Charge and the topology of spacetime

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Abstract. A new class of electrically charged wormholes is described in which the outer two sphere is not spanned by a compact coorientable hypersurface. These wormholes can therefore display net electric charge from the source free Maxwell's equation. This extends the work of Sorkin on non-space orientable manifolds, to spacetimes which do not admit a time orientation. The work is motivated by the suggestion that quantum theory can be explained by modelling elementary particles as regions of spacetime with non-trivial causal structure. The simplest example of an electrically charged spacetime carries a spherical symmetry.

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1. Introduction

It is known that the source free equations of electromagnetism can display apparent charge in regions of spacetime with a non-trivial topology. The simplest examples are Wheeler's wormholes [1], where one mouth has positive electric charge and the other mouth has the same amount of negative electric charge. These topological structures of spacetime are commonly known as geons. Sorkin constructed an example in which the entire wormhole exhibits magnetic charge [2]. This wormhole does not admit a space orientation. A purely differential topological discussion of the notion of magnetic and electric charge in spacetime is presented. It contains a review and extension of the work of Wheeler, Misner and Sorkin.

It has been suggested by one of the authors that quantum mechanics could be explained by modelling elementary particles as 4-geons (geons with a non-trivial casual structure) [3]. A second paper relates topology changing interactions between geons with a reversal of the time orientation, resulting in spacetimes that are not time orientable [4]. Previously spacetimes that lacked a time orientation were considered unphysical; now, from this radical perspective, they are not only physically relevant but very important.

We apply the source free Maxwell equations and the definition of electric and magnetic charge to spacetimes which lack space or time orientations. Various wormholes are constructed with each type of orientability. In these examples net magnetic charge can appear when space is not orientable and electric charge when spacetime is not time orientable. Finally a new spacetime is described which lacks a time orientation, has spherical symmetry and the outward appearance of a point source of electric charge - an electric monopole. Topological obstructions prevent the construction of an analogous magnetic monopole.

The organisation of the article is as follows: The next section is applicable to forms and multi vector densities on manifolds of any dimension. It contains the definition of magnetic and electric charge and the corresponding conservation laws. In section 3 we define wormhole topologies and construct physically relevant examples of four dimensional spacetimes. They carry two forms and bi vector densities, which display electric and magnetic charge from the source free Maxwell equations. A more general topological construction is defined in section 4 which leads to the construction of a spacetime possessing an electric monopole with spherical symmetry.

2. Electromagnetic charge

We are concerned here with a closed k-form F and a divergence free $(n-k)$-multi vector density H on a smooth n-dimensional manifold M. In formulas we deal with $F \in C^\infty(M, \Lambda^k T^*M)$ and $H \in C^\infty(M, L^{-n} \otimes \Lambda^{n-k} TM)$ subject to the equations $dF = 0$ and $div H = 0$. (A review of exterior calculus is given in the appendix).

For $n = 4$ and $k = 2$ this reduces to the relativistic version of the first and second Maxwell equations for electromagnetism. In electromagnetism F is called the

Faraday two form (electric field plus magnetic flux) and H is called the *Faraday bi vector* (dielectric displacement plus magnetic loop tension). F and H have to be related by the so called *constitutive relation* of the matter in question. In vacuum this relation is simply given by a Lorentzian metric g. In index notation this relation reads $g_{ac}g_{bd}H^{cd} = \sqrt{|\det g|}F_{ab}$. This formula is invariant under rescaling the metric, which means that the identification only depends upon the conformal class of g. (This point of view goes back to Mie and is taken from [5]).

If a Riemannian or Lorentzian metric is given on M, some authors replace the multi vector density H by another k-form H' and use a (local) Hodge star operator to require $*d*H' = 0$. However, we will take advantage of the clear geometric distinction between forms and multi vector densities, leading to a clear distinction between the notion of magnetic and electric charge. This distinction can already be seen in the following definitions: (For the meaning of the following integrals and the notion of a coorientation, see the appendix)

Definition 2.1 (Magnetic charge) *Given a closed k-form F as above we define the* magnetic charge *(represented by F) contained in an (immersed) oriented k-dimensional sphere $S^k \hookrightarrow M$ by*

$$Q_m := \int_{S^k,\mathrm{or}} F.$$

Definition 2.2 (Electric charge) *Similarly, if H denotes a divergence free $(n-k)$-multi vector density we define the* electric charge *(represented by H) contained in an (immersed) cooriented k-dimensional sphere $S^k \hookrightarrow M$ by:*

$$Q_e := \int_{S^k,\mathrm{coor}} H.$$

The definition of magnetic (or electric) charge can easily be extended to an integral over any oriented (cooriented) compact k-manifold, but we will restrict the examples to spheres for convenience. As is well known, these two quantities satisfy conservation laws, which follow from Stoke's theorem and the Divergence theorem: Combining the definition of magnetic charge 2.1 with the first Maxwell equation $dF = 0$, and Stoke's theorem (see Appendix A.3), gives a vanishing result, or conservation law for magnetic charges:

