THE GEOMETRY OF THE MAP $\eta : Q\,BU(1) \to BU$

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CONTENTS

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SUMMARY

We study several geometric constructions associated to the map $\eta : Q\,BU(1) \to BU$. Using techniques of infinite loop spaces, we define a set of universal characteristic classes $\hat{c}_k$, proving that they agree with the Chern classes. We get geometric interpretations of them in terms of the k-tuple points of immersions. Also, we show some cases where this description is useful.

INTRODUCTION

In this dissertation we study the geometry of the map $\eta : Q\,BU(1) \to BU$, following the ideas developped by B.J. Sanderson in [26] to study Mahowald's map $\Omega^2 S^3 \to BU$. We describe the contents chapter by chapter.

Chapter 1 gives a slight introduction to the main categories used in the work. In the first paragraph, we look into Adams description of the category of spectra (see [1]), including the definition of the homology and cohomology theories associated to a spectrum.

In the second paragraph we give May's description of the category of infinite loop spaces (see [21]).

Chapter 2 contains the general theory of infinite loop spaces to construct the map η. In the first paragraph we state May's recognition principle for infinite loop spaces ([20]), proving that BU is an infinite loop, as in [21] . Then, η is defined as the unique infinite loop map extending the inclusion $BU(1) \subset BU$. The last paragraph gives the approximation of $Q\,BU(1)$ by $c(BU(1))$ for a convenient coefficient system $\mathcal{C}$ as in [9] .

The main goal of chapter 3 is to define universal characteristic classes $\hat{c}_k \in H^{2k}(BU,\mathbb{Z})$. The first paragraph follows Snaith ([27]) and Becker ([3]) to construct a map $\tau : BU \to Q\,BU(1)$ that is the right homological inverse of η. We begin the second paragraph by stating the stable splitting of QX got by F.Cohen-P. May and L. Taylor in [9] . We are then able to identify their space $D_r(F(\mathbb{R}^\infty), BU(1))$ as $T\gamma^{(r)}$, the Thom space of the vector bundle $\gamma^{(k)} = E\Sigma_k \times_{\Sigma_k} \gamma^k$. We prove also that $B\gamma^{(k)} = E\Sigma_k \times_{\Sigma_k} BU(1)^k$ is homotopic to the classifying space of $\Sigma_k \int U(1)$-principal bundles, $B\Sigma_k \int U(1)$. Then, the map induced by the inclusion of groups

$$p_k = Bi_k : B\,\Sigma_k \int U(1) \to BU(k)$$

classifies $\gamma^{(k)}$ and $\hat{c}_k$ can be defined as the composition

$$\Sigma^\infty BU \xrightarrow{\tau} \Sigma^\infty Q\, BU(1) \xrightarrow{h_k} \Sigma^\infty T\gamma^{(k)} \xrightarrow{Tp_k} \Sigma^\infty MU(k) \hookrightarrow \Sigma^{2k} MU \xrightarrow{t} \Sigma^{2k} H\mathbb{Z}$$

where t is the universal Thom class, so

$$\hat{c}_k = \tau^* h_k^* (t\gamma^{(k)})$$

$t\gamma^{(k)}$ being the Thom class of the vector bundle $\gamma^{(k)}$.

Chapter 4 is devoted to identifying $\hat{c}_k$ in terms of the universal Chern classes c_k. We do this by evaluating the Kronecker product $<\hat{c}_k, a>$ for any $a \in H_{2k}(BU; \mathbb{Z})$. Since $H_*(BU; \mathbb{Z})$ is a polynomial ring ring on the classes $\{a_i\}_{i \in \mathbb{N}}$, we only need to know

$$< \hat{c}_k, a_{i_1}, \dots, a_{i_n} > = < t\gamma^{(k)}, h_{k_*} \tau_* (a_{i_1}, \dots, a_{i_n})>$$

If we reduce the coefficients mod p for a prime p, we can use the calculations done in [8] and [24] to get that τ_* is the inclusion and h_{k_*} annihilates all monomials but $a_1^{\,k}$. Using [16], we prove that $<t\gamma^k, a_1^{\,k}> = 1$ so $\hat{c}_k = c_k$ as elements of $H^*(BU; \mathbb{Z}_p)$. As this is true for any p, an easy argument shows that $\hat{c}_k = c_k$ in $H^*(BU; \mathbb{Z})$.

The three following chapters have the general aim of getting a geometric description of $\hat{c}_k(\xi)$, for ξ a complex vector bundle over a weakly complex compact manifold M.

It is known that such ξ is classified by a homotopy class of maps

$$f_\xi : M \to BU \ .$$

Using τ, we define the composite map

$$f'_\xi = \tau \circ f_\xi : M \to Q\,BU(1) ,$$

and we study such maps in chapter 5. The first paragraph recalls the work of Koschorke and Sanderson [17], classifying the homotopy class of f' by a bordism class of triples $[(N,g,\tilde{g})]$, where g is an embedding of N into $M \times \mathbb{R}^\infty$ projecting to an immersion $f: N \to M$ and $\tilde{g}$ classifies the normal bundle of f, ν, as a complex line bundle. In the second paragraph, we study the manifold of k-tuple points, N_k, proving that the map $f_k: N_k \to M$ induced by f is an immersion with normal bundle $\nu_k = \nu^k/\Sigma_k$. The last paragraph begins with some properties of the manifold of based k-tuple points, N'_k, proving that the map $f'_k: N'_k \to M$ induced by the projection is an immersion with normal bundle $\nu'_k =$ $\nu'_k = \nu^{k-1} \times \{0\}/\Sigma_k$. Then, we define an extension of f to ν in good position, $\bar{f}$, to be one that has, for any k, maps $\bar{f}_k, \bar{f}'_k, \bar{\bar{f}}_k$ extending f_k, f'_k, making commutative the diagram

$$\begin{tikzcd}
\nu \arrow[rr, bend left, "\bar{f}"] & N \arrow[l] \arrow[r, "f"] & M \\
 & \nu'_k \arrow[u, "\bar{f}'_k"] \arrow[r] \arrow[dl] & \nu_k \arrow[u, "\bar{f}_k"'] \\
\pi_k^*(\nu_k) \arrow[uu, "\bar{\bar{f}}_k"] \arrow[urr] & &
\end{tikzcd}$$

We prove that the map $h_k \circ f_\xi |_{M - \mathrm{Im}\, f_{k+1}}$ is homotopic to the Thom-Pontrjagin construction of a bundle $M_k \to N_k$.

Chapter 6 is the tedious proof of the existence and uniqueness, up to isotopy, of extensions in good position. First, we prove the existence of a very particular kind of chart and, then, we glue them inductively by using a isotopy result similar to the one in Mather's notes [19] .

In chapter 7 we complete the geometric description of $\hat{c}_k(\xi)$. The first paragraph contains an exposition of complex bordism and cobordism theory as sketched by Quillen in [25] getting, in particular, the interpretation of the duality theorems of Lefschetz and Poincaré and the elements representing the Thom class and the fundamental class of a weakly complex manifold M. Using it, in the last paragraph we get that

$$i_*(P\,D(\hat{c}_k(\xi))) = i_*(f_{k*}([N_k]))$$

where i is the inclusion map $i : M \to (M, \operatorname{Im} \bar{f}_{k+1})$ and $[N_k]$ is the fundamental class of $[N_k]$. As i_* is a monomorphism we get the appropiate description of $\hat{c}_k(\xi)$.

In the last chapter we try to recover some information about ξ from the triple $(N,g,\bar{g})$. In the first paragraph we have a closer look at the map η, and we use it to give, in the second paragraph, a general description of ξ together with some interesting particular cases.

The work ends with an appendix on BU where we prove a result used in proving that the map τ is the right homotopical inverse of η .

CHAPTER 1: A description of the categories used.

We work most of the time in the category, Top, of compactly generated weakly Hausdorff topological spaces and continuous maps between them. The associated based category Top_* has as objects spaces of Top together with a non degenerate base point, and as morphisms, maps of Top that preserve the base point. Then, when we say a space, we mean an object of Top and a based space is an object of Top_* .

These categories have been studied recently (see [18] , [28],[31]) and all the usual constructions (like taking suspensions or loops) are well defined in them. Also, they are good categories for doing homotopy theory and the associated homotopy categories are H Top and H Top_* .

There are two interesting functors between them :
the forgetful functor

$$F : Top_* \to Top \quad (\text{or } F: H\, Top_* \to H\, Top)$$

that forgets the base point, and the functor

$$(\ \ )^+ : Top \to Top_* \ (\text{or } (\ \ )^+ : H\, Top \to H\, Top_*)$$

that adds a disjoint base point to each space (i.e. $X^+ = X \amalg \{*\}$) .

Sometimes, we shall use the categories, CW, of CW complexes and cellular maps and, H CW, of spaces having the homotopy type of a CW complex and homotopy classes of maps between them. This category was studied in [22] and it is closed under the usual constructions. The associated pointed categories are CW_* and H CW_* .

§1.1 The stable category

In this paragraph we describe in short Boardman's stable category as done in [1], [2].

1.1 DEFINITION.- A CW-spectrum E is given by

i) A sequence of based complexes $\{E_n\}$,

ii) A sequence of cellular inclusions $\{\varepsilon_n\}$ where $\varepsilon_n: SE_n \to E_{n+1}$, and SE_n is the suspension of E_n.

Notice that the existence of ε_n is equivalent to the existence of adjoints

$$\hat{\varepsilon}_n : E_n \to \Omega E_{n+1}$$

and the spectrum E is called an Ω-spectrum if the maps $\hat{\varepsilon}_n$ are weak equivalences for every n. The index set may run over the integers or over $\mathbb{N} = \{0,1,2,3,\ldots,\}$.

1.2 NOTE.- Given a sequence of based complexes $\{E_n\}_{n\in N}$ and maps $\{\varepsilon_n\}_{n\in N}$, where $\varepsilon_n: SE_n \to E_{n+1}$ we can replace them inductively by homotopy equivalent complexes $\{E'_n\}_{n\in N}$ and cellular inclusions $\{\varepsilon'_n\}_{n\in N}$ giving a spectrum.

1.3 EXAMPLES.- a) Let G be a group. We say that a based complex X is the n^{th} Eilenberg-MacLane space of G if the homotopy groups of X are

$$\pi_i(X,*) = \begin{cases} G & \text{if } i=n \\ 0 & \text{otherwise.} \end{cases}$$

This complex is unique up to homotopy (see [30]) and we call it $K(G,n)$.

Then, the Eilenberg - MacLane spectrum of G, called $\mathbb{H}G$, is defined as follows:

i) The spaces are the Eilenberg-Maclane spaces

$$\mathbb{H}G_n = K(G,n)$$

ii) The weak homotopy equivalences

$$\hat{\varepsilon}_n : K(G,n) \to \Omega(K(G,n+1))$$

are given by the uniqueness of the $n^{\underline{th}}$ Eilenberg-Maclane space up to homotopy since.

$$\pi_i(\Omega(K(G,n+1))) = \pi_{i+1}(K(G,n+1)) = \begin{cases} G & \text{if } i=n \\ 0 & \text{otherwise.} \end{cases}$$

b) Let $BU(n)$ be the classifying space of n-dimensional complex vector bundles and γ^n the universal n-dimensional complex vector bundle over it. For a definition see Chapter 2.

We define the spectrum MU as follows:

i) The spaces are

$MU_{2n} = MU(n)$ the Thom space of γ^n for $n \in N$

$MU_{2n+1} = S\, MU(n)$

ii) The maps are:

$\varepsilon_{2n} : S\, MU(n) \to S\, MU(n)$ the identity

$\varepsilon_{2n+1} : S^2 MU(n) \to MU(n+1)$ the map of Thom spaces

associated to the bundle map induced by the inclusion

$$\begin{array}{ccc} \varepsilon^1_{\mathbb{C}} \oplus \gamma^n & \longrightarrow & \gamma^{n+1} \\ \downarrow & & \downarrow \\ BU(n) & \longrightarrow & BU(n+1) \end{array}$$

where $\varepsilon^1_{\mathbb{C}}$ is the trivial 1-dimensional complex vector bundle.

