DYNKIN VARIETIES

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Thesis submitted for the degree of Doctor of Philosophy

at the University of Warwick

Mathematics Institute September 1977

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Acknowledgements.

Firstly I wish to thank my supervisor, Professor George Lusztig, for his constant help and encouragement during my stay at Warwick. I am also very grateful to Professor Roger Carter for many interesting discussions.

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Summary.

Let G be a linear algebraic group. The Dynkin variety $\mathcal{B}_x$ of an element x of G is the fixed point set of x on the variety $\mathcal{B}$ of all Borel subgroups of G. We show that all irreducible components of this variety have the same dimension, and that $\mathcal{B}_x$ is connected if x is unipotent.

Suppose now that G is reductive (but not necessarily connected) and that x is unipotent. We generalize an inequality linking $\dim \mathcal{B}_x$ and $\dim Z_G(x)$ and some results on the action of $A_0(x)$ on the set $S(x)$ of all irreducible components of $\mathcal{B}_x$, where $A_0(x)$ is the group of components of $Z_{G^0}(x)$. We consider also regular and sub-regular elements in non-connected reductive groups. For classical groups we get a combinatorial description for $S(x)$ and the action of $A_0(x)$ on $S(x)$ and a formula for $\dim \mathcal{B}_x$. We generalize to non-connected reductive groups a theorem of Richardson which associates to each conjugacy class of parabolic subgroups of G a unipotent class of G and for classical groups we get a combinatorial description of this map.

There is also some material on unipotent classes in arbitrary reductive groups.

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INTRODUCTION.

The main problem considered in this work is the study of Dynkin varieties. Let G be an algebraic group and let x be an element of G. Then the Dynkin variety $\mathcal{B}_x$ of x is the fixed point set of x on the variety $\mathcal{B}$ of all Borel subgroups of G. The problem can be reduced to the following one : G is reductive (or semisimple, or G^o is simple) and x is unipotent. In general we do not assume that G is connected.

Chapter 0 introduces some notations and collects results which shall be used frequently. The definition we use for the Weyl group W of G is similar to the one given in [7].

We need some informations about unipotent classes in reductive groups. As the literature deals mostly with connected groups (e.g. [16], [20], [2], [6], etc.), chapter I is devoted to this problem. In characteristic 2 (resp. 3) some new unipotent classes arise from the symmetries of order 2 (resp. 3) in the Dynkin diagrams of type A_n, D_n, E_6 (resp. D_4). For example in characteristic 2 we get unipotent classes in $O_{2n} \backslash SO_{2n}$. This case has been studied in [23].

In I.1 we show how the study of unipotent classes in reductive groups can be reduced to the case where G^o is simple. We state the following result : a reductive group has only finitely many unipotent classes. This result was already known for connected reductive groups ([6], and [12] with some mild restrictions on the characteristic). The proof depends on a case by case study which is carried out in I.2, I.3 and I.4 and on the proof for connected groups (it is therefore a result due to George Lusztig since the proof in I.4 and the proof for the connected case are his). We fix also notations for the parametrization of unipotent classes in the classical groups.

In I.2 we study the case of the symmetry of order 3 in the Dynkin diagram of type D_4 in characteristic 3.

In I.3 we define unipotent bilinear forms in characteristic 2. The study of classes of such forms provides a parametrization of the unipotent classes arising from the symmetry of order 2 in the diagrams of type A_n. The results are very similar to the results obtained for the symplectic groups in characteristic 2 in [23].

The results in I.4 are due to George Lusztig. The aim of this section is to show that there are only finitely many unipotent classes arising from the symmetry of order 2 in the Dynkin diagram of type E_6 in characteristic 2.

In chapter II we study Dynkin varieties and we consider a few applications. Many results in this chapter are generalizations of results known for connected groups.

In II.1 we consider some projective lines contained in $\mathcal{B}_u$ (u unipotent). These lines where introduced first by Tits for connected groups. We use these lines to prove that $\mathcal{B}_u$ is connected and that all irreducible components of $\mathcal{B}_u$ have the same dimension. The last result is true even if u is not unipotent. We give also a proof (using Dynkin varieties) of the fact that the centralizer of a semisimple element in a reductive group is reductive.

In II.2 we use an idea introduced in [20] (and developped in [21]) to generalize a relation between dim $\mathcal{B}_u$ and dim $Z_G(u)$ and some results on a natural application $S(u) \times S(u) \longrightarrow W^u$, where $S(u)$ is the set of irreducible components of $\mathcal{B}_u$ (u unipotent, G reductive). 2.10 was also obtained for connected groups by Steinberg [21], M. Cross (unpublished) and by Springer [17] (with some restrictions on the characteristic) and 2.11 and 2.12 are obtained in [21] (with equality) essentially for GL_n. We consider then unipotent quasisemisimple elements and we get several characterizations for quasisemisimple elements in reductive groups.

In II.3 we generalize a theorem of Richardson [13] and we define P-regular (unipotent) classes (P a parabolic subgroup of G^o, G reductive). The proof given here is close to the proof of Richardson's theorem given in [21]. We consider then regular and subregular elements and we show that these elements can be defined by the properties dim $\mathcal{B}_x = 0$ and dim $\mathcal{B}_x = 1$ respectively. We look also at sub-subregular unipotent elements.

In II.4 we study a natural morphism $p : \mathcal{B}_u \longrightarrow \mathcal{P}_u^o$, where $\mathcal{P}_u^o$ is the fixed point set of u on some G^o-conjugacy class of parabolic subgroups of G. We attach also to each irreducible component X_σ of $\mathcal{B}_u$ a set I_σ of fundamental reflections. We show how to use the morphism p to get informations about $\mathcal{P}_u^o$ if we know enough about $\mathcal{B}_u$ and the sets $(I_\sigma)_{\sigma \in S(u)}$. We show also how p can be used to study $\mathcal{B}_u$ if the conjugacy class of parabolic subgroups is choosen in a suitable way.

In II.5 we consider Dynkin varieties for classical groups (we include here the groups G(V) defined in I.3). We get a combinatorial description of S(u) and of the action of $Z_G(u)/Z_G(u)^o$ on S(u). If $G = Sp_{2n}$ or O_n and the characteristic is not 2, Hesselink has also obtained such a combinatorial description (unpublished). The Dynkin varieties for these groups have also been studied by B. Srinivasan in connection with the representation theory of the finite classical groups. The best results are obtained for GL_n. If u is unipotent, then S(u) can be parametrized by standard tableaux [14], [21], and $\mathcal{B}_u$ can be decomposed into a union of affine spaces [10], [14]. We compute also the sets I_σ ($\sigma \in S(u)$) in all cases. For GL_n we can then reverse the proof connecting the varieties $\mathcal{P}_u$ (for various conjugacy classes of parabolic subgroups) and the representation theory of $S_n \cong W$ given by Steinberg in [21]. We compute also dim $\mathcal{B}_u$ in all cases and we check that we have actually an equality in the formula of II.2.5.

In II.6 we consider the same groups as in II.5. Let P be a parabolic subgroup of G^o. Using the results of II.5 we determine which unipotent class is P-regular. We look then at some questions related to the parametrization of unipotent classes given by Bala and Carter [1].

In II.7 we consider some equivalence relations in the Weyl groups. These relations have properties similar to those of an equivalence relation arising in the theory of primitive ideals in enveloping algebras. Using results of [11], we can then prove that in the case of GL_n the application $S(u) \times S(u) \longrightarrow W \cong S_n$ is the inverse of a combinatorial map introduced by Robinson and Schensted.

In II.8 we give a few examples which are actually counterexamples.

In II.9 we give tables for the Dynkin varieties of unipotent elements for some of the groups considered in II.5. These tables give in particular the dimension and the number of irreducible components of $\mathcal{B}_u$. It is also possible to deduce from the table how the group $Z_G(u)/Z_G(u)^o$ acts on S(u) (this group is described in I.1.12 or I.3.23). Examination of the tables suggests that the results of Springer [17] should be true for all reductive groups and for all characteristics.

CHAPTER 0.

NOTATIONS.

0.1. Unless otherwise stated, k will always be an algebraically closed field. p denotes its characteristic (0 or a prime).

0.2. All algebraic groups considered here are affine algebraic groups defined over k.

0.3. G will always be an algebraic group and $\mathcal{B} = \mathcal{B}(G)$ is its variety of Borel subgroups. G acts on $\mathcal{B}$ by conjugation and the Dynkin variety $\mathcal{B}_x$ of $x \in G$ is the fixed point set of x on $\mathcal{B}$. $\mathcal{B}_x$ is never empty [19, thm 7.2]. $Z(x) = Z_G(x)$ acts on $\mathcal{B}_x$ and therefore $A(x) = Z(x)/Z(x)^o$ acts on the set S(x) of all irreducible components of $\mathcal{B}_x$. Unless otherwise stated we shall also write these components as $(X_\sigma)_{\sigma \in S(x)}$. Let $Z_o(x) = Z(x) \cap G^o$, $A_o(x) = Z_o(x)/Z(x)^o$.

If G' is a closed subgroup of G normalized by x, then we can define $\mathcal{B}(G')_x$ in a similar way.

0.4. Let W be the set of G^o-orbits in $\mathcal{B} \times \mathcal{B}$ (for the action of G given by $g.(B_1, B_2) = ({}^gB_1, {}^gB_2)$). W is finite and is a group in a natural way. We shall often write w for an element of W considered as a finite group and O(w) for the same element considered as a subvariety of $\mathcal{B} \times \mathcal{B}$. The length of w is $\ell(w) = \dim O(w) - \dim \mathcal{B}$. The elements of length 1 are called fundamental reflections and we denote by Π the set of all fundamental reflections. The composition law in W is given by :

a) $w^2 = 1$ if $w \in \Pi$.

b) If w, $w' \in W$ are such that $\dim (O(w) \circ O(w')) - \dim \mathcal{B} = \ell(w) + \ell(w')$, then $O(w) \circ O(w')$ is the G^o-orbit $O(ww')$ in $\mathcal{B} \times \mathcal{B}$ (where $O(w) \circ O(w') = \{(B_o, B_2) \mid \exists B_1 \in \mathcal{B}$ such that $(B_o, B_1) \in O(w)$ and $(B_1, B_2) \in O(w')\}$).

With this composition law, W will be called the Weyl group of G (or G^o).

Π generates W and for any $w \in W$, $\ell(w)$ is the smallest $j \in \mathbb{N}$ such that w can be expressed as $w = s_1 s_2 \ldots s_j$ for some $s_1, \ldots, s_j \in \Pi$.

0.5. The action of G on $\mathcal{B} \times \mathcal{B}$ induces an action of G (or G/G^o) on W which we shall denote $(g,w) \longmapsto g.w$. For any $x \in G$, $w \longmapsto x.w$ is an automorphism of W and it permutes the fundamental reflections. Denote by $\overline{w}$ the G-orbit of $w \in W$ and let $\overline{W} = \{\overline{w} \mid w \in W\}$.

0.6. Suppose now that we are considering a fixed element $x \in G$ (or a fixed component xG^o of G). Let o(s) be the x-orbit of $s \in \Pi$ (i.e. the orbit under the action of the cyclic group generated by x). Let $\tilde{s}$ be the element of maximal length in the subgroup of W generated by o(s). W^x, the fixed point set of x in W, can be regarded as a Coxeter group with $\{\tilde{s} \mid s \in \Pi\}$ as a system of generators. Its length function is $\ell_x(w) = \min \{j \in \mathbb{N} \mid w = \tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_j$ for some $s_1, \ldots, s_j \in \Pi\}$. Such a decomposition $w = \tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_j$ has a minimal number of terms if and only if $\ell(w) = \ell(\tilde{s}_1) + \ldots + \ell(\tilde{s}_j)$.

0.7. If $(B_o, B_2) \in O(ww')$ and $\ell(w) + \ell(w') = \ell(ww')$, there is a unique $B_1 \in \mathcal{B}$ such that $(B_o,B_1) \in O(w)$ and $(B_1,B_2) \in O(w')$, and this defines a morphism $O(ww') \longrightarrow \mathcal{B}$. If $w = s_1 s_2 \dots s_j$ ($j = \ell(w)$, $s_1,\dots,s_j \in \Pi$) and $(B_o,B_j) \in O(w)$, then we can find $B_1, B_2, \dots, B_{j-1} \in \mathcal{B}$ such that $(B_{i-1},B_i) \in O(s_i)$ $(1 \le i < j)$ and these Borel subgroups are unique. Similarly, if $w \in W^x$ and $w = \tilde{s}_1 \dots \tilde{s}_j$ ($j = \ell_x(w)$, $s_1,\dots,s_j \in \Pi$) and $(B_o,B_j) \in O(w)$, then we can find $B_1,\dots,B_{j-1} \in \mathcal{B}$ such $(B_{i-1},B_i) \in O(\tilde{s}_i)$ $(1 \le i < j)$ and these are unique.

0.8. If X,Y are irreducible subvarieties of $\mathcal{B}$, define $\varphi(X,Y)$ to be the unique element w of W such that $(X \times Y) \cap O(w)$ is dense in $X \times Y$. In particular, we get for each $x \in G$ an application φ: $S(x) \times S(x) \longrightarrow W$, $(\sigma,\tau) \longmapsto \varphi(\sigma,\tau) = \varphi(X_\sigma, X_\tau)$. Define also $\overline{\varphi}(\sigma,\tau) \in \overline{W}$ to be the orbit of $\varphi(\sigma,\tau)$ in W.

0.9. Unless otherwise stated, B is a fixed Borel subgroup of G and T is a fixed maximal torus of B. U is the maximal unipotent subgroup of B and $N = N_G(B)$.

0.10. Suppose now that G is reductive. W can be identified with $N_{G^o}(T)/T$. If $n \in N_{G^o}(T)$, nT corresponds to the G^o-orbit of $(B, {}^nB)$ in $\mathcal{B} \times \mathcal{B}$. In this way W is a normal subgroup of $N_G(T)/T$ and the action of G on W defined in 0.5 corresponds to the action by conjugation of $N_G(T)/T$ on W.

Let $\Delta(G^o)$ be the Dynkin diagram of G^o. Its nodes are in bijective correspondence with Π. G/G^o acts on $\Delta(G^o)$ and it will be convenient to use the following definition. The Dynkin diagram $\Delta(G)$ of G consists of $\Delta(G^o)$, of the finite group G/G^o and of the natural homomorphism $G/G^o \longrightarrow \Gamma(G^o)$, where $\Gamma(G^o)$

is the group of automorphisms of $\triangle(G^o)$.

Let ϕ be the root system of G^o (with respect to T). For each $\alpha\in\phi$, let X_α be the corresponding unipotent 1-dimensional subgroup of G. We choose fixed isomorphisms $x_\alpha : \mathbb{G}_a \longrightarrow X_\alpha$, where $\mathbb{G}_a$ is the additive group of k viewed as an algebraic group. ϕ^+ will be the set of positive roots (with respect to B). $s_\alpha \in W$ is the reflection corresponding to $\alpha\in\phi$. The fundamental roots correspond bijectively to the fundamental reflections and Π will also denote the set of fundamental roots of G^o. If $\alpha = \sum_{1\leq i\leq r} n_i\alpha_i$ $(\alpha_1,\dots,\alpha_r \in \Pi)$, let $ht(\alpha) = \sum_{1\leq i\leq r} n_i$.

0.11. Let P,Q,... be parabolic subgroups of G. We denote by the corresponding letter $\mathcal{P}, \mathcal{Q},\dots$ the conjugacy class of P, Q,... respectively and by $\mathcal{P}^o$, $\mathcal{Q}^o$,... the G^o-conjugacy class of P,Q,... respectively. There is in every G^o-conjugacy class of parabolic subgroups of G a unique subgroup containing B. Let W_P be the Weyl group of the parabolic subgroup P of G. The inclusion $\mathcal{B}(P)\times\mathcal{B}(P)\subset\mathcal{B}\times\mathcal{B}$ induces a natural homomorphism $W_P \longrightarrow W$. This homomorphism is injective and we shall regard W_P as a subgroup of W. $P \longmapsto W_P\cap\Pi$ induces a bijection between the set of G^o-conjugacy classes of parabolic subgroups of G^o and the collection of all subsets of Π. If P is a parabolic subgroup of G^o and $I = W_P\cap\Pi$, let $\mathcal{P}_I = \mathcal{P}$, $\mathcal{P}_I^o = \mathcal{P}^o$. If $x\in G$, ${}^xP\in\mathcal{P}^o$ if and only if I is x-stable.

Suppose now that G is reductive. If $P\supset B$ is a parabolic subgroup of G, the corresponding subset of Π can also be defined as $I = \{\alpha\in\Pi | X_{-\alpha}\subset P\}$. Let $L\supset T$ be a Levi subgroup of P. Then $N_L(T)/T = N_P(T)/T \subset N_{G^o}(T)/T$ and this inclusion is compatible with the inclusion $W_P\subset W$.

<u>0.12</u>. R_G is the radical of G and U_G is the unipotent radical of G. If $x \in G$, C(x) is the conjugacy class of x and $C^o(x)$ is the G^o-conjugacy class of x. If x = su is the Jordan decomposition of x, s will always be the semisimple part of x and u the unipotent part of x.

<u>0.13</u>. |X| is the cardinal of the set X and if A is a group acting on X, X/A is the set of A-orbits in X.

<u>0.14</u>. We shall frequently use the following easy result. Let $f : X \longrightarrow Y$ be a surjective morphism of algebraic varieties. Suppose that f is open or closed and that all fibres of f are irreducible and have the same dimension. Then the inverse image of any irreducible subvariety of Y is irreducible.

CHAPTER I.

COMPLEMENTS ON UNIPOTENT CLASSES IN REDUCTIVE GROUPS.

1. General results.

1.1. Let x be any element of G. We define $\mathrm{rank}_x(G)$ to be the following integer.

a) If G^o is a torus, $\mathrm{rank}_x(G) = \dim Z_G(x)$.

b) If G^o is soluble, $\mathrm{rank}_x(G) = \mathrm{rank}_{xU_G}(G/U_G)$.

c) In general, let $N_1 = N_G(B_1)$, where B_1 is any element of $\mathcal{B}_x$. Then $\mathrm{rank}_x(G) = \mathrm{rank}_x(N_1)$.

It is easy to check in each case that $\mathrm{rank}_x(G) = \mathrm{rank}_y(G)$ if $y \in xG^o$, that in (c) $\mathrm{rank}_x(N_1)$ is independent of the choice of $B_1 \in \mathcal{B}_x$ and that (b) generalizes (a) and (c) generalizes (b).

Remark 1.2. a) If $x \in G^o$ it is clear that $\mathrm{rank}_x(G) = \mathrm{rank}_1(G)$ is the rank of G, i.e. the dimension of a maximal torus of G.
b) $\mathrm{rank}_x(G) = \mathrm{rank}_{xU_G}(G/U_G)$. This is clear from the definition.
c) If x is semisimple, $\mathrm{rank}_x(G)$ is the rank of $Z(x)$. To show that, assume first that $G^o = B$ is soluble. For every torus S in G, the natural homomorphism $S \longrightarrow G/U_G$ is injective. It follows that if S is a maximal torus in $Z(x)$, then $\mathrm{rank}_x(G) \geq \dim S = \mathrm{rank}_1(Z(x))$. On the other hand, G contains a maximal torus T normalized by x [19, p.51] and if $t \in T$ is such that $xtx^{-1}U_G = tU_G$, we must have $xtx^{-1} = t$. Hence $\dim Z_T(x) \geq \mathrm{rank}_x(G)$. This shows that if G^o is soluble, then $\mathrm{rank}_x(G) = \mathrm{rank}_1(Z(x))$ and that for any maximal torus T of G normalized by x, $Z_T(x)^o$ is a maximal torus of $Z(x)$.

This shows that in the general case $\mathrm{rank}_x(G) = \mathrm{rank}_x(N_1) = \mathrm{rank}_1(Z_{B_1}(x))$ for every B_1 as in 1.1 (c). Choose B_1 in a closed $Z(x)$-orbit in $\mathfrak{B}_x$. Then $Z_{B_1}(x)^o = B_1 \cap Z(x)^o$ is a Borel subgroup of $Z(x)$ and therefore $\mathrm{rank}_1(Z(x)) = \mathrm{rank}_1(Z_{B_1}(x))$. This proves that if x is semisimple and G is any algebraic group, then $\mathrm{rank}_x(G) = \mathrm{rank}_1(Z(x))$.

Proposition 1.3. Let Z be a normal subgroup of G^o consisting of semisimple elements. If u and uz are both unipotent ($g \in G$, $z \in Z$) then they are Z-conjugate. In particular the canonical morphism $G \longrightarrow G/Z$ induces a bijection between the set of unipotent classes of G and the set of unipotent classes of G/Z.

Proof. Since $Z \subset G^o$ is normal and consists of semisimple elements, it is contained in every maximal torus of G. In particular Z is commutative and it is contained in some $B_1 \in \mathfrak{B}_u$. It is sufficient to prove the proposition for $N_G(B_1)$. So we may as well assume that G^o is soluble.

Let q be the order of uG^o in G/G^o. q is a power of p. Suppose that u and uz are unipotent. Then u^q and $(uz)^q$ are unipotent elements of G^o and hence are contained in U_G. But $(uz)^q = (uzu^{-1})(u^2zu^{-2})\ldots(u^qzu^{-q})u^q$ and $(uzu^{-1})\ldots(u^qzu^{-q}) \in Z$. Therefore $(uzu^{-1})\ldots(u^qzu^{-q}) = 1$.

Define $f : Z \longrightarrow Z$, $z \longmapsto uzu^{-1}$. Then f^q is the identity since $u^q \in U_G$. Define also $\varphi : Z \longrightarrow Z$, $z \longmapsto zf(z)f^2(z)\ldots f^{q-1}(z)$ and $\psi : Z \longrightarrow Z$, $z \longmapsto zf(z)^{-1}$. Since f^q is the identity, $\varphi \circ f = f \circ \varphi = \varphi$ and therefore $\varphi^2(z) = \varphi(z)^q$ for all $z \in Z$. But $z \longmapsto z^q$ is a bijective endomorphism of Z and therefore $\mathrm{Im}\varphi \cap \mathrm{Ker}\varphi = 1$. This shows that Z is the direct product of $\mathrm{Im}\varphi$ and $\mathrm{Ker}\varphi$. Also

$\varphi(\psi(z)) = \psi(\varphi(z)) = 1$ for all $z \in Z$. Hence $\mathrm{Im}\varphi \subset \mathrm{Ker}\psi$ and $\mathrm{Ker}\varphi \supset \mathrm{Im}\psi$. Moreover, if $z \in \mathrm{Ker}\psi$, then $z = z'^q$ for some $z' \in \mathrm{Ker}\psi$ and then $\varphi(z') = z'^q = z$. Hence $\mathrm{Ker}\psi \subset \mathrm{Im}\varphi$. Since Z is the direct product of $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$, these inclusions show that $\mathrm{Ker}\varphi = \mathrm{Im}\psi$ and $\mathrm{Im}\,\varphi = \mathrm{Ker}\psi$.

We can now prove the proposition. If u is unipotent, uz is unipotent if and only if $z \in \mathrm{Ker}\varphi$. But then $z = tf(t)^{-1}$ for some $t \in Z$ and $f(t)uf(t)^{-1} = utf(t)^{-1} = uz$.

Corollary 1.4. Let uG^o be a unipotent component of G. Let V be the variety of all unipotent elements contained in uG^o. Then V is a closed irreducible subvariety of G and $\mathrm{codim}_G V = \mathrm{rank}_u(G)$.

Proof. V is clearly closed. For the other statements, we consider three cases.

a) If G^o is a torus, this follows immediately from 1.3 with $Z = G^o$.

b) An element of G is unipotent if and only if its image in G/U_G is so. If G^o is soluble, the result follows then from (a), 1.2 (b) and 0.14.

c) In the general case, we may clearly assume that $B \in \mathcal{B}_u$. Let V' be the variety of all unipotent elements in uB. Then V is the image of $G^o \times V'$ under the morphism $(g,x) \longmapsto gxg^{-1}$. This shows that V is irreducible and in order to prove that $\mathrm{codim}_G V = \mathrm{rank}_u(G)$ it is sufficient to prove that for some $v \in V$, $|\mathcal{B}_v| < \infty$. This will be done in II.1.8.

1.5. Suppose now that G is reductive. The group $H = \mathrm{Aut}(G^o/R_G)$

can be considered as an algebraic group with identity component isomorphic to the adjoint group of G^o. There is a natural isomorphism $H/H^o \cong \Gamma(G^o)$. We also have a homomorphism $G/G^o \longrightarrow \Gamma(G^o)$ and we can form the fibre product $G^* = \{(gG^o,h) \in G/G^o \times H \mid g$ and h have the same image in $\Gamma(G^o)\}$. This fibre product depends only on $\Delta(G)$ (as defined in 0.10) and there is a natural epimorphism $G \longrightarrow G^*$.

<u>Proposition 1.6</u>. Let G be a reductive group. Then the unipotent classes of G depend only on $\Delta(G)$ (for a given k). More precisely the natural epimorphism $G \longrightarrow G^*$ induces a bijection between the set of unipotent classes of G and the set of unipotent classes of G^*.

<u>Proof</u>. Let Z be the centre of G^o. $G \longrightarrow G^*$ factors as $G \longrightarrow G/Z \longrightarrow G^*$. $G/Z \longrightarrow G^*$ is bijective and therefore induces a bijection from the set of unipotent classes of G/Z to the set of unipotent classes of G^* and the same is true for $G \longrightarrow G/Z$ by 1.3.

<u>1.7</u>. Suppose now that G^o is an adjoint semisimple group. To classify the unipotent classes in G we must solve two problems.

a) Determine the unipotent classes of G/G^o.

b) Choose a component uG^o in each unipotent class of G/G^o and determine the unipotent classes of H contained in uG^o, where $G^o \subset H \subset G$ and $H/G^o = Z_{G/G^o}(uG^o)$.

We shall not consider (a) here.

<u>1.8</u>. Suppose that G^o is semisimple and adjoint. Let uG^o be a unipotent component of G/G^o. We assume that uG^o is central in

G/G^o.

Let $G_1,\dots,G_s$ be the minimal connected normal subgroups of G. We have homomorphisms $G/G^o \longrightarrow \Gamma(G_i)$ and as in 1.5 we can use these homomorphisms to get fibre products G_i^* with $G_i^{*o} \cong G_i$ and $G_i^*/G_i^{*o} \cong G/G^o$. Let u_i be an element in the component of G_i^* corresponding to uG^o. G is naturally isomorphic to $\{(g_1,\dots,g_s) \in \prod_{1\le i\le s} G_i^* \mid g_1,\dots,g_s$ correspond to the same component of $G/G^o\}$. Let X_i be the set of unipotent G_i^{*o}-classes in $u_iG_i^{*o}$. G/G^o acts on X_i. The set X of all unipotent G^o-classes in uG^o can be identified with $\prod_{1\le i\le s} X_i$ and the unipotent classes correspond to the G/G^o-orbits in X or in $\prod_{1\le i\le s} X_i$. So we need only to determine the sets X_i and the action of G/G^o on these sets.

<u>1.9</u>. Assume that G^o is semisimple and adjoint, that uG^o is a central unipotent element of G/G^o and that 1 and G^o are the only normal connected subgroups of G. Let H be the subgroup of G generated by G^o and u. Let $H_1,\dots,H_r$ be the minimal connected normal subgroups of H. Using the homomorphisms $H/H^o \longrightarrow \Gamma(H_i^o)$, we get fibre products H_i^* and H can be considered as a subgroup of $\prod_{1\le i\le r} H_i^*$. Notice that $H_1 \cong \dots \cong H_r$ and $H_1^* \cong \dots \cong H_r^*$ and to each $g \in G$ corresponds a family of isomorphisms $H_i^* \longrightarrow H_{\sigma(i)}^*$, where σ is some permutation of $\{1,\dots,r\}$. Let X_i be the set of all H_i^{*o}-classes of unipotent elements in the component of H_i^* corresponding to uG^o. For each component gG^o of G we get a family of bijections $X_i \longrightarrow X_{\sigma(i)}$ and this defines an action on $X = \prod_{1\le i\le r} X_i$. X can be identified with the set of all unipotent G^o-classes in uG^o and the unipotent classes in uG^o correspond to the G/G^o-orbits in X. The bijections

$x_i \longrightarrow x_{\sigma(i)}$ attached to a component gG^o of G can be determined from $\triangle(G)$ if we can solve the problem in the case where $r = 1$.

So assume that 1 and G^o are the only connected u-stable normal subgroups of G^o. Let $G_1,\dots,G_m$ be the minimal normal subgroups of G^o. We may assume that ${}^uG_i = G_{i+1}$ for $1 \leqslant i \leqslant m-1$ and ${}^uG_m = G_1$. It is easily checked that the unipotent G^o-classes in uG^o correspond bijectively to the unipotent G_1-classes in the component u^mG_1 of the subgroup of G generated by G_1 and u^m, and to the action of G/G^o on the set of unipotent G^o-classes in uG^o corresponds an action on the set of unipotent G_1-classes in u^mG_1 which can be determined from $\triangle(G)$ if we can solve the same problem when $m = 1$. But in this case we can clearly replace G by its image in $Aut(G^o)$. It is therefore sufficient to determine the set X of unipotent G^o-classes in uG^o and the action of G/G^o on X in the following case. G^o is an adjoint simple group, G is a subgroup of $Aut(G^o)$ and $G/G^o = Z_{Aut(G^o)/G^o}(uG^o)$. This problem is not yet completely solved.

<u>Proposition 1.10</u>. A reductive group has only finitely many unipotent classes.

<u>Proof</u>. A connected reductive group has only finitely many unipotent classes [6]. If $p = 2$, O_{2n} has finitely many unipotent classes [23]. This shows that the symmetry of order 2 in the diagrams of type D_n gives only finitely many unipotent classes. We shall show in 3.18 (resp. 4.7, 2.4) that the same is true for the symmetry of order 2 of A_n (resp. the symmetry of order 2 of E_6, the symmetries of order 3 of D_4 if $p = 3$). The result follows then from 1.6, 1.8 and 1.9.

1.11. Let X be the set of unipotent G^0-classes contained in G^0, where G^0 is a simple adjoint group and $G = Aut(G^0)$. If p is 0 or is large, X can be described by weighted Dynkin diagrams [2] or equivalently by G^0-conjugacy classes of pairs (L,P) where L is a Levi subgroup of some parabolic subgroup of G^0 and P is a distinguished parabolic subgroup of L [1]. If X is described by weighted Dynkin diagrams, the action of G/G^0 on X is given by the obvious action of $\Gamma(G^0)$ on the set of weighted Dynkin diagrams corresponding to $\Delta(G^0)$. This action is trivial except in the following cases.

a) D_4. Then $\Gamma(G^0)$ is isomorphic to S_3, and there are two orbits consisting of three unipotent classes. $\Gamma(G^0)$ acts trivially on the other G^0-classes.

b) D_{2n}, $n \geqslant 3$. There are p(n) orbits consisting of two classes, where p(n) is the number of partitions of n. $\Gamma(G^0)$ acts trivially on the other G^0-classes.

1.12. We shall use the following notations for the unipotent classes in GL_n, $Sp_{2n} \subset GL_{2n}$, $O_n \subset GL_n$ (in the last case we suppose that n is even if p = 2).

The class of a unipotent element $u \in GL_n$ will be described by the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ (infinite sequence) where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ are 0 or the dimensions of the Jordan blocks of u. Let $C_\lambda = C(u)$. C_λ and the partition λ will also be represented by the Young diagram d_λ (sometimes denoted d_n) with lines of length $\lambda_1, \lambda_2, \ldots$. In such a situation ℓ_i will be the length of the i^{th} column of d_λ and c_i will be the number of lines of length i (i.e. the number of Jordan blocks of u of dimension i). Each of the sequences $\ell_1, \ell_2, \ldots$ and c_1,

$c_2, \dots$ determines the class of u completely.

If $p \neq 2$, each unipotent class of Sp_{2n} (resp. O_n) is the intersection of Sp_{2n} (resp. O_n) with a unipotent class of GL_{2n} (resp. GL_n). The unipotent classes of Sp_{2n} (resp. O_n) correspond in this way to the partitions of 2n (resp. n) for which c_i is even if i is odd (resp. if i is even). A unipotent class of O_n for which all λ_i and all c_i are even consists of two classes in SO_n. In all other cases the unipotent classes of O_n and SO_n coincide.

It will be convenient to attach to a unipotent class in Sp_{2n} (resp. O_n) with partition λ an application $\varepsilon : \mathbb{N} \longrightarrow \{\omega, 0, 1\}$ defined as follows. $\varepsilon_i = 1$ if i is even (resp. odd) and $c_i \neq 0$, and $\varepsilon_i = \omega$ otherwise. This is to simplify the notations in paragraph II.5. We denote by C_λ or $C_{\lambda,\varepsilon}$ the unipotent class corresponding to λ.

To describe the unipotent classes in Sp_{2n} and $O_{2n} \subset Sp_{2n}$ if $p = 2$, we shall use a set with 3 elements $\{\omega, 0, 1\}$. It will be convenient to give it an ordering $\omega < 0 < 1$. The unipotent classes of Sp_{2n} are in bijective correspondence with the pairs (λ, ε) satisfying the following conditions.

a) λ is a partition of 2n with c_i even if i is odd.

b) $\varepsilon : \mathbb{N} \longrightarrow \{\omega, 0, 1\}$ is an application satisfying :

b_1) $\varepsilon_i = \omega$ if i is odd or if $c_i = 0$ and $i \geqslant 1$.

b_2) $\varepsilon_i = 1$ if i is even and c_i is odd.

b_3) $\varepsilon_i \neq \omega$ if i is even and $c_i \geqslant 1$.

b_4) $\varepsilon_0 = 1$.

The correspondence is as follows. A unipotent element $u \in Sp_{2n}$ determines a class in GL_{2n}, hence a partition, and this partition has the required property. Moreover, if $i \geqslant 2$ is even and $c_i \geqslant 1$, we put $\varepsilon_i = 0$ if and only if $f((u-1)^{i-1}(x),x) = 0$ for all x such that $(u-1)^i(x) = 0$ (here f is the bilinear form used to define Sp_{2n}). With condition (b) this defines ε completely.

Each unipotent class of Sp_{2n} intersects O_{2n} in a single class of O_{2n} ($p = 2$). The unipotent classes in O_{2n} can therefore be represented by pairs (λ, ε) as above. In this case however we replace (b_4) by (b_4') :

$b_4')$ $\varepsilon_0 = 0$.

The unipotent classes of O_{2n} contained in SO_{2n} are those with ℓ_1 even. A unipotent class such that all ${}_i$ and c_i are even and ${}_i \neq 1$ for all i consists of two classes of SO_{2n}. All other unipotent classes of SO_{2n} are classes in O_{2n}.

We shall denote by $C_{\lambda,\varepsilon}$ the unipotent class of Sp_{2n} or O_{2n} corresponding to the pair (λ, ε).

The results in this paragraph are contained in [23].

<u>1.13</u>. Let $a_1, a_2, \ldots$ be a sequence of distinct elements. Let G be one of the groups GL_n, Sp_{2n} or O_n (if $p = 2$ and $G = O_n$, we assume that n is even). If u is a unipotent element of G and if the class of u is represented by λ or (λ, ε), the group A(u) can be described in the following way :

a) $G = GL_n$. Then $A(u) = 1$.

b) $G = Sp_{2n}$ (resp. O_n) and $p \neq 2$. Then A(u) is the abelian group generated by $\{a_i \mid \lambda_i$ is even (resp. odd)$\}$ with the following relations. $a_i a_j = 1$ if $\lambda_i = \lambda_j$ and $a_i = 1$ if $\lambda_i = 0$. In the case of O_n, the subgroup $A_0(u)$ of A(u) consists of the elements which are the product of an even number of generators.

c) $G = Sp_{2n}$ (resp. O_{2n}) and $p = 2$. Then $A(u)$ is the abelian group generated by $\{a_i \mid \varepsilon_{\lambda_i} \neq 0\}$ with the following relations. $a_i a_j = 1$ if $\lambda_i = \lambda_j$, or $\lambda_i = \lambda_j + 1$, or if $\lambda_i = \lambda_j + 2$ and λ_i is even, and $a_i = 1$ if $\lambda_i = 0$. In the case of O_{2n}, $A_o(u)$ is the subgroup of $A(u)$ consisting of elements which are the product of an even number of generators.

This description of $A(u)$ can be deduced from [2] and [23]. Another method is to adapt the proof of 3.21.

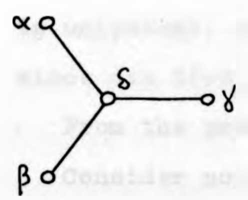
2. Groups of type D_4.

Here G is a reductive group such that G^o is of type D_4. We shall use results and notations of chapter II to simplify some proofs and some statements.

2.1. The classification of unipotent classes of G^o given by Bala and Carter [1] works for all characteristics and G/G^o acts as indicated in 1.11.

2.2. If p = 2, the classification of unipotent classes of G corresponding to elements of order 2 in $\Gamma(G^o)$ is essentially contained in the case of O_8 (1.12).

2.3. Up to the end of 2.5 $p = 3$, $|G/G^o| = 3$ and the homomorphism $G/G^o \longrightarrow \Gamma(G^o)$ is injective. We assume also that G^o is adjoint. We denote the fundamental roots as in the



picture. We can choose T, B, $\sigma \in G$ and the isomorphisms x_λ $(\lambda \in \phi)$ in such a way that ${}^\sigma T = T$, ${}^\sigma B = B$, $\sigma.\alpha = \beta$, $\sigma.\beta = \gamma$, $\sigma.\gamma = \alpha$ and $\sigma(x_\lambda(1))\sigma^{-1} = x_{\sigma.\lambda}(1)$ for all $\lambda \in \phi$. We write the positive roots in the following order : $\alpha, \beta, \gamma, \delta, \alpha+\delta, \beta+\delta, \gamma+\delta, \alpha+\beta+\delta, \beta+\gamma+\delta, \gamma+\alpha+\delta, \alpha+\beta+\gamma+\delta, \alpha+\beta+\gamma+2\delta$. We have now a fixed isomorphism $U = \prod_{\lambda \in \phi^+} X_\lambda \cong A^{12}$. We shall consider the following situation. We conjugate an element $\sigma u \in \sigma U$ by some suitable element $x \in G^o$ to get an element $\sigma u' \in \sigma U$. In this situation we denote by $a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, c_2, c_3, d_1, e_1$ the coordinates of u and by $a_1', a_2', \ldots, e_1'$ the coordinates of u'.

Let $T_o = \{t \in T \mid \alpha(t)\beta(t)\gamma(t) = \delta(t) = 1\}$. T_o is a torus and $\sigma T_o U$ is the variety of all unipotent elements contained in σB.

Every element in $\sigma T_0 U$ is T-conjugate to an element in σU and for this reason we shall concentrate on elements contained in σU. Let $S = \{t \in T \mid \alpha(t) = \beta(t) = \gamma(t)\}$. S is connected and acts on σU by conjugation.

Let $u_0 = x_\alpha(1)x_\delta(1)$. It is easy to check from the commutation formulae that $f : SU \longrightarrow \sigma U$, $su \longmapsto su\sigma u_0(su)^{-1}$ has a surjective differential at 1. It follows easily that the SU-class of σu_0 is dense in σU and that the class C_0 of σu_0 in G is dense in the variety of all unipotent elements in σG^0 (it is the regular unipotent class of σG^0). Therefore $\dim Z(\sigma u_0) = 2$.

Let $U' = \{u \in U \mid a_1+a_2+a_3 = a_4 = 1\}$. If u is any element of U such that $a_1+a_2+a_3 \neq 0$ and $a_4 \neq 0$, then σu is S-conjugate to an element of $\sigma U'$ and is therefore conjugate to σu_0 since $\sigma U'$ is the U-class of σu_0 (the U-class of σu_0 is closed since U is unipotent, is contained in $\sigma U'$ and has dimension $\geq \dim U - 2$ since $\dim Z(\sigma u_0) = 2$).

From the proof of II.1.8 $\mathcal{B}_{\sigma u_0} = \{B\}$.

Consider now an element u for which $a_4 = 0$. It is easy to check that there is a line of type δ through B contained in $\mathcal{B}_{\sigma u}$ and therefore $\dim \mathcal{B}_{\sigma u} \geq 1$ and $\sigma u \notin C_0$.

Similarly if $a_1+a_2+a_3 = 0$, then conjugating σu by $x_\beta(a_2+a_3)x_\gamma(a_3)$ we get $\sigma u'$ with $a_1' = a_2' = a_3' = 0$ and $x_{-\alpha}(t)x_{-\beta}(t)x_{-\gamma}(t)B \in \mathcal{B}_{\sigma u'}$ for all $t \in k$. This shows that there is a line of type α through B contained in $\mathcal{B}_{\sigma u}$. Hence $\dim \mathcal{B}_{\sigma u} \geq 1$ and $\sigma u \notin C_0$.

Let $U_3 = \prod_{ht(\chi)\geq 2} X_\chi$, $U_1 = X_\alpha X_\beta X_\gamma U_3$, $U_2 = X_\delta U_3$. We have proved that every element in $\sigma U \setminus C_0$ is B-conjugate to an element in

$\sigma U_1 \cup \sigma U_2$.

Suppose that $u \in U_1 \setminus U_3$. We can assume that $b_1+b_2+b_3 \neq 0$ (if this is not the case we replace σu by its conjugate by $x_\alpha(r)x_\beta(r)x_\gamma(r)$ for some suitable $r \in k$). Take $\sigma u' = x_{-\delta}(t)\sigma u x_{-\delta}(-t)$. Then $a_1'+a_2'+a_3'$ is a function of the form $at + b$ with $a \neq 0$. Therefore $a_1'+a_2'+a_3' = 0$ for exactly one $t \in k$. This shows that every element in σU_1 is conjugate to an element in σU_3, that every line of type δ in $\mathcal{B}_v$ (v any unipotent element in σG^0) meets a line of type α in $\mathcal{B}_v$ and that the elements $u \in U_1$ such that the line of type δ through B in $\mathcal{B}_{\sigma u}$ meets exactly one line of type α in $\mathcal{B}_{\sigma u}$ are dense in U_1.

The same method works if $u \in U_2 \setminus U_3$. We can assume that $d_4 \neq 0$ (if not we replace σu by $x_\delta(r)\sigma u x_\delta(-r)$ for some suitable $r \in k$). Conjugating then σu by $x_{-\alpha}(t)x_{-\beta}(t)x_{-\gamma}(t)$ we get $\sigma u'$ with $a_4' = at^3 + bt^2 + ct + d$ for some $a,b,c,d \in k$ and $a \neq 0$. This shows that every element in σU_2 is conjugate to some element in σU_3, that every line of type α in $\mathcal{B}_v$ (v any unipotent element in σG^0) meets a line of type δ contained in $\mathcal{B}_v$ and that the elements $u \in U_2$ such that the line of type α through B contained in $\mathcal{B}_{\sigma u}$ meets exactly three lines of type δ in $\mathcal{B}_{\sigma u}$ are dense in U_2.

This shows that every element in $\sigma U \setminus C_0$ is conjugate to an element in σU_3.

Let $U_4 = \prod_{R(\lambda)=3n5} X_\lambda$. Suppose that $u \in U_3$ is such that $b_1+b_2+b_3 \neq 0$. Conjugating by $x_{\beta+\delta}(b_2+b_3)x_{\gamma+\delta}(b_3)$ and by a suitable element of S we can arrange to have $b_1 = 1$ and $b_2 = b_3 = 0$. So assume this is the case. If $\sigma u'$ is obtained by conjugating σu by

$x_\alpha(t)x_\beta(t)x_\gamma(t)$, we get $d_1' = at^3 + bt^2 + ct + d$ for some $a,b,c,d \in k$ and $a \neq 0$. Therefore σu is conjugate to an element in $\sigma x_{\alpha+\delta}(1)U_4$.