Corollary 2.3 *If S^k is the boundary of an (immersed) compact oriented $(k+1)$-dimensional submanifold Σ with boundary $\partial\Sigma = S^k$, then $Q_m = 0$.*

Consequently, if S^k, or and $S^{k'}, \mathrm{or}'$ are two immersed spheres, such that there is an (immersed) compact oriented $(k+1)$-dimensional submanifold Σ with these two spheres as boundary: $\partial\Sigma, \mathrm{or} = S^k, \mathrm{or} \cup S^{k'}, -\mathrm{or}'$, then S^k and $S^{k'}$ contain the same amount of magnetic charge.

Analogously combining the definition of electric charge 2.2 with the second Maxwell equation $div H = 0$, the Divergence theorem (see Appendix A.5), gives a vanishing result, or conservation law for electric charges:

Corollary 2.4 *If S^k, coor is the boundary of an (immersed) compact cooriented $(k+1)$-dimensional submanifold Σ with boundary $\partial\Sigma = S^k$, then $Q_e = 0$.*

Consequently, if S^k, coor and $S^{k'}, \mathrm{coor}'$ are two immersed spheres, such that there is an (immersed) compact cooriented $(k+1)$-dimensional submanifold Σ, coor with these two spheres as boundary: $\partial\Sigma, \mathrm{coor} = S^k, \mathrm{coor} \cup S^{k'}, -\mathrm{coor}'$, then S^k and $S^{k'}$ contain the same amount of electric charge.

It is highly significant that while the vanishing of magnetic charge requires Σ to be orientable, the vanishing of electric charge requires it to be coorientable. A simple $(1+1)$-dimensional example of a submanifold that is orientable but not coorientable is a circle going round a Möbius strip. The circle S^1 is orientable, but on the Möbius strip a consistent normal vector cannot be defined along S^1. More generally: a surface $\Sigma \hookrightarrow M$ of dimension $n-1$ is coorientable in M if there is a vector field along Σ which is everywhere transversal to $T\Sigma$ (hence the boundary of a manifold is always coorientable). If the manifold M itself is orientable, then a k-dimensional Σ is coorientable if and only if Σ is orientable. Similarly, if a Lorentzian manifold $M^{n-1,1}$ is time orientable and if Σ is a spacelike surface of dimension $n-1$ then Σ is always coorientable.

Note, that the above definitions and claims have a pure differential topological nature. It is irrelevant whether Σ is spacelike, or timelike in places, as in some of the examples in [4]. However when the assumptions for the two vanishing results are not satisfied then examples exist of apparent net charge arising from the source free equations. Such examples will be constructed in the following two sections.

Remark 2.5 *As Sorkin pointed out [2], one could twist Maxwell's theory by the orientation bundle (bundle of pseudoscalars): real numbers would turn into pseudo scalars (and vice versa), k-forms F would turn into $(n-k)$-multi vector densities F', multi vector densities H would turn into forms H'. Maxwell's equation then would read $divF' = 0$ and $dH' = 0$, which shows that it is not clear how to distinguish between those two theories. The integral of H' over an oriented k-sphere would now define electric charge. Sorkin uses this freedom of writing Maxwell's equations, so he defines electric and magnetic charge using a convention that differs from the more common one presented here.*

A clear correspondence between the definition of charge given above and the notion of a point charge with an associated worldline is given by the following two examples: The *Coulomb field* of an electric point charge $q_e \in \mathbb{R}$ with a straight worldline $\mathbb{R} \times \{0\}$ in Minkowski space $\mathbb{R} \times \mathbb{R}^3$ becomes singular along the worldline. On $\mathbb{R} \times \mathbb{R}^{>0} \times S^2$ it is given by $E(t, r, \sigma) = 1/(4\pi r^2)(0, 1, 0)$. The field of observers at rest with the electric charge is $N(t, r, \sigma) := (1, 0, 0)$.