1.4 DEFINITION.- A <u>strict map</u> between two spectra,

$$f : E \to E' ,$$

is given by a sequence of maps $\{f_n\}$, where

$$f_n : E_n \to E'_n ,$$

and the diagram,

$$\begin{array}{ccc} S\,E_n & \xrightarrow{Sf_n} & S\,E'_n \\ \varepsilon_n \downarrow & & \downarrow \varepsilon'_n \\ E_{n+1} & \xrightarrow{f_{n+1}} & E'_{n+1} \end{array}$$

commutes for each n.

1.5 EXAMPLE.- It is known that, for any based space X, $\tilde{H}^k(X;\mathbb{Z})$ is isomorphic to the group $[X, K(\mathbb{Z}, k)]$, so the Thom class of the complex vector bundle γ^n, $t(\gamma^n)$, can be interpreted as a homotopy class of maps.

$$t(\gamma^n) : MU(n) \to K(\mathbb{Z}, 2n) .$$

Chosing inductively the maps t_n so that

$t_{2n} : MU(n) \to K(\mathbb{Z}, 2n)$ is in the above class. and

$t_{2n+1} : S\,MU(n) \to K(\mathbb{Z}, 2n+1)$ is given by the suspension of t_{2n}.

we get a strict map of spectra,

$$t : MU \to \mathbb{H}\,\mathbb{Z} .$$

1.6 DEFINITION.- We say that a spectrum E' is a <u>subspectrum</u> of E if for any n, E'_n is a subcomplex of E_n and ε'_n is the restriction of ε_n. We say that E' is <u>cofinal</u> in E if for any cell e_α in E_n there is some N such that the N^{th} suspension of e_α lies in E'_{n+N} .

1.7 DEFINITION.- <u>A map between spectra</u>

$$f : E \to E'$$

is a class of couples $(\bar{E},\bar{f})$, where $\bar{E}$ is a cofinal subspectrum of E and,

$$f : \bar{E} \to E' ,$$

is a strict map of spectra. Two couples $(\bar{E}_1, \bar{f}_1)$ and $(\bar{E}_2, \bar{f}_2)$ are equivalent if the restrictions of both maps to $\bar{E}_1 \cap \bar{E}_2$ agree.

With these objects and morphisms, we define the category Sp . See [1] or [30] for further details.

1.8 DEFINITION.- Let E be a spectrum and X a based space. We define the spectrum $E \wedge X$ by

i) The spaces $(E \wedge X)_n = E_n \wedge X$

ii) The maps $S(E_n \wedge X) \to E_{n+1} \wedge X$

are given by ϵ_n.

We can define a <u>homotopy between maps</u> of spectra

$$F : f_0 \sim f_1$$

as a map $F: E \wedge I^+ \to E'$ such that the composites $F \circ i_\epsilon = f_\epsilon$ for $\epsilon = 0,1$ where $i_\epsilon : E \to E \wedge I^+$ are the obvious inclusions.

We denote by $[E, E']$ the set of homotopy classes of maps from E to E' , and define the category $H\,Sp$, whose objects are those of Sp and whose morphism from E to E' are the elements of $[E, E']$.

1.9 DEFINITION.- We define the k^{th} <u>traslation functor.</u>

$$\Sigma^k : Sp \to Sp$$

on objects by

i) $\Sigma^k(E)_n = E_{n+k}$

ii) $\epsilon(\Sigma^k(E))_n = \epsilon_{n+k}$,

and on morphisms by

$$\Sigma^k ([(\bar{E}, \bar{f})]) = [(\Sigma^k(\bar{E}), \Sigma^k(\bar{f}))]$$

where $\Sigma^k(\bar{f})_n = \bar{f}_{n+k}$.

The induced functor on $H\,Sp$ is still called Σ^k

1.10 DEFINITION.- The <u>suspension functor</u>

$$\Sigma^\infty : CW_* \to Sp$$

is defined as follows:

For any complex X, $\Sigma^\infty(X)$ is the spectrum given by

i) $(\Sigma^\infty(X))_n = S^n X$

ii) $\epsilon_n : S(S^n X) \to S^{n+1}(X)$ is the associativity homeomorphism.

For any map $f: X \to Y$, the strict map of spectra.

$$\Sigma^\infty f : \Sigma^\infty(X) \to \Sigma^\infty(Y)$$

is given by $(\Sigma^\infty f)_n = S^n f$.

As in 1.9 , we also denote the induced funtor in homotopy by Σ^∞.

We say that f is <u>a stable map</u> between two based complexes X and Y , if it is a map of the associated suspension spectra

$$f : \Sigma^\infty(X) \to \Sigma^\infty(Y).$$

Notice that a stable map induces maps $f : E \wedge X \to E \wedge Y$, for any spectrum E.

1.11 DEFINITION.- Let E by a spectrum. We define the associated n^{th} reduced homology and cohomology groups of a based complex, X, to be

$$\tilde{E}_n(X) = [\Sigma^\infty S^0, \Sigma^{-n}(E \wedge X)]$$

$$\tilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n(E)] \qquad \text{for any} \quad n \in \mathbb{Z}.$$

Then, a stable map, f, from X to Y induces, by composition, the homomorphism,

$$f_n : \tilde{E}_n(X) \to \tilde{E}_n(Y)$$

$$f^n : \tilde{E}^n(X) \to \tilde{E}^n(Y) \qquad \text{for any} \quad n \in \mathbb{N}.$$

In this way $\{\tilde{E}_n\}$ and $\{\tilde{E}^n\}$ give homology and cohomology theories, from the category of CW complexes and stable maps.

Similarly, by composition, any map of spectra

$$f : E \to E'$$

induces natural transformations between the associated reduced homology and cohomology theories.

We define the (unreduced) homology and cohomology of a complex X as:

$$E_n(X) = \tilde{E}_n(X^+) \quad , \quad \text{and}$$

$$E^n(X) = \tilde{E}^n(X^+) \ .$$

1.12 REMARK.- i) If E is an Ω-spectrum there is an isomorphism

$$\tilde{E}_n(X) \xrightarrow{\approx} [X, E_n] \qquad \text{for any} \quad n \quad \text{(see [30])}.$$

Then, the reduced cohomology associated to the spectrum $\mathbb{H}G$ is isomorphic to the singular cohomology with coeffi--tients in G.

ii) The reduced homology and cohomology theories associated to the spectrum MU are the complex bordism and cobordism as we shall see in chapter 7 .

§1.2 Infinite loop spaces .

In this paragraph, we study the category of infinite loop spaces.

1.13 DEFINITION.- An <u>infinite loop space</u> X is a sequence $\{X_n, \partial_n\}_{n \in N}$ where, for each n :

i) X_n is a based space, and

ii) ∂_n is a homeomorphism, $\partial_n : \Omega X_{n+1} \xrightarrow{\approx} X_n$.

Obviously, each infinite loop space has an associated Ω-spectrum.

1.14 DEFINITION.- A <u>morphism of infinite loop spaces</u>

$$f : X \to Y$$

is a sequence of maps $\{f_n\}_{n \in \mathbb{N}}$, where, $f_n : X_n \to Y_n$ and for each n, the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] & Y_n \\ \Omega X_{n+1} \arrow[u, "\partial_n", "\wr"'] \arrow[r, "\Omega f_{n+1}"] & \Omega Y_{n+1} \arrow[u, "\partial_n"', "\wr"] \end{tikzcd}$$

commutes.

We have defined the category I_∞ of infinite loop spaces and morphisms

1.15 DEFINITION.- The functor

$$(\quad)_0 : I_\infty \to Top_*$$

is defined as follows:

i) On objects, $(X)_0 = X_0$

ii) On morphisms, $(f)_0 = f_0$

We say that a space is an infinite loop space if it is in the image of $(\quad)_0$ and similarly with maps.

1.16 DEFINITION.- We define the functor

$$\Omega^{\infty} : Sp \to I_{\infty}$$

as follows:

For any spectrum E we define $\Omega^{\infty}(E)_n = \lim \Omega^{m-n} E_m$ where the limit is taken with respect to the maps

$$\Omega^{m-n} E_m \xrightarrow{\Omega^{m-n}\varepsilon_m} \Omega^{m-n} \Omega E_{m+1} \xrightarrow{\simeq} \Omega^{(m+1)-n} E_{m+1}$$

the homeomorphisms $\Omega(\Omega^{\infty}(E)_{n+1}) \to \Omega^{\infty}(E)_n$ are given by the limit of the identities.

For any morphism $f = [(\bar{E}, \bar{f})]$ the associated map is defined cellularly as follows:

Let e_{α} be a cell of E_m and let N be an index such that $S^N e_{\alpha}$ lies in $\bar{E}_{m+N}$. Then, there is defined the map

$$\hat{\bar{f}} : e_{\alpha} \to \Omega^N E'_{m+N} ,$$

giving the restriction of the map $\Omega^{\infty}(f)_n$ to $\Omega^{m-n} e_{\alpha}$. In the limit, these restrictions glue together, giving maps

$$\Omega^{\infty} (f)_n : \Omega^{\infty}(E)_n \to \Omega^{\infty}(E')_n$$

that produce a morphism of infinite loop spaces.

By abuse of notation we denote also by Ω^{∞} the composite functor

$$Sp \to CW_*$$

and also the induced functor in the homotopy categories.

1.17 PROPOSITION [2] Σ^{∞} and Ω^{∞} are adjoint functors.

□

If we define $Q = \Omega^{\infty} \Sigma^{\infty}$, then it is easy to prove the following result,

1.18 PROPOSITION.-

i) $Q(X)$ is the infinite loop space generated by X ; i.e., X is included in $Q(X)$ and any map to an infinite loop space $f: X \to Y$ has a unique extension to a map of infinite loop spaces

$$\bar{f} : QX \to Y$$

ii) For any based CW complexes X, Y, there is a 1:1 correspondence

$$[\Sigma^{\infty} X, \Sigma^{\infty} Y] \xrightarrow{\sim} [X, Q(Y)]$$

given by adjuntion.

□

Notice that by i) there is a unique map of infinite loop spaces

$$c : Q\, QX \to QX$$

extending the identity; it is called the "collapsing map".

CHAPTER 2 : The map $\eta : QBU(1) \to BU$

The object of this chapter is the definition and study of the map

$$\eta : Q\,BU(1) \to BU$$

, the unique map of infinite loop spaces that extends the inclusion. To do this, we need two standard results of infinite loop space theory, the "recognition principle", to know when a space is homotopic to an infinite loop space, and the "approximation theorem", giving the structure of QX for a connected space X. These results are given in the next two paragraphs.

§2.1 BU as infinite loop space.

To state and use the "recognition principle", we need some familiarity with the concept of operad and examples of it.

2.1 DEFINITION.- An <u>operad</u> $\mathcal{C}$ is

i) A sequence of spaces $\{\mathcal{C}(n)\}_{n\geq 0}$ with $\mathcal{C}(0)=\{*\}$

ii) A composition law for any $n, j_1, \ldots, j_n$

$$\phi : \mathcal{C}(n)\times \mathcal{C}(j_1)\times \ldots \times \mathcal{C}(j_n) \to \mathcal{C}(j)$$

where $j=\sum_{i=1}^{n} j_i$; and an element $1\in\mathcal{C}(1)$ satisfying:

A) For each n and $c\in \mathcal{C}(n)$

$$\phi\,(1;c) = c$$

B) For each n and $c\in\mathcal{C}(n)$

$$\phi\,(c;\underbrace{1,\ldots,1}_{n \text{ copies}}) = c$$

iii) For each n ; a right Σ_n-action on $\mathcal{C}(n)$

$$\alpha : \mathcal{C}(n) \times \Sigma_n \to \mathcal{C}(n)$$

where Σ_n is the symmetric group of n letters. We denote this action by $\alpha(c,\sigma) = c\sigma$, and it satisfies:

A) For each $c \in \mathcal{C}(n)$, $c_i \in \mathcal{C}(j_i)$, $i=1,\dots,n$ and $\sigma \in \Sigma_n$

$\phi(c\sigma\,;\, c_1,\dots,c_n) = \phi(c;\, c_{\sigma^{-1}(1)},\dots,c_{\sigma^{-1}(n)})\ \sigma(j_1,\dots,j_n)$

where $\sigma(j_1,\dots,j_n)$ acts on $(1,\dots,j)$ by permuting the blocks $B_1 = (1,\dots,j_1)$ $B_2 = (j_1+1,\dots,j_1+j_2)\dots$ $B_n = (\sum_{i=1}^{n-1} j_i+1,\dots, j)$ as σ does with $(1,\dots,n)$.

