SOME RESULTS IN THE CONNECTIVE K-THEORY OF LIE GROUPS

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SUMMARY

In this thesis we study the connective K-theory of compact, connected Lie groups. We use mainly Borel's results in their ordinary cohomology, L. Hodgkin's paper [21] about their K-theory, the Atiyah-Hirzebruch spectral sequence and L. Smith's exact sequence relating the connective K-theory with the integral cohomology.

We have divided it in four chapters, as follows:

I - We construct the bu spectrum and prove that it is an associative, commutative, ring Ω-spectrum, after we define a ring spectra map from bu to $H\mathbb{Z}$; we show that $k^*(\ ;\mathbb{Z}_q)$ is a multiplicative cohomology theory defined in the homotopy category of CW complexes; we prove L. Smith's Theorem [34] for $k^*(X;L)$, X any CW complex, $L = \mathbb{Z},\mathbb{Z}_q$ or any free abelian group; finally we work out the Atiyah-Hirzebruch spectral sequence converging to $k^*(X)$ (X compact) and we compare it with that one converging to $K^*(X)$ to obtain some results that we will need later. We show that: If $K^*(X)$ is torsion free then $k^*(X)$ has t^{-1} torsion if and only if it has $\mathbb{Z}$ torsion. This together with the dual of a proposition from [15] : "If $k_*(X)$ is a free $\mathbb{Z}[t]$ module then $H_*(X;Z)$ is a free $\mathbb{Z}$ module", implies that for a compact connected Lie group $k^*(G)$ is a free abelian group if and only if $H^*(G;\mathbb{Z})$ is.

II - We give a small survey about the classification of compact, connected Lie groups, their K-theory and ordinary cohomology. We prove the following theorem: "Let G be a compact, connected Lie group, L a ring of type Q(P) so that $H^*(G;L)$ is torsion free. Then: (i) $k^*(G;L)=\Lambda_{L[t-1]}(y_1,\dots,y_m)$ where y_j has degree i_j for all $1\leq j\leq r$, $n = \sum_{j=1}^{n} i_j$, (ii) The y_j can be choosen so that they are primitive in the Hopf algebra $k^*(G;L)$"

III - We calculate $k^*(G_2;L)\big(L = \mathbb{Z},\mathbb{Z}_2$ and $Q(2)\big)$.

IV - We calculate $k^*(\mathrm{Spin}(n);Q(2)), k^*(\mathrm{Spin}(n);\mathbb{Z}_2)/\{x\in k^*(\mathrm{Spin}(n);\mathbb{Z}_2)/t^{-1}x=0\}$ and we give some properties of $k^*(\mathrm{Spin}(n))$.

Those two last chapters are applications of all the results obtained before. The cases of F_4, E_6, E_2, E_8 are referred to in the Appendix.

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INTRODUCTION

Through this thesis the cohomology theories are defined in the homotopy category of (compact when stated) C.W. complexes. "Space" always means a space with the homotopy type of a C.W. complex.

In 1.2, when applying Araki and Toda's results [6] to connective K-theory we have omitted the condition that obliged every space to be compact since if the cohomology theories are defined in the homotopy category of CW complexes every construction and statement remains true. Compactness was only needed to take a representative of stable $\{X,Y\}$ (homotopy classes of based maps from X to Y) but it is only used when X and Y are compact spaces ($S^n, M_q, N_q, M_q \wedge M_q$). Also in 1.3 we omitted the compactness condition on the spaces since we defined k^* in the homotopy category of CW complexes. We rewrite the proof of L. Smith's Theorem [34] for $k^*(\ ;L)$ defined in that category and $L = Z, Z_q$ or a free abelian group. In 1.4 we deal with compact spaces to avoid problems with the limits of the spectral sequences as we work with compact Lie groups afterwards.

In the two last chapters we only deal with the simply-connected representatives of the distinct classes of locally isomorphic Lie groups, case covered by [21], although $K^*(SO(n))$ has been calculated by [19,22]. But as it is not torsion free our main propositions do not apply.

We note that the proof of Proposition 4.1.3. with corrected generators was suggested by Dr. A. Robinson.

The notations more frequently used are:

$\mathbb{Z}$ - the integers

$\mathbb{Z}_p (p \geq 1)$ - the integer mod p, $\mathbb{Z}/p\mathbb{Z}$.

$\mathbb{N}$ - the positive integers

$\mathbb{Q}$ - the rationals

Q(P) (P a possibly empty set of primes) - ring of fractions whose denominators are in the lowest term, prime to p for any $p \in P$.

$\wedge$ - smash product

$S^n (n \in \mathbb{N})$ - n-th sphere.

CX - cone of X

[,] - based homotopy classes of maps.

bu - connective K spectrum

$H\mathbb{Z}$ - Eilenberg-MacLane spectrum

Let L be an abelian group

$K^*(\;; L)$ - K cohomology with coefficients in L.

$k^*(\;; L)$ - connective K cohomology with coefficients in L.

$H^*(\;; L)$ - ordinary cohomology with coefficients in L.

CHAPTER 1 - GENERAL RESULTS IN CONNECTIVE K-THEORY

In the first paragraph of this chapter we show how to construct the ring spectrum bu for connective K-theory and the ring spectrum map from bu to $H\mathbb{Z}$, the spectrum for ordinary cohomology with integer coefficients. We work in the stable category $\mathcal{SP}$ of CW spectra as it is defined in [1,35]

In the second paragraph we introduce $\mathbb{Z}_q$ coefficients (q integer) in the connective K-cohomology. We reformulate the results of [6] and show how they apply to connective K-theory.

In the third paragraph we relate the connective K-cohomology with the singular cohomology in the same way as L. Smith in [34].

Finally in the fourth paragraph we work out the Atiyah-Hirzebruch spectral sequence for connective K-cohomology and prove some results that we will need later.

1. Connective K-theory's spectrum

Let us consider the spectrum $K = (K_n, \sigma_n)_{n \in \mathbb{Z}}$ for K-theory. It is a periodic Ω-spectrum, $K_{2i} = BU \times \mathbb{Z}$ and $K_{2i+1} = U$ $(i \in \mathbb{Z})$ where $BU = \varinjlim_n BU(n)$, $BU(n)$ is the classifying space of the unitary group $U(n)$, $U = \varinjlim_n U(n)$. Bott's periodicity theorem says that exists a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega^2 BU$.

K^*, K-cohomology, is a multiplicative cohomology theory whose product is naturally induced by the tensor product of vector bundles. K can be made a ring spectrum in a unique way with a multiplication that induces the former one of K^* [7,17.35].

1.1.1. <u>Definition</u>:

Let $E = \{E_n, \varepsilon_n\}_{n\in\mathbb{Z}}$ be a spectrum. We say that the spectrum $\bar{E} = \{\bar{E}_n, \bar{\varepsilon}_n\}$ is the connective E spectrum if:

(i) $\bar{E}_n = E_n$ for $n \leq 0$.

(ii) $\forall\, n > 0 \quad \pi_i(\bar{E}_n) = 0$ for $0 \leq i < n$.

(iii) There is a function $f:\bar{E} \to E$ such that it induces isomorphisms $f_{n*}:\pi_i(\bar{E}_n) \to \pi_i(E_n)$ for all $n > 0$ and all $i \geq n$ and $f_n:\bar{E}_n \to E_n$ is the identity for $n \leq 0$.

Given a spectrum $E = \{E_n, \varepsilon_n\}$ there exists $\bar{E} = \{\bar{E}_n . \bar{\varepsilon}_n\}$, unique up to equivalence satisfying the above conditions [37].

1.1.2. <u>Remark</u>:

We recall that for $n > 0$ $\bar{E}_n$ is the fibre of a fibration $p_n:E_n \to G_n$ where G_n is a space whose homotopy groups are 0 in dimensions greater or equal to n and p_n induces isomorphisms $p^*:\pi_i(E_n) \to \pi_i(G_n)$ for $0 \leq i < n$. □

1.1.3. <u>Lemma</u>:

Given an (n-1)-connected space X, $n \geq 2$ and $g:X \to E_n$ it is possible to lift it to $\bar{E}_n$, i.e., there exists $\tilde{g}:X \to \bar{E}_n$ such that the diagram

$$\begin{tikzcd} & \bar{E}_n \arrow[d, "f_n"] \\ X \arrow[ur, "\exists\tilde{g}"] \arrow[r, "g"'] & E_n \end{tikzcd}$$

homotopy commutes. This map $\tilde{g}$ is unique up to homotopy.

Proof:

We can assume without loss of generality that f is an inclusion of a subcomplex of E_n and that X has cells only in dimension $\geq n$.

$\pi_r(E_n, \overline{E}_n) = 0$ for $r \geq n$ since we have the homotopy exact sequence:

$$\cdots \to \pi_r(\overline{E}_n) \underset{r \geq n}{\overset{\approx}{\to}} \pi_r(E_n) \longrightarrow \pi_r(E_n, \overline{E}_n) \to \underset{\substack{\| \\ 0 \\ r-1<n}}{\pi_{r-1}(\overline{E}_n)} \xrightarrow[r-1 \geq n]{\approx} \pi_{r-1}(E_n) \to \cdots$$

Moreover $\pi_0(\overline{E}_n) \approx 0$. Hence, g is homotopic to $\tilde{g}$, $\tilde{g}: X \to E_n$ mapping X in $\overline{E}_n$, and $\tilde{g}$ is unique up to homotopy [22]. $\square$

1.1.4. Proposition:

Let $E = \{E_n, \varepsilon_n\}$ be a ring spectrum with identity $\iota: S \to E$ and product $\mu: E \wedge E \to E$. Then the connective E-spectrum $\overline{E} = \{\overline{E}_n, \overline{\varepsilon}_n\}$ admits a unique, up to homotopy, structure of ring spectrum such that $f: \overline{E} \to E$ is a map of ring spectra.

Proof:

We have to prove the existence and unicity, up to homotopy, of the maps of ring spectra:

$\overline{\iota}: S \to \overline{E}$ (S denotes the sphere spectrum), $\overline{\mu}: \overline{E} \wedge \overline{E} \to \overline{E}$ such that the diagrams:

$$
\begin{tikzcd}
S \arrow[r, "\tau"] \arrow[dr, "\iota"'] & \bar{E} \arrow[d, "f"] \\
 & E
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{E}\wedge\bar{E} \arrow[r] \arrow[d, "f\wedge f"'] & \bar{E} \arrow[d, "f"] \\
E\wedge E \arrow[r] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
S\wedge\bar{E} \arrow[r, "\tau\wedge 1"] \arrow[dr, "\simeq"'] & \bar{E}\wedge\bar{E} \arrow[d] & \bar{E}\wedge S \arrow[l, "1\wedge\tau"'] \arrow[dl, "\simeq"] \\
 & E &
\end{tikzcd}
$$

(1) (2) (3)

homotopy commute where $\simeq$ denotes the natural homotopy equivalences.

The unit $\iota: S \to E$ is a function of spectra since S has no cofinal subspectrum contained in itself. Thus we lift each map $\iota_n: S^n \to E_n$ to $\bar{E}_n$. Such lifting exists and is unique up to homotopy [Lemma 1.1.3]. It defines a function of spectra $\tau: S \to \bar{E}$ that makes (1) homotopy commutative.

To define $\bar{\mu}$ we have to construct a function of spectra from a cofinal spectrum Γ of $\bar{E}\wedge\bar{E}$ to $\bar{E}$ so that $\mu\circ(f\wedge f)/_{\Gamma} = f\circ\bar{\mu}$.

As for all $n \in \mathbb{Z}$ $\pi_i(\bar{E}_n) = 0$ if $i < n$ there is a spectrum $G = (G_n, \sigma_n)$ with the (n-1) skeleton of G_n reduced to a point for all $n \geq 1$ and a function of spectra $\lambda: G \to \bar{E}$ that is a homotopy equivalence. We note that $\lambda\wedge\lambda : G\wedge G \to \bar{E}\wedge\bar{E}$ is still a homotopy equivalence. Now we take a function $\mu': F \to E$ defined on a cofinal subspectrum F of $E\wedge E$ representing μ. Then there exists a cofinal subspectrum $H = (H_n, C_n)$ of $G\wedge G$ that is mapped by $(f\wedge f)\circ(\lambda\wedge\lambda)$ in F. It can be chosen to have cells only in dimensions greater or equal to zero because $G\wedge G$ is equivalent to the naive smash product $G\wedge_{BC} G$ (B,C are infinite sets that form a partition of an ordered set A isomorphic to $\mathbb{N}\cup\{0\}$, i.e., $B\cup C = A$ and $B\cap C = \emptyset$) and the set of the stable cells of $G\wedge_{BC} G$ is the product of the set of stable cells of G by itself [36]. Hence, for all $r \in \mathbb{Z}$, $\pi_n(H_r) = 0$ if $r < n$ and so we can lift $(f\wedge f)\circ(\lambda\wedge\lambda)_r$ to $\bar{E}_r$ for each r, i.e.

exists a unique map up to homotopy, θ_r, such that $f_r \circ \theta_r = (\mu' \circ (f \wedge f) \circ (\lambda \wedge \lambda))_r$.

$$\begin{tikzcd}
H \arrow[rrrr, dashed, "\theta"] \arrow[rrr, "(f\wedge f)\,(\lambda\wedge\lambda)|_H"] \arrow[d, hook] & & & F \arrow[d, hook] & \overline{\overline{E}} \arrow[d] \\
G\wedge G \arrow[r, "\lambda\wedge\lambda", "\cong"'] & \overline{E}\wedge\overline{E} \arrow[rr, "f\wedge f"] & & E\wedge E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$\theta = \{\theta_r\}_{r \in \mathbb{Z}}$

For $r \leq 0$, $\theta_r = (\mu' \circ (f \wedge f) \circ (\lambda \wedge \lambda))_r$ because $\overline{\overline{E}}_r = E_r$. Thus, we can inductively change those maps (for $r > 0$) to get a function of spectra $\sigma = \{\sigma_r\}_{r \in \mathbb{Z}}$ that is, $\forall n \in \mathbb{Z}$, $\overline{\varepsilon}_n \circ \Sigma\sigma_n = \sigma_{n+1} \circ C_n$.

Let $\nu : G \wedge G \to \overline{\overline{E}}$ be the map of spectra that is the equivalence class of θ. We define:

$$\overline{\mu} = \nu \circ (\lambda \wedge \lambda)^{-1}$$

Then (2) commutes.

Obviously $\overline{\mu}$ is unique up to homotopy since all the constructions made are unique up to homotopy or equivalence of spectra.

To prove that (3) is commutative first we note that $S \wedge \overline{E}$ ($\overline{E} \wedge S$ as well) can be replaced by an homotopic equivalent spectrum with cells in dimensions greater or equal to zero (same method as above). Projecting the diagram (1) over E we have a prism with all faces commutative but (a) and (b):

$$\begin{tikzcd}
S\wedge\overline{E} \arrow[rr, "\tau\wedge 1"] \arrow[dr, "\overline{h}"'] \arrow[dd] \arrow[r, phantom, "(a)"] & & \overline{E}\wedge\overline{E} \arrow[dl, "\overline{\mu}"] \arrow[dd, dashed] \arrow[r, phantom, "(b)"] & \overline{E}\wedge S \arrow[l, "1\wedge\tau"'] \arrow[dll, "\overline{h}'"] \arrow[dd] \\
& \overline{E} \arrow[dd] & & \\
S\wedge E \arrow[rr, dashed, "\tau\wedge 1"] \arrow[dr, "h"'] & & E\wedge E \arrow[r, dashed, "1\wedge\tau"] \arrow[dl, "\mu"] & E\wedge S \arrow[dll, "h'"] \\
& E & &
\end{tikzcd}$$

$h, h', \overline{h}, \overline{h}'$ are natural homotopy equivalences

Composing the maps we obtain:

$$f\circ\overline{h} \simeq f\circ\overline{\mu}\circ(\tau\wedge 1)$$

$$f\circ\overline{h}' \simeq f\circ\overline{\mu}\circ(1\wedge\tau)$$

Using the unicity property, up to homotopy, of the liftings we get (a) and (b) homotopy commutative as desired. □

1.1.5. Corollary:

(i) If E is associative so is $\overline{E}$.

(ii) If E is commutative so is $\overline{E}$.

Proof:

We have to show that the diagrams (1) and (2)

$$\begin{tikzcd}
\overline{E}\wedge\overline{E}\wedge\overline{E} \arrow[r, "\overline{\mu}\wedge 1"] \arrow[d, "1\wedge\overline{\mu}"'] & \overline{E}\wedge\overline{E} \arrow[d, "\overline{\mu}"] \\
\overline{E}\wedge\overline{E} \arrow[r, "\overline{\mu}"] & \overline{E}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{E}\wedge\overline{E} \arrow[rr, "c"] \arrow[dr, "\overline{\mu}"'] & & \overline{E}\wedge\overline{E} \arrow[dl, "\overline{\mu}"] \\
& \overline{E} &
\end{tikzcd}$$

(1) (2)

homotopy commute to prove (i) and (ii) respectively.

C denotes a homotopy equivalence part of the smash product structure that interchanges factors. The method used in Proposition 1.1.5 to prove that (3) was commutative applies here straightforward. □

J.P. May proved in [2.8] a more general result in his category HS of spectra:

For a spectrum E, there exists one, and up to equivalence only one connective spectrum D (i.e., $\pi_{-i}(D) = 0$ for $i>0$) and a map $\theta : D \to E$ in HS such that $\pi_i(\theta)$ is an isomorphism for $i \geq 0$. If E is a ring spectrum then D admits a unique structure of ring spectrum so that θ is a map of ring spectra.

Furthermore he proved another result that we shall only prove for the spectrum bu:

If E is a connective ring spectrum, then the unique map $d : E \to H\,\pi_0 E$ in HS which realizes the identity map of $\pi_0 E$ is a map of ring spectra.

1.1.6. Definition:

$bu = (bu_n, \bar{\sigma}_n)_{n \in \mathbb{Z}}$ is the connective K spectrum, $j : bu \to K$ the associated map of spectra, k^* the connective K-cohomology. We note that $bu_0 = BU \times \mathbb{Z}$, $bu_1 \simeq U$, $bu_2 \simeq BU$ □

1.1.7. Proposition:

bu is a commutative, associative ring Ω-spectrum.

Proof:

It follows from Proposition 1.1.4 and Corollary 1.1.5 that bu is a commutative, associative ring spectrum with a multiplication inherited from $K = \{K_n, \sigma_n\}_{n \in \mathbb{Z}}$.

It remains to show that the adjoints of the structure maps $(\bar{\sigma}_n)_{n\in\mathbb{Z}}$ are homotopy equivalences. But since K is an Ω-spectrum and $(bu_n,\bar{\sigma}_n) = (K_n,\sigma_n)$ for $n \leq 0$ this is true for $n < 0$.