Consider an element $u \in x_{\alpha+\delta}(1)U_4$. Suppose that $c_1+c_2+c_3 \neq 0$. Conjugating σu by $x_{\beta+\gamma+\delta}(c_2+c_3)x_{\gamma+\alpha+\delta}(c_3)$ and by a suitable element of S, we can arrange $c_1 = 1$, $c_2 = c_3 = 0$. Therefore σu is conjugate to some element in $\sigma x_{\alpha+\delta}(1)x_{\alpha+\beta+\delta}(1)X_{\alpha+\beta+\gamma+2\delta}$. But every element in $\sigma x_{\alpha+\delta}(1)x_{\alpha+\beta+\delta}(1)X_{\alpha+\beta+\gamma+2\delta}$ is conjugate to $\sigma u_1 = \sigma x_{\alpha+\delta}(1)x_{\alpha+\beta+\delta}(1)$ by an element of the form $x_{\alpha+\beta+\delta}(t)x_{\beta+\gamma+\delta}(t)x_{\gamma+\alpha+\delta}(t)$. This shows that the class C_1 of σu_1 is dense in the variety of all unipotent elements of σG^0 not contained in C_0.

If $u \in x_{\alpha+\delta}(1)U_4$ is such that $c_1+c_2+c_3 = 0$, then the same argument shows that σu is conjugate to σu_2 where $u_2 = x_{\alpha+\delta}(1)$. Let C_2 be the class of σu_2.

If $u \in U_3$ is such that $\sigma u \notin C_1$, then $\sigma u \in C_2$ or $b_1+b_2+b_3 = 0$. If $b_1+b_2+b_3 = 0$ and $c_1+c_2+c_3 \neq 0$, then we can arrange to have $b_1 = b_2 = b_3 = 0$ and $c_1 = 1$, $c_2 = c_3 = 0$. Conjugating then by $x_\alpha(t)x_\beta(t)x_\gamma(t)$ we can get $d_1 = 0$ (for some $t \in k$). Conjugating by $x_{\alpha+\delta}(t)x_{\beta+\delta}(t)x_{\gamma+\delta}(t)$ we get also $e_1 = 0$ (for some $t \in k$). Therefore σu is conjugate to $x_{\alpha+\beta+\delta}(1)$. It is easy to check that for some $n \in N_{G^0}(T)$ representing $\tilde{s}_\alpha$ in W we have $n\sigma x_{\alpha+\beta+\delta}(1)n^{-1} = \sigma x_{\alpha+\delta}(1)$. Hence $\sigma u \in C_2$.

If $u \in U_3$ is such that $b_1+b_2+b_3 = c_1+c_2+c_3 = 0$ then we can arrange $b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = 0$. Hence σu is conjugate to an element in $\sigma X_{\alpha+\beta+\gamma+\delta}X_{\alpha+\beta+\gamma+2\delta}$. Let $u_3 = x_{\alpha+\beta+\gamma+2\delta}(1)$ and let C_3 be the class of $\sigma x_{\alpha+\beta+\gamma+2\delta}(1)$. If $u \in \prod_{R(\lambda)\geq 4} X_\lambda$ and $e_1 \neq 0$, then conjugation by $x_{-\delta}(t)$ (some $t \in k$) and some element of S shows that σu is conjugate to σu_3. If $d_1 \neq 0$, then we can arrange $e_1 \neq 0$ by conjugating by $x_\delta(t)$ for some $t \in k$. It follows that any unipotent element in σG^o not contained in $C_o \cup C_1 \cup C_2 \cup C_3$ is conjugate to σ. Let C_4 be the class of σ.

Proposition 2.4. In the situation of 2.3 there are exactly 5 unipotent classes in σG^o.

Proof. We know already that there are at most 5 classes. We have to show that C_o, C_1, C_2, C_3 and C_4 are distinct.

C_o has to be the regular class, C_1 the subregular class, C_4 the quasisemisimple class. Therefore $\dim \mathcal{B}_{\sigma u_o} = 0$, $\dim \mathcal{B}_{\sigma u_1} = 1$ and $\dim \mathcal{B}_\sigma = 6$ (W^u is of type G_2).

In 2.3 we have seen that $x = \sigma x_{\alpha+\beta+\delta}(1) \in C_2$. It is easily checked that the conjugate of B by $x_{-\alpha}(s)x_{-\beta}(s)x_{-\gamma}(s)x_{-\delta}(t)$ is contained in $\mathcal{B}_x$ for all $s,t \in k$. Therefore $\dim \mathcal{B}_x \geq 2$. It is easily checked from the computations of 2.3 that $\operatorname{codim}_{\overline{C}_o \cap \sigma B}(C_2 \cap \sigma B) \leq 2$. By II.2.7 this implies $\dim \mathcal{B}_x = 2$.

Look now at $\mathcal{B}_{\sigma u_3}$. The conjugate of B by $x_{-\alpha}(r)x_{-\beta}(r)x_{-\gamma}(r)x_{-\delta}(s)x_{-\alpha}(t)x_{-\beta}(t)x_{-\gamma}(t)$ belongs to $\mathcal{B}_{\sigma u_3}$ for all $r,s,t \in k$. This shows that $\dim \mathcal{B}_{\sigma u_3} \geq 3$. But the computations of 2.3 show that $\operatorname{codim}_{\overline{C}_o \cap \sigma B}(C_3 \cap \sigma B) \leq 3$. Therefore $\dim \mathcal{B}_{\sigma u_3} = 3$ and this shows that C_o, C_1, C_2, C_3 and C_4 are distinct.

2.5. The following results are consequences of 2.4 and of the computations made in 2.3.

a) Every line of type α (resp. δ) contained in $\mathcal{B}_x$ ($x \in \sigma G^o$ unipotent) meets a line of type δ (resp. α) contained in $\mathcal{B}_x$.

b) If $x \in C_1$, then $\mathcal{B}_x$ consists of 3 lines of type δ meeting a line of type α (see II.3.12).

c) With the notations of II.2.4, we have :

$Q(C_o) = \{1\}.$

$Q(C_1) = \{\tilde{s}_\alpha,\ \tilde{s}_\delta,\ \tilde{s}_\alpha \tilde{s}_\delta,\ \tilde{s}_\delta \tilde{s}_\alpha,\ \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta\}.$

$Q(C_2) = \{\tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha,\ \tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta,\ \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha,\ \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta\}.$

$Q(C_3) = \{\tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha\}$

$Q(C_4) = \{\tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta \tilde{s}_\alpha \tilde{s}_\delta\}.$

3. Unipotent bilinear forms.

3.1. Let k be any field (not necessarily algebraically closed) and let V be a finite dimensional vector space over k. Let $G_0 = G_0(V) = GL(V)$ and let $G_1 = G_1(V)$ be the set of all bilinear forms $f : V\times V \longrightarrow k$ which are non-singular (i.e. $f(x,y) = 0$ for all $y \in V \Rightarrow x = 0$). Let $G(V) = G_0 \cup G_1$. G(V) can be made into a group in the following way. If $a,b \in G_0$ and $f,g \in G_1$, $ab \in G_0$ is the usual product in GL(V), $af \in G_1$ is the bilinear form $(x,y) \longmapsto f(a^{-1}x,y)$, $fa \in G_1$ is the bilinear form $(x,y) \longmapsto f(x,ay)$ and $fg \in G_0$ is the automorphism of V such that $fg(x) = y$ if and only if $f(y,v) = g(v,x)$ for all $v \in V$.

For any $f \in G_1$, f^{-1} is the bilinear form $(x,y) \longmapsto f(y,x)$. $f^2 = 1$ if and only if f is symmetric.

If M is a subset of V, we can define $fM = \{v \in V | f(v,m) = 0$ for all $m \in M\}$. This is clearly a subspace of V. This, together with the usual action of GL(V) on subspaces of V, defines an action of G(V) on the set of all subspaces of V. In particular we get an action of G(V) on the set $\mathcal{F}$ of all complete flags of V. A flag $F \in \mathcal{F}$ is <u>isotropic</u> for f if $fF = F$ ($f \in G_1$).

3.2. If k is algebraically closed, $G = G(V)$ is an algebraic group in a natural way. A bilinear form f has a Jordan decomposition $f = su$. If $p \neq 2$, $s \in G_1$ and $u \in G_0$. If $p = 2$, $s \in G_0$ and $u \in G_1$. The same result holds if k is perfect (in particular for finite fields).

G acts by conjugation on the variety $\mathcal{B}$ of all its Borel subgroups. Identifying $\mathcal{B}$ with $\mathcal{F}$ (each flag corresponding to its stabilizer), we get an action of G on $\mathcal{F}$. This action is easily seen to be the same as the action defined in 3.1. This shows that G_1 acts non-trivially on the Dynkin diagram of GL(V) (if dim $V > 3$). For every $f \in G_1$, $\mathcal{B}_f$ can be identified with the variety of all flags isotropic for f.

3.3. From now on we assume that p = 2. We assume also that k is algebraically closed or finite. If k is finite, we denote by $\bar{k}$ an algebraically closed field containing k.

In this situation there are unipotent bilinear forms. We want now to determine the conjugacy classes of such forms and to get informations about their centralizers.

$f \in G_1$ is unipotent if and only if $u = f^2 \in G_o$ is so. As the unipotent G-classes and G_o-classes in G_1 coincide, it is sufficient to determine :

a) The unipotent classes of GL(V) which arise in this way.

b) If u is an element of such a class, what is the action of $Z_o(u)$ on the variety of all $f \in G_1$ such that $f^2 = u$.

Lemma 3.4. If M is a subspace of V fixed by $u = f^2$, then so is $fM = f^{-1}M$.

Proof. $ufM = fuM = fM$. Also $fM = f^{-1}(f^2)M = f^{-1}M$.

Lemma 3.5. If $e_o=0, e_1, \dots, e_n$ and $e'_o=0, e'_1, \dots, e'_m$ are elements of V such that $u(e_i) = e_i+e_{i-1}$ $(1 \leq i \leq n)$ and $u(e'_i) = e'_i+e'_{i-1}$ $(1 \leq i \leq m)$, then $f(e_i, e'_j) = 0$ if $i+j < \max(m,n)$ (where $u = f^2$).

Proof. Let $p_{ij} = f(e_i, e'_j)$ and $q_{ij} = f(e'_i, e_j)$. Since $u = f^2$ and $u(e_i) = e_i+e_{i-1}$, $f(e_i+e_{i-1}, v) = f(v, e_i)$ for all $v \in V$. In particular $p_{ij} + p_{i-1,j} = q_{ji}$. Similarly $q_{ij} + q_{i-1,j} = p_{ji}$. Hence $p_{ij} = q_{ji} + q_{j-1,i} = (p_{ij} + p_{i-1,j}) + (p_{i,j-1} + p_{i-1,j-1})$

and therefore $p_{i-1,j} + p_{i,j-1} + p_{i-1,j-1} = 0$ ($1 \le i \le n$, $1 \le j \le m$). Since $p_{ij} = 0$ if $i = 0$ or $j = 0$, this implies by induction $p_{ij} = 0$ if $i+j \leq \max(m,n)$.

<u>Lemma 3.6</u>. Let $V'_1 \oplus V'_2 \oplus \dots \oplus V'_n = V$ be a decomposition of V as a direct sum of u-stable subspaces. Suppose that for each i, all Jordan blocks of the restriction of u to V'_i have dimension i. Then the restriction of f to $V'_i \times V'_i$ is non-singular for all i ($u = f^2$ unipotent).

<u>Proof</u>. In each V'_i choose a basis such that the restriction of u to V'_i has a matrix of the form

$$\begin{pmatrix} I & I & 0 & \dots & & & \\ 0 & I & I & 0 & \dots & & \\ \vdots & 0 & I & I & 0 & \dots & \\ \vdots & \vdots & 0 & & & & \\ & \vdots & & & I & I & 0 \\ & & \dots & & 0 & I & I \\ & & & \dots & & 0 & I \end{pmatrix}$$

where I is the $c_i \times c_i$ identity matrix if $\dim V'_i = ic_i$. Let M_i be the matrix of the restriction of f to $V'_i \times V'_i$ and let M be the matrix of f. It is easy to deduce from 3.5 that $\det(M) = \prod_{1 \le i \le n} \det(M_i)$ (with $\det(M_i) = 1$ if $\dim V'_i = 0$). This proves the lemma.

<u>Corollary 3.7</u>. We can find u-stable subspaces $V_1, V_2, \dots, V_n$ such that :

a) $f(x,y) = 0$ if $x \in V_i$, $y \in V_j$ and $i \neq j$.

b) $V = V_1 \oplus \dots \oplus V_n$ and for all i ($1 \le i \le n$) all Jordan blocks of the restriction of u to V_i have dimension i.

Moreover if $V'_1, \dots, V'_n$ are as in 3.6 and i_0 is a given integer, we can arrange to have $V_{i_0} = V'_{i_0}$.

Proof. This is a consequence of the following remark. If V_1', $\dots, V_n'$ are as in 3.6, then for every i $V = V_i' \oplus fV_i'$ (by 3.6) and $fV_i' = f^{-1}V_i'$ is u-stable (by 3.4).

3.8. Assume now that all Jordan blocks of $u = f^2$ (f a unipotent bilinear form) have dimension n. Choose a basis for V as in 3.6. The matrix P of f with respect to this basis has a block decomposition $P = (P_{ij})$ $(1 \leq i,j \leq n)$ where each P_{ij} is a $c_n \times c_n$ matrix ($c_n \neq 0$ being the number of Jordan blocks of u). Then $f^2 = u$ is equivalent to :

a) $P_{i-1,j} + P_{i,j-1} + P_{i-1,j-1} = 0$ $(1 \leq i,j \leq n)$.

b) $P_{ii} = 0$ if $2i \leq n$ and $P_{ii} = {}^tP_{ii}$ if $2i = n+1$.

c) $P_{i-1,i} = \delta P_{ii}$ $(1 \leq i \leq n)$, where for any square matrix A $\delta A = A + {}^tA$.

d) $\det(P_{1n}) \neq 0$.

These conditions imply in particular that $P_{ij} = 0$ if $i+j \leq n$ and $P_{1n} = P_{2,n-1} = \dots = P_{n1}$.

If $n = 2m$, $P_{1n} = P_{m,m+1} = \delta P_{m+1,m+1}$ is a non-singular antisymmetric matrix and therefore c_n has to be even. It is easy to check that the matrix P is completely determined by the matrices P_{ii} for $m+1 \leq i \leq n$, and that any choice of $(P_{ii})_{m+1 \leq i \leq n}$ with $\delta P_{m+1,m+1}$ non-singular occurs for some f.

If $n = 2m-1$, $P_{1n} = P_{\underline{mm}}$ is a non-singular symmetric matrix. It is easy to check that P is completely determined by the matrices P_{ii} for $m \leq i \leq n$ and that any choice for $(P_{ii})_{m \leq i \leq n}$ with $P_{\underline{mm}}$ symmetric and non-singular occurs for some f.

3.9. Let now f be any unipotent element in G_1. It follows from 3.8 that the class of $u = f^2$ has the following property. If c_i

is the number of Jordan blocks of u of dimension i, then c_i is even if i is even. So we get a partition λ of $n = \dim V$ which represents a unipotent class in $O_n(\mathbb{C})$. Let $u_\lambda \in O_n(\mathbb{C})$ be an element in this class and let $z_\lambda = \dim Z_{O_n(\mathbb{C})}(u_\lambda)$.

Suppose now that i is odd and $c_i > 0$. Define ε_i to be 0 if $(u+1)^i(x) = 0 \Rightarrow f(x,(u+1)^{i-1}(x)) = 0$, and to be 1 otherwise. Choose a decomposition of V as in 3.7 and a basis for V_i as in 3.8. From 3.8 we get a non-singular symmetric $c_i \times c_i$ matrix P_{1i}. It is easy to check that P_{1i} is antisymmetric if $\varepsilon_i = 0$ and is not antisymmetric if $\varepsilon_i = 1$.

For all other $i \in \mathbb{N}$, put $\varepsilon_i = \omega$, where $\{\omega, 0, 1\}$ is as in 1.12.

In this way we attach to each class of unipotent bilinear forms a pair (λ, ε) such that :

a) λ is a partition of dim V with c_i even if i is even.

b) $\varepsilon : \mathbb{N} \longrightarrow \{\omega, 0, 1\}$ is an application satisfying :

  b_1) $\varepsilon_i = \omega$ if i is even or if $c_i = 0$.

  b_2) $\varepsilon_i = 1$ if i is odd and c_i is odd.

  b_3) $\varepsilon_i \neq \omega$ if i is odd and $c_i \neq 0$.

Let $C_{\lambda,\varepsilon}$ be the subset of G_1 consisting of all unipotent bilinear forms corresponding to (λ, ε).

<u>3.10</u>. Let $W_1 \oplus W_2 \oplus \dots \oplus W_m$ be a decomposition of V as a direct sum. Suppose that we are given non-singular bilinear forms $f_i : W_i \times W_i \longrightarrow k$ $(1 \leq i \leq m)$. Then there is a unique bilinear form $f : V \times V \longrightarrow k$ which coincides with f_i on $W_i \times W_i$ (all i) and such that $f(x,y) = 0$ if $x \in W_i$, $y \in W_j$ and $i \neq j$. F is non-singular and we write $f = f_1 \oplus f_2 \oplus \dots \oplus f_m$. If $f \in C_{\lambda,\varepsilon}$ and $f_i \in C_{\lambda^i,\varepsilon^i} \subset G(W_i)$, the parts of λ are the parts of $\lambda^1, \lambda^2, \dots, \lambda^m$

and $\varepsilon_j = \max_{1\le i\le m} \varepsilon_j^i$ (all $j \geq 0$). We write also $(\lambda,\varepsilon) = (\lambda^1,\varepsilon^1)\oplus\ldots\oplus(\lambda^m,\varepsilon^m)$.

3.11. Assume that we are in the situation of 3.8 and that $f \in C_{\lambda,\varepsilon}$. We fix the following notations.

a) If $n = 2m-1$ and $\varepsilon_n = 1$, then M is the identity $c_n\times c_n$ matrix.

b) If $n = 2m-1$ and $\varepsilon_n = 0$, then $M = (m_{ij})$ is the $c_c\times c_n$ matrix such that $m_{ij} = 1$ if $i+j = c_n+1$ and $m_{ij} = 0$ otherwise.

In cases (a) and (b), we shall say that f is split if for some choice of the basis $P_{mm} = M$ and $P_{ii} = 0$ for $m+1 \leq i \leq n$.

c) If $n = 2m$ (and then $\varepsilon_n = \omega$), let $K = (k_{ij})$ be the $c_n\times c_n$ matrix defined by $k_{ij} = 1$ if $i+j = c_n+1$ and $i \leq c_n/2$, and $k_{ij} = 0$ otherwise, and let $M = \delta K$.

In case (c) we say that f is split if for some choice of the basis $P_{m+1,m+1} = K$ and $P_{ii} = 0$ for $m+2 \leq i \leq n$.

In each of the cases (a), (b), (c), the split bilinear forms contained in $C_{\lambda,\varepsilon}$ form a single conjugacy class in G.

In each case we denote by Q the matrix obtained for this particular choice of the matrices P_{ii}.

3.12. Let $f \in G_1$ be any unipotent bilinear form. We shall say that f is split if for some decomposition $V = V_1\oplus\ldots\oplus V_n$ as in 3.7 the restriction f_i of f to $V_i\times V_i$ is split (in the sense of 3.11) for each i such that $V_i \neq 0$.

Let $C_{\lambda,\varepsilon,1} = \{f \in C_{\lambda,\varepsilon} \mid f \text{ is split}\}$. $C_{\lambda,\varepsilon,1}$ is a single conjugacy class in G. If $V = W_1\oplus\ldots\oplus W_m$ and $f_i \in G(W_i)$ is split for each i, then $f = f_1\oplus\ldots\oplus f_m$ is also split.

3.13. Suppose that k is finite. Let $\bar{V} = V\otimes_k\bar{k}$. Choose a basis for V. This is a $\bar{k}$-basis for $\bar{V}$. Write $\bar{G}_0$, $\bar{G}_1$, $\bar{G}$ for $G_0(\bar{V})$,

$G_1(\overline{V})$, $G(\overline{V})$ respectively. Define $F: \overline{G} \longrightarrow \overline{G}$ by $F(a_{ij}) = (a_{ij}^q) \in \overline{G}_o$ if $(a_{ij}) \in \overline{G}_o$, $F(f_{ij}) = (f_{ij}^q) \in \overline{G}_1$ if $(f_{ij}) \in \overline{G}_1$, where $q = |k|$ (the elements of $\overline{G}_o$ and $\overline{G}_1$ being represented by their matrices). F is a morphism and $\overline{G}^F = \{g \in \overline{G} | F(g) = g\}$ can be identified with $G = G(V)$.

We write $C_{\lambda,\varepsilon}$ (resp. $\overline{C}_{\lambda,\varepsilon}$) for the unipotent bilinear forms $f \in G$ (resp. $f \in \overline{G}$) corresponding to (λ, ε). Similarly we write $C_{\lambda,\varepsilon,1}$ (resp. $\overline{C}_{\lambda,\varepsilon,1}$) for the elements of $C_{\lambda,\varepsilon}$ (resp. $\overline{C}_{\lambda,\varepsilon}$) which are split as elements of G (resp. $\overline{G}$).

Fix $g \in C_{\lambda,\varepsilon,1}$. Let $\overline{Z}(g) = Z_{\overline{G}}(g)$, $\overline{Z}_o(g) = \overline{Z}(g) \cap \overline{G}_o$, $\overline{A}_o(g) = \overline{Z}_o(g)/\overline{Z}(g)^o$. From [2 , p. E-7] we get for each $a \in \overline{A}_o(g)$ a G-conjugacy class $C_{\lambda,\varepsilon,a}$ contained in $\overline{C}_{\lambda,\varepsilon,1} \cap \overline{G}^F$. It is the class of xgx^{-1}, where $x \in \overline{G}_o$ is such that $x^{-1}F(x)$ is an element of $\overline{Z}_o(g)$ representing a. The definition of $C_{\lambda,\varepsilon,a}$ is independent of the choice of $g \in C_{\lambda,\varepsilon,1}$. $(\overline{C}_{\lambda,\varepsilon,1}) \cap \overline{G}^F = \bigcup_{a \in \overline{A}_o(g)} C_{\lambda,\varepsilon,a}$. If F acts trivially on $\overline{A}_o(g)$ and $\overline{A}_o(g)$ is commutative, then $C_{\lambda,\varepsilon,a}$ and $C_{\lambda,\varepsilon,b}$ are distinct if $a \neq b$ $(a, b \in \overline{A}_o(g))$.

Suppose that $V = W_1 \oplus \dots \oplus W_m$ and $f = f_1 \oplus \dots \oplus f_m$ as in 3.10. Suppose also that $f \in \overline{C}_{\lambda,\varepsilon,1} \cap \overline{G}^F$ and $f_i \in C_{\lambda^i,\varepsilon^i,a_i}$ for all i (a_i in the group of components of $Z_{\overline{G}_o(\overline{W}_i)}(g_i)$, where $g_i \in C_{\lambda^i,\varepsilon^i,1}$). Then $f \in C_{\lambda,\varepsilon,a}$ where a is the image of $(a_1, \dots, a_m)$ under the composite homomorphism $\prod_{1 \leq i \leq m} Z_{G_o(\overline{W}_i)}(g_i) \longrightarrow \overline{Z}_o(g) \longrightarrow \overline{A}_o(g)$ (with $g = g_1 \oplus \dots \oplus g_m$). This is clear from the definitions.

3.14. Suppose that $f \in C_{\lambda,\varepsilon}$ is such that all Jordan blocks of $u = f^2$ have dimension $n = 2m$. We use the notations of 3.8 and 3.11. Choose a basis (e_s^r) $(1 \leq s \leq n,\ 1 \leq r \leq c_n)$ such that $e_{n-i}^r = (u+1)^i(e_n^r)$ (all i, r such that $(1 \leq i \leq n-1,\ 1 \leq r \leq c_n)$. The matrix P of f has a block decomposition (P_{ij}). We show now that we can choose another basis (f_s^r) (with $f_{n-i}^r = (u+1)^i(f_n^r)$, $1 \leq i \leq n-1$, $1 \leq r \leq c_n$) for which the matrix P' of f (with block decomposition (P'_{ij})) is closer to Q (Q as in 3.11).

We write $P_i = P_{ii}$, $P'_i = P'_{ii}$.

Take first $f_n^s = \sum_r x_{rs}e_n^r + \sum_r y_{rs}e_{n-1}^r$, where $X = (x_{rs})$ and $Y = (y_{rs})$ are $c_n \times c_n$ matrices. Then
$P'_{m+1} = \delta({}^tY(\delta P_{m+1})X) + {}^tXP_{m+1}X$.

We want to have $P'_{m+1} = K$. We must therefore have ${}^tXP_{m+1}X = K + A$ for some antisymmetric matrix A. If k is algebraically closed, this equation has a solution X_o say, and the solutions are the matrices of the form X_oX', where $X' \in O_{c_n}(k)$. If k is a finite field, then we can choose a matrix K' such that $\delta K' = M$ and ${}^tXK'X \neq K + A$ for all matrices X, A with A antisymmetric. It is then possible to solve exactly one of the equations ${}^tXP_{m+1}X = K + A$, ${}^tXP_{m+1}X = K' + A$ for some antisymmetric matrix A. This follows from the classification of quadratic forms in characteristic 2 [5, p. 197-199].

Fix now a matrix X_o such that ${}^tX_oP_{m+1}X_o = K + A_o$ (or $K' + A_o$ if k is finite) with A_o antisymmetric. Let B_o be a fixed matrix such that $B_o = A_o$. Then taking $Y = (\delta P_{m+1})({}^tX^{-1})B_o$, we get $P'_{m+1} = K$ (or $P'_{m+1} = K'$ if k is finite).

Suppose now that we already have $\delta P_{m+1} = M$ and $P_{m+j} = 0$ for $2 \leq j \leq i-1$. Take $f_n^s = e_n^s + \sum_r x_{rs}e_{n-2i+2}^r + \sum_r y_{rs}e_{n-2i+1}^r$. We get then $P'_{m+j} = P_{m+j}$ for $1 \leq j \leq i-1$ and $P'_{m+i} = \delta(MY) + {}^tP_{m+i,m-i+2}X + {}^tXP_{m-i+2,m+i} + P_{m+i}$.

But $P_{m+i,m-i+2} = P_{m-i+2,m+i} = P_{m+1}$ or ${}^tP_{m+1}$ (this follows from 3.8). Hence $P'_{m+i} = \delta(MY) + \delta({}^tP_{m+i,m-i+2}X) + MX + P_{m+i}$.

In order to have $P'_{m+1} = 0$ we must take $X = M(P_{m+1} + A)$, where A is any antisymmetric matrix, and then Y can be chosen in such a way as to have $P'_{m+1} = 0$.

If k is algebraically closed these computations show that f is split. Hence $C_{\lambda,\varepsilon}$ is a single class in G. We find also that $A_o(f)$ has two elements. It is easy to check that $u \in Z(g)^o$.

If k is finite, $C_{\lambda,\varepsilon}$ consists of two classes. If $|k| = 2$, and $c_n = 2$, then a form f with $P_{m+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is split if ad = 0 and is not if ad = 1.

3.15. Let $W_1 = \mathrm{Ker}(u+1)$, where $u = f^2$ and f is a unipotent bilinear form. $Z_o(f)$ acts on $\mathbb{P}(W_1)$. Suppose we are in the situation of 3.14 with k algebraically closed. Let $C_o = \{kw \in \mathbb{P}(W_1) \mid f(v,v) = 0 \text{ if } (u+1)^{m+1}(v) = w\}$. Let $C_1 = \mathbb{P}(W_1) \setminus C_o$. Then C_o and C_1 are the $Z_o(f)$-orbits in $\mathbb{P}(W_1)$ and $\dim C_o = \dim \mathbb{P}(W_1) - 1$. This is easy to see if $c_n = 2$. In this case $|C_o| = 2$. If $c_n \geq 4$, this can be proved by using decompositions $f = f_1 \oplus \ldots \oplus f_{c_n/2}$, where each f_i has two Jordan blocks. C_o is irreducible if $c_n > 4$.

3.16. Suppose that $f \in C_{\lambda,\varepsilon}$ is such that all Jordan blocks of $f^2 = u$ have dimension $n = 2m-1$. We use the notations of 3.8 and 3.11. (e_s^r), (f_s^r), $P = (P_{ij})$, $P' = (P'_{ij})$, P_i, P'_i are defined as in 3.14.

Take first $f_n^s = \sum_r x_{rs} e_n^r$. Then $P'_m = {}^tXP_mX$ and we can arrange to have $P'_m = M$. It is also easy to check that if $\varepsilon_n = 1$ and $P'_m = M$, then $\mathrm{Tr}({}^tP'_{m+1}P'_{m+1})$ is independent of the choice of X. Notice that if k is algebraically closed, then the solutions X for $P'_m = M$ form an irreducible variety.

If n = 1 this shows that f is split and therefore $C_{\lambda,\varepsilon}$ is a single conjugacy class. If k is algebraically closed, then $Z_o(f)$ is connected.

We suppose now that $n \geqslant 3$. If k is finite, choose an element $\alpha \in k$ such that $\alpha \neq x^2 + x$ for all $x \in k$ and let K' be the $c_n \times c_n$ matrix such that $k'_{11} = \alpha$ and $k'_{ij} = 0$ if $(i,j) \neq (1,1)$.

We suppose now that $P_m = M$ and that for some integer $i \geqslant 1$ we have :

a) if $i > 1$, then $P_{m+1} = 0$ or $P_{m+1} = K'$ if k is finite and $\varepsilon_n = 1$.

b) $P_{m+j} = 0$ if $2 \leqslant j \leqslant i-1$.

Take $f_n^s = e_n^s + \sum_r x_{rs} e^r_{n-2i+1} + \sum_r y_{rs} e^r_{n-2i}$.

Then $P'_{m+j} = P_{m+j}$ if $j \leqslant i-1$ and $P'_{m+i} = \delta(MY) + {}^tXP_{m-i+1}X + P_{m+i,m-i+1}X + {}^tXP_{m-i+1,m+i} + P_{m+i}$. From 3.8 we get $P_{m+i,m-i+1} = {}^tP_{m+i,m-i+1} = P_{m-i+1,m+i} + M$. Hence

$P'_{m+i} = \delta(MY) + \delta(P_{m-i+1,m+i}X) + {}^tXP_{m-i+1}X + MX + P_{m+i}$.

Suppose first that $\varepsilon_n = 0$. Then $\delta(MY) + \delta(P_{m-i+1,m+i}X) + {}^tXP_{m-i+1}X$ is antisymmetric. We can choose X in such a way as to make $MX + P_{m+i}$ antisymmetric and then we can find Y to have $P'_{m+i} = 0$. This shows that f is split. $C_{\lambda,\varepsilon}$ is therefore a single class in G. If k is algebraically closed, $Z_o(f)$ is easily seen to be connected.

If $\varepsilon_n = 1$ and $i \geqslant 2$, the same method works to arrange $P'_{m+i} = 0$. We have therefore only to look at the case i = 1. In this case M = I and $P_{m-i+1} = P_m = I$. Therefore $P'_{m+1} = \delta(MY) + \delta(P_{m,m+1}X) + {}^tXX + X + P_{m+1}$. We have to see if it is possible to make ${}^tXX + X + P_{m+1}$ antisymmetric. As ${}^tXX$ is symmetric, we must have $X = P_{m+1} + S$, where S is a symmetric matrix, and we want $S^2 + S + {}^tP_{m+1}P_{m+1}$ to be antisymmetric. As ${}^tP_{m+1}P_{m+1}$ is symmetric, we need only to look at the diagonal.

If k is algebraically closed, it is easy to find a solution with S a diagonal matrix. This shows that in this case f is split and therefore $C_{\lambda,\varepsilon}$ is a single class in G. Moreover $Z_o(f)$ has two components. This will follow from the discussion of the case where k is finite. If $c_n = 1$ it is easily checked that $u \in Z_o(f) \setminus Z(f)^o$.

Suppose now that k is finite. Since S is symmetric, $Tr(S^2) = Tr(S)^2$. It follows then easily that there is a solution if and only if $Tr({}^tP_{m+1}P_{m+1}) = x^2 + x$ for some $x \in k$. If there is no solution, then the same argument shows that for some X, ${}^tXX + X + P_{m+1} = K' + A$, where A is antisymmetric. Then by choosing Y in a suitable way we can arrange to have $P'_{m+1} = 0$ or $P'_{m+1} = K'$. It follows that $C_{\lambda,\varepsilon}$ contains two classes of G.

Notice that if $|k| = 2$ and $c_n = 1$, then f is split if $P_{m+1} = 0$ and is not if $P_{m+1} = 1$ $(n \geq 3)$.

<u>3.17</u>. Suppose that we are in the situation of 3.16 with k algebraically closed. Let $W_1 = Ker(u+1)$. If $\varepsilon_n = 0$, then $Z_o(f)$ acts transitively on $\mathbb{P}(W_1)$.

If $\varepsilon_n = 1$, let $H = \{w \in W_1 | f(v,w) = 0 \text{ if } (u+1)^{n-1}(v) = w\}$. H is a hyperplane in W_1. $L = (u+1)^{n-1}(fH)$ is a 1-dimensional subspace of W_1. Let $C_o = \{L\}$, $C_1 = \mathbb{P}(H) \setminus C_o$, $C_2 = \mathbb{P}(W_1) \setminus (C_o \cup C_1)$. $C_2 \neq \emptyset$ if $c_n \geq 2$ and $C_1 \neq \emptyset$ if $c_n \geq 3$. Each C_i is empty or is a single $Z_o(f)$-orbit. This is clear if $c_n \leq 2$. The general case follows by using suitable decompositions $f = f_1 \oplus f_2$ for which f_1^2 has only one or two Jordan blocks.

Proposition 3.18. If k is algebraically closed, then every unipotent bilinear form is split. The unipotent classes of bilinear forms are exactly the varieties $C_{\lambda,\varepsilon}$, where (λ,ε) is any pair satisfying (a) and (b) of 3.9.

Proof. Let f be a unipotent bilinear form. Take a decomposition $V = V_1 \oplus \ldots \oplus V_n$ as in 3.7. By 3.14 and 3.16 the restriction of f to any V_i is split. Hence f is split and every $C_{\lambda,\varepsilon}$ is a single class in G.

Corollary 3.19. If k is finite, the classes of unipotent bilinear forms are the subsets $C_{\lambda,\varepsilon,a}$, where (λ,ε) is any pair satisfying (a) and (b) of 3.9 and $a \in \bar{A}_0(g)$, g being a fixed element of $C_{\lambda,\varepsilon,1}$ (with the notations of 3.13).

Proof. This follows from 3.13 and 3.18.

Remark 3.20. If k is finite, the number of classes of unipotent bilinear forms is equal to the number of partitions of dim V. This will be proved in 4.5.

Proposition 3.21. If k is algebraically closed and $f \in C_{\lambda,\varepsilon}$ is a unipotent bilinear form, then $A_0(f)$ is naturally isomorphic to the abelian group generated by $\{a_i \mid \varepsilon_{\lambda_i} \neq 0\}$ $(a_1, a_2, \ldots$ as in 1.13) with the following relations. $a_i a_j = 1$ if $\lambda_i = \lambda_j$, or if $\lambda_i = \lambda_j + 1$, or if $\lambda_i = \lambda_j + 2$ and λ_i is odd, and $a_i = 1$ if $\lambda_i = 0$.

Proof. It is easy to check that we can find subspaces $W_1, W_2, \ldots$ $\ldots, W_m$ (for some m) with the following properties.

a) The restriction f_i of f to W_i is non-singular for all i and

$f = f_1 \oplus \dots \oplus f_m$.

b) if $f_i \in C_{\lambda^i, \varepsilon^i} \subset G(W_i)$, then one of the following holds.

b_1) $\lambda_1^i = n_i$ is odd, $\lambda_2^i = 0$ (then $\varepsilon_{n_i}^i = 1$).

b_2) $\lambda_1^i = \lambda_2^i = n_i$ are odd, $\lambda_3^i = 0$ and $\varepsilon_{n_i}^i = 0$.

b_3) $\lambda_1^i = \lambda_2^i = n_i \neq 0$ are even, $\lambda_3^i = 0$ (then $\varepsilon_{n_i}^i = \omega$).

Moreover in each case $\varepsilon_{n_i}^i = \varepsilon_{n_i}$.

c) $n_1 > n_2 \geqslant \dots \geqslant n_m$.

$X = \{(W_1, \dots, W_m) | (a), (b) \text{ and } (c) \text{ are satisfied}\}$ is an algebraic variety and $Z_o(f)$ acts transitively on X (by 3.18). X is irreducible and therefore the stabilizer S of $x \in X$ meets all components of $Z_o(f)$. The natural homomorphism $S/S^o \longrightarrow A_o(f)$ is surjective. S/S^o can be computed from 3.14 and 3.16 and we clearly can take $\{a_i | \varepsilon_{\lambda_i} \neq 0\}$ as a set of generators. Some of the relations listed in the proposition are already true in S/S^o ($a_i = 1$ if $\lambda_i = 0$, $a_i a_j = 1$ if $\lambda_i = \lambda_j + 1$ and $\lambda_i = 0$, $a_i^2 = 1$, some of the relations $a_i a_j = 1$ if $\lambda_i = \lambda_j$ when $i = j+1$). We need to check that the kernel of $S/S^o \longrightarrow Z_o(f)$ is given by the relations in the proposition. The relations $a_i a_j = 1$ if $\lambda_i = \lambda_j$ follow from 3.14 and 3.16. By 3.13 and 3.19 we need only to classify the unipotent bilinear forms over one finite field. The proposition is then a consequence of the following lemma.

<u>Lemma 3.22</u>. Suppose that $|k| = 2$ and that $f \in C_{\lambda, \varepsilon}$. Suppose also that there is a decomposition $V = W_1 \oplus W_2$ such that (a), (b) and (c) of the proof of 3.21 hold (with m = 2). Then we have :

a) If $n_1 = 2$ or 3, $n_2 = 1$ and $\varepsilon_1 = 1$, then f is split.

b) If $\lambda_1 = \lambda_2 + 2 \geqslant 5$ and $\lambda_3 = 0$, or if $\lambda_1 = \lambda_2 = \lambda_3 + 1 \geqslant 4$ and

$\lambda_4 = 0$, or if $\lambda_1 = \lambda_2+1 = \lambda_3+1$, or if $n_1 = n_2$, then f is split if and only if f_1 and f_2 are both split or non-split.

c) In all other cases, f is split if and only if f_1 and f_2 are split.

<u>Proof</u>. Suppose that $\lambda_1 = \lambda_2+2 \geqslant 5$ and $\lambda_3 = 0$. Then λ_1 is odd. Say $\lambda_1 = 2m+1 = n$. Choose a basis $e_1^1,\dots,e_n^1$ of W_1 and a basis $e_1^2,\dots,e_{n-2}^2$ of W_2 as in 3.8. If $V = W_1' \oplus W_2'$ satisfies (a), (b) and (c) of the proof of 3.21, we can take a basis $f_1^1,\dots,f_n^1$ of W_1' and a basis $f_1^2,\dots,f_{n-2}^2$ of W_2' with $f_n^1 = e_n^1 + x_1e_{n-2}^2 + x_2e_{n-3}^2 + \dots$ and $f_{n-2}^2 = e_{n-2}^2 + y_1e_{n-2}^1 + y_2e_{n-3}^1 + \dots$. By 3.16, we have to look at $p = f(e_{m+2}^1,e_{m+2}^1)$, $q = f(e_{m+1}^2,e_{m+1}^2)$, $p' = f(f_{m+2}^1,f_{m+2}^1) = p + x_1^2$, $q' = f(f_{m+1}^2,f_{m+1}^2) = q + y_1^2$. It is sufficient to prove that $x_1 = y_1$. This is true since $x_1 + y_1 = f(f_3^1,f_{n-2}^2) = 0$.

If $\lambda_1 = \lambda_2 = \lambda_3+1 \geqslant 4$ and $\lambda_4 = 0$, then $\lambda_1 = 2m = n$ is even. Choose a basis $e_1^1,e_1^2,\dots,e_n^1,e_n^2$ of W_1 and a basis $e_1^3,\dots,e_{n-1}^3$ of W_2. If $V = W_1' \oplus W_2'$ satisfies (a), (b) and (c) of the proof of 3.21, then we can take a basis $f_1^1,f_1^2,\dots,f_n^1,f_n^2$ of W_1' and a basis $f_1^3,\dots,f_{n-1}^3$ (as in 3.8) with $f_n^1 = e_n^1 + x_1e_{n-1}^3 + x_2e_{n-2}^3 + \dots$, $f_n^2 = e_n^2 + y_1e_{n-1}^3 + y_2e_{n-2}^3 + \dots$, $f_{n-1}^3 = e_{n-1}^3 + z_1e_{n-1}^1 + t_1e_{n-1}^2 + z_2e_{n-2}^1 + t_2e_{n-2}^2 + \dots$. By 3.14 and 3.16, we have to look at :

$r = p_1p_2$, where $p_1 = f(e_{m+1}^1,e_{m+1}^1)$, $p_2 = f(e_{m+1}^2,e_{m+1}^2)$.

$q = f(e_{m+1}^3,e_{m+1}^3)$.

$r' = p_1'p_2'$, where $p_1' = f(f_{m+1}^1,f_{m+1}^1) = p_1 + x_1^2$ and $p_2' = f(f_{m+1}^2,f_{m+1}^2) = p_2 + y_1^2$.

$q' = f(f_{m+1}^3,f_{m+1}^3) = q + p_1z_1^2 + p_2t_1^2 + z_1t_1$. Since $z_1^2 = z_1$ and $t_1^2 = t_1$, it is sufficient to prove that $x_1 = t_1$ and $y_1 = z_1$

to get $q + r + q' + r' = 0$ (this proves the lemma in this case). This is true since $x_1 + t_1 = f(f_2^1, f_{n-1}^3) = 0$ and $y_1 + z_1 = f(f_2^2, f_{n-1}^3) = 0$.

The proof for the other cases is similar.

Corollary 3.23. In the situation of 3.21, $A(f)$ is naturally isomorphic to the abelian group generated by $\{a_0\} \cup \{a_i \mid \varepsilon_{\lambda_i} \neq 0\}$ with the relations given in 3.21 and with $a_0^2 = \prod_{i \in I} a_i$, where $I = \{i \mid \varepsilon_{\lambda_i} \neq 0 \text{ and } \lambda_i \geq 1\}$.

Proof. $A(f)$ is obtained by adjoining $fZ(f)^o$ to $A_0(f)$. By definition $fZ(f)^o$ is central in $A_0(f)$. We need only to prove that $uZ(f)^o$ ($u = f^2$) corresponds to $\prod_{i \in I} a_i$. From the proof of 3.21, it is sufficient to check this in the following cases :

a) u has only one Jordan block.

b) u has two Jordan blocks, both of even dimension.

In these cases the formula results from the computations of 3.14 and 3.16.

Proposition 3.24. If k is algebraically closed and $f \in C_{\lambda, \varepsilon}$ is a unipotent bilinear form, then $\dim Z(f) = z_\lambda + \sum_{\varepsilon_i = 0} c_i$ (z_λ, c_i as in 3.9)

Proof. Let Y be the variety of all decompositions $V = V_1 \oplus \dots \oplus V_\ell$ as in 3.7. $Z_0(f)$ acts transitively on Y (by 3.18). Let Y_λ be the variety of all decompositions $\mathbb{C}^n = V_1' \oplus \dots \oplus V_\ell'$, where $V_1', \dots, V_\ell'$ are orthogonal u_λ-stable subspaces and all Jordan blocks of the restriction of u_λ to V_i' have dimension i (with the notations of 3.9). $\dim Y_\lambda = \dim Y$ and $Z_{O_n(\mathbb{C})}(u_\lambda)$ acts transitively on Y_λ. Taking stabilizers of $y \in Y$ and $y_\lambda \in Y_\lambda$, we reduce the computation of dim $Z(f) - z_\lambda$ to the case where all Jordan blocks of u have the the same dimension.