B) For each $c \in \mathcal{C}(n)$, $c_i \in \mathcal{C}(j_i)$ and $\sigma_i \in \Sigma_{j_i}$ $i=1,\dots,n$

$\phi(c;\, c_1\sigma_1,\dots,c_n\sigma_n) = \phi(c;c_1,\dots,c_n)\ (\sigma_1 \oplus \dots \oplus \sigma_n)$ where $(\sigma_1 \oplus \dots \oplus \sigma_n)$ acts on $(1,\dots,j)$ leaving the blocks B_i fixed and permuting the letters in each B_i as σ_i does.

2.2 EXAMPLES.- 1) Let X be a space. We define the endomorphisms operad of X, ξ_X, as follows:

i) $\xi_X(n) = \{f: X^n \to X : f$ is a continuous map $\}$

ii) $\phi: \xi_X(n) \times \xi_X(j_1) \times \dots \times \xi_X(j_n) \to \xi_X(j)$ is given by composition: i.e $\phi(f;\, g_1,\dots,g_n) = f\circ(g_1 \times \dots \times g_n)$

iii) The Σ_n-action on $\xi_X(n)$ is given by composition; i.e. for any $\sigma \in \Sigma_n$ we define $\sigma : X^n \to X^n$ by the formula $\sigma(x_1,\dots,x_n) =$ $= (x_{\sigma(1)},\dots,x_{\sigma(n)})$. Then the action is given by $f\sigma = f\circ\sigma$

2) Let $\mathbb{R}^\infty$ be the limit of $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ with respect to the maps

$$\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \times \{0\} \to \mathbb{R}^{n+1} .$$

We define the isometries operad, $\mathcal{L}_\infty$, as follows:

i) $\mathcal{L}_\infty(n) = \{\psi: (\mathbb{R}^\infty)^n \to \mathbb{R}^\infty ;\ \psi$ is an injective linear isometry$\}$

ii) and iii). As in 1), the composition law and the Σ_n-action are given by composition.

3) Let $I^r = [0,1]^r$ be the standard cube in $\mathbb{R}^r$. We call an <u>r-little cube</u> a map $c\colon \operatorname{int} I^r \to \operatorname{int} I^r$, of the type $X_1 \times, \ldots, \times X_r$, where $X_i\colon (0,1) \to (0,1)$ is a strictly increasing affine map.

We define the <u>r-little cubes operad</u>, $\mathcal{C}_r$ as follows:

i) $\mathcal{C}_r(n) = \{(c_1,\ldots,c_n)\colon c_i$ are r-cubes with disjoint closures$\}$

ii) $\phi\colon \mathcal{C}_r(n) \times \mathcal{C}_r(j_1) \times,\ldots,\times \mathcal{C}_r(j_n) \to \mathcal{C}_r(j)$ is given by composition;

$$\phi\,((c_1,\ldots,c_n);\ (d^1_1,\ldots,d^1_{j_1}),\ldots,\ (d^n_1,\ldots,d^n_{j_n})) =$$

$$=(c_1\circ d^1_1,\ldots,\ c_1\circ d^1_{j_1},\ldots,\ c_n\circ d^n_1,\ldots,\ c_n\circ d^n_{j_n})$$

iii) $(c_1,\ldots,c_n)\sigma = ((c_{\sigma(1)},\ldots,c_{\sigma(n)})$

In all three cases is easy to show that they are operads and the identity map is the 1.

2.3 DEFINITION.- A <u>morphism between two operads</u>

$$f\colon \mathcal{C} \to \mathcal{C}',$$

is a sequence of maps, $\{f_n\}_{n\in\mathbb{N}}$, with $f_n\colon \mathcal{C}(n) \to \mathcal{C}'(n)$ and satisfying

i) For each $n, j_1,\ldots,j_n$, the diagrams

$$\begin{tikzcd}
\mathcal{C}(n) \times \mathcal{C}(j_1) \times,\ldots,\times \mathcal{C}(j_n) \arrow[r, "\phi"] \arrow[d, "f_n \times f_{j_1} \times \ldots \times f_{j_n}"] & \mathcal{C}(j) \arrow[d, "f_j"] \\
\mathcal{C}'(n) \times \mathcal{C}'(j_1) \times,\ldots,\times \mathcal{C}'(j_n) \arrow[r, "\phi"] & \mathcal{C}'(j)
\end{tikzcd}$$

commute.

ii) $f_1(1_{\mathcal{C}}) = 1_{\mathcal{C}'}$

iii) For each n, the diagrams

$$\begin{tikzcd} \mathcal{C}(n) \times \Sigma_n \arrow[r, "\alpha"] \arrow[d, "f_n \times 1"] & \mathcal{C}(n) \arrow[d, "f_n"] \\ \mathcal{C}'(n) \times \Sigma_n \arrow[r, "\alpha'"] & \mathcal{C}'(n) \end{tikzcd}$$

commute.

2.4 EXAMPLE.- If we identify the r-little cube c_i, with the (r+1)-little cube $c_i \times 1$, we get a morphism of operads.

$$\mathcal{C}_r \to \mathcal{C}_{r+1}$$

given by the inclusions $\mathcal{C}_r(n) \subset \mathcal{C}_{r+1}(n)$. We can define a new operad $\mathcal{C}_\infty$ by $\mathcal{C}_\infty(n) = \bigcup_{r=1}^{\infty} \mathcal{C}_r(n)$ and we get inclusions $\mathcal{C}_r \subset \mathcal{C}_\infty$, for any r.

2.5 DEFINITION.- Let $\mathcal{C}$ be an operad and X a space. We said that X is a $\mathcal{C}$-space if there is a morphism of operads

$$\Theta : \mathcal{C} \to \xi_X ,$$

that we call a $\mathcal{C}$-action on X.

Notice that Θ is given by a sequence of maps

$$\Theta_n : \mathcal{C}(n) \times X^n \to X$$

commuting with the composition law and the Σ_n-action.

2.6 DEFINITION.- An operad $\mathcal{C}$ is an E_∞-operad if, for any n, $\mathcal{C}(n)$ is contractible and the Σ_n-action on $\mathcal{C}(n)$ is free.

A space X is an <u>E_∞-space</u> if it has a $\mathcal{C}$-action for some E_∞-operad $\mathcal{C}$.

2.7 EXAMPLES 1) $\mathcal{L}_\infty$ is an E_∞-operad since the Σ_n-action is obviously free and $\mathcal{L}_\infty(n)$ is contractible for any n (see [21]).

2) $\mathcal{C}_\infty$ is an E_∞-operad since the Σ_n-action is free and $\mathcal{C}_\infty(n)$ has the homotopy type of the space,

$$F(\mathbb{R}^\infty, n) = \{(x_1,\dots,x_n) \in (\mathbb{R}^\infty)^n : x_i \neq x_j \text{ for } i \neq j\} \text{(see [17])}$$

and

$$F(\mathbb{R}^\infty, n) \text{ is contractible } ([11]).$$

2.8 PROPOSITION.- Let X be an infinite loop space. Then, it has a natural $\mathcal{C}_\infty$-action (in particular, any infinite loop space is an E_∞-space).

<u>Proof</u>.- Let Y_r be such that $X = \Omega^r Y_r$. We define the maps

$$\Theta_n : \mathcal{C}_r(n) \times (\Omega^r Y_r)^n \to \Omega^r Y_r ,$$

where $\Theta_n((c_1,\dots,c_n); f_1,\dots,f_n) : (I^r, \partial I^r) \to (Y_r, *)$ is the map sending any x, lying in $\operatorname{Im} c_i$ to $f_i(c_i^{-1}(x))$ and any point outside $\bigcup_{i=1}^{n} \operatorname{Im} c_i$ to the base point.

These maps produce a $\mathcal{C}_r$-action on X. So, X is a $\mathcal{C}_r$-space for any r, and, as such actions are compatible, X is a $\mathcal{C}_\infty$-space.

□

The recognition principle is a partial converse of this result.

2.9 THEOREM.- [20] Let X be a connected E_∞-space. Then X has the weak homotopy type of an infinite loop space.

□

Now, we define the space BU, in a way appropriate to prove that it is an infinite loop spaces.

2.10 DEFINITION.- The <u>Stiefel manifold</u> of k-frames in $\mathbb{C}^n$ is defined as the space

$$V_{k,n} = \{(X_1,\dots,X_k) \in (\mathbb{C}^n)^k : \{X_i\}_{i=1}^{k} \text{ is orthonormal }\},$$

with the topology induced by $(\mathbb{C}^n)^k$.

The Grassmann manifold of k-planes in $\mathbb{C}^n$ is the space

$$G_{k,n} = \{V \subset \mathbb{C}^n : V \text{ is a k-dimensional complex subspace of } \mathbb{C}^n\},$$

with the quotient topology of the map

$$\psi : V_{k,n} \to G_{k,n}$$

that sends each k-frame into the subspace that it generates.

Over $G_{k,n}$, we define the k-dimensional complex vector bundle $\gamma_{k,n}$ by $E\gamma_{k,n} = \{(x,V) : x \in V \in G_{k,n}\}$

Using the isomorphism $\mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}$, we can consider $G_{k,n}$ as the set of k-planes in $\mathbb{C}^k \times \mathbb{C}^{n-k}$.

Identifying a k-plane in $\mathbb{C}^k \times \mathbb{C}^{n-k}$ with the one in $\mathbb{C}^k \times \mathbb{C}^{n+1-k}$ given by inclusion

$$\mathbb{C}^{n-k} \xrightarrow{\cong} \mathbb{C}^{n-k} \times \{0\} \to \mathbb{C}^{n+1-k},$$

we have an inclusion $G_{k,n} \subset G_{k,n+1}$, and we define $BU(k) = \bigcup_{n=k}^{\infty} G_{k,n}$. In the same way we have $\gamma^k = \bigcup_{n=k}^{\infty} \gamma_{k,n}$

2.11 THEOREM [23]. BU(k) classifies classes of isomorphic k-dimensional complex bundles over any paracompact space, and γ^k is its universal bundle.

□

Identifying a k-plane $V \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ with the (k+1)-plane $\mathbb{C} \oplus V \subset \mathbb{C}^{k+1} \times \mathbb{C}^{n-k}$ we get an inclusion $G_{k,n} \subset G_{k+1,n+1}$ and in the limit we have $BU(k) \subset BU(k+1)$, so we can define $BU = \bigcup_{k=1}^{\infty} BU(k)$.

2.12 THEOREM [23]. BU classifies stable classes of complex bundles over any paracompact space. □

Let $U(n)$ be the group of unitary automorphisms of $\mathbb{C}^n$ and $U(k,\ell)$ the group of unitary automorphisms of $\mathbb{C}^k \times \mathbb{C}^\ell$. Then we have the following results.

2.13 PROPOSITION.- The map

$$\alpha : \frac{U(k,n-k)}{U(k)\times U(n-k)} \to G_{k,n}$$

given by $\alpha([g]) = g(\mathbb{C}^k \times \{0\})$ is a homeomorphism

Proof.- As both spaces are compact Hausdorff, it is enough to see that α is a continuous bijection.

α is continuous since it is the map induced by the continuous map

$$\bar{\alpha} : U(k,\ n-k) \to V_{k,n}$$

given by $\bar{\alpha}(g)=(g(e_1),\ldots,g(e_k))$ where $(e_1,\ldots,e_n)$ is the standard basis of $\mathbb{C}^k \times \mathbb{C}^{n-k}$.

The bijectivity of α is immediate.