Suppose now $n \geq 0$. As $j:bu \to K$ is a function of spectra we have the following commutative diagram:

$$\begin{tikzcd} bu_n \arrow[r, "\bar{\sigma}'_n"] \arrow[d, "jn"] & \Omega bu_{n+1} \arrow[d, "\Omega j_{n+1}"] \\ K_n \arrow[r, "\sigma'_n"] & \Omega K_{n+1} \end{tikzcd}$$

where σ'_n is the adjoint of σ_n, $\Omega_{j_{n+1}}$ is induced by j_{n+1} in the obvious way. It induces the commutative diagram:

$$\begin{tikzcd} \pi_r(bu_n) \arrow[r, "(\bar{\sigma}'_n)_*"] \arrow[d, "(j_n)_*"] & \pi_r(\Omega bu_{n+1}) \arrow[d, "(\Omega j_{n+1})_*"] \\ \pi_r(K_n) \arrow[r, "(\sigma'_n)_*"] & \pi_r(\Omega K_{n+1}) \end{tikzcd}$$

For $r \geq n$ $(j_n)_*$, $(\Omega j_{n+1})_*$ are isomorphisms, for $0 \leq r < n$ $\pi_r(bu_n) = \pi_r(\Omega bu_{n+1}) = 0$ and for all $r \geq 0$ $(\sigma'_n)_*$ is an isomorphism. Hence, for all $r \geq 0$ $(\bar{\sigma}'_n)_*$ is an isomorphism.

Since the loop space of a CW complex has the homotopy type of a CW complex [30], $\bar{\sigma}'_n$ is an homotopy equivalence. □

1.1.8. Remark:

(i) $K^*(pt) = \mathbb{Z}[t,t^{-1}]$, the polynomial ring generated by the class of the reduced Hopf bundle $t^{-1} \in K^{-2}(pt)$ and its inverse [7]. Then $k^*(pt) = \mathbb{Z}[t^{-1}]$

(ii) Bott periodicity theorem says that there exists an homotopy equivalence $\nu: BU \times \mathbb{Z} \to \Omega^2 BU$ [7]. Taking the adjoint we have a map $\tilde{\nu}: S^2 \wedge BU \to BU$ that induces an isomorphism $K^i(X) \to K^{i-2}(X)$, X a space. $\tilde{\nu} \circ (1 \wedge j): S^2 \wedge bu \to BU$ lifts to a map $m_{t^{-1}}: S^2 \wedge bu \to bu$ that induces $m^*_{t^{-1}}: k^*(X) \to k^*(X)$, for any space X, a map of degree-2 that is the multiplication by t^{-1}. □

1.1.9. Lemma:

Let X be a based CW complex. Then $\tilde{k}^i(X) \approx \tilde{K}^i(X/X^{i-2})$, where X^{i-2} denotes the (i-2)th skeleton of X.

Proof:

Let us consider the k-cohomology long exact sequence of the pair (X^{i-2},X):

$$\cdots \to \tilde{k}^{i-1}(X^{i-2}) \to \tilde{k}^i(X/X^{i-2}) \to \tilde{k}^i(X) \to \tilde{k}^i(X^{i-2}) \to \cdots$$

$\tilde{k}^r(X^{i-2}) = 0$ for $r \geq i-1$ because bu_r is $(r-1)$ connected. By the exactness of the sequence we get $\tilde{k}^i(X/X^{i-2}) \approx \tilde{k}^i(X)$.

Since we are dealing with Ω-spectra it remains to prove that $[X/X^{i-2}, bu_i] \approx [X/X^{i-2}, K_i]$ ([,] denotes the based homotopy classes of based maps). As in 1.1.3. we consider the homotopy exact sequence of $bu_i \hookrightarrow K_i$:

$$\cdots \to \pi_r(bu_i) \underset{r \geq i}{\overset{\approx}{\to}} \pi_r(K_i) \to \pi_r(bu_i,K_i) \to \underset{\substack{\| \\ 0 \\ r-1<i}}{\pi_{r-1}(bu_i)} \underset{r-1 \geq i}{\overset{\approx}{\to}} \pi_{r-1}(K_i) \to \cdots$$

As before we get $\pi_r(bu_i,K_i)=0$ for $r \geq i$. For $r=i-1$, $\pi_r(K_i)=0$ and $\pi_r(bu_i)=0$ thus $\pi_{i-1}(bu_i,K_i)=0$. Using the same result as in 1.1.3. we obtain the required isomorphism. □

1.1.10. Proposition:

There is a map of ring spectra $\eta: bu \to H\mathbb{Z}$, $H\mathbb{Z}$ denotes the Eilenberg Maclane spectrum with integer coefficients, such that induces the homomorphism $\eta^*: k^*(pt) \to H^*(pt;\mathbb{Z})$ given by

$$\eta^*(a.t^{-n}) = \begin{cases} 0 & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}, \quad a \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\}.$$

Proof:

Since $\pi_i(bu) = 0$ for $i < 0$ and $\pi_0(bu) = \mathbb{Z}$, $H_0(bu) \approx \pi_0(bu) \approx \mathbb{Z}$ by the Hurewicz isomorphism. We take the cohomology class dual to the generator of $H_0(bu)$ image of the generator of $\pi_0(bu)$. It is represented by a map $\eta: bu \to H\mathbb{Z}$ that induces η^* as required.

It remains to show that it is a map of ring spectra. η(bu-unit) = $H\mathbb{Z}$-unit since in both the unit gives the generator of $\pi_0(bu)$ and $\pi_0(H\mathbb{Z})$ respectively. Let $\varepsilon: S \to bu$ be the bu-unit. Then $\eta\varepsilon$ is the $H\mathbb{Z}$ unit.

We need the following diagram to homotopy commute:

$$\begin{tikzcd} bu_\wedge bu \arrow[r, "\bar{\mu}"] \arrow[d, "\eta_\wedge \eta"'] & bu \arrow[d, "\eta"] \\ H\mathbb{Z}_\wedge H\mathbb{Z} \arrow[r, "\mu_H"] & H\mathbb{Z} \end{tikzcd}$$

μ_H denotes the ring product of $H\mathbb{Z}$.

Or, equivalently, $\bar{\mu}^*[\eta] = [\eta \times \eta] \in H^0(bu_\wedge bu)$.

We consider the following diagram:

$$\begin{tikzcd}
S_\wedge S \arrow[r, "\mu_1"] \arrow[d, "\varepsilon_\wedge\varepsilon"'] & S \arrow[d, "\varepsilon"] \\
bu_\wedge bu \arrow[r, "\bar{\mu}"] \arrow[d, "\eta_\wedge\eta"'] & bu \arrow[d, "\eta"] \\
H\mathbb{Z}_\wedge H\mathbb{Z} \arrow[r, "\bar{\mu}_H"] & H\mathbb{Z}
\end{tikzcd}$$

Where $\mu_1 : S_\wedge S \to S$ is the product map of the sphere spectrum, S, that is an isomorphism. The upper square commutes since the unit of a ring spectrum is a ring spectra map.

$\mu_1^*[\eta\varepsilon] = [\eta\varepsilon\times\eta\varepsilon]$ because μ_1^* is an isomorphism and each element generates $H^o(S^o)$. Then $\bar{\mu}^*[\eta] = [\eta\times\eta]$ since $\varepsilon^* : H^o(bu) \to H^o(S^o)$ is a ring isomorphism. □

2. Connective K-theory with $\mathbb{Z}_q$ coefficients.

We shall need later to consider k^* with $\mathbb{Z}_p$ coefficients where p is a prime. We want a natural multiplicative transformation $T : k^*(\ ) \to k^*(\ ;\mathbb{Z}_p)$ and a universal coefficient formula relating the two theories. Through this paragraph we recall the results of S. Araki and H. Toda [6] omitting the compactness condition on the space X and show how they work for connective K-theory.

Let h be a cohomology theory defined in the category of (finite) CW complexes, $\tilde{h}$ the corresponding reduced cohomology theory defined in the category of (finite) CW complexes with base point. We recall that there is a bijective correspondence between $h \leftrightarrow \tilde{h}$ [36]. To give an (associative, commutative) multiplication in h is equivalent to give an (associative, commutative) multiplication in $\tilde{h}$.

2.1. Definition:

Let X be a based CW complex, A a subcomplex. We define for all $i \in \mathbb{Z}$, $q \in \mathbb{N}$:

$$h^i(X,A;\mathbb{Z}_q) = h^{i+2}(X \times M_q, X \times * \cup A \times M_q)$$
$$\tilde{h}^i(X;\mathbb{Z}_q) = \tilde{h}^{i+2}(X_\wedge M_q),$$

where $*$ is the basepoint, M_q is the space obtained by attaching a 2-cell e^2 to S^1 by a map of degree q, i.e., $M_q = S^1 \cup_q e^2$. □

1.2.2. Definition:

Let L be a torsion free abelian group, E^* a cohomology theory. $E^* \otimes L$ is still a cohomology theory since, tensoring by L preserves the exact sequences. We define $E^*(-;L)$ to be $E^*(-) \otimes L$. □

Notation:

1. $\pi_q : M_q \to S^2$ is the map collapsing S^1 to a point

$\iota_q : S^1 \to M_q$ is the inclusion map.

The suspension map $\sigma_q : \tilde{h}^i(X;\mathbb{Z}_q) \to \tilde{h}^{i+1}(SX,\mathbb{Z}_q)$ is the composite of: $\tilde{h}^{i+2}(X\wedge M_q) \xrightarrow[q]{\sigma} \tilde{h}^{i+3}(X\wedge M_q \wedge S^1) \xrightarrow{1\wedge T} \tilde{h}^{i+3}(X\wedge S^1 \wedge M_q) \to \tilde{h}^{i+3}(SX\wedge M_q)$

The reduction mod q $\rho_q : \tilde{h}^i(X) \to \tilde{h}^i(X;\mathbb{Z}_q)$ is the composite of: $\tilde{h}^i(X) \xrightarrow{\sigma^2} \tilde{h}^{i+2}(X\wedge S^2) \xrightarrow{(1\wedge \pi_q)^*} \tilde{h}^{i+2}(X\wedge M_q)$

The Bockstein homomorphism $\delta_q : \tilde{h}^i(X,\mathbb{Z}_q) \to \tilde{h}^{i+1}(X)$ is the composite of: $\tilde{h}^{i+2}(X\wedge M_q) \xrightarrow[q]{(1\wedge i)^*} \tilde{h}^{i+1}(X\wedge S^1) \xrightarrow{\sigma^{-1}} \tilde{h}^i(X)$

2. Let Y,Z be two based spaces. {Y,Z} denote the stable homotopy classes of maps from Y to Z preserving the base point.

Let X be a based space, $\alpha \in \{Y,Z\}$. Suppose that Y,Z are compact. α induces a map $\alpha^{**} : \tilde{h}^*(X\wedge Z) \to \tilde{h}^*(X\wedge Y)$ defined as follows: α is represented by a map $f : Y\wedge S^i \to Z\wedge S^i$ for some $i \in \mathbb{N}$. α^{**} is the composite of:

$h^r(X\wedge Z) \xrightarrow{\sigma^i} \tilde{h}^{r+i}(X\wedge Z\wedge S^i) \xrightarrow{(1\wedge f)^*} \tilde{h}^{r+i}(X\wedge Y\wedge S^i) \xrightarrow{(\sigma^i)^{-1}} \tilde{h}^r(X\wedge Y)$ for all $r \geq 0$.

3. $\eta \in \{S^2,S^1\}$ and $\nu \in \{S^4,S^1\}$ are the stable classes of the Hopf maps $\eta : S^3 \to S^2$ and $\nu : S^7 \to S^4$ respectively.

4. $T : X\wedge Y \to Y\wedge X$ is the map "switching factors" $\square$

Let X be a finite CW complex. The cofibration

$$X \wedge S^1 \xrightarrow{1 \wedge i_q} X \wedge M_q \xrightarrow[q]{1 \wedge \pi} X \wedge S^2$$

induces the long exact sequence:

$$\ldots \to \tilde{h}^i(X) \xrightarrow{q} \tilde{h}^i(X) \xrightarrow[q]{\rho} \tilde{h}^i(X;\mathbb{Z}_q) \xrightarrow[q]{\delta} \tilde{h}^{i+1}(X) \to \ldots$$

where q denotes the homomorphism "multiplication by q". It splits in short exact sequences:

$$0 \to \tilde{h}^i(X) \otimes \mathbb{Z}_q \xrightarrow{\rho'} \tilde{h}^i(X;\mathbb{Z}_q) \xrightarrow{\delta'} \mathrm{Tor}(\tilde{h}^{i+1}(X);\mathbb{Z}_q) \to 0$$

where the maps ρ' and δ' are induced in the obvious way by ρ_q and δ_q

A - A sufficient condition for the splitting of those short exact sequences is that $\eta^{**} = 0$ in $\tilde{h}^*$ or $q \not\equiv 2 \pmod 4$. Then we have an universal coefficient formula:

$$\tilde{h}^i(X;\mathbb{Z}_q) \cong \tilde{h}^i(X) \otimes \mathbb{Z}_q \oplus \mathrm{Tor}\,(\tilde{h}^{i+1}(X,\mathbb{Z}_q)), X \text{CW complex}.$$

It implies too that $\tilde{h}^*(X;\mathbb{Z}_q)$ is a $\mathbb{Z}_q$ module.

Suppose that $\tilde{h}$ is a multiplicative cohomology theory, i.e. we have a map $\mu: \tilde{h}^i(X) \otimes \tilde{h}^j(Y) \to \tilde{h}^{i+j}(X \wedge Y)$ for all $i,j \in \mathbb{Z}$, X, Y based CW complexes such that:

(i) μ is linear.

(ii) μ is a natural with respect to both variables.

(iii) μ has a bilateral unit $1 \in \tilde{h}^{o}(S^{o})$, that is,

$\mu(1 \otimes x) = x = \mu(x \otimes 1)$ $(x \in \tilde{h}^{i}(X)$.

(iv) μ is compatible with the suspension isomorphism σ, that is, $\sigma(\mu(X \otimes y) = (1_\wedge T)^{*}\mu(\sigma \times \otimes y) = (-1)^{i}\mu(x \otimes \sigma y), x \in \tilde{h}^{i}(X)$, $y \in \tilde{h}^{*}(Y)$, $T: S'_\wedge Y \to Y_\wedge S'$.

Moreover μ is;

(v) associative if $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$.

(vi) commutative if $T^{*}\mu(x \otimes y) = (-1)^{ij}\mu(y \otimes x), x \in \tilde{h}^{i}(X)$, $y \in \tilde{h}^{j}(Y)$.

This multiplication induces two multiplications:

$\mu_R : \tilde{h}^{i}(X;\mathbb{Z}_q) \otimes \tilde{h}^{j}(Y) \to \tilde{h}^{i+j}(X_\wedge Y;\mathbb{Z}_q)$ given by the composite

$\tilde{h}^{i+2}(X_\wedge M_q) \otimes \tilde{h}^{j}(Y) \xrightarrow{\mu} \tilde{h}^{i+j+2}(X_\wedge M_{q\wedge} Y) \xrightarrow{(1_\wedge T)^{*}} \tilde{h}^{i+j+2}(X_\wedge Y_\wedge M_q)$

$T: M_{q\wedge} Y \rightleftarrows Y_\wedge M_q$.

$\mu_* : \tilde{h}^{i}(X) \otimes \tilde{h}^{j}(Y;\mathbb{Z}_q) \to \tilde{h}^{i+j}(X_\wedge Y;\mathbb{Z}_q)$ given by μ:

$\tilde{h}^{i}(X) \otimes \tilde{h}^{j+2}(Y_\wedge M_q) \to \tilde{h}^{i+j+2}(X_\wedge Y_\wedge M_q)$.

They satisfy similar properties [6].

We want a multiplication $\mu_q : \tilde{h}^{i}(X;\mathbb{Z}_q) \otimes \tilde{h}^{j}(Y;\mathbb{Z}_q) \to \tilde{h}^{i+j}(X_\wedge Y;\mathbb{Z}_q)$ satisfying

1). (i) - (iv)

2). compatible with μ_R and μ_L through reduction mod q, i.e.

$$\mu_R = \mu_q (1 \otimes \rho_q),\ \mu_L = \mu_q (\rho_q \otimes 1).$$

We note that the properties of μ and, hence, of μ_R, μ_L imply that $\mu_q(\rho_q \otimes \rho_q) = \rho_q\, \mu$, that is:

$$\begin{tikzcd} \tilde{h}^i(X) \otimes \tilde{h}^j(Y) \arrow[r, "\mu"] \arrow[d, "\rho_q \otimes \rho_q"'] & \tilde{h}^{i+j}(X_\wedge Y) \arrow[d, "\rho_q"] \\ \tilde{h}^i(X;\mathbb{Z}_q) \otimes \tilde{h}^j(Y;\mathbb{Z}_q) \arrow[r, "\mu_q"] & \tilde{h}^{i+j}(X_\wedge Y;\mathbb{Z}_q) \end{tikzcd}$$

commutes.

3). δ_q is a derivation, i.e. $\delta_q\, \mu_q(x \otimes y) =$

$$\mu_q(\delta_q x \otimes y) + (-1)^i \mu_q(x \otimes \delta_q y),\ x \in \tilde{h}^i(X),\ y \in h^*(Y).$$

μ_q doesn't always exist and when it exists it is not unique.

B - A sufficient condition for the existence of an associative multiplication μ_q in $\tilde{h}^*(\ ;\mathbb{Z}_q)$ compatible with a given associative, commutative multiplication μ in $\tilde{h}$ is that $\eta^{**} = 0$ and $\nu^{**} = 0$ [6]

Applying A and B to connective K-theory we obtain:

1.2.3. <u>Proposition</u>:

(i) Let X be a CW complex. Then for all $q \geq 1$. $\tilde{k}^*(X;\mathbb{Z}_q)$ is a $\mathbb{Z}_q$-module and $k^i(X;\mathbb{Z}_q) = k^i(X) \otimes \mathbb{Z}_q \oplus \mathrm{Tor}\,(\tilde{k}^{i+1}(X),\mathbb{Z}_q)$ for all $i\in\mathbb{Z}$.

(ii) The multiplication of $\tilde{k}^*$ induces an associative multiplication in $\tilde{k}^*(\ ;\mathbb{Z}_q)$

Proof:

By A and B it is enough to prove that given $\alpha \in \{S^{n+r},S^n\}$, $n \geq 0$, $r > 0$ then $\alpha^{**}:k^*(X_\wedge S^r) \to \tilde{k}^*(X_\wedge S^{n+r})$ is the zero map.