Assume now that all Jordan blocks of u have dimension i. In this case $z_\lambda = (ic_i^2 - c_i)/2$ if i is odd and $z_\lambda = ic_i^2/2$ if i is even. Also dim $Z(f)$ = dim $Z(u)$ - dim $\{g \in C_{\lambda,\varepsilon} | g^2 = u\}$ can be computed from 3.8. We get :

$$\dim Z(f) = \begin{cases} (ic_i^2 - c_i)/2 & \text{if } i \text{ is odd and } \varepsilon_i = 1 \\ (ic_i^2 + c_i)/2 & \text{if } i \text{ is odd and } \varepsilon_i = 0 \\ ic_i^2/2 & \text{if } i \text{ is even.} \end{cases}$$

Hence
$$\dim Z(f) = \begin{cases} z_\lambda & \text{if } \varepsilon_i \neq 0 \\ z_\lambda + c_i & \text{if } \varepsilon_i = 0. \end{cases}$$

This proves the proposition.

4. Groups of type E_6.

In this paragraph p = 2. We prove that there are only finitely many unipotent classes arising from the symmetry of order 2 in the Dynkin diagram of type E_6. Without loss of generality we may assume that k is an algebraic closure of $\mathbb{F}_q$ ($q = 2^e$). These results are due to George Lusztig.

4.1. Let G be a connected reductive group defined over $\mathbb{F}_q$. Let $F : G \longrightarrow G$ be the corresponding Frobenius endomorphism. G has a dual group G* defined up to isomorphism. G* is defined over $\mathbb{F}_q$ and its Frobenius endomorphism is also denoted by F. If G has a connected centre there is a natural partition of the set $(G^F)^\wedge$ of all irreducible representations of G^F (isomorphism classes of irreducible complex representations of G^F) indexed by semisimple classes in G^{*F} [4]. Let X_s be the subset of $(G^F)^\wedge$ corresponding to the class of $s \in G^{*F}$ (s semisimple). In this situation $Z_{G^*}(s)$ is always connected and it is known that if $Z_{G^*}(s)$ is a Levi subgroup of some parabolic subgroup of G*, then X_s can be parametrized in a natural way by the set of irreducible unipotent representations of $Z_{G^*}(s)^{*F}$.

Let $r(G) = |(G^F)^\wedge|$ and let $r_u(G) = |\{\theta \in (G^F)^\wedge | \theta \text{ is unipotent}\}|$. Then $r(G) = \sum_s |X_s|$, where the summation is taken over representatives of the semisimple classes of G^{*F}, and if $Z_{G^*}(s)$ is a Levi subgroup of some parabolic subgroup of G* then $|X_s| = r_u(Z_{G^*}(s)^*)$.

4.2. Let $\tilde{G}$ be a reductive group defined over $\mathbb{F}_q$ such that $G = \tilde{G}^o$ has a connected centre and $\tilde{G}/G$ has two elements. $\tilde{G}/G$ acts on the canonical torus of G (defined as in [4]) and therefore on its dual which is the canonical torus of G*. This gives an involution σ on the set of semisimple classes of G*. We write $s \equiv \sigma s$ if the class of s is fixed by σ. $\tilde{G}/G$ acts also on $(G^F)^\wedge$ and this gives an involution on $(G^F)^\wedge$ (also denoted by σ). If $s \in G^{*F}$, the subset X_s of $(G^F)^\wedge$ is σ-stable if and only if $s \equiv \sigma s$. In fact $\sigma X_s = X_{\sigma s}$. This follows from the definition of X_s. If $\theta \in (G^F)^\wedge$ is such that $\theta = \sigma\theta$, then θ is the restriction of 2 irreducible representations of $\tilde{G}^F$. If $\theta \neq \sigma\theta$, then there is one irreducible representation of $\tilde{G}^F$ such that θ and $\sigma\theta$ are the components of its restriction to G^F. We get in this way all irreducible representations of $\tilde{G}^F$. Therefore $r(\tilde{G}) = |(\tilde{G}^F)^\wedge| = 2|\{\theta \in (G^F)^\wedge | \theta = \sigma\theta\}| + \frac{1}{2}|\{\theta \in (G^F)^\wedge | \theta \neq \sigma\theta\}|$.

If $s \in G^F$ is semisimple and $s \equiv \sigma s$, let $\tilde{X}_s = \{\Upsilon \in (\tilde{G}^F)^\wedge |$ the components of the restriction of Υ to G^F are in $X_s\}$. Then $r(\tilde{G}) = \sum_{s \equiv \sigma s} |\tilde{X}_s| + \frac{1}{2} \sum_{s \not\equiv \sigma s} |X_s|$, where the summations are taken over representatives of the semisimple classes of G^{*F} such that $s \equiv \sigma s$ and $s \not\equiv \sigma s$ respectively. Clearly $|\tilde{X}_s| = 2|X_s^\sigma| + \frac{1}{2}|X_s \setminus X_s^\sigma|$.

We shall say that $\Upsilon \in (\tilde{G}^F)^\wedge$ is unipotent if some component (or all components) of its restriction to G^F is unipotent. Let $r_u(\tilde{G}) = |\{\Upsilon \in (G^F)^\wedge | \Upsilon \text{ is unipotent}\}|$. We also have $r_u(\tilde{G}) = 2|\{\theta \in (G^F)^\wedge | \theta \text{ is unipotent and } \theta = \sigma\theta\}| + \frac{1}{2}|\{\theta \in (G^F)^\wedge | \theta \text{ is unipotent and } \theta \neq \sigma\theta\}|$.

4.3. If G is any algebraic group defined over $\mathbb{F}_q$, let $c_u(G)$ be the number of unipotent classes of G^F and let c(G) be the total number of conjugacy classes of G^F. If $\tilde{G}$ and G are as in 4.2, then $c(G) = \sum_{s} c_u(Z_G(s))$ and $c_u(\tilde{G}) = \sum_{s=\sigma s} c_u(Z_{\tilde{G}}(s)) + \frac{1}{2} \sum_{s \neq \sigma s} c_u(Z_G(s))$. The summations are taken over representatives of the semisimple classes of G^F, with the restrictions as indicated (notice that $p = 2$).

Proposition 4.4. Let $\tilde{G}$ and G be as in 4.2. Suppose that all components of $\Delta(G)$ are of type A_n or D_n (for various values of n) and that G and G* have connected centres. Then $c_u(\tilde{G}) = r_u(\tilde{G})$.

Proof. Consider the group $H = \prod GL_{n_i} \times \prod SO_{2m_j}$ ($n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ such that $\Delta(H) = \Delta(G)$) with a rational structure corresponding to that of G and choose an automorphism σ of H (over $\mathbb{F}_q$) of order 2 which acts on $\Delta(H)$ as $\tilde{G}/G$ on $\Delta(G)$. Let $\tilde{H}$ be the semidirect product of H and $\{1, \sigma\}$. Then it follows from [4] that $r_u(\tilde{H}) = r_u(\tilde{G})$. Also $c_u(\tilde{H}) = c_u(\tilde{G})$. It is therefore sufficient to prove the proposition fo $\tilde{H}$.

H is isomorphic to H* and the partition of $(H^F)^\wedge$ can therefore be indexed by semisimple classes in H^F. Since $r(\tilde{H}) = c(\tilde{H})$, we have :

$$\sum_{s=\sigma s} c_u(Z_{\tilde{H}}(s)) + \frac{1}{2} \sum_{s \neq \sigma s} c_u(Z_H(s)) = \sum_{s=\sigma s} |\tilde{X}_s| + \frac{1}{2} \sum_{s \neq \sigma s} |X_s|.$$

All the groups $Z_H(s)$ are Levi subgroups of parabolic subgroups of H. We have therefore $|X_s| = r_u(Z_H(s))$ and it follows also from computations in [8] that $|\tilde{X}_s| = r_u(Z_{\tilde{H}}(s))$. From [8] $c_u(Z_H(s)) = r_u(Z_H(s))$ and all the terms with $s \neq \sigma s$ cancel. The groups $Z_H(s)$ are all of the type considered in the proposition and therefore we may assume by induction on dim G that $c_u(Z_{\tilde{H}}(s)) = r_u(Z_{\tilde{H}}(s))$ if $s = \sigma s$ and $s \notin Z(H)$. So all the terms with $s \notin Z(\tilde{H})$ cancel and we get $|Z(\tilde{H})^F| c_u(\tilde{H}) =$

$|Z(\tilde{H})^F| r_u(\tilde{H})$. Hence $c_u(\tilde{H}) = r_u(\tilde{H})$.

Corollary 4.5. Let V be a vector space of dimension n over $\mathbb{F}_q$. Then there are p(n) equivalence classes of unipotent bilinear forms on V, where p(n) is the number of partitions of n.

Proof. Let G(V) be defined as in 3.1. G(V) has p(n) unipotent classes contained in GL(V). From 4.4 it is therefore sufficient to prove that there are 2p(n) unipotent representations in $G(V)^\wedge$. Since $GL(V)^\wedge$ contains p(n) unipotent representations, it is sufficient to show that G(V)/GL(V) fixes these unipotent representations. This comes from the fact that the $\mathbb{Z}$-module generated by their characters is also generated by characters of the form $\mathrm{Ind}_P^{GL(V)}(1)$ (P a parabolic subgroup of GL(V)) and G(V)/GL(V) acts trivially on the characters of this form.

4.6. Let H be a simply connected semisimple group of type E_7 defined over $\mathbb{F}_q$. H has a parabolic subgroup defined over $\mathbb{F}_q$ with a Levi subgroup G of type E_6 defined over $\mathbb{F}_q$. G is unique up to conjugation by an element of H^F.

H* can be taken to be the adjoint group of H and we have therefore a bijective homomorphism $f : H \longrightarrow H^*$ (since p = 2 H has a trivial centre). The image of G in H* is defined over $\mathbb{F}_q$, has type E_6 and is a Levi subgroup of some parabolic subgroup of H* defined over $\mathbb{F}_q$. From [8, 7.2] G* is isomorphic to some subgroup of H* with these properties. Since such subgroups form a single H^{*F}-conjugacy class, this shows that we can take $G^* = f(G)$. In particular the partition of $(G^F)^\wedge$ can be indexed by semisimple classes in G^F.

$|N_H(G)/G| = 2$ and $N_H(G)/G$ acts non-trivially on the Dynkin diagram of G. We can use 4.2 with $G = N_G(H)$. σ can be taken to be the conjugation by some suitable $x \in N_H(T)$ representing the longest element in the Weyl group of H ($T \subset G$ a maximal F-stable torus contained in some F-stable Borel subgroup of G). Since

$2c(\tilde{G}) = 2r(\tilde{G})$, we have :

$$2 \sum_{\Delta \equiv \sigma(\Delta)} c_u(Z_{\tilde{G}}(s)) + \sum_{\Delta \not\equiv \sigma(\Delta)} c_u(Z_G(s)) = 2 \sum_{\Delta \equiv \sigma(\Delta)} |\tilde{X}_s| + \sum_{\Delta \not\equiv \sigma(\Delta)} |X_s|.$$

Also, since $c(G) = r(G)$, we have :

$$\sum_{\Delta \equiv \sigma(\Delta)} c_u(Z_G(S)) + \sum_{\Delta \not\equiv \sigma(\Delta)} c_u(Z_G(s)) = \sum_{\Delta \equiv \sigma(\Delta)} |X_s| + \sum_{\Delta \not\equiv \sigma(\Delta)} |X_s|.$$

Substracting and rearranging terms, we get :

$$2 \sum_{\Delta \equiv \sigma(\Delta)} (c_u(Z_{\tilde{G}}(s)) - |\tilde{X}_s|) - \sum_{\Delta \equiv \sigma(\Delta)} (c_u(Z_G(s)) - |X_s|) = 0.$$

Examination of the different possibilities for s shows [3] :

a) If $s = 1$, $Z_{\tilde{G}}(s) = \tilde{G}$.

b) There is a semisimple element $s_o \in G^F$ such that s_o and $\sigma(s_o)$ are conjugate in G and $Z_G(s_o)$ has type $A_2 \times A_2 \times A_2$.

c) Suppose that $s \in G^F$ is semisimple and is conjugate in G to $\sigma(s)$ but not to 1 or s_o. Then $Z_G(s)$ is a Levi subgroup of some parabolic subgroup of G and is of the type considered in 4.4. Therefore $c_u(Z_{\tilde{G}}(s)) = |\tilde{X}_s|$ and $c_u(Z_G(u)) = |X_s|$ (as in 4.4).

Since in general $|\tilde{X}_s| \leq 2|X_s|$ and $c_u(Z_G(s)) \leq 2c_u(Z_{\tilde{G}}(s))$, the terms remaining after the cancellations due to (c) give :

$2c_u(\tilde{G}) \leq 3|X_1| + 3|X_{s_o}| + c_u(G)$.

It follows from computations in [6] that the right hand side has a bound independent of q. Since for some q every unipotent class of G contains rational points, this shows :

<u>Proposition 4.7</u>. The group $\tilde{G}$ of 4.6 has only finitely many unipotent classes.

CHAPTER II.

DYNKIN VARIETIES.

1. Equidimensionality of components.

We consider here a fixed unipotent element $u \in G$. We use in particular the notations of 0.6 with this element.

1.1. For many questions concerning $\mathcal{B}_u$, we can assume that G is reductive and generated by G^o and u. In this case, let G_1, ..., G_r be the minimal connected normal subgroups of G. Then $\mathcal{B}_u \cong \prod_{1 \leq i \leq r} \mathcal{B}(G_i)_u$. So we can also assume in many cases that G has no non-trivial connected normal subgroup. Suppose this is the case and let $G_1, \ldots, G_s$ be the minimal connected normal subgroups of G^o. It is easily checked that $\mathcal{B}_u \cong \mathcal{B}(G_1)_{(u^s)}$. This shows that we can often assume that $\Delta(G^o)$ is connected.

Another reduction (when G is reductive) is obtained by replacing G by $\mathrm{Aut}(G^o)$ via the morphism $\mathrm{ad} : g \longmapsto \mathrm{ad}(g)|_{G^o}$. We have clearly $\mathcal{B}_u \cong \mathcal{B}(\mathrm{Aut}(G^o))_{\mathrm{ad}(u)}$.

Lemma 1.2. Suppose that $B_o, B_j \in \mathcal{B}_u$ and $(B_o, B_j) \in O(w)$. Then
a) $w \in W^u$.
b) If $\ell_u(w) = j$ and $w = \tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_j$ $(s_1, \ldots, s_j \in \Pi)$, then the Borel subgroups $B_1, \ldots, B_{j-1}$ of 0.7 all belong to $\mathcal{B}_u$.

Proof. a) This is obvious since B_o and B_j are fixed by u.
b) This follows from the unicity property in 0.7.

Lemma 1.3. Suppose that $s \in \Pi$ and $P \in \mathcal{P}_{o(s)}$.

a) If $\mathcal{B}_u \cap \mathcal{B}(P)$ is not empty, then it is a single point or a projective line.

b) Suppose moreover that $B \subset P$ and $B \in \mathcal{B}_u$. Let V be the variety of all unipotent elements in uB. Define $V_s = \{v \in V \mid \dim(\mathcal{B}_v \cap \mathcal{B}(P)) = 1\}$. Then V_s is a hypersurface in V.

__Proof__. Without loss of generality, we may assume that $B \in \mathcal{B}_u$ and $B \subset P$.

Let H be the subgroup of G generated by P and u. Then for any $v \in V$, $\mathcal{B}_v \cap \mathcal{B}(P) \cong \mathcal{B}(H)_v$. Using the methods of 1.1, it is easy to see that it is sufficient to prove the lemma in the following two cases.

a) G is connected semisimple of type A_1.

b) $p = 2$, $G = \mathrm{Aut}(SL_3)$ and $u \notin G^o$.

This is done in 1.4 and 1.5.

__1.4__. If G is connected semisimple of type A_1, then $V = U$ is isomorphic to G_a. If $u \in U \setminus \{1\}$, then $\mathcal{B}_u = \{B\}$. If $u = 1$, $\mathcal{B}_u = \mathcal{B} \cong \mathbb{P}^1$.

__1.5__. Assume that $p = 2$ and let V be a k-vector space of dimension 3. Then $\mathrm{Aut}(SL(V)) \cong G(V)/Z$, where G(V) is defined as in I.3.1 and $Z = Z(GL(V))$.

It is sufficient to prove (a) of 1.3 for unipotent bilinear forms on V. Let f be such a form. If f^2 has only one Jordan block, then $\mathcal{B}_{f^2}$ is a single point (this can be checked directly) and therefore $\mathcal{B}_f \subset \mathcal{B}_{f^2}$ is also a single point. If f^2 has more than one Jordan block, then $f^2 = 1$, i.e. f is symmetric. It is easily checked that the isotropic flags for f form a variety isomorphic to $\mathbb{P}^1$. This proves (a) of 1.3.

Choose now a basis (e_1,e_2,e_3) of V. Let B be the subgroup of upper triangular matrices. A bilinear form $f \in G(V)$ is in $N_{G(V)}(B)$ if and only if its matrix has the form

$$\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

where $f_{ij} = f(e_i,e_j)$. It is easily checked that f is unipotent if and only if $f_{13} = f_{31}$ and that $f^2 = 1$ if and only if f is unipotent and $f_{23} = f_{32}$. (b) of 1.3 for Aut(SL(V)) follows then from the fact that $f \longmapsto (f_{23}+f_{32})/f_{22}$ is a regular function on $(N_{G(V)}(B) \setminus GL(V))/Z$.

Definition 1.6. Let s be a fundamental reflection and $P \in \mathcal{P}_{o(s)}$. A one-dimensional subvariety of $\mathcal{B}(P)$ will be called a line of type s if it is of the form $\mathcal{B}(P) \cap \mathcal{B}_x$ for some unipotent element $x \in uG^o$. If G is reductive and $s = s_\alpha$, we shall also call it a line of type α.

Corollary 1.7. Any two points in $\mathcal{B}_u$ can be connected in $\mathcal{B}_u$ by a sequence of arcs of lines of the kind described above (for various fundamental reflections). In particular $\mathcal{B}_u$ is connected.

Proof. This follows immediately from 1.2 and 1.3.

Corollary 1.8. uG^o contains a unipotent element v such that $\mathcal{B}_v$ consists of a single element.

Proof. We may assume that G is reductive. uG^o contains a unipotent element x which normalizes B and T. Choose one fundamental root $\alpha_1, \alpha_2, \dots, \alpha_m$ in each x-orbit in Π. Then $u' = x \prod_{1 \le i \le m} x_{\alpha_i}(1)$ is unipotent and it is easy to check that there are no lines of type α (any $\alpha \in \Pi$) through B contained in $\mathcal{B}_{u'}$. Therefore $\mathcal{B}_{u'} = \{B\}$ is reduced to a single element.

1.9. Let x be any element of G. We consider here a natural correspondence between the irreducible components $C_1, \dots, C_n$ of $C^o(x) \cap N$ and the components $(X_\sigma)_{\sigma \in S(x)}$ of $\mathcal{B}_x$.

Consider the morphisms $\pi_1 : G^o \longrightarrow \mathcal{B}$, $g \longmapsto {}^gB$, $\pi_2 : G^o \longrightarrow C^o$, $g \longmapsto g^{-1}xg$. Let $Y = \pi_1^{-1}(\mathcal{B}_x) = \pi_2^{-1}(C^o(x) \cap N)$, $Y_\sigma = \pi_1^{-1}(X_\sigma)$, $Y_i = \pi_2^{-1}(C_i)$ ($\sigma \in S(x)$, $1 \le i \le n$). Y_σ and Y_i are closed in Y. $Z_o(x)YB = Y$, $Z(x)^o Y_\sigma B = Y_\sigma$ and $Z_o(x)Y_iB = Y_i$. For any $\sigma \in S(x)$, Y_σ is irreducible since X_σ and B are so (0.14). As $Y_\sigma \not\subset \bigcup_{\tau \ne \sigma} Y_\tau$, $(Y_\sigma)_{\sigma \in S(x)}$ is the family of irreducible components of Y.

For any $\sigma \in S(x)$, $Z_o(x)Y_\sigma = \bigcup_{a \in A_o(x)} Y_{a\sigma}$ is closed in Y and is a union of fibres of π_2. Since π_2 is open, it follows that $\pi_2(Y_\sigma) = \pi_2(\bigcup_{a \in A_o(x)} Y_{a\sigma})$ is closed in $C^o(x) \cap N$. In particular, for each C_i there is a $\tau \in S(x)$ such that $\pi_2(Y_\tau) = C_i$.

Suppose that $\pi_2(Y_\sigma) \subset C_i$. Then Y_σ is an irreducible component of Y_i. But for some $\tau \in S(x)$, $C_i = \pi_2(Y_\tau)$ and therefore $Y_i = \bigcup_{a \in A_o(x)} Y_{a\tau}$. So the irreducible components of Y_i are $(Y_{a\tau})_{a \in A_o(x)}$. In particular $\sigma = a\tau$ for some $a \in A_o(x)$ and $\pi_2(Y_\sigma) = C_i$.

This gives a natural surjection $\pi : S(x) \longrightarrow \{C_1, \dots, C_n\}$ and for each i, $S_i = \pi^{-1}(C_i)$ is a single $A_o(x)$-orbit in S(x).

More generally, the same argument gives a bijection between the set of $Z_o(x)$-orbits in $\mathcal{B}_x$ and the set of B-orbits in

$C^o(x) \cap N$, and to each B-stable irreducible subvariety of $C^o(x) \cap N$ of codimension r we can associate a subvariety of $\mathcal{B}_x$ which is $Z_o(x)$-stable and whose components are permuted transitively by $A_o(x)$, and each component has codimension r in $\mathcal{B}_x$. A similar argument gives a bijective correspondence between the set of G/G^o-orbits in the set of irreducible components of $C(x) \cap N$ and the set of A(x)-orbits in S(x) (we use here the natural isomorphism $G/G^o \cong N/B$).

__1.10__. We use the notations of 1.9. If $\sigma \in S_i$, then $\dim Y_\sigma = \dim Y_i$ and therefore :
$\dim X_\sigma + \dim B = \dim C_i + \dim Z(x)$.

__Lemma 1.11__. Let X be a closed irreducible subvariety of $\mathcal{B}_u$. Suppose that for some $s \in \Pi$ there is a line of type s contained in $\mathcal{B}_u$ through all $x \in X$. Suppose that these lines are not all contained in X. Then the union Y of these lines is a closed irreducible subvariety of $\mathcal{B}_u$ and $\dim Y = \dim X + 1$.

__Proof__. Let $\bar{O}(s)$ be the closure of O(s) in $\mathcal{B} \times \mathcal{B}$. Let $Z = (X \times \mathcal{B}_u) \cap \bar{O}(s)$. Z is a projective variety, and since X is irreducible and all the fibres of $pr_2 : Z \longrightarrow X$ are projective lines, Z is irreducible (0.14) and $\dim Z = \dim X + 1$. Y is the projection of Z on $\mathcal{B}_u$ and is therefore closed and irreducible. $\dim Z \geq \dim Y \geq \dim X + 1$ since $X \subset Y$ and $X \neq Y$. Hence $\dim Y = \dim X + 1$.

__Proposition 1.12__. All irreducible components of $\mathcal{B}_u$ have the same dimension.

Proof. By 1.7, it is sufficient to prove that if X_σ is a component of maximal dimension and $L \subset \mathcal{B}_u$ is a line of type s ($s \in \Pi$) meeting X_σ, then L is contained in a component of maximal dimension.

With the notations of 1.3, V_s is a hypersurface in V. We use the correspondence of 1.9 with x = u. Choose $B' \in L \cap X_\sigma$. If $\sigma \in S_i$, let $X \ni B'$ be an irreducible component of the subvariety of $\mathcal{B}_u$ corresponding to $C_i \cap V_s$. If X_σ is a union of lines of type s, there is nothing to prove. Assume this is not the case. Then X has codimension 1 in $\mathcal{B}_u$ since $C_i \cap V_s$ has codimension 1 in $C^o(x) \cap N$. If X is a union of lines of type s, then $L \subset X \subset X_{a\sigma}$ for some $a \in A_o(x)$ and $\dim X_{a\sigma} = \dim X_\sigma$. We may therefore assume that X is not a union of lines of type s. We can now apply 1.11 and we get a closed irreducible subvariety $Y \subset \mathcal{B}_u$ such that $L \subset Y$ and $\dim Y = \dim X + 1 = \dim \mathcal{B}_u$. This proves the proposition.

1.13. In [19] Steinberg considers elements $x \in G$ such that x normalizes some $B' \in \mathcal{B}$ and some maximal torus T' of B'. Such an element is called quasisemisimple. We shall use the following result of Steinberg [19, p. 51] : every Borel subgroup normalized by a semisimple element x contains a maximal torus normalized by x. Thus semisimple elements are quasisemisimple. He shows also that if G^o is a simply connected semisimple algebraic group and $x \in G$ is quasisemisimple, then $Z_o(x)$ is a connected reductive group [19, p. 52] (this is stronger than 1.16). We consider here quasisemisimple elements in the framework of Dynkin varieties.

Lemma 1.14. Suppose that G is reductive and that $x \in G$ normalizes B and T. Then $Z_T(x)$ contains a one-dimensional torus T_0 independent of x with the following property. For all $c \in k^*$, there exists $t \in T_0$ such that $\alpha(t) = c$ for all $\alpha \in \Pi$.

Proof. Take for T_0 the identity component of $\{t \in T \cap [G^0, G^0] \mid \alpha(t) = \beta(t)$ for all $\alpha, \beta \in \Pi\}$. This is clearly a one-dimensional torus. T_0 is x-stable. x normalizes T and B and therefore leaves Φ^+ invariant. Since $\mathrm{Aut}(T_0) = \{+1, -1\}$, this shows that $T_0 \subset Z_T(x)$.

Corollary 1.15. a) $Z_T(x)^0$ is a regular torus in G, i.e. T is the only maximal torus of G containing it.

b) $Z_T(x)^0$ is a maximal torus of $Z(x)$ and every maximal torus of $Z(x)$ contained in B is of the form $Z_{T'}(x)^0$ for a unique maximal torus T' of B and x normalizes T'.

c) $C^0(x) \cap N_N(T)$ is a single $N_{G^0}(T)$-conjugacy class.

d) The B-conjugacy class of x is closed.

e) $C^0(x)$ and $C(x)$ are closed.

Proof. $Z_T(x) \supset T_0$ clearly contains regular elements [20, p. 96]. This proves (a). (b) is an immediate consequence of (a).

c) If $y = gxg^{-1} \in N_N(T)$ $(g \in G^0)$, then $Z_T(y)$ and $Z_{({}^gT)}(y)$ are maximal tori in $Z(y)$. Hence for some $z \in Z(y)^0$, $Z_T(y) = Z_{({}^{zg}T)}(y)$. By (b), $T = {}^{zg}T$. This proves (c) since $zg \in G^0$ and $y = (zg)x(zg)^{-1}$.

d) If $t \in T$ and $v \in U$, $(tv)x(tv)^{-1} = x(x^{-1}txt^{-1})(t(x^{-1}vxv^{-1})t^{-1})$. $t(x^{-1}vxv^{-1})t^{-1} \in U$ and $T' = \{x^{-1}txt^{-1} \mid t \in T\}$ is a subtorus of T. Hence the B-class of x is contained in $xT'U$ and contains xT'.

<u>Lemma 1.14</u>. Suppose that G is reductive and that $x \in G$ normalizes B and T. Then $Z_T(x)$ contains a one-dimensional torus T_o independent of x with the following property. For all $c \in k^*$, there exists $t \in T_o$ such that $\alpha(t) = c$ for all $\alpha \in \Pi$.

<u>Proof</u>. Take for T_o the identity component of $\{t \in T \cap [G^o, G^o] \mid \alpha(t) = \beta(t)$ for all $\alpha, \beta \in \Pi\}$. This is clearly a one-dimensional torus. T_o is x-stable. x normalizes T and B and therefore leaves Φ^+ invariant. Since $Aut(T_o) = \{+1, -1\}$, this shows that $T_o \subset Z_T(x)$.

<u>Corollary 1.15</u>. a) $Z_T(x)^o$ is a regular torus in G, i.e. T is the only maximal torus of G containing it.

b) $Z_T(x)^o$ is a maximal torus of Z(x) and every maximal torus of Z(x) contained in B is of the form $Z_{T'}(x)^o$ for a unique maximal torus T' of B and x normalizes T'.

c) $C^o(x) \cap N_N(T)$ is a single $N_{G^o}(T)$-conjugacy class.

d) The B-conjugacy class of x is closed.

e) $C^o(x)$ and C(x) are closed.

<u>Proof</u>. $Z_T(x) \supset T_o$ clearly contains regular elements [20, p. 96]. This proves (a). (b) is an immediate consequence of (a).

c) If $y = gxg^{-1} \in N_N(T)$ $(g \in G^o)$, then $Z_T(y)$ and $Z_{(^gT)}(y)$ are maximal tori in Z(y). Hence for some $z \in Z(y)^o$, $Z_T(y) = Z_{(^{zg}T)}(y)$. By (b), $T = {}^{zg}T$. This proves (c) since $zg \in G^o$ and $y = (zg)x(zg)^{-1}$.

d) If $t \in T$ and $v \in U$, $(tv)x(tv)^{-1} = x(x^{-1}txt^{-1})(t(x^{-1}vxv^{-1})t^{-1})$. $t(x^{-1}vxv^{-1})t^{-1} \in U$ and $T' = \{x^{-1}txt^{-1} \mid t \in T\}$ is a subtorus of T. Hence the B-class of x is contained in xT'U and contains xT'.

If $y = xt\prod_{\alpha\in\Phi} x_\alpha(c_\alpha) \in xT'U$, then xt is in the closure of the T_o-class of y. The B-class of x is therefore closed since it is contained in the closure of any B-class contained in xT'U.

e) This follows from (d) since G^o/B is complete.

__Proposition 1.16__. Let G be a reductive algebraic group. Suppose that $x \in G$ is such that every $B' \in \mathcal{B}_x$ contains a maximal torus normalized by x (semisimple elements have this property). Then $H = Z(x)$ is reductive. Moreover there is a natural morphism $f : \mathcal{B}_x \longrightarrow \mathcal{B}(H)$ given by $B' \longmapsto B' \cap H^o$. f is H-equivariant and its restriction to any component of $\mathcal{B}_x$ is an isomorphism.

__Proof__. We may assume that x normalizes B and T. Since any $y \in C^o(x) \cap N$ normalizes a maximal torus of B, every element of $C^o(x) \cap N$ is B-conjugate to some element in $N_N(T)$. By 1.15 (c) and (d), $C^o(x) \cap N$ is therefore a finite union of closed B-conjugacy classes of elements in $N_N(T)$. So B acts transitively on each irreducible component of $C^o(x) \cap N$. By 1.9, H^o acts transitively on each irreducible component of $\mathcal{B}_x$. Since $\mathcal{B}_x$ is complete, the stabilizer $H^o \cap B$ of $B \in \mathcal{B}_x$ is parabolic in H and therefore $H^o \cap B$ is a Borel subgroup of H. This gives an equivariant bijective morphism f' from $\mathcal{B}(H)$ to the component of $\mathcal{B}_x$ containing B.

Let $U^- = \prod_{\alpha\in\Phi} X_{-\alpha}$, $B^- = TU^-$. $B^- \in \mathcal{B}_x$ and $U^-TU \cap H = Z_{U^-}(x)Z_T(x)Z_U(x) = Z_{U^-}(x)(B \cap H)$ is a neighbourhood of 1 in H. Hence $\dim \mathcal{B}(H) = \dim Z_{U^-}(x)$. Also f' is an isomorphism since $\dim df'(T(H)_1) \geq \dim df'(T(Z_{U^-}(x))_1) = \dim Z_{U^-}(x) = \dim \mathcal{B}(H)$. Replacing now B by B^-, we get $\dim \mathcal{B}(H) = \dim Z_U(x)$. Therefore $\dim H = \dim Z_{U^-}(x) + \dim Z_T(x) + \dim Z_U(x) = 2\dim \mathcal{B}(H) +$

dim $Z_T(x)$. This shows that H is reductive. The remaining statements in the proposition follow from the fact that B can be any element of $\mathcal{B}_x$.

Proposition 1.17. Let G be any algebraic group and let x be any element of G. Then all irreducible components of $\mathcal{B}_x$ have the same dimension and all irreducible components of $C(x) \cap N$ have the same dimension.

Proof. Because of 1.10, we need only to prove that all irreducible components of $\mathcal{B}_x$ have the same dimension. We may also assume that G is reductive. Let x = su be the Jordan decomposition of x. By 1.16, $\mathcal{B}_x$ is isomorphic to a union of varieties isomorphic to $\mathcal{B}(Z(s))_u$. The proposition follows then from 1.12.

2. Dimension of $\mathcal{B}_u$ and relative positions.

In this paragraph G is always a reductive group and $u \in G$ is a fixed unipotent element. We write C^o, C, Z, A_o, A and S for $C^o(u)$, C(u), Z(u), $A_o(u)$, A(u) and S(u) respectively.

2.1. In 0.7, we have defined $\varphi : S\times S \longrightarrow W$ as follows. $\varphi(\sigma,\tau) = w$ if and only if $(X_\sigma \times X_\tau) \cap O(W)$ is dense in $X_\sigma \times X_\tau$. The following properties are clear :

a) $\varphi(\sigma,\tau) \in W^u$.

b) $\varphi(\tau,\sigma) = \varphi(\sigma,\tau)^{-1}$.

c) $\varphi(a\sigma,a\tau) = \varphi(\sigma,\tau)$ for all $a \in A_o$.

2.2. Let $E = \{(B_1,B_2,x) | B_1, B_2 \in \mathcal{B}_x$ and $x \in C^o\}$. G^o acts on E by $g.(B_1,B_2,x) = ({}^gB_1, {}^gB_2, gxg^{-1})$. For any $\sigma, \tau \in S$, let $E_{\sigma,\tau} = G^o.(X_\sigma \times X_\tau \times \{u\})$. $E_{\sigma,\tau}$ is closed in E and irreducible and by 1.12 $\dim E = \dim E_{\sigma,\tau} = 2 \dim \mathcal{B}_u + \dim C^o$. $E_{\sigma,\tau} = E_{\sigma',\tau'}$ if and only if $(\sigma',\tau') = (a\sigma,a\tau)$ for some $a \in A_o$. Since $E = \bigcup_{\sigma,\tau \in S} E_{\sigma,\tau}$, the varieties $E_{\sigma,\tau}$ are the irreducible components of E.

2.3. For any $w \in W$, let $E_w = E \cap (O(w)\times C^o)$. $E_w = \emptyset$ if $w \notin W^u$. If $\dim E_w = \dim E$, $\overline{E}_w$ contains some component $E_{\sigma,\tau}$ of E and then $\varphi(\sigma,\tau) = w$. Conversely, if $\varphi(\sigma,\tau) = w$, then $\overline{E}_w$ contains $E_{\sigma,\tau}$ and $\dim E_w = \dim E$. Therefore $\varphi(S\times S) = \{w \in W^u | \dim E_w = \dim E\}$.

2.4. For any $w \in W$, let $N_w = N \cap {}^wN$. $N_w \cap uG^o = \emptyset$ if $w \notin W^u$. For $w \in W^u$, let V_w be the variety of all unipotent elements in $N_w \cap uG^o$. From I.1.4, V_w is irreducible and $\dim V_w = \dim B - \text{rank}_u(G) - \ell(w) = \dim V_1 - \ell(w)$.

Since there are only finitely many unipotent classes in G, there is a unique G^o-class C_w^o such that $C_w^o \cap V_w$ is dense in V_w. We have therefore a natural map $w \longmapsto C_w^o$ from W^u to the set of unipotent G^o-classes in uG^o.

For any subvariety X of uG^o, define $Q(X) = \{w \in W^u | X \cap V_w$ is dense in $V_w\}$. We shall consider in particular the sets $Q(C^o)$ and $Q(C_i)$ $(1 \leqslant i \leqslant n)$, where $C_1, \ldots, C_n$ are the irreducible components of $C^o \cap N$. Clearly $Q(C^o) = \{w \in W^u | C^o = C_w^o\}$.

Proposition 2.5. $\dim Z \geqslant 2 \dim \mathcal{B}_u + \text{rank}_u(G)$ and there is equality if and only if $Q(C^o) \neq \emptyset$.

Proof. For any $w \in W^u$, $\dim E_w = \dim O(w) + \dim (V_w \cap C^o) \leqslant \dim G - \text{rank}_u(G)$, and there is equality if and only if $w \in Q(C^o)$. Therefore $\dim E \leqslant \dim G - \text{rank}_u(G)$, and there is equality if and only if $Q(C^o) \neq \emptyset$. But from 2.2, $\dim E = \dim C^o + 2 \dim \mathcal{B}_u$. This proves the proposition since $\dim G = \dim Z + \dim C^o$.

2.6. If G is connected, Bala and Carter have proved that $Q(C^o) \neq \emptyset$ if $p = 0$ or if $p \geqslant 4m + 3$, where $m = \max_{\alpha \in \Phi^+} ht(\alpha)$ [1] If G^o is of type A_n, B_n, C_n or D_n, $\dim Z$ and $\dim \mathcal{B}_u$ can be computed and we find that $\dim Z = 2 \dim \mathcal{B}_u + \text{rank}_u(G)$ (5.7 and 5.21) and therefore $Q(C^o) \neq \emptyset$ (actually the computations in paragraph 5 don't deal with the case where G^o is of type D_4 and the image of u in $\Gamma(G^o)$ has order 3 but this case has been considered in I.2.5). In paragraph 6 we shall show how to find elements in $Q(C^o)$ when G^o is of type B_n or C_n (this is the method sketched by Steinberg in [21] to prove that $Q(C^o) \neq \emptyset$ for connected groups of these types). If G^o is of type G_2, direct computations show that $Q(C^o) \neq \emptyset$.

Collecting these results, we find that the application $W^u \longrightarrow \{\text{unipotent } G^o\text{-classes in } uG^o\}$, $w \longmapsto C^o_w$ is surjective at least if one of the following conditions holds :

a) $p = 0$ or $p \geqslant 120$.

b) $\triangle(G^o)$ has no components of type E_6, E_7, E_8 or F_4.

c) $\triangle(G^o)$ has no components of type E_7 or E_8 and $p \geqslant 47$.

d) $\triangle(G^o)$ has no components of type E_8 and $p \geqslant 71$.

Corollary 2.7. Let x be any element of G. Then :

a) $\dim Z(x) \geqslant 2 \dim \mathcal{B}_x + \text{rank}_x(G)$.

b) $\dim \mathcal{B}_x + \dim (C(x) \cap N) \leqslant \dim B - \text{rank}_x(G)$.

c) $\dim C(x) \geqslant 2 \dim (C(x) \cap N) - (\text{rank}_1(G) - \text{rank}_x(G))$.

If one of (a), (b) or (c) is an equality, then they are all equalities. If one of the conditions of 2.6 holds, then they are all equalities.

Proof. By 1.10, $\dim \mathcal{B}_x + \dim B = \dim (C(x) \cap N) + \dim Z(x)$. (b) is obtained from (a) by replacing $\dim Z(x)$ by $\dim \mathcal{B}_x + \dim B - \dim (C(x) \cap N)$. To get (c) from (a), replace $\dim \mathcal{B}_x$ by $\dim (C(x) \cap N) + \dim Z(x) - \dim B$ and use $\dim G = \dim Z(x) + \dim C(x) = 2 \dim B - \text{rank}_1(G)$.

So we need only to consider (a). Let $x = su$ be the Jordan decomposition of x. By 1.16 and 2.5, $\dim Z_{Z(s)}(u) \geqslant 2 \dim \mathcal{B}(Z(s))_u + \text{rank}_u(Z(s))$. $Z_{Z(s)}(u) = Z(x)$ and by 1.16 $\dim \mathcal{B}(Z(s))_u = \dim \mathcal{B}_x$. Hence $\dim Z(x) \geqslant 2 \dim \mathcal{B}_x + \text{rank}_u(Z(s))$. We have to show that $\text{rank}_u(Z(s)) = \text{rank}_x(G)$. We may assume that $B \in \mathcal{B}_x = \mathcal{B}_s \cap \mathcal{B}_u$. By 1.16, $B \cap Z(s)^o$ is a Borel subgroup of $Z(s)$. Therefore $\text{rank}_u(Z(s)) = \text{rank}_u(N \cap Z(s)) =$

$\dim (Z_{Z_B(s)/Z_U(s)}(uZ_U(s)))$. In the proof of I.1.2 (c) we have shown that $Z_B(s)/Z_U(s) \cong Z_{B/U}(sU)$. It follows that $Z_{Z_B(s)/Z_U(s)}(uZ_U(s)) \cong Z_{Z_{B/U}(sU)}(uU) = Z_{B/U}(suU) = Z_{B/U}(xU)$. Hence $\mathrm{rank}_u(Z(s)) = \dim (Z_{B/U}(xU)) = \mathrm{rank}_x(G)$. This proves (a). If one of the conditions of 2.6 holds for G, it holds also for Z(s) and we have equality in (a). This proves the corollary.

2.8. Suppose that $Q(C^o) \neq \emptyset$. From the proof of 2.5, $\dim E_w = \dim E$ if and only if $w \in Q(C^o)$. By 2.3 we have then $\varphi(S\times S) = Q(C^o)$. Moreover, as V_w is irreducible, $E_w = G^o.(\{B\}\times\{^wB\}\times (C^o \cap V_w))$ is irreducible if $w \in Q(C^o)$. By 2.3 $\varphi(\sigma,\tau) = w$ is then equivalent to $E_{\sigma,\tau} = \overline{E}_w$. Collecting previous results, we get :

Proposition 2.9. Assume that $Q(C^o) \neq \emptyset$. Then $\varphi : S\times S \longrightarrow W^u$ has the following properties.

a) $\varphi(\tau,\sigma) = \varphi(\sigma,\tau)^{-1}$.

b) $\varphi(\sigma',\tau') = \varphi(\sigma,\tau)$ if and only if $(\sigma',\tau') = (a\sigma,a\tau)$ for some $a \in A_o$.

c) $\varphi(S\times S) = Q(C^o)$.

Corollary 2.10. Let $u_1,u_2,\ldots,u_m$ be a complete set of representatives for the unipotent G^o-classes contained in uG^o which are of the form C^o_w for some $w \in W^u$. Then

$$\sum_{1\leq i\leq m} |(S(u_i)\times S(u_i))/A_o(u_i)| = |W^u|.$$

Proof. W^u is the disjoint union of the sets $Q(C^o(u_i))$ $(1\leq i\leq m)$. The corollary follows then from 2.9.

Proposition 2.11. Suppose that $Q(C^o) \neq \emptyset$. Then $|S| \geq |\{w \in Q(C^o) | w^2 = 1\}|$. There is equality if $a^2 = 1$ for all $a \in A_o$.

Proof. Let $S_1,\ldots,S_n$ be the A_o-orbits in S. If $w \in Q(C^o)$, $w = \varphi(\sigma,\tau)$ for some $\sigma \in S_i, \tau \in S_j$. If $w^2 = 1$, $\varphi(\sigma,\tau) = \varphi(\sigma,\tau)^{-1} = \varphi(\tau,\sigma)$. Hence $(\tau,\sigma) = (a\sigma,a\tau)$ for some $a \in A_o$. So i=j and $w \in \varphi(S_i \times S_i)$.

Suppose that $a^2 = 1$ for all $a \in A_o$. Then $\varphi(\sigma,a\sigma) = \varphi(a\sigma,a^2\sigma) = \varphi(a\sigma,\sigma) = \varphi(\sigma,a\sigma)^{-1}$. Hence $\varphi(\sigma,\tau)^2 = 1$ if $\sigma,\tau \in S_i$.