□

2.14 THEOREM.- In the limit, the maps

$$\alpha : \frac{U(k,\infty)}{U(k)\times U(\infty)} \to BU(k) \quad , \quad \text{and}$$

$$\alpha : \frac{U(\infty,\infty)}{U(\infty)\times U(\infty)} \to BU$$

are homeomorphisms.

Proof.- It is an immediate consequence of 2.13 and the fact that the maps α commute both with the inclusions $G_{k,n} \subset G_{k,n+1}$ and $G_{k,n} \subset G_{k+1,n+1}$.

□

2.15 REMARK.- By 2.14, a point $x \in BU$, can be interpreted either as a subspace of $\mathbb{C}^\infty \times \mathbb{C}^\infty$ of the type $\ldots \oplus \mathbb{C} \oplus \ldots \oplus \mathbb{C} \oplus V \oplus \{0\} \oplus \ldots \oplus \{0\} \oplus \ldots$, ,where V is a k-dimensional subspace of some $\mathbb{C}^k \times \mathbb{C}^{n-k}$, or as a class of unitary automorphisms of $\mathbb{C}^\infty \times \mathbb{C}^\infty$.

Also, by fixing a bijective linear isometry $\mathbb{C}^\infty \times \mathbb{C}^\infty \overset{\cong}{\to} \mathbb{C}^\infty$ we have both interpretations with $\mathbb{C}^\infty$ instead of $\mathbb{C}^\infty \times \mathbb{C}^\infty$.

2.16 THEOREM [21].- BU has the homotopy type of an infinite loop space

Proof.- We want to prove that BU is an $\mathcal{L}_\infty$-space, so we need to define maps $h_n : \mathcal{L}_\infty(n) \times (BU)^n \to BU$.

Let be $g \in \mathcal{L}_\infty(n)$ and $x_i \in BU$ for $i=1,\ldots,n$. Chose automorphism

$$f_i : \mathbb{C}^\infty \times \mathbb{C}^\infty \to \mathbb{C}^\infty \times \mathbb{C}^\infty$$

representing x_i. Then, $h_n(g; x_1,\ldots,x_n)$ is the class of the automorphism

$$f : \mathbb{C}^\infty \times \mathbb{C}^\infty \to \mathbb{C}^\infty \times \mathbb{C}^\infty$$

defined as follows:

: $g_{\mathbb{C}}$ the complexification of g. On $\mathrm{Im}(g_{\mathbb{C}} \times g_{\mathbb{C}})$, f is the unique map that makes the following diagram commute

$$\begin{tikzcd}
(\mathbb{C}^\infty \times \mathbb{C}^\infty)^n \arrow[r, "sh"] \arrow[d, "f_1 \times \ldots \times f_n"] & (\mathbb{C}^\infty)^n \times (\mathbb{C}^\infty)^n \arrow[r, "g_{\mathbb{C}} \times g_{\mathbb{C}}"] & \mathbb{C}^\infty \times \mathbb{C}^\infty \arrow[d, "f"] \\
(\mathbb{C}^\infty \times \mathbb{C}^\infty)^n \arrow[r, "sh"] & (\mathbb{C}^\infty)^n \times (\mathbb{C}^\infty)^n \arrow[r, "g_{\mathbb{C}} \times g_{\mathbb{C}}"] & \mathbb{C}^\infty \times \mathbb{C}^\infty
\end{tikzcd}$$

, where sh is the appropiate reordering map.

On the complementary subspace of $\mathrm{Im}(g_{\mathbb{C}} \times g_{\mathbb{C}})$ we define f as the identity.

It is easy to prove that the class of f is independent of the choice of f_i and that the maps h_n so defined, commute with the composition law and the Σ_n-action.

$\square$

Now, we define the map

$$\eta : Q\,BU(1) \to BU$$

as the unique map of infinite loop spaces extending the inclusion.

§2.2 The space Q BU(1)

We give a sketch of the proof of the approximation theorem for the space QX, when X is a connected space, and look with some detail into the case X= BU(1). To do it we need to develop the concept of a coefficient system (see [9]).

2.17 DEFINITION.- The category Λ has as objects the finite sets:

$$\underline{n} = \{0,1,\ldots, n\}$$

based at 0, and as morphism from $\underline{n}$ to $\underline{m}$, all the injective based maps

$$\phi : \underline{n} \to \underline{m} .$$

It is easy to see that any such map ϕ decompose as the product of a permutation $\sigma \in \Sigma_n$ and an injective order-preserving based map from $\underline{n}$ to $\underline{m}$, $\bar{\phi}$. Also, any injective order-preserving based map, $\bar{\phi}$, decomposes as a finite product of "degeneracy" maps

$$\sigma_{q,n} : \underline{n} \to \underline{n+1} \qquad \text{for } q= 0,\ldots, n$$

where

$$\sigma_{q,n}(i) = \begin{cases} i & \text{if } i \le q \\ i+1 & \text{if } i > q \end{cases}$$

2.18 DEFINITION.- A coefficient system, $\mathcal{C}$, is a contravariant functor.

$$\mathcal{C} : \Lambda \to Top_*$$

where $\mathcal{C}(\underline{0}) = \{*\}$

From now on, we will denote $\mathcal{C}(\underline{n})$ simply by $\mathcal{C}_n$, and $\mathcal{C}_n$ has an obvious Σ_n-action.

Notice that to know $\mathcal{C}$ on morphisms we only need to know it on the elements of Σ_n and the "degeneracy" maps.

2.19 EXAMPLES.- 1) Any operad, $\mathcal{C}$, has associated a natural structure of coefficient system taking $\mathcal{C}_n = \mathcal{C}(n)$. Then $\mathcal{C}(\sigma)$ is given by the Σ_n-action on $\mathcal{C}(n)$ for any $\sigma \in \Sigma_n$ and $\sigma_{q,n} = \mathcal{C}(\sigma_{q,n})$ is the map given by

$$\sigma_{q,n}(c) = \phi(c, s_q) \qquad \text{for any} \qquad c \in \mathcal{C}(n)$$

where $\quad s_q = (1,\ldots,1,*,1,\ldots,1) \in \mathcal{C}(1)^q \times \mathcal{C}(0) \times \mathcal{C}(1)^{n-q}$.

2) <u>The Stiefel coefficient system.</u> $\mathcal{V}_\infty$ is defined as follows:

On objects

$$\mathcal{V}_{\infty,n} = V_{n,\infty} = \{(v_1,\ldots,v_n) \in (\mathbb{R}^\infty)^n : \{v_i\} \text{ is orthonormal }\}$$

$$= \{\phi : \mathbb{R}^n \to \mathbb{R}^\infty : \phi \text{ is an injective linear isometry}\}$$

For any injective based map $\phi \in \Lambda(\underline{n},\underline{m})$ we define the associated injective linear isometry

$$\phi : R^n \to R^m$$

given by

$$\phi(x_1,\ldots,x_n) = (0,\ldots,\underset{\substack{\uparrow \\ \phi(1)}}{x_1},\ldots,\underset{\substack{\uparrow \\ \phi(2)}}{x_2},\ldots,\underset{\substack{\uparrow \\ \phi(n)}}{x_n},\ldots,0\ )$$

and the map $\phi = \mathcal{V}_\infty(\phi)$ is given by composition.

3) Let X be a space. The <u>coefficient system</u> $\mathcal{F}(X)$ <u>of configurations in</u> X is defined as follows:

On objects

$$\mathcal{F}(X)_n = F(X,n) = \{(x_1,\ldots,x_n) \in X^n : \ x_i \neq x_j \ \text{ for any } \ i \neq j\ \}$$

For any injective based map $\phi \in \Lambda(\underline{n},\underline{m})$ we define the map

$$\phi : F(X,m) \to F(X,n)$$

by

$$\phi(x_1,\ldots,x_m) = (x_{\phi(1)},\ldots,x_{\phi(n)})$$

2.20 DEFINITION.- A <u>morphism</u> <u>between</u> <u>coefficient</u> <u>system</u> $f: \mathcal{C} \to \mathcal{C}'$ is a natural transformation between the functors $\mathcal{C}$ and $\mathcal{C}'$.

Notice that the associated maps $f_n: \mathcal{C}_n \to \mathcal{C}'_n$ are maps of Σ_n-spaces.

2.21 EXAMPLES 1) Any morphism between operads induces a morphism between the associated coefficient systems.

2) We define the morphism.

$$\kappa : \mathcal{C}_\infty \to \mathcal{F}(\mathbb{R}^\infty)$$

given by the maps

$$\kappa : \mathcal{C}_{\infty,n} \to F(\mathbb{R}^\infty, n)$$

that sends each little r-cube to its center and the configuration we get in int I^∞ to R^∞ by a chosen homeomorphism

3) We define the morphism

$$\beta: \mathcal{V}_\infty \to F(\mathbb{R}^\infty)$$

given by the maps

$$\beta_n: V_{n,\infty} \to F(\mathbb{R}^\infty, n)$$

that send the n-frame $\{x_i\}_{i=1}^n$ to the configuration in $\mathbb{R}^\infty$, $\{x_i\}_{i=1}^n$

[illegible]

4) We define the morphism

$$\alpha : \mathcal{L}_\infty \to \mathcal{V}_\infty$$

given by the maps

$$\alpha_n: \mathcal{L}_{\infty,n} \to V_{n,\infty}$$

where

$$\alpha_n(f) = (f(e_{11}), \ldots, f(e_{1n}))$$

where $\{e_{j\,i}\}_{j=1}^{\infty}$ is the standard basis of the i^{th} factor of $(\mathbb{R}^\infty)^n$.

Now we associate, in a natural way, to each coefficient system, $\mathcal{C}$, a construction on topological spaces denoted c.

2.22 DEFINITION.- Let $\mathcal{C}$ be a coefficient system. We define the functor.

$$c\colon \mathrm{Top}_* \to \mathrm{Top}_*$$

as follows:

For any based space X

$$c\,X = \frac{\coprod_{r=0}^{\infty} \mathcal{C}_r \times X^r}{\sim}$$

, the quotient of the space $\coprod_{r=0}^{\infty} \mathcal{C}_r \times X^r$ by the equivalence relation generated by the relations

$$(c\,\phi;\, x_1,\dots,x_r) \sim (c;\, x_{\phi^{-1}(1)},\dots,x_{\phi^{-1}(r)}) \quad \text{for any} \quad \phi \in \Lambda(\underline{r},\underline{s})$$

where $x_{\phi^{-1}(i)} = *$ if $i \notin \mathrm{Im}\,\phi$

For any morphism $g\colon X \to Y$, the map

$$c\,g : c\,X \to c\,Y$$

is given by

$$c\,g(\,[(c;\, x_1,\dots,x_r)\,]) = [(c;\, g(x_1),\dots,g(x_r))\,].$$

Notice that for any morphism of coefficient systems $f\colon \mathcal{C} \to \mathcal{C}'$ we have a natural transformation $f\colon c \to c'$ given, for any space X, by the map

$$f_X : c\,X \to c'\,X$$

defined by

$$f_X(\,|(c\colon x_1,\dots,x_r)\,|\,) = |(f_r(c);\, x_1,\dots,x_r)|$$

2.23 REMARK.- If we define

$$F_n\, c\,X = \mathrm{Im}\,\Big(\coprod_{r=0}^{r} \mathcal{C}_r \times X^r \to cX \Big)$$

We get a natural filtration of $c\,X$, that is preserved by $c\,g$ for any map $g\colon X \to Y$ so we can consider the functor c taking values in the category of filtered topological spaces.

Also, for any map of coefficient systems $f: \mathcal{C} \to \mathcal{C}'$, f_X preserves the filtration, so the natural transformation f can be considered too in the category of filtered topological spaces.

2.24 LEMMA [9].- i) The inclusion $F_{n-1} c X \to F_n c X$ is a cofibration

ii) the diagram

$$\begin{array}{ccc} \mathcal{C}_n \times_{\Sigma_n} \sigma_n X & \longrightarrow & F_{n-1} c X \\ \downarrow & & \downarrow \\ \mathcal{C}_n \times_{\Sigma_n} X^n & \longrightarrow & F_n c X \end{array}$$

is a push out, where $\sigma_n X = \bigcup_{i=0}^{n-1} X^i \times \{*\} \times X^{n-i-1}$

Now it is easy to prove inductively the following

2.25 PROPOSITION [9] .- Let $f: \mathcal{C} \to \mathcal{C}'$ be a morphism of coefficient systems. Then,

i) if, for any n, Σ_n acts freely on $\mathcal{C}_n$ and $\mathcal{C}'_n$ and f_n is a weak homotopy equivalence, then f_X is a weak homotopy equivalence for any X.

ii) if, for any n, f_n is an equivariant homotopy equivalence, then f_X is a homotopy equivalence for any X.