As $\tilde{k}^*(S^n)$ is a $Z[t^{-1}]$ free module, using a special case of the Künneth theorem for generalized multiplicative cohomology theories [35], we have $\tilde{k}^i(X_\wedge S^r) \cong (\tilde{k}^*(X) \otimes k^*(S^r))^i$; thus we only have to show that $\alpha^*:\tilde{k}^i(S^r) \to \tilde{k}^i(S^{n+r})$ is the zero map. This follows from the fact that $\{S^{n+r},S^n\}$ has finite order [33] and $\tilde{k}^i(S^{n+r})$ is $\mathbb{Z}$ or 0. □

3. L. Smith's exact sequence

L. Smith proved in [34] that given a finite CW complex X exists a natural exact sequence.

$$0 \to Z \otimes_{\mathbb{Z}[t]} k_*(X) \xrightarrow[\tilde{\eta}_*]{} H_*(X;\mathbb{Z}) \to \mathrm{Tor}_{1,*}^{\mathbb{Z}[t]}(\mathbb{Z},k_*(X)) \to 0$$

where k_* is the connective K-homology, $\tilde{\eta}_*$ is the map induced by $\eta_*:k^*(X) \to H_*(X;\mathbb{Z})$ and $\mathbb{Z}$ is viewed as a $\mathbb{Z}[t]$ module via the augmentation $\eta_*:\mathbb{Z}[t] = k_*(pt) \to H_*(pt) = \mathbb{Z}$. We are going to reformulate the result for k^* with R coefficients, R a torsion free abelian group or $R = \mathbb{Z}_q$ $(q \geq 1)$.

1.3.1. Theorem

Let X be a CW complex. Then there is an exact sequence:

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}[t^{-1}]} k^*(X) \xrightarrow[\tilde{\eta}_*]{} H^*(X;\mathbb{Z}) \to \mathrm{Tor}_{1,*}^{\mathbb{Z}[t^{-1}]}(\mathbb{Z};k^*(X)) \to 0$$

where $\tilde{\eta}^*$ is induced by $\eta^*: k^*(X) \to H^*(X;\mathbb{Z})$ and $\mathbb{Z}$ is viewed as a $\mathbb{Z}[t^{-1}]$ module via the augmentation $\mathbb{Z}[t^{-1}] = k^*(pt) \xrightarrow{\eta^*} H^*(pt,\mathbb{Z}) = \mathbb{Z}$

<u>Proof</u>

Let $m_{t^{-1}}: S^2{\wedge}bu \longrightarrow bu$ be the map given by Bott periodicity theorem. We consider the cofibration sequence of spectra:

$$S^2{\wedge}bu \xrightarrow{m_{t^{-1}}} bu \xrightarrow{\phi} X$$

where X is a spectrum homotopy equivalent to $bu \cup_{m_{t^{-1}}} C(S^2{\wedge}bu)$, ϕ is the "inclusion" map.

<u>Claim</u>:

X is homotopy equivalent to $H\mathbb{Z}$ and we have a homotopy commutative diagram:

$$\begin{tikzcd} & X \arrow[d, "\psi"] \\ bu \arrow[ur, "\phi"] \arrow[r, "\eta"'] & H\mathbb{Z} \end{tikzcd}$$

I

<u>Proof of the claim</u>:

Considering the exact sequence of the cofibration we get the exact triangle:

$$\begin{tikzcd} \pi_*(S^2{\wedge}bu) \arrow[rr, "(m)_{t^{-1}}"] & & \pi_*(bu) \arrow[dl, "\phi_*"] \\ & \pi_*(X) \arrow[ul, "\delta_*"] & \end{tikzcd} \quad \text{or, equivalently,} \quad \begin{tikzcd} \mathbb{Z}[t^{-1}] \arrow[rr, "m_{t^{-1}}"] & & \mathbb{Z}[t^{-1}] \arrow[dl, "\phi_*"] \\ & \pi_*(X) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

$m_{t^{-1}}$ is injective and the cokernel is $\mathbb{Z}$. Hence

$$\pi_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

By Hurewicz's theorem $\pi_0(X) \approx H_0(X) \approx \mathbb{Z}$. We take a map $\psi : X \to H\mathbb{Z}$ representative of the cohomology class dual to the generator of $H_0(X)$ corresponding to that one of $\pi_0(bu)$. ψ induces isomorphisms in the homotopy groups of the two spectra, hence is a homotopy equivalence.

The diagram I homotopy commutes since the diagram

$$\begin{tikzcd} \pi_*(bu) \arrow[rr, "\phi_*"] \arrow[dr, "\eta_*"'] & & \pi_*(X) \arrow[dl, "\psi_*"] \\ & \pi_*(H\mathbb{Z}) & \end{tikzcd}$$

commutes and by the definition of the two maps ψ and η.

We have got a cofibration: $S^2 \wedge bu \xrightarrow[t^{-1}]{m} bu \xrightarrow{\eta} H\mathbb{Z}$

that for every CW complex X induces an exact triangle.

$$\begin{tikzcd} k^*(X) \arrow[rr, "m"', "t^{-1}"] & & k^*(X) \arrow[dl, "\eta^*"] \\ & H^*(X;\mathbb{Z}) \arrow[ul, "\delta^*"] & \end{tikzcd}$$

that gives the long exact sequence for $i \geq 2$:

$$\text{II} \quad \ldots \to k^i(X) \xrightarrow[t^{-1}]{m^i} k^{i-2}(X) \xrightarrow{\eta^*} H^{i-2}(X;\mathbb{Z}) \xrightarrow{\delta^*} k^{i+1}(X) \quad \ldots$$

It splits in short exact sequences:

$$0 \longrightarrow \operatorname{coKer} m^i_{t^{-1}} \xrightarrow{\tilde{\eta}^*} H^{i-2}(X;\mathbb{Z}) \xrightarrow{\delta^*} \operatorname{Ker} m^{i+3}_{t^{-1}} \longrightarrow 0$$

The theorem follows from the following lemma:

1.3.2. Lemma:

Let M be a $\mathbb{Z}[t^{-1}]$ module and $m_{t^{-1}}: M \longrightarrow M$ multiplication by t^{-1}. Then $\operatorname{coKer} m_{t^{-1}} = \mathbb{Z} \otimes_{\mathbb{Z}[t^{-1}]} M$, $\operatorname{Ker} m_{t^{-1}} = \operatorname{Tor}_{1,*}^{\mathbb{Z}[t^{-1}]}(\mathbb{Z},M)$ where $\mathbb{Z}$ is a $\mathbb{Z}[t^{-1}]$ module via $\alpha:\mathbb{Z}[t^{-1}] \to \mathbb{Z}$ given by

$$\alpha(rt^{-i}) = \begin{cases} r & i = 0 \\ 0 & i > 0 \end{cases} \qquad r \in \mathbb{Z},\ i \in \mathbb{N} \cup \{0\}.$$

Proof:

The exact sequence

$$0 \to \mathbb{Z}[t^{-1}] \xrightarrow{m_{t^{-1}}} \mathbb{Z}[t^{-1}] \xrightarrow{\eta^*} \mathbb{Z} \to 0$$

yields tensoring by M the following exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow \operatorname{Tor}_{1,*}^{\mathbb{Z}[t^{-1}]}(\mathbb{Z},M) & \longrightarrow & \mathbb{Z}[t^{-1}] \otimes_{\mathbb{Z}[t^{-1}]} M & \longrightarrow & \mathbb{Z}[t^{-1}] \otimes_{\mathbb{Z}[t^{-1}]} M & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}[t^{-1}]} M & \to & 0 \\ & & \downarrow \wr & & \downarrow \wr & & & & \\ & & M & \xrightarrow{m_{t^{-1}}} & M & & & & \end{array}$$

Thus, the result follows. □

It splits in short exact sequences:

$$0 \longrightarrow \text{coKer } m^i_{t^{-1}} \xrightarrow{\tilde{\eta}^*} H^{i-2}(X;\mathbb{Z}) \xrightarrow{\delta^*} \text{Ker } m^{i+3}_{t^{-1}} \longrightarrow 0$$

The theorem follows from the following lemma:

1.3.2. Lemma:

Let M be a $\mathbb{Z}[t^{-1}]$ module and $m_{t^{-1}}: M \longrightarrow M$ multiplication by t^{-1}. Then $\text{coKer } m_{t^{-1}} = \mathbb{Z} \otimes_{\mathbb{Z}[t^{-1}]} M$, $\text{Ker } m_{t^{-1}} = \text{Tor}_{1,*}^{\mathbb{Z}[t^{-1}]}(\mathbb{Z},M)$ where $\mathbb{Z}$ is a $\mathbb{Z}[t^{-1}]$ module via $\alpha:\mathbb{Z}[t^{-1}] \to \mathbb{Z}$ given by

$$\alpha(rt^{-i}) = \begin{cases} r & i = 0 \\ 0 & i > 0 \end{cases} \qquad r \in \mathbb{Z},\ i \in \mathbb{N} \cup \{0\}.$$

Proof:

The exact sequence

$$0 \to \mathbb{Z}[t^{-1}] \xrightarrow{m_{t^{-1}}} \mathbb{Z}[t^{-1}] \xrightarrow{\eta^*} \mathbb{Z} \to 0$$

yields tensoring by M the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \text{Tor}_{1,*}^{\mathbb{Z}[t^{-1}]}(\mathbb{Z},M) \arrow[r] & \mathbb{Z}[t^{-1}] \otimes_{\mathbb{Z}[t^{-1}]} M \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}[t^{-1}] \otimes_{\mathbb{Z}[t^{-1}]} M \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \otimes_{\mathbb{Z}[t^{-1}]} M \arrow[r] & 0 \\ & & M \arrow[r, "m_{t^{-1}}"'] & M & & \end{tikzcd}$$

Thus, the result follows. □

1.3.3. Corollary:

Let X be a CW complex. Then we have the following exact sequences:

(i) $O \to \mathbb{Z}_q \otimes_{\mathbb{Z}_q[t-1]} \tilde{k}^*(X;\mathbb{Z}_q) \xrightarrow[\tilde{\eta}_{q*}]{} \tilde{H}^*(X;\mathbb{Z}_q) \to \mathrm{Tor}_{1,*}^{\mathbb{Z}_q[t^{-1}]}(\mathbb{Z}_q, \tilde{k}^*(X;\mathbb{Z}_q)) \to O$

where $q \geq 1$, $\tilde{\eta}_q^*$ is induced by $\eta^*: \tilde{k}^*(X \wedge M_q) \to \tilde{H}^*(X \wedge M_q)$

(ii) $O \to L \otimes_{L[t^{-1}]} k^*(X;L) \xrightarrow[\tilde{\eta}_L^*]{} H^*(X;L) \longrightarrow \mathrm{Tor}_{1,*}^{L[t^{-1}]}(L; k^*(X;L)) \longrightarrow O$

where L is a free abelian group, $\tilde{\eta}_L^*$ is induced by $\eta_* \otimes 1: k^*(X) \otimes L \to H^*(X;\mathbb{Z}) \otimes L$.

Proof:

(i) We can rewrite the sequence II of the proof of Theorem 1.3.1 for the space $X \wedge M_q$ and the reduced cohomology theories. Then we obtain:

$$\cdots \to \tilde{k}^{i+2}(X \wedge M_q) \xrightarrow{m_{t^{-1}}^{i+2}} \tilde{k}^i(X \wedge M_q) \xrightarrow{\eta^*} \tilde{H}^i(X \wedge M_q) \xrightarrow{\delta^*} \tilde{k}^{i+3}(X \wedge M_q) \to \cdots$$

that, as before, splits in short exact sequences:

$$O \to \mathrm{coKer}\, m_{t^{-1}}^{i+2} \longrightarrow \tilde{H}^i(X \wedge M_q) \to \mathrm{Ker}\, m_{t^{-1}}^{i+3} \longrightarrow O$$

As we have seen in last paragraph (1.2.3) $\tilde{k}^*(X;\mathbb{Z}_q)$ is a $\mathbb{Z}_q$ module, hence a $\mathbb{Z}_q[t^{-1}]$ module. Then the Lemma 1.3.2 applies here and we get the required exact sequence.

(ii) As L is torsion free, tensoring by L preserves exact sequences. Then we have:

$$O \to \mathrm{coKer}\, m_{t^{-1}}^* \otimes L \xrightarrow{\eta^* \otimes 1} H^*(X;\mathbb{Z}) \otimes L \xrightarrow{\delta^* \otimes 1} \ker m_{t^{-1}}^* \otimes L \to O$$

exact or, equivalently,

$$0 \to \mathrm{coKer}(m^L_{t^{-1}})^* \xrightarrow{\eta^*_L} H^*(X;L) \xrightarrow{\delta^*_L} \mathrm{Ker}\ (m^L_{t^{-1}})^* \to 0$$

where $(m^L_{t^{-1}})^*: k^*(X;L) \to k^*(X;L), \tilde{\eta}^*_L, \delta^*_L$ are the obvious maps induced by $m^*_{t^{-1}}$, $\tilde{\eta}^*$, δ^* respectively.

Using the exact sequence:

$$0 \to L[t^{-1}] \xrightarrow{m^L_{t^{-1}}} L[t^{-1}] \to L \to 0$$

we get, as before:

$$L \otimes_{L[t^{-1}]} k^*(X;L) \approx \mathrm{coKer}\ (m^L_{t^{-1}})^*$$

$$\mathrm{Tor}^{L[t^{-1}]}_{1,*}\ (L, k^*(X;L)) \approx \mathrm{Ker}(m^L_{t^{-1}})^* \qquad \square$$

4. Spectral sequences and connective K-theory

Let X be a compact CW complex of dimension n.

We are going to consider the H(p,q) system [14] associated to the filtration of X by its skeleta for a cohomology theory h^*. Then we have:

$$\phi = X^{-1} \subset X^0 \subset \ldots \subset X^n = X$$

$$H(p,q) = h^*(X^{q-1}, X^{p-1}),\ q \geq p$$

There is a natural (in the space X and in the cohomology theory h^*) spectral sequence $(E^{**}_r(X), d_r)_{r\geq 1}$ bigraded, with $d_r: E^{p,q}_r(X) \longrightarrow E^{p+r,q-r+1}_r(X)$ a differential convergent to $h^*(X)$.

We have:

$$E_{r+1}^{p,q}(X) = \mathrm{Ker}\,(d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X) / \mathrm{Im}(d_r : E_r^{p-r,q+r-1}(X) \to E_r^{p,q}(X))$$

$$E_2^{p,q}(X) \approx H^p(X; h^q(pt)) \ [9,34]$$

As dim $X = N$, $d_r = 0$ for $r > N$ and $E_{N+1}^{**} = \ldots = E_\infty^{**}(X)$

$h^m(X)$, $m \in \mathbb{Z}$, has a decreasing filtration given by:

$$0 = F_{N+1}(h^m(X)) \subset \ldots \subset F_1(h^m(X)) \subset F_0(h^m(X)) = h^m(X)$$

where $F_q(h^m(X)) = \mathrm{Ker}\,[h^m(X) \to h^m(X^{q-1})]$. Moreover,

$E_\infty^{p,q}(X) \approx F_p(h^{p+q}(X)) / F_{p+1}(h^{p+q}(X))$ and we have the following extension short exact sequences:

$$0 \to F_{p+1}(h^{p+q}(X)) \to F_p(h^{p+q}(X)) \to E_\infty^{p,q}(X) \to 0$$

Since $F_{N+1}(h^m(X)) = 0$, $F_N(h^m(X)) \approx E_\infty^{N,m-N}(X)$

This is called the Atiyah-Hirzebruch spectral sequence. All the differentials in this spectral sequence are torsion-valued [9,18]. It behaves well with respect to products in the sense that if h^* is a multiplicative cohomology theory its multiplication induces a multiplication "$\times$" in $(E_r^{**}, d_r)_{r\geq 1}$ so that E_r^{**} is a bigraded ring, d_r a derivation (for all $r \in \mathbb{N}$), that is:

X,Y compact CW complexes, $E_r^{p,q}(X) \otimes E_r^{p',q'}(Y) \xrightarrow{\times} E_r^{p+p',q+q'}(X \times Y)$

For $x \in E_r^{p,q}(X)$, $y \in E_r^{p',q'}(Y)$, $d_r(x \times y) = d_r(x) \times y + (-1)^{p+q} x \times d_r(y)$

In the E_2-term this map is the usual cohomology cross product. It respects also the filtration structure of h^*, that is, gives a map:

$$F_p(h^i(X)) \otimes F_{p'}(h^j(Y)) \to F_{p+p'}(h^{i+j}(X \times Y))$$

that agrees with the h^* product [35,16]

1.4.1. Remarks:

(i) The spectral sequence mentioned above was first considered by Atiyah and Hirzebruch in [9] for K-theory. It is compatible with the Bott isomorphism. This means that multiplication by t^{-1}, the canonical generator of $K^{-2}(pt)$, induces an isomorphism in the spectral sequence. Its behaviour with respect to products was first conjectured in [9]. Furthermore this spectral sequence can be extended to the category of CW complexes [21].

(ii) From now on we shall work on the homotopy category of *compact* CW complexes unless otherwise stated. Suppose that h^* is a cohomology theory defined in this category, associated to a spectrum $h = (h_n)_{n\in\mathbb{Z}}$. Let $'h^*$ be the connective h-cohomology, $'h = ('h_n)_{n\in\mathbb{Z}}$ the connective h-spectrum, $f: 'h \to h$ the map given in 1.1.1. We denote the two Atiyah-Hirzebruch spectral sequences of converging to $'h^*$, h^* by $('E_r^{**}, d_r')_{r\geq 1}$, $(E_r^{**}, d_r)_{r\geq 1}$ and the filtrations by $('F_p^*)_{p\in\mathbb{Z}}$, $(F_p^*)_{p\in\mathbb{Z}}$ respectively. The map $f: 'h \to h$

induces the maps $f^*: {}'h^* \longrightarrow h^*$, $f_r^{**}: {}'E_r^{**} \longrightarrow E_r^{**}$ $(r \geq 2)$. In particular for a space X, ${}'E_r^{p,q}(X) = 0$ for $q > 0$, $r \geq 2$ since ${}'E_2^{p,q}(X) = H^p(X; {}'h^q(pt))$. This implies $F_i(h^i(X) = h^i(X)$. We have also ${}'h^i(X) = 0$ for $i > \dim X$. □

Notation:

When there will be no possible confusion about the space X we shall write E_r^{**}, F_*^n for $E_r^{**}(X), F_*^n(X)$ respectively.

1.4.2. Proposition:

Let X be a compact CW complex. Then:

(i) $f_s^{**}: {}'E_s^{p,q} \to E_s^{p,q}$ is an isomorphism for $q \leq -\dim X + 1$

(ii) If $d_r = 0$ for $r > s$ then $f^*/_{F_n(h^m(X))}$ is an isomorphism onto $F_n(h^m(X))$ for all $m \in \mathbb{Z}$, $n \quad m+s-1$.

Proof:

(i) We are going to prove by induction on $r \geq 2$ a more general result:

* $f_r^{**}: {}'E_r^{p,q} \longrightarrow E_r^{p,q}$ is surjective for $-r+3 \leq q \leq 0$, isomorphism for $q \leq -r+2$

This and the fact that the differentials d_r are zero for $r > \dim X$ gives (i).