Therefore $\{w \in Q(C^o) \mid w^2 = 1\} \subset \bigcup_{1 \leq i \leq n} \varphi(S_i \times S_i)$, and there is equality if $a^2 = 1$ for all $a \in A_o$.

Also $|\varphi(S_i \times S_i)| = |(S_i \times S_i)/A_o| \leqslant |S_i|$ since A_o acts transitively on S_i, and there is equality if A_o is abelian. This proves the proposition.

Proposition 2.12. Let $u_1,\ldots,u_m$ be a complete set of representatives for the unipotent G^o-classes contained in uG^o. Then $\sum_{1 \leq i \leq m} |S(u_i)| \geqslant |\{w \in W^u \mid w^2 = 1\}|$. There is equality if the following condition is realized. $\triangle(G^o)$ has no components of type E_6, E_7, E_8, F_4 or G_2 and there is no power of u acting by an automorphism of order 3 on a component of type D_4.

Proof. $(Q(C^o(u_i)))_{1 \leq i \leq m}$ is a partition of W^u. The inequality follows then from 2.11. For the equality, notice that if the given condition is verified, then $Q(C^o(u_i)) \neq \emptyset$ for all i and that $A_o(u_i)$ is a product of cyclic groups of order 2 for all i. We can therefore use the equality in 2.11.

Remark 2.13. It is known that $|\{w \in W^u \mid w^2 = 1\}|$ is the sum of the degrees of the irreducible complex characters of W^u. If G is connected and the characteristic is good (in the sense of [20, p. 106]), 2.11 and 2.12 are consequences of the results of Springer on the representations of W in the cohomology of $\mathcal{B}_u$ [17]. It follows from Springer's results that in this situation there is equality in 2.11 if A_o is commutative (however no example is known in which A_o is abelian but fails to be a product of cyclic groups of order 2).

2.14. There are similar results for the application $\bar{\varphi} : S\times S \longrightarrow \bar{W}$ (0.5 and 0.8). Assume that uG^o is central in G/G^o. Then $\bar{\varphi}(\sigma,\tau) \in \{\bar{w} | w \in W^u\}$ and if $\bar{\varphi}(\sigma,\tau) = \bar{w}$, then $\bar{\varphi}(\tau,\sigma) = \overline{w^{-1}}$. Assume moreover that $Q(C) \neq \emptyset$. Then $\bar{\varphi}(S\times S) = \bar{Q}(C) = \{\bar{w} | w \in Q(C)\}$ and $\bar{\varphi}(\sigma,\tau) = \bar{\varphi}(\sigma',\tau')$ if and only if $(\sigma',\tau') = (a\sigma, a\tau)$ for some $a \in A$. If $\bar{w} = \overline{w^{-1}} \in \bar{Q}(C)$, then $\bar{w} = \bar{\varphi}(\sigma, a\sigma)$ for some $\sigma \in S$ and $a \in A$. If A acts on S via a quotient isomorphic to a product of groups of order 2, then $\bar{\varphi}(\sigma, a\sigma) = \bar{w} \Rightarrow \bar{w} = \overline{w^{-1}}$. It follows that $|S| \geq |\{\bar{w} \in \bar{Q}(C) | \bar{w} = \overline{w^{-1}}\}|$ and there is equality if A acts on S via a quotient isomorphic to a product of groups of order 2. The proofs are essentially the same as the proofs for the corresponding statements for φ.

Proposition 2.15. Suppose that $Q(C^o) \neq \emptyset$. Let C_i be a component of $C^o \cap N$ and let S_i be the corresponding A_o-orbit in S. Then $Q(C_i) = \varphi(S_i \times S) = \{w \in W^u | \overline{C_i} = \overline{{}^{B}V_w}\}$.

Proof. Suppose that $w = \varphi(\sigma,\tau)$ with $\sigma \in S_i$ and $\tau \in S$. Then $X' = \{B_1 \in X_\sigma | \exists\ B_2 \in \mathcal{B}_u \text{ such that } (B_1,B_2) \in O(w)\} \supset pr_1(X_\sigma \times X_\tau \cap O(w))$ contains a dense open subset of X_σ. This shows that $C_i' = \{v \in C_i | \exists\ B' \in \mathcal{B}_v \text{ such that } (B,B') \in O(w)\}$ contains a dense open subset of C_i. If $v \in C_i'$ and $B' \in \mathcal{B}_v$ are such that $(B,B') \in O(w)$, then for some $b \in B$, ${}^bB' = {}^wB$ and therefore $b^{-1}vb \in V_w$.

Hence $\overline{C_i} \subset \overline{{}^B V_w}$. But $C^o \cap V_w$ is dense in V_w and therefore $\overline{{}^B V_w} \subset \overline{C^o} \cap N$. Therefore $\overline{C_i} = \overline{{}^B V_w}$ since ${}^B V_w$ is irreducible and C_i is an irreducible component of $C^o \cap N$.

Hence $\varphi(S_i \times S) \subset \{w \in W^u \mid \overline{C_i} = \overline{{}^B V_w}\}$. Every $w \in W^u$ is of the form $\varphi(\sigma, \tau)$ for some $v \in V$ and $\sigma, \tau \in S(v)$ (with $v \in C_w^o$) and $\overline{{}^B V_w}$ is an irreducible component of $C_w^o \cap N$. It is then easy to check the remaining inclusions of the proposition.

<u>Corollary 2.16</u>. If $Q(C^o)$ is non-empty, then so is each $Q(C_i)$ and $Q(C^o)$ is the disjoint union of $Q(C_1), \ldots, Q(C_n)$. Each $Q(C_i)$ contains an involution.

<u>Proof</u>. This follows from 2.15. For the involution in $Q(C_i)$, take $\varphi(\sigma, \sigma)$, where $\sigma \in S_i$.

<u>Proposition 2.17</u>. $\dim \mathcal{B}_u \leq \min \{\ell_u(w) \mid w \in \varphi(S \times S)\}$.

<u>Proof</u>. We prove first that for every $B' \in \mathcal{B}_u$ and $w \in W^u$, $\dim (\{B'\} \times \mathcal{B}_u \cap O(w)) \leq \ell_u(w)$ (if u acts trivially on W this is clear since $\dim O(w) = \ell(w) + \dim \mathcal{B}$). We prove this by induction on $\ell_u(w)$. The result is obvious if $\ell_u(w) = 0$. Assume that it is true for w and that $s \in \Pi$ is such that $\ell_u(w\tilde{s}) = \ell_u(w) + 1$. There is a natural morphism $(\{B'\} \times \mathcal{B}_u) \cap O(w\tilde{s}) \longrightarrow (\{B'\} \times \mathcal{B}_u) \cap O(w)$ induced by the morphism $O(w\tilde{s}) \longrightarrow O(w)$ of 0.7 and its fibres have dimension ≤ 1 by 1.3. Therefore
$\dim (\{B'\} \times \mathcal{B}_u \cap O(w\tilde{s})) \leq \dim (\{B'\} \times \mathcal{B}_u \cap O(w)) + 1 \leq$
$\ell_u(w) + 1 = \ell_u(w\tilde{s})$.

Consider now an element $w \in \varphi(S\times S)$ such that $\ell_u(w) = \min\{\ell_u(w') \mid w' \in \varphi(S\times S)\}$. There exists $B' \in \mathcal{B}_u$ and $\sigma \in S$ such that $w = \varphi(\{B'\}, X_\sigma)$. Then $\dim \mathcal{B}_u = \dim X_\sigma = \dim(\{B'\}\times\mathcal{B}_u \cap O(w)) \leqslant \ell_u(w) = \min\{\ell_u(w') \mid w' \in \varphi(S\times S)\}$.

<u>Proposition 2.18</u>. Let w_o be the element of maximal length in W and let $\ell = \ell_u(w_o)$. Then the following conditions are equivalent.

a) $w_o \in \varphi(S\times S)$.

b) $\varphi(S\times S) = \{w_o\}$.

c) $w_o \in Q(C^o)$.

d) $\dim \mathcal{B}_u = \ell$.

e) Some $B' \in \mathcal{B}_u$ contains a maximal torus normalized by u.

f) Every $B' \in \mathcal{B}_u$ contains a maximal torus normalized by u.

<u>Proof</u>. (b) $\Longrightarrow$ (a) and (f) $\Longrightarrow$ (e) are obvious. (c) $\Longrightarrow$ (a) is a consequence of 2.9. (d) $\Longrightarrow$ (b) is a consequence of 2.17. (a) $\Longrightarrow$ (e) is clear since $(B_1, B_2) \in O(w_o)$ if and only if $B_1 \cap B_2$ is a maximal torus.

We suppose now that (e) holds and we prove that (b), (c), (d) and (f) hold also. We may assume that u normalizes B and T. If $g \in G^o$ is such that ug normalizes B and T, then $g \in T$. If moreover ug is unipotent, then u and ug are T-conjugate (by I.1.3 applied to the group generated by T and u). This shows that the unipotent elements of uG^o satisfying (e) form a single G^o-class. As (c) $\Longrightarrow$ (e), this class must be C_{w_o}. In particular (e) $\Longrightarrow$ (c). This shows also that ${}^BV_{w_o}$ is the B-orbit of u. By 1.15 this orbit is closed. By 2.15 ${}^{B}V_{w_o}$ is therefore an irreducible component of $C^o \cap N$. $Z_o(u)$ acts transitively on the union of the corresponding components of $\mathcal{B}_u$. Since $\mathcal{B}_u$ is connected, this implies that $\mathcal{B}_u$ is irreducible. In particular $|\varphi(S\times S)| = 1$ and therefore $\varphi(S\times S) = \{w_o\}$. So (e) $\Longrightarrow$ (b). Since $Z_o(u)$ acts transitively on $\mathcal{B}_u$, we have also (e) $\Longrightarrow$ (f). It is easy to check that for every $s \in \Pi$, $\mathcal{B}_u$ contains a line of type s through B (as in the proof of 1.3 it is sufficient to check that when G is connected of type A_1 and when $G = Aut(SL_3)$, $u \in G^o$ and $p = 2$). Since $Z_o(u)$ acts transitively on $\mathcal{B}_u$, $\mathcal{B}_u$ contains a line of type s through every $B' \in \mathcal{B}_u$ (any $s \in \Pi$). The demonstration of 2.17 gives then $\dim(\{B\} \times \mathcal{B}_u \cap O(w)) = \rho_u(w)$ for every $w \in W^u$. Taking $w = w_o$, we get $\dim \mathcal{B}_u = \ell$. So (e) $\Longrightarrow$ (d). This proves the proposition.

Corollary 2.19. Every unipotent component of G contains a unique unipotent quasisemisimple G^o-conjugacy class which is characterized by any one of the equivalent conditions of 2.18. If u is an element in such a class, then $H = Z_G(u)$ is reductive, $\mathcal{B}_u$ is irreducible, H^o acts transitively on $\mathcal{B}_u$ and $B' \longrightarrow B' \cap H^o$ defines an H-equivariant isomorphism $\mathcal{B}_u \longrightarrow \mathcal{B}(H)$.

Proof. This follows from 1.16 and 2.18.

Lemma 2.20. Every unipotent G^o-class contains the corresponding quasisemisimple unipotent G^o-class in its closure.

Proof. This follows from the proof of 1.15 (d).

Proposition 2.21. Let g be any element of G and let $g = su$ be its Jordan decomposition. Then the following conditions are equivalent.

a) g normalizes a Borel subgroup of G and a maximal torus of this subgroup (i.e. g is quasisemisimple).

b) Every Borel subgroup normalized by g contains a maximal torus normalized by g.

c) u is quasisemisimple in Z(s).

d) C(g) is closed in G.

Proof. (b) $\Longrightarrow$ (a) is clear. (a) $\Longrightarrow$ (d) has been proved in 1.15. (d) $\Longrightarrow$ (c) follows from 2.20.and 1.16. So we need only to prove that (c) $\Longrightarrow$ (b). Suppose that u is quasisemisimple in Z(s). If $B' \in \mathcal{B}_g = \mathcal{B}_s \cap \mathcal{B}_u$, then $B' \cap Z(s)^o$ is a Borel subgroup of Z(s) normalized by u. By 2.18 $B' \cap Z(s)^o$ contains a maximal torus of Z(s) normalized by u. By 1.15 this torus is contained in a unique maximal torus T' of G and by unicity T' is normalized by s and u, hence by g. This proves the proposition.

3. Some special classes and some special components.

In this paragraph G is supposed to be reductive, $x \in G$ is a unipotent quasisemisimple element normalizing B and T and $T_o = \{t \in T \mid xt \text{ is unipotent}\}$.

3.1. Let $P \supset B$ be an x-stable parabolic subgroup of G^o. $\mathcal{P}^o$ is characterized by $I = \{\alpha \in \Pi \mid X_{-\alpha} \subset P\}$ and I is x-stable. $L = \langle T, X_{\pm\alpha} \mid \alpha \in I\rangle$ is a Levi subgroup of P and its Weyl group W_P may be identified with the subgroup of W generated by $\{s_\alpha \mid \alpha \in I\}$. Let w_P be the element of maximal length in W_P. $w_P \in W^x$ and from 2.4 we get a unipotent G^o-class $C_P^o = C_{w_P}^o$ contained in xG^o. An element $u \in C_P^o$ will be called P-regular (or $\mathcal{P}^o$-regular) and C_P^o is the P-regular (or $\mathcal{P}^o$-regular) G^o-class. We get in the same way a unipotent class C_P meeting xG^o, $u \in C_P$ will be called $\mathcal{P}$-regular and C_P is the $\mathcal{P}$-regular class (meeting xG^o).

Notice that $w_P \in Q(C_P^o)$ and therefore $Q(C_P^o) \neq \emptyset$. Notice also that $V_P = V_{w_P}$ (defined as in 2.4) consists of all elements of the form xtu with $t \in T_o$ and $u \in U_P$.

3.2. Let $X = \{v \in xP \mid vU_P \text{ is unipotent quasisemisimple in } N_G(P)/U_P\}$. By 2.18 $X = \{v \in xP \mid v \text{ is unipotent and } \dim(\mathcal{B}_v \cap \mathcal{B}(P)) = \ell\}$, where $\ell = \ell_x(w_P)$. X is closed in G by 1.15 and $X \cap xB \cap x^{(w_P)}B = V_P$. Therefore $X \cap xB = {}^BV_P$ and ${}^BV_P$ is closed in G. This shows that $C_P^o \cap {}^BV_P$ is an irreducible component of $C_P^o \cap N$.

x is unipotent quasisemisimple in $\langle L, x\rangle$ and by 2.18

$\dim \mathcal{B}(L)_x = \ell$ and $L \cap B$ acts transitively on the intersection of N with the L-class of x. By 2.7 this intersection has dimension $\dim (B \cap L) - \mathrm{rank}_x(L) - \ell = \dim (B \cap L) - \dim T + \dim T_o - \ell$. It follows easily that $\dim {}^B V_P \geqslant \dim U + \dim T_o + \ell$. We have therefore $\dim \mathcal{B}_u + \dim (C_P^o \cap N) \geqslant \ell + (\dim U + \dim T_o) - \ell) = \dim B - \mathrm{rank}_u(G)$ if $u \in C_P^o$. By 2.7 we must have equality and therefore $\dim \mathcal{B}_u = \ell$ and $\dim {}^B V_P = \dim U + \dim T_o - \ell$.

If $u \in C_P^o \cap X$, $\mathcal{B}_u \cap \mathcal{B}(P)$ is isomorphic to $\mathcal{B}(L)_x$ and has dimension $\ell = \dim \mathcal{B}_u$. It is therefore an irreducible component of $\mathcal{B}_u$ and it corresponds to the component $C_P^o \cap {}^B V_P$ of $C_P^o \cap N$. It follows then from 1.9 that there are only finitely many $P' \in \mathcal{P}^o$ such that $\dim (\mathcal{B}_u \cap \mathcal{B}(P')) = \ell$ and that all such parabolic subgroups are conjugate under $Z_o(u)$. Also $\mathcal{B}_u \cap \mathcal{B}(P) \subset \{{}^P B \mid p \in P\}$ and therefore $C_P^o \cap {}^B V_P \subset \{p^{-1} u p \mid p \in P\}$. It follows that $X \cap C_P^o$ is a single P-orbit. Since G/P is complete, ${}^{G^o}X$ is closed in xG^o. We have proved :

<u>Proposition 3.3</u>. Let $X = \{v \in xP \mid vU_P$ is unipotent quasisemisimple in $N_G(P)/U_P\}$. Then ${}^{G^o}X$ is closed in xG^o, $C_P^o \cap X$ is a single P-orbit and is dense in X. If $u \in C_P^o \cap X$, then $\mathcal{B}_u \cap \mathcal{B}(P)$ is isomorphic to $\mathcal{B}(L)_x$ and is a component of $\mathcal{B}_u$. $\dim \mathcal{B}_u = \ell = \ell_x(w_P)$. There are only finitely many $P' \in \mathcal{P}^o$ such that $\dim (\mathcal{B}_u \cap \mathcal{B}(P')) = \ell$, and all such parabolic subgroups are conjugate to P under $Z_o(u)$.

<u>Example 3.4</u>. If $P = G^o$, the P-regular elements are the

unipotent quasisemisimple ones.

Definition 3.5. An element $g \in G$ is regular if $\dim Z_G(g) = \mathrm{rank}_g(G)$.

Remark 3.6. If g is any element of G, 2.7 shows that $\dim Z(g) \geqslant \mathrm{rank}_g(G)$.

Proposition 3.7. An element $g \in G$ is regular if and only if $|\mathcal{B}_g| < \infty$.

Proof. If g is regular, then 2.7 shows that $\dim \mathcal{B}_g = 0$, i.e. $\mathcal{B}_g$ is a finite set. Suppose conversely that $\dim \mathcal{B}_g = 0$. Let $g = su$ be the Jordan decomposition of g. Then $\dim \mathcal{B}(Z(s))_u = 0$, $Z_G(g) = Z_{Z(s)}(u)$ and $\mathrm{rank}_g(G) = \mathrm{rank}_u(Z(s))$ (the last statement has been proved in the proof of 2.7). It is therefore sufficient to prove the following : if $u \in xG^o$ is unipotent and $\dim \mathcal{B}_u = 0$, then $\dim Z(u) = \mathrm{rank}_u(G)$.

The unipotent elements in xG^o form a closed irreducible subvariety of codimension $\mathrm{rank}_u(G)$ and the number of unipotent G^o-classes is finite. It follows that there is exactly one unipotent G^o-class in xG^o which is dense in this subvariety and if u_o is an element in this class then $\dim Z(u_o) = \mathrm{rank}_{u_o}(G)$. Therefore u_o is regular (in particular this shows that every unipotent component of G contains exactly one unipotent regular G^o-class). It is sufficient to prove that u and u_o are conjugate. We may assume that u and u_o normalize B and that G^o is adjoint (by I.1.6). We consider two cases.

a) If x acts transitively on Π, there are two unipotent classes in xG^o, the regular one (with $|\mathcal{B}_v| = 1$ if v is regular) and the quasisemisimple one (with $\mathcal{B}_v \cong \mathbb{P}^1$ if v is quasisemisimple). This follows from I.1.6, 1.4 and 1.5. In this case u and u_o are certainly conjugate. Notice that u and u_o are even B-conjugate since Z(u) acts transitively on $\mathcal{B}_u = \{B\}$. It is also easy to check that $\mathrm{rank}_x(G) = 1$ and $\dim Z_U(u) = 1$.

b) In the general case we can assume that $u, u_o \in xU$. For each $\alpha \in \phi$, let $U_\alpha = \langle x^i X_\alpha x^{-i} \mid i \in \mathbb{Z}\rangle$. Let $R = \{\lambda \in \phi^+ \mid X_\lambda \not\subset U_\alpha$ for all $\alpha \in \Pi\}$ and let $U' = \prod_{\lambda \in R} X_\lambda$. Let $\alpha_1, \ldots, \alpha_n$ be a system of representatives for the x-orbits in Π. Notice that n = $\mathrm{rank}_x(G)$. u can be written uniquely as $u = xu_1 \ldots u_n v$ with $u_i \in U_{\alpha_i}$ $(1 \leq i \leq n)$ and $v \in U'$. Since there is no line of type α_i through B contained in $\mathcal{B}_u$, xu_i is regular in $\langle x, U_{-\alpha_i}, U_{\alpha_i}\rangle$ (in particular by case (a)). $xx_{\alpha_i}(1)$ is also regular in this group and by case (a) xu_i is conjugate to $xx_{\alpha_i}(1)$ under $B \cap \langle U_{-\alpha_i}, U_{\alpha_i}\rangle$. It is then clear that u is B-conjugate to an element of the form $xx_{\alpha_1}(1)\ldots x_{\alpha_n}(1)v'$ with $v' \in U'$. We may as well assume that $u = xx_{\alpha_1}(1)\ldots x_{\alpha_n}(1)v'$. Similarly we may assume that $u_o = xx_{\alpha_1}(1)\ldots x_{\alpha_n}(1)v_o'$ with $v_o' \in U'$. Let $Z_i = Z_{U_{\alpha_i}}(x_{\alpha_i}(1))$. As noted in case (a) $\dim Z_i = 1$. Hence $U'' = Z_1 \ldots Z_n U'$ is a subgroup of U of dimension $\dim U' + n$. It is easily checked that the U''-orbit of u_o is contained in $xx_{\alpha_1}(1)\ldots x_{\alpha_n}(1)U'$. Since u_o is regular and $\mathrm{rank}_x(G) = n$ this orbit has dimension $\geqslant$ dim U' and it is closed since U'' is unipotent. It is therefore exactly $xx_{\alpha_1}(1)\dots x_{\alpha_n}(1)U' \ni u$. This proves that u and u_o are conjugate.

Corollary 3.8. A unipotent element $u \in G$ is regular if and only if $|\mathcal{B}_u| = 1$. Every unipotent element is contained in the closure of a unipotent regular class. Every unipotent component of G contains exactly one unipotent regular G^o-class.

Proof. This follows from 1.7, 3.7 and the proof of 3.7.

Corollary 3.9. Let $g = su$ be the Jordan decomposition of an element $g \in G$. g is regular if and only if u is regular in Z(s). Every conjugacy class in G is contained in the closure of a regular class of G.

Proof. This follows from 3.8 and the proof of 3.7.

Definition 3.10. An element $g \in G$ is subregular if dim $Z(g) = \mathrm{rank}_g(G) + 2$.

Remark 3.11. 3.7 and 2.7 show that if g is not regular, then dim $Z(g) \geqslant \mathrm{rank}_g(G) + 2$.

Lemma 3.12. Suppose that Π consists of exactly two x-orbits $o(\alpha)$ and $o(\beta)$. Suppose also the $u \in xG^o$ is unipotent and that $L \subset \mathcal{B}_u$ is a line of type α. Then the following hold :

a) If $\tilde{s}_\alpha$ and $\tilde{s}_\beta$ do not commute, then L meets a line of type β

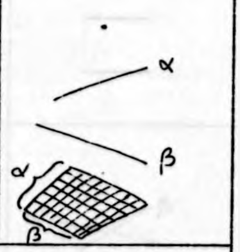
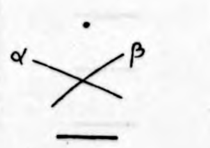
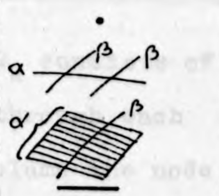
contained in $\mathfrak{B}_u$.

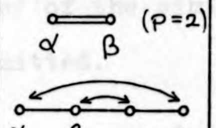
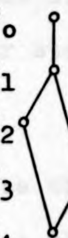
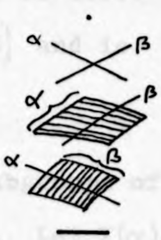
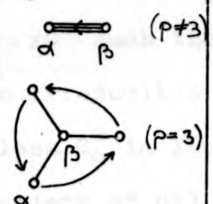
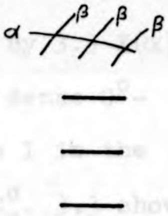
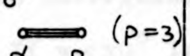
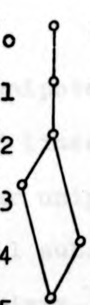
b) If $\mathfrak{s}_\alpha$ ans $\mathfrak{s}_\beta$ commute, then one of the following holds :

b_1) there is a line of type β contained in $\mathfrak{B}_u$ through every point of L.

b_2) L does not meet any line of type β contained in $\mathfrak{B}_u$.

c) Moreover, in the following cases the unipotent classes contained in xG^o and the corresponding Dynkin varieties are as follows (the action of x on $\triangle(G^o)$ is indicated by arrows if it is not trivial).

$\triangle(G^o)$	unipotent classes		dim $\mathfrak{B}_u$	$\lvert S(u)\rvert$	picture of $\mathfrak{B}_u$
c_1) α β	C_o C_1 C_2 C_3	regular subregular subregular quasisemi-simple	0 1 1 2	1 1 1 1	
c_2) α β	C_o C_1 C_2	regular subregular quasisemi-simple	0 1 3	1 2 1	
c_3) α β $(p\neq 2)$ α β $(p=2)$	C_o C_1 C_2 C_3	regular subregular quasisemi-simple	0 1 2 4	1 3 1 1	

$\Delta(G^o)$	unipotent classes		dim $\mathcal{B}_u$	$\lvert S(u)\rvert$	picture of $\mathcal{B}_u$
c_4)  $(\rho=2)$ $(\rho=2)$	C_o 	regular	0	1	
	C_1	subregular	1	2	
	C_2		2	1	
	C_3		2	1	
	C_4	quasisemi-simple	4	1	
c_5)  $(\rho\neq3)$ $(\rho=3)$	C_o 	regular	0	1	
	C_1	subregular	1	4	
	C_2		2	2	
	C_3		3	1	
	C_4	quasisemi-simple	6	1	
c_6)  $(\rho=3)$	C_o 	regular	0	1	
	C_1	subregular	1	2	
	C_2		2	2	
	C_3		3	1	
	C_4		3	1	
	C_5	quasisemi-simple	6	1	

A picture like  **means that $\mathcal{B}_u$ consists of one line of type α and of one line of type β through each point of this line of type α. In the second column the node representing C_i is joined to the node representing C_j $(i > j)$**

if $\overline{C}_i \subset \overline{C}_j$ and $\overline{C}_i \subset \overline{C}_k \subset \overline{C}_j \Rightarrow k = i$ or $k = j$.

<u>Proof</u>. The proof is similar to the demonstration of I.2.5 and of the similar statements in [20, p. 140-145] and is omitted.

<u>3.13</u>. Let $\mathcal{P}^o_{o(\alpha)}$ be the G^o-class of parabolic subgroups of G^o corresponding to the x-orbit $o(\alpha) \subset \Pi$ $(\alpha \in \Pi)$. Let $X(\alpha) = \{u \in xG^o \mid \text{for some } P \in \mathcal{P}^o_{o(\alpha)},\ {}^uP = P \text{ and } uU_P \text{ is quasisemisimple unipotent in } N_G(P)/U_P\}$. $X(\alpha)$ is the variety of all unipotent $u \in xG^o$ such that $\mathcal{B}_u$ contains a line of type α. By 3.3 $X(\alpha)$ is irreducible and closed in xG^o and there is a dense G^o-class C^o_α in $X(\alpha)$. By 1.3 $X(\alpha) \cap N$ has codimension 1 in the variety of all unipotent elements of xB. If $u \in C^o_\alpha$, 3.3 shows that $\dim \mathcal{B}_u = 1$ and $\dim Z(u) = \mathrm{rank}_u(G) + 2$ (by 2.7). Hence u is subregular.

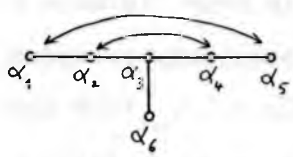
If $v \in xG^o$ is unipotent but not regular, 1.7 shows that $\mathcal{B}_v$ is a union of lines of various types. Hence the variety of all non-regular unipotent elements in xG^o is $\bigcup_{\alpha\in\Pi} X(\alpha)$ and the variety of all subregular unipotent elements of xG^o is dense in this variety.

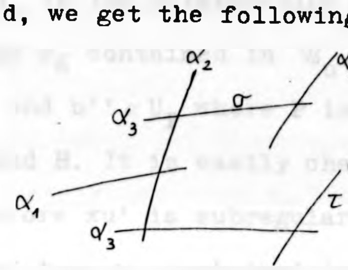
3.12 shows that if $\tilde{s}_\alpha$ and $\tilde{s}_\beta$ do not commute, then $X(\alpha) = X(\beta)$ and in particular $C^o_\alpha = C^o_\beta$, and if $o(\alpha)$ does not meet the component of $\triangle(G^o)$ containing β, then $X(\alpha) \neq X(\beta)$. We get therefore :

<u>Proposition 3.14</u>. There is one subregular unipotent G^o-class

in xG^o for each x-orbit in the set of connected components of $\triangle(G^o)$. The subregular unipotent elements are dense among the non-regular unipotent elements.

<u>3.15</u>. If dim $\mathcal{B}_u = 1$ (u unipotent), 3.12 used repeatedly gives a description of $\mathcal{B}_u$. For example, if G^o is of type E_6 and u acts as indicated, we get the following pattern of lines :





In this case 1.3 gives an easy method to compute φ: $S(u)\times S(u) \longrightarrow W^u$. We just need to take the shortest path from one line to the other. For example, if σ, τ are as shown in the picture, $\varphi(\sigma,\tau) = s_3\tilde{s}_2 s_3 s_6$ ($s_i = s_{\alpha_i}$).

<u>Proposition 3.16</u>. An element $g \in G$ is subregular if and only if dim $\mathcal{B}_g = 1$.

<u>Proof</u>. By 3.7 and 2.7 dim $\mathcal{B}_g = 1$ if g is subregular. As in the proof of 3.7 we need only to prove the converse when g is unipotent. If G is connected, this is proved in [20]. If G^o is of type A_n or D_n and g acts by an automorphism of order 2 on $\triangle(G^o)$, then dim $\mathcal{B}_g$ can be computed (5.19) and it is clear that dim $\mathcal{B}_g = 1$ defines a unique unipotent class. If G^o is of type D_4 and g acts by an automorphism of order 3 on $\triangle(G^o)$, the proof of I.2.4 shows that g is subregular if

$\dim \mathcal{B}_g = 1$. It follows then easily that it is sufficient to consider the following situation. $p = 2$, G^o is of type E_6, $|G/G^o| = 2$, $u \in G \setminus G^o$ is unipotent and acts on $\Delta(G^o)$ by an automorphism of order 2, $\dim \mathcal{B}_u = 1$ and we have to prove that u is subregular. We use the same notations as in 3.15.

Let $H = \langle x, T, X_{\pm\alpha_i} | 1 \leq i \leq 5 \rangle$. H^o is of type A_5. We may assume that $u \in xU$ and that $B \in \mathcal{B}_u$ is the intersection of a line of type α_3 and a line of type α_6 contained in $\mathcal{B}_u$. We may write $u = xu'u''$ with $u' \in U \cap H$ and $u'' \in U_P$ where P is the parabolic subgroup generated by B and H. It is easily checked that $\dim \mathcal{B}(H)_{xu'} = 1$ and therefore xu' is subregular in H. $B \cap H \in \mathcal{B}(H)_{xu'}$ is on a line of type α_3 contained in $\mathcal{B}(H)_{xu'}$ but not on a line of type α_2.

Let now $u_o \in xU$ be a subregular element of G such that $B \in \mathcal{B}_{u_o}$ is the intersection of a line of type α_3 and a line of type α_6 contained in $\mathcal{B}_{u_o}$. We have $u_o = xu_o'u_o''$ with $u_o' \in U \cap H$ and $u_o'' \in U_P$. xu_o' is subregular in H and $B \cap H \in \mathcal{B}(H)_{xu_o'}$ is on a line of type α_3 contained in $\mathcal{B}(H)_{xu_o'}$ but not on a line of type α_2. Direct computations show that xu' and xu_o' are $(B \cap H)$-conjugate. So we may assume that $u' = u_o'$.

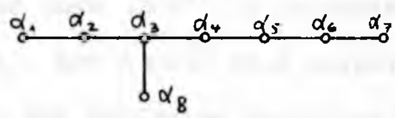
u'' and u_o'' can be expressed as $u'' = \prod_{\lambda\in\phi^+} x_\lambda(c_\lambda)$ and $u_o'' = \prod_{\lambda\in\phi^+} x_\lambda(d_\lambda)$. It is clear that $c_{\alpha_6} = d_{\alpha_6} = 0$, $c_{\alpha_3+\alpha_6} \neq 0$, $d_{\alpha_3+\alpha_6} \neq 0$. Conjugating by suitable elements of T we can arrange to have $c_{\alpha_3+\alpha_6} = d_{\alpha_3+\alpha_6} = 1$ (with $u' = u_o'$).

Let $R = \{\lambda \in \phi^+ | X_\lambda \subset U_P \text{ and } \lambda \neq \alpha_6, \lambda \neq \alpha_3+\alpha_6\}$ and let $U' =$

$\prod_{\lambda \in R} x_\lambda$. dim U' = 19.

Direct computations show that dim $Z_{H \cap U}(xu') = 4$. Let $U'' = Z_{H \cap U}(xu')U_P$. This is a subgroup of U and dim U'' = 25. It is easily checked that the U''-orbit of u_o is contained in $xu'x_{\alpha_3+\alpha_6}(1)U'$ and has dimension $\geqslant 19$ since dim $Z_G(u_o) = 6$. Since U'' is unipotent, this orbit is closed and hence is exactly $xu'x_{\alpha_3+\alpha_6}(1)U' \ni xu'u'' = u$. Therefore u and u_o are conjugate and u is subregular.

<u>Remark 3.17</u>. The same method works for other groups. For example if G is connected adjoint of type E_8 (with the fundamental roots labelled as in the picture) we take H =



$\langle T, X_{\pm\alpha_i} \mid 1 \leqslant i \leqslant 7 \rangle$ and P is the parabolic subgroup generated by B and H. If u is a unipotent element such that dim $\mathcal{B}_u = 1$, we may assume that B is the intersection of the line of type α_3 and the line of type α_8 in $\mathcal{B}_u$. $u = u'u''$ with $u' \in U \cap H$ and $u'' = \prod_{\lambda \in \phi^+} x_\lambda(c_\lambda) \in U_P$. We have $c_{\alpha_8} = 0$ and $c_{\alpha_3+\alpha_8} \neq 0$. Conjugating by a suitable element of T we can arrange $c_{\alpha_3+\alpha_8} = 1$. u' can be seen to be unipotent as in 3.16. We choose a unipotent subregular element u_o in the same way. $u_o = u_o'u_o''$ with $u_o' \in U \cap H$ and $u_o'' = \prod_{\lambda \in \phi^+} x_\lambda(d_\lambda) \in U_P$ and we can arrange to have $u_o' = u'$, $d_{\alpha_8} = 0$ and $d_{\alpha_3+\alpha_8} = 1$. Let now R = $\{\lambda \in \phi^+ \mid X_\lambda \subset U_P \text{ and } \lambda \neq \alpha_8, \lambda \neq \alpha_3+\alpha_8\}$ and let $U' = \prod_{\lambda \in R} X_\lambda$. dim U' = 90. Direct computations show that dim $Z_{H \cap U}(u')$ =

8. $U'' = Z_{U \cap H}(u')U_P$ is a unipotent group of dimension 100. Since $\dim Z_G(u_o) = 10$, we find that the U''-orbit of u_o is $u'x_{\alpha_3+\alpha_8}(1)U' \ni u$. This proves that u is subregular.

3.18. If $g \in G$ is neither regular nor subregular, then $\dim \mathcal{B}_g \geq 2$ and $\dim Z(g) \geq \mathrm{rank}_g(G) + 4$. As in [9] we shall say that g is sub-subregular if $\dim Z(g) = \mathrm{rank}_g(G) + 4$. If $g = su$ is the Jordan decomposition of g, g is sub-subregular if and only if u is sub-subregular in Z(s). If g is unipotent and $Q(C^o(g)) \neq \emptyset$, g is sub-subregular if and only if $\dim \mathcal{B}_g = 2$.

3.19. Assume now that $\Delta(G^o)$ is connected and that G^o is not of type A_1 or A_2. Let $u \in xG^o$ be a unipotent element. If $\dim \mathcal{B}_u \geq 2$, then the following condition holds for some $\alpha, \beta \in \Pi$ (α, β in distinct x-orbits).

(*) There is a line of type β in $\mathcal{B}_u$ such that through each of its points there is a line of type α contained in $\mathcal{B}_u$.

If (*) holds and $(\tilde{s}_\alpha \tilde{s}_\beta)^2 = 1$, then u is in the closure of the $\mathcal{P}^o_{o(\alpha) \cup o(\beta)}$-regular class (by 3.3) and (*) holds with α and β permuted. The $\mathcal{P}^o_{o(\alpha) \cup o(\beta)}$-regular class is sub-subregular. If $\gamma \in \Pi$ is such that $\tilde{s}_\alpha \tilde{s}_\gamma$ has order ≥ 3 and $\tilde{s}_\beta \tilde{s}_\gamma$ has order 2, then (*) holds with (α, β) replaced by (β, γ) (by 3.12). In particular the $\mathcal{P}^o_{o(\alpha) \cup o(\beta)}$-regular class and the $\mathcal{P}^o_{o(\beta) \cup o(\gamma)}$-regular class coincide.

If (*) holds and $\tilde{s}_\alpha \tilde{s}_\beta$ has order 3, then 3.3 and 3.12 show that (*) holds with α and β permuted, that $\dim \mathcal{B}_u \geq 3$ and that u is in the closure of the $\mathcal{P}^{o}_{o(\alpha)\cup o(\beta)}$-regular class. If $\gamma \in \Pi$ is such that $\tilde{s}_\alpha \tilde{s}_\gamma$ has order $\geqslant 3$ and $\tilde{s}_\beta \tilde{s}_\gamma$ has order 2, then 3.12 implies that u is also in the closure of the $\mathcal{P}^{o}_{o(\beta)\cup o(\gamma)}$-regular class.

If (*) holds and $\tilde{s}_\alpha \tilde{s}_\beta$ has order > 4 and $\Pi = o(\alpha) \cup o(\beta)$, then 3.12 shows that there is one unipotent sub-subregular G^o-class in xG^o if we are in case (c_3), (c_5) or (c_6) of 3.12 and there are two unipotent sub-subregular classes in case (c_4). The sub-subregular classes contain in their closure all unipotent elements of xG^o which are neither regular nor subregular.

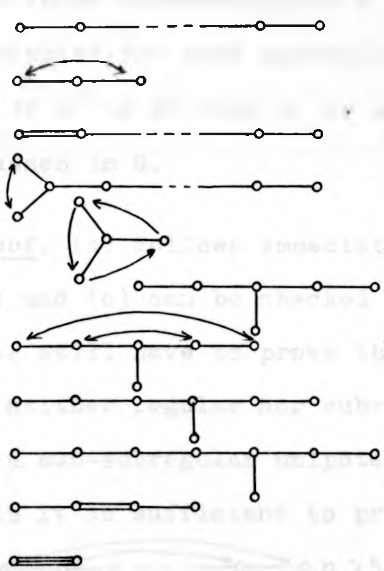
If (*) holds and $\tilde{s}_\alpha \tilde{s}_\beta$ has order 4 and $\gamma \in \Pi$ is such that $\tilde{s}_\alpha \tilde{s}_\gamma$ has order 2 and $\tilde{s}_\beta \tilde{s}_\gamma$ has order 3, then 3.12 shows that (*) holds for u with (α,β) replaced by (α,γ). In particular u is contained in the closure of the $\mathcal{P}^{o}_{o(\alpha)\cup o(\gamma)}$-regular class.

These remarks show that in most cases a unipotent element which is neither regular nor subregular is contained in the closure of a unipotent sub-subregular G^o-class which is $\mathcal{P}^o$-regular for some G^o-class of parabolic subgroups of G^o and that in many cases the $\mathcal{P}^o_1$-regular class and the $\mathcal{P}^o_2$-regular class in xG^o coincide if they are sub-subregular ($\mathcal{P}^o_1$, $\mathcal{P}^o_2$ G^o-classes of parabolic subgroups of G^o).

<u>Proposition 3.20</u>. If $\Delta(G^o)$ has no components of type A_2 the sub-subregular unipotent elements of G are dense in the

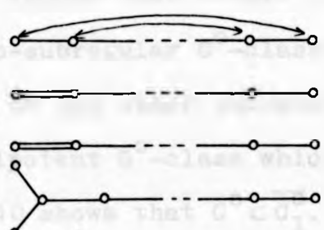
variety of all unipotent elements which are neither regular nor subregular. Moreover if $\triangle(G^{o})$ is connected, then the sub-subregular classes are as follows (the action of x is indicated by arrows if it is not trivial).

a) In the following cases there is only one sub-subregular G^{o}-class in xG^{o} :



- A_n — $n \geqslant 3$
- A_3 — $p = 2$
- B_n — $n \geqslant 2$, $p \neq 2$
- D_n — $n \geqslant 4$, $p = 2$
- D_4 — $p = 3$
- E_6
- E_6 — $p = 2$
- E_7
- E_8
- F_4
- G_2

b) In the following cases there are exactly two sub-subregular G^{o}-classes in xG^{o} :



- A_n — $n \geqslant 4$, $p = 2$
- B_n — $n \geqslant 2$, $p = 2$
- C_n — $n \geqslant 3$
- D_n — $n \geqslant 5$

If G^{o} is of type D_n $(n \geqslant 5)$ and x acts trivially on $\triangle(G^{o})$ both sub-subregular classes are P-regular for some parabolic subgroup of G^o. If $p = 2$ and G^o is of type B_2 or A_4 with a non-trivial action action of x, then none of the sub-subregular classes of xG^o is P-regular for some P. In the other cases one of the sub-subregular classes is P-regular for some P and the other is not.

c) If G^o is of type D_4 and x acts trivially on $\Delta(G^o)$, there are three sub-subregular G^o-classes in xG^o anf they are all P-regular for some parabolic subgroup P of G^o.

If G^o is of type A_1 or A_2 there are no sub-subregular classes in G.

<u>Proof</u>. (a) follows immediately from the discussion in 3.19. (b) and (c) can be checked from 3.19, 5.19, 6.2.

We still have to prove that every unipotent element which is neither regular nor subregular is contained in the closure of a sub-subregular unipotent class. From the discussion in 3.19 it is sufficient to prove this in the following cases :

A_n  $n \geq 5$, $p = 2$

C_n  $n \geq 3$.

Assume that we are in one of these cases. Let C_1^o be the sub-subregular G^o-class which is P-regular for some P and let C_2^o be the other sub-subregular class. Then if $C^o \neq C_2^o$ is a unipotent G^o-class which is neither regular nor subregular 5.40 shows that $C^o \subset \overline{C_1^o}$. This proves the proposition.

4. Parabolic subgroups fixed by u.

In this paragraph G is reductive and $x \in G$ is a unipotent quasisemisimple element normalizing B and T.

4.1. Let $P \supset B$ be a parabolic subgroup of G normalized by x and let I be the corresponding subset of Π. For every $g \in xG^o$ $\mathcal{P}^o_g = \{P' \in \mathcal{P}^o \mid {}^gP' = P'\}$ is non-empty and $p : \mathcal{B}_g \longrightarrow \mathcal{P}^o_g$, $B' \longmapsto$ (unique $P' \in \mathcal{P}^o$ such that $B' \subset P'$) is a surjective morphism. $Z_o(g)$ acts naturally on $\mathcal{P}^o_g$ and p is $Z_o(g)$-equivariant. $Z(g)$ acts also on $\mathcal{P}^o_g$ if I is $Z(g)$-stable and G^oP/G^o is normal in G/G^o. In this case p is $Z(g)$-equivariant. Notice that for each $P' \in \mathcal{P}^o_g$, $p^{-1}(P') = \mathcal{B}(P')_g \cong \mathcal{B}(P'/U_{P'})_{gU_{P'}}$.