2.26 DEFINITION.- Let X be a based space. The map

$$\alpha_n: c_n X \to \Omega^n S^n X$$

is given by:

$$\alpha_n([(c_1,\dots,c_r);(x_1,\dots,x_r)]):(I^n,\partial I^n) \to (S^n X, *)$$

is the map that sends a point $t \in \operatorname{Im} c_i$ to $(c_i^{-1}(t),x)$ and any point $t \notin \bigcup_{i=1}^{r} \operatorname{Im} c_i$ to the base point.

As the α_n are compatible, in the limit we get

$$\alpha_\infty: c_\infty X \to QX$$

2.27 THEOREM ([20]).- If X is connected, then α_n is a weak homotopy equivalence for any n (even $n=\infty$).

2.28 COROLLARY.- If X is connected and $\mathcal{C}$ is any of $\mathcal{C}_\infty$, $F(\mathbb{R}^\infty)$, $\mathcal{V}_\infty$ or $\mathcal{L}_\infty$, $c\,X$ has the weak homotopy type of QX.

Proof.- The maps α,β,k of 2.21 satisfy 2.25 so any two of the cX have the same weak homotopy type. Thus by 2.27 each one has the weak homotopy type of QX.

In particular, we want to study the space $F(\mathbb{R}^\infty)(BU(1))$ since it has the homotopy type of $Q\,BU(1)$.

2.29.- DEFINITION.- Let G be a group. Then, $\Sigma_n \int G$ is the group of $(n\times n)$-matrices with entries in G such that each row or column has a unique non-zero entry. The composition law is given by the product of matrices and the law in G.

If $G=U(1)$, we have an obvious inclusion

$$\Sigma_n \int U(1) \subset U(n),$$

and using it, we can interpret the elements of $\Sigma_n \int U(1)$ as unitary transformations of $\mathbb{C}^n$ that act by permuting the elements of the standard basis and multiplying them by modulus one scalars.

2.30 PROPOSITION.- If we define $B(\Sigma_n \int U(1)) = \dfrac{U(n,\infty)}{(\Sigma_n \int U(1))\times U(\infty)}$, then it is the classifying space of $\Sigma_n \int U(1)$-principal bundles.

Proof.- With this definition, it is easy to see that $B(\Sigma_n \int U(1))$ is the quotient of $EU(n)$, the total space of the universal $U(n)$-principal bundle, by the $\Sigma_n \int U(1)$-action induced by the inclusion $\Sigma_n \int U(1) \subset U(n)$.

Since $EU(n)$ is contractible and the $\Sigma_n \int U(1)$ action is free the quotient is the classifying space of $\Sigma_n \int U(1)$-principal bundles ([30]).

We define

$$p_n : B\,\Sigma_n \int U(1) \to BU(n) \quad \text{as the limit map of}$$

$$p_{n,k} : \frac{U(n,m)}{\Sigma_n \int U(1) \times U(m)} \to \frac{U(n,m)}{U(n) \times U(m)}$$

Let

$$q_n : F(\mathbb{R}^\infty, n) \times_{\Sigma_n} BU(1)^n \to B(\Sigma_n \int U(1))$$

be a map classifying the $\Sigma_n \int U(1)$-principal bundle

$$F(\mathbb{R}^\infty, n) \times EU(1)^n \to F(\mathbb{R}^\infty, n) \times_{\Sigma_n} BU(1)^n .$$

then we have the following results.

2.31 PROPOSITION.- q_n is a homotopy equivalence.

Proof.- $EU(1)$ and $F(\mathbb{R}^\infty, n)$ are contractible , so $F(\mathbb{R}^\infty, n) \times EU(1)^n$ is a contractible space. As the obvious $\Sigma_n \int U(1)$ action is free, the quotient space $F(\mathbb{R}^\infty, n) \times_{\Sigma_n} BU(1)^n$ is a classifying space for $\Sigma_n \int U(1)$-principal bundles and $F(\mathbb{R}^\infty, n) \times EU(n)$ is the universal bundle ([30]). Thus the map q_n that classifies it has to be an equivalence.

$\square$

2.32 PROPOSITION.- The diagram

$$\begin{tikzcd}
F(\mathbb{R}^\infty, n) \times_{\Sigma_n} BU(1)^n \arrow[r, "q_n"] & B(\Sigma_n \int U(1)) \\
F(\mathbb{R}^\infty, n-1) \times_{\Sigma_{n-1}} BU(1)^{n-1} \arrow[r, "q_{n-1}"] \arrow[u, "T_n"] & B(\Sigma_{n-1} \int U(1)) \arrow[u]
\end{tikzcd}$$

commutes up to homotopy , where the map τ_n is given by a chosen Σ_{n-1} -equivariant inclusion of $F(\mathbb{R}^\infty, n-1)$ in $F(\mathbb{R}^\infty, n)$ and the right hand vertical map is induced by the inclusion of groups.

Proof.- It is easy to see that the $\Sigma_n / U(1)$-principal bundle classified by the map $q_n \circ \tau_n$ has a reduction to the pull-back by q_{n-1} of the universal $\Sigma_{n-1} / U(1)$-principal bundle.

By induction, we choose maps

$$\bar{q}_n : B(\Sigma_n / U(1)) \longrightarrow F(\mathbb{R}^\infty, n) \times_{\Sigma_n} BU(1)^n$$

that are homotopy inverses of q_n and ,also, they commute the above diagram .

Now , we define i_n as the composition

$$B(\Sigma_n / U(1)) \xrightarrow{\bar{q}_n} F(\mathbb{R}^\infty, n) \times_{\Sigma_n} BU(1)^n \longrightarrow F_n(F(\mathbb{R}^\infty)(BU(1)))$$

2.33 PROPOSITION.- The diagram

$$\begin{tikzcd} B(\Sigma_n / U(1)) \arrow[r, "i_n"] & F_n(F(\mathbb{R}^\infty)(BU(1))) \\ B(\Sigma_{n-1} / BU(1)) \arrow[r, "i_{n-1}"] \arrow[u] & F_{n-1}(F(\mathbb{R}^\infty)(BU(1))) \arrow[u] \end{tikzcd}$$

commutes up to homotopy.

Proof.- It follows immediatly from the definition of i_n .

We choose a map

$$i_\infty : B(\Sigma_\infty / U(1)) \longrightarrow F(\mathbb{R}^\infty)(BU(1))$$

such that $i_\infty|_{B(\Sigma_n / U(1))} \sim i_n$

CHAPTER 3 . Definition of the Characteristic Classes $\hat{c}_k$.

As seen in chapter 1, to define $\hat{c}_k \in H^{2k}(BU;\mathbb{Z})$, it is enough to give the corresponding maps:

$$\hat{c}_k : \Sigma^\infty BU \to \Sigma^{2k} H\mathbb{Z}.$$

They are the composite maps

$$\Sigma^\infty BU \xrightarrow{\tau} \Sigma^\infty Q\, BU(1) \xrightarrow{\hat{h}_k} \Sigma^\infty T\gamma^{(k)} \xrightarrow{\hat{t}_k} \Sigma^{2k} H\mathbb{Z},$$

where τ is defined in §3.1 following [27], $\hat{h}_k$ is the splitting map of §3.2 (see [9]) and $\hat{t}_k$ is the Thom class of the bundle $\gamma^{(k)} = F(\mathbb{R}^\infty, k)\times_{\Sigma_k} (\gamma)^k$.

§3.1 The map $\tau : BU \to Q\, BU(1)$

The construction is done by inductive use of the "transfer".

3.1 THEOREM ([4]).-Let ξ be a fibre bundle over a finite complex B, with fibre a compact G-manifold, where G is a compact Lie group. Then, there is a stable map, called the transfer map,

$$\tau(\xi) : B^+ \to E^+$$

such that the composite of the maps induced in singular cohomology, with coefficient in a ring R,

$$H^*(B;R) \xrightarrow{p^*} H^*(E;R) \xrightarrow{\tau^*} H^*(B;R)$$

is the multiplication by $\chi(F)$, the Euler characteristic of F.

Moreover, the construction is natural with respect to morphisms of fibre bundles.

3.2 THEOREM.- Let

$$p_n : B\Sigma_n \int U(1) \to BU(n)$$

be the map defined in 2.30. Then, there is a stable map

$$\tau_n : BU(n)^+ \to B\Sigma_n \int U(1)^+$$

that is the right homological inverse of p_n.

<u>Proof</u>.- The bundles,

$$p_{n,k}: \frac{U(n,k)}{(\Sigma_n \int U(1)) \times U(k)} \to \frac{U(n,k)}{U(n) \times U(k)}$$

have fibre the $U(n)$ manifold $\dfrac{U(n)}{\Sigma_n \int U(1)}$.

By 3.1 there are stable maps,

$$\tau_{n,k}: \left(\frac{U(n,k)}{U(n)\times U(k)}\right)^+ \to \left(\frac{U(n,k)}{\Sigma_n \int U(1)\times U(k)}\right)^+,$$

such that $p_{n,k} \circ \tau_{n,k}$ induces in singular cohomology multiplication by $\chi(U(n)/\Sigma_n \int U(1)) = 1$ (by [15]).

As the $\tau_{n,k}$ commute with the inclusions, there is in the limit a stable map,

$$\tau_n : BU(n)^+ \to B\Sigma_n \int U(1)^+$$

such that the map $p_n \circ \tau_n$ induces the identity in singular cohomology. Thus τ_n is the right homological inverse of p_n.

To be able to take the limit of the τ_n, we use the following

3.3 THEOREM [27].- Let ξ be a fibre bundle with fibre the compact G-manifold F, for G a compact Lie group, and ξ_G the associated G-principal bundle. Let F_1 be a G-submanifold of F and N an equivariant tubular neighbourhood of F_1 in F. If there is an equivariant vector field on F such that the induced vector field on ∂N is homotopic to the outward normal field through a homotopy of non-zero vector fields, and the vectors have moduli 1 outside N, the diagram

$$\begin{tikzcd} B^+ \arrow[r, "\tau(p')"] \arrow[dr, "\tau"'] & (E\,\xi_G \times_G F_1)^+ \arrow[d] \\ & (E\,\xi_G \times_G F)^+ = E\xi^+ \end{tikzcd}$$

commutes.

3.4 THEOREM.- Let

$$p_\infty : B\Sigma_\infty \wr U(1) \to BU$$

be the limit of p_n. Then, there is a stable map,

$$\tau_\infty : BU^+ \to B\Sigma_\infty \wr U(1)^+$$

that is the right homological inverse of p_∞.

<u>Proof</u>.- Snaith proved, in [27], that the inclusion

$$\frac{U(n)}{\Sigma_n \wr U(1)} \to \frac{U(n+1)}{\Sigma_{n+1} \wr U(1)}$$

satisfies the hypothesis of 3.3, so the diagram.

$$\begin{tikzcd} & \left(EU(n+1)\big|_{BU(n)} \times_{U(n)} \frac{U(n)}{\Sigma_n \wr U(1)}\right)^+ = B\Sigma_n \wr U(1)^+ \arrow[d] \\ BU(n)^+ \arrow[ur, "\tau_n'"] \arrow[r, "\tau_n'"'] & \left(EU(n+1)\big|_{BU(n)} \times_{U(n)} \frac{U(n+1)}{\Sigma_{n+1} \wr U(1)}\right)^+ \end{tikzcd}$$

commutes.