The inductive hypothesis is trivially verified for $r = 2$

since $'h^q(pt) = \begin{cases} h^q(pt) & \text{for } q \le 0, \\ 0 & \text{for } q > 0 \end{cases}$ $'E_2^{p,q} = \begin{cases} H^p(X;'h^q(pt)) & \text{for } q \le 0 \\ 0 & \text{for } q > 0 \end{cases}$

$$= \begin{cases} H^p(X;h^q(pt)) = E_2^{p,q} & \text{if } q \le 0 \\ 0 & \text{if } q > 0 \end{cases}$$

Suppose now that ⊛ is true for $r = s$. We have the commutative diagram:

$$\begin{tikzcd}
'E_s^{p-s,q+s-1} \arrow[r, "d'_s"] \arrow[d, "f_s^{**}"] & 'E_s^{p,q} \arrow[r, "d'_s"] \arrow[d, "f_s^{**}"] & 'E_s^{p+s,q-s+1} \arrow[d, "f_s^{**}"] \\
E_s^{p-s,q+s-1} \arrow[r, "d_s"] & E_s^{p,q} \arrow[r, "d_s"] & E_s^{p+s,q-s+1}
\end{tikzcd}$$

If $q \le -(s+1) + 2 = -s + 1$, the two right-hand vertical arrows are isomorphisms and the left-hand arrow is surjective $(q+s-1 \le 0)$ by induction. Then $\mathrm{Ker}\,[d'_s:'E_s^{p,q} \to 'E^{p+s,q-s+1}]$ is mapped isomorphically onto $\mathrm{Ker}\,[d_s:E_s^{p,q} \to E_s^{p+s,q-s+1}]$ and
$f_s^{**}[\mathrm{Im}\, d'_s : 'E_s^{p-s,q+s-1} \longrightarrow 'E_s^{p,q}] = \mathrm{Im}[d_s : E_s^{p-s,q+s-1} \longrightarrow E_s^{p,q}]$
Thus $f_{s+1}^{**}:\ 'E_{s+1}^{p,q} \to E_{s+1}^{p,q}$ is an isomorphism.

If $-(s+1)+3 \le q \le 0$ $'E_s^{p,q+s-1} = 0 (q+s-1 > 0)$, $f_s^{**}:'E_s^{p,q} \to E_s^{p,q}$ is surjective and $f_s^{**}:'E_s^{p+s,q-s+1} \to E_s^{p+s,q-s+1}$ is an isomorphism by induction. Then f_s^{**} maps $'E_{s+1}^{p,q} = \mathrm{Ker}\,[d'_s:'E_s^{p,q} \to 'E^{p+s,q-s+1}]$ onto $\mathrm{Ker}\,[d_s:E_s^{p,q} \to E_s^{p+s,q-s+1}]$. Hence $f_{s+1}^{**}:'E_{s+1}^{p,q} \to E_{s+1}^{p,q}$ is surjective.

(ii) If $d_r = 0$ for $r > s$ the proof of (i) implies that $f_\infty^{**}: {}'E_\infty^{p,q} \to E_\infty^{p,q}$ is surjective for $-s+2 \le q \le 0$ and it is an isomorphism for $q \le -s+1$.

Now we consider the extension exact sequences and the commutative diagram for all $m \in Z$.

$$\begin{tikzcd}
0 \arrow[r] & {}'F^m_{p+1} \arrow[r] \arrow[d, "f^*"] & {}'F^m_p \arrow[r] \arrow[d, "f^*"] & {}'E_\infty^{p,m-p} \arrow[r] \arrow[d, "f_\infty^{**}"] & 0 \\
0 \arrow[r] & F^m_{p+1} \arrow[r] & F^m_p \arrow[r] & E_\infty^{p,m-p} \arrow[r] & 0
\end{tikzcd}$$

f_∞^{**} is an isomorphism for $m - p \le - s +1$ or, equivalently, $p \ge m + s - 1$. Since $F^m_N \approx E_\infty^{N,m-N}$ $(N = \dim X)$ $f^*: {}'F^m_N \to F^m_N$ is an isomorphism if $m - N \le - s + 1$. Using decreasing induction on $p \le N$, supposing it always greater or equal to $m + s - 1$, and the 5-lemma we get the result. □

1.4.3. Remark:

In the cases of connective and usual K-theory we have a special case of the Atiyah-Hirzebruch spectral sequence: for all $p \in \mathbb{Z}$, $r \ge 2$, $E_r^{p,q} = 0$ if q is odd and all the differentials of even degrees are zero since $K^q(pt) = 0 = k^q(pt)$ for q odd. Then we have $F^i_{n-1} = F^i_n$ if $n-i$ even, $F^i_n = F^i_{n+1}$ if $n-i$ odd, where $F^i_* = F_*(h^i(X))$ with $h^* = K^*$ or k^*. $t^{-1} \in K^{-2}(pt) = k^{-2}(pt)$ acts on the following way:

$$m_{t^{-1}}(F^i_j) \subset F^{i-2}_j.$$

1.4.4. Remark:

When we consider $\mathbb{Z}_q$ coefficients (in k^* or K^*) we have a multiplicative map of spectral sequences $\rho_q^{**}:E_r^{**}(\ ) \to E_r^{**}(\ ;\mathbb{Z}_q)$ induced by the reduction homomorphism $\rho_q:h^*(\ ) \to h^*(\ ;\mathbb{Z}_q)$. For $r = 2, \rho_q^{**}$ is the usual "reduction mod q" map for ordinary cohomology.

Also we will need to consider $k^*(\ ;L)$ for $L = Q(P)$ where P is a set of prime numbers, Q(P) is the quotient ring of $\mathbb{Z}$ with respect to the multiplicative subset generated by P. Let L be a torsion free abelian group. We have defined $h^*(-;L) = h^*(-) \otimes L$(1.2.2). The Atiyah-Hirzebruch spectral sequence for $h^*(\ ;L)$ is obtained from that one for $h^*(\ )$ tensoring by L, i.e., suppose that $(F_r^{**},d_r)_{r\geq 1}$ is the spectral sequence converging to $h^*(X)$, X a compact CW complex then $d_r \otimes 1_L:E_r^{p,q} \otimes L \to E_r^{p+r,q-r+1} \otimes L$ is a differential and $(E_r^{**},d_r)_{r\geq 1}$ converges to $h^*(X) \otimes L = h^*(X;L)$ since $H^p(X;h^q(pt;L)) = H^p(X;h^q(pt)) \otimes L$. The idea of taking L is to "kill" the torsion of $k^*(\ )$ when suitable.

1.4.5. Proposition

Let X be a compact CW complex such that $K^*(X)$ is torsion free and the differentials d_r in the Atiyah-Hirzebruch spectral sequence $(E_r^{**},d_r)_{r\geq 1}$ converging to $K^*(X)$ are zero for $r > s$ (We can suppose s odd since the differentials of even degree are zero). Then $\{y \in k^*(X)/t^{\frac{-s+1}{2}}\ y = 0\} = \{y \in k^*(X)/\lambda\ y=0$ for some $\lambda \in Z - \{0\}\}$.

<u>Proof</u>:

We consider the Atiyah-Hirzebruch spectral sequences $(E_r^{**}, d_r)_r$ $('E_r^{**}, d'_r)_r$ converging to $K^*(X)$, $k^*(X)$ respectively, and the extension exact sequences. We have the following commutative diagram for all $m \in \mathbb{Z}$, $i \geq 0$:

$$\begin{tikzcd}
0 \arrow[r] & 'F^m_{m+2i+2} \arrow[r] \arrow[d, "j^*"'] & 'F^m_{m+2i} \arrow[r, "p'_i"] \arrow[d, "j^*"'] & 'E_\infty^{m+2i,-2i} \arrow[r] \arrow[d, "j_\infty^{**}"'] & 0 \\
0 \arrow[r] & F^m_{m+2i+2} \arrow[r] & F^m_{m+2i} \arrow[r] & E_\infty^{m+2i,-2i} \arrow[r] & 0
\end{tikzcd}$$

j^*, j_∞^{**} are the maps induced by $j: bu \to K$, p'_i the map of the extension exact sequence.

Let $y \in k^m(X)$ such that $\lambda y = 0$ for some $\lambda \in \mathbb{Z} - \{0\}$ $t^{\frac{-s+1}{2}} y \in {}'F_m^{m-s+1}$ because $k^m(X) = {}'F^m_m$. But ${}'F_m^{m-s+1} = F_m^{m-s+1}$ by Proposition 1.4 .2. Since $K^*(X)$ is torsion free, so is F_m^{m-s+1}. Hence $t^{\frac{-s+1}{2}} y = 0$.

Suppose now that $y \in k^m(X)$ and $t^{\frac{-s+1}{2}} y = 0$. $j^* y = 0$ because $\{y \in k^*(X) / t^{-i} y = 0 \text{ for some } i \in \mathbb{N}\}$ = Ker $[j^*: k^*(X) \to K^*(X)]$. Then it is enough to prove that there exists $\lambda \in \mathbb{Z} - \{0\}$ so that $\lambda y \in {}''F^m_{m+s-1}$ because on the one hand, $j^*: {}'F^m_{m+s-1} \to F^m_{m+s-1}$, is an isomorphism; on the other hand, $j^*(\lambda y) = 0$ for all $\lambda \in \mathbb{Z}$.

We are going to prove, by induction on $i \geq 0$, that:

(*) There exists $\lambda \in \mathbb{Z} - \{0\}$ such that $\lambda y \in {}'F^m_{m+2i}$.

For $i = 1$ we have $j_\infty^{**}\, p_0'\, y=0$. Then $p_0'\, y=0$ or $p_0'\, y \in \operatorname{Im} d_r$ for some $2 \le r \le s$. If $p_0'\, y = 0$ then $y \in {}'F^m_{m+2}$ by the exactness of the top row. In the other case, since all the differentials have torsion, there exists $\alpha \in \mathbb{Z}-\{0\}$ so that $\alpha p_0'\, y = 0$. Thus $\alpha y \in {}'F^m_{m+2}$ as required and the induction hypothesis is true.

If ⊛ is true for $i=j$ then there exists $\beta \in \mathbb{Z}-\{0\}$ such that $\beta y \in {}'F^m_{m+2j}$. Proceeding exactly as above we conclude that $\nu(\beta y) \in {}'F^m_{m+2j+2}$ for some $\nu \in \mathbb{Z} - \{0\}$ as required. □

1.4.6. Proposition:

Let X be a finite CW complex of dimension N with $H^*(X;\mathbb{Z})$ torsion free. Then $k^m(X)$ is isomorphic, as a $\mathbb{Z}$ module, to $\bigoplus_{i=0}^{[\frac{N-m}{2}]} H^{m+2i}(X;\mathbb{Z})$ for all $m \in \mathbb{Z}$.

Proof:

The Atiyah-Hirzebruch spectral sequence $({}'E_r^{**}, d_r')_{r\ge1}$ is trivial because as $H^*(X;\mathbb{Z})$ is torsion free the differentials can't be torsion-valued, thus they are zero for $r \ge 2$.

The extension exact sequences:

$$0 \to {}'F^m_{i+1} \to {}'F^m_i \to {}'E_\infty^{i,m-i} \to 0$$

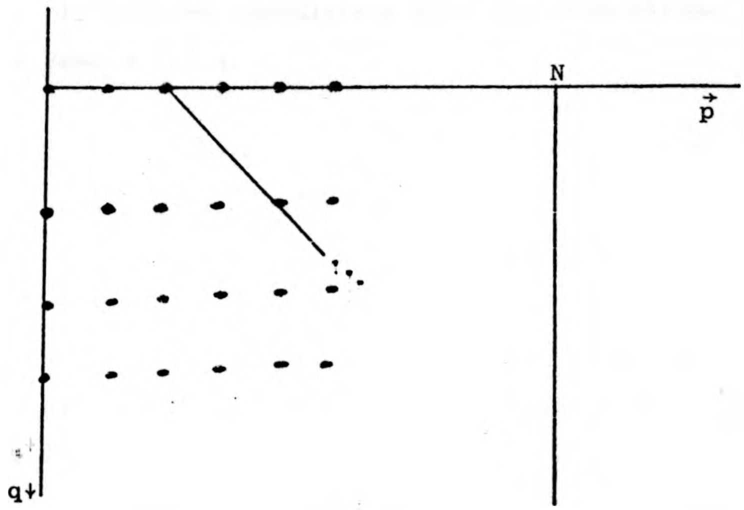
split. This can be proved by decreasing induction on $m \le i \le N$ since ${}'E_\infty^{**}$ is a free Z module and ${}'F^m_N \approx {}'E_\infty^{N,n-N}$. Then

$${}'F^m_i \approx {}'F^m_{i+1} \oplus E_\infty^{i,m-i} \approx \bigoplus_{j=i}^{N} E_\infty^{j,m-j}.$$

In particular, $k^m(X) \approx \bigoplus_{j=m}^{N} E^{j,-j}_{\infty}$. The result follows from:

$$E_{\infty}^{j,m-j} = E_2^{j,m-j} = \begin{cases} H^j(X;\mathbb{Z}) & \text{if } m-j \text{ is even} \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

This is illustrated by the picture below:



□

1.4.7. <u>Corollary</u>:

Let X be a compact CW complex of dimension N.

(i) Let L be a commutative ring which is an torsion free abelian group so that $H^*(X;L)$ is torsion free. Then the Atiyah-Hirzebruch spectral sequence converging to $k^*(X;L)$ collapses and $k^m(X,L) \approx \bigoplus_{i=0}^{\left[\frac{N-m}{2}\right]} H^{m+2i}(X,L)$, $m \le N$, that is, as $L[t^{-1}]$ modules $k^*(X,L) \approx H^*(X;L) \otimes L[t^{-1}]$.

(ii) Let q be a prime so that the Atiyah-Hirzebruch spectral sequence converging to $k^*(X;\mathbb{Z}_q)$ is trivial. Then $k^m(X;\mathbb{Z}_q) \approx \bigoplus_{i=0}^{[\frac{N-m}{2}]} H^{m+2i}(X;\mathbb{Z}_q)$, $m \leq n$.

Proof:

It follows immediately from the Proposition 1.4.6 and the Remark 1.4.4. □

1.4.8. Proposition

Let X be a compact CW complex, $(E_r^{**},d_r)_r$ the Atiyah-Hirzebruch spectral sequence converging to $k^*(X;L)$ where L is a ring of type Q(P) or $\mathbb{Z}_p$ (p prime). Then $x \in H^p(X;L)$ lies in the image of $\eta^*: k^*(X;L) \to H^*(X;L)$ if and only if x is an infinite cycle in the spectral sequence, i.e., $d_r x = 0$ for all $r \geq 2$.

Proof

We consider the spectral sequence $(F_r^{**},e_r)_{r\geq 1}$ converging to $H^*(X;L)$. All the differentials are zero for $r \geq 2$ and

$$F_\infty^{p,q} = \begin{cases} H^p(X;L) & \text{for } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\eta_L^*:k^*(X;L) \to H^*(X;L)$ induces a map of spectral sequences (η_r^{**}) since it is a natural transformation of cohomology theories.

For $r = 2$, $\eta_2^{**}:E_2^{p,q} = H^p(X;k^q(pt;L) \to H^p(X;H^q(pt;L)) = F_2^{p,q}$ is induced by the map $\eta^*:k^q(pt;L) \to H^q(pt;L)$ defined on the coefficient groups. Hence η_2^{**} is, under the usual identifications, the identity for $q = 0$, the zero map for $q \neq 0$. We recall that as $E_r^{p,q} = 0$ for $q > 0$, $E_{r+1}^{p,o} = \operatorname{Ker} d_r (r \geq 2)$. Thus, $E_\infty^{p,o}$ can be considered as the subgroup of $H^p(X;L)$ consisting of the infinite cycles.

$\eta_\infty^{**}:E_\infty^{p,q} \to F_\infty^{p,q} = H^p(X;H^q(pt;L)$ is the zero map for $q \neq 0$ and for $q = 0$ is the inclusion map using the above identification.

On the other hand, $E_\infty^{p,o}$ is isomorphic to coKer $[m_{t^{-1}}^{p+2} : k^{p+2}(X) \to k^p(X)]$. Since $E_\infty^{p,o} = F_p(k^p(X) / F_{p+1}(k^p(X))$ and

$F_{p+1}(k^p(X)) = F_{p+2}(k^p(X)) = \operatorname{Im}[m_{t^{-1}}^{p+2} \; k^{p+2}(X) \to k^p(X)]$ (Remark 1.4.3).

Hence, we get the isomorphism $\tilde{\eta}^*$: coKer $m_{t^{-1}}^{p+2} \to F_\infty^{p,o} = H^p(X;L)$ induced by η. The result follows immediately. □

CHAPTER II - LIE GROUPS: SMALL SURVEY AND $k^*(G;R)$

Through this chapter we consider only compact, connected Lie groups over $\mathbb{R}$.

In the first paragraph we mention some well-known results of their classification, representation ring and its relation to their K-cohomology, ordinary cohomology with $\mathbb{Z}, \mathbb{Z}_p$ (p prime) and $\mathbb{Q}$ coefficients. The main references for this paragraph are [9, 10, 11, 12, 21].

In the second paragraph we give the structure of $k^*(G;Q(P))$ whenever $H^*(G;Q(P))$ is torsion free.

1. General results in Lie groups

A. Classification of Lie groups

2.1.1. Definition:

Let G be a compact, connected Lie group. We say that:

(i) G is simple if it has no proper closed invariant subgroup of dimension greater than zero.

(ii) G is semi-simple if its centre is finite.

2.1.2. Theorem [38]

Any compact Lie group is locally isomorphic to the direct product of simple non abelian groups and tori. □

2.1.3. [11]

We have the following different classes of locally isomorphic compact connected simple Lie groups that contain a unique (up to global isomorphism) simply connected representative:

(i) Classical structure

$A_r (r \geq 1)$ - represented by the group $SU(r+1)$ of $(r+1) \times (r+1)$ complex unitary matrices of determinant + 1. It has dimension $r(r+2)$ and rank r. $B_r (r \geq 2)$ - represented by the group $SO(2r+1)$ of real orthogonal $(2r+1) \times (2r+1)$ matrices of determinant + 1 or by the spinor group Spin $(2r+1)$. They have dimension $r(2r+1)$, rank r. $C_r (r \geq 3)$-represented by the group $Sp(r)$ of $r \times r$ quaternionic matrices. It has dimension $r(2r+1)$, rank r. $D_r (r \geq 4)$ - represented by the group $SO(2r)$ of real orthogonal $2r \times 2r$ matrices of determinant + 1 or by the Spinor group Spin $(2r)$. They have dimension $r(2r-1)$, rank r.

(ii) Exceptional structures

G_2 - the group of all automorphisms of the Cayley numbers system. Has dimension 14, rank 2. Its centre has order 1.

F_4 - has dimension 52, rank 4.