By 2.7 $\dim \mathcal{P}^o_g \leqslant \dim \mathcal{B}_g \leqslant \frac{1}{2}(\dim Z(g) - \operatorname{rank}_g(G))$. We shall denote by $S_{\mathcal{P}^o}(g)$ the set of all irreducible components of $\mathcal{P}^o_g$ which have dimension $\frac{1}{2}(\dim Z(g) - \operatorname{rank}_g(G))$. This set is empty if g is unipotent and $Q(C^o(g)) = \emptyset$. The action of $Z_o(g)$ on $\mathcal{P}^o_g$ induces an action of $A_o(g)$ on $S_{\mathcal{P}^o}(g)$.

4.2. There is a generalization of the application φ defined in 0.8. We shall use it only when G is connected. So assume that G is connected and consider two parabolic subgroups $P \supset B$, $Q \supset B$. The set of G-orbits in $\mathcal{P} \times \mathcal{Q}$ corresponds bijectively to $W_P \backslash W / W_Q$, where W_P and W_Q are the Weyl groups of P and Q respectively. Let u be a unipotent element. We

get an application $\varphi_{\mathcal{P},\mathcal{Q}} : S_{\mathcal{P}}(u)\times S_{\mathcal{Q}}(u) \longrightarrow W_P\backslash W/W_Q$.
This application has the following properties.

a) $\varphi_{\mathcal{P},\mathcal{Q}}(\sigma,\tau) = \varphi_{\mathcal{P},\mathcal{Q}}(\sigma',\tau')$ if and only if $(\sigma',\tau') = (a\sigma,a\tau)$ for some $a \in A(u)$ $(\sigma,\sigma' \in S_{\mathcal{P}}(u),\ \tau,\tau' \in S_{\mathcal{Q}}(u))$.

b) If $u_1,\dots,u_n$ is a complete set of representatives for the unipotent classes of G, then the sets $\varphi_{\mathcal{P},\mathcal{Q}}(S_{\mathcal{P}}(u_i)\times S_{\mathcal{Q}}(u_i))$ form a partition of $W_P\backslash W/W_Q$. In particular we have :

$$\sum_{1\le i\le n} \left|(S_{\mathcal{P}}(u_i)\times S_{\mathcal{Q}}(u_i))/A(u_i)\right| = \left|W_P\backslash W/W_Q\right|.$$

These results are proved in [21]. The proofs are similar to the proofs of the corresponding statement for φ.

4.3. Let $u \in xG^0$ be a unipotent element. By 1.3 the union of all lines of type α contained in $\mathcal{B}_u$ is closed $(\alpha \in \Pi)$. For each component X_σ of $\mathcal{B}_u$ exactly one of the following holds.

a) X_σ is a union of lines of type α.

b) There is a dense open subset of X_σ which is not met by any line of type α contained in $\mathcal{B}_u$.

Let $I_\sigma = \{\alpha \in \Pi \mid X_\sigma \text{ is a union of lines of type } \alpha\}$. I_σ is x-stable. $I_\sigma \neq \emptyset$ unless u is regular. It is easy to check that with the notations of 4.1 u is $\mathcal{P}^0$-regular if and only if the following conditions hold :

a) $Q(C^0(u)) \neq \emptyset$.

b) $\dim \mathcal{B}_u = \ell_u(w_P)$, where w_P is the element of maximal length in W_P.

c) For some $\sigma \in S(u)$, $I_\sigma = I$.

Proposition 4.4. Let P, I be as in 4.1 and let $u \in xG^o$ be a unipotent element. Let $S_I(u) = \{\sigma \in S(u) \mid I_\sigma \cap I = \emptyset\}$. Then $S_I(u) = \{\sigma \in S(u) \mid \dim p(X_\sigma) = \dim \mathcal{B}_u\}$, where p is the natural morphism $\mathcal{B}_u \longrightarrow \mathcal{P}^o_u$. If $Q(C^o(u)) \neq \emptyset$, p induces an $A_o(u)$-equivariant bijection $S_I(u) \longrightarrow S_{\mathcal{P}^o}(u)$.

Proof. If $\alpha \in I_\sigma \cap I$, then each fibre of the restriction of p to $X_\sigma \longrightarrow \mathcal{P}^o_u$ contains a line of type α. Hence $\dim p(X_\sigma) < \dim \mathcal{B}_u$.

Suppose now that $I_\sigma \cap I = \emptyset$. We can choose $B' \in \overset{\circ}{X}_\sigma$ in such a way that for every $\alpha \in I$ there is no line of type α contained in $\mathcal{B}_u$ through B'. $uU_{P'}$ is therefore regular in $N_G(P')/U_{P'}$, where $P' = p(B')$, and $p^{-1}(P') = \{B'\}$. This shows that $\dim p(X_\sigma) = \dim X_\sigma = \dim \mathcal{B}_u$ and $p(X_\sigma) \not\subset (\bigcup_{\tau \neq \sigma} X_\tau)$. This proves the proposition.

4.5. If $Q(C^o(u)) \neq \emptyset$ we shall identify $S_{\mathcal{P}^o}(u)$ and $S_I(u)$.

4.6. We assume now that xG^o is central in G/G^o and that P is a parabolic subgroup of G^o with the following properties :

a) P^o is normalized by x.

b) $P = N_G(P)$.

c) $\mathcal{P} = \mathcal{P}^o$.

(b) and (c) imply that P meets all components of G. Let $u \in xG^o$ be a unipotent element. Then $\mathcal{P}^o_u = \mathcal{P}_u = \{P' \in \mathcal{P} \mid$

$u \in P'\}$. $Z(u)$ acts on $\mathcal{P}_u$ and $p : \mathcal{B}_u \longrightarrow \mathcal{P}_u$ is $Z(u)$-equivariant.

If $P' = {}^gP \in \mathcal{P}_u$, the class of $g^{-1}ugU_P$ in P/U_P (or the class of $g^{-1}ugR_P$ in P/R_P) depends only on P' and not on the choice of g. This gives an application $f : \mathcal{P}_u \longrightarrow \{$unipotent classes of P/U_P contained in $xP^o/U_P\}$. Let $f(\mathcal{P}_u) = \{C_1', \ldots, C_j'\}$. Let $Y_i = f^{-1}(C_i') \subset \mathcal{P}_u$, $X_i = p^{-1}(Y_i) \subset \mathcal{B}_u$, $Y_i^* = f^{-1}(\{C_h' | C_h' \subset \overline{C_i'}\}) \subset \mathcal{P}_u$, $X_i^* = p^{-1}(Y_i^*)$ $(1 \leq i \leq j)$.

<u>Lemma 4.7</u>. For every $C_i' \in f(\mathcal{P}_u)$, X_i^* is closed in $\mathcal{B}_u$, Y_i^* is closed in $\mathcal{P}_u$, X_i is locally closed in $\mathcal{B}_u$ and Y_i is locally closed in $\mathcal{P}_u$.

<u>Proof</u>. For every $g \in G^o$, $\varphi_g : U^- \longrightarrow \mathcal{B}$, $v \longmapsto {}^{gv}B$ gives an isomorphism between U^- and an open neighbourhood of ${}^gB$ in $\mathcal{B}$. Let Ψ be the morphism $\varphi_g(U^-) \longrightarrow G/U_P$, ${}^{gv}B \longmapsto v^{-1}g^{-1}ugvU_P$. Then $X_i^* \cap \varphi_g(U^-) = \Psi^{-1}(\overline{C_i'})$. This shows that $X_i^* \cap \varphi_g(U^-)$ is closed in $\varphi_g(U^-)$. Therefore X_i^* is closed in $\mathcal{B}$. The lemma follows easily from this property.

<u>4.8</u>. Consider a fixed component X_σ of $\mathcal{B}_u$ (in the situation of 4.6). Since $X_1, \ldots, X_j$ are locally closed in $\mathcal{B}_u$ there is a unique i such that $X_\sigma \cap X_i$ is dense in X_σ. If $P' \in p(X_\sigma \cap X_i)$, $\dim X_\sigma \leq \dim p(X_\sigma \cap X_i) + \dim (X_\sigma \cap \mathcal{B}(P')) \leq \dim Y_i + \dim \mathcal{B}(P')_u \leq \dim \mathcal{B}_u$. Since $\dim X_\sigma = \dim \mathcal{B}_u$, we have $\dim Y_i = \dim p(X_\sigma \cap X_i)$, $\dim \mathcal{B}(P')_u = \dim (X_\sigma \cap \mathcal{B}(P'))$ and $X_\sigma \cap \mathcal{B}(P')$ is a union of irreducible components of $\mathcal{B}(P')_u$

and since $p(X_\sigma \cap X_i) = p(X_\sigma) \cap Y_i$ is closed in Y_i, $p(X_\sigma \cap X_i)$ is an irreducible component of Y_i (of maximal dimension).

If $P' \in \mathcal{P}_u$ and $X'_{\sigma'}$ is an irreducible component of $\mathcal{B}(P')_u$ we can define $I_{\sigma'}$ as in 4.3. In this case $I_{\sigma'} = \{\alpha \in I \mid \mathcal{B}(P')_u$ is a union of lines of type $\alpha\}$. If $\alpha \in I$, there is a line of type α through $B' \in X_\sigma$ contained in X_σ if and only if there is a line of type α through B' contained in $X_\sigma \cap \mathcal{B}(P')_u$ ($P' = p(B')$). It follows easily that there is a dense open subset in $p(X_\sigma \cap X_i)$ such that for every P' in this subset and every irreducible component $X'_{\sigma'}$ of $X_\sigma \cap \mathcal{B}(P')$, $I_{\sigma'} = I \cap I_\sigma$.

Let $Q \supset B$ be an x-stable parabolic subgroup of G^0. Let J be the corresponding subset of Π. Suppose that $X_\sigma = \mathcal{B}(Q)_u$ and that uU_Q is quasisemisimple in $N_G(Q)/U_Q$. If $P' \in p(X_\sigma \cap X_i)$, then $X_\sigma \cap \mathcal{B}(P') = \mathcal{B}(P' \cap Q)_u$ and $uU_{P' \cap Q}$ is quasisemisimple in $N_G(P' \cap Q)/U_{P' \cap Q}$ (this can be checked by considering lines in $\mathcal{B}(P' \cap Q)$). If $Q(C^0(uU_{P'}))$ (defined as in 2.4 and computed in $P'/U_{P'}$) is not empty, this shows that $uU_{P'} \in P'/U_{P'}$ is $(P' \cap Q)/U_{P'}$-regular (by 4.3).

<u>4.9</u>. Suppose that we are in the situation of 4.6 and that $Z(u)$ acts transitively on Y_i for some fixed i ($1 \leq i \leq j$). Then $Z(u)^0$ acts transitively on each component of Y_i. Let Y'_i, Y''_i, ... be these components. For simplicity assume that $P \in Y'_i$. Let G' be one of the groups P/U_P, P/R_P and let u' be the image of u in G'. Let S_i be the set of irreducible

components of X_i and let S_i' be the set of irreducible components of $\mathcal{B}(P)_u \cong \mathcal{B}(G')_{u'}$. We write also $(X'_{\sigma'})_{\sigma' \in S_i'}$ for the components of $\mathcal{B}(P)_u$. Let $Z'(u') = Z_{G'}(u')$, $A'(u') = Z'(u')/Z'(u')^o$. $A'(u')$ acts on S_i' and $A(u)$ acts on S_i. Let $H = Z(u) \cap P$ and $K = Z(u)^o \cap P$. The natural homomorphisms $H \longrightarrow Z(u)$, $H \longrightarrow Z'(u')$ and $K \longrightarrow Z'(u')$ induce homomorphisms $H \longrightarrow A(u)$, $H \longrightarrow A'(u')$ and $K \longrightarrow A'(u')$. Let A_P be the image of H in $A(u)$ and let A_P' be the image of K in $A'(u')$. The components of Y_i are in bijective correspondence with $A(u)/A_P$. If $\sigma', \tau' \in S_i'$, $Z(u)^o X'_{\sigma'}$ is an irreducible component of X_i and $Z(u)^o X'_{\sigma'} = Z(u)^o X'_{\tau'}$ if and only if σ' and τ' are in the same A_P'-orbit in S_i'. Let $\overline{\sigma'}$ be the A_P'-orbit of σ' and let $\overline{S}_i' = \{\overline{\sigma'} \mid \sigma' \in S_i'\}$. H acts on S_i' and also on $\overline{S}_i'$ and this action coincides with the action of H on the set of irreducible components of $p^{-1}(Y_i')$. K acts trivially on $\overline{S}_i'$ and therefore $A_P \cong H/K$ acts on $\overline{S}_i'$. Since all components of $\mathcal{B}(P)_u$ have the same dimension, we get easily :

<u>Proposition 4.10</u>. In the situation of 4.9 $S_i = \{aX'_{\sigma'} \mid a \in A(u), \sigma' \in S_i'\}$ (by definition $a \in A(u)$ is of the form $zZ(u)^o$ for some $z \in Z(u)$). Moreover $aX'_{\sigma'} = bX'_{\tau'}$ if and only if $b^{-1}a \in A_P$ and $b^{-1}a\overline{\sigma'} = \overline{\tau'}$. All components of X_i have the same dimension. If D is a subgroup of $A(u)$ such that $A(u) = D \times A_P$, then $D \times \overline{S}_i' \longrightarrow S_i$, $(d, \overline{\sigma'}) \longmapsto dX'_{\sigma'}$ is an equivariant bijection (for the action of $A(u) = D \times A_P$).

<u>4.11</u>. Suppose that in 4.6 $\dim X_i = \dim \mathcal{B}_u$ if $1 \leq i \leq h$ and

$\dim X_i < \dim \mathcal{B}_u$ if $h < i \leq j$ and that $Z(u)$ acts transitively on Y_i for $1 \leq i \leq h$. Since all components of $\mathcal{B}_u$ have the same dimension, 4.10 shows that the closure of any component of X_i is a component of $\mathcal{B}_u$ if $1 \leq i \leq h$ and that every component of $\mathcal{B}_u$ is of this form, and we have an $A(u)$-equivariant bijection $\bigcup_{1\leq i\leq h} S_i \longrightarrow S(u)$. We shall use this bijection and 4.10 to give a descrition of $S(u)$ and of the action of $A(u)$ on $S(u)$ for the classical groups.

4.12. Suppose that we are in the situation of 4.9 and that X is a $Z(u)$-stable subvariety of X_i. Then the same method shows that the irreducible components of X have all the same dimension (resp. are disjoint) if the irreducible components of $X \cap p^{-1}(P)$ have all the same dimension (resp. are disjoint) and the action of $A(u)$ on the set of irreducible components of X can be deduced in the same way from the action of $A'(u')$ on the set of irreducible components of $X \cap p^{-1}(P)$.

5. Dynkin varieties for classical groups.

Let V be a finite dimensional vector space over k. We shall consider the following situations.

I) $G = GL(V)$, $\dim V = n$.

II) $G = G(V)$ (I.3.1), $\dim V = 2n+1$ and we consider unipotent elements in $G \setminus G^o$ ($p = 2$).

III) $G = O_{2n+1}(k) = Z_{GL(V)}(f)$, where $f \in G(V)$ is a symmetric bilinear form, $\dim V = 2n+1$ ($p \neq 2$).

IV) $G = G(V)$, $\dim V = 2n$ and we consider unipotent elements in $G \setminus G^o$ ($p = 2$).

V) $G = O_{2n}(k) = Z_{GL(V)}(f)$, where $f \in G(V)$ is a symmetric bilinear form, $\dim V = 2n$ ($p \neq 2$).

VI) $G = Sp_{2n}(k) = Z_{GL(V)}(f)$, where $f \in G(V)$ is a symplectic form, $\dim V = 2n$ ($p \neq 2$).

VII) Same as in (VI), but with $p = 2$.

VIII) $G = O_{2n}(k) = \{g \in GL(V) | Q \circ g = Q\}$, where $Q : V \longrightarrow k$ is a quadratic form such that $f : V \times V \longrightarrow k$, $(x,y) \longmapsto Q(x) + Q(y) + Q(x+y)$ is non-degenerate. $\dim V = 2n$ and we consider unipotent elements contained in G^o ($p = 2$).

IX) Same as in (VIII) but we consider unipotent elements in $G \setminus G^o$.

5.1. We shall denote by $\mathcal{F}$ the following variety. In cases (I), (II) and (IV), $\mathcal{F}$ is the variety of all complete flags of V. In cases (II), (V), (VI) and (VII), $\mathcal{F}$ is the variety of all complete flags of V which are isotropic for f (I.3.1). In cases (VIII) and (IX), $\mathcal{F}$ consists of all isotropic flags $F = (F_o, \ldots,$

F_{2n}) of V such that Q vanishes on F_n. $\mathcal{F}$ is complete and there is a natural morphism $\mathcal{F} \longrightarrow \mathcal{B}$ which associates to each flag its stabilizer in G°. Its restriction to any component of $\mathcal{F}$ is an isomorphism. $\mathcal{F}$ is irreducible except in cases (V), (VIII) and (IX) where it has two components which are permuted by G/G^0. If we are in one of these cases and $F = (F_0, \ldots, F_{n-1}, F_n, F_{n+1}, \ldots, F_{2n}) \in \mathcal{F}$, then the other element of $\mathcal{F}$ corresponding to the same Borel subgroup is of the form $F' = (F_0, \ldots, F_{n-1}, F'_n, F_{n+1}, \ldots, F_{2n})$. We get a variety isomorphic to $\mathcal{B}$ by identifying flags in such a pair.

We shall use the following notations for the fundamental roots.

(I)

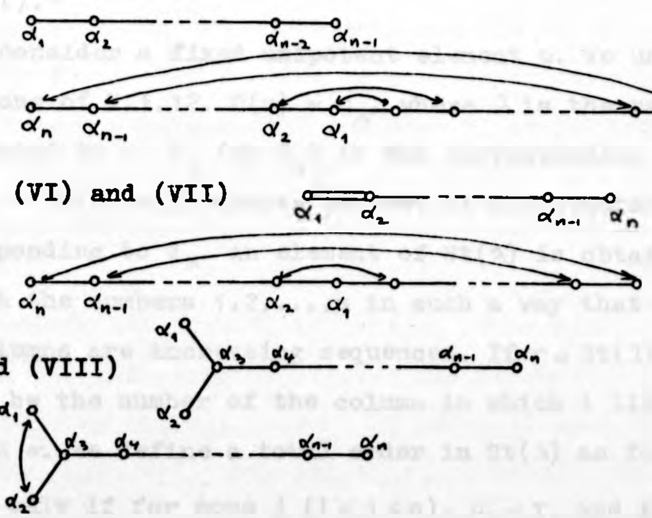
(II)

(III), (VI) and (VII)

(IV)

(V) and (VIII)

(IX)



In each case we have labelled only one root in each x-orbit (where xG^0 is the component we consider) except in case (IX).

The lines of type α ($\alpha \in \Pi$) have a geometric description in $\mathcal{F}$. In case (I), the line of type α_i trough $F = (F_0, \ldots, F_n)$ consists of all flags of the form $(F_0, \ldots, F_{i-1}, F'_i, F_{i+1}, \ldots, F_n)$

(notice that this removes the ambiguity in the labelling of the fundamental roots which existed in this case). There are similar descriptions for the other cases. In case (V) it is easy to see that the two components of $\mathcal{F}$ correspond to α_1 and α_2 in a natural way. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the components. Let $F = (\ldots,F_{n-2},F_{n-1},F_n,F_{n+1},F_{n+2},\ldots) \in \mathcal{F}$, then the set of all flags of the form $(\ldots,F_{n-2},F'_{n-1},F_n,(F'_{n-1})^{\perp},F_{n+2},\ldots)$ is a line of type α_1 if $F \in \mathcal{F}_1$ and a line of type α_2 if $F \in \mathcal{F}_2$ (if the components of $\mathcal{F}$ are labelled in a suitable way). The same remark applies in case (VIII).

5.2. Until the end of 5.17, $G = GL_n = GL(V)$, i.e. we are in case (I).

We consider a fixed unipotent element u. We use the notations of I.1.12. $C(u) = C_{\lambda}$, where λ is the partition associated to u. d_{λ} (or d_n) is the corresponding Young diagram. $St(d_{\lambda}) = St(\lambda)$ will denote the set of all standard tableaux corresponding to d_{λ}. An element of $St(\lambda)$ is obtained by filling d_{λ} with the numbers $1,2,\ldots,n$ in such a way that all lines and all columns are increasing sequences. If $\sigma \in St(\lambda)$ and $1 \leqslant i \leqslant n$, let σ_i be the number of the column in which i lies in the tableau σ. We define a total order in $St(\lambda)$ as follows. $\sigma < \tau$ if and only if for some j $(1 \leqslant j \leqslant n)$, $\sigma_j < \tau_j$ and for all i such that $j < i \leqslant n$, $\sigma_i = \tau_i$.

If $F = (F_0,F_1,\ldots,F_n) \in \mathcal{F}_u$, let $d_i(F)$ be the diagram corresponding to the class of the restriction of u to $F_i \longrightarrow F_i$. $d_n(F) = d_n$ and the sequence $d_0(F),d_1(F),\ldots,d_n(F)$ is such that

$d_{i-1}(F)$ and $d_i(F)$ differ by one square which is a corner of $d_i(F)$ $(1 \leq i \leq n)$. Putting i in this corner of $d_i(F)$ for $1 \leq i \leq n$ we get a standard tableau, and this defines an application $\pi : \mathcal{F}_u \longrightarrow St(\lambda)$.

Lemma 5.3. For each $\sigma \in St(\lambda)$, $\pi^{-1}(\sigma)$ is locally closed in $\mathcal{F}_u$ and $\bigcup_{\tau > \sigma} \pi^{-1}(\tau)$ is closed in $\mathcal{F}_u$.

Proof. This is a consequence of 4.6.

5.4. Notations. Up to the end of 5.17, we write $X_\sigma = \pi^{-1}(\sigma)$ for each $\sigma \in St(\lambda)$. The similar notation introduced in 0.3 is not used here.

5.5. $Z_{G(V)}(u)$ has two components and one of them consists of bilinear forms. Let $f \in G(V)$ be a bilinear form commuting with u. If M is a subspace of V and uM = M, it is easy to check that the Young diagram of the restriction of u to M is the same as the Young diagram of the automorphism induced by u on V/fM. Define $\pi_0 : \mathcal{F}_u \longrightarrow St(\lambda)$, $F \longmapsto \pi(fF)$. π_0 is independent of the choice of f. Let $Y_\sigma = \pi_0^{-1}(\sigma) = f(X_\sigma)$ $(\sigma \in St(\lambda))$.

With the notations of I.1.12, dim Ker(u-1) = ℓ_1. Choose a complete flag $(W_0, W_1, \ldots, W_{\ell_1})$ of Ker(u-1) such that $W_{\ell_i} = Ker(u-1) \cap Im(u-1)^{i-1}$ for all $i \geq 1$. It is easy to check that if $\sigma = \pi_0(F)$ $(F \in \mathcal{F}_u)$, then $\sigma_n = i$ if and only if $F_1 \subset \mathbb{P}(W_{\ell_i}) \setminus \mathbb{P}(W_{\ell_{i+1}})$.

Proposition 5.6. Each X_σ $(\sigma \in St(\lambda))$ is irreducible and dim $X_\sigma = \sum_{i \geq 1} \ell_i(\ell_i - 1)/2$. In particular dim $\mathcal{B}_u = \sum_{i \geq 1} \ell_i(\ell_i - 1)/2$ and the irreducible components of $\mathcal{F}_u$ are the subvarieties X_σ $(\sigma \in St(\lambda))$.

<u>Proof</u>. We may replace X_σ by Y_σ in the proof. We use induction on n. If $\ell_i \neq \ell_{i+1}$, $\mathbb{P}(W_{\ell_i}) \setminus \mathbb{P}(W_{\ell_{i+1}})$ is irreducible and has dimension $\ell_i - 1$. It is easy to check that $Z(u)$ acts transitively on $\mathbb{P}(W_{\ell_i}) \setminus \mathbb{P}(W_{\ell_{i+1}})$. The result follows then immediately from 4.9 (P being the stabilizer of any line $L \in \mathbb{P}(V)$).

<u>Corollary 5.7</u>. $\dim Z(u) = 2\dim \mathcal{B}_u + \mathrm{rank}(G)$.

<u>Proof</u>. $\dim \mathcal{B}_u$ and $\dim Z(u)$ depend only on λ. By 2.6 and 2.7, we know that the formula is true if $p = 0$. It is therefore true for all characteristics.

<u>5.8</u>. Let $B_i = \mathbb{P}(W_i) \setminus \mathbb{P}(W_{i-1})$ and let $Y_i = p^{-1}(B_i)$, where p is the projection $\mathcal{F} \longrightarrow \mathbb{P}(V)$, $F \longmapsto F_1$. Each Y_i is locally closed and $\bigcup_{j \geq i} Y_j$ is closed in $\mathcal{F}$. If $L \in B_i$, then $p^{-1}(L) \cong \mathcal{F}(V/L)$, the flag variety of V/L, and $p^{-1}(L) \cap \mathcal{F}_u \cong \mathcal{F}(V/L)_{u'}$, where u' is the automorphism of V/L induced by u. If λ' is the partition corresponding to u', we have a natural inclusion $St(\lambda') \subset St(\lambda)$ and we get therefore a natural application π'_0 : $\mathcal{F}(V/L)_{u'} \longrightarrow St(\lambda)$.

<u>Lemma 5.9</u>. There is an isomorphism $\theta : Y_i \longrightarrow B_i \times \mathcal{F}(V/L)$ which gives by restriction an isomorphism $\theta_0 : Y_i \cap \mathcal{F}_u \longrightarrow B_i \times \mathcal{F}(V/L)_{u'}$ and such that the following diagrams commute.

$$\begin{array}{ccc} Y_i & \xrightarrow{\theta} & B_i \times \mathcal{F}(V/L) \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & B_i & \end{array} \qquad \begin{array}{ccc} Y_i \cap \mathcal{F}_u & \xrightarrow{\theta_0} & B_i \times \mathcal{F}(V/L)_{u'} \\ & {\scriptstyle \pi_0} \searrow \quad \swarrow {\scriptstyle \pi_0' \circ pr_2} & \\ & St(\lambda) & \end{array}$$

<u>Proof</u>. We choose a basis $(e^h_{j,k})$ $(1 \leq j \leq h,\ 1 \leq k \leq c_h)$ (c_h as in I.1.12) such that $(u-1)(e^h_{j,k}) = e^h_{j-1,k}$ if $j \neq 1$, 0 if $j = 1$,

and such that :

a) L is generated by a vector $e^{h_o}_{1,m_o}$ of the basis.

b) W_{i-1} is generated by the vectors $e^{h}_{1,m}$ such that $h > h_o$ or $h = h_o$ and $m > m_o$.

In the proof, we replace B_i by W_{i-1} via the isomorphism $W_{i-1} \longrightarrow B_i$, $w \longmapsto k(e^{h_o}_{1,m_o} + w)$.

If $w = \sum a^h_m e^h_{1,m} \in W_{i-1}$, define w_j by $w_j = \sum a^h_m e^h_{j,m}$ $(1 \leq j \leq h_o)$. Let g_w be the automorphism of V leaving $e^h_{j,m}$ fixed if $(h,m) \neq (h_o,m_o)$ and such that $g_w(e^{h_o}_{j,m_o}) = e^{h_o}_{j,m_o} + w_j$.

It is then easy to check that we can take θ to be the inverse of the isomorphism $g : W_{i-1} \times \mathcal{F}(V/L) \longrightarrow Y_i$, $(w,F) \longmapsto g_w(F)$.

Proposition 5.10. There exists a partition $(A_i)_{1 \leq i \leq m}$ (for some $m \in \mathbb{N}$) of $\mathcal{F}$ with the following properties.

a) each A_i is isomorphic to an affine space.

b) each A_i is contained in some X_σ.

c) $\bigcup_{i \leq j} A_i$ is closed in $\mathcal{F}$ for all j $(1 \leq j \leq m)$.

Proof. It is sufficient to prove this with (b) replaced by

b') each A_i is contained in some Y_σ.

The proposition follows then from 5.9 by induction on n.

Lemma 5.11. Let $L \in \mathcal{F}_u$ be a line of type α_i $(1 \leq i \leq n)$. If $\sigma \in St(\lambda)$ is such that $X_\sigma \cap L$ is dense in L, then $\sigma_i > \sigma_{i+1}$ (if the tableau is written as in 8.3 , we shall say that i is above i+1 in the tableau σ) and $\bar{X}_\sigma$ is a union of lines of type α_i.

Proof. This is equivalent to the following statement : if $L \subset \mathcal{F}_u$ is a line of type α_i and $\sigma \in St(\lambda)$ is such that $L \cap Y_\sigma$

is dense in L, then n-i is above n-i+1 in the tableau σ, and $\overline{Y}_\sigma$ is a union of lines of type α_i. We prove this statement.

Choose $F = (F_0,\dots,F_n) \in L \cap Y_\sigma$. By 4.8 in particular, we can replace V and u by V/F_i and the automorphism u' of V/F_i induced by u. This reduces the problem to the case i = 1.

In this case $L \subset \mathcal{F}_u$ means that the restriction of u to F_2 is the identity. Let j be the largest integer such that $F_2 \subset W_{\rho_j}$. We certainly have $\sigma_n \geq j$, $\sigma_{n-1} \geq j$. If $\sigma_n \geq j+1$, then we must have $F_1 = F_2 \cap W_{\rho_{j+1}}$ and then $L \cap Y_\sigma$ is reduced to a point, a contradiction. Hence $\sigma_n = j$ and therefore n-1 is above n in σ. Let now F' be any flag contained in Y_σ. Since n-1 is above n and $\sigma_n = j$, it is clear that $F'_2 \subset W_{\rho_j}$. This shows that the line of type α_1 through F' is contained in $\mathcal{F}_u$. This implies that $\overline{Y}_\sigma$ is a union of lines of type α_1. This proves the lemma.

Proposition 5.12. a) For any $\sigma \in St(\lambda)$, $I_\sigma = \{\alpha_i \mid \overline{X}_\sigma$ is a union of lines of type $\alpha_i\}$ is given by $I_\sigma = \{\alpha_i \mid 1 \leq i \leq n-1$ and i is above i+1 in the tableau $\sigma\}$.

b) Let P be any parabolic subgroup of G. If $\mathcal{B}(P) \subset \mathcal{F}_u$, then for some $\sigma \in St(\lambda)$, $\mathcal{B}(P) \subset \overline{X}_\sigma$ and $\overline{X}_\sigma$ is a union of subvarieties of the form $\mathcal{B}({}^gP)$ $(g \in G)$.

Proof. This is an immediate consequence of 5.11. For (b), take σ to be the unique element of $St(\lambda)$ such that $\mathcal{B}(P) \cap X_\sigma$ is dense in $\mathcal{B}(P)$.

Remark 5.13. (b) of 5.12 fails for the line of type β in the Dynkin variety of elements of the class C_2 in 3.12 (c_3). It is therefore not possible to replace GL(V) by an arbitrary semisimple group in this statement.

Corollary 5.14. Let $\mathcal{P}$ be a conjugacy class of parabolic subgroups of G. Then all irreducible components of the variety $\{P \in \mathcal{P} | u \in U_P\}$ have dimension $\dim \mathcal{B}_u - \dim \mathcal{B}(P)$.

Proof. This follows from 5.12 and 1.12.

5.15. We know from 4.4 and 5.12 which components of $\mathcal{B}_u$ have their dimension preserved by $p : \mathcal{B}_u \longrightarrow \mathcal{P}_u$, where $\mathcal{P}$ is a conjugacy class of parabolic subgroups.

Let λ be a partition of n. We denote by λ^* the dual partition. The parts of λ^* are $\ell_1, \ell_2, \ldots$, the length of the columns of d_λ. For each $\sigma \in St(\lambda)$ we have a dual standard tableau σ^*. The lignes of σ^* are the columns of σ. Clearly $(\lambda^*)^* = \lambda$, $(\sigma^*)^* = \sigma$. From 5.12, $I_{\sigma^*} = \Pi \setminus I_\sigma$ for all $\sigma \in St(\lambda)$.

For each partition λ of n, choose an element $u_\lambda \in C_\lambda$. The set of all partitions of n is partially ordered by $\lambda \leq \mu$ if $\sum_{i \leq j} \lambda_i \leq \sum_{i \leq j} \mu_i$ for all $j \geq 1$. It is known that $\lambda \leq \mu \Leftrightarrow \overline{C}_\lambda \subset \overline{C}_\mu$. It is easy to check that $\lambda \leq \mu \Longleftrightarrow \mu^* \leq \lambda^*$.

For each partition λ of n, we choose also a conjugacy class of parabolic subgroups $\mathcal{P}_\lambda$ such that P/U_P is of type $A_{\ell_1} \times A_{\ell_2} \times \ldots$ if $P \in \mathcal{P}_\lambda$. We denote by W_λ the corresponding subgroup of W and by I_λ the corresponding subset of Π. The complex representation $\mathrm{Ind}_{W_\lambda}^{W}(1)$ depends only on λ. We shall denote it by θ_λ. For any unipotent element x, we write $S_\lambda(x)$ for $S_{\mathcal{P}_\lambda}(x)$, and we write $S_\lambda(\mu)$ for $S_\lambda(u_\mu)$.

From 6.2 (or by direct computations) the $\mathcal{P}_\lambda$-regular class is C_{λ^*}.

Lemma 5.16. a) $S_\mu(\lambda) \neq \emptyset \Longleftrightarrow \mu \leq \lambda$.

b) $|S_\lambda(\lambda)| = 1$.

Proof. a) We have the following equivalences.

$S_\mu(\lambda) \neq \emptyset \iff I_\mu \cap I_\sigma = \emptyset$ for some $\sigma \in St(\lambda)) \iff$
$I_\mu \subset I_{\sigma *}$ for some $\sigma \in St(\lambda)) \iff I_\mu \subset I_\tau$ for some $\tau \in St(\lambda^*)$
$\iff \overline{C}_{\lambda *} \subset \overline{C}_{\mu *}$ (in particular because of 5.12 (b)) $\iff$
$\lambda^* \leq \mu^* \iff \mu \leq \lambda$.

b) We take $\lambda = \mu$ in these equivalences. If $\tau \in St(\lambda^*)$ is such that $I_\mu = I_\lambda \subset I_\tau$, then for dimension reasons $I_\lambda = I_\tau$ and the component $\overline{X}_\tau$ of $\mathcal{F}_{u_{\lambda *}}$ is of the form $\mathcal{B}(P)$ for some $P \in \mathcal{P}_\lambda$. Since $Z(u_{\lambda *})$ is connected, there is at most one component with this property. Therefore $\sigma = \tau^*$ is the unique element of $S_\lambda(\lambda)$.

Proposition 5.17. a) For any partition λ of n, there exists a unique (up to isomorphism) complex irreducible representation ρ_λ of $W \cong S_n$ such that $\theta_\lambda = \rho_\lambda + \sum_{\mu < \lambda} n_{\mu\lambda} \rho_\mu$ for some suitable integers $n_{\mu\lambda}$.

b) $|S_\mu(\lambda)| = n_{\mu\lambda}$ for any pair of partitions λ, μ.

Proof. For any pair of partitions μ, ν, we have

1) $\sum |S_\mu(\lambda)||S_\nu(\lambda)| = |W_\mu \backslash W / W_\nu|$ (4.2).

Choose a total ordering $\lambda^1 < \lambda^2 < \ldots$ on the set of all partitions of n, compatible with the partial order defined in 5.15. Let $s_{ij} = |S_{\lambda^i}(\lambda^j)|$ and let S be the matrix (s_{ij}).

Let $\rho_1, \rho_2, \ldots$ be the irreducible complex representations of S_n. Their number is the number of partitions of n. From the definition of θ_λ, we get

2) $\sum_j \langle \theta_\mu, \rho_j \rangle \langle \theta_\nu, \rho_j \rangle = \langle \theta_\mu, \theta_\nu \rangle = |W_\mu \backslash W / W_\nu|$ for all pairs of partitions μ, ν of n.

Let $n_{ij} = \langle \theta_{\lambda^i}, \rho_j \rangle$ and let N be the matrix (n_{ij}).

(1) and (2) show that $N({}^tN) = S({}^tS)$. Hence $(S^{-1}N)({}^t(S^{-1}N)) =$

1. By 5.16, S is a unipotent triangular matrix. S^{-1} is a matrix with coefficients in $\mathbb{Z}$. Therefore $S^{-1}N$ corresponds to a permutation of the basis with possible sign changes. Changing the numbering of the irreducible representations, we can arrange to have $S^{-1}N$ triangular, and its eigenvalues are in $\{-1,+1\}$. But S is unipotent triangular and $N = S(S^{-1}N)$ has no negative coefficients. This shows that $S^{-1}N = 1$, i.e. $S = N$. This clearly proves the proposition.

5.18. We assume now that we are in one of the cases (II) to (IX). We use the parametrization of unipotent classes given in I.1.12 and I.3.18 and we consider a fixed unipotent element $u \in C_{\lambda,\varepsilon}$. We shall say that we are in the odd orthogonal case if we are in cases (II) or (III), in the even orthogonal case if we are in cases (IV) or (V) and in the symplectic case if we are in cases (VI), (VII), (VIII) or (IX). This refers to the kind of diagrams occuring in the parametrization of unipotent classes.

Let N = dim V. N = 2n+1 in the odd orthogonal case and N = 2n in the other cases. d_λ (or d_N) is the Young diagram corresponding to λ and as in I.1.12 c_i is the number of lines of length i in d_λ and ℓ_i is the length of the i^{th} column of d_λ. We associate to λ integers b_λ, z_λ in the following way. If we are in the odd orthogonal case (resp. the even orthogonal case, the symplectic case), let u_λ be an element of $O_{2n+1}(\mathbb{C})$ (resp. $O_{2n}(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$) in the unipotent class corresponding to λ. Then :

a) In the odd orthogonal case,

$b_\lambda = \frac{1}{4}(1 + \sum_{i\geq 1}(\ell_{2i-1}(\ell_{2i-1}-2)) + \ell_{2i}^2))$,

$z_\lambda = \dim Z_{O_{2n+1}}(u_\lambda)$.

b) In the even orthogonal case,

$b_\lambda = \frac{1}{4}(\sum_{i\geq 1}(\ell_{2i-1}(\ell_{2i-1}-2) + \ell_{2i}^2)$,

$z_\lambda = \dim Z_{O_{2n}}(u_\lambda)$.

c) In the symplectic case,

$b_\lambda = \frac{1}{4}\sum_{i\geq 1}(\ell_{2i-1}^2 + \ell_{2i}(\ell_{2i}-2))$,

$z_\lambda = \dim Z_{Sp_{2n}}(u_\lambda)$.

Notice that a partition λ may occur in the even orthogonal case and in the symplectic case and that the integers b_λ and z_λ depend on the case we consider.

Proposition 5.19. $\dim \mathcal{B}_u = b_\lambda - \left[\frac{1}{2}\sum_{\varepsilon_i=0}(\ell_{i+1}-\ell_i)\right]$. In this formula $\ell_0 = 0$ and $[x] = \max\{y \in \mathbb{Z} \mid y \leq x\}$.

Proof. We shall give the proof for cases (II) and (IV) in 5.30. In the other cases the proof is similar.

Remark 5.20. In cases (III), (V) and (VI), $\varepsilon_i \neq 0$ for all i and therefore $\dim \mathcal{B}_u = b_\lambda$. In cases (II), (IV) and (VII), $\varepsilon_0 \neq 0$ and therefore $\ell_{i+1}-\ell_i = -c_i < 0$ is even if $\varepsilon_i = 0$. In case (VIII) $\varepsilon_0 = 0$ and $\ell_1 - \ell_0 = \ell_1$ is even. In case (IX) $\varepsilon_0 = 0$ and $\ell_1 - \ell_0 = \ell_1$ is odd.

Corollary 5.21. $\dim Z(u) = 2 \dim \mathcal{B}_u + \mathrm{rank}_u(G)$.

Proof. We know already that this is true if p = 0 (2.5 and

2.6). Hence $z_\lambda = 2b_\lambda + n$. It can be checked that dim $Z(u) = z_\lambda - \sum_{\varepsilon_i=0} (\ell_{i+1}-\ell_i)$ (for cases (II) and (IV) this is done in I.3.24). This shows that dim $Z(u) = 2 \dim \mathcal{B}_u + n - 1$ in case (IX) and dim $Z(u) = 2 \dim \mathcal{B}_u + n$ in the other cases. This proves the lemma since $\mathrm{rank}_u(G) = n-1$ in case (IX) and $\mathrm{rank}_u(G) = n$ in the other cases.

5.22. Assume that $n \geqslant 2$ if we are in case (V), (VIII) or (IX) and $n \geqslant 1$ in the other cases. Let L be a subspace of dimension 1 in V and let $H \supset L$ be a hyperplane in V. If we are in case (III), (V), (VI) or (VII) we assume that $H = L^\perp$. If we are in case (VIII) or (IX) we assume that Q vanishes on L and $H = L^\perp$. Let P be the stabilizer of (L,H) in G (in cases (II) and (IV) we say that a bilinear form f G stabilizes (L,H) if $(fL, fH) = (H,L)$). P is a parabolic subgroup of G and the conditions of 4.6 are satisfied. We want to use 4.9 with this parabolic subgroup. If $P' \in \mathcal{P}_u$, the class of $uU_{P'}$ in $P'/U_{P'}$ is parametrized by a pair (λ', ε') and we can write $f(P') = (\lambda', \varepsilon')$ (f as in 4.6 ; this should not create any confusion with the bilinear forms denoted by the same letter). Consider now a fixed unipotent class C' in P/U_P parametrized by a pair (λ', ε'). Let $Y = f^{-1}(C')$ and $X = p^{-1}(Y)$ (as in 4.6). If $u' \in C'$ the group $A'(u')$ (as in 4.9) can be described as in I.1.13 and I.3.23 with a subset of $\{a_0, a_1, \dots\}$ as a system of generators. The relations for $A(u)$ and $A'(u')$ need not to be the same.

Proposition 5.23. In the situation of 5.22 Z(u) acts transitively on Y. In the orthogonal case (resp. the symplectic case) dim X = dim $\mathcal{B}_u$ if and only if (λ', ε') is obtained from (λ, ε) as follows. $d_{\lambda'}$ must be deduced from d_λ by one of the following operations.

a) if $c_i \geqslant 2$, remove two squares from the i^{th} column.

b) if $i \geqslant 2$ is odd (resp. even), $\varepsilon_i = 1$ and $\ell_{i-1} = \ell_i$, remove one square from the i^{th} column and one square from the $(i-1)^{th}$ column. In cases (VIII) and (IX) we must have $i \geqslant 4$.

ε' is any application $\mathbb{N} \longrightarrow \{\omega, 0, 1\}$ such that :

i) (λ', ε') represents a unipotent class in P/U_P.

ii) $\varepsilon'_j = \varepsilon_j$ if the number of lines of length j is unchanged.

iii) $\varepsilon'_j \neq 1$ if $\varepsilon_j = 0$ and $\varepsilon'_j \neq 0$ if $\varepsilon_j = 1$.

The groups A_P and A'_P of 4.9 are the following. A_P is the subgroup of A(u) generated by the elements in the system of generators of A(u) which are also in the system of generators of A'(u') (as described in I.1.13 or I.3.23). A'_P is the smallest subgroup of A'(u') such that the application from the system of generators of A_P to A'(u') gives rise to a homomorphism $A_P \longrightarrow A'(u')/A'_P$. This homomorphism is the one considered in 4.9.

Proof. We give the proof for cases (II) and (IV) in several steps (from 5.25 to 5.33). The proof for the other cases is similar.