Example 1 (Electric charge on an incomplete manifold) *On Minkowski space with a removed worldline $\mathbb{R} \times \mathbb{R}^{>0} \times S^2$, the radial field E together with the observer field N induce a smooth Faraday bi vector $H := q_e E \wedge N$. The electric charge contained in the two sphere S^2 linking the removed worldline corresponds to q_e.*

In contrast a magnetic (monopole) point charge can be modelled by a pseudo scalar $q_m \in L^4 \otimes \Lambda^4\mathbb{R}^{4*}$ (see appendix) along a worldline.

Example 2 (Magnetic charge on an incomplete manifold) *On Minkowski space with removed worldline* $\mathbb{R} \times \mathbb{R}^{>0} \times S^2$*, the radial field* E *and the observer field* N *induce a smooth Faraday two form* $F := \langle q_m, E \wedge N\rangle$*. The magnetic charge contained in the two sphere* S^2 *linking the removed worldline corresponds to* q_m*.*

Since the linking two sphere S^2 on the resultant manifold $\mathbb{R} \times \mathbb{R}^{>0} \times S^2$ is not the boundary of a compact hypersurface, the vanishing theorems from above do not apply to the two examples. However, such spacetimes are geodesically incomplete.

3. Wormhole topology

In order to model an elementary particle as a region of non-trivial topology we define a *topological geon*. A point particle traces out a world line in spacetime, but if the particle has some size and structure we can consider a worldtube such that the particle is always inside the tube. Some properties of a particle may depend upon the internal structure but others, such as mass, electric charge and magnetic charge can be determined, and defined, entirely by measurements made outside the worldtube. It will be shown that certain topological structures inside the worldtube together with the source free Maxwell equations exhibit net charge, exactly as if the worldtube contained a point charge.

Denote by $B^{k+1} \subset \mathbb{R}^{k+1}$ the unit ball in $\mathbb{R}^{k+1}$ with boundary S^k. (The open Ball will be denoted by $\dot{B}^{k+1}$). We now restrict our attention to n-manifolds M containing a cylinder diffeomorphic to $\mathbb{R}^{n-k-1} \times S^k$, for $1 \le k \le (n-2)$. The region outside the cylinder is assumed to be "topologically trivial" in the sense that it is diffeomorphic to $\mathbb{R}^n$ minus a solid cylinder. The region inside the cylinder can be highly non-trivial, but has to be "complete" in the sense that the k-sphere S^k is the boundary of a compact $(k+1)$-surface. Physically relevant are the cases $k = 2$ and $n = 4$. In this case the inner region of non-trivial topology can be considered as a model of an elementary particle a so called 4-*geon* [3].

Definition 3.1 (Topological n-geon) *We will call a smooth n-dimensional manifold M a* topological n-geon*, if it contains an open set $T \subset M$ (the* worldtube*) such that $M - T$ is diffeomorphic to $\mathbb{R}^n$ with an open tube removed: $M - T \cong \mathbb{R}^{n-k-1} \times (\mathbb{R}^{k+1} - \dot{B}^{k+1})$ and if the k-sphere S^k linking this tube T is the boundary of some compact (immersed) $(k+1)$-dimensional surface $\Sigma \hookrightarrow \bar{T}$ going through the worldtube T. Such a Σ will be called a* spanning surface*.*

If M is a topological n-geon, then the k-sphere S^k linking the world tube T is orientable and coorientable. Hence it can be used to define the magnetic and electric charge of the n-geon.

Definition 3.2 (Net charge) *If a closed k-form or a divergence free $(n-k)$-multi vector density is given on M, the charge inside a k-sphere S^k linking the worldtube T is also called the* net charge *of the n-geon.*

Topological n-geons with non-trivial topology can be explored using spacetimes with wormholes. A two dimensional (space like) wormhole is simply a surface with a handle attached. Mathematically a wormhole may be constructed by cutting two regions out of spacetime and joining the edges according to a given rule. A Möbius strip is constructed from a cylinder by cutting and then joining the edges, after a rotation of π. The rule applied for joining the excised regions allows all combinations of space and time orientations to be constructed. The construction differs fundamentally from examples in the literature because we allow for a non-trivial identification of time as well as space coordinates. The resulting structures are *topological n-geons* which carry all combinations of net magnetic and net electric charge, as will be seen in the rest of this section.