E_6 - has dimension 78 and rank 6.

E_7 - has dimension 133 and rank 7.

E_8 - has dimension 248 and rank 8.

B. Representations and K^* of Lie groups

Let G be a compact Lie group. R(G) denotes the representation ring, that is, the free abelian group on the isomorphism classes of irreducible complex representations of G with a multiplication induced by the tensor product of representations. We can only consider unitary representations $\rho: G \to U(n)$.

2.1.4. Proposition [21]:

Let G be a semi-simple simply-connected compact Lie group of rank ℓ. Then R(G) is a polynomial algebra $\mathbb{Z}[\rho_1,\ldots,\rho_\ell]$ where $\rho_1,\ldots,\rho_\ell$ are the basic representations whose maximal weights $\lambda_1,\ldots,\lambda_n$ form a basis for the character group $\hat{T}$ of the maximal torus T($\hat{T}$ with an order given in the usual way). □

There are two homomorphisms $\alpha: R(G) \to K^0(BG)$ (BG is the classifying space of G) and $\beta: R(G) \to K^1(G)$ [9,21]. α is constructed by:

Let $\rho: G \to U(n)$ be an irreducible representation, $\gamma: EG \to BG$ the universal G-bundle. $\alpha(\rho)$ is the class of the vector bundle over BG obtained from the universal G-bundle changing its structure through $\rho: G \to U(n)$.

β is obtained by looking at $K^1(G)$ as the set of homotopy classes of maps $G \to U$ and taking $\beta(\rho) = [i_n\rho]$ where $i_n: U(n) \to U$ is the usual inclusion map and $\rho: G \to U(n)$ as before.

We note that we can define a map $\alpha(\xi): R(G) \to K^0(B_\xi)$ for any principal G-bundle $\pi_\xi: E_\xi \to B_\xi$ in a similar way [9,24].

Let us consider the augmentation map $\varepsilon: R(G) \to R(1) = Z$ (1 is the trivial group) given by $\varepsilon(\rho) = \dim \rho$. We denote the kernel of ε by I(G)

2.1.5. Proposition:

Let π_ξ: $E_\xi \to B_\xi$ be a principal G-bundle. Then the diagram:

$$\begin{tikzcd} K^1(G) \arrow[r, "\delta"] & K^0(E_\xi, G) \\ I(G) \arrow[u, "\beta"] \arrow[r, "\alpha", "\xi"'] & K^0(B_\xi, pt) \arrow[u, "\pi!"'] \end{tikzcd}$$

is anti-commutative, i.e, $\pi' \circ \alpha + \delta \circ \beta = 0$ □

Now we are going to enunciate the main theorem of [21].

2.1.6. <u>Theorem</u>

Let G be a compact connected Lie group with $\pi_1(G)$ torsion free. Then:

(i) $K^*(G)$ is torsion free.

(ii) $K^*(G)$ can therefore be given the structure of a Hopf algebra over the integers, graded by $\mathbb{Z}_2$.

(iii) Regarded as Hopf algebra $K^*(G)$ is the exterior algebra on the module of primitive elements, which are of degree 1.

(iv) A unitary representation $\rho: G \to U(n)$, by composition with the inclusion $U(n) \subset U$ defines a homotopy class $\beta(\rho)$ in $[G,U] = K^1(G)$. The module of primitive elements in $K^1(G)$ is exactly the module generated by all the classes of this type.

(v) In particular, if G is semi-simple of rank ℓ, the ℓ "basic representations" $\rho_1, \ldots, \rho_\ell$ are defined and the classes $\beta(\rho_1), \ldots, \beta(\rho_\ell)$ form a basis for the above set of primitive elements; we can write:

$$K^*(G) = \Lambda_{\mathbb{Z}}(\beta(\rho_1), \ldots, \beta(\rho_\ell)) \qquad \square$$

2.1.7. Remark:

(i) Atiyah proved in [8] that if G is a compact connected and simply connected Lie group then $K^*(G)/_{\mathrm{Tor}K^*(G)}$ is the exterior algebra $\Lambda_Z(\beta'(\rho_1),\ldots,\beta'(\rho_\ell))$ where $\rho_1,\ldots,\rho_\ell$ are the "basic representations" of G, $\beta'(\rho_i) = \beta(\rho_i) \bmod \mathrm{Tor}\, K^*(G)$. Araki proved in [2] that $K^*(G)$ is torsion free, G as before.

(ii) In Husemoller's book [24] can be found a good description of the representation rings of the classical groups.

(iii) Atiyah and Hirzebruch have several results on the relation between $\hat{R}(G)$, the completed representation ring of G, and $\mathcal{K}^0(BG)$, inverse limit of $K^0(BG(i))$ [9].

C. Ordinary cohomology of Lie groups

2.1.8. Transgression map [12]

Let $E \longrightarrow B$ be a fibre bundle with fibre F, base B where B, F are connected spaces, E compact. We consider the ordinary cohomology $H^*(\ ;A)$ with coefficients in an abelian group A and the spectral sequence (E_r, d_r) associated to the fibre bundle, assuming that the fibre bundle is A-orientable. We have $E_2^{p,q} \approx H^p(B;H^q(F;A))$. Therefore we can identify $H^{s+1}(B;A)$ with $E_2^{s+1,0}$ and $H^s(F;A)$ with $E_2^{0,s}$. Then we obtain

$$H^s(F;A) \approx E_2^{0,s} \xleftarrow{\alpha_{0,s}} E_{s+1}^{0,s} \xrightarrow{d_{s+1}} E_{s+1}^{s+1,0} \xleftarrow{\alpha_{s+1,0}} E_2^{s+1,0} \approx H^{s+1}(B;A)$$

where $\alpha_{s+1,0}$ is the composite of the projections $E_2^{**} \to E_3^{**} \to \dots \to E_{s+1}^{**}$ and $\alpha_{0,s}$ is the inclusion $E_{s+1}^{0,s} = \operatorname{Ker} d_s \subset \dots \subset \operatorname{Ker} d_2 \subset E_2^{0,s}$. We say that $x \in H^s(F;A)$ is *transgressive* if $x \in E_{s+1}^{0,s}$ and $d_{s+1}\,{}_{0,s}(x) \in \operatorname{Im} \alpha_{s+1,0}$. Thus we have obtained a map from the subgroup T of the transgressive elements of $H^s(F;A)$ to a quotient L of $H^{s+1}(B;A)$. This map $C:T \to L$ is called the *transgression map*.

Alternatively, it can be described by:

Let $\delta:\tilde{H}^s(F;A) \to H^s(E,F;A)$ be the coboundary homomorphism associated to the cohomology exact sequence of the pair (E,F), $\pi^*:H^{s+1}(B,*;A) \to H^{s+1}(E,F;A)$ be the map induced by the projection where $* = \pi(F)$. We define

$$C:T \xrightarrow{\delta} H^{s+1}(E,F;A) \xrightarrow{p\circ(\pi^*)^{-1}} \tilde{H}^{s+1}(B;A)/\operatorname{Ker} \pi^*$$

where $T = \delta^{-1}(\operatorname{Im} \pi^*)$, p is the projection map.

The definitions coincide for connected fibre bundles [12,35]. We shall consider the special case of the universal G-bundle: $G \to EG \to BG$. In this case, as EG is contractible, $\delta:\tilde{H}^s(G;A) \to H^{s+1}(EG,G;A)$ is an isomorphism. We say that an element is *universally transgressive* if it is transgressive in this fibration. □

2.1.9. Notations

(i) Let K be a free abelian group or a field. $\Lambda_K(x_1,\dots,x_s)$ denotes the K exterior algebra generated by x_i of degree $n_i \in \mathbb{Z}$.

(ii) Let R be a ring. $R[x_1,\dots,x_n]$ denotes the ring of polynomials with indeterminate x_i and-coefficients in R.

(iii) Let R be a ring. $\Delta(x_1,\dots,x_s)$ denotes an algebra generated by a simple system of generators $x_1,\dots,x_s$ of degree $n_i \in \mathbb{Z}$, that is, it is the weak direct sum of the R modules generated by the unit (if any) and by the elements $x_{i_1}\dots x_{i_K}$, $1 \le i_1 < \dots < i_K \le s$. □

2.1.10. Hopf proved the following theorem:

Let X be a finite H-complex. Then X is rationally the product of odd dimensional spheres, i.e., exists a map $\nu: X \to \prod_{i=1}^{r} S^{2n_i-1}$ $(n_i \in \mathbb{N})$ which is a rational homotopy equivalence. The set $\{n_i\}$, type of X, is a homotopy invariant.

2.1.11. Kumpel [26] and Serre [33] proved:

Let G be a compact simply connected simple Lie group. Then G is p regular, p prime, if and only if $p \ge \frac{\dim G}{\text{rank } G} - 1$

(G is p regular if there exists a map $\nu: X \to \prod_{i=1}^{r} S^{2n_i-1}$ that induces an isomorphism $\nu^*: H^*(G;\mathbb{Z}_p) \to H^*(\prod_{i=1}^{r} S^{2n_i-1};\mathbb{Z}_p)$, $n_i \in \mathbb{N}$, r = rank G) [31]. □

2.1.12. Borel proved [11] for a compact connected Lie group G:

(i) If $H^*(G;\mathbb{Z}_p)$, p odd prime or p=1, is the exterior algebra of a subspace graded by odd degrees, then $H^*(G;\mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(x_1,\dots,x_m)$, with x_i universally transgressive of odd degrees, and $H^*(BG;\mathbb{Z}_p) = \mathbb{Z}_p[y_1,\dots,y_m]$ with the $y_i = C(x_i)$, $1 \le i \le m$, C transgression map for the universal fibre bundle.

Conversely, if $H^*(BG;\mathbb{Z}_p) = \mathbb{Z}_p\,[y_1,\ldots,y_m]$ with the y_i's of even degrees, then $H^*(G;\mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(x_1,\ldots,x_m)$ with the x_i's universally transgressive and $y_i = C(x_i)$ $(1 \le i \le m)$.

(ii) If $H^*(G;\mathbb{Z}_2)$ has a simple system (x_i) of universally transgressive generators then $H^*(BG;\mathbb{Z}_2) = \mathbb{Z}_2\,[y_1,\ldots,y_m]$ with $y_i = C(x_i)$, $1 \le i \le m$, and conversely.

(iii) Let T be a maximal torus of G. We have the natural projection map $EG_{/T} \to EG_{/G}$ that induces $\rho(T,G):BT \to BG$. The Weyl group W(G) group of inner automorphisms of G that leave T invariant, operates on T and,hence, on $H^*(T;\mathbb{Z})$ and $H^*(BT;\mathbb{Z})$. Let I_G be the ring of polynomials contained in $H^*(BT;\mathbb{Z})$ invariant under that action. As $H^*(BT;\mathbb{Z})$ is torsion free, $I_G \otimes \mathbb{Z}_p$ (p prime) is cannonically embedded in $H^*(BT;\mathbb{Z}_p)$. We are now in conditions to enunciate the third theorem:

Assume that $H^*(G;\mathbb{Z}_p)$ is an exterior algebra of an s-dimensional subspace graded by odd degrees. Then $s = \dim T$ and $\rho^*(T,G)$ maps $H^*(BG;\mathbb{Z}_p)$ isomorphically onto $I_G \otimes \mathbb{Z}_p$. □

2.1.13. <u>Hopf algebra structure</u>

The group product $m:G \times G \to G$ induces a map $m^*:H^*(G;A)\to H^*(G\times G;A)$, A a ring. If $H^*(G;A)$ is a flat module then the cross product map $H^*(G;A) \otimes H^*(G;A) \to H^*(G \times G;A)$ is an isomorphism. Composing the inverse of m^* with it we get a diagonal $H^*(G;A)\to H^*(G;A)\otimes H^*(G;A)$ that gives a co-algebra structure to $H^*(G;A)$. One can prove that $H^*(G;A)$ is a Hopf algebra over A [12].

We have an analogous situation for any cohomology theory E^* derived from a ring spectra E, defined on the homotopy category of based compact CW complexes. But now we need $E^*(G)$ to be a flat finitely generated $E^*(pt)$ module to have the isomorphism [35]:

$$\tilde{E}^*(G) \otimes_{\tilde{E}^*(S^0)} \tilde{E}^*(G) \to \tilde{E}^*(G \wedge G)$$

Thus, we get a diagonal $\psi_E : \tilde{E}^*(G) \to \tilde{E}^*(G) \otimes_{\tilde{E}^*(S^0)} \tilde{E}^*(G)$ that gives a $E^*(pt)$ co-algebra structure to $E^*(G)$. Moreover, $E^*(G)$ is an $E^*(pt)$ Hopf algebra. □

2.1.14. References for the calculation of $H^*(G;A)$, G simple, simply-connected Lie group.

Borel has described the Hopf algebra structure of $H^*(G;A)$ in the cases covered by the results mentioned in 2.1.12. and when $G = G_2, F_4$, $A = \mathbb{Z}_2$ [11]. He gave the algebra structure of $H^*(G;A)$ for $G = G_2$ and $A = \mathbb{Z}$; $G = F_4$ and $A = \mathbb{Z}, \mathbb{Z}_3$ [10]. SU(r) and Sp(r) are torsion free groups ($r \geq 1$). He has also determined the prime numbers p for which G has p-torsion [13], and the action of the Steenrod algebra.

In the case G = Spin (n) Borel determined the Hopf algebra structure of $H^*(\text{Spin}(n);\mathbb{Z}_2)$ for $n \leq 9$. the algebra structure for all $n \geq 1$ and the action of the Steenrod algebra. Furthermore, he obtained some results in its integer cohomology such as that the torsion coefficients of $H^*(\text{Spin}(n);\mathbb{Z})$ are 2 [10]. The Hopf algebra structure of $H^*(\text{Spin}(n);\mathbb{Z}_2)$ has been completely determined by [25,29].

For the exceptional Lie groups E_6, E_7, E_8 we have: the algebra structure of $H^*(G;A)$ in [3,4,5,13] for $A = \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ and the action of the Steenrod algebra; the Hopf algebra structure for $A = \mathbb{Z}_2$ [references of 25].

2. Connective K-theory of compact conneeted Lie groups with Q(P) coefficients.

Through this paragraph G denotes a compact, connected Lie group of rank r, dimension n; Q(P) is the ring defined in 1.4.4.

2.1. Theorem:

Let L be a ring of type Q(P) (P any subset of the set of all prime numbers) so that $H^*(G;L)$ is torsion free. Then

(i) $k^*(G;L) \approx \Lambda_{L[t^{-1}]}(y_1,\dots,y_r)$ where y_j has odd degree i_j for all $1 \le j \le r$, $n = \sum_{j=1}^{r} i_j$.

(ii) The y_j can be choosen so that they are primitive in the Hopf algebra $k^*(G,L)$.

Proof

(i) By the results mentioned in the previous paragraph $H^*(G;L) \approx \Lambda_L(x_1,\dots,x_r)$ where x_j has odd degree i_j, $1 \le j \le r$, $\sum_{i=1}^{r} i_j = n$. Therefore, the Atiyah-Hirzebruch spectral sequence (E_r^{**},d_r) converging to $k^*(G;L)$ is trivial. Then Corollary 1.4.7(ii) applies and we have an isomorphism of $L[t^{-1}]$ modules: $k^*(G;L) \approx H^*(G;L) \otimes_L L[t^{-1}]$. Moreover, we note that $k^*(G;L)$ is a free $k^*(pt;L)$ module. Thus $j^*: k^*(G;L) \to K^*(G;L)$ is injective.

We take elements $y_1, \dots, y_r$ in $k^*(G;L)$ so that $\eta^*(y_j) = x_j \; \forall 1 \le j \le r$ (to simplify the notation η^* denotes the map $\eta^*_L : k^*(\;;L) \to H^*(\;;L)$ defined in I.3). Those elements exist, since $\tilde{\eta}^*$ is surjective, and they don't lie in Im $m^*_{t^{-1}}$. They are unique modulo Im $m^*_{t^{-1}}$

$\forall 1 \le j \le r \; (y_j)^2 = 0$ since every element in $K^1(G;L)$ has square zero (this is true for $K^1(X)$, X any CW complex, [7] and, hence, for $K^1(X;L) = K^1(X) \otimes L$) and $j^* : k^*(G;L) \to K^*(G;L)$ is an injective ring homomorphism. Therefore, we have an algebra homomorphism:

$$f : \Lambda_{L[t^{-1}]}(y_1, \dots, y_r) \to k^*(G;L)$$

It is an isomorphism. To show it, it is enough to prove the following:

<u>Claim</u>:

The $(y_j)_{1 \le j \le r}$ form a $L[t^{-1}]$ basis of the $L[t^{-1}]$algebra $k^*(G;L)$.

<u>Proof of the claim</u>:

Since $k^i(G;L) \approx \text{coKer } m^{i+2}_{t^{-1}} \oplus \text{Im } m^{i+2}_{t^{-1}} \approx H^i(G;L) \oplus \text{Im } m^{i+2}_{t^{-1}}$ and η^* is a multiplicative epimorphism it follows that the $(y_j)_{1 \le j \le r}$ generate $k^*(G;L)$ as an $L[t^{-1}]$ algebra.

Now it remains to show that they are linearly independent. Suppose not. Then there exists a sum:

$$\sum_{1\le j_1<\dots<j_p\le r} a_{j_1\dots p}\, y_{j_1}\cdots y_{j_p} = 0, \quad \text{where } a_{j_1\dots p} \in L[t^{-1}] \text{ not all zero.}$$

We can write it as:

$$\sum_{i\in A} t^{-i} \sum_{1\le j_{1,i}<\dots<j_{p,i}\le r} b_{j_1\dots p,i}\, y_{j_{1,i}}\cdots y_{j_{p,i}} = 0,$$ where A denotes a finite subset of the non-negative integers, $b_{j_1,\dots p,i} \in L$.

Let ℓ be the minimum of A. Since $m^*_{t^{-1}}: k^*(G;L) \to k^*(G;L)$ is a monomorphism $((m^*_{t^{-1}})^{-1})^{\ell}\left(\sum_{i\in A} t^{-i} \sum b_{j_1\dots p}\, y_{j_{1,i}}\cdots y_{j_{p,i}}\right) = 0$

that is:

$$\sum_{i\in A} t^{-i+\ell} \sum b_{j_1\dots p,i}\, y_{j_{1,i}} \cdots y_{j_{p,i}} = 0.$$

Applying η^* we obtain:

$$\sum_{j_{1,\ell}<\dots<j_{p,\ell}} b_{j_1,\dots p,\ell}\, x_{j_1,\ell}\cdots x_{j_p,\ell} = 0$$

This implies that all the $b_{j_1,\dots,p,\ell}$ are zero because $H^*(G;L) = \Lambda_L(x_1,\dots,x_r)$.

We proceed equally with all $j \in A$. As A is a finite set we conclude that all the L coefficients are zero which contradicts our assumption. This finishes the proof of the claim and, therefore, of (i).