The diagram of $U(n+1)/\Sigma_{n+1} \wr U(1)$ bundles

$$\begin{tikzcd}
EU(n+1)\big|_{BU(n)} \times_{U(n)} \dfrac{U(n+1)}{\Sigma_{n+1}\wr U(1)} \arrow[r] \arrow[d] & BU(n) \arrow[d] \\
EU(n+1) \times_{U(n+1)} \dfrac{U(n+1)}{\Sigma_{n+1}\wr U(1)} \arrow[r] & BU(n+1)
\end{tikzcd}$$

commutes; so, by naturality of the transfer, the diagram

$$\begin{tikzcd}
BU(n)^+ \arrow[r, "\tau_n'"] \arrow[d, dash] & \left(EU(n+1)\big|_{BU(n)} \times_{U(n+1)} \dfrac{U(n+1)}{\Sigma_{n+1} \wr U(1)}\right)^+ \arrow[d] \\
BU(n+1)^+ \arrow[r, "\tau_{n+1}"] & \left(EU(n+1) \times_{U(n+1)} \dfrac{U(n+1)}{\Sigma_{n+1} \wr U(1)}\right)^+ \simeq B\Sigma_{n+1} \wr U(1)^+ .
\end{tikzcd}$$

commutes.

Thus, the diagram

$$\begin{tikzcd}
BU(n)^+ \arrow[r, "\tau_n"] \arrow[d] & B\,\Sigma_n \wr U(1)^+ \arrow[d] \\
BU(n+1)^+ \arrow[r, "\tau_{n+1}"] & B\,\Sigma_{n+1} \wr U(1)^+
\end{tikzcd}$$

commutes, and we get in the limit a stable map

$$\tau_\infty : BU^+ \to B\Sigma_\infty \wr U(1)^+$$

that is, obviously, the right homological inverse of p_∞ .

□

As proved in [3], the same holds for

$$\tau_\infty : BU \to B\Sigma_\infty \wr U(1)$$

3.5 DEFINITION.- We define the map $\tau : BU \to Q\,BU(1)$ as the composite

$$BU \xrightarrow{\hat{\tau}_\infty} Q\,B\Sigma_\infty \int U(1) \xrightarrow{Q(i_\infty)} Q\,Q\,BU(1) \xrightarrow{c} Q\,BU(1),$$

where $\hat{\tau}_\infty$ is the adjoint of the stable map τ_∞.

3.6 LEMMA [27] .- The maps

$$B\,\Sigma_n \int U(1) \xrightarrow{i_n} Q\,BU(1) \xrightarrow{\eta} BU \qquad \text{and}$$

$$B\,\Sigma_n \int U(1) \xrightarrow{p_n} BU(n) \subset BU$$

induce the same homomorphism in homology.

□

3.7 THEOREM [27] .- τ is the right homological inverse of η.

<u>Proof</u>.- By 3.6, for any n, $\eta \circ \tau|_{BU(n)}$ induces in homology the same map as $p_n \circ \tau_n$, i.e. the identity.

□

3.8 COROLLARY.- τ is the right homotopical inverse of η.

<u>Proof</u>.- It follows directly from 3.7 and the Appendix.

§3.2 Stable splitting of the space c X .

In this paragraph, we study the splitting of the space c X in the wedge of less complicated spaces $D_n(\mathcal{C},X)$, for any coefficient system $\mathcal{C}$.

3.9 DEFINITION.- Given a coefficient system $\mathcal{C}$ and a space X, we define

$$D_n(\mathcal{C},X) = \frac{F_n\,c\,X}{F_{n-1}c\,X} .$$

As any morphism between coefficient systems $f: \mathcal{C} \to \mathcal{C}'$ induces a map $f_X : cX \to c'X$ preserving the filtration, it induces also maps

$$f_n : D_n(\mathcal{C}, X) \to D_n(\mathcal{C}', X)$$

3.10 PROPOSITION [9].- $D_n(\mathcal{C}, X)$ is homeomorphic to the quotient $\mathcal{C}_n \times_{\Sigma_n} X^n / \mathcal{C}_n \times_{\Sigma_n} \sigma_n X$.

Proof.- The map $\mathcal{C}_n \times_{\Sigma_n} X^n \to F_n\, c\, X \to D_n(\mathcal{C}, X)$ sends $\mathcal{C}_n \times_{\Sigma_n} \sigma_n X$ to the base point, so it induces a map

$$\frac{\mathcal{C}_n \times_{\Sigma_n} X^n}{\mathcal{C}_n \times_{\Sigma_n} \sigma_n X} \to D_n(\mathcal{C}, X)$$

It is easy to see that it is a homeomorphism.

□

As there is an obvious homeomorphism,

$$\frac{\mathcal{C}_n \times_{\Sigma_n} X^n}{\mathcal{C}_n \times_{\Sigma_n} \sigma_n X} \to \frac{\mathcal{C}_n \times_{\Sigma_n} X^{(n)}}{\mathcal{C}_n \times_{\Sigma_n} \{*\}} ,$$

where $X^{(n)} = \bigwedge_{i=1}^{n} X$ is the n^{th} smash power of X , $D_n(\mathcal{C}, X)$ is also homeomorphic to the latter space.

3.11 PROPOSITION.- [?] Let $f: \mathcal{C} \to \mathcal{C}'$ be a morpnism of coefficient systems. Then ,

i) if , for any n , Σ_n acts freely on $\mathcal{C}_n$ and $\mathcal{C}'_n$, and f_n is a weakhomotopy equivalence, so is the induced map f_n .

ii) if , for any n , f_n is an equivariant honotopy equivalence, then the induced map f_n is a homotopy equivalence.

3.12 DEFINITION.- Let $\mathcal{C}$ be a coefficient system, and we define $\beta_k = \mathcal{C}_k/\Sigma_k$. If $r = \binom{n}{k}$ and the set of ordered injections $\underline{k} \to \underline{n}$ is $\{\psi_1,\dots,\psi_r\}$, we define the map

$$\xi_{k,n} : \mathcal{C}_n \to (\beta_k)^r$$

by

$$\xi_{k,n}(c) = ([c\,\psi_1] ,\dots, [c\,\psi_r]).$$

We say that $\mathcal{C}$ is <u>separated</u> if $\operatorname{Im} \xi_{k,n} \subset F(\beta_k, r)$ for any k,n .

3.13 EXAMPLES .- $\mathcal{C}_\infty$, $\mathcal{U}_\infty$, $F(R^\infty)$ and $\mathcal{L}_\infty$ are separated.

3.14 DEFINITION.- Let $\mathcal{C}$ be a separated coefficient system. We define the map

$$j_{k,n} : \mathcal{C}_n \times X^n \to F(\beta_k, r) \times D_k(\mathcal{C}, X)^r$$

as

$$j_{k,n}(c;x_1,\dots,x_n)=(\xi_{k,n}(c);\ [c\psi_1;(x_{\psi_1(1)},\dots,x_{\psi_1(k)})],\dots,$$

$$,\dots,\ [c\psi_r;(x_{\psi_r(1)},\dots,x_{\psi_r(k)})] .$$

As it is equivariant with respect to the actions of Σ_n on the domain and Σ_r on the range, we have in the limit the map

$$j_k : c\,X \to F(\beta_k)(D_k(\mathscr{C}, X)).$$

To get the splitting maps h_k, we need a technical result of [9]

3.15 THEOREM.- There is a covariant functor, W, from the category of coefficient systems to the category of pointed topological spaces, with the following natural maps

i) A contraction of $W(\mathscr{C})$: i.e. a homotopy $d: 0 \sim 1$.

ii) For each k, an injection $e_k : \beta_k \to W(\mathscr{C})$

iii) An inclusion $i : \coprod_{k=1}^{\infty} W(\mathscr{C}) \to W(\mathscr{C})$

iv) An inclusion $j : R^{\infty} \to W(\mathscr{C})$

$\square$

Notice that by i) $F(W(\mathscr{C}))(X)$ has the weak homotopy type of QX, for any connected space X. We denote the equivalence as w.

3.16 DEFINITION.- Let $\mathscr{C}$ be a separated coefficient system. We define the map

$$h_k : c\,X \xrightarrow{j_k} F(\beta_k)(D_k(\mathscr{C},X)) \xrightarrow{F(e_k)} F(W(\mathscr{C}))(D_k(\mathscr{C},X)) \xrightarrow{w} Q(D_k(\mathscr{C},X))$$

and $\hat{h}_k$ is the adjoint stable map.

By abuse of notation h_k and $\hat{h}_k$ represent also the restrictions to $F_n\, c\,X$, for any n.

3.17 PROPOSITION.- The map

$$\hat{h}_n : \Sigma^{\infty} F_n\, c\,X \to \Sigma^{\infty} D_n(\mathscr{C}, X)$$

is induced by the obvious identification map.

Proof.- It is immediate since the map

$$j_{n,n} : \mathscr{C}_n \times X^n \to F(\beta_n,1) \times D_n(\mathscr{C},X)$$

is given by the identification.

□

Now we can state the splitting theorem.

3.18 THEOREM [9].-Let $\mathscr{C}$ be a separated coefficient system. Then, the sum of the maps $\hat{h}_1,\dots,\hat{h}_n$

$$\hat{k}_n : \Sigma^\infty F_n c X \to \Sigma^\infty \bigvee_{k=1}^{n} D_k(\mathscr{C},X)$$

is a stable homotopy equivalence for any X and any n (even $n=\infty$).

3.19 COROLLARY.- Let X be a connected space and $\mathscr{C}$ a separated coefficient system with $\mathscr{C}_n$ contractible for any n. Then QX splits stably as the wedge of $D_k(\mathscr{C},X)$.

Proof.- It follows from 3.18 since in this case QX has the stable homotopy type of $c X$.

□

§ 3.3 Definition of $\hat{c}_k$.

We study the splitting in the case $\mathcal{F} = F(\mathbb{R}^\infty)$ and $X = BU(1)$. We drop the index $\mathcal{F}$ when there is no possible confusion.

3.20 THEOREM.- Let $X = T(\xi)$ be the Thom space of the vector bundle ξ, Then $D_k(X)$ is homeomorphic to $T(\xi^{(k)})$ where $\xi^{(k)} = F(\mathbb{R}^\infty,k) \times_{\Sigma_k} (\xi)^k$.

Proof.- Choose a riemannian metric on ξ and let $D(\xi)$, $S(\xi)$ be the associated disc and sphere bundle.

On $\xi^{(k)}$ we have an obvious metric induced by the one on ξ. With this metric we have.

$$D(\xi^{(k)}) = F(\mathbb{R}^\infty, k) \times_{\Sigma_k} D(\xi)^k$$

$$S(\xi^{(k)}) = F(\mathbb{R}^\infty, k) \times_{\Sigma_k} \Big(\bigcup_{i=1}^{k} D(\xi)^{i-1} \times S(\xi) \times D(\xi)^{k-i} \Big)$$

and the identification map

$$F(\mathbb{R}^\infty, k) \times_{\Sigma_k} D(\xi)^k \to \frac{F(\mathbb{R}^\infty, k) \times_{\Sigma_k} T(\xi)^k}{F(\mathbb{R}^\infty, k) \times_{\Sigma_k} \{*\}}$$

induces the homeomorphism

$$T(\xi^{(k)}) \simeq D_k(T(\xi)) .$$

$\square$

3.21 COROLLARY.- $D_k(BU(1))$ has the homotopy type of $T(\gamma^{(k)})$

Proof.- As $BU(1)$ has the homotopy type of $T(\gamma)$, $D_k(BU(1))$ has the homotopy type of $D_k(T(\gamma))$ that is, by 3.21, such of $T(\gamma^{(k)})$.

$\square$

As $\gamma^{(k)}$ is classified by the map.

$$F(\mathbb{R}^{\infty} \times k) \times_{\Sigma_k} BU(1)^k = B\Sigma_k \int U(1) \xrightarrow{p_k} BU(k)$$

it has a standard orientation and Thom class, t_k, induced by its complex structure.

3.22 DEFINITION.- The characteristic class $\hat{c}_k \in H^{2k}(BU; \mathbb{Z})$ is the one represented by the stable map

$$\Sigma^{\infty} BU \xrightarrow{\tau} \Sigma^{\infty} Q\, BU(1) \xrightarrow{\hat{h}_k} \Sigma^{\infty} D_k(BU(1)) \simeq \Sigma^{\infty} T\gamma^{(k)} \xrightarrow{t_k} \Sigma^{2k} \mathbb{H}\mathbb{Z}$$

CHAPTER 4 : Characterization of $\hat{c}_k$.