A *wormhole* is constructed out of $\mathbb{R}^n$ by removing two open solid cylinders and identifying points on the boundaries: Choose a space time split $\mathbb{R}^n = \mathbb{R}^{n-k-1} \times \mathbb{R}^{k+1}$ and let $B_1 := B^{k+1}(-a) \subset \mathbb{R}^{k+1}$ and $B_2 := B^{k+1}(+a)$ be the unit balls centered at $-a$ and $a \in \mathbb{R}^{k+1}$ (with $|a| > 1$). The identifications at the cylindrical boundaries of $\mathbb{R}^{n-k-1} \times (\mathbb{R}^{k+1} - (\dot{B}_1 \cup \dot{B}_2))$ will be determined by orthogonal maps $\phi \in \mathbf{O}(\mathbb{R}^{n-k-1})$ (time identification) and $\psi \in \mathbf{O}(\mathbb{R}^{k+1})$ (space identification): for all $\vec{r} \in S^k$ we identify $(t, \vec{r} - a) \sim (\phi(t), \psi(\vec{r}) + a)$. This defines a topological manifold. The smooth structure should be such that a curve which moves into one cylinder comes out of the other: i.e. a velocity vector $(\dot{t}, \dot{q})$ at $(t, \vec{r} - a)$ will be identified with $(\phi(\dot{t}), \psi(\dot{q} - 2\langle \dot{q}, \vec{r} \rangle \vec{r}))$ (mirror image rule).

Remark 3.3 *A Lorentzian metric on a manifold induces a light cone structure for which it makes sense to define space and time orientability of the manifold. In our topological discussion we don't have a lightcone structure, so we define space and time orientation for the particular class of examples we have in mind. However, the examples have been constructed to facilitate the addition of a Lorentzian metric, and if one is added then the two definitions should coincide.*

Definition 3.4 (Wormhole) *The resulting quotient manifold $M := \mathbb{R}^{n-k-1} \times (\mathbb{R}^{k+1} - (\dot{B}_1 \cup \dot{B}_2))/(\phi, \psi)$ will be called a* wormhole of signature $(n-k-1, k+1)$. *M is called* time orientable *if $\det \phi = +1$ and M is called* space orientable *if $\det \psi = -1$.*

This is clearly an example of an n-geon. The sphere $S^k(R) \subset \mathbb{R}^{k+1}$ centered at the origin with large radius $R > |a| + 1$ links both removed tubes and is the boundary of $\Sigma := B^{k+1}(R) - (\dot{B}_1 \cup \dot{B}_2)/\psi$. The vanishing results 2.3 and 2.4 can be applied to give:

Proposition 3.5 *If the wormhole M is space orientable, then M does not contain net magnetic charge, since the spanning Σ is orientable. If M is time orientable, then M does not contain net electric charge, since the spanning Σ is coorientable.*

Wormholes of physical interest have $n = 4$ and $k = 2$. Those described in the literature have been restricted to $\phi(t) = t$ and are therefore time orientable, hence they do not carry electric net charge (in the sense of definition 2.2). All possible orientability characteristics are treated in the following examples. Most cases are well known, they are included for completeness. The examples are labelled by the static multipole they naturally carry.

Example 3 (Dipole) *Misner and Wheeler [1] used a reflection for the space identification:* $\psi(\vec{r}) := \vec{r} - 2\frac{\langle \vec{r},a\rangle}{\langle a,a\rangle}a$ *and* $\phi(t) = t$*. This wormhole naturally carries static dipole fields, i.e. a Faraday bi vector induced by two opposite electric charges and a Faraday two form induced by two opposite magnetic charges. It cannot display electric nor magnetic net charge in the sphere* $S^k(R)$ *which links both tubes.*

Electric and magnetic fields are well-defined throughout the above dipole: field lines going into one mouth of the wormhole and coming out of the other mouth create pairs of equal and opposite charges. One disadvantage of this construction for modelling particles is that each charge is connected to one distant opposite charge - apparently in contradiction to the observed indistinguishability of particles [2].

Example 4 (Magnetic monopole with quadrupole) *Sorkin [2] used the identity for both identifications:* $\phi(t) = t$ *and* $\psi(\vec{r}) = \vec{r}$*. This wormhole is non-space orientable, but time orientable. It naturally carries a Faraday two form induced by two equal magnetic charges, leading to a monopole with additional higher moments.*

Note that in order to have a non-zero net magnetic charge it is necessary for *every* spanning Σ with S^2 as its boundary to be non-orientable. A wormhole which appeared and then disappeared at $t = \tau$ could not display net charge. This can be seen by applying Stoke's theorem to the orientable three manifold $S^2 \times [0, \tau] \cup B^3_{t=\tau}$. The fact that this three manifold is timelike in places is irrelevant to the application of Stoke's theorem.