<u>Remark 5.24</u>. Suppose that in 5.23 dim Y = dim $\mathcal{B}_u$. Then Y is connected (and $A_P = A(u)$) or Y has two components (and $|A(u)/A_P| = 2$). Y has two components in the following cases :

a) In case (III), (V) or (VI), Y has two components if $\varepsilon_i = 1$, $c_i = 2$ and $d_{\lambda'}$ is obtained from d_λ by removing two squares from the i^{th} column.

b) In case (II), (IV), (VII), (VIII) or (IX), Y has two components if $\varepsilon_i = \omega$, $c_i = 2$, $\varepsilon_{i+1} \neq 1$, $\varepsilon'_{i-1} = 0$ and $d_{\lambda'}$ is obtained from d_λ by removing two squares from the column i.

In both cases $D = \{1, a_{\ell_i}\} = \{1, a_{\ell_i - 1}\} \subset A(u)$ is such that $A(u) = D \times A_P$.

We also have $A'_P = 1$ or $|A'_P| = 2$. $A'_P \neq 1$ in the following cases.

a') In case (III), (V) or (VI), $A'_P = \{1, a_{\ell_i} a_{\ell_i - 1}\} \subset A'(u')$ if $\varepsilon_i = 1$, $c_i \geqslant 2$ and $d_{\lambda'}$ is obtained from d_λ by removing one square from the i^{th} column and one square from the $(i-1)^{th}$ column.

b') In case (II), (IV), (VII), (VIII) or (IX), $A'_P = \{1, a_{\ell_i} a_{\ell_i - 1}\} \subset A'(u')$ if one of the following conditions holds.

b'_1) $\varepsilon_i = 1$, $c_i = 2$ and ($c_{i+1} \neq 0$ or $\varepsilon_{i+2} = 1$) and $d_{\lambda'}$ is obtained from d_λ by removing two squares from the column i.

b'_2) $\varepsilon_i = 1$, $c_i = 1$ and ($c_{i+1} \neq 0$ or $\varepsilon_{i+2} = 1$) and $d_{\lambda'}$ is obtained from d_λ by removing one square from the column i and one square from the column (i-1).

These results are consequences of 5.23.

5.25. Until the end of 5.33 we suppose that we are in case (II) or (IV) and that $f \in C_{\lambda,\varepsilon}$ is a unipotent bilinear form in $G = G(V)$. Let $u = f^2$ and let $W_i = \mathrm{Ker}(u-1) \cap \mathrm{Im}(u-1)^{i-1}$.

A subspace M of V will be called an f-submodule if it is u-stable and the restriction f_M of f to $M \times M$ is non-singular. The conjugacy class of f_M in $G(M)$ is determined by a diagram and by a sequence $(\varepsilon_i(M))_{i\geq 0}$. Here we characterize the diagram by the sequence $(c_i(M))_{i\geq 1}$, where $c_i(M)$ is the number of Jordan blocks of dimension i of $f_M^2 \in GL(M)$.

A line $L \in \mathbb{P}(V)$ is isotropic (for f) if the restriction of f to $L \times L$ is 0 and L is stabilized by u. There is an isotropic flag $(F_0, F_1, \ldots, F_N)$ ($N = \dim V$) fixed by f such that $F_1 = L$ if and only if L is isotropic (for f).

If M is a u-stable subspace of V, we shall write $M^{\perp} = fM = \{v \in V \mid f(v,m) = 0 \text{ for all } m \in M\} = \{v \in V \mid f(m,v) = 0 \text{ for all } m \in M\}$ (I.3.4). $M^{\perp}$ is u-stable. If M is an f-submodule, $V = M \oplus M^{\perp}$.

Suppose that M is an f-submodule and that $L \in \mathbb{P}(M)$ is isotropic. f induces a unipotent bilinear form on $(L^{\perp} \cap M)/L$. Its conjugacy class will be represented by $(c_i'(M), \varepsilon_i'(M))_{i\geq 1}$. We have $c_i(V) = c_i$, $\varepsilon_i(V) = \varepsilon_i$ and we write $c_i' = c_i'(V)$, $\varepsilon_i' = \varepsilon_i'(V)$. Clearly $c_i = c_i(M) + c_i(M^{\perp})$, $\varepsilon_i = \max(\varepsilon_i(M), \varepsilon_i(M^{\perp}))$, $c_i' = c_i'(M) + c_i(M^{\perp})$, $\varepsilon_i' = \max(\varepsilon_i'(M), \varepsilon_i(M^{\perp}))$.

We shall say that two f-submodules M_1 and M_2 are equivalent if $c_i(M_1) = c_i(M_2)$ and $\varepsilon_i(M_1) = \varepsilon_i(M_2)$ for all $i \geq 1$ and we write $M_1 \sim M_2$. There exists $z \in Z_0(f)$ such that $zM_1 = M_2$ if and only if $M_1 \sim M_2$ and $M_1^{\perp} \sim M_2^{\perp}$ (in particular because of I.3.18).

<u>5.26</u>. Choose P as in 5.22. $\mathcal{P}_f$ is naturally isomorphic to the variety of all isotropic lines in V and we shall identify these varieties. Let Y be a subvariety of $\mathcal{P}_f$ as in 5.22. It is clear from 5.5 that there is a unique i such that $Y \subset \mathbb{P}(W_i) \setminus \mathbb{P}(W_{i+1})$. If $L \in Y$ is an isotropic line, $(c_j')_{j\geq 1}$ and $(\varepsilon_j')_{j\geq 1}$ depend only on Y and not on the choice of L (by definition of Y).

Let $X(L)^*$ be the set of all f-submodules of V containing L such that the following conditions hold.

a) M has no proper f-submodule containing L.

b) If $\varepsilon_i(M) \neq \omega$ for some i, then $\varepsilon_i(M) = \varepsilon_i$.

The elements of $X(L)^*$ can be obtained as follows. Fix an element $v \in L$, $v \neq 0$. Let $V_v = \{w \in V \mid (u-1)^{i-1}(w) = v\}$. $f(v,w)$ is constant on V_v (by I.3.5). There are two cases.

1) if $f(v,w) \neq 0$ on V_v, the u-stable submodule generated by any $w \in V_v$ is in $X(L)^*$ and every element of $X(L)^*$ is of this form.

2) if $f(v,w) = 0$ on V_v, choose first an element $w \in V_v$. By I.3.6 we can choose $v' \in W_i \setminus W_{i+1}$ such that $f(w,v') \neq 0$. If $\varepsilon_i = 1$, we can choose $v' \in W_i \setminus W_{i+1}$ such that $f(v',w') \neq 0$ on $V_{v'}$. Choose now an element $w' \in V_{v'}$. The u-stable submodule

generated by w and w' is in X(L)* and every element of X(L)* is of this form.

It is then easy to check that X(L)* is not empty and that it is an irreducible subset of some Grassmannian variety. Moreover in case (1) we have for all $M \in X(L)^*$:

$$c_j(M) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad \varepsilon_j(M) = \begin{cases} 1 & \text{if } j = i \\ \omega & \text{if } j \neq i \end{cases}$$

$$c'_j(M) = \begin{cases} 1 & \text{if } j = i-2 \\ 0 & \text{if } j \neq i-2 \end{cases} \qquad \varepsilon'_j(M) = \begin{cases} 1 & \text{if } j = i-2 \\ \omega & \text{if } j \neq i-2 \end{cases}$$

and in case (2) we have for all $M \in X(L)^*$:

$$c_j(M) = \begin{cases} 2 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad \varepsilon_j(M) = \begin{cases} \varepsilon_i & \text{if } j = i \\ \omega & \text{if } j \neq i \end{cases}$$

$$c'_j(M) = \begin{cases} 2 & \text{if } j = i-1 \\ 0 & \text{if } j \neq i-1 \end{cases} \qquad \varepsilon'_j(M) = \begin{cases} ? & \text{if } j = i-1 \\ \omega & \text{if } j \neq i-1 \end{cases}$$

This shows in particular that the equivalence class of M depends only on Y. This is also the case for $M^{\perp}$. Since $c_j = c_j(M) + c_j(M^{\perp})$ and $\varepsilon_j = \max(\varepsilon_j(M), \varepsilon_j(M^{\perp}))$, we find that $c_j(M^{\perp})$ is independent of the choice of $M \in X(L)^*$ and of the choice of $L \in Y$ for all $j \geq 1$. The same holds for $\varepsilon_j(M\ )$ if $j \neq i$. But $\varepsilon'_i = \max(\varepsilon'_i(M), \varepsilon_i(M^{\perp})) = \max(\omega, \varepsilon_i(M^{\perp})) = \varepsilon_i(M^{\perp})$ depends only on Y. This shows that the equivalence class of $M^{\perp}$ depends only on Y.

Let $\tilde{Y}^* = \{(L,M) \mid L \in Y \text{ and } M \in X(L)^*\}$. We have shown that $Z_o(f)$ acts transitively on the second projection $pr_2(\tilde{Y}^*)$.

<u>5.27</u>. We assume now that u has one Jordan block of dimension

generated by w and w' is in X(L)* and every element of X(L)* is of this form.

It is then easy to check that X(L)* is not empty and that it is an irreducible subset of some Grassmannian variety. Moreover in case (1) we have for all $M \in X(L)^*$:

$$c_j(M) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad \varepsilon_j(M) = \begin{cases} 1 & \text{if } j = i \\ \omega & \text{if } j \neq i \end{cases}$$

$$c'_j(M) = \begin{cases} 1 & \text{if } j = i-2 \\ 0 & \text{if } j \neq i-2 \end{cases} \qquad \varepsilon'_j(M) = \begin{cases} 1 & \text{if } j = i-2 \\ \omega & \text{if } j \neq i-2 \end{cases}$$

and in case (2) we have for all $M \in X(L)^*$:

$$c_j(M) = \begin{cases} 2 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad \varepsilon_j(M) = \begin{cases} \varepsilon_i & \text{if } j = i \\ \omega & \text{if } j \neq i \end{cases}$$

$$c'_j(M) = \begin{cases} 2 & \text{if } j = i-1 \\ 0 & \text{if } j \neq i-1 \end{cases} \qquad \varepsilon'_j(M) = \begin{cases} ? & \text{if } j = i-1 \\ \omega & \text{if } j \neq i-1 \end{cases}$$

This shows in particular that the equivalence class of M depends only on Y. This is also the case for $M^\perp$. Since $c_j = c_j(M) + c_j(M^\perp)$ and $\varepsilon_j = \max(\varepsilon_j(M), \varepsilon_j(M^\perp))$, we find that $c_j(M^\perp)$ is independent of the choice of $M \in X(L)^*$ and of the choice of $L \in Y$ for all $j \geq 1$. The same holds for $\varepsilon_j(M\ )$ if $j \neq i$. But $\varepsilon'_i = \max(\varepsilon'_i(M), \varepsilon_i(M^\perp)) = \max(\omega, \varepsilon_i(M^\perp)) = \varepsilon_i(M^\perp)$ depends only on Y. This shows that the equivalence class of $M^\perp$ depends only on Y.

Let $\tilde{Y}^* = \{(L,M) \mid L \in Y \text{ and } M \in X(L)^*\}$. We have shown that $Z_o(f)$ acts transitively on the second projection $pr_2(\tilde{Y}^*)$.

__5.27__. We assume now that u has one Jordan block of dimension

2n+1 and we are in case (II) or two blocks of dimension n and we are in case (IV). The possibilities for Y are as follows.

a) if $c_{2n+1} = 1$, $\mathcal{P}_f$ consists of a single point $L \in \mathbb{P}(W_1)$ and $V \in X(L)^*$.

b) if $c_n = 2$ and $\varepsilon_n = 0$, then $\mathcal{P}_f = \mathbb{P}(W_1)$ consists of a single $Z_o(f)$-orbit. If $L \in \mathcal{P}_f$, then $V \in X(L)^*$.

c) if $c_n = 2$ and $\varepsilon_n = 1$, then $\mathcal{P}_f \subset \mathbb{P}(W_1)$ consists of two $Z_o(f)$-orbits if $n \geqslant 2$ and is a single orbit if $n = 1$. $Y_1 = \{L \in \mathbb{P}(W_1) \mid f(v,w) = 0 \text{ if } v \in L \text{ and } (u-1)^{n-1}(w) = v\}$ consists of a single point. If $n \geqslant 2$ the other orbit in $\mathcal{P}_f$ is $Y_2 = \mathbb{P}(W_1) \setminus Y_1$. $V \in X(L)^*$ if $L \in Y_1$ and $V \notin X(L)^*$ if $L \in Y_2$.

d) If $c_n = 2$ and $\varepsilon_n = \omega$, then $\mathcal{P}_f = \mathbb{P}(W_1)$ consists of two $Z_o(f)$-orbits. $Y_1 = \{L \in \mathbb{P}(W_1) \mid \varepsilon'_{i-1} = 0\}$ consists of two points and $Y_2 = \mathbb{P}(W_1) \setminus Y_1 = \{L \in \mathbb{P}(W_1) \mid \varepsilon'_{i-1} = 1\}$. $V \in X(L)^*$ for all $L \in \mathbb{P}(W_1)$.

These results can be deduced easily from I.3.15 and I.3.17.

<u>5.28</u>. We can now prove that $Z_o(f)$ acts transitively on the variety Y of 5.26. With the notations of 5.26, we have already proved that $Z_o(f)$ acts transitively on $pr_2(\tilde{Y}^*)$. Consider the action of Z_M on Y^*_M, where $Z_M = \{z \in Z_o(f) \mid zM = M\}$ and $Y^*_M = \{L \in Y \mid M \in X(L)^*\}$. From 5.27 it is clear that Z_M acts transitively on Y^*_M if we are in cases (a), (b) or (c) and there are at most two orbits if we are in case (d) of 5.27

for $f_M \in G(M)$ (clearly $Z_M \cong Z_{GL(M)}(f_M) \times Z_{GL(M^\perp)}(f_{M^\perp})$). Hence $Z_o(f)$ acts transitively on Y if we are not in case (d).

Suppose now that we are in case (d). Then $c'_{i-1}(M) = 2$. Therefore $c'_{i-1} \geq 2$ and $\varepsilon'_{i-1} = 0$ or 1. If $\varepsilon_{i-1} = \omega$ or 0, we must have $\varepsilon'_{i-1}(M) = \varepsilon'_{i-1}$ and 5.27 shows that Y^*_M is a single Z_M-orbit. This shows that unless $\varepsilon_{i-1} = 1$, $Z_o(f)$ acts transitively on Y.

Suppose that we are in case (d) with $\varepsilon_{i-1} = 1$. Then Y^*_M consists of two Z_M-orbits and $\tilde{Y}^*$ consists of two $Z_o(f)$-orbits. Let $\tilde{Y}^*_1$, $\tilde{Y}^*_2$ be the orbits in $\tilde{Y}^*$ corresponding to Y_1, Y_2 of 5.27 (d) respectively. Choose $(L,M) \in \tilde{Y}^*_1$ and pick $v \in L$, $v \neq 0$. M is the u-stable submodule of V generated by some w, w' such that $w \in V_v$ and $w' \in V_{v'}$, where $v' \in W_i$ is such that $f(w,v') \neq 0$ (5.26). Since $\varepsilon_{i-1} = 1$, there exists $x \in M^\perp$ such that $(u-1)^{i-1}(x) = 0$ and $f(x,(u-1)^{i-2}(x)) = 1$. For every $\lambda \in k$ let $w_\lambda = w + \lambda x$. We still have $w_\lambda \in V_v$ and $f(w_\lambda, v') \neq 0$. The u-stable submodule M_λ generated by w_λ and w' is an element of $X(L)^*$. Since $f(w_\lambda,(u-1)^{i-2}(w_\lambda)) = f(w,(u-1)^{i-2}(w)) + \lambda^2 = \lambda^2$, $(L,M_\lambda) \in \tilde{Y}^*_2$ for all $\lambda \neq 0$. This shows that $\tilde{Y}^*_2$ is dense in $\tilde{Y}^*$ and that $pr_1(\tilde{Y}^*_1)$ and $pr_1(\tilde{Y}^*_2)$ both contain L. This implies that $Y = pr_1(\tilde{Y}^*_1) = pr_1(\tilde{Y}^*_2)$ is a single $Z_o(f)$-orbit.

We have therefore shown that in all cases $Z_o(f)$ acts transitively on Y. We have shown also that $X(L)^*$ contains a dense Z_M-orbit and that $\tilde{Y}^*$ contains a dense $Z_o(f)$-orbit. Let $X(L) \subset X(L)^*$ and $\tilde{Y} \subset \tilde{Y}^*$ be these dense orbits. Clearly $\tilde{Y} =$

$\{(L,M) \mid L \in Y \text{ and } M \in X(L)\}$. Let also $Y_M = \{L \in Y \mid M \in X(L)\}$. Y_M is dense in Y_M^* and Z_M acts transitively on Y_M.

5.29. $\dim \tilde{Y} = \dim pr_2(\tilde{Y}) + \dim Y_M$. $\dim Y_M$ can be computed from 5.27 (this is always 0 or 1). Since $Z_o(f)$ acts transitively on $pr_2(\tilde{Y})$, $\dim pr_2(\tilde{Y}) = \dim Z(f) - \dim Z_M =$ $\dim Z(f) - \dim Z_{GL(M)}(f_M) - \dim Z_{GL(M^\perp)}(f_{M^\perp})$.

Suppose that $\varepsilon_i = 1$ and $\varepsilon_i' = 0$. Let $\overline{Y}$ be the subvariety of $\mathcal{P}_f$ corresponding to $(\overline{\lambda}', \overline{\varepsilon}')$, where $\overline{\lambda}' = \lambda'$ and $\overline{\varepsilon}'$: $\mathbb{N} \longrightarrow \{\omega, 0, 1\}$ is defined by $\overline{\varepsilon}_j' = \varepsilon_j'$ if $j \neq i$ and $\overline{\varepsilon}_i' = 1$. It is easily checked that $\overline{Y} \neq \emptyset$. Let X (resp. $\overline{X}$) be the subvariety of $\mathcal{P}_f$ corresponding to Y (resp. $\overline{Y}$). By induction on n, we may assume that 5.19 is true for $f_{M^\perp} \in G(M^\perp)$. By I.3.24 we get then easily $\dim X = \dim \overline{X} - [(c_i-1)/2] < \dim \mathcal{B}_f$.

5.30. Let E be the variety of all $k[u]$-submodules M of V such that M is a direct factor of V as $k[u]$-modules and :

1) if we are in case (1) of 5.26, the restriction of u to M has only one Jordan block and $\dim M = i$.

2) if we are in case (2) of 5.26, the restriction of u to M has exactly two Jordan blocks, both of dimension i.

E is irreducible. $E_o = \{M \in E \mid M \text{ is an f-submodule}\}$ is open dense in E. $pr_2(\tilde{Y})$ is a subvariety of E_o and (by considering all possible cases) 5.29 shows that $pr_2(\tilde{Y})$ is dense in E_o if $\dim X = \dim \mathcal{B}_f$ (X being the subvariety of $\mathcal{B}_f$ corresponding to Y). In particular, if $\dim X = \dim \mathcal{B}_f$, then $pr_2(\tilde{Y})$ is irreducible and $\dim pr_2(\tilde{Y}) = \dim E = \sum_{j<i}(\ell_j-1)$ in case (1), $2\sum_{j<i}(\ell_j-2)$ in case (2). $\dim \tilde{Y} = \dim pr_2(\tilde{Y}) + \dim Y_M$ can be computed in this case from 5.27. Since $\dim Y = \dim \tilde{Y} - \dim X(L)$, we need only to compute $\dim X(L) = \dim X(L)^*$ to determine $\dim Y$ when $pr_2(\tilde{Y})$ is dense in E_o $((L,M) \in \tilde{Y})$. From the descrition of $X(L)^*$ given in 5.26, we get $\dim X(L)^* = \sum_{j<i}(\ell_j-1)$ in case (1), $(\ell_i-2) + 2\sum_{j<i}(\ell_j-2)$ in case (2). Therefore $\dim Y = \ell_i-1$ in case (1), $(\ell_i-2) + \dim Y_M$ in case (2) (if $pr_2(\tilde{Y})$ is dense in E_o).

By induction on n we can assume that 5.19 is true for P/U_P. We can then compute $\dim X = \dim Y + \dim \mathfrak{B}(P/U_P)_{u'}$, where $u' \in P/U_P$ is in the unipotent class corresponding to (λ', ε'), at least if $pr_2(\tilde{Y})$ is dense in E_o. This is certainly the case if $\dim X = \dim \mathfrak{B}_f$. In fact we get a method to compute $\dim X$ in all cases since 5.29 reduces the problem to the case considered here.

The different possibilities for (λ', ε') can be obtained from 5.27 and the formulae $c'_j = c'_j(M) + c_j(M^{\perp})$, $\varepsilon'_j = \max(\varepsilon'_j(M), \varepsilon_j(M^{\perp}))$. By considering all possible cases, it is then easy to check (by induction on n) that $\dim \mathfrak{B}_f$ is given by the formula of 5.19 and that $\dim X = \dim \mathfrak{B}_f$ if and only if (λ', ε') is obtained from (λ, ε) by one of the operations described in 5.23.

<u>5.31</u>. We turn now to the proof of the statements concerning A_P and A'_P in 5.23. In I.3.21 we use a system of generators

for $A_o(f)$ corresponding to some lines of d_λ. Suppose that (λ',ε') is such that $\dim X = \dim \mathcal{B}_f$. Fix $(L,M) \in \tilde{Y}$. We may assume that the parabolic subgroup P is the stabilizer of $(L,L^\perp)$ in $G(V)$. We have already shown that $d_{\lambda'}$ is obtained from d_λ by removing two squares from one line of d_λ or by removing two squares from one column of d_λ, i.e. from two consecutive lines of d_λ (of the same length). The Young diagram of $f_{M^\perp} \in G(M^\perp)$ consists of the remaining lines of d_λ. The corresponding group $A_o(f_{M^\perp})$ can be described with $\{a_j \mid \varepsilon_{\lambda_j} \neq 0 \text{ and } \lambda_j = \lambda'_j\}$, i.e. the elements of the system of generators for $A_o(f)$ which correspond to lines which are not touched by the operation giving $d_{\lambda'}$ from d_λ. The relations for $A_o(f_{M^\perp})$ are all true in $A_o(f)$ but the obvious homomorphism $A_o(f_{M^\perp}) \longrightarrow A_o(f)$ is not always injective. Similarly $A_o(f_M)$ can be described with $\{a_j \mid \varepsilon_{\lambda_j} \neq 0 \text{ and } \lambda_j \neq \lambda'_j\}$ as a system of generators. From the proof of I.3.21 the homomorphism $A_o(f_M) \times A_o(f_{M^\perp}) \longrightarrow A_o(f)$ induced by $Z_M \subset Z_o(f)$ is the obvious homomorphism given by the generators. It is surjective.

We have a similar situation with $A'_o(f')$ with a surjective homomorphism $A'_o(f'_M) \times A_o(f_{M^\perp}) \longrightarrow A'_o(f')$ (where f'_M is the bilinear form induced by f on $(M \cap L^\perp)/L$).

5.32. In the situation of 5.31 $pr_2(\tilde{Y})$ is irreducible (5.30) and $Z_o(f)$ acts transitively on $\tilde{Y}$ and $pr_2(\tilde{Y})$. It follows that the irreducible components of $\tilde{Y}$ are in bijective correspondence with the $(Z(f)^o \cap Z_M)$-orbits in Y_M. $Z_o(f_{M^\perp})$ acts trivially on Y_M. It follows that the image of $A_o(f_{M^\perp})$ in $A(f)$ is contained in A_P. It is easily checked from 5.27 that Y_M is connected unless the given system of generators for $A_o(f_M)$ is not contained in the given system of generators for $A_o'(f_M')$. Also $fZ(f)^o \in A(f)$ is clearly contained in A_P. It follows easily that $A_P = A(f)$ (and Y is connected) unless i is even, $c_i = 2$, $\varepsilon_{i+1} \neq 1$ and $\varepsilon_{i-1}' = 0$. In this case the image of $A_o(f_{M^\perp})$ in $A(f)$ has index 2 in $A_o(f)$. From 5.27 Y_M consists of two points and it is easy to see that $Z(f)^o \cap Z_M$ acts trivially on Y_M (in particular from the relations of I.3.21 for $A_o(f)$). $\check{Y}$ has therefore two components and the same is true for Y since all fibres of $pr_1 : \check{Y} \longrightarrow Y$ are connected (X(L) is connected). A_P is generated by $fZ(f)^o$ and the image of $A_o(f_{M^\perp})$ in $A(f)$ and $|A(f)/A_P| = 2$.

This shows that the description of A_P given in 5.23 is correct if we are in case (II) or (IV).

<u>5.33</u>. Let $Z_L = \{z \in Z_o(f) \mid zL = L\}$. The homomorphism $A_P \longrightarrow A'(f')/A_P'$ is induced by the natural homomorphism $Z_L/Z_L^o \longrightarrow A'(f')$ (with $fZ(f)^o \longmapsto f'Z(f')^o$). Since X(L) is irreducible, $Z_{L,M} = Z_L \cap Z_M$ meets all components of Z_L. It is therefore sufficient to study the natural homomorphism $Z_{L,M}/Z_{L,M}^o \longrightarrow A'(f')$. Let $H = \{z \in Z_{GL(M)}(f_M) \mid zL = L\}$. $Z_{L,M} = H \times Z_{GL(M^\perp)}(f_{M^\perp})$ and $Z_{L,M}/Z_{L,M}^o = (H/H^o) \times A_o(f_{M^\perp})$. The homomorphism $A_o(f_{M^\perp}) \longrightarrow A'(f')$ is the one considered in 5.31. In order to prove 5.23

(in cases (II) and (IV)) we need only to prove (a) and (b) :

a) the natural homomorphism $H/H^{o} \longrightarrow A_{o}(f_{M})$ is injective and its image is the subgroup of $A_{o}(f_{M})$ generated by $\{a_j \mid \varepsilon_{\lambda_j} \neq 0, \varepsilon'_{\lambda_j} \neq 0$ and $\lambda_j \neq \lambda'_j\}$.

b) the homomorphism $H/H^{o} \longrightarrow A'(f'_{M})$ is the obvious one given by the systems of generators.

This reduces the problem to the case where $V \in X(L)$.

(a) can be deduced from the proofs of I.3.14 and I.3.16. (b) can be checked easily if we use the following facts. In 5.27 (a), $u \in Z_{o}(f) \setminus Z(f)^{o}$ if $n \geq 1$ and in 5.27 (d) an element $z \in Z_{o}(f)$ is in $Z(f)^{o}$ if and only if it acts trivially on Y_1 (with the notations of 5.27). In the different cases of 5.27 it is then easy to identify the homomorphism $H/H^{o} \longrightarrow A'(f')$ and to see that it is given by (b).

This gives a complete description of the homomorphism $Z_{L,M}/Z^{o}_{L,M} \longrightarrow A'(f')$ and it clearly follows from this description that A'_{P} and the homomorphism $A_{P} \longrightarrow A'(f')/A'_{P}$ are as descibed in 5.23. The proof of 5.23 in cases (II) and (IV) is now complete.

<u>5.34</u>. Using 4.10, 4.11 and 5.23 it is possible to get by induction on n a description of S(u) and of the action of A(u) on S(u).($u \in G$ unipotent). We give now a combinatorial description of S(u) and the action of A(u) on S(u) which is closer to the parametrization by standard tableaux obtained for GL_n.

Consider a flag $F \in \mathcal{F}$ fixed by u (or corresponding to a Borel subgroup fixed by u if we are in case (IX)). Let $G_i(F)$ be the following group $(0 \leq i \leq n)$:

a) In case (II) or (IV), $G_i(F) = G(F_{N-n+i}/F_{n-i})$ (defined as in I.3.1).

b) In case (III), (V), (VIII) or (IX), $G_i(F) = O(F_{N-n+i}/F_{n-i})$ (with respect to the bilinear form (resp. quadratic form) induced by f (resp. Q) in cases (III) and (V) (resp. (VIII) and (IX))).

c) In case (VI) or (VII), $G_i(F) = Sp(F_{N-n+i}/F_{n-i})$ (with respect to the bilinear form induced by f).

u has an image u_i in $G_i(F)$. We get then a unipotent class C_i in $G_i(F)$, hence a pair $(\lambda^i, \varepsilon^i)$ (depending on F). In this way we attach to each $B' \in \mathcal{B}_u$ (and to each $F \in \mathcal{F}_u$ if we are not in case (IX)) a sequence $(\lambda^o, \varepsilon^o)$, $(\lambda^1, \varepsilon^1)$, ..., $(\lambda^n, \varepsilon^n)$. Clearly $(\lambda^n, \varepsilon^n) = (\lambda, \varepsilon)$. Let D be the set of all sequences obtained in this way. We have defined an application π : $\mathcal{B}_u \longrightarrow D$. For each $d \in D$, let $X_d = \pi^{-1}(d)$. X_d is Z(u)-stable and it follows easily from 4.7 that it is locally closed in $\mathcal{B}_u$. 4.12 shows by induction on n that all components of X_d have the same dimension and are disjoint. Let $D^* = \{d \in D | \dim X_d = \dim \mathcal{B}_u\}$. If $d \in D^*$, the closure of any irreducible component of X_d is an irreducible component of $\mathcal{B}_u$ and every irreducible component of $\mathcal{B}_u$ is of this form (for a unique $d \in D^*$ and a unique irreducible component of X_d). This gives a partition $S(u) = \bigcup_{d \in D^*} S_d$. Each S_d is stable for the action of A(u) since X_d is stable for Z(u).

The sequence $(\lambda^o, \varepsilon^o)$, $(\lambda^1, \varepsilon^1)$, ..., $(\lambda^n, \varepsilon^n)$ is in D* if and only if $(\lambda^{i-1}, \varepsilon^{i-1})$ is deduced from $(\lambda^i, \varepsilon^i)$ by one of the operations described in 5.23 for all i ($1 \leq i \leq n$ in cases (II) to (VIII), $2 \leq i \leq n$ in case (IX)). Hence D* can be described in a combinatorial way. We want to get also a combinatorial description of S_d and the action of A(u) on S_d ($d \in D^*$).

Consider now a fixed $d \in D^*$, i.e. a sequence $(\lambda^o, \varepsilon^o)$, $(\lambda^1, \varepsilon^1)$, ..., $(\lambda^n, \varepsilon^n)$. If N is even (resp. odd) let d_o, d_2, ..., d_N (resp. d_1, d_3, ..., d_N) be the Young diagrams corresponding to λ^o, λ^1, ..., λ^n. If d_j is one of these diagrams ($j \geq 2$), d_{j-2} is obtained from d_j by removing a 'box' consisting of two adjacent squares. Call this box the box i if j = 2i or 2i+1. If N is even, d_N consists then of n boxes labelled 1, 2, ..., n. If N is odd, d_N consists of one square for d_1 and n boxes labelled 1, 2, ..., n.

Suppose that the image of $F \in \mathcal{F}$ in $\mathcal{B}$ is in X_d. In I.1.13 and I.3.23 we use a subset of $\{a_o, a_1, \ldots\}$ to describe $A(u_i)$ (u_i the image of u in $G_i(F)$). We shall say that the box i is special if the subset of $\{a_o, a_1, \ldots\}$ associated to $(\lambda^i, \varepsilon^i)$ in this way is strictly larger than the subset associated to $(\lambda^{i-1}, \varepsilon^{i-1})$. In other words :

a) if $p \neq 2$, the box i is special if it consists of two squares in the column h (some h), $\varepsilon^i_h = 1$ and $\varepsilon^{i-1}_{h-1} = \omega$.

a partition $S(u) = \bigcup_{d \in D^*} S_d$. Each S_d is stable for the action of $A(u)$ since X_d is stable for $Z(u)$.

The sequence $(\lambda^o, \varepsilon^o)$, $(\lambda^1, \varepsilon^1)$, ..., $(\lambda^n, \varepsilon^n)$ is in D^* if and only if $(\lambda^{i-1}, \varepsilon^{i-1})$ is deduced from $(\lambda^i, \varepsilon^i)$ by one of the operations described in 5.23 for all i ($1 \leq i \leq n$ in cases (II) to (VIII), $2 \leq i \leq n$ in case (IX)). Hence D^* can be described in a combinatorial way. We want to get also a combinatorial description of S_d and the action of $A(u)$ on S_d ($d \in D^*$).

Consider now a fixed $d \in D^*$, i.e. a sequence $(\lambda^o, \varepsilon^o)$, $(\lambda^1, \varepsilon^1)$, ..., $(\lambda^n, \varepsilon^n)$. If N is even (resp. odd) let d_o, d_2, ..., d_N (resp. d_1, d_3, ..., d_N) be the Young diagrams corresponding to λ^o, λ^1, ..., λ^n. If d_j is one of these diagrams ($j \geq 2$), d_{j-2} is obtained from d_j by removing a 'box' consisting of two adjacent squares. Call this box the box i if j = 2i or 2i+1. If N is even, d_N consists then of n boxes labelled 1, 2, ..., n. If N is odd, d_N consists of one square for d_1 and n boxes labelled 1, 2, ..., n.

Suppose that the image of $F \in \mathcal{F}$ in $\mathcal{B}$ is in X_d. In I.1.13 and I.3.23 we use a subset of $\{a_o, a_1, \ldots\}$ to describe $A(u_i)$ (u_i the image of u in $G_i(F)$). We shall say that the box i is special if the subset of $\{a_o, a_1, \ldots\}$ associated to $(\lambda^i, \varepsilon^i)$ in this way is strictly larger than the subset associated to $(\lambda^{i-1}, \varepsilon^{i-1})$. In other words :

a) if $p \neq 2$, the box i is special if it consists of two squares in the column h (some h), $\varepsilon_h^i = 1$ and $\varepsilon_{h-1}^{i-1} = \omega$.

b) if $p = 2$, the box i is special if it consists of two squares in the column h (some h), $\varepsilon^{i}_{h} = \omega$ and $\varepsilon^{i-1}_{h-1} = 0$.

We take the special boxes as a basis for a vector space $\tilde{S}_d$ over $\mathbb{Z}/2\mathbb{Z}$. We consider $\tilde{S}_d$ as an abelian group. An element of $\tilde{S}_d$ will be represented by the diagram d_N with signs + or - in the special boxes.

If $p \neq 2$ (resp. $p = 2$) let $\tilde{A}$ be the abelian group generated by $\{a_i | \varepsilon_{\lambda_i} = 1\}$ (resp. $\{a_i | \varepsilon_{\lambda_i} \neq 0\}$) with the relations $a_i^2 = 1$ for all generators and $a_i = 1$ if $\lambda_i = 0$. The description of $A(u)$ given in I.1.13 and I.1.23 gives a homomorphism $\tilde{A} \longrightarrow A(u)$. This homomorphism is surjective if we are not in case (II) or (IV). If we are in case (II) or (IV) its image is $A_o(u)$. This will be sufficient since in this case $A(u)$ is generated by $A_o(u)$ and a_o, and a_o acts trivially on $S(u)$.

Let φ_d be the homomorphism $\tilde{A} \longrightarrow \tilde{S}_d$ such that $\varphi_d(a_i)$ is the diagram d_N with signs - in the special boxes meeting the line i and with signs + in the other special boxes (for all a_i in the system of generators of $\tilde{A}$). This gives an action of $\tilde{A}$ on $\tilde{S}_d$: $\tilde{A} \times \tilde{S}_d \longrightarrow \tilde{S}_d$, $(a,s) \longrightarrow \varphi_d(a)s$.

Let d' be the sequence $(\lambda^o, \varepsilon^o)$, $(\lambda^1, \varepsilon^1)$, ..., $(\lambda^{n-1}, \varepsilon^{n-1})$. We get in a similar way a group $\tilde{S}_{d'}$. An element of $\tilde{S}_{d'}$ consists of the diagram d_{N-2} with signs + or - in the special boxes. By adding to the elements of $\tilde{S}_{d'}$ a box with a sign + if the box n is special or an empty box if the box n is not special, we make $\tilde{S}_{d'}$ into a subgroup of $\tilde{S}_d$.

We define now a subgroup R_d of $\tilde{S}_d$. If $n \geqslant 2$ we may assume by induction on n that a subgroup $R_{d'}$ of $\tilde{S}_{d'} \subset \tilde{S}_d$ has already been defined. If $n = 1$ we define $R_{d'} = \tilde{S}_d$ in cases (V) and (VIII) and $R_{d'} = 1$ in the other cases. Then $R_d = R_{d'}\varphi_d(K_d)$, where K_d is the kernel of the homomorphism $\tilde{A} \longrightarrow A(u)$.

If we are not in case (II) or (IV), the action of $\tilde{A}$ on $\tilde{S}_d$ induces an action of $A(u)$ on $\tilde{S}_d/R_d$. If we are in case (II) or (IV) we get an action of $A_o(u)$ on $\tilde{S}_d/R_d$ but this action can be extended to an action of $A(u)$ by saying that a_o acts trivially provided that $a_o^2 \in A_o(u)$ acts trivially.

<u>Proposition 5.35</u>. In the situation of 5.34 there exists a family $(f_d)_{d\in D^*}$ of $A(u)$-equivariant bijections f_d : $\tilde{S}_d/R_d \longrightarrow S_d$. This family can be choosen in such a way that the resulting $A(u)$-equivariant bijection $f : \bigcup_{d\in D^*} \tilde{S}_d/R_d \longrightarrow S(u)$ is well-defined up to composition with a bijection f_a : $S(u) \longrightarrow S(u)$, $\sigma \longmapsto a\sigma$ $(a \in A(u))$.

<u>Proof</u>. Using 4.12, 5.23 and the definitions of $\tilde{S}_d$ and R_d, it is possible to check by induction on n that there exists an equivariant bijection $f_d : \tilde{S}_d/R_d \longrightarrow S_d$. If we are in case (II) or (IV) it is sufficient to consider the action of $A_o(u)$ in the proof.

What we want to do is to choose this bijection in the best possible way. This will be especially important in cases (V) and (VIII). For this reason we give the construction of f_d for these cases. In the other cases the method is similar.

So assume that we are in case (V) or (VIII). For each $j \geq 1$ let V_j be a vector space over k of dimension n_j, where n

$$n_j = \begin{cases} j & \text{if } \varepsilon_j = 1 \\ j+1 & \text{if } \varepsilon_j = \omega \\ j+2 & \text{if } _j = 0. \end{cases}$$

If we are in case (V) (resp. (VIII)), we choose a bilinear form f_j (resp. a quadratic form Q_j with associated bilinear form f_j) such that f_j is symmetric and non-degenerate. We get an orthogonal group $O(V_j)$. Choose a unipotent element $u_j \in O(V_j)$ such that u_j has a single Jordan block. u_j is regular. Choose now a basis $e_1^j,\dots,e_{n_j}^j$ for V_j such that $(u_j-1)(e_r^j) = e_{r-1}^j$ $(1 \leq r \leq n_j$, with $e_o^j = 0)$, and such that :

a) in case (V), $f_j(e_r,e_r) = 1$ if $2r = n_j+1$ and $f_j(e_r,e_s) = 0$ if $(n_j+1)/2 \leqslant r \leqslant n_j$, $(n_j+1)/2 < s < n_j$.

b) in case (VIII), $f_j(e_1,e_{n_j}) = 1$, $f_j(e_r,e_s) = 0$ if $n_j/2 < r,s \leq n_j$ and $Q_j(e_{n_j}) = 0$.

Let also $\bar{a}_j$ be the following element of $Z_{O(V_j)}(u_j)$. In case (V), $\bar{a}_j = -1$. In case (VIII), $\bar{a}_j = u_j$.

For every $h \geqslant 1$, let $J_h = \{j \in \mathbb{N} \mid \lambda_j = h\}$ if $\varepsilon_h = 1$ and $J_h = \{j \in \mathbb{N} \mid \lambda_j = h \text{ and } j-\ell_{h+1} \text{ is even}\}$ if $\varepsilon_h \neq 1$. Let $J = \bigcup_{h \geq 1} J_h$. We define for each $j \in J$ a vector space M_j with a bilinear form (in case (V)) or a quadratic form (in case (VIII)) and a unipotent element $v_j \in O(M_j)$.

a) if $\varepsilon_h = 1$, $M_j = V_j$ with f_j or Q_j and $v_j = u_j$.

b) if $\varepsilon_h = \omega$, $M_j = L^{\perp}/L$, where L is the subspace of $V_j \oplus V_{j+1}$ generated by $e_1^j + e_1^{j+1}$. We use on $V_j \oplus V_{j+1}$ the bilinear form or quadratic form which extends the given forms and for which V_j and V_{j+1} are orthogonal. We get also an element $u_j \oplus u_{j+1} \in O(V_j \oplus V_{j+1})$. The bilinear form or quadratic form on M_j is induced by this bilinear or quadratic form on $V_j \oplus V_{j+1}$ and v_j is induced by $u_j \oplus u_{j+1}$.

c) if $\varepsilon_h = 0$, $M_j = N^{\perp}/N$, where N is the subspace of $V_j \oplus V_{j+1}$ generated by $e_1^j + e_1^{j+1}$ and $e_2^j + e_2^{j+1}$. The bilinear form or quadratic form on M_j and $v_j \in O(M_j)$ are defined as in (b).

Let now $\overline{V} = \bigoplus_{j \in J} M_j$. It is easily seen that $\overline{V}$ inherits a bilinear form (resp. a quadratic form) if we are in case (V) (resp. in case (VIII)) which defines an orthogonal group $O(\overline{V})$ and $(v_j)_{j \in J}$ gives a unipotent element $\bar{u} \in O(\overline{V})$. The conjugacy class of $\bar{u}$ is parametrized by (λ, ε). There exists therefore an isomorphism θ: $V \longrightarrow \overline{V}$ such that $\theta \circ u = \bar{u} \circ \theta$ and such that the bilinear forms or quadratic forms on V and $\overline{V}$ correspond via θ. Moreover θ is unique up to composition with an automorphism $g \in Z(u)$ of V. In order to prove the proposition, it is sufficient to show that if such a θ is given, then we have a procedure to define the bijections f_d $(d \in D^*)$.

We use θ to identify V and $\overline{V}$.

If we are in case (V) (resp. in case (VIII)) and $\varepsilon_{\lambda_j} = 1$ (resp. $\varepsilon_{\lambda_j} \neq 0$), the automorphism of $\bigoplus_{j \in J} V_m$ defined by $v \longmapsto v$

if $v \in V_m$ $(m \neq j)$ and $v \longmapsto \bar{a}_j(v)$ if $v \in V_j$ induces an automorphism of V which is actually an element of Z(u). We denote also by $\bar{a}_j$ this element of Z(u). Notice that the image of $\bar{a}_j$ in A(u) is a_j.

Consider now a fixed $d \in D^*$. Let $(\lambda^o,\varepsilon^o)$, $(\lambda^1,\varepsilon^1)$, ..., $(\lambda^n,\varepsilon^n)$ be this sequence and let d' be the sequence $(\lambda^o,\varepsilon^o)$, $(\lambda^1,\varepsilon^1)$, ..., $(\lambda^{n-1},\varepsilon^{n-1})$. We write also (λ',ε') for $(\lambda^{n-1}, \varepsilon^{n-1})$. We define now by induction on n a family $(F_s)_{s \in \tilde{S}_d}$ of elements of $\mathcal{F}_u$. There will be several cases. In case (1) the box n consists of two squares of the line j, in the column h and h-1. In the other cases the box n consists of two squares of the column h, in the lines j and j+1. Suppose first that $s \in \tilde{S}_{d'} \subset \tilde{S}_d$. Then in each case F_s will be a flag $F = (F_o, F_1, \ldots, F_{2n})$ defined as follows. We fix F_1 in each case. Then Θ induces an isomorphism $\Theta' : F_1^{\perp}/F_1 = V' \cong \bar{V}' = \bigoplus_{m \in J'} M'_m$. Since $s \in \tilde{S}_{d'}$ we get by induction a flag of $F_1^{\perp}/F_1$. This defines the flag $F_s \in \mathcal{F}_u$. To get Θ' we start with vector spaces V'_m. We take $V'_m = V_m$ except in the following cases. In cases (1) to (3) V'_j is the subspace of V_j/ke_1^j generated by $e_2^j+ke_1^j$, ..., $e_{h-1}^j+ke_1^j$ and we take these vectors as a basis. In cases (2) and (3) V'_{j+1} is the subspace of V_{j+1}/ke_1^{j+1} generated by $e_2^{j+1}+ke_1^{j+1}$, ..., $e_{h-1}^{j+1}+ke_1^{j+1}$ and we take these vectors as a basis. The remaining data to get Θ' are obtained in a similar way. If $s \in \tilde{S}_{d'}$ and if the box n is as specified above for the different cases, we choose F_1 as follows.