In this chapter, we characterize the elements $\hat{c}_k \in H^{2k}(BU; \mathbb{Z})$ in terms of the universal Chern classes, $\{c_n\}_{n \in \mathbb{N}}$. To do it, we evaluate the Kronecker product $< \hat{c}_k, a >$ on a basis of $H_{2k}(BU; \mathbb{Z})$, and we use the duality between the singular homology and cohomology of BU.

The best way to evaluate $< \hat{c}_k, a >$, is to reduce it mod p, for any prime p, and to use the calculations of the singular homology with $\mathbb{Z}_p$ coefficients for iterated loop spaces, as stated in the first paragraph.

§4.1 Behaviour of the map $\hat{h}_k$ in homology.

For any space X, and any E_∞-operad $\mathcal{C}$, cX has a natural structure of H-space ; so, $H_*(c\,X, \mathbb{Z}_p)$ is a $\mathbb{Z}_p$-algebra with the associated Pontrjagin product. Our aim, in this paragraph, is to describe it as it is done in [8] for any p.

4.1 THEOREM [8], [10].- Let $\mathcal{C}$ be an E_∞-operad and X a $\mathcal{C}$-space. Then,

a) If p=2, for any, i,n, there are natural homeomorphisms,

$$Q^i : H_n(X; \mathbb{Z}_2) \to H_{n+i}(X; \mathbb{Z}_2)$$

such that

i) $Q^i(x) = 0$ if $\deg x > i$

ii) $Q^i(x) = x^2$ if $\deg x = i$.

b) If p≠2, for any i,n, there are natural homeomorphisms

$$Q^i : H_n(X; \mathbb{Z}_p) \to H_{n+2i(p-1)}(X; \mathbb{Z}_p)$$

such that

i) $Q^i(x) = 0$ if $\deg x > 2i$

ii) $Q^i(x) = x^p$ if $\deg x = 2i$

□

For use in the next definition, let

$$\beta : H_n(X; \mathbb{Z}_p) \to H_{n-1}(X; \mathbb{Z}_p)$$

be the Bockstein map associated to the short exact sequence,

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$$

4.2 DEFINITION.- a) If $p=2$, for any finite sequence of natural numbers, $I=(i_1,\dots, i_k)$, we define its <u>length</u> as $\ell(I)=k$, and the associated homomorphism

$$Q^I = Q^{i_1} \circ \dots \circ Q^{i_k} .$$

We say that I is <u>admissible</u> if, for any j, $2i_j \geq i_{j-1}$.

b) If $p\neq 2$, for any finite sequence of natural numbers $I=(\varepsilon_1, i_1, \varepsilon_2, i_2, \dots, \varepsilon_k, i_k)$, where, for any j, ε_j is equal to 0 or 1 and $i_j > \varepsilon_j$, we define its <u>length</u> as $\ell(I)=k$ and the associated homomorphism

$$Q^I = \beta^{\varepsilon_1} \circ Q^{i_1} \circ \beta^{\varepsilon_2} \circ Q^{i_2} \circ \dots \circ \beta^{\varepsilon_k} \circ Q^{i_k} .$$

We say that I is <u>admissible</u> if, for any j, $pi_j - \varepsilon_j \geq i_{j-1}$.

Now, we can state a result giving the structure of $H_*(c\, X, \mathbb{Z}_p)$ in terms of $H_*(X; \mathbb{Z}_p)$, valid for any prime p.

4.3 THEOREM [8] , [10].-Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a basis of $H_*(X; \mathbb{Z}_p)$ as $\mathbb{Z}_p$-module. Identifying x_α with its image in $H_*(cX; \mathbb{Z}_p)$ under the inclusion $X \subset c\, X$; $H_*(c\, X; \mathbb{Z}_p)$ is the free graded commutative algebra generated by the set

$$\{Q^I(x_\alpha)\ \alpha\in\Lambda \text{ and } I \text{ is admissible}\} .$$

□

So, $H_*(C X; \mathbb{Z}_p)$ is generated as $\mathbb{Z}_p$-module by the set

$$\{Q^{I_1}(x_{\alpha_1})\, Q^{I_2}(x_{\alpha_2}) \ldots Q^{I_n}(x_{\alpha_n}) : n \in \mathbb{N},\ \alpha_i \in \Lambda \text{ and } I_i \text{ is admissible, for any, } i\}.$$

To state a similar result for $F_k(C X)$, we need to define the <u>height</u> of such monomials. For any $x = Q^{I_1}(x_{\alpha_1}) \ldots Q^{I_n}(x_{\alpha_n})$, its height is

$$h(x) = \sum_{i=1}^{n} p^{\ell(I_i)}.$$

4.4 THEOREM [24]- Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be, as before, a basis of $H_*(X; \mathbb{Z}_p)$ as $\mathbb{Z}_p$-module. Then, the inclusion induces a monomorphism in homology

$$H_*(F_k C X; \mathbb{Z}_p) \to H_*(C X; \mathbb{Z}_p)$$

with image the $\mathbb{Z}_p$-module generated by the monomials of height less , or equal than k.

$\square$

With this, it is easy to prove.

4.5 THEOREM.- $H_*(D_k(\mathcal{C}, X); \mathbb{Z}_p)$ is isomorphic, as $\mathbb{Z}_p$-module, to the one generated by the monomials of height k.

<u>Proof</u>.- As the inclusion $F_{k-1}(C X) \subset F_k(C X)$ is a cofibration, we have

$$\tilde{H}_*(D_k(\mathcal{C}, X); \mathbb{Z}_p) \tilde{\leftarrow} H_*(F_k(C X), F_{k-1}(C X); \mathbb{Z}_p).$$

By 4.4, this inclusion induces a monomorphism in homology, so the long exact sequence splits, giving.

$$0 \to H_*(F_{k-1}(C X); \mathbb{Z}_p) \to H_*(F_k(CX); \mathbb{Z}_p) \to H_*(F_k(CX), F_{k-1}(CX); \mathbb{Z}_p) \to 0.$$

As $\mathbb{Z}_p$ is a field, this exact sequence splits giving an isomorphism,

$$\tilde{H}_*(D_k(\mathcal{C},X);\mathbb{Z}_p) = \frac{H_*(F_k(cX);\mathbb{Z}_p)}{H_*(F_{k-1}(cX);\mathbb{Z}_p)}$$

, and the $\mathbb{Z}_p$-module on the right is generated by the set of monomials of height k.

□

4.6 THEOREM.- The stable map $\hat{h}_k$ induces in homology the map $(\hat{h}_k)_*$ which sends each monomial of height k to itself and all the others to zero.

<u>Proof</u>.- As the triangle.

$$\begin{tikzcd} F_k(cX) \arrow[d] \arrow[dr, "p_k"] & \\ cX \arrow[r, "\hat{h}_k"] & D_k(\mathcal{C},X) \end{tikzcd}$$

commutes, all we need is to know the action of p_k.

By 4.5, it is obvious that p_k sends the monomials of height k to themselves and any other to zero.

□

§4.2 Evaluation of $< \hat{c}_k, a >$.

We are going to use the classical result on the homology and cohomology of BU:

4.7 THEOREM [30].- Let $c_n \in H^{2n}(BU; \mathbb{Z})$ be the universal Chern class. Then

i) $H^*(BU; \mathbb{Z})$ is the free graded commutative $\mathbb{Z}$-algebra generated by them, i.e.

$$H^*(BU; \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n, \ldots] .$$

If $\rho_p : H^*(\ ; \mathbb{Z}) \to H^*(\ ; \mathbb{Z}_p)$ is the natural transformation induced by the projection $\mathbb{Z} \to \mathbb{Z}_p$, we have.

ii) $H^*(BU; \mathbb{Z}_p)$ is the free graded commutative $\mathbb{Z}_p$-algebra generated by the images $\rho_p(c_k)$; i.e.

$$H^*(BU; \mathbb{Z}_p) = \mathbb{Z}_p \left[\rho_p(c_1), \ldots, \rho_p(c_n), \ldots \right] , \quad \text{and}$$

iii) The induced map,

$$\rho_p : H^*(BU; \mathbb{Z}) \to H^*(BU; \mathbb{Z}_p),$$

sends c_n to $\rho_p(c_n)$ and reduces the coefficients mod p .

$\square$

4.8 THEOREM [30].-There are elements $a_n \in H_{2n}(BU(1); \mathbb{Z})$ such that

i) $H_*(BU(1); \mathbb{Z})$ is the $\mathbb{Z}$-module generated by $\{a_n\}_{n \in \mathbb{N}}$

ii) $H_*(BU; \mathbb{Z})$ is the free graded commutative $\mathbb{Z}$-algebra generated by the images under the inclusion $BU(1) \subset BU$, that we call also a_n; i.e.

$$H_*(BU; \mathbb{Z}) \cong \mathbb{Z}[a_1, \ldots, a_n, \ldots] .$$

The same results are true with homology with coefficients in $\mathbb{Z}_p$ and the elements $\rho_p(a_n)$, and the map ρ_p, is induced by reduction of coefficients mod p, as before.

□

The action of the map

$$\tau : BU \to Q\, BU(1)$$

in homology, is easy to calculate using that τ is a map of H-spaces ([27]).

4.9 THEOREM.- The map

$$\tau_* : H_*(BU; \mathbb{Z}_p) \to H_*(Q\, BU(1); \mathbb{Z}_p)$$

is the inclusion ; i.e., $\tau_*(\rho_p(a_{r_1}), \ldots, \rho_p(a_{r_n})) = \rho_p(a_{r_1}) \cdots \rho_p(a_{r_n})$.

<u>Proof</u>.- As τ is an H-space map, all we need is to prove that $\tau_*(\rho_p(a_n)) = \rho_p(a_n)$, since τ_* commutes with the Pontrjagin product.

Recall that τ was constructed inductively starting with

$$\tau_1 = 1_{BU(1)} : BU(1) \to BU(1)$$

so the diagram

$$\begin{tikzcd} BU(1) \arrow[r] \arrow[dr] & BU \arrow[d, "\tau"] \\ & Q\, BU(1) \end{tikzcd}$$

commutes, thus $\tau_*(\rho_p(a_n)) = \rho_p(a_n)$.

□

The last calculation we need is $< \rho(t_{\gamma^{(k)}}), \rho_p(a^k{}_*) >$, where $t_{\gamma^{(k)}}$ is the Thom class of the vector bundle $\gamma^{(k)}$.

4.10 PROPOSITION.- Let ξ be an oriented vector bundle over a complex B with a unique 0-cell, $\{*\}$. Giving to $T\xi$ a cell structure with the the suspensions of the cells of B, the Thom class $t(\xi) \in H^q(T\xi, \mathbb{Z})$ is represented by the cochain that has value 1 over the q-cell $p^{-1}(\{*\})$ and 0 on any other cell.

<u>Proof</u>.- By definition, if

$$j : S^q \to T(\xi)$$

is induced by the inclusion $p^{-1}(\{*\}) \to E\xi$, $t(\xi)$ is the Thom class if $j^*(t(\xi))$ is a generator of $H^*(S^q;\mathbb{Z})$. As j is a cellular map, $t(\xi)$ is 1 evaluated on $p^{-1}(\{*\})$ and this is the only q-cell of $T(\xi)$.

□

4.11 PROPOSITION.- The element $\rho_p(a_1)^k \in H_{2k}(T\gamma^{(k)}, \mathbb{Z}_p)$ is represented by the cell $\{*\} \times_{\Sigma_k} D(\gamma|_{S^2})^k$.

<u>Proof</u>.- By [15] this cell represents $\rho_p(a_1)^k \in H_{2k}(D_k(\mathcal{E}_1, S^2); \mathbb{Z}_p)$. So, taking cellular maps

$$D_k(\mathcal{E}_1, S^2) \to D_k(\mathcal{E}_\infty, BU(1)) \simeq T\gamma^{(k)}$$

we can consider $\rho_p(a_1)^k$ represented by the same cell, in $H_{2k}(T\gamma^{(k)};\mathbb{Z}_p)$

□

4.12 COROLLARY.- For any ρ,

$$<\rho_p(t\gamma^{(k)}), \rho_p(a_1^k)> = 1$$

<u>Proof</u>.- It is immediate, since by 4.10 $\rho_p(t\gamma^{(k)})$ is one on the cell representing $\rho_p(a_1^k)$.