A wormhole where all spanning surfaces Σ of $S^k(R)$ are non-coorientable is constructed as follows:

Example 5 (Electric monopole with quadrupole) *We use a reflection for the space identification* $\psi(\vec{r}) := \vec{r} - 2\frac{\langle \vec{r},a\rangle}{\langle a,a\rangle}a$ *and a reflection for the time identification* $\phi(t) = -t$ *(figure 1). This wormhole naturally carries a static Faraday bi vector induced by two equal electric charges, leading to a monopole with an additional higher moment. It may also carry a Faraday two form which exhibits a magnetic dipole moment as in the case of Wheeler's wormhole.*

The result that non-time orientable wormholes have net electric charge but not magnetic charge has added significance because it is known [3] that the lack of a time orientation could be used to explain quantum phenomena. By contrast there is no physical reason in support of (or contrary to) the existence of spacetimes which do not have a space orientation.

The example constructed possesses closed timelike curves. Although spacelike slices can be taken they are not hypersurfaces, in the sense that each point of the surface is crossed once and once only by a timeline through each point of M. The last example of this section can display both types of net charge:

Example 6 (Electric and magnetic monopoles with quadrupoles) *If one uses the identity for the space identification $\psi(\vec{r}) = \vec{r}$ and a reflection for the time identification $\phi(t) = -t$, the spanning surface is neither orientable nor coorientable.*

4. Line bundle construction

The previous method for constructing 4-geons with interesting topologies has a limited repertoire plus the problem of establishing that the structures are indeed smooth manifolds. A sphere with a point removed is diffeomorphic to $\mathbb{R}^2$, the stereographic projection is the diffeomorphism, and the point removed corresponds to infinity. This technique allows any compact manifold to be transformed into a non-compact one which includes a region of non-trivial topology. Time will be modelled as a vector bundle - for one time dimension this corresponds to attaching a timeline at every point on the manifold. The line bundle is trivial if and only if the spacetime is time orientable.

To construct other examples of n-geon topologies as defined in 3.1, recall that a vector bundle L of rank $(n-k-1)$ over some $(k+1)$-dimensional manifold Q is a n-dimensional manifold with a smooth projection $\pi\colon L \to Q$, such that the preimages of points $q \in Q$ are real $(n-k-1)$-dimensional vector spaces $L_q := \pi^{-1}(q) := \{l \in L \,|\, \pi(l) = q\}$. The manifold Q is immersed into L as zero section. In the physically relevant case $k = 2$ and $n = 4$ the total space L, the base space Q and the fibres L_q can be thought of as spacetime, space and worldlines through q respectively.

Proposition 4.1 (Special class of n-geon) *Let Q be some compact $(k+1)$-dimensional manifold and let $L \to Q$ be some real vector bundle of rank $(n-k-1)$ over Q. Now choose a point $q_\infty \in Q$ and define M by removing the fibre L_{q_∞} from L. Then $M := L - L_{q_\infty}$ defines a topological n-geon.*

Proof: To see that M satisfies the properties of the n-geon definition 3.1 we remark that the vector bundle L is locally trivialisable: (Notation: for any $U \subset Q$ define $L|_U := \pi^{-1}(U) = \{L_q \,|\, q \in U\}$). Around q_∞ we can choose a small closed disc $D \subset Q$ (D is diffeomorphic to a $(k+1)$-ball in $\mathbb{R}^{k+1}$) over which L becomes trivial, i.e. $L|_D \cong \mathbb{R}^{n-k-1} \times D$ as vector bundles. Then $T := L|_{Q-D} \subset M$ defines a worldtube. Indeed, $M - T = L|_{D-\{q_\infty\}} \cong \mathbb{R}^{n-k-1} \times (D - \{q_\infty\})$ is diffeomorphic to $\mathbb{R}^{n-k-1} \times (\mathbb{R}^{k+1} - B^{k+1})$ as required. The k-sphere $S^k := \partial D \subset M$ links T and is clearly the boundary of $\Sigma := Q - \dot{D}$, which is a compact $(k+1)$-surface through T. $\square$

The above special class of wormhole topologies 4.1 reduces for $k = 2$ and $n = 4$ to a real line bundle $L \to Q$ (space time) over a compact three manifold Q (space) with one worldline removed $M := L - L_{q_\infty}$. This removed worldline plays the role of the observer at infinity.

As with the topological definition of wormholes, the line bundle construction has been designed to facilitate the addition of a Lorentzian metric but it does not carry a metric (see remark 3.3). If a Lorentzian metric is added to the line bundle then the definition of a space and time orientation should coincide with the following definition.