(ii) Let $G \to EG \xrightarrow{p} BG$ be the universal G-bundle. We have a commutative diagram:

$$\begin{tikzcd}
\tilde{K}^m(G;L) \arrow[r, "\delta_K", "\approx"'] & K^{m+1}(EG,G;L) & \tilde{K}^{m+1}(BG;L) \arrow[l, "p^*_K"'] \\
\tilde{k}^m(G;L) \arrow[u, "j^*"'] \arrow[d, "\eta^*"] \arrow[r, "\delta_k", "\approx"'] & k^{m+1}(EG,G;L) \arrow[u, "j^*"'] \arrow[d, "\eta^*"] & \tilde{k}^{m+1}(BG;L) \arrow[l, "p^*_k"'] \arrow[u, "j^*"'] \arrow[d, "\eta^*"] \\
\tilde{H}^m(G;L) \arrow[r, "\delta_H", "\approx"'] & H^{m+1}(EG,G;L) & \tilde{H}^{m+1}(BG;L) \arrow[l, "p^*_H"']
\end{tikzcd}$$

where δ_K, δ_k, δ_H are the coboundary homomorphisms and p^*_K, p^*_k, p^*_H are the homomorphisms induced by the projection $p:EG \to BG$ considering the cohomology theories K^*, k^*, H^* respectively, η^*, j^* as before.

By the Borel's result 2.1.11, $H^*(G;L) = \Lambda_L(x_1,\dots,x_r)$ where the x's are universally transgressive and $H^*(BG;L) = L[z_1,\dots,z_r]$, $z_j = C(x_j)$ where C is the transgression map, degree $C(x_j) = i_j+1$.

We take elements $w_j \in k^{j+1}(BG;L)$ so that $\eta^*(w_j) = z_j$. They exist and are non-zero since $H^*(BG;L)$ is torsion free and so is $k^*(BG;L)$. As EG is contractible, δ^*_K, δ^*_k and δ^*_H are isomorphisms and we can consider elements $y_j = \delta_k^{-1}(p^*_k(w_j))$. By commutativity of diagram and because δ^*_H is an isomorphism, $\eta^*(y_j) = x_j$. Thus the y_j are as in part (i), of this theorem.

It remains to show:

Claim:

The $(y_j)_{1 \le j \le r}$ are primitive.

Proof of the claim:

By the commutativity of diagram (I) we have:

$$j^*(y_j) = \delta_K^{-1}(p^*_K(j^*(w_j))) \text{ for all } 1 \le j \le r$$

Let $j^{\#}: k^* \to K^{\#}$, $K^{\#}$ is the $\mathbb{Z}_2$-graded K-cohomology, be the natural transformation of cohomology theories induced by $j^*: k^* \to K^*$. If we replace K^* by $K^{\#}$ and respective maps in the diagram (I) it still commutes. L. Hodgkin proved in [21] that $\delta_K^{-1}(p^*_K(\tilde{K}^0(BG;L)))$ is the module of primitive elements in $K^{\#}(G;L)$. Then $j^{\#}(y_j)$ is primitive in the L module $K^{\#}(G;L)$ for all $1 \le j \le r$. But this is equivalent to say that $j^*(y_j)$ is primitive in the $L[t,t^{-1}]$ module $K^*(G;L)$. Thus, since the following diagram:

$$\begin{tikzcd}
k^*(G;L) \arrow[r, "\psi^*_k"] \arrow[d, "j^*"] & k^*(G;L) \otimes_{L[t^{-1}]} k^*(G;L) \arrow[d, "j^* \otimes j^*"] \\
K^*(G;L) \arrow[r, "\psi^*_K"] & K^*(G;L) \otimes_{L[t,t^{-1}]} K^*(G;L)
\end{tikzcd}$$

(ψ^*_k, ψ^*_K are the diagonals for $k^*(G;L)$, $K^*(G;L)$ respectively)

commutes we have:

$$(j^* \otimes j^*)\ \psi^*_k\ (y_j) = 1 \otimes j^*(y_j) + j^*(y_j) \otimes 1$$

This implies that $\psi^*(y_j) = 1 \otimes y_j + y_j \otimes 1$ because j^* is injective. Therefore the $(y_j)_{1 \le j \le r}$ are primitive. □

2.2.2. Corollary:

(i) $k^*(SU(n)) = \Lambda_{\mathbb{Z}[t^{-1}]}\ (y_3, \ldots, y_{2n-1})$ where degree $y_i = i$, $y_3, \ldots, y_{2n-1}$ are primitive

(ii) $k^*(Sp(n)) = \Lambda_{\mathbb{Z}[t^{-1}]}\ (y_3, \ldots, y_{4n-1})$ where degree $y_i = i$, $y_3, \ldots, y_{4n-1}$ are primitive.

Proof:

We have [10]:

$H^*(SU(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_3, \ldots, x_{2n-1})$, degree $x_i = i$, $x_3, \ldots, x_{2n-1}$ universally transgressive.

Hence, applying the theorem we get the result. □

2.2.3. Remark:

In general we don't have $j^{\#}(y_j) = \beta(\rho_i)$, $1 \le j \le r$, $1 \le i \le r$ $(\beta : R(G) \to K^1(G)$, $(\rho_i)_{1\le i\le r}$ basic representations, defined in 2.1.B).

Suppose $H^*(G;\mathbb{Z})$ torsion free. Given $x \in K^*(G)$, x lies in $F_p(K^*(G))$ if and only if $ch_i(x) = 0$ for $i < p$ where ch_i denotes the i-component of the Chern character [18]. Then $x \in K^{\varepsilon}(G)$ $(\varepsilon = 0,1)$ is equal to $j^{\#}(y)$, $y \in k^p(G)$, if and only if $x \in F_p(K^{\varepsilon}(G))$ since by 1.4.2 $k^p(G) = F_p(k^p(G)) = F_p(K^p(G)) = F_p(K^{\varepsilon}(G))$ (p odd or even whether $\varepsilon = 1$ or 0). Hence, $x = j^{\#}(y)$ if and only if $ch_i(y) = 0$ for $i < p$.

Now let G denote a simple simply-connected Lie group with basic representations $(\rho_i)_{1\le i\le r}$ of highest weight λ_i. B. Harris proved in [20] that

$$ch_3(\beta(\rho_i)) = \eta_i x_3; \text{ where } \eta_i = \frac{2(\lambda_i,\lambda_i+2\delta)}{(\alpha,\alpha)} \cdot \frac{\dim(\lambda_i)}{\dim(G)} \in \mathbb{Z},$$

$\delta = \lambda_1 + \ldots + \lambda_r$, α root of maximal length.

The conclusion follows from the fact that the η_i's are greater or equal to 1. □

CHAPTER III - The exceptional Lie group G_2.

In the first paragraph we enunciate a proposition from [21] that describes the differentials in Atiyah-Hirzebruch spectral sequence converging to $K^*(X,\mathbb{Z}_p)$ (p prime, X compact).

In the second paragraph we calculate the algebra structure of $k^*(G_2;\mathbb{Z}_2)$ mainly by working out the Atiyah-Hirzebruch, spectral sequence converging to it, using that proposition and 1.4.2, which relates the two spectral sequences, and L. Smith's exact sequence.

Finally we determine the algebra structure of $k^*(G_2)$, our main tools being the universal coefficient theorem and L. Smith's exact sequence.

1. Differentials in the Atiyah-Hirzebruch Spectral Sequence

3.1.1. Proposition [21]:

Let X be a compact CW complex. Then in the Atiyah-Hirzebruch spectral sequence $\{E_r(X;\mathbb{Z}_p),d_r\}$ (p prime) converging to $K^*(X;\mathbb{Z}_p)$:

(i) $d_r = 0$ for $2 \le r \le 2p-2$, so that for $2 \le r \le 2p-1$ $E_r^q(X;\mathbb{Z}_p)$ can be identified with $H^q(X;\mathbb{Z}_p)$.

(ii) Using the above identification, d_{2p-1} is equal (up to multiplication by a non-zero element of $\mathbb{Z}_p$) to Milnor's stable cohomology operation $Q_1:H^q(X;\mathbb{Z}_p) \to H^{q+2p-1}(X;\mathbb{Z}_p)$. □

We note that $Q_1 = P^1\delta - \delta P^1$ where $P^1 : H^q(X;\mathbb{Z}_p) \to H^{q+2p-2}(X;\mathbb{Z}_p)$ is the first power operation and δ is the coboundary homomorphism. For $p = 2$, $d_3 = Sq^1Sq^2 + Sq^2Sq^1$ where the Sq^i's are the Steenrod squares. By the results in [21], for $G = G_2$, Spin(n), F_4, E_6 and $p = 2$ d_3 is the only non-zero differential; for $G = F_4$, E_6, E_7, E_8 and $p = 3$, d_5 is the only non-zero differential, for $G = E_8$ and $p = 5$ d_9 is the only non-zero differential.

2. <u>$k^*(G_2;\mathbb{Z}_2)$</u>

3.2.1. <u>Proposition</u>:

$k^*(G_2;Q(2)) = \Lambda_{Q(2)[t^{-1}]}(y_3,y_{11})$; $y_3;y_{11}$ primitive elements of degree 3,11 respectively.

<u>Proof</u>:

$H^*(G_2;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3,x_{11})$; x_3,x_{11} primitive generators of degree 3,11 respectively. Since $H^*(G_2;\mathbb{Z})$ has only 2-torsion $H^*(G_2;Q(2)) = \Lambda_{Q(2)}(x_3',x_{11}')$ where x_3', x_{11}' are primitive generators of degree 3,11 respectively. Now the result follows from Theorem 2.2.1. □

3.2.2. <u>Proposition</u>:

$k^*(G_2;\mathbb{Z}_2)$ is a $\mathbb{Z}_2[t^{-1}]$ module generated by elements y_0, y_5, y_6, y_9, y_{11}, y_{14} in which the subscript denotes the degree, subjected to the relations: $t^{-1}y_6 = t^{-1}y_{11} = 0$.

<u>Proof</u>:

$H^*(G_2;Z_2)$ is a Z_2 algebra with a simple system of generators x_3, x_5, x_6, degree $x_i = i$, such that $Sq^2\, x_3 = x_5$ $Sq^1\, x_5 = x_6$, $Sq^i(x_j) = 0$ otherwise.

Let $\{E_r^*, d_r\}$ be the Atiyah-Hirzebruch Spectral sequence converging to $k^*(G_2;\mathbb{Z}_2)$. The only possibly non-zero differential $(r \geq 2)$ is $d_3 = Sq^2\, Sq^1 + Sq^1 Sq^2 : H^i(G_2;\mathbb{Z}_2) \to H^{i+3}(G_2;\mathbb{Z}_2)$ using the identification $E_r^{p,2q} = H^p(G_2;\mathbb{Z}_2)$ $(q \leq 0)$ by 3.1.1 and 1.4.2. Thus we have: $d_3(x_3) = x_6$, $d_3(x_3x_5) = x_5x_6$, and d_3 is zero otherwise.

The pictures for (E_3^{**}, d_3) and E_∞^{**} are:

$q\downarrow$ \ $p\to$	3	5	6	8	9	11	14
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
-1-	0	0	0	0	0	0	0
-2-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
-3-	0	0	0	0	0	0	0
-4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

(E_3^{**}, d_3)

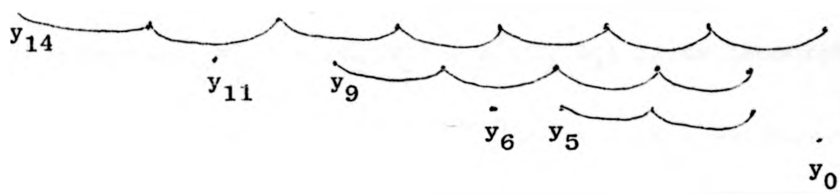
$q\downarrow$ \ $p\to$	3	5	6	8	9	11	14
0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
-1-	0	0	0	0	0	0	0
-2-	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-3-	0	0	0	0	0	0	0
-4-	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

(E_∞^{**})

Considering the extension short exact sequences, as they split we get the following table:

η	>14	14	13	12	11	10	9	8	7	6	5	4	3	2	1	≤ 0
$k^*(G_2;\mathbb{Z}_2)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$

stable range



The lines under the table indicate the non-trivial $\mathbb{Z}_2[t^{-1}]$ action on $k^*(G_2;\mathbb{Z}_2)$. It has been obtained, as it will be detailed, using the L. Smith's exact sequence and the fact:

⊕ If $a \in k^*(X;L)$ projects to $\bar{a} \in E_\infty^{**}$, and $t^{-1}\,\bar{a} \neq 0$ then $t^{-1}a \neq 0$, where X is a compact CW complex, L a ring.

This is trivially verified by looking at the extension exact sequences.

We recall that given a space X and a ring L $k^i(X;L) = K^i(X;L)$ and $m^i_{t^{-1}}:k^i(X;L) \to k^{i-2}(X;L)$ is the Bott isomorphism for any i less or equal to 1.

By ⊕ , $m^{2i}_{t^{-1}}:k^{2i}(G_2;\mathbb{Z}_2) \to k^{2i-2}(G_2,\mathbb{Z}_2)$ is: an isomorphism for $5 \leq i \leq 7, i=2$

a monomorphism for $i=4,1$

a epimorphism for $i=3$

$m^{11}_{t^{-1}} : k^{11}(G_2;\mathbb{Z}_2) \to k^{9}(G_2;\mathbb{Z}_2)$ is the O-map since L. Smith's exact sequence

$$0 \to \text{coKer } m^{11}_{t^{-1}} \to \underset{\substack{\parallel \\ \mathbb{Z}_2}}{H^9(G_2;\mathbb{Z}_2)} \to \underset{\substack{\parallel \\ 0}}{\text{Ker } m^{12}_{t^{-1}}} \to 0$$

implies $\text{coKer } m^{11}_{t^{-1}} = \mathbb{Z}_2$.

Also, by ⊛ we have: $m^{9}_{t^{-1}}:k^{9}(G_2;\mathbb{Z}_2) \to k^{7}(G_2;\mathbb{Z}_2)$ is an isomorphism

$m^{7}_{t^{-1}}:k^{7}(G_2;\mathbb{Z}_2) \to k^{5}(G_2;\mathbb{Z}_2)$ is a monomorphism

$m^{5}_{t^{-1}}:k^{5}(G_2;\mathbb{Z}_2) \to k^{3}(G_2;\mathbb{Z}_2)$ is an isomorphism since

$\text{Ker } m^{5}_{t^{-1}} = 0$ by:

$$0 \to \text{coKer } m^{4}_{t^{-1}} \to \underset{\substack{\parallel \\ 0}}{H^2(G_2;\mathbb{Z}_2)} \to \text{Ker } m^{5}_{t^{-1}} \to 0$$

Similarly, considering:

$$0 \to \text{coKer } m^{3}_{t^{-1}} \to \underset{\substack{\parallel \\ 0}}{H^1(G_2;\mathbb{Z}_2)} \to \text{Ker } m^{4}_{t^{-1}} \to 0$$

we conclude that $m^{3}_{t^{-1}}:k^{3}(G_2;\mathbb{Z}_2) \to k^{1}(G_2;\mathbb{Z}_2)$ is an isomorphism.

Using L. Smith's exact sequence we see that there exist elements $\bar{y}_j \in k^j(G_2;\mathbb{Z}_2)/\text{Im } m^{j+2}_{t^{-1}}$, $j = 5,6,9,11,14$, such that

$\bar{\eta}^*(\bar{y}_j) = x_j$ for $j = 5,6$; $\bar{\eta}^*(\bar{y}_9) = x_3x_6$, $\bar{\eta}^*(\bar{y}_{11}) = x_5x_6$,

$\bar{\eta}^*(\bar{y}_{14}) = x_3x_5x_6$. Furthermore those elements are unique.

We take a representative of each class $\bar{y}_j$ that will be denoted by y_j for $j = 5,9,11,14$ and y'_6 for $j = 6$. y_o denotes the element of $k^o(G_2;\mathbb{Z}_2)$ corresponding to the algebra unit. As Ker $m^6_{t^{-1}} = \mathbb{Z}_2$ and y'_6, $t^{-4}y_{14}$ are generators of $k^6(G_2;\mathbb{Z}_2)$, $t^{-1}y'_6 = 0$ or $t^{-1}y'_6 = t^{-5}y_{14}$. In the first case we take $y_6 = y'_6$, in the second one we take $y_6 = y'_6 + t^{-4}y_{14}$.

By the above results and the choice of the elements we have:

1). $t^{-1}y_{11} = 0$

2). $t^{-1}y_6 = 0$

3). $t^{-i}y_k, y_j$ form a $\mathbb{Z}_2$ basis of $k^j(G_2;\mathbb{Z}_2)$ for $k \in \{14,9,5,0\}$, $j \in \{0,5,6,9,11,14\}$, $i \geq 1$, $i+k=j$

This gives the $\mathbb{Z}_2[t^{-1}]$ module structure of $k^*(G_2;\mathbb{Z}_2)$. □

3.2.3. Proposition

Considering the $\mathbb{Z}_2[t^{-1}]$ algebra structure of $k^*(G_2;\mathbb{Z}_2)$ we have the following relations: $y_{14} = y_5\, y_9$, $y_{11} = y_5\, y_6$, all other products are zero.

Proof:

Since $\eta^*: k^*(G_2;\mathbb{Z}_2) \to H^*(G_2;\mathbb{Z}_2)$ is a ring homomorphism, $\eta^*(y_5y_9) = x_3x_5x_6$. Then $y_5\, y_9 = y_{14}$ because $\eta^*: k^{14}(G_2,\mathbb{Z}_2) \to H^{14}(G_2;\mathbb{Z}_2)$ is an isomorphism. A similar argument applies for y_{11}.

As $k^i(G_2;\mathbb{Z}_2) = 0$ for $i > 14$ it remains to prove that $y_5^2 = y_6^2 = 0$.

$y_6^2 = 0$ because otherwise it would be equal to $t^{-1}y_{14}$. But this is impossible since $t^{-1}\, y_6^2 = 0$.

$y_5^2 = 0$ since all the elements of $K^1(G^2;\mathbb{Z}_2)$ have zero square and $j^*:k^{10}(G_2;\mathbb{Z}_2) \to K^{10}(G_2;\mathbb{Z}_2)$ is injective. □

Putting together the two propositions we get:

3.2.4. <u>Theorem</u>

$k^*(G_2;\mathbb{Z}_2)$ is a $\mathbb{Z}_2[t^{-1}]$ algebra generated by $y_i \in k^i(G_2,\mathbb{Z}_2)$ $i = 5,6,9$ with $t^{-1}\, y_6 = 0$, $y_6 y_9 = 0$, $y_i^2 = 0$. □

3. <u>$k^*(G_2)$</u>

3.3.1. <u>Proposition</u>:

$k^*(G_2)$ is a $\mathbb{Z}[t^{-1}]$ module generated by Z_o, Z_3, Z_6, Z_9, Z_{11}, Z_{14} such that degree $Z_i = i$ and $2\,Z_6 = t^{-1}Z_6 = 0$, $t^{-1}Z_{11} = 2Z_9$.