1) $F_1 = ke_1^j$.

2) $\varepsilon_h = 0$. Then $F_1 = (ke_1^j \oplus ke_1^{j+1} \oplus k(e_2^j + e_2^{j+1}))/N$.

3) $\varepsilon_h = \omega$ and $\varepsilon'_{h-1} = 1$. Then $F_1 = (ke_1^j \oplus ke_1^{j+1})/L$.

4) $\varepsilon_h = 1$. Then $F_1 = L$.

5) $\varepsilon_h = \omega$ and $\varepsilon'_{h-1} = 0$. Then $F_1 = N/L$.

We have used the same notations as in the definition of the vector spaces M_m, $m \in J$. $L = k(e_1^j + ie_1^{j+1})$ $(i \in k,\ i^2 = -1)$ and $N = k(e_1^j + e_1^{j+1}) \oplus k(e_2^j + e_2^{j+1})$.

In cases (1), (2) and (3), and in case (4) if $p = 2$, $\tilde{S}_{d'} = \tilde{S}_d$ and therefore the family $(F_s)_{s \in \tilde{S}_d}$ is completely defined. In case (5), and in case (4) if $p \neq 2$ (i.e. if the box n is special), $\tilde{S}_{d'} \neq \tilde{S}_d$. If $s \in \tilde{S}_d \setminus \tilde{S}_{d'}$, then we define $F_s = \bar{a}_j F_{a_j s}$. This makes sense since $a_j s \in \tilde{S}_{d'}$.

The family $(F_s)_{s \in \tilde{S}_d}$ has the following property. For every $a_i \in \tilde{A}$ and every $s \in \tilde{S}_d$, $\bar{a}_i F_s = F_{a_i s}$. This can be proved by induction on n. There are many cases to consider. For example if the box n corresponds to the case (5) above, there is a box consisting of two squares of the column h-1, in the lines j and j+1. This can be used to prove that $\bar{a}_j \bar{a}_{j+1} F_s = F_{a_j a_{j+1} s}$. The complete proof of this property is omitted.

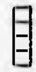
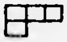
By construction the image of F_s in $\mathcal{B}_u$ is contained in X_d. Since all irreducible components of X_d are disjoint, and since the closure of any irreducible component of X_d is an irreducible component of $\mathcal{B}_u$ (an element of S_d by definition of S_d), we get an application $\tilde{f}_d : \tilde{S}_d \longrightarrow S_d$. The property of

$(F_s)_{s \in \tilde{S}_d}$ mentionned above shows that f_d is $\tilde{A}$-equivariant. 5.23 shows by induction on n that f_d induces an application $f_d : \tilde{S}_d/R_d \longrightarrow S_d$ and that f_d is bijective. Moreover f_d is A(u)-equivariant since f_d is $\tilde{A}$-equivariant.

<u>Remarks 5.36</u>. a) In the proof of 5.35 the image of the flag F_s in $\mathcal{B}_u$ is contained in a unique irreducible component of X_d, but it may be contained in several components of $\mathcal{B}_u$.
b) It follows from 5.35 that for every $d \in D^*$ there is a well-defined bijection $\tilde{S}_d/R_d \varphi_d(\tilde{A}) \longrightarrow S_d/A(u)$. This makes the set of all A(u)-orbits contained in S_d into a group. Suppose that $P \ni u$ is a parabolic subgroup of G such that uU_P is quasisemisimple in P/U_P and $\mathcal{B}(P)_u = X_\sigma$ is an irreducible component of $\mathcal{B}_u$. Suppose also that $\sigma \in S_d$. It will be clear from later computations that the A(u)-orbit of σ is the identity element in $S_d/A(u)$. I don't know in general if the group structure of $S_d/A(u)$ has a geometric meaning.

<u>5.37</u>. In 4.3 we have attached to each $\sigma \in S(u)$ a subset I_σ of Π. I_σ is u-stable.

Suppose first that n = 1. If we are in case (II) or (III), $I_\sigma = o(\alpha_1)$ if $d_3 = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$, and $I_\sigma = \emptyset$ if $d_3 = \square\square\square$. If we are in case (IV) $d_2 = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$, and $I_\sigma = \{\alpha_1\}$ if $\varepsilon_1 = 0$, $I_\sigma = \emptyset$ if $\varepsilon_1 = 1$. If we are in case (VI) or (VII), $I_\sigma = \{\alpha_1\}$ if $d_2 = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$ and $I_\sigma = \emptyset$ if $d_2 = \square\square$.

Suppose now that n = 2. If we are in case (V) (resp. in case (VIII)), $I_\sigma = \{\alpha_1, \alpha_2\}$ if $d_4 =$ , $I_\sigma = \{\alpha_1\}$ or $\{\alpha_2\}$ if $d_4 =$  (resp. $d_4 =$  and $\varepsilon_2 = 0$) and $I_\sigma = \emptyset$ if $d_4 =$  (resp. $d_4 =$  and $\varepsilon_2 = 1$). If we are in case (IX), $I_\sigma = o(\alpha_1) = \{\alpha_1, \alpha_2\}$ if $d_4 =$  and $I_\sigma = \emptyset$ if $d_4 =$ .

If $n \geq 2$ we shall say that (for a given $d \in D^*$) the box n-1 is above (resp. below) the box n if for some j the box n-1 (resp. n) is contained in the union of the lines 1,2,...,j and the box n (resp. n-1) is contained in the union of the lines j+1,j+2,... .

Proposition 5.38. Suppose that we are in the situation of 5.37. Suppose also that $n \geq 2$, and $n \geq 3$ if we are in case (V), (VIII) or (IX). Let $(\lambda^0, \varepsilon^0)$, ..., $(\lambda^n, \varepsilon^n)$ be a sequence $d \in D^*$ and let $\sigma \in S_d$. Then $\alpha_n \in I_\sigma$ (resp. $\alpha_n \notin I_\sigma$) if the box n-1 is above (resp. below) the box n. If the box n-1 is neither above nor below the box n, then $\alpha_n \in I_\sigma$ if for some h and j the box n consists of two squares in the column h, in the lines j and j+1, the box n-1 consists of two squares in the column h, in the lines j and j+1, and one of the following conditions holds.

a) $\varepsilon^n_h = \varepsilon^{n-2}_{h-2}$, the box n-1 is special and contains a sign + for some (or every) $s \in \tilde{S}_d$ such that $f_d(s) = \sigma$.

b) $p = 2$ and $\varepsilon^{n-1}_{h-1} = 1$.

Proof. The proof is similar to the proof of 5.11 (in particular if the box n-1 is above or below the box n) but with more cases and is omitted.

5.39. 4.8 and 5.38 give an inductive method to determine I_σ if $S(u)$ is described as in 5.35. We have however to be careful if we are in case (V) or (VIII) because G/G^o permutes α_1 and α_2 which are not in the same u-orbit.

Suppose that we are in case (V) or (VIII). Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ has two components and we can define lines of type α_1 in $\mathcal{F}_1$ and lines of type α_2 in $\mathcal{F}_2$ (this is clear from 5.1).

Consider a fixed $d \in D^*$ such that $d_{\lambda^2} = \boxplus$ and $\varepsilon_2^2 \neq 1$ (where d is the sequence $(\lambda^o, \varepsilon^o), \ldots, (\lambda^n, \varepsilon^n)$). Then for each $\sigma \in S_d$, $\alpha_1 \in I_\sigma$ or $\alpha_2 \in I_\sigma$, and $\{\alpha_1, \alpha_2\} \not\subset I_\sigma$. In order to determine which possibility occurs, we need only to solve the following problems.

a) For which $s \in \tilde{S}_d$ is there a line of type α_1 or α_2 through F_s contained in $\mathcal{F}_u$?

b) For each $s \in \tilde{S}_d$, which is the component of $\mathcal{F}$ containing F_s ?

The answer to (a) can be obtained by induction on n. If $n = 2$, there is a line of type α_1 or α_2 through F_s contained in $\mathcal{F}_u$ if and only if $s = \boxed{+}\boxed{}$. If $n \geqslant 3$, we just have to look at the construction of the family $(F_s)_{s \in \tilde{S}_d}$ in the proof of 5.35 and use induction on n.

For (b), notice first that if $F = (F_o, F_1, \ldots, F_{2n}) \in \mathcal{F}$, then $\mathcal{F}' = \{F' = (F'_o, \ldots, F'_{2n}) \in \mathcal{F} \mid F'_1 = F_1\}$ has two components (one in $\mathcal{F}_1$ and one in $\mathcal{F}_2$) and therefore F and F' are in the same component of $\mathcal{F}$ if and only if they are in the same component of $\mathcal{F}'$. We can therefore determine by induction on n which component of $\mathcal{F}$ contains F_s if we can solve the same problem with $s = 1$.

For every $d \in D^*$ let F_d be the element of $(F_s)_{s \in \tilde{S}_d}$ indexed by $1 \in \tilde{S}_d$. We show now that the flags $(F_d)_{d \in D^*}$ belong to the same component of $\mathcal{F}$. If $A(u) \neq 1$, this component depends on the choice of θ in the proof of 5.35. If $A(u) = 1$ this component is completely determined by u.

Suppose that $d, e \in D^*$ and let $F_d = (F_o, F_1, \ldots, F_{2n})$ and $F_e = (F'_o, F'_1, \ldots, F'_{2n})$. We have to show that F_d and F_e belong to the same component of $\mathcal{F}$. This is clear by induction on n if $F_1 = F'_1$. If $F_1 \neq F'_1$, then we can find $\bar{d}, \bar{e} \in D^*$ such that (with $F_{\bar{d}} = (\bar{F}_o, \bar{F}_1, \ldots, \bar{F}_{2n})$ and $F_{\bar{e}} = (\bar{F}'_o, \bar{F}'_1, \ldots, \bar{F}'_{2n})$) $\bar{F}_1 = F_1$, $\bar{F}'_1 = F'_1$, $\bar{F}_2 = \bar{F}'_2 = F_1 \oplus F'_1$ and $\bar{F}_i = \bar{F}'_i$ if $i = 1, 2n-1$ (this can be proved by considering the different possibilities for the box n in d and e and by using the definition of F_d). F_d and $F_{\bar{d}}$ are in the same component of $\mathcal{F}$ and F_e and $F_{\bar{e}}$ are in the same component of $\mathcal{F}$. We need only to prove that $F_{\bar{d}}$ and $F_{\bar{e}}$ are in the same component of $\mathcal{F}$. It is easy to check that in all possible cases $\{(\bar{F}_o, L, \bar{F}_2, \ldots, \bar{F}_{2n-2}, L, \bar{F}_{2n}) \in \mathcal{F}\}$ is contained in $\mathcal{F}_u$ and contains $F_{\bar{d}}$ and $F_{\bar{e}}$. This implies that $F_{\bar{d}}$ and $F_{\bar{e}}$ (and hence F_d and F_e) are contained in the same component of $\mathcal{F}$.

We have also shown that if $C(u) \neq C^o(u)$ (in case (V) or

(VIII)), then we can attach to u or $C^o(u)$ a component of $\mathcal{F}$, (or an element of $\{\alpha_1,\alpha_2\}$) in a canonical way. We take the component of $\mathcal{F}$ containing the flags F_d ($d \in D^*$).

Lemma 5.40. Suppose that we are in one of the cases (I) to (IX). If a unipotent element $u \in G$ is such that $\dim \mathcal{B}_u \geqslant 3$, then $|I_\sigma| \geqslant 2$ for some $\sigma \in S(u)$.

Proof. This can be checked easily in each case. In case (I) we use 5.6 and 5.12. In the other cases we use 5.19 and 5.38.

6. $\mathcal{P}$-regular classes.

6.1. In this paragraph we consider the same cases as in paragraph 5 and we use the notations of 5.1. We use also the notations of 3.1. The problem we consider here is the following. Let $P \supset B$ be a parabolic subgroup of G^o and let I be the corresponding subset of Π. Determine λ (in case (I)) or (λ, ε) (in cases (II) to (IX)) such that $C_P = C_\lambda$ (in case (I)) or $C_P = C_{\lambda,\varepsilon}$ (in cases (II) to (IX)).

The subset $I \subset \Pi$ will be described as follows.

a) In case (I), I is characterized by integers $n_1,\dots,n_s$ such that $\sum_{1\leq r\leq s} n_r = n$ and $I = \bigcup_{1\leq r\leq s}\{\alpha_i \mid n_1+\dots+n_{r-1} < i < n_1+\dots+n_r\}$.

b) In cases (II), (III), (IV), (VI) and (VII), I is described by integers m, $n_1,\dots,n_s$ such that $m+n_1+\dots+n_s = n$ and $I = (\bigcup_{1\leq r\leq s}\{o(\alpha_i) \mid m+n_1+\dots+n_{r-1}+2 \leq i \leq m+n_1+\dots n_r\}) \cup \{o(\alpha_i) \mid 1 \leq i \leq m\}$.

c) In cases (V) and (VIII), notice first that every G-conjugacy class of parabolic subgroups of G^o contains a parabolic subgroup $P' \supset B$ corresponding to a subset I' of Π such that $I' \cap \{\alpha_1,\alpha_2\} \neq \{\alpha_1\}$. So we may assume that $I \cap \{\alpha_1,\alpha_2\} \neq \{\alpha_1\}$. Then I will be characterized by integers m, $n_1,\dots,n_s$ such that $m+n_1+\dots+n_s = n$ and $I = (\bigcup_{1\leq r\leq s}\{\alpha_i \mid m+n_1+\dots+n_{r-1}+2 \leq i \leq m+n_1+\dots+n_r\}) \cup \{\alpha_i \mid 1 \leq i \leq m\}$.

d) In case (IX), if $o(\alpha_1) \subset I$, I will be characterized by integers m, $n_1,\dots,n_s$ such that $m+n_1+\dots+n_s = n-1$ and $I = (\bigcup_{1\leq r\leq s}\{\alpha_i \mid m+n_1+\dots+n_{r-1}+3 \leq i \leq m+n_1+\dots+n_r+1\}) \cup \{\alpha_i \mid 1 \leq i \leq m+1\}$. If $o(\alpha_1) \not\subset I$, then I will be characterized by integers m, n_1,

$\dots,n_s$ with $m = 0$, $m+n_1+\dots+n_s = n-1$ and $I = \bigcup_{1\leq r\leq s}\{\alpha_i \mid m+n_1+\dots+n_{r-1} \leq i \leq m+n_1+\dots+n_r+1\}$.

In each case let $n_1',\dots,n_s'$ be the integers $n_1,\dots,n_s$ arranged in decreasing order (i.e. for some permutation σ of $\{1,\dots,s\}$, $n_i' = n_{\sigma_i}$ for all i, $1 \leq i \leq s$, and $n_1' \geq n_2' \geq \dots \geq n_s'$).

Define also $E_j = \{i \mid n_i' = j\}$ $(j \geq 1)$.

<u>Proposition 6.2</u>. In the situation of 6.1 the Φ-regular class is the unipotent class parametrized by λ (in case (I)) or (λ,ε) (in cases (II) to (IX)), where λ or (λ,ε) is defined as follows.

a) In case (I), $\ell_i = n_i'$ for all i, $1 \leq i \leq s$.

b) In cases (II) and (III), d_λ satisfies :

b_1) if $j \leq 2m+1$, then $\ell_{2i} = \ell_{2i+1} = j$ if $i \in E_j$.

b_2) if $j > 2m+1$ is odd, then $\ell_{2i-1} = \ell_{2i} = j$ if $i \in E_j$.

b_3) if $j > 2m+1$ is even and $E_j = \{p,p+1,\dots,q\}$, then $\ell_{2p-1} = j+1$, $\ell_{2p} = \ell_{2p+1} = \dots = \ell_{2q-1} = j$, $\ell_{2q} = j-1$.

b_4) if $n_p' > 2m+1$ and $n_{p+1}' \leq 2m+1$, then $\ell_{2p+1} = 2m+1$.

c) In case (IV), d_λ and ε satisfy :

c_1) if $j \leq 2m$ is odd and $E_j = \{p,p+1,\dots,q\}$, then $\ell_{2p} = j+1$, $\ell_{2q+1} = j-1$ and if $p \neq q$, $\ell_{2p+1} = j+1$, $\ell_{2p+2} = \dots = \ell_{2q-1} = j$, $\ell_{2q} = j-1$. Moreover $\varepsilon_{2q+1} = 0$ if $n_{q+1}' \leq j-3$.

c_2) if $j \leq 2m$ is even and $i \in E_j$, then $\ell_{2i} = \ell_{2i+1} = j$. Moreover $\varepsilon_{2i+1} = 0$ if $n_{i+1}' \leq j-2$.

c_3) if $j > 2m$ is odd and $E_j = \{p,p+1,\dots,q\}$, then $\ell_{2p-1} = j+1$, $\ell_{2p} = \dots = \ell_{2q-1} = j$, $\ell_{2q} = j-1$.

c_4) if $j > 2m$ is even and $i \in E_j$, then $\ell_{2i-1} = \ell_{2i} = j$.

c_5) if $n'_p > 2m$ and $n'_{p+1} \leqslant 2m$, then $\ell_{2p+1} = 2m$. Moreover $\varepsilon_{2p+1} = 0$ if n'_{p+1} $2m-2$.

d) In case (V), d_λ satisfies :

d_1) if $j \leqslant 2m$ and $i \in E_j$, then $\ell_{2i} = \ell_{2i+1} = j$.

d_2) if $j > 2m$ is odd and $E_j = \{p, p+1, \ldots, q\}$, then $\ell_{2p-1} = j+1$, $\ell_{2p} = \ldots = \ell_{2q-1} = j$, $\ell_{2q} = j-1$.

d_3) if $j > 2m$ is even and $i \in E_j$, then $\ell_{2i-1} = \ell_{2i} = j$.

d_4) if $n'_p > 2m$ and $n'_{p+1} \leqslant 2m$, then $\ell_{2p+1} = 2m$.

e) In cases (VI) and (VII), d_λ satisfies :

e_1) if $j < 2m$ is odd and $E_j = \{p, p+1, \ldots, q\}$, then $\ell_{2p} = j+1$, $\ell_{2p+1} = \ldots = \ell_{2q} = j$, $\ell_{2q+1} = j-1$.

e_2) if $j < 2m$ is even and $i \in E_j$, then $\ell_{2i} = \ell_{2i+1} = j$.

e_3) if $j \geqslant 2m$ and $i \in E_j$, then $\ell_{2i-1} = \ell_{2i} = j$.

e_4) if $n'_p \geqslant 2m$ and $n'_{p+1} < 2m$, then $\ell_{2p+1} = 2m$.

f) In case (VIII), d_λ and ε satisfy :

f_1) if $j \leqslant 2m$ is odd and $E_j = \{p, p+1, \ldots, q\}$, then $\ell_{2p} = j+1$, $\ell_{2p+1} = \ldots = \ell_{2q} = j$, $\ell_{2q+1} = j-1$.

f_2) if $j \leqslant 2m$ is even and $i \in E_j$, then $\ell_{2i} = \ell_{2i+1} = j$.

f_3) if $j > 2m$ is odd and $E_j = \{p, p+1, \ldots, q\}$, then $\ell_{2p-1} = j+1$, $\ell_{2q} = j-1$ and if $p \neq q$, $\ell_{2p} = j+1$, $\ell_{2p+1} = \ldots = \ell_{2q-2} = j$, $\ell_{2q-1} = j-1$. Moreover $\varepsilon_{2q} = 0$ if $j \geqslant 2m+3$ and $n'_{q+1} \leqslant j-3$.

f_4) if $j > 2m$ is even and $i \in E_j$, then $\ell_{2i-1} = \ell_{2i} = j$. Moreover $\varepsilon_{2i} = 0$ if $n'_{i+1} \leqslant j-2$.

f_5) if $n'_p > 2m$ and $n'_{p+1} \leqslant 2m$, then $\ell_{2p+1} = 2m$.

g) In case (IX), d_λ and ε satisfy :

g_1) if $j \leq 2m+1$ is odd and $i \in E_j$, then $\ell_{2i} = \ell_{2i+1} = j$.

g_2) if $j \leq 2m+1$ is even and $E_j = \{p, p+1, \ldots, q\}$, then $\ell_{2p} = j+1$, $\ell_{2p+1} = \ldots = \ell_{2q} = j$, $\ell_{2q+1} = j-1$.

g_3) if $j > 2m+1$ is odd and $i \in E_j$, then $\ell_{2i-1} = \ell_{2i} = j$. Moreover $\varepsilon_{2i} = 0$ if $n'_{i+1} \leq j-2$.

g_4) if $j > 2m+1$ is even and $E_j = \{p, p+1, \ldots, q\}$, then $\ell_{2p-1} = j+1$, $\ell_{2q} = j-1$ and if $p \neq q$, $\ell_{2p} = j+1$, $\ell_{2p+1} = \ldots = \ell_{2q-2} = j$, $\ell_{2q-1} = j-1$. Moreover $\varepsilon_{2q} = 0$ if $j \geq 2m+4$ and $n'_{q+1} \leq j-3$.

g_5) if $n'_p > 2m+1$ and $n'_{p+1} \leq 2m+1$, then $\ell_{2p+1} = 2m+1$.

g_6) $\ell_{2s+2} = 1$.

In cases (II) to (IX) $\varepsilon_i \neq 0$ unless otherwise stated.

<u>Proof</u>. For the proof we modify the notations as follows. $\mathcal{Q}$ is a G-conjugacy class of parabolic subgroups of G^o corresponding to the subset I of Π and C_λ or $C_{\lambda,\varepsilon}$ is the $\mathcal{Q}$-regular class. P is defined as in 5.22.

If u is $\mathcal{Q}$-regular, $\mathcal{B}_u$ has an irreducible component X_σ of the form $\mathcal{B}(Q)_u$ for some $Q \in \mathcal{Q}$ such that uU_Q is quasisemisimple in $N_G(Q)$. Suppose that we are not in case (I). We can choose (λ', ε') in 5.23 in such a way that $X_\sigma \cap X$ is dense in X_σ. Consider the projection $p : X \longrightarrow Y$ used in 5.23. It is easy to check (from 5.19 or 5.30) that dim Y = $\ell_i - 1$ if operation (b) of 5.23 is used to get (λ', ε') from (λ, ε), or if operation (a) is used with $\varepsilon_i = 0$ or $\varepsilon_i = \omega$ and

$\varepsilon'_{i-1} = 1$, and $\dim Y = \ell_i - 2$ if operation (a) is used with $\varepsilon_i = 1$ or $\varepsilon_i = \omega$ and $\varepsilon'_{i-1} = 0$. It follows easily that if (λ', ε') and $\dim Y$ are given, then (λ, ε) is uniquely determined.

By 4.8 the unipotent class of P/U_P parametrized by (λ', ε') is the $\mathcal{Q}'$-regular class, where $\mathcal{Q}'$ is the conjugacy class of parabolic subgroups of P^o/U_P corresponding to the subset $I \setminus o(\alpha_n)$ of $\Pi \setminus o(\alpha_n)$. By induction on n we may therefore assume that (λ', ε') is given by the proposition.

By 3.3, $\dim \mathcal{B}_u = m^2 + 1/2 \sum_{1 \leq r \leq s} n_r(n_r - 1)$ in cases (II), (III), (IV), (VI), (VII) and (IX) and $\dim \mathcal{B}_u = m(m-1) + 1/2$ $1/2 \sum_{1 \leq r \leq s} n_r(n_r - 1)$ in cases (V) and (VIII) (and $\dim \mathcal{B}_u = 1/2 \sum_{1 \leq r \leq s} n_r(n_r - 1)$ in case (I)). A similar formula gives the dimension of the fibres of $p : X \longrightarrow Y$. Substracting, we get $\dim Y$. For example, if $s \geq 1$, we get $\dim Y = n_s - 1$. Knowing (λ', ε') and $\dim Y$, it is then easy to check in each case that (λ, ε) is given by the proposition.

The proof in case (I) is similar.

6.3. Suppose that we are in case (V) (resp. (VIII)). Let u be a unipotent element such that $C^o(u) \neq C(u)$. This is equivalent to $A(u) = 1$. The elements with this property are those corresponding to pairs (λ, ε) such that $c_i = 0$ if i is odd and c_i is even if i is even (resp. $c_i = 0$ if i is odd, c_i is even if i is even and $\varepsilon_i \neq 1$ for all i). It follows from 6.2 that $C(u)$ is the $\mathcal{P}$-regular class if and only if $\mathcal{P}$ corresponds to a subset I_2 of Π (with $I_2 \cap \{\alpha_1, \alpha_2\} \neq \{\alpha_1\}$)

characterized by the integers $m = 0$ and $n_i' = \ell_{2i}$ $(1 \leq i \leq s)$. Notice that $\ell_1, \ldots, \ell_{2s}$ are even. In particular $\alpha_2 \in I_2$ and $\alpha_1 \notin I_2$ since $m = 0$. Let $I_1 = (I_2 \setminus \{\alpha_2\}) \cup \{\alpha_1\}$. Let P_1 (resp. P_2) be the parabolic subgroup of G^o containing B which corresponds to I_1 (resp. I_2) (with the notations of 6.1 we have $P_2 = P$). Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $C(u) = C_P = C^o_{P_1} \cup C^o_{P_2}$. We show now that $u \in C^o_{P_1}$ if and only if the root $\alpha_i \in \{\alpha_1, \alpha_2\}$ attached to $C^o(u)$ in 5.39 is α_1.

Let X_σ be a component of $\mathcal{B}_u$ of the form $\mathcal{B}(P')$ for some $P' \in \mathcal{P}$. Then $\sigma \in S_d$ for some $d \in D^*$ and $I_\sigma = I_1$ or I_2. For dimension reasons, the $A_o(u)$-orbit of σ in $S(u)$ is $\{\tau \in S(u) \mid I_\tau = I_\sigma\} = \{\tau \in S_d \mid I_\tau \supset I_\sigma\}$. Let τ be the image in S_d of $1 \in \tilde{S}_d$. From 5.38 $I_\tau \supset I_\sigma$. Hence $\sigma = \tau$ since $A_o(u) = 1$. Since σ is the image of $1 \in \tilde{S}_d$, 5.38 and 5.39 show that $I_\sigma = I_1$ (i.e. $P' \in \mathcal{P}_1$) if and only if the fundamental root corresponding to $C^o(u)$ is α_1.

The argument used here works in a more general situation and proves a statement made in 5.36.

<u>6.4</u>. Every unipotent class in a connected reductive group of type A_ℓ is $\mathcal{P}$-regular for some conjugacy class of parabolic subgroups. This is not true in other groups. Suppose that we are in one of the cases (II) to (IX). Then it is possible to deduce from 6.2 a method to determine whether a unipotent class $C_{\lambda,\varepsilon}$ is $\mathcal{P}$-regular for some $\mathcal{P}$, and if this is the case for which conjugacy classes of parabolic subgroups. A special

case has been considered in 6.3. In general the main problem is to determine the possible values of m. For example if we are in case (VI) or (VII) the smallest possible value of m can be obtained as follows. If ℓ_1 is odd, the problem has no solution. If $\ell_1 \neq \ell_2$ and ℓ_1 is even, let $m_o = \ell_1/2$. If ℓ_1 is even and $\ell_1 = \ell_2$, remove the columns 1 and 2 from d_λ and start again unless the new diagram is empty, in which case we put $m_o = 0$. In this way either we find that there is no solution or we find an integer m_o. Suppose we have obtained this integer m_o. Then it is easy to see if it gives a solution. If this is the case, the possible values for m are m_o, m_o+1, ..., m_o+j if d_λ has columns of length $2m_o+2$, $2m_o+4$, ..., $2m_o+2j$ but no column of length $2m_o+2j+2$.

It will be convenient now to allow some of the integers $n_1,\dots,n_s$ to be 0.

Suppose that we are in case (II), (III), (IV), (VI), (VII) or (IX). Let $P \supset B$ be the parabolic subgroup of G^o corresponding to the sequence m, $n_1,\dots,n_s$. Suppose that $n_i = 2m+2$ (resp. $n_i = 2m-1$). Replace m by m+1 (resp. m-1) and n_i by n_i-1 (resp. n_i+1) in the sequence m, $n_1,\dots,n_s$. Let $Q \supset B$ be the parabolic subgroup of G^o corresponding to this new sequence. Then it is easy to check that $C_Q = C_P$. Moreover if $P' \supset B$ is a parabolic subgroup of G^o such that $C_{P'} = C_P$, then P' can be obtained by repeated operations of the type above and by permutation of the integers $n_1,\dots,n_s$.

If we are in case (V) or (VIII) and $n_i = 2m+1 \geqslant 5$ (resp. $n_i = 2m-2 \geqslant 4$) then we get the same unipotent class if we replace m by m+1 (resp. m-1) and n_i by n_i-1 (resp. n_i+1). If $m = 0$, $n_i = 1$ and $n_j = 3$ (resp. $m = 2$, $n_i = 0$ and $n_j = 2$) then we can replace m by 2, n_i by 0 and n_j by 2 (resp. m by 0, n_i by 1 and n_j by 3) without changing the unipotent class. If P, $P' \supset B$ are parabolic subgroups of G^o such that $C_P = C_{P'}$ and if the corresponding subsets I, I' of Π are such that $I \cap \{\alpha_1, \alpha_2\} \neq \{\alpha_1\}$ and $I' \cap \{\alpha_1, \alpha_2\} \neq \{\alpha_1\}$, then I' can be obtained from I by repeated operations of the type above and by permutation of the integers $n_1, \dots, n_s$.

6.5. Let G be a connected reductive algebraic group, let P be a parabolic subgroup of G and let M be a Levi subgroup of P. Then P is distinguished in G if $\dim U_P/U_{P'} = \dim M'$, where $U_{P'}$ and M' are the derived subgroups of U_P and M respectively.

Consider a pair (L,P) where L is a Levi subgroup of some parabolic subgroup of G and P is a parabolic subgroup of L. Associate to this pair (L,P) the unipotent class of G which contains the P-regular class of L. This induces an application from the set of conjugacy classes of such pairs to the set of unipotent classes of G.

Consider first the restriction of this application to the set of conjugacy classes of pairs (L,P) with P distinguished in L. Bala and Carter have proved that if p = 0 or if p is large (i.e. $p \geqslant 4m+3$, where $m = \max_{\alpha \in \phi} ht(\alpha)$), then this restriction is a bijection. The condition on p can be weakened. For example if G is of type A_n it is true for all p. If G is of type B_n, C_n or D_n it is true if $p \neq 2$. If G is of type G_2 it is true if $p \neq 3$. This restriction is injective but not surjective if G is of type B_n $(n \geq 2)$, C_n $(n \geq 2)$ or D_n $(n \geq 6)$ and $p = 2$. If G is of type G_2 and $p = 3$, then it is neither injective nor surjective (notice that in this case every parabolic subgroup P of G is distinguished, except G itself, and we get therefore twice the subregular elements).

We shall show that if we consider all pairs (L,P) (with L a Levi subgroup of some parabolic subgroup of G and P a parabolic subgroup of L) then we get all unipotent classes if G is of type B_n or C_n, or if G is of type D_n and $p \neq 2$. This application is not surjective if G is of type D_n $(n \geq 8)$ and $p = 2$ or if G is of type G_2 and $p = 3$.

Suppose now that we are in one of the cases (II) to (IX). Let Q be a parabolic subgroup of G corresponding to a subset I of Π stable under the action of the component of G we consider. Let B' be a Borel subgroup of Q and let T' be a maximal torus of B'. Let x be a unipotent quasisemisimple element of G (in the component we consider) normalizing B' and T'. The Levi subgroup $L \supset T'$ of Q is also normalized by x. Let P be a parabolic subgroup of L corresponding to an x-stable subset J of I. Associate to the pair (L,P) the unipotent class of G (contained in xG^o) which contains the P-regular class of $\langle L,x \rangle$ contained in xL. We get in this way an application from the set of all G-conjugacy classes of such pairs to the set of unipotent classes of G contained in the component we consider. This application is surjective if we are in case (II), (III), (V), (VI) or (VII). We sketch here the proof for case (VII).

Suppose that we are in case (VII). We need only to understand which class of G correspond to (L,P). If $J \subset I$ are as above and if I is described by integers m, $n_1, \dots, n_s$, let $I_o = \{\alpha_i \mid 1 \leq i \leq m\}$, $I_r = \{\alpha_i \mid m+n_1+\dots+n_{r-1}+2 \leq i \leq m+n_1+\dots+n_r\}$ $(1 \leq r \leq s)$, $J_r = J \cap I_r$ $(0 \leq r \leq s)$.

If m = n (and s = 0), then L = G and (L,P) gives the P-regular class. In this case we can just use 6.2. If $m \neq n$, the group $\langle X_{\pm\alpha_i} \mid i \in I_o \rangle$ is a symplectic group Sp_{2m} and the subset $J_o \subset I_o$ gives a unipotent class $C_{\lambda^o, \varepsilon^o}$ in this group. $(\lambda^o, \varepsilon^o)$ can be computed from 6.2.

If m = 0 and $n_1 = n$ (and s = 1), then $J = J_1$ can be described by integers $q_1, \dots, q_h$ such that $q_1+\dots+q_h = n$ and $J = \bigcup_{1 \leq j \leq h} \{\alpha_i \mid q_1+\dots+q_{j-1}+2 \leq i \leq q_1+\dots+q_j\}$. The class $C_{\lambda, \varepsilon}$ corresponding to (L,P) is the following. d_λ is the diagram such that $\ell_i = 2q'_i$ $(1 \leq i \leq h)$, where $q'_1, \dots, q'_h$ are the integers $q_1, \dots, q_h$ arranged in decreasing order, and $\varepsilon_i \neq 1$ for all $i \geq 1$. If $m \geq 0$ and $1 \leq r \leq s$, $\langle X_{\pm\alpha_i} \mid i \in I_r \rangle$ can be thought of as a subgroup of a symplectic group Sp_{2n_r} and the subset J_r of I_r gives a unipotent class $C_{\lambda^r, \varepsilon^r}$ in Sp_{2n_r}.

We get in this way pairs $(\lambda^o, \varepsilon^o)$, ..., $(\lambda^s, \varepsilon^s)$. Then the unipotent class $C_{\lambda,\varepsilon}$ associated to (L,P) is the class such that d_λ is obtained by taking all the lines of the diagrams $d_{\lambda^0}, \ldots, d_{\lambda^s}$, and $\varepsilon_i = \max_{0 \leq r \leq s} \varepsilon_i^r$ for all i.

Using 6.2 it is easy to see that a unipotent class $C_{\lambda,\varepsilon}$ is φ-regular for some φ if (λ,ε) satisfies :

a) All lines of d_λ have even length.

b) For each $i > 0$ there are at most two lines of length i in d_λ.

c) $\varepsilon_i \neq 0$ for all $i \geq 1$.

It follows then easily that every unipotent class $C_{\lambda,\varepsilon}$ of G can be obtained for some choice of I and some subset J of I_0.

A similar proof works for cases (II), (III), (V) and (VI).

Suppose that we are in case (IV) (resp. (VIII), (IX)) and n = 6 (resp. n = 8, n = 4). Then the unipotent class $C_{\lambda,\varepsilon}$ cannot be obtained from a pair (L,P) if $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (5,3,3,1)$ (resp. (6,4,4,2), (3,3,2,0)) and $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5) = (1,\omega,1,\omega,1)$ (resp. $(\omega,1,\omega,1,\omega)$, $(\omega,1,\omega,1,\omega)$).

If a unipotent class C^0 is obtained for some pair (L,P), then $w_P w_L w_0 \in Q(C^0)$ (defined as in 2.4), where $w_P \in W_P \subset W_L \subset W$ (resp. $w_L \in W_L \subset W$, $w_0 \in W$) is the element of maximal length in the Weyl group of P (resp. L, G). In particular, if we are in one of the cases (I), (II), (III), (V), (VI) or (VII), this gives a method to find elements in $Q(C^0)$ for every unipotent class of G.

7. Equivalence relations in the Weyl groups.

Unless otherwise stated G is a reductive group and x is a unipotent element of G normalizing B and T. We consider unipotent elements contained in xG^o.

7.1. For every $w \in W^x$ let V_w be as in 2.4. Let also $U_w = U \cap {}^wU$. Then $V_w \subset xTU_w$. Write $w \prec w'$ if $\overline{{}^BV_w} \supset \overline{{}^BV_{w'}}$ $(w, w' \in W^x)$. This is a preorder in W^x. Write $w \simeq w'$ if $w \prec w'$ and $w' \prec w$. This defines and equivalence relation $\mathcal{R} = \mathcal{R}_G$ in W^x and the preorder $w \prec w'$ makes $W^x/\mathcal{R}$ into an ordered set. By 2.15 $w \simeq w'$ if and only if there exists a unipotent G^o-class C and an irreducible component C_i of $C \cap N$ such that $w, w' \in Q(C_i)$. The equivalence classes for $\mathcal{R}$ are the sets of the form $Q(C_i)$. They can also be described as the sets of the form $\varphi(\{\sigma\} \times S(u))$ for some unipotent $u \in xG^o$ such that $Q(C^o(u)) \neq \emptyset$ and some $\sigma \in S(u)$. Each equivalence class contains at least one involution which can be choosen in a canonical way ($\varphi(\sigma, \sigma)$ if u and σ are as above).

Proposition 7.2. For every $w \in W^x$ define $I_w = \{\alpha \in \Pi \mid \ell_x(w\tilde{s}_\alpha) < \ell_x(w)\}$. Let $u \in xG^o$ be a unipotent element and let $\sigma, \tau \in S(u)$. Then $I_{\varphi(\sigma,\tau)} = I_\tau$.

Proof. Consider a fundamental root α. Let $w = \varphi(\sigma, \tau)$ and $w' = w\tilde{s}_\alpha$.

Suppose that $\alpha \in I_w$. Then $w = w'\tilde{s}_\alpha$ and $\ell_x(w) = \ell_x(w') +$

$\ell_x(\tilde{s}_\alpha)$. Therefore for each pair $(B_o,B_2)\in(X_\sigma\times X_\tau)\cap O(w)$ there is a unique $B_1\in\mathcal{B}$ such that $(B_o,B_1)\in O(w')$ and $(B_1,B_2)\in O(\tilde{s}_\alpha)$. By uniqueness $B_1\in\mathcal{B}_u$ and therefore there is a line of type α through B_2 contained in $\mathcal{B}_u$ (by 1.3). This shows that the elements of X_τ contained in a line of type α contained in $\mathcal{B}_u$ are dense in X_τ. Therefore X_τ is a union of lines of type α and $\alpha\in I_\tau$.

Suppose conversely that $\alpha\in I_\tau$. Choose $(B_o,B_1)\in(X_\sigma\times X_\tau)\cap O(w)$. Let L be the line of type α through B_1 contained in $\mathcal{B}_u$. Then $\{B_2\in L\,|\,(B_o,B_2)\in O(w)\}$ is open since $(X_\sigma\times X_\tau)\cap O(w)$ is open in $X_\sigma\times X_\tau$ and contains B_1. If $\ell_x(w\tilde{s}_\alpha)=\ell_x(w)+1$, then $(B_o,B_2)\in O(w\tilde{s}_\alpha)$ for all $B_2\in L\setminus\{B_1\}$, a contradiction. Therefore $\ell_x(w\tilde{s}_\alpha)=\ell_x(w)-1$ and $\alpha\in I_w$.

<u>Corollary 7.3</u>. The application $W^x\longrightarrow\mathcal{P}(\Pi)$, $w\longmapsto I_{w^{-1}}$ is constant on each equivalence class for $\mathcal{R}$.

<u>Proof</u>. Every equivalence class for $\mathcal{R}$ is of the form $\varphi(\{\sigma\}\times S(u))$ for some unipotent $u\in xG^o$ and some $\sigma\in S(u)$. On this class the application takes the value I_σ.

<u>7.4</u>. The relation $\mathcal{R}$ can also be defined by : $w\simeq w'$ if there exist a unipotent element $u\in xG^o$ such that $Q(C^o(u))\neq\emptyset$ and irreducible components σ,τ,τ' of $\mathcal{B}_u$ such that $w=\varphi(\sigma,\tau)$ and $w'=\varphi(\sigma,\tau')$. This definition makes sense even if G is not reductive. In particular let P be a parabolic subgroup of the reductive group G^o such that P and ${}^xP$ are G^o-conjugate.

By considering the component $N_G(P) \cap xG^o$ of $N_G(P)$, we get an equivalence relation $\mathcal{R}_P$ on $W_P^x \subset W^x$. If $v, v' \in W^x$, we write $v \simeq_P v'$ if there exist $w_o \in W^x$ such that $\ell_x(w_o) = \min\{\ell_x(w_o w) \mid$ such that $v = w_o w$, $v' = w_o w'$ and w and w' are equivalent for $\mathcal{R}_P$. This extends $\mathcal{R}_P$ to an equivalence relation on W^x which we shall also denote by $\mathcal{R}_P$.

<u>Lemma 7.5</u>. Let P be a parabolic subgroup of G^o corresponding to an x-stable subset of Π. If $v, v' \in W^x$ are such that $v \simeq_P v'$, then $v \simeq v'$.

<u>Proof</u>. Let w_o, w, w' be as in 7.4. Let w_P be the element of maximal length in W_P and let $w_1 = w_o w_P$.

We may assume that $P \supset B$. Then x normalizes P and the Levi subgroup $L \supset T$ of P. $B_L = B \cap L$ is a Borel subgroup of L and $V_L = xL \cap V_1$ is the variety of all unipotent elements in xB_L. For every $w \in W_P^x$ let $V_{L,w} = V_L \cap {}^w V_L = xL \cap V_w$. W_P can be identified with the Weyl group of L.

The conditions of 7.4 on w_o, w and w' imply $V_v = V_{w_o w} = U_{w_1}{}^{(w_o)}V_{L,w}$, $V_{v'} = V_{w_o w'} = U_{w_1}{}^{(w_o)}V_{L,w'}$ and $\overline{\left({}^{(B_L)}U_{L,w}\right)} = \overline{\left({}^{(B_L)}U_{L,w'}\right)}$. It is easily checked that ${}^{(w_o)}B_L \subset B$ and ${}^{(w_o)}B_L$ normalizes U_{w_1}. Therefore $\overline{\left({}^{({}^{w_o}B_L)}V_v\right)} = \overline{\left({}^{({}^{w_o}B_L)}V_{v'}\right)}$. This shows that $\overline{{}^B V_v} = \overline{{}^B V_{v'}}$, i.e. $v \simeq v'$.

7.6. For each integer n ($n \geq 1$) we define a new equivalence relation $\mathcal{R}_n$ on W^x. We write $w \simeq_n w'$ for this relation. $\mathcal{R}_n$ is the finest equivalence relation on W^x such that for every parabolic subgroup P of G^o corresponding to a subset of Π which is the union of at most n x-orbits, $w \simeq_P w' \Longrightarrow w \simeq_n w'$. Clearly $w \simeq_1 w' \Longleftrightarrow w = w'$ and $w \simeq_n w' \Longrightarrow w \simeq_{n+1} w' \Longrightarrow w \simeq w'$ (by 7.5).

Proposition 7.7. Suppose that $G = GL_n$. Then $\mathcal{R} = \mathcal{R}_2$.