Definition 4.2 (Space and time orientability) *If the n-geon is given by the above construction $M := L - L_{q_\infty}$, M is called* space orientable *if Q is orientable and M is called* time orientable *if the vector bundle $L \to Q$ is orientable.*

As in the previous section, the vanishing results 2.3 and 2.4 can be applied to give:

Proposition 4.3 *If the n-geon M is space orientable, then M does not contain magnetic charge, since the spanning Σ is orientable. If M is time orientable, then M does not contain electric charge, since the spanning Σ is coorientable.*

Example 7 (Minkowski space) *Minkowski four space is easily rediscovered by taking the trivial line bundle over the three sphere: $Q = S^3$ and $L := \mathbb{R} \times S^3$ since stereographic projection from a point $q_\infty \in S^3$ defines a diffeomorphism $L - L_{q_\infty} \cong \mathbb{R} \times \mathbb{R}^3$. It cannot contain net charge.*

Remark 4.4 (Wormhole examples) *The examples of the previous sections are rediscovered as line bundle constructions as follows: Wheeler's wormhole example 3 is based upon the trivial line bundle $L := \mathbb{R} \times Q$ over $Q := S^1 \times S^2$. Sorkin's wormhole example 4 is a trivial line bundle $L := \mathbb{R} \times Q$ over a three dimensional Klein bottle, i.e. a non-orientable twofold subquotient of $S^1 \times S^2 \to Q$. In example 5 L is the pull back of the Möbius strip $M\ddot{o} \to S^1$ to $Q := S^1 \times S^2$. In the last example 6 L is the pull back of the Möbius strip to the three dimensional Klein bottle.*

The next aim is to use the line bundle construction to produce a spacetime carrying an electric monopole with no higher moments. Recall, that a topological 4-geon $L - L_{q_\infty}$ can contain electric net charge only if the manifold is not time orientable, i.e. only if the line bundle is non-trivial 4.3. Since S^3 is simply connected, all line bundles over S^3 are trivialisable and the bundle constructions gives back Minkowski space. The simplest quotient of S^3 is *real projective three space* $Q := \mathbb{R}P^3 = S^3/\{\pm \mathrm{id}\}$. It can be viewed as the set of real lines through the origin in four space: $\mathbb{R}P^3 = \{\mathbb{R}x \,|\, x \in \mathbb{R}^4 - \{0\}\}$. Recall, that the *canonical line bundle* $L \to \mathbb{R}P^3$ has as fibre over $\mathbb{R}x$ the line $L_{\mathbb{R}x} := \mathbb{R}x$ itself.

Example 8 (Electric monopole) *The canonical line bundle $L \to \mathbb{R}P^3$ over real projective three space is non-trivial and the resultant 4-geon $M := L - L_{q_\infty}$ is a simple spacetime obtained by the bundle construction that can carry electric net charge.*

To see how the static electric Coulomb field of an electric charge $q_e \in \mathbb{R}$ defines a smooth bi vector density on M, we do a variation of the wormhole construction of the previous section. Similar to the first example 1 we deal with Minkowski space with an open solid worldtube removed: $\mathbb{R} \times \mathbb{R}^{\geq 1} \times S^2$. The radial field $E(t, r, \sigma) = 1/(4\pi r^2)(0, 1, 0)$ together with the observer field $N(t, r, \sigma) := (1, 0, 0)$ induce a smooth Faraday bi vector $H := q_e E \wedge N$. The electric charge contained in the two sphere S^2 linking the removed worldtube corresponds to q_e. Now we identify events on the cylindrical boundary by flipping the time coordinate (time identification) and using the antipodal map (space identification): $(t, 1, \sigma) \sim (-t, 1, -\sigma)$. The smooth structure should identify the velocity vector $(\dot{t}, \dot{r}, \dot{\sigma})$ at $(t, 1, \sigma)$ with $(-\dot{t}, -\dot{r}, -\dot{\sigma})$ at $(-t, 1, -\sigma)$ (mirror image rule). Notice, that neither the static observer vector field N nor the radial field E are well defined with these identifications. However, the observer independent wedge product $H := q_e N \wedge E$ clearly is.

The one point compactification of $\mathbb{R}^{\geq 1} \times S^2$ with q_∞ (using stereographic projection) can be identified with the upper half three sphere. The identification on the equator are given by the antipodal map, showing that the underlying Q can be identified with real projective three space. The time line bundle $L \to Q$ must be the non-trivial canonical bundle.

The $\mathbf{O}(3)$ symmetry of the Coulomb field carries over to the bundle construction.

This construction is simpler than the wormholes of section 3 and has higher symmetry. Essentially a ball is removed from space (a cylinder from spacetime) and then joined up by identifying opposite points, but swapping the time direction. Externally this is spherically symmetric and has the same electric field as a point electric charge. The Faraday tensor is well-defined throughout. Any path through the reidentified region undergoes a time reversal and a reversal of the radial electric field. Unlike the wormhole examples this is a simple electric monopole, with no higher moments.