<u>Proof</u>:

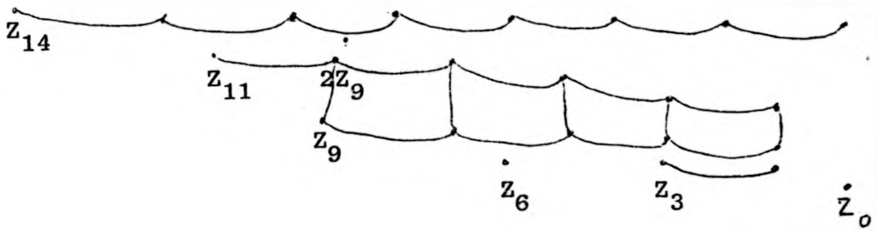
$H^*(G_2;\mathbb{Z})$ is an algebra with two generators h_3, h_{11} of degree 3,11 respectively, subjected to the relations:

$$2\, h_3^2 = h_3^4 = h_{11}^2 = h_3^2\, h_{11} = 0.$$

Using 3.2.1, 3.2.2 and applying the universal coefficient theorem we get the following table:

n	$n>14$	14	13	12	11	10	9	8	7	6	5	4	3	2	1	≤ 0
$k^*(G_2)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

Stable range



As in 3.2.2 the lines denote the non-trivial multiplication by t^{-1}. We have used L. Smith's exact sequence to calculate it.

$H^i(G_2;\mathbb{Z}) = 0$ for $i = 12, 10, 8$ implies that we have the isomorphisms:

$$k^{14}(G_2) \xrightarrow[t^{-1}]{m^{14}} k^{12}(G_2) \xrightarrow[t^{-1}]{m^{12}} k^{10}(G_2) \xrightarrow[t^{-1}]{m^{8}} k^{8}(G_2).$$

$m^{11}_{t^{-1}}: k^{11}(G_2) \longrightarrow k^9(G_2)$ is a monomorphism with coKernel $\mathbb{Z}_2$

by the two exact sequences:

$$0 \to \mathrm{coKer}\ m^{10}_{t^{-1}} \to \underset{\underset{0}{\|}}{H^8(G_2;\mathbb{Z})} \to \mathrm{Ker}\ m^{11}_{t^{-1}} \to 0$$

$$0 \to \mathrm{coKer}\ m^{11}_{t^{-1}} \to \underset{\underset{\mathbb{Z}_2}{\|}}{H^9(G_2;\mathbb{Z})} \to \mathrm{Ker}\ \underset{\underset{0}{\|}}{m^{12}_{t^{-1}}} \to 0$$

Both maps, $m^9_{t^{-1}}:k^9(G_2) \to k^7(G_2)$ and $m^7_{t^{-1}}:k^7(G_2) \to k^5(G_2)$ are isomorphisms since $H^5(G_2;\mathbb{Z}) = H^7(G_2;\mathbb{Z}) = 0$.

$m^8_{t^{-1}}:k^8(G_2) \to k^6(G_2)$ is a monomorphism with coKernel $\mathbb{Z}_2$ by the two exact sequences:

$$0 \to \mathrm{coKer}\ m^{8}_{t^{-1}} \to \underset{\underset{\mathbb{Z}_2}{\|}}{H^6(G_2;\mathbb{Z})} \to \mathrm{Ker}\ \underset{\underset{0}{\|}}{m^{9}_{t^{-1}}} \to 0$$

$$0 \to \mathrm{coKer}\ m^{7}_{t^{-1}} \to \underset{\underset{0}{\|}}{H^5(G_2;\mathbb{Z})} \to \mathrm{Ker}\ m^{8}_{t^{-1}} \to 0$$

$m^6_{t^{-1}}:k^6(G_2) \to k^4(G_2)$ is onto with Kernel $\mathbb{Z}_2$ by the exact sequence:

$$0 \to \mathrm{coKer}\ m^{6}_{t^{-1}} \to \underset{\underset{0}{\|}}{H^4(G_2;\mathbb{Z})} \to \mathrm{Ker}\ m^{7}_{t^{-1}} \to 0$$

$m^5_{t^{-1}}:k^5(G_2) \to k^3(G_2)$ is a monomorphism with coKernel $\mathbb{Z}$; by the two exact sequences.

$$0 \to \mathrm{coKer}\ m^4_{t^{-1}} \to \underset{\underset{0}{\parallel}}{H^2(G_2;\mathbb{Z})} \to \mathrm{Ker}\ m^5_{t^{-1}} \to 0$$

$$0 \to \mathrm{coKer}\ m^5_{t^{-1}} \to \underset{\underset{\mathbb{Z}}{\parallel}}{H^3(G_2;\mathbb{Z})} \to \underset{\underset{\mathbb{Z}_2}{\parallel}}{\mathrm{Ker}\ m^6_{t^{-1}}} \to 0$$

$m^4_{t^{-1}}:k^4(G_2) \to k^2(G_2)$ and $m^3_{t^{-1}}:k^3(G_2) \to k^1(G_2)$ are isomorphisms since $H^2(G_2;\mathbb{Z}) = H^1(G_2;\mathbb{Z}) = 0$.

Finally, $m^2_{t^{-1}}:k^2(G_2) \to k^0(G_2)$ is a monomorphism with coKernel $\mathbb{Z}$. This result can be obtained either by looking at L. Smith's exact sequence or by noting that $m^2_{t^{-1}}$ is the natural map from reduced K-theory to K-theory, i.e., from $\tilde{K}^0(G_2)$ to $K^0(G_2)$.

Looking again at L. Smith's exact sequences we can see that there exist unique elements $\overline{Z}_3$, $\overline{Z}_6$, $\overline{Z}_{11}$, $\overline{Z}_{14}$ in $k^*(G_2)/_{\mathrm{Im}\ m^*_{t^{-1}}}$ so that $\tilde{\eta}^*(Z_3) = 2h_3, \tilde{\eta}(\overline{Z}_6) = h_3^2$, $\tilde{\eta}^*(\overline{Z}_{11}) = h_{11}$, $\tilde{\eta}^*(\overline{Z}_{14}) = h_{14}$. We choose an element Z_i in each class $\overline{Z}_i$ for $i = 3,11,14$. As $k^6(G_2) \approx \mathbb{Z} \oplus \mathbb{Z}_2$ we can take the element of order $2, Z_6$, representative of class $\overline{Z}_6$. This element is uniquely determined and is killed by $m^6_{t^{-1}}$ (1.4.5 or L. Smith's exact sequence). We consider also an element $Z_9 \in k^9(G_2)$ so that $2Z_9 = t^{-1}Z_{11}$. It exists since coKer $m^{11}_{t^{-1}} = \mathbb{Z}_2$ and $m^{11}_{t^{-1}}$ is injective. Finally we take the algebra unit $Z_0 \in k^0(G_2)$ which corresponds to the algebra unit of $H^*(G_2,\mathbb{Z})$.

Therefore we have:

1). $2Z_6 = 0 = t^{-1}Z_6$.

2). $t^{-1}Z_{11} = 2Z_9$

3). Z_j, $t^{-i}Z_K$ form a $\mathbb{Z}$ basis of $k^j(G_2)$ where $j \in \{0,3,6,9,11,14\}$,

$i \geq 1$, $i+K=j$ and $K \in \{0,3,9,14\}$.

This gives the $\mathbb{Z}[t^{-1}]$ module structure of $k^*(G_2)$. □

3.3.2. Proposition:

Considering the $\mathbb{Z}[t^{-1}]$ algebra structure of $k^*(G_2)$ we have the following relations: $2Z_{14} = Z_3Z_{11}$, $Z_3Z_9 = t^{-1}Z_{14}$, all other products are zero.

Proof:

$\eta^*: k^{14}(G_2) \to H^{14}(G_2;\mathbb{Z})$ is an isomorphism since $\operatorname{Im} m^{16}_{t^{-1}} = 0$.

Then $\eta^*(2Z_{14}) = 2h_3\, h_{11} = \eta^*(Z_3Z_{11})$ implies $2Z_{14} = Z_3Z_{11}$ and $2t^{-1}Z_{14} = t^{-1}Z_3Z_{11} = 2Z_3Z_9$. Hence the second equality follows.

For the last statement it is enough to show that $Z_j^2 = 0$ for $j = 3,6$ and $Z_3Z_6 = 0$ since $k^i(G_2) = 0$ for $i > 14$.

$Z_6^2 = 0$ and $Z_3Z_6 = 0$ because Z_6 has order 2 and both $k^{12}(G_2)$ and $k^9(G_2)$ are torsion free.

$Z_3^2 = 0$ because $j^*: k^3(G_2) \to K^3(G_2)$ is an isomorphism and all the elements of $K^3(G_2)$ have zero square. □

Putting the two propositions together we get:

3.3.3. <u>Theorem</u>

$k^*(G_2)$ is a $\mathbb{Z}[t^{-1}]$ algebra generated by $Z_i \in k^i(G_2)$, $i = 3,6,9,11,14$, so that $2Z_6 = t^{-1}Z_6 = Z_3Z_6 = 0$; $t^{-1}Z_{11} = 2Z_9$; $Z_3Z_9 = t^{-1}Z_{14}$; $2Z_{14} = Z_3Z_{11}$; $Z_i^2 = 0$ for all i; $Z_iZ_j = 0$ for $i + j > 14$. □

Putting the two propositions together we get:

3.3.3. Theorem

$k^*(G_2)$ is a $\mathbb{Z}[t^{-1}]$ algebra generated by $Z_i \in k^i(G_2)$, $i = 3,6,9,11,14$, so that $2Z_6 = t^{-1}Z_6 = Z_3Z_6 = 0$; $t^{-1}Z_{11} = 2Z_9$; $Z_3Z_9 = t^{-1}Z_{14}$; $2Z_{14} = Z_3Z_{11}$; $Z_i^2 = 0$ for all i; $Z_iZ_j = 0$ for $i + j > 14$. $\square$

CHAPTER IV - $k^*(Spin(n)$

We are going to apply the same techniques as in the last chapter to calculate $k^*(Spin(n))$. In this case it is more difficult since the only non-zero differential in the Atiyah-Hirzebruch spectral sequence converging to $k^*(Spin(n);\mathbb{Z}_2)$, d_3, is non-zero in a number of generators increasing with n. We couldn't get a complete general description for $k^*(Spin(n);\mathbb{Z}_2)$ although it is possible to work it out giving particular values to n as we show in the example.

1. Preliminary

4.1.1. Proposition [11]

(i) $H^*(Spin(n);\mathbb{Z}_2)$ is an algebra with a simple system of generators x_i, x; degree $x_i = i \in S = \{i \le n-1/i$ is not a power of $2\}$, degree $x = 2^{s(n)} - 1$ where $s(n)$ is the integer determined by the inequality $2^{s(n)-1} < n \le 2^{s(n)}$. Moreover for all $i \in \mathbb{N}$, $Sq^i(x_j) = \binom{j}{i}x_{i+j}$ if $j \in S$ and $i + j \in S$; $Sq^i\, x_j = 0$ otherwise; $Sq^i\, x = 0$.

(ii) $H^*(Spin(n);\mathbb{Z})$ has only 2-torsion and its torsion coefficients are equal to 2, i.e., as an abelian group $H^*(Spin(n);\mathbb{Z})$ is isomorphic to the direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s.

(iii) $$H^*(Spin(n);L) \approx \begin{cases} \Lambda_L(x_3,x_7,\dots,x_{2n-3}) & \text{if n is odd ,} \\ \Lambda_L(x_3,x_7,\dots,x_{2n-5},u_{n-1}) & \text{if n is even} \end{cases}$$

$\deg x_i = i$, degree $u_{n-1} = n-1$, $L = \mathbb{Z}_p$ (p odd prime) or $\mathbb{Q}$. □

By 3.1.1 and the results mentioned above, it follows:

4.1.2. Proposition:

In the Atiyah-Hirzebruch spectral sequence (E_r^{**}, d_r) converging to $k^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ the only non-zero differential is d_3. Using the usual identifications of $E_3^{p,q}$ with $H^p(\mathrm{Spin}(n);\mathbb{Z}_2)$ (q even ≤ 0), d_3 with $Sq^1\, Sq^2 + Sq^2 Sq^1$ we have:

(i) $d_3 x_j = \begin{cases} x_{j+3} & \text{if } j \text{ is odd} \in S,\ j+3 \in S \\ 0 & \text{otherwise} \end{cases}$

(ii) If $2j \in S$ then $2j-3 \in S$ and $x_{2j} = d_3\, x_{2j-3}$

(iii) If $x_j^{\,2} \neq 0$ then $x_j^{\,2} = x_i$ for some $i \in S$.

Proof:

L. Hodgkin proved it for $K^*(\mathrm{Spin}(n);\mathbb{Z}_2)$. Then by 1.4.2 it is still true for $k^*(\mathrm{Spin}(n);\mathbb{Z}_2)$.

4.1.3. Proposition:

Considering d_3 as a map in $H^*(\mathrm{Spin}(n);\mathbb{Z}_2)$, $\ker d_{3/\mathrm{Im}\, d_3}$

is a Z_2 exterior algebra generated by :

$(\bar{x}_i)_{i\in S_1}$, $\bar{x}$, $(\bar{z}_j)_{j\in S_2}$ where $S_1 = \{i \text{ odd} \in S / i+3 \notin S\}$,

$S_2 = \{i \text{ odd} \in S / i+3 \in S\}$, $z_j = \begin{cases} x_j\, x_{j+3} + x_{2j+3} & \text{if } 2j+6 \in S \\ x_j\, x_{j+3} & \text{if } 2j+6 \notin S \end{cases}$

and $\overline{u}$ denotes the image under the projection $\mathrm{Ker}\ d_3 \to \mathrm{Ker}\ d_3/\mathrm{Im}d_3$ of any element $u \in \mathrm{Ker}\ d_3$.

Proof:

We consider the differential algebra $A = (H^*(\mathrm{Spin}(n);\mathbb{Z}_2),d_3)$. We are going to prove the result by induction on n.

First we shall prove that:

⊛ $H_*(A)$ is an exterior algebra on the given generators if and only if $H_*(A/_{(x)})$ is an exterior algebra on $(\overline{x}_i)_{i\in S_1}$, $(\overline{z}_j)_{j\in S_2}$ ((x) denotes the ideal of A generated by x).

Proof of ⊛:

We have an isomorphism of differential graded algebras

$$A \approx A_1 \otimes_{\mathbb{Z}_2} \Lambda_{\mathbb{Z}_2}(x),$$

where A_1 is the subalgebra of A generated by $(x_i)_{i\in S}$, since $d_3x = 0$ and $x \notin \mathrm{Im}\ d_3$. Then: $H(A) \approx H(A_1) \otimes_{\mathbb{Z}_2} \Lambda_{\mathbb{Z}_2}(x)$

But A_1 is isomorphic to $A/_{(x)}$. Hence the result follows.

Let B_n denote $H^*(\mathrm{Spin}(n);\mathbb{Z}_2)/_{(x)}$. We are going to prove by induction on n that $H_*(B_n)$ is an exterior algebra generated by $(\overline{x}_i)_{i\in S_1^n},(\overline{z}_j)_{j\in S_2^n}$ where S_1^n, S_2^n are the subsets S_1, S_2 of S associated to $H^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ (we note that we use the same notation for the generators x_i of B_n and of $H^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ since B_n is isomorphic to the sub-algebra of $H^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ generated by $(x_i)_{i\in S}$).

This is obviously true for $n=6$ because $\mathrm{Spin}(6) \approx SU(4)$.

Assume now it true for $B_{n-1}(n-1 \geq 6)$. We have three cases:

(i) $n-1$ is a power of 2. Then $B_n = B_{n-1}$.

(ii) $n-1$ is odd. Then B_n has one more generator, x_{n-1}, than B_{n-1}. In this case $d_3\, x_{n-1} = 0$ and $x_{n-1} \notin \operatorname{Im} d_3$ (d_3 viewed as a map of B_n). Using the same proof as in (*) we get the result.

(iii) $n-1$ is even and it is not a power of 2. B_n has one more generator, x_{n-1}, than B_{n-1}. $x_{n-1} = d_3\, x_{n-4}$ $(d_3 : B_n \to B_n)$. We consider the following exact sequence:

$$0 \longrightarrow (x_{n-1}) \xrightarrow{i} B_n \xrightarrow{p} B_{n/(x_{n-1})} \longrightarrow 0$$

where i is the inclusion of the ideal (x_{n-1}) of B_n, p is the projection. It induces the exact triangle:

$$\begin{tikzcd}
H_*((x_{n-1})) \arrow[rr, "i_*"] & & H_*(B_n) \arrow[dl, "p_*"] \\
& H_*(B_{n/(x_{n-1})}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

$B_{n/(x_{n-1})}$ is isomorphic, as a differential algebra, to B_{n-1}. Therefore we can replace it by B_{n-1}.

Clearly, $d_3 x_i = 0$ for $i \in S_1^{\,n}$ and $d_3 z_j = \begin{cases} x_{j+3}^2 + x_{2j+6} & \text{if } 2j+6 \in S \text{ which} \\ x_{j+3}^2 & \text{if } 2j+6 \notin S \end{cases}$

is zero.

$$H_*(B_{n-1}) = \Lambda_{\mathbb{Z}_2}((\bar{x}_i)_{i\in S_1^{\,n-1}}, (\bar{z}_j)_{j\in S_2^{\,n-1}})$$ by the inductive

hypothesis. But $S^{n-1} = \{i \le n-2/i$ is not a power of $2\}$. Therefore, $S_1^{\,n-1} = S_1^{\,n} \cup \{n-4\}$ and $S_2^{\,n-1} = S_2^{\,n}\setminus\{n-4\}$. As the definition of the elements z_j depends on n we will denote $z_j \in B_r$ by $z_j^{\,r}$.

Let us assume that $\frac{n-1}{2}$ is even. Then if $j = \frac{n-7}{2}$ (odd number), $2j+6 = n-1 \in S^n$. We have: $z_j^{\,n} = x_{\frac{n-7}{2}} \cdot x_{\frac{n-1}{2}} + x_{n-4}$ and $z_j^{\,n-1} = x_{\frac{n-7}{2}} \cdot x_{\frac{n-1}{2}}$

If $j \in S_2^n - \{\frac{n-7}{2}, n-4\}$ then $z_j^{\,n-1} = z_j^{\,n}$.