Proof. If $u \in GL_n$ and $\sigma, \tau \in S(u)$ are such that $\varphi(\sigma,\tau) = \varphi(\tau,\sigma)$, then $\sigma = \tau$ since $A(u) = 1$ (by 2.9). This shows that every equivalence class for $\mathcal{R}$ contains exactly one involution. Each equivalence class for $\mathcal{R}_2$ contains an involution [11, cor. 7.10]. Since $w \simeq_2 w' \Longrightarrow w \simeq w'$, this shows that $\mathcal{R} = \mathcal{R}_2$.

Remarks 7.8. a) If G is connected of type C_3 and $p \neq 2$, $\mathcal{R} = \mathcal{R}_2$. If G is connected of type B_3, $\mathcal{R} \neq \mathcal{R}_2$. If G is connected of type D_4, $\mathcal{R}_2 = \mathcal{R}_3 \neq \mathcal{R}$. It would be interesting to know if there is an integer n such that $\mathcal{R}_n = \mathcal{R}$ for all reductive groups.

b) Consider the order relation in W^x generated by all relations of the form $w < w\tilde{s}_\alpha$, where $w \in W^x$, $\alpha \in \Pi$ and $\ell_x(w) = \ell_x(w\tilde{s}_\alpha) - 1$. We write $w < w'$ for this relation. $w < w' \Longrightarrow w \prec w'$. Consider the following equivalence relation S on W^x. (written $w \sim w'$). S is the finest equivalence relation such that $w \sim w\tilde{s}_\alpha$ if $w < w\tilde{s}_\alpha$ and $I_w \setminus I_{w\tilde{s}_\alpha} \neq \emptyset$ ($w \in W^x$, $\alpha \in \Pi$). If G is connected and all roots of G have the same length, then $\mathcal{S} = \mathcal{R}_2$. Let $\mathcal{S}'$ be the finest equivalence relation on W^x such that $\mathcal{S}$ is finer than $\mathcal{S}'$ and the order $w < w'$ in W^x induces an ordering on $W^x/\mathcal{S}'$. If G is connected of type D_4 $\mathcal{S} \neq \mathcal{S}'$. If G is connected of type A_n, then $\mathcal{S} = \mathcal{S}'$ since $\mathcal{R} = \mathcal{R}_2 = \mathcal{S}$, $w \prec w'$ induces an ordering on $W^x/\mathcal{R}$ and $w < w' \Longrightarrow w \prec w'$. If G is connected and all roots of G have the same length, then $\mathcal{S}'$ is finer than $\mathcal{R}$.

<u>7.9</u>. Consider a diagram consisting of lines of length $\lambda_1, \lambda_2, \dots$ with $\sum_{i \geq 1} \lambda_i = n$. Here we do not assume that the sequence $\lambda_1, \lambda_2, \dots$ is decreasing (the diagram is therefore not a Young diagram in general). A tableau is obtained by filling the diagram with the numbers $1, 2, \dots, n$. Consider a tableau t. If $\lambda_i \geq j$, t_{ij} will be the integer in the line i and in the column j. Suppose that at least one line of t is not an increasing sequence. Let i be the largest integer such that the sequence $t_{i1}, t_{i2}, \dots, t_{i\lambda_i}$ is not increasing. Let j be the smallest integer such that $t_{i1}, t_{i2}, \dots, t_{ij}$ is not increasing. Let h be the smallest integer such that $t_{ih} > t_{ij}$. We define a new tableau t' with lines of length $\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}+1, \lambda_{i+2}, \dots$. t' is obtained by replacing the line i in t by the sequence $t_{i1}, \dots, t_{i,h-1}, t_{ij}, t_{i,h+1}, \dots, t_{i,j-1}, t_{i,j+1}, \dots$ and the line (i+1) in t by the sequence $t_{i+1,1}, \dots, t_{i+1,\lambda_{i+1}}, t_{ih}$.

The operation $t \mapsto t'$ can be repeated a finite number of times until we get a tableau f(t) in which all lines are increasing sequences.

Let $w : i \longmapsto w_i$ be a permutation of $\{1,\dots,n\}$. We take the sequence $w_1,\dots,w_n$ as the first line of a tableau with lines of length n,0,0,... and we identify w with this tableau. Then f(w) is a standard tableau and f(w) and $f(w^{-1})$ are standard tableaux corresponding to the same Young diagram [11, 7.5].

Suppose now that $G = GL_n$. Identify W with S_n, the group of permutations of $\{1,\dots,n\}$.(s_{α_i} being identified with the permutation $h \longmapsto h$ if $h \neq i,i+1$, $i \longmapsto i+1$, $i+1 \longmapsto i$). If $u \in C_\lambda$ is unipotent, we use the parametrization of S(u) given in 5.6 to identify S(u) and St(λ).

<u>Proposition 7.10</u>. Suppose that $G = GL_n$ and that $u \in C_\lambda$ is a unipotent element of G. Let $Q(C_\lambda)$ be as in 2.4. Then $w \longmapsto (f(w), f(w^{-1}))$ gives a bijection $Q(C_\lambda) \longrightarrow S(u)\times S(u)$ and this bijection is the inverse of $\varphi : S(u)\times S(u) \longrightarrow Q(C_\lambda)$.

<u>Proof</u>. Let $\tau \in St(\lambda)$ be the unique standard tableau such that $\tau_1,\dots,\tau_n$ (defined as in 5.2) is an increasing sequence. $\overline{X}_\tau$ is the irreducible component of $\mathfrak{B}_u$ which is of the form $\mathfrak{B}(P)$ for some parabolic subgroup P corresponding to the subset I of Π characterized by the integers $\ell_1,\dots,\ell_s$ ($s = \lambda_1$) (with the notations of 6.1). Then $\varphi(\tau,\tau) = w_P$, where w_P is the element of maximal length in W_P. It is easily checked from the definition that $f(w_P) = \tau$.

If $F = (F_o,\dots,F_n) \in \mathcal{F}$, $F' = (F'_o,\dots,F'_n) \in \mathcal{F}$ and $(F,F') \in O(w)$, then w is the unique permutation of $\{1,\dots,n\}$ such that $w(i) = j$ if and only if $F'_i \cap F_{j-1} = F'_{i-1} \cap F_{j-1}$ and $F'_i \cap F_j \neq F'_{i-1} \cap F_j$ $(1 \leq i \leq n)$.

Consider the component $\overline{Y}_\tau$ of $\mathcal{B}_u$ (Y_τ defined as in 5.5). $\overline{Y}_\tau = \{F \in \mathcal{F} \mid F_{n-\ell_1-\dots-\ell_r} = \mathrm{Im}(u-1)^r$ for all $r \geq 0\}$. If $\sigma \in St(\lambda)$, then it is easily checked that $\varphi(\overline{Y}_\tau,\overline{Y}_\sigma) = \varphi(\overline{Y}_\tau,Y_\sigma) = w$ is the following permutation. If $(n-i+1)$ is in the column r and in the line s of σ, then $w(i) = n-\ell_1-\dots-\ell_r+s$ $(1 \leq i \leq n)$. Let w_o be the element of maximal length in W. Then $\varphi(\tau,\sigma) = w_o\varphi(\overline{Y}_\tau,\overline{Y}_\sigma)w_o$. Therefore $\varphi(\tau,\sigma)$ is the following permutation w'. If i is in the column r and in the line s of σ, then $w'(i) = \ell_1+\dots+\ell_r-s+1$. It follows then easily from the definition of f that $f(\varphi(\sigma,\tau)) = f(\varphi(\tau,\sigma)^{-1}) = \sigma$.

Let w, w' be any elements of W. Then $w \simeq w'$ if and only if $f(w) = f(w')$ [11,7.9]. If $\sigma_1,\sigma_2 \in St(\lambda)$, we have therefore $f(\varphi(\sigma_1,\sigma_2)) = f(\varphi(\sigma_1,\tau)) = \sigma_1 \in St(\lambda)$ and $f(\varphi(\sigma_1,\sigma_2)^{-1}) = f(\varphi(\sigma_2,\sigma_1)) = \sigma_2 \in St(\lambda)$. This shows that $w \longmapsto (f(w),f(w^{-1}))$ defines an application $Q(C_\lambda) \longrightarrow St(\lambda)\times St(\lambda)$ and that this application is the inverse of $\varphi : S(u)\times S(u) \longrightarrow Q(C_\lambda)$ (with the suitable identifications).

8. Examples.

G is reductive and $u \in G$ is unipotent.

8.1. Suppose that u is $\mathcal{P}$-regular for some conjugacy class of parabolic subgroups of G. Then it is not alwways possible to find a conjugacy class $\mathcal{Q}$ of parabolic subgroups of G such that :

a) u is $\mathcal{Q}$-regular.

b) There is only one $Q \in \mathcal{Q}$ such that $\mathcal{B}(Q)_u$ is an irreducible component of $\mathcal{B}_u$.

For example take $G = Sp_6$, $p \neq 2$ and u such that $(u-1)^2 = 0$, $\dim \operatorname{Ker}(u-1) = 4$. Then u is $\mathcal{P}$-regular for a unique conjugacy class of parabolic subgroups (the class corresponding to $\{\alpha_1, \alpha_2\} \in \Pi$ with the notations of 5.1) and there are exactly two parabolic subgroups $P_1, P_2 \in \mathcal{P}$ such that $\mathcal{B}(P_1)$ and $\mathcal{B}(P_2)$ are irreducible components of $\mathcal{B}_u$.

8.2. Let C' be a unipotent class such that $C' \subset \overline{C(u)}$. It is not always possible to find $v \in C'$ such that $\mathcal{B}_u \subset \mathcal{B}_v$.

For example take u subregular in $G = GL_4$. Let v be a unipotent element such that $(v-1)^2 = 0$, $\dim \operatorname{Ker}(v-1) = 2$. Then $C(v) \subset \overline{C(u)}$. It follows easily from 5.12 that every line of type α_1 in $\mathcal{B}_v$ meets every line of type α_3 in $\mathcal{B}_v$. This clearly shows that $\mathcal{B}_u \not\subset \mathcal{B}_v$.

8.3. The irreducible components of $\mathcal{B}_u$ may have singular points. Let $G = GL_6$ and suppose that $(u-1)^2 = 0$, $\dim \operatorname{Ker}(u-1) = 4$. Consider the component $\overline{X}_\sigma$ corresponding to the standard tableau σ (we use the notation introduced in 5.4) :

1	3
2	5
4	
6	

This component contains singular points [14], [22]. Let $W_2 = \mathrm{Im}(u-1)$, $W_4 = \mathrm{Ker}(u-1)$. Then $W_2 \subset W_4$ and $\overline{X}_\sigma = \{F = (F_0,\dots,F_6) \in \mathcal{F} \mid \dim (F_2 \cap W_2) \geq 1, \dim (F_4 \cap W_4) \geq 3, (u-1)(F_4) \subset F_2 \subset W_4, F_4 \supset W_2\}$. $F \in \overline{X}_\sigma$ is a singular point of $\overline{X}_\sigma$ if and only if $F_2 = W_2$ and $F_4 = W_4$. (Vargas has pointed out to me that the condition $F_4 \supset W_2$ is missing in [14]).

8.4. All irreducible components of $\mathcal{B}_u$ have the same dimension. This is not true in general for $\mathcal{P}_u$, where $\mathcal{P}$ is a conjugacy class of parabolic subgroups of G. For example take $G = GL_4$. If $\mathcal{P}$ corresponds to $\{\alpha_1\} \subset \Pi$ and $\dim \mathrm{Ker}(u-1) = 3$, $\mathcal{P}_u$ has one irreducible component of dimension 2 and one irreducible component of dimension 3 (this follows from the fact that $O(s_1s_2s_1) \not\subset \overline{O(s_2s_3s_2s_1)}$, with $s_i = s_{\alpha_i}$, $i = 1,2,3$).

9. Tables.

In the following tables we suppose that we are in one of the cases (II) to (IX) of paragraph 5 and $u \in C_{\lambda,\varepsilon}$. Unless otherwise stated $\varepsilon_1 \neq 0$ for all $i \geqslant 1$. $d(u) = \dim \mathcal{B}_u$, $s(u) = |S(u)|$, $s_a(u) = |S(u)/A_0(u)|$, $s_a'(u) = |S(u)/A(u)|$ and $q(u) = |Q(C^0(u))|$. Except in cases (V) and (VIII), we have always $s_a'(u) = s_a(u)$. If we are in case (II) or (IV) let $B(u) = \{a \in A_0(u) \mid a$ acts trivially on $S(u)\}$. In the other cases let $B(u) = \{a \in A(u) \mid a$ acts trivially on $S(u)\}$. If the column giving $B(u)$ is empty, $A(u)$ acts trivially on $S(u)$. If $|A_0(u)/B(u)| > 2$ in cases (II) and (IV) or if $|A(u)/B(u)| > 2$ in the other cases, more indications are given on the action of $A(u)$ on $S(u)$.

The total in the column giving $s(u)$ (resp. $q(u)$, $s_a(u)$) is $|\{w \in W^u \mid w^2 = 1\}|$ (resp. $|W^u|$, the number of equivalence classes for $\mathcal{R}$). In cases (V) and (VIII) the classes of O_{2n} which split into two classes of SO_{2n} are repeated to get the correct totals.

Case (II). $G = G(V)$, $\dim V = 2n+1$, $p = 2$, $u \in G \setminus G^0$.

λ	ε	$d(u)$	$s(u)$	$s_a(u)$	$B(u)$	$q(u)$
$n = 2$, $\dim V = 5$.						
5		0	1	1		1
3 1^2		1	2	2		4
3 1^2	$\varepsilon_1 = 0$	2	1	1		1
2^2 1		2	1	1		1
1^4		4	1	1		1
Total			6	6		8

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
<u>n = 3</u>, dim V = 7.						
7		0	1	1		1
$5\ 1^2$		1	3	3		9
$5\ 1^2$	$\varepsilon_1 = 0$	2	2	2		4
$3^2\ 1$		2	3	3		9
$3^2\ 1$	$\varepsilon_3 = 0$	3	1	1		1
$3\ 2^2$		3	3	3		9
$3\ 1^4$		4	3	3		9
$3\ 1^4$	$\varepsilon_1 = 0$	6	1	1		1
$2^2\ 1^3$		5	2	2		4
1^7		9	1	1		1
Total			20	20		48
<u>n = 4</u>, dim V = 9.						
9		0	1	1		1
$7\ 1^2$		1	4	4		16
$7\ 1^2$	$\varepsilon_1 = 0$	2	3	3		9
5 3 1		2	6	6		36
$5\ 2^2$		3	10	8	$\langle a_1 \rangle$	68
$5\ 1^4$		4	6	6		36
$5\ 1^4$	$\varepsilon_1 = 0$	6	3	3		9
$4^2\ 1$		3	5	4	$\{1\}$	17
3^3		4	6	6		36
$3^2\ 1^3$		5	8	8		64
$3^2\ 1^3$	$\varepsilon_3 = 0$	6	3	3		9
$3\ 2^2\ 1^2$		6	6	6		36
$3\ 2^2\ 1^2$	$\varepsilon_1 = 0$	7	4	4		16
$3\ 1^6$		9	4	4		16

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$3\ 1^6$	$\varepsilon_1 = 0$	12	1	1		1
$2^4\ 1$		8	2	2		4
$2^2\ 1^5$		10	3	3		9
1^9		16	1	1		1
Total			76	73		384

<u>n = 5</u>, dim V = 11.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
11		0	1	1		1
$9\ 1^2$		1	5	5		25
$9\ 1^2$	$\varepsilon_1 = 0$	2	4	4		16
7 3 1		2	10	10		100
$7\ 2^2$		3	20	15	$\langle a_1 \rangle$	250
$7\ 1^4$		4	10	10		100
$7\ 1^4$	$\varepsilon_1 = 0$	6	6	6		36
$5^2\ 1$		3	11	10	$\{1\}$	101
$5^2\ 1$	$\varepsilon_5 = 0$	4	5	5		25
$5\ 3^2$		4	20	20		400
$5\ 3^2$	$\varepsilon_3 = 0$	5	10	10		100
$5\ 3\ 1^3$		5	20	20		400
$5\ 2^2\ 1^2$		6	20	20		400
$5\ 2^2\ 1^2$	$\varepsilon_1 = 0$	7	20	15	$\langle a_1 \rangle$	250
$5\ 1^6$		9	10	10		100
$5\ 1^6$	$\varepsilon_1 = 0$	12	4	4		16
$4^2\ 3$		5	10	10		100
$4^2\ 1^3$		6	19	15	$\{1\}$	241
$3^3\ 1^2$		7	20	20		400
$3^3\ 1^2$	$\varepsilon_1 = 0$	8	10	10		100
$3^2\ 2^2\ 1$		8	10	10		100

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$3^2\ 2^2\ 1$	$\varepsilon_3 = 0$	9	5	5		25
$3^2\ 1^5$		10	15	15		225
$3^2\ 1^5$	$\varepsilon_3 = 0$	11	6	6		36
$3\ 2^4$		10	10	10		100
$3\ 2^2\ 1^4$		11	10	10		100
$3\ 2^2\ 1^4$	$\varepsilon_1 = 0$	13	5	5		25
$3\ 1^8$		16	5	5		25
$3\ 1^8$	$\varepsilon_1 = 0$	20	1	1		1
$2^4\ 1^3$		13	5	5		25
$2^2\ 1^7$		17	4	4		16
1^{11}		25	1	1		1
Total			312	297		3840

<u>Case (III)</u>. $G = O_{2n+1}$. $p \neq 2$.

λ	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
<u>n = 2</u>, $G = O_5$.					
5	0	1	1		1
$3\ 1^2$	1	3	2	$\langle a_1 \rangle$	5
$2^2\ 1$	2	1	1		1
1^5	4	1	1		1
Total		6	5		8
<u>n = 3</u>, $G = O_7$.					
7	0	1	1		1
$5\ 1^2$	1	5	3	$\langle a_1 \rangle$	13
$3^2\ 1$	2	4	3	$\langle \varepsilon_3 \rangle$	10
$3\ 2^2$	3	3	3		9
$3\ 1^4$	4	4	3	$\langle a_1 \rangle$	10

λ	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$2^2\ 1^3$	5	2	2		4
1^7	9	1	1		1
Total		20	16		48

<u>n = 4</u>, $G = O_9$.

9	0	1	1		1
$7\ 1^2$	1	7	4	$\langle a_1\rangle$	25
5 3 1	2	9	6	(*)	41
$5\ 2^2$	3	8	8		64
$5\ 1^4$	4	9	6	$\langle a_1\rangle$	45
$4^2\ 1$	3	4	4		16
3^3	4	6	6		36
$3^2\ 1^3$	5	11	8	$\langle a_3\rangle$	73
$3\ 2^2\ 1^2$	6	10	6	$\langle a_1\rangle$	52
$3\ 1^6$	9	5	4	$\langle a_1\rangle$	17
$2^4\ 1$	8	2	2		4
$2^2\ 1^5$	10	3	3		9
1^9	16	1	1		1
Total		76	59		384

(*) $\langle a_1a_2,a_3\rangle$ is the stabilizer of two components, $\langle a_1,a_2a_3\rangle$ is the stabilizer of 4 components and A(u) stabilizes 3 components.

<u>n = 5</u>, $G = O_{11}$.

11	0	1	1		1
$9\ 1^2$	1	9	5	$\langle a_1\rangle$	41
7 3 1	2	16	10	(*)	126
$7\ 2^2$	3	15	15		225
$7\ 1^4$	4	16	10	$\langle a_1\rangle$	136

λ	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$5^2\ 1$	3	15	10	$\langle a_3 \rangle$	125
$5\ 3^2$	4	30	20	$\langle a_1 \rangle$	500
$5\ 3\ 1^3$	5	29	20	(**)	441
$5\ 2^2\ 1^2$	6	35	20	$\langle a_1 \rangle$	625
$5\ 1^6$	9	14	10	$\langle a_1 \rangle$	116
$4^2\ 3$	5	10	10		100
$4^2\ 1^3$	6	15	15		225
$3^3\ 1^2$	7	30	20	$\langle a_1 \rangle$	500
$3^2\ 2^2\ 1$	8	15	10	$\langle a_5 \rangle$	125
$3^2\ 1^5$	10	21	15	$\langle a_3 \rangle$	261
$3\ 2^4$	10	10	10		100
$3\ 2^2\ 1^4$	11	15	10	$\langle a_1 \rangle$	125
$3\ 1^8$	16	6	5	$\langle a_1 \rangle$	26
$2^4\ 1^3$	13	5	5		25
$2^2\ 1^7$	17	4	4		16
1^{11}	25	1	1		1
Total		312	226		3840

(*) $\langle a_1 a_2, a_3 \rangle$ is the stabilizer of 2 elements, $\langle a_1, a_2 a_3 \rangle$ is the stabilizer of 10 elements and A(u) stabilizes 4 elements.

(**) $\langle a_1 a_2, a_3 \rangle$ is the stabilizer of 8 elements, $\langle a_1, a_2 a_3 \rangle$ is the stabilizer of 10 elements and A(u) stabilizes 9 elements.

Case (IV). $G = G(V)$, $\dim V = 2n$, $p = 2$, $u \in G \setminus G^0$.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
n = 2, dim V = 4.						
$3\ 1$		0	1	1		1
2^2		1	3	2	$\{1\}$	5

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
1^4		2	1	1		1
1^4	$\varepsilon_1 = 0$	4	1	1		1
Total			6	5		8
n = 3, dim V = 6.						
5 1		0	1	1		1
3^2		1	4	3	{1}	10
3^2	$\varepsilon_3 = 0$	2	3	3		9
$3\ 1^3$		2	2	2		4
$2^2\ 1^2$		3	3	3		9
$2^2\ 1^2$	$\varepsilon_1 = 0$	4	5	3	{1}	13
1^6		6	1	1		1
1^6	$\varepsilon_1 = 0$	9	1	1		1
Total			20	17		48
n = 4, dim V = 8.						
7 1		0	1	1		1
5 3		1	5	4	$\langle a_1 a_2 \rangle$	17
$5\ 1^3$		2	3	3		9
4^2		2	10	6	{1}	52
$3^2\ 1^2$		3	8	8		64
$3^2\ 1^2$	$\varepsilon_3 = 0$	4	6	6		36
$3^2\ 1^2$	$\varepsilon_1 = 0$	4	9	6	{1}	45
$3^2\ 1^2$	$\varepsilon_1 = \varepsilon_3 = 0$	5	8	8		64
$3\ 2^2\ 1$		4	2	2		4
$3\ 1^5$		6	3	3		9
2^4		6	8	6	{1}	40
$2^2\ 1^4$		7	4	4		16
$2^2\ 1^4$	$\varepsilon_1 = 0$	9	7	4	{1}	25

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
1^8		12	1	1		1
1^8	$\varepsilon_1 = 0$	16	1	1		1
Total			76	63		384

<u>n = 5</u>, dim V = 10.

9 1		0	1	1		1
7 3		1	6	5	$\langle a_1 a_2 \rangle$	26
7 1^3		2	4	4		16
5^2		2	15	10	$\{1\}$	125
5^2	$\varepsilon_5 = 0$	3	10	10		100
5 3 1^2		3	15	15		225
5 3 1^2	$\varepsilon_1 = 0$	4	14	10	$\langle a_1 a_2 \rangle$	116
5 2^2 1		4	5	5		25
5 1^5		6	6	6		36
4^2 1^2		4	30	20	$\{1\}$	500
4^2 1^2	$\varepsilon_1 = 0$	5	35	20	$\{1\}$	625
3^3 1		5	10	10		100
3^2 2^2		6	25	20	$\{1\}$	425
3^2 2^2	$\varepsilon_3 = 0$	7	30	20	$\{1\}$	500
3^2 1^4		7	15	15		225
3^2 1^4	$\varepsilon_3 = 0$	8	10	10		100
3^2 1^4	$\varepsilon_1 = 0$	9	16	10	$\{1\}$	136
3^2 1^4	$\varepsilon_1 = \varepsilon_3 = 0$	10	15	15		225
3 2^2 1^3		8	5	5		25
3 1^7		12	4	4		16
2^4 1^2		10	10	10		100
2^4 1^2	$\varepsilon_1 = 0$	11	15	10	$\{1\}$	125
2^2 1^6		13	5	5		25

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
1^8		12	1	1		1
1^8	$\varepsilon_1 = 0$	16	1	1		1
Total			76	63		384

<u>n = 5</u>. dim V = 10.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
9 1		0	1	1		1
7 3		1	6	5	$\langle a_1 a_2 \rangle$	26
7 1^3		2	4	4		16
5^2		2	15	10	$\{1\}$	125
5^2	$\varepsilon_5 = 0$	3	10	10		100
5 3 1^2		3	15	15		225
5 3 1^2	$\varepsilon_1 = 0$	4	14	10	$\langle a_1 a_2 \rangle$	116
5 2^2 1		4	5	5		25
5 1^5		6	6	6		36
4^2 1^2		4	30	20	$\{1\}$	500
4^2 1^2	$\varepsilon_1 = 0$	5	35	20	$\{1\}$	625
3^3 1		5	10	10		100
3^2 2^2		6	25	20	$\{1\}$	425
3^2 2^2	$\varepsilon_3 = 0$	7	30	20	$\{1\}$	500
3^2 1^4		7	15	15		225
3^2 1^4	$\varepsilon_3 = 0$	8	10	10		100
3^2 1^4	$\varepsilon_1 = 0$	9	16	10	$\{1\}$	136
3^2 1^4	$\varepsilon_1 = \varepsilon_3 = 0$	10	15	15		225
3 2^2 1^3		8	5	5		25
3 1^7		12	4	4		16
2^4 1^2		10	10	10		100
2^4 1^2	$\varepsilon_1 = 0$	11	15	10	$\{1\}$	125
2^2 1^6		13	5	5		25

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
1^8		12	1	1		1
1^8	$\varepsilon_1 = 0$	16	1	1		1
Total			76	63		384

n = 5. dim V = 10.

9 1		0	1	1		1
7 3		1	6	5	$\langle a_1 a_2 \rangle$	26
7 1^3		2	4	4		16
5^2		2	15	10	$\{1\}$	125
5^2	$\varepsilon_5 = 0$	3	10	10		100
5 3 1^2		3	15	15		225
5 3 1^2	$\varepsilon_1 = 0$	4	14	10	$\langle a_1 a_2 \rangle$	116
5 2^2 1		4	5	5		25
5 1^5		6	6	6		36
4^2 1^2		4	30	20	$\{1\}$	500
4^2 1^2	$\varepsilon_1 = 0$	5	35	20	$\{1\}$	625
3^3 1		5	10	10		100
3^2 2^2		6	25	20	$\{1\}$	425
3^2 2^2	$\varepsilon_3 = 0$	7	30	20	$\{1\}$	500
3^2 1^4		7	15	15		225
3^2 1^4	$\varepsilon_3 = 0$	8	10	10		100
3^2 1^4	$\varepsilon_1 = 0$	9	16	10	$\{1\}$	136
3^2 1^4	$\varepsilon_1 = \varepsilon_3 = 0$	10	15	15		225
3 2^2 1^3		8	5	5		25
3 1^7		12	4	4		16
2^4 1^2		10	10	10		100
2^4 1^2	$\varepsilon_1 = 0$	11	15	10	$\{1\}$	125
2^2 1^6		13	5	5		25

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$2^2\ 1^6$	$\varepsilon_1 = 0$	16	9	5	$\{1\}$	41
1^{10}		20	1	1		1
1^{10}	$\varepsilon_1 = 0$	25	1	1		1
Total			312	247		3840

Case (V). $G = O_{2n}$, $p \neq 2$.

λ	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
n = 2, $G = O_4$.						
$3\ 1$	0	1	1	1		1
2^2	1	1	1	1		1
2^2	1	1	1	1		1
1^4	2	1	1	1		1
Total		4		4		4
n = 3, $G = O_6$.						
$5\ 1$	0	1	1	1		1
3^2	1	3	2	3	$\{1\}$	9
$3\ 1^3$	2	2	2	2		4
$2^2\ 1^2$	3	3	2	3	$\{1\}$	9
1^6	6	1	1	1		1
Total		10		10		24
n = 4, $G = O_8$.						
$7\ 1$	0	1	1	1		1
$5\ 3$	1	4	3	4	$\langle a_1 a_2 \rangle$	16
$5\ 1^3$	2	3	3	3		9
4^2	2	3	3	3		9
4^2	2	3	3	3		9

λ	d(u)	s(u)	$s_a'(u)$	$s_a(u)$	B(u)	q(u)
$3^2\ 1^2$	3	14	6	8	(*)	100
$3\ 2^2\ 1$	4	2	2	2		4
$3\ 1^5$	6	3	3	3		9
2^4	6	3	3	3		9
2^4	6	3	3	3		9
$2^2\ 1^4$	7	4	3	4	$\{1\}$	16
1^8	12	1	1	1		1
Total		44		38		192

(*) $\{1\}$ is the stabilizer of 8 elements, $\langle a_1\rangle$ is the stabilizer of 2 elements, $\langle a_3\rangle$ is the stabilizer of 2 elements and A(u) stabilizes 2 elements.

<u>n = 5</u>, $G = O_{10}$.

9 1	0	1	1	1		1
7 3	1	5	4	5	$\langle a_1 a_2\rangle$	25
$7\ 1^3$	2	4	4	4		16
5^2	2	10	6	10	$\{1\}$	100
$5\ 3\ 1^2$	3	25	12	15	(*)	325
$5\ 2^2\ 1$	4	5	5	5		25
$5\ 1^5$	6	6	6	6		36
$4^2\ 1^2$	4	20	12	20	$\{1\}$	400
$3^3\ 1$	5	10	10	10		100
$3^2\ 2^2$	6	20	12	20	$\{1\}$	400
$3^2\ 1^4$	7	25	12	15	(**)	325
$3\ 2^2\ 1^3$	8	5	5	5		25
$3\ 1^7$	12	4	4	4		16
$2^4\ 1^2$	10	10	6	10	$\{1\}$	100
$2^2\ 1^6$	13	5	4	5	$\{1\}$	25

λ	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
1^8	20	1	1	1		1
Total		156		136		1920

(*) $\langle a_1 a_2 \rangle$ is the stabilizer of 12 elements, $\langle a_1 a_2, a_3 \rangle$ is the stabilizer of 2 elements, $\langle a_1, a_2 \rangle$ is the stabilizer of 6 elements and A(u) stabilizes 5 elements.

(**) $\{1\}$ is the stabilizer of 12 elements, $\langle a_1 \rangle$ is the stabilizer of 2 elements, $\langle a_3 \rangle$ is the stabilizer of 6 elements and A(u) stabilizes 5 elements.

Case (VI). $G = Sp_{2n}$, $p \neq 2$.

λ	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
n = 2. $G = Sp_4$.					
4	0	1	1		1
2^2	1	3	2	$\{1\}$	5
2 1^2	2	1	1		1
1^4	4	1	1		1
Total		6	5		8
n = 3. $G = Sp_6$.					
6	0	1	1		1
4 2	1	4	3	$\langle a_1 a_2 \rangle$	10
4 1^2	2	2	2		4
3^2	2	3	3		9
2^3	3	3	3		9
2^2 1^2	4	5	3	$\langle a_1 a_2 \rangle$	13
2 1^4	6	1	1		1
1^6	9	1	1		1
Total		20	17		48

λ	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
n = 4, G = Sp_8.					
8	0	1	1		1
6 2	1	5	4	$\langle a_1 a_2 \rangle$	17
6 1^2	2	3	3		9
4^2	2	10	6	$\{1\}$	52
4 2^2	3	10	8	$\langle a_1 \rangle$	68
4 2 1^2	4	9	6	$\langle a_1 a_2 \rangle$	45
4 1^4	6	3	3		9
3^2 2	4	6	6		36
3^2 1^2	5	8	8		64
2^4	6	8	6	$\{1\}$	40
2^3 1^2	7	4	4		16
2^2 1^4	9	7	4	$\{1\}$	25
2 1^6	12	1	1		1
1^8	16	1	1		1
Total		76	61		384
n = 5, G = Sp_{10}.					
10	0	1	1		1
8 2	1	6	5	$\langle a_1 a_2 \rangle$	26
8 1^2	2	4	4		16
6 4	2	15	10	$\langle a_1 a_2 \rangle$	125
6 2^2	3	20	15	$\langle a_1 \rangle$	250
6 2 1^2	4	14	10	$\langle a_1 a_2 \rangle$	116
6 1^4	6	6	6		36
5^2	3	10	10		100
4^2 2	4	30	20	$\langle a_3 \rangle$	500

λ	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$4^2\ 1^2$	5	35	20	$\{1\}$	625
$4\ 3^2$	5	10	10		100
$4\ 2^3$	6	25	20	$\langle a_1a_2\rangle$	425
$4\ 2^2\ 1^2$	7	20	15	$\langle a_1\rangle$	250
$4\ 2\ 1^4$	9	16	10	$\langle a_1a_2\rangle$	136
$4\ 1^6$	12	4	4		16
$3^2\ 2^2$	7	30	20	$\{1\}$	500
$3^2\ 2\ 1^2$	8	10	10		100
$3^2\ 1^4$	10	15	15		225
2^5	10	10	10		100
$2^4\ 1^2$	11	15	10	$\{1\}$	125
$2^3\ 1^4$	13	5	5		25
$2^2\ 1^6$	16	9	5	$\{1\}$	41
$2\ 1^8$	20	1	1		1
1^{10}	25	1	1		1
Total		312	237		3840

<u>Case (VII)</u>. $G = Sp_{2n}$, $p = 2$.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
<u>n = 2</u>, $G = Sp_4$.						
4		0	1	1		1
2^2		1	2	2		4
2^2	$\varepsilon_2 = 0$	2	1	1		1
$2\ 1^2$		2	1	1		1
1^4		4	1	1		1
Total			6	6		8

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$\underline{n = 3}$, $G = Sp_6$.						
6		0	1	1		1
4 2		1	3	3		9
$4\ 1^2$		2	2	2		4
3^2		2	4	3	{1}	10
2^3		3	3	3		9
$2^2\ 1^2$		4	3	3		9
$2^2\ 1^2$	$\varepsilon_2 = 0$	5	2	2		4
$2\ 1^4$		6	1	1		1
1^6		9	1	1		1
Total			20	19		48
$\underline{n = 4}$, $G = Sp_8$.						
8		0	1	1		1
6 2		1	4	4		16
$6\ 1^2$		2	3	3		9
4^2		2	7	6	{1}	37
4^2	$\varepsilon_4 = 0$	3	4	4		16
$4\ 2^2$		3	8	8		64
$4\ 2^2$	$\varepsilon_2 = 0$	4	2	2		4
$4\ 2\ 1^2$		4	6	6		36
$4\ 1^4$		6	3	3		9
$3^2\ 2$		4	6	6		36
$3^2\ 1^2$		5	11	8	{1}	73
2^4		6	6	6		36
2^4	$\varepsilon_2 = 0$	8	2	2		4
$2^3\ 1^2$		7	4	4		16
$2^2\ 1^4$		9	4	4		16

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$2^2\ 1^4$	$\varepsilon_2 = 0$	10	3	3		9
$2\ 1^6$		12	1	1		1
1^8		16	1	1		1
Total			76	72		384
$\underline{n = 5}$, $G = Sp_{10}$.						
10		0	1	1		1
8 2		1	5	5		25
$8\ 1^2$		2	4	4		16
6 4		2	11	10	{1}	101
$6\ 2^2$		3	15	15		225
$6\ 2^2$	$\varepsilon_2 = 0$	4	5	5		25
$6\ 2\ 1^2$		4	10	10		100
$6\ 1^4$		6	6	6		36
5^2		3	15	10	{1}	125
$4^2\ 2$		4	20	20		400
$4^2\ 2$	$\varepsilon_4 = 0$	5	10	10		100
$4^2\ 1^2$		5	24	20	{1}	416
$4^2\ 1^2$	$\varepsilon_4 = 0$	6	15	15		225
$4\ 3^2$		5	10	10		100
$4\ 2^3$		6	20	20		400
$4\ 2^2\ 1^2$		7	15	15		225
$4\ 2^2\ 1^2$	$\varepsilon_2 = 0$	8	5	5		25
$4\ 2\ 1^4$		9	10	10		100
$4\ 1^6$		12	4	4		16
$3^2\ 2^2$		7	20	20		400
$3^2\ 2^2$	$\varepsilon_2 = 0$	8	15	10	{1}	125
$3^2\ 2\ 1^2$		8	10	10		100

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
$3^2\ 1^4$		10	21	15	$\{1\}$	261
2^5		10	10	10		100
$2^4\ 1^2$		11	10	10		100
$2^4\ 1^2$	$\varepsilon_2 = 0$	13	5	5		25
$2^3\ 1^4$		13	5	5		25
$2^2\ 1^6$		16	5	5		25
$2^2\ 1^6$	$\varepsilon_2 = 0$	17	4	4		16
$2\ 1^8$		20	1	1		1
1^{10}		25	1	1		1
Total			312	291		3840

Case (VIII). $G = O_{2n}$, $p = 2$, $u \in G^0$.

$n = 2$, $G = O_4$.

λ	ε	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
2^2		0	1	1	1		1
2^2	$\varepsilon_2 = 0$	1	1	1	1		1
2^2	$\varepsilon_2 = 0$	1	1	1	1		1
1^4		2	1	1	1		1
Total			4		4		4

$n = 3$, $G = O_6$.

λ	ε	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
$4\ 2$		0	1	1	1		1
3^2		1	3	2	3	$\{1\}$	9
$2^2\ 1^2$		2	2	2	2		4
$2^2\ 1^2$	$\varepsilon_2 = 0$	3	3	2	3	$\{1\}$	9
1^6		6	1	1	1		1
Total			10		10		24

λ	ε	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
<u>n = 4</u>, $G = O_8$.							
6 2		0	1	1	1		1
4^2		1	4	3	4	{1}	16
4^2	$\varepsilon_4 = 0$	2	3	3	3		9
4^2	$\varepsilon_4 = 0$	2	3	3	3		9
4 2 1^2		2	3	3	3		9
3^3 1^2		3	14	6	8	(*)	100
2^4		4	2	2	2		4
2^4	$\varepsilon_2 = 0$	6	3	3	3		9
2^4	$\varepsilon_2 = 0$	6	3	3	3		9
2^2 1^4		6	3	3	3		9
2^2 1^4	$\varepsilon_2 = 0$	7	4	3	4	{1}	16
1^8		12	1	1	1		1
Total			44		38		192

(*) {1} is the stabilizer of 8 elements, $\langle a_1 \rangle$ is the stabilizer of 2 elements, $\langle a_3 \rangle$ is the stabilizer of 2 elements and A(u) stabilizes 2 elements.

λ	ε	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
<u>n = 5</u>, $G = O_{10}$.							
8 2		0	1	1	1		1
6 4		1	5	4	5	{1}	25
6 2 1^2		2	4	4	4		16
5^2		2	10	6	10	{1}	100
4^2 1^2		3	25	12	15	(*)	325
4^2 1^2	$\varepsilon_4 = 0$	4	20	12	20	{1}	400
4 2^3		4	5	5	5		25
4 2 1^4		6	6	6	6		36

λ	ε	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
$3^2\ 2^2$		5	10	6	10	$\{1\}$	100
$3^2\ 2^2$	$\varepsilon_2 = 0$	6	20	12	20	$\{1\}$	400
$3^2\ 1^4$		7	25	12	15	(**)	325
$2^4\ 1^2$		8	5	5	5		25
$2^4\ 1^2$	$\varepsilon_2 = 0$	10	10	6	10	$\{1\}$	100
$2^2\ 1^6$		12	4	4	4		16
$2^2\ 1^6$	$\varepsilon_2 = 0$	13	5	4	5	$\{1\}$	25
1^{10}		20	1	1	1		1
Total			156		136		1920

(*) $\{1\}$ is the stabilizer of 12 elements, $\langle a_1\rangle$ is the stabilizer of 6 elements, $\langle a_3\rangle$ is the stabilizer of 2 elements and A(u) stabilizes 5 elements.

(**) $\{1\}$ is the stabilizer of 12 elements, $\langle a_1\rangle$ is the stabilizer of 2 elements, $\langle a_3\rangle$ is the stabilizer of 6 elements and A(u) stabilizes 5 elements.

<u>Case (IX)</u>. $G = O_{2n}$, $p = 2$, $u \in G^o$.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
<u>n = 2</u>. $G = O_4$.						
4		0	1	1		1
$2\ 1^2$		1	1	1		1
Total			2	2		2
<u>n = 3</u>. $G = O_6$.						
6		0	1	1		1
$4\ 1^2$		1	3	2	$\langle a_1\rangle$	5
2^3		2	1	1		1
$2\ 1^4$		4	1	1		1

λ	ε	d(u)	s(u)	$s'_a(u)$	$s_a(u)$	B(u)	q(u)
$3^2\ 2^2$		5	10	6	10	$\{1\}$	100
$3^2\ 2^2$	$\varepsilon_2 = 0$	6	20	12	20	$\{1\}$	400
$3^2\ 1^4$		7	25	12	15	(**)	325
$2^4\ 1^2$		8	5	5	5		25
$2^4\ 1^2$	$\varepsilon_2 = 0$	10	10	6	10	$\{1\}$	100
$2^2\ 1^6$		12	4	4	4		16
$2^2\ 1^6$	$\varepsilon_2 = 0$	13	5	4	5	$\{1\}$	25
1^{10}		20	1	1	1		1
Total			156		136		1920

(*) $\{1\}$ is the stabilizer of 12 elements, $\langle a_1\rangle$ is the stabilizer of 6 elements, $\langle a_3\rangle$ is the stabilizer of 2 elements and A(u) stabilizes 5 elements.

(**) $\{1\}$ is the stabilizer of 12 elements, $\langle a_1\rangle$ is the stabilizer of 2 elements, $\langle a_3\rangle$ is the stabilizer of 6 elements and A(u) stabilizes 5 elements.

<u>Case (IX)</u>. $G = O_{2n}$, $p = 2$, u G G^o.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
<u>n = 2</u>. $G = O_4$.						
4		0	1	1		1
$2\ 1^2$		1	1	1		1
Total			2	2		2
<u>n = 3</u>. $G = O_6$.						
6		0	1	1		1
$4\ 1^2$		1	3	2	$\langle a_1\rangle$	5
2^3		2	1	1		1
$2\ 1^4$		4	1	1		1

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
Total			6	5		8

<u>n = 4</u>, $G = O_8$.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
8		0	1	1		1
$6\ 1^2$		1	5	3	$\langle a_1\rangle$	13
$4\ 2^2$		2	3	3		9
$4\ 2^2$	$\varepsilon_2 = 0$	3	3	3		9
$4\ 1^4$		4	4	3	$\langle a_1\rangle$	10
$3^2\ 2$		3	1	1		1
$2^3\ 1^2$		5	2	2		4
$2\ 1^6$		9	1	1		1
Total			20	17		48

<u>n = 5</u>, $G = O_{10}$.

λ	ε	d(u)	s(u)	$s_a(u)$	B(u)	q(u)
10		0	1	1		1
$8\ 1^2$		1	7	4	$\langle a_1\rangle$	25
$6\ 2^2$		2	8	6	$\langle a_1\rangle$	40
$6\ 2^2$	$\varepsilon_2 = 0$	3	8	8		64
$6\ 1^4$		4	9	6	$\langle a_1\rangle$	45
$4^2\ 2$		3	4	4		16
$4^2\ 2$	$\varepsilon_4 = 0$	4	1	1		1
$4\ 3^2$		4	6	6		36
$4\ 2^2\ 1^2$		5	8	8		64
$4\ 2^2\ 1^2$	$\varepsilon_2 = 0$	6	10	6	$\langle a_1\rangle$	52
$4\ 1^6$		9	5	4	$\langle a_1\rangle$	17
$3^2\ 2\ 1^2$		6	3	3		9
2^5		8	2	2		4
$2^3\ 1^4$		10	3	3		9
$2\ 1^8$		16	1	1		1
Total			76	63		384

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