A similar construction to create a magnetic monopole is not possible:

Remark 4.5 *The Lefschetz fixed point theorem shows that the three sphere S^3 can only cover orientable three manifolds $S^3 \to Q$. Hence, the line bundle construction based upon a discrete subquotient Q of S^3 can never carry a magnetic monopole. Similarly, for the magnetic monopole field from example 2, we cannot find an identification free of fixed points linking the two sphere S^2 with itself, which leaves the magnetic monopole two form invariant. The Lefschetz fixed point theorem determines such an identification to be orientation reversing, which rules out the possibility of a magnetic monopole.*

5. Conclusion

New topological structures for spacetime are given which can exhibit an apparent net electric charge without any apparent source. Spacetimes with non-orientable immersed surfaces were known to exhibit magnetic charge (in the sense of definition 2.1). The spaces with non-coorientable immersed surfaces described exhibit the opposite type of charge (electric by definition 2.2). These spacetimes are not time orientable; this may seem unphysical, but they are the type of classical structure which would be required to exhibit quantum mechanical effects [3]. So the classical, gravitational, model for quantum mechanics is also seen to lead naturally to the existence of electric charge and the absence of magnetic charge.

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Appendix A. Densities and orientations

In this appendix we recall the basic definitions from the theory of integration along submanifolds.

Definition Appendix A.1 (Density) *Let V be a real n-dimensional vector space (here V will play the role of the tangent space at a point of a manifold or submanifold). A homogeneous map $\mu\colon \Lambda^n V - \{0\} \to \mathbb{R}$ with the property $\mu(\lambda\omega) = |\lambda|\mu(\omega)$ for all $\lambda \in \mathbb{R} - \{0\}$ and $\omega \in \Lambda^n V$ is called a* density *(of weight $-n$).*

The set of all densities builds a one dimensional oriented vector space denoted by $L^{-n}(V)$ or simply L^{-n}. A positive density assigns naturally a real number as volume to an n-dimensional parallelepiped. The tensor product $L^{-n} \otimes \Lambda^n V$ is the space of *pseudo scalars*. This one dimensional space naturally carries a norm given by $||\mu \otimes \omega|| := |\mu(\omega)|$. Using the induced inner product, the dual space $L^n \otimes \Lambda^n V^*$ can naturally be identified with $L^{-n} \otimes \Lambda^n V$. The two orientations of V are in one to one correspondence with the two unit elements of $L^{-n} \otimes \Lambda^n V$.

The above algebraic constructions can be done pointwise on any n-dimensional manifold M. In particular at $x \in M$ we simply write $L_x^{-n} := L^{-n}(T_x M)$. The resulting line bundle $L^{-n} \to M$ is trivialisable, but does not come with a canonical trivialisation. The sections of L^{-n} can naturally be integrated over M.

Proposition Appendix A.2 (Integration over oriented submanifolds) *Let $S \hookrightarrow M$ denote a k-dimensional immersed submanifold. If S is orientable and* $\mathrm{or} \in L^{-k} \otimes \Lambda^k TS$ *a choice of orientation then any k-form F on M can be turned into a density $\langle F, \mathrm{or}\rangle$ over S, which can therefore be integrated naturally.*

On every manifold M we have the deRham sequence of exterior derivatives on differential forms $d\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ as a natural differential operator.

Theorem Appendix A.3 (Stoke's theorem) *If $F \in C^\infty(M, \Lambda^k T^*M)$ is any smooth k-form on an n-manifold M, and $\Sigma \hookrightarrow M$ an (immersed) compact oriented* $(k+1)$*-dimensional submanifold with boundary $\partial\Sigma$ then*

$$\int_{\partial\Sigma,\mathrm{or}} F = \int_{\Sigma,\mathrm{or}} dF.$$

(for a proof see [6]).

We recall that a *coorientation* of an immersed k-manifold $S \hookrightarrow M$ is an orientation of the normal bundle (quotient bundle) $TM/TS \to S$ along S. Whether the normal bundle is orientable or not depends upon how S is immersed into M.