Hence, $H_*(B_{n-1}) = \Lambda_{\mathbb{Z}_2}((\bar{x}_i)_{i\in S_1^{\,n}}, \bar{x}_{n-4}, (\bar{z}_j^{\,n})_{\substack{j\in S_2^{\,n} \\ j\neq n-4 \\ j\neq \frac{n-7}{2}}}, \bar{x}_{\frac{n-7}{2}} \cdot \bar{x}_{\frac{n-1}{2}})$

As we are dealing with $\mathbb{Z}_2$ algebras we can write:

$$H_*(B_{n-1}) = \Lambda_{\mathbb{Z}_2}((\bar{x}_i)_{i\in S_1^n}, (\bar{z}_j^{\,n})_{\substack{j\in S_2^n \\ j\neq n-4 \\ j\neq \frac{n-7}{2}}}, \overline{x_{\frac{n-7}{2}} x_{\frac{n-1}{2}}} + \bar{x}_{n-4} \cdot \bar{x}_{n-4})$$

(we recall that $z^n_{\frac{n-7}{2}} = x_{\frac{n-7}{2}} \cdot x_{\frac{n-1}{2}} + x_{n-4}$)

It is clear that all these generators except $\bar{x}_{n-4}$ belong to Im p_* and $\partial_* \bar{x}_{n-4} = \bar{x}_{n-1} \neq 0$. Hence, $\bar{x}_{n-4} \notin$ Im p_* since by exactness of the triangle Im p_* = Ker ∂_*. As $d_3(\alpha . x_{n-4}) = \alpha x_{n-1}$ in B_n if α is a cycle we have $\partial_*(\overline{\alpha x}_{n-4}) = \bar{\alpha} . \bar{x}_{n-1}$ if $\bar{\alpha} \in H_*(B_{n-1})$. Thus

$$\text{Im } p_* = \Lambda_{\mathbb{Z}_2}((\bar{x}_i)_{i\in S_1^n}, (z_j^n)_{\substack{j\in S_2^n \\ j\neq n-4}})$$

Let R denote Im p_* = Ker ∂_*. $\partial_* : H_*(B_{n-1}) \to H_*((x_{n-1}))$ is R-linear since R = Ker ∂_* and ∂_* is a derivation. As an R-module $H_*(B_{n-1})$ is free on two generators 1, $\bar{x}_{n-4}$. Moreover, since

$d_3x_{n-1} = 0$, $d_3x_{n-4} = x_{n-1}$, the map $B_{n-1} \to (x_{n-1})$, $\alpha \mapsto \alpha\, x_{n-1}$ is an isomorphism of differential B_{n-1}-modules and $\partial_*(1)=0$, $\partial_*(\bar{x}_{n-4}) = \bar{x}_{n-1}$. Then $H_*((x_{n-1}))$ is a free R module on $\bar{x}_{n-1};\ \bar{x}_{n-1}\cdot\bar{x}_{n-4}$.

It follows from the exact triangle that

$$0 \to \underset{\substack{\| \\ R.\bar{x}_{n-4}\bar{x}_{n-1}}}{\text{coKer}\ \partial_*} \to H_*(B_n) \to \underset{\substack{\| \\ R.1}}{\text{Im}\ p_*} \to 0$$

is exact. Hence $H_*(B_n) \approx R.1 \oplus R.\bar{x}_{n-4}\bar{x}_{n-1}$ as a $\mathbb{Z}_2$-vector space. To check that $H_*(B_n)$ is an exterior algebra on the given elements it is enough to show that all the squares of those elements are zero. But we have:

(i) $(x_{n-4}x_{n-1})^2 = 0$ (trivial)

(ii) $(x_i)^2 = x_{2i} = \begin{cases} 0 \text{ if } 2i \notin S \\ d_3x_{2i-3} \text{ if } 2i \in S \end{cases}$

(iii) $(z_j^n)^2 = \begin{cases} x_{2j}.x_{2j+6} + x_{4j+6} = 0 & \text{if } 2j+6 \in S. \\ 0 & \text{if } 2j+6 \notin S. \end{cases}$ If $4j+6 \in S$

$d_3x_{4j+3} = x_{4j+6}$, otherwise $x_{4j+6} = 0$. On the other hand, $d_3(x_{2j}.x_{2j+3}) = x_{2j}.x_{2j+6}$. This completes the proof.

The case $\frac{n-1}{2}$ odd is easier since $z_j^{\,n} = z_j^{\,n-1}$ for $j \in S_2^{\,n-1}-\{n-4\}$ and so we do not need to change the basis of $H_*(B_{n-1})$. ⊟

2. $k^*(\text{Spin}(n);L)$

4.2.1. __Proposition__:

$$k^*(\text{Spin}(n);\mathbb{Q}) = \begin{cases} \Lambda_{Q(2)[t^{-1}]}(y_3, y_7, \ldots, y_{2n-3}) \text{ if } n \text{ is odd} \\ \Lambda_{Q(2)[t^{-1}]}(y_3, y_7, \ldots, y_{2n-5}, u_{n-1}) \text{ if } n \text{ is even.} \end{cases}$$

Proof:

It follows from 2.2.1 and 4.1.1. □

4.2.2. Lemma:

$\mathrm{Ker}\,[j^*:k^*(\mathrm{Spin}(n);\mathbb{Z}_2) \to K^*(\mathrm{Spin}(n);\mathbb{Z}_2)] = \{y \in k^*(\mathrm{Spin}(n);\mathbb{Z}_2)/ t^{-1} y = 0\}$

Proof:

Given $x \in k^r(\mathrm{Spin}(n);\mathbb{Z}_2), j^*(x) = t^{[\frac{r}{2}]}(t^{-[\frac{r}{2}]}x), t^{-[\frac{r}{2}]}x \in k^\varepsilon(\mathrm{Spin}(n);\mathbb{Z}_2)$ $= K^\varepsilon(\mathrm{Spin}(n);\mathbb{Z}_2)$ where $\varepsilon = 0, 1$ for r even or odd respectively.

Thus, if $t^{-1}x = 0$ then $j^*(x) = 0$. Conversely if $x \in \mathrm{Ker}\, j^*$, $t^{-1}x \in F_r(k^{r-2}(\mathrm{Spin}(n);\mathbb{Z}_2))$. By 1.4.2 $F_r(k^{r-2}(\mathrm{Spin}(n);\mathbb{Z}_2)) = F_r(K^{r-2}(\mathrm{Spin}(n);\mathbb{Z}_2))$. This implies $t^{-1}x = 0$. □

4.2.3. Proposition:

The Kernel of $j^*:k^*(\mathrm{Spin}(n);\mathbb{Z}_2) \to K^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ is mapped isomorphically onto $\mathrm{Im}\, d_3$ by the map $\eta^*:k^*(\mathrm{Spin}(n);\mathbb{Z}_2) \to H^*(\mathrm{Spin}(n);\mathbb{Z}_2)$.

Proof:

First we will show that $\eta^*(\mathrm{Ker}\, j^*) \subset \mathrm{Im}\, d_3$. We have a commutative diagram:

$\circledast$

$$\begin{tikzcd}
0 \arrow[r] & F_{i+2}(k^i(\mathrm{Spin}(n);\mathbb{Z}_2)) \arrow[r] \arrow[d, "\wr"] & F_i(k^i(\mathrm{Spin}(n);\mathbb{Z}_2) \arrow[r, "\alpha"] \arrow[d, "j^*"] & \mathrm{Ker}\, d_3 \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & F_{i+2}(K^i(\mathrm{Spin}(n);\mathbb{Z}_2) \arrow[r] & F_i(K^i(\mathrm{Spin}(n);\mathbb{Z}_2) \arrow[r, "\alpha'"] & \mathrm{Ker}\, d_3/\mathrm{Im}\, d_3 \arrow[r] & 0
\end{tikzcd}$$

where p is the projection, α is η^*(1.4.8) and the rows are exact

$j^*(y) = 0$ implies $\eta^*(y) \in \mathrm{Im}\, d_3$ since the diagram is commutative.

Now we will prove that $\eta^*(\mathrm{Ker}\, j^*) \supset \mathrm{Im}\, d_3$. Let $x \in \mathrm{Im}\, d_3$. Then there exists $y \in k^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ so that $\eta^*(y) = x$. By diagram $\circledast$, $\alpha' \cdot j^*(y) = 0$ since $\eta^*(y) \in \mathrm{Im}\, d_3$. Therefore there exists $z \in F_{i+2}(K^i(\mathrm{Spin}(n);\mathbb{Z}_2) = F_{i+2}(k^i(\mathrm{Spin}(n);\mathbb{Z}_2)$ and $\eta^*(z) = 0$ by exactness of the rows. Hence, $\eta^*(y - z) = x$ and $j^*(y - z) = 0$. $y - z$) is the required element of Ker j^*.

It remains to show that $\eta^*/_{\mathrm{Ker}\, j^*}$ is injective. Let $y \in \mathrm{Ker}\, j^*$ and $\eta^*(y) = 0$. Then $y \in \mathrm{Im}\, m_{t^{-1}}$. Since $j^*/\mathrm{Im}\, m_{t^{-1}}$ is injective, $y = 0$. $\square$

4.2.4. Proposition:

$k^*(Spin(n);\mathbb{Z}_2)/_{Ker j^*}$ is a $\mathbb{Z}_2[t^{-1}]$ exterior algebra generated by $(\bar{y}_k)_{k\in S_1}$ $(\bar{v}_{2j+3})_{j\in S_2}$ and $\bar{y}$ where degree $\bar{y}_k = k$, degree $\bar{v}_j = j$, degree $\bar{y} = 2^{s(n)-1}$, S_1, S_2, $s(n)$ as in 4.1.3 and 4.1.1.

Proof:

We have seen in 1.4.8 that the image of $\eta^*:k^*(Spin(n);\mathbb{Z}_2) \to H^*(Spin(n);\mathbb{Z}_2)$ is Ker d_3 and η^*:Ker $j^* \to$ Im d_3 is an isomorphism. Hence, η^* induces a surjective map:

$$(1)\quad \bar{\eta}^*:k^*(Spin(n);\mathbb{Z}_2)/_{Ker\ j^*} \to Ker\ d_3/_{Im\ d_3}$$

and an isomorphism:

$$(2)\quad \bar{\tilde{\eta}}^*:coKer\ m_{t^{-1}}/_{Ker\ j^*} \to Ker\ d_3/_{Im\ d_3}$$

Considering the Atiyah-Hirzebruch spectral sequence converging to $k^*(Spin(n):\mathbb{Z}_2)$, all the extension short exact sequences split since we are dealing with $\mathbb{Z}_2$-vector-spaces. Then we obtain:

$$(3)\quad k^r\,(Spin(n);\mathbb{Z}_2) \approx A^r \oplus \bigoplus_{i=0}^{[\frac{N-r}{2}]} B_r^{2i},\ \text{where } A^r = Im\ d_3^{\,r-3},$$

$B_r^{2i} = Ker\ d_3^{\,r+2i}/_{Im\ d_3^{\,r+2i-3}}$ with $d_3^{\,j}:H^j(Spin(n);\mathbb{Z}_2) \to H^{j+3}(Spin(n);\mathbb{Z}_2)$

$N = \dim\ (Spin(n))$.

(4) Given an element of odd degree $z \in k^*(\mathrm{Spin}(n);\mathbb{Z}_2)$, $\bar{z}^2 = 0$ where $\bar{z}$ is the corresponding element in $k^*(\mathrm{Spin}(n);\mathbb{Z}_2)/\mathrm{Ker}\, j^*$ (as $j^*(z)$ has odd degree $j^*(z)^2 = 0$).

Now we take elements $(\bar{y}_k)_{k\in S_1}$, $(\bar{v}_{2j+3})_{j\in S_2}$ and $\bar{y}$ in $k^*(\mathrm{Spin}(n);\mathbb{Z}_2)/\mathrm{Ker}\, j^*$ such that $\bar{\eta}^*(\bar{y}_k) = \bar{x}_k$, $\eta^*(\bar{v}_{2j+3}) = \bar{z}_j$, $\bar{\eta}^*(\bar{y}) = \bar{x}$. They are uniquely determined module $\mathrm{Im}\, m_{t^{-1}}$. Furthermore all of them have zero square, by (4). Then there exists an algebra homomorphism $g\colon \Lambda_{\mathbb{Z}_2}((\bar{y}_k)_{k\in S_1}, (\bar{y}_{2j+3})_{j\in S_2}\, \bar{y}) \to k^*(\mathrm{Spin}(n);\mathbb{Z}_2)/\mathrm{Ker}\, j^*$.

By (1), (2), (3) a similar method to that one used in the proof of the claim in 2.2.1 applies here to prove that g is an isomorphism. This finishes the proof of the proposition. □

4.2.5. <u>Proposition</u>

The torsion coefficients of $k^*(\mathrm{Spin}(n))$ are 2 and for all $y \in k^*(\mathrm{Spin}(n)$, $2y = 0$ if and only if $t^{-1}\, y = 0$.

<u>Proof</u>:

First we note that as with $k^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ the only non-zero differential in the Atiyah-Hirzebruch spectral sequence (E_r^{**}, d_r) for $k^r(\mathrm{Spin}(n)$ is d_3. Then the second part of the proposition follows from 1.4.5.

In the extension exact sequences

$$0 \longrightarrow F_{r-2}(k^r(\mathrm{Spin}(n))) \longrightarrow F_r(k^r(\mathrm{Spin}(n))) \rightarrow E_\infty^{r,0} \longrightarrow 0$$

$F_{r-2}(k^r(\mathrm{Spin}(n))) \simeq F_{r-2}(K^r(\mathrm{Spin}(n)))$ is torsion free and $E_\infty^{r,0}$ is the direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s by 4.1.1. This gives the first statement. □

4.2.6. <u>Example</u>: $k^*(\mathrm{Spin}(14);\mathbb{Z}_2)$

$H^*(\mathrm{Spin}(14);Z_2) = \Delta\,(x_3,x_5,x_6,x_7,x_9,x_{11},x_{12},x_{13},x_{15})$ where the subscripts indicate the degree (x_{15} was called x before).

We have: $d_3(x_3) = x_6$, $d_3(x_7) = x_{10}$, $d_3(x_9) = x_{12}$, $d_3(x_i) = 0$ otherwise. Also $d_3(x_3x_7) = x_6x_7 + x_3x_{10} = y_{13}$, $d_3(x_3x_9) = x_6x_9 + x_3x_{12} = y_{15}$, $d_3(x_7x_9) = x_9x_{10} + x_7x_{12} = y_{19}$, $d_3(x_3x_7x_9) = x_6x_7x_9 + x_3x_{10}x_9 + x_3x_7x_{12} = y_{22}$, $d_3(x_3x_6) = x_{12}$, $d_3(x_7x_{10}) = 0$, $d_3(x_9x_{12}) = 0$, $d_3(x_3x_6 + x_9) = 0$ (we put $z_3 = x_3x_6 + x_9$, $z_7 = x_7x_{10}$, $z_9 = x_9x_{12}$).

Im d_3 is the ideal of Ker d_3 generated by x_6, x_{10}, x_{12} and y_{13}, y_{19}, y_{22} since if $\alpha \in \operatorname{Im} d_3$ $\alpha = d_3(\beta)$ where β is the sum of simple monomials on the x_i's. Then $\alpha = \sum_{i_1<\dots<i_k} d_3(x_{i_1}\dots x_{i_k})$.

If we reorder each monomial as $x_{j_1}\dots x_{j_r}\dots x_{j_k}$ so that $x_{j_1}\dots x_{j_r}$ are all the elements in Ker d_3 then $d_3(x_{j_1}\dots x_{j}) = x_{j_1}\dots x_{j_r}z$

where z is one of the monomials y_{13}, y_{19}, y_{22}.

$$\mathrm{Ker}\, d_3 /_{\mathrm{Im}\, d_3} = \Lambda_{\mathbb{Z}_2}(\bar{x}_5, \bar{x}_{11}, \bar{x}_{13}, \bar{x}_{15}, \bar{z}_3, \bar{z}_7, \bar{z}_9)$$

Hence Ker d_3 is the subalgebra of $H^*(\mathrm{Spin}(n);\mathbb{Z}_2)$ generated by x_5, x_{11}, x_{13}, x_{15}, z_3, z_7, z_9, x_6, x_{10}, x_{12}, y_{13}, y_{19}, y_{22}. There is no easy way of describing the algebra structure of Ker d_3. We have, for example, $x_{10}\, y_{22} = y_{15}\, z_{17}$, $y_{19}\ \ y_{22} = 0$. Doing all the calculations we can find out all the products. The only element that has non-zero square is x_6.

$k^*(\mathrm{Spin}(14);\mathbb{Z}_2)$ is a $\mathbb{Z}_2[t^{-1}]$ algebra generated by elements u_i $i = (5,11,13,15)$ $v_j (j = 3,7,9)$, $w_k (k=6,10,12,13,19,22)$ such that

$$\eta^*(u_i) = x_i,\ \eta^*(v_j) = z_j,\ \eta^*(w_k) = \begin{cases} x_k & \text{for } k = 6,10,12 \text{ and } t^{-1}w_k = 0 \\ y_k & \text{for } k = 13,19,22 \end{cases}$$

Those elements w_k are uniquely determined. The products in Ker j^* are uniquely determined by the products in Im d_3 and

$$k^*(\mathrm{Spin}(14);\mathbb{Z}_2)/_{\mathrm{Ker}\, j^*} \approx \Lambda_{\mathbb{Z}_2[t^{-1}]}(u_5, u_{11}, u_{13}, u_{15}, v_3, v_7, v_9). \quad \square$$

APPENDIX

For $G = F_4, E_6$, $L = Z_2$ it is possible to calculate $k^*(G;\mathbb{Z}_2)$ using the same methods as with $k^*(G_2;\mathbb{Z}_2)$ since d_3 is the only non-zero differential in the spectral sequence converging to $K^*(G;\mathbb{Z}_2)$. For $G = F_4, E_6, E_7, E_8$ and $L = \mathbb{Z}_3$, $G = E_7, E_8$ and $L = \mathbb{Z}_2$, $G = E_8$ and $L = \mathbb{Z}_5$ it is more complicated because we have non-zero differentials in degrees greater or equal to 5 and then we can't apply the same method to detect whether a product is zero or not. However, working out the spectral sequences a lot of information can be obtained.

In the case $G = F_4$ it is possible to find without further complications an almost complete description of the algebra structure of $k^*(F_4;\mathbb{Z}_3)$ and $k^*(F_4;\mathbb{Z})$ (we could not calculate two squares) The latter can be calculated using the universal coefficient theorem, L. Smith's exact sequence and the Atiyah-Hirzebruch spectral sequence. We note that we have a complete description of $H^*(F_4;\mathbb{Z})$ and it is possible to find the Atiyah-Hirzebruch spectral sequence converging to $k^*(F_4;\mathbb{Z})$ by applying the "reduction mod q" map with $q = 2,3$ to the spectral sequence converging to $k^*(F_4;\mathbb{Z}_2)$ and $k^*(F_4;\mathbb{Z}_3)$ respectively. I do not put here those calculations because they are rather long and the methods used are exactly the same as for $k^*(G_2)$.

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