such that P is not star-covered. Further, it is asserted that P must lie in L^+.

If $L_{23} = 1$, then $G = (LL_1)(L_2L_3) = (L^+L_1)(\tilde{L}L_2L_3)$ and so if $P \leq \tilde{L}$, it would follow that $O_p(LL_1)^G$ is a non-trivial proper normal α-invariant subgroup of G. Hence, when $L_{23} = 1$, $P \leq L^+$.

Suppose $L_{23} \neq 1$ and that $P \leq \tilde{L}$. If $PL_{23} = L_{23}P$, then a non-trivial proper normal α-invariant subgroup of G may be constructed as in the previous paragraph. Thus $PL_{23} \neq L_{23}P$ and so $\mathcal{M}_{p,\pi_3} = \{L_{23}N_P(L_{23}), P\}$ with $[L_{23}, P_\rho] = 1$. If $Z(J(P))_\rho \neq 1$, then $J(P) \leq N_P(L_3)$ and hence $G = (L_2L_3L_{23})(LL_1) = N_G(L_{23})N_G(D^{(LL_1)})$ where $D = Z(P) \cap O_p(LL_1)$ $(\neq 1)$. Since $D^{(LL_1)} \leq Z(J(P)) \leq N_G(L_{23})$, G cannot satisfy hypothesis D. Whereas, if $Z(J(P))_\rho = 1$, Glauberman's factorization theorem combined with lemma 2.12 gives $LL_1 = C_{LL_1}(D)L_\rho$ with $D = O_p(LL_1) \cap Z(P)$ $(\neq 1)$. Therefore, as $Z(P)$, $G_\rho \leq N_G(L_{23})$, $G = N_G(L_{23})\, C_G(D)$ and again hypothesis D is violated. Thus it has been shown that $P \leq L^+$.

If $L_1L_2 = L_2L_1$, then $G = (L_3L_{23})(LL_1L_2)$ which has been dealt with in theorem 9.1. So it may be assumed that $L_1L_2 \neq L_2L_1$. Since P is not star-covered, P permutes with L_1 and L_2 but does not permute with L_3, lemma 6.10 is applicable to give that $Z(J(P)) \leqslant N_P(L_3)$. Clearly $P_\rho, P_\sigma \leqslant N_P(L_3)$ must hold and, using lemma 6.1 on L_3, L_2 and P, it may be deduced that $O_{\pi_2}(L_2L_3) = 1$ so yielding that $L_3 \trianglelefteq L_2L_3$. Consequently $G = (LL_1)(L_2L_3L_{23}) = N_G(D^{(LL_1)})N_G(L_3)$ (again $D = O_p(LL_1) \cap Z(P)$ ($\neq 1$)) and because $D^{(LL_1)} \leq Z(J(P)) \leqslant N_G(L_3)$ hypothesis D is again impossible. This finishes the proof of the above stated lemma.

Sufficient information has now been amassed to prove the main result of this work.

Theorem 10.6 Let G be a finite group admitting a fixed-point-free coprime automorphism α of order rst, where r,s, and t are distinct primes and rst is a non-Fermat number. Then G is soluble.

Proof Suppose the theorem is false and choose G to be a counterexample of minimal order. Clearly all proper α-invariant subgroups of G must be soluble. Further, if H is a non-trivial proper normal α-invariant subgroup of G then, by (1.1), α induces a fixed-point-free automorphism on G/H and so G/H would then have to be soluble. Thus G does not possess any non-trivial proper normal α-invariant subgroups. The assumption that rst is coprime to $|G|$ combined with the Feit, Thompson theorem ([2]) means that rst may be taken to be an odd number. Consequently G satisfies hypothesis D.

However, taken together, theorem 9.1 and lemmas 10.4 and 10.5 show that there do not exist any finite groups for which hypothesis D holds. Therefore no minimal counterexample exists and the theorem is now proven.

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