Proposition Appendix A.4 (Integration over cooriented submanifolds) *If $S \hookrightarrow M$ is coorientable and* coor *is a choice of coorientation then any $(n-k)$-multi vector density H on M can be turned into a density $\langle H, \mathrm{coor}\rangle$ on S, which therefore can naturally be integrated.*

Proof: At each point of S this density $\langle H, \mathrm{coor}\rangle$ is defined on k-vectors $v \in \Lambda^k T\Sigma$ as $\langle H, \mathrm{coor}\rangle(v) := \langle \mathrm{n}^*, H\rangle(\mathrm{n} \wedge v)$, where $\mathrm{n} \in \Lambda^{n-k}(TM/TS)$ is a positive normal (according to the coorientation), and $\mathrm{n}^* \in \Lambda^{n-k}(TM/TS)^*$ is its dual $\langle \mathrm{n}^*, \mathrm{n}\rangle = 1$. This definition is independent of the choice of n. $\square$

Adjoint to the deRham sequence is the sequence of exterior divergences of multi vector density fields $div\colon C^\infty(M, L^{-n} \otimes \Lambda^{n-k}TM) \to C^\infty(M, L^{-n} \otimes \Lambda^{n-k-1}TM)$. If μ denotes a non-vanishing density and X a vector field, the divergence of $\mu \otimes X$ can invariantly be defined as a Lie derivative: $div(\mu \otimes X) := \mathcal{L}_X \mu$. Similarly, on a decomposable bi vector density field we have $div(\mu \otimes X \wedge Y) = (\mathcal{L}_X \mu) \otimes Y - (\mathcal{L}_Y \mu) \otimes X + \mu \otimes [X, Y]$. Exterior derivative d and exterior divergence div are related using a local orientation $\mathrm{or} \in L^{-n} \otimes \Lambda^n TM$ and its dual $\langle \mathrm{or}^*, \mathrm{or}\rangle = 1$: an orientation turns a $n-k$ multi vector density into a k-form and vice versa: $divH = \langle \mathrm{or}, d\langle \mathrm{or}^*, H\rangle\rangle$.

Theorem Appendix A.5 (Divergence theorem) *If $H \in C^\infty(M, L^{-n} \otimes \Lambda^{n-k}TM)$ is any smooth $(n-k)$-multi vector density on an n-manifold M, and $\Sigma \hookrightarrow M$ an (immersed) compact cooriented $(k+1)$-dimensional submanifold with boundary $\partial\Sigma$ then*

$$\int_{\partial\Sigma, \mathrm{coor}} H = \int_{\Sigma, \mathrm{coor}} divH.$$

(for a proof see [6], also [2]).

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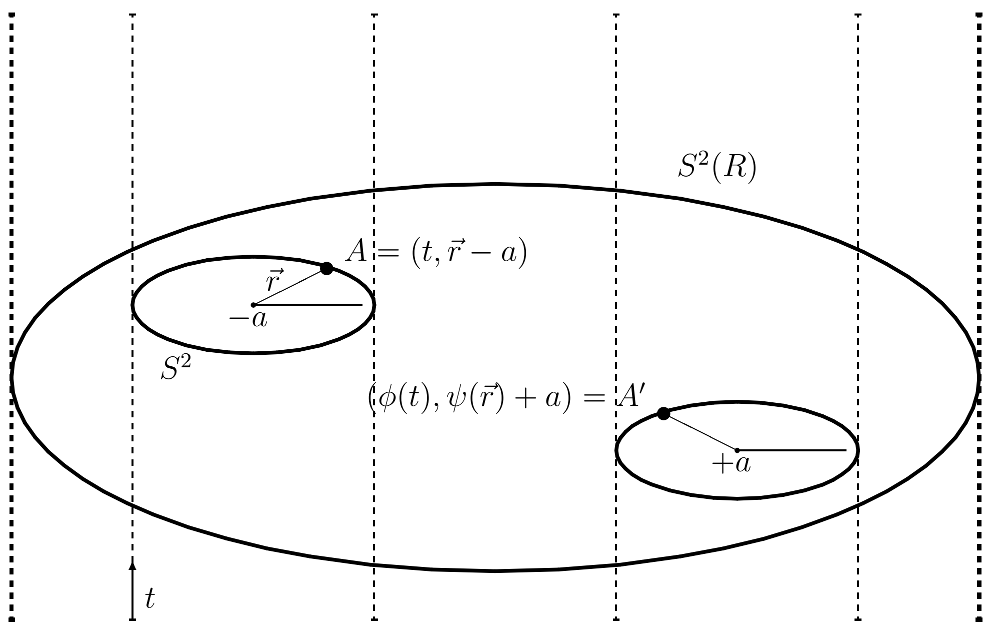

Figure 1. A generalized wormhole construction with one space dimension suppressed. Unit world tubes are removed from $\mathbb{R}^3$; points on the boundaries S^2 are identified according to the orthogonal maps ψ and ϕ. Net charge is defined to be contained in the outer two sphere $S^2(R)$, linking the large worldtube.