□

4.13 THEOREM.- For any prime p

$$\rho_p(\hat{c}_k) = \rho_p(c_k) \in H^{2k}(BU, \mathbb{Z}_p)$$

<u>Proof</u>.- We evaluate $\rho_p(\hat{c}_k)$ on the basis of monomials in $\{a_n\}_{n\in\mathbb{N}}$

$$< \rho_p(\hat{c}_k), \rho_p(a_{i_1})\dots \rho_p(a_{i_n})> = <\tau^* h_k^*(t_\gamma^{(k)}), \rho_p(a_{i_1}),\dots, \rho_p(a_{i_n})>$$

$$= \text{(by naturality of the Kronecker product)}$$

$$= <t(\gamma^k), h_{k*}\, \tau_*(\rho_p(a_{i_1}),\dots,\rho_p(a_{i_n}))> =$$

$$\text{(by 4.9)} = <t(\gamma^k), h_{k*}(\rho_p(a_{i_1}) \quad \rho_p(a_{i_n})>$$

To be non-zero the last product, we need,

$$h(\rho_p(a_{i_1}),\dots,\rho_p(a_{i_n})) = k \quad , \quad \text{so} \quad n=k$$

and

$$\deg\,(\rho_p(a_{i_1}),\dots,\rho_p(a_{i_k})) = 2k \text{ , so } \rho_p(a_{i_1})\dots\rho_p(a_{i_k}) = \rho_p(a_i)^k$$

Then

$$< \rho_p(\hat{c}_k), \rho_p(a_{i_1}),\dots,\rho_p(a_{i_n})> = \begin{cases} 1 & \text{if } \rho_p(a_{i_1})\dots\rho_p(a_{i_n}) = \rho_p(a_1)^k \\ & \qquad \text{by 4.12} \\ 0 & \text{otherwise} \end{cases}$$

As $\rho_p(\hat{c}_k)$ is the dual of $\rho_p(a_1)^k$ with respect to the basis of monomials in $\rho_p(a_n)$, it is $\rho_p(c_k)$. (see [6]).

4.14 THEOREM.- $\hat{c}_k = c_k$ in $H^{2k}(BU;\mathbb{Z})$.

Proof.- By 4.8, $\hat{c}_k$ is a homogeneous polynomial of degree 2k in $\{c_n\}_{n\in\mathbb{N}}$ and the coefficient $\alpha_{i_1 \ldots i_n}$ of $c_{i_1} \ldots c_{i_n}$ satisfies

i) $\alpha_{i_1 \ldots i_n} \equiv 0 \mod p$, for any p, if $c_{i_1} \ldots c_{i_n} \neq c_k$,
so $\alpha_{i_1 \ldots i_n} = 0$,

ii) $\alpha_k \equiv 1 \mod p$, for any p, so $\alpha_k = 1$,
thus $\hat{c}_k = c_k$.

$\square$

CHAPTER 5.- Some results on immersions.

The next goal of this work, is to get a geometric interpretation of the elements

$$\hat{c}_k(\xi) \in H^{2k}(M; \mathbb{Z})$$

,for any weakly complex manifold M. In order to do that, we assume we have a complex bundle on M, ξ, classified by a map

$$f_\xi : M \to BU ,$$

That lifts to a map

$$f'_\xi : M \xrightarrow{f} BU \xrightarrow{\tau} Q\,BU(1)$$

that has a nice geometric interpretation, given in [17], that we analyse in the first paragraph. The second one deals with the action of the map h_k^* on this interpretation. The last paragraph looks into the advantages of working with extensions in good position.

From now on, we shall work, when required, in the category of smooth (or C^∞) manifolds and maps. For each manifold M, $T\,M$ is its tangent bundle and T_xM is the fibre over $x \in M$, and for any smooth map $f: N \to M$, $d\,f$ is the differential and it is a morphism of vector bundles

$$d\,f : T\,N \to T\,M$$

§5.1 Immersions and $[M, F(\mathbb{R}^\infty, T(\xi))]$.

First, we recall some facts about immersions.

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First, we recall some facts about immersions.

5.1 DEFINITION.- A map $f: N \to M$ is said to be an immersion if, for any $x \in N$, the map

$$df_x : T_x N \to T_{f(x)} M$$

is a monomorphism.

Sometimes, it is useful to use the following characterization.

5.2 PROPOSITION.- A map $f: N \to M$ is an immersion iff the induced map

$$TN \to f^* TM$$

is a vector bundle monomorphism .

Proof.- It follows immediatly from the definition.

$\square$

Also, we shall use the existence and uniqueness of tubular neighbourhoods for immersions.

5.3 THEOREM.- [14].-Let $f: N \to M$ be an immersion. We define its normal bundle as

$$\nu = \frac{f^* TM}{TN}$$

Then,

i) There is an extension of f to an immersion

$$\bar{f} : \nu \to M$$

ii) Any two extensions are regularly homotopic relative to f.

Now, we can state the geometric interpretation of $[M, F(\mathbb{R}^\infty, T(\zeta))]$ in terms of immersions.

5.4 DEFINITION.- Let ζ be an n-dimensional vector bundle over B. We define $\mathfrak{I}(M, \zeta)$ as the set of bordism classes of triples $(N, g, \bar{g})$, where

i) The map $g = (f,e)$ is an embedding such that $f: N \to M$ is an immersion and $e : N \to \mathbb{R}^\infty$ is a map.

ii) Let ν be the normal bundle of the immersion f. Then, the map $\tilde{g}$ is a morphism of vector bundles

$$\begin{array}{ccc} E_\nu & \xrightarrow{\tilde{g}} & E_\zeta \times \mathbb{R}^\infty \\ \downarrow & & \downarrow \\ N & \xrightarrow{\tilde{g}_1} & B_\zeta \times \mathbb{R}^\infty \end{array}$$

and the projection

$$N \xrightarrow{\tilde{g}_1} B_\zeta \times \mathbb{R}^\infty \to \mathbb{R}^\infty$$

is the map e .

$\mathfrak{J}(M,\zeta)$ is made an abelian group with the composition law induced by the disjoint union of manifolds.

5.5 DEFINITION.- Let $\hat{M}$ be the one point compactification of M . We define the map

$$\beta : \mathfrak{J}(M,\zeta) \to [\hat{M}, F(\mathbb{R}^\infty, T(\zeta))]$$

as follows:

For any element of $\mathfrak{J}(M,\zeta)$ we choose a representative $(N,g,\tilde{g})$ and then we extend f to an immersion $\bar{f} : \nu \to M$ satisfying,

i) The map $(\bar{f}, e \circ \pi_\nu) : E_\nu \to M \times \mathbb{R}^\infty$ is an embedding, and

ii) There is an integer n such that, for any $m \in M$, $\bar{f}^{-1}(m)$ has at most n-points.

Now, the map $\beta([N,g,\tilde{g}])$ is given by

$$\beta([(N,g,\tilde{g})])(m) = \begin{cases} \{ e(\pi_\nu(x)) : x \in \bar{f}^{-1}(m)\} & \text{if } m \in \operatorname{Im} \bar{f} . \\ * & \text{otherwise.} \end{cases}$$

i) The map $g = (f,e)$ is an embedding such that $f: N \to M$ is an immersion and $e : N \to \mathbb{R}^\infty$ is a map.

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5.6 THEOREM [17].- β is a group morphism, with the group structure on $[\hat{M}, F(\mathbb{R}^\infty, T(\xi))]$ given by the H-space structure of $F(\mathbb{R}^\infty, T(\xi))$.
$\square$

§5.2 k-tuple points and the action of h_k^*.

First we describe the space of k-tuple points.

5.7 DEFINITION.- Let $f: X \to Y$ be a map. The *space of ordered k-tuple points of f* is the subspace of $F(X,k)$.

$$\tilde{X}_k = \{(x_1,\dots,x_k) \in F(X,k) : \text{For any } i,j \;\; f(x_i)= f(x_j) \} .$$

We define the map $\tilde{f}_k : \tilde{X}_k \to Y$ by $\tilde{f}_k(x_1,\dots,x_k)= f(x_1)$.

Then, the following property is a direct consequence of the definition.

5.8 PROPOSITION.- Let $f^k: F(X,k) \to Y^k$ be the restriction of the k^{th} power of f. Then,

i) $\tilde{X}_k = (f^k)^{-1} \; (\mathrm{diag}_k \, Y)$

ii) $\tilde{f}_k$ is the composite, $\tilde{X}_k \xrightarrow{f^k|} \mathrm{diag}_k \, Y \cong Y$.
$\square$

5.9 DEFINITION.- The space $\tilde{X}_k$ has an obvious Σ_k-action, given by restriction of the action on X^k given by permuting factors. The *space of k-tuple points of f* is the quotient

$$X_k = \tilde{X}_k / \Sigma_k .$$

The map $\tilde{f}_k$ is Σ_k-invariant, so, it induces a map $f_k: X_k \to Y$.

To repeat these constructions in the category of manifolds, we have to use "self-transverse" maps.

5.10 DEFINITION.- Let $f: N \to M$ be a map between manifolds, and M' a submanifold of M. We say that f is transverse to M' at $x \in N$ if it satisfies either

i) $f(x) \notin M'$

or ii) $T_{f(x)}M = d\,f(T_x N) + T_{f(x)} M'$.

We say that f is transverse to M' if f is transverse to M' at x, for any $x \in N$.

5.11 PROPOSITION [12].- Let f be transverse to M', then $f^{-1}(M')$ is a submanifold of N and it has the same codimension as M' in M.

□

5.12 THEOREM [12].-Let $C^\infty(N,M)$ be the space of all smooth maps from N to M, with the C^∞-Whitney topology. Then, for any M', submanifold of M, the set of maps transverse to M' is dense in $C^\infty(N,M)$.

□

5.13 DEFINITION.- Let $f: N \to M$ be a map. We say that f is selftransverse, if , for any k, the map f^k is transverse to $\mathrm{diag}_k(M)$.

5.14 NOTE.- As in 5.12, the set of selftransverse map is dense in $C^\infty(N,M)$ ([12]). Since the set $\mathrm{Imm}(N,M)$, of immersions of N in M, is open in $C^\infty(N,M)$, the set of all self-transverse immersions is dense in $\mathrm{Imm}(N,M)$.

Now, we relate the normal bundle of an immersion with the normal bundle of its k-tuple points.

5.15 PROPOSITION.- Let f be a self-transverse map. Then, for any k, $\tilde{N}_k$ is a manifold and $\tilde{f}_k$ is a map of manifolds.

<u>Proof</u>.- By 5.11, $\tilde{N}_k$ is a submanifold of $F(N,k)$, that has the same codimension as $\mathrm{diag}_k\ M$ in M^k. The map $\tilde{f}_k$ is smooth since it is the restriction of a smooth map .

□

5.16 THEOREM.- Let f be a self-transverse immersion and ν its normal bundle. Then, the map $\tilde{f}_k$ is an immersion, whose normal bundle $\tilde{\nu}_k$ is isomorphic to the restriction to $\tilde{N}_k$ of ν^k.

<u>Proof</u>.- The tangent bundle of a product of manifolds is the product of tangent bundle, so

$$T(M^k) \simeq (T\ M)^k \quad \text{and} \quad T(N^k) \simeq (T\ N)^k$$

Since $F(N,k)$ is open in N^k, its tangent bundle is the restriction of $(T\ N)^k$.

Also, the pull back commutes with the product, so

$$(f^k)^*((TM)^k) \simeq (f^*(TM))^k \ .$$

Then, the normal bundle of the immersion f^k is

$$\frac{(f^k)^*(T(M^k))}{T(N^k)} \simeq \frac{(f^*(TM))^k}{(TN)^k} \simeq \nu^k$$

The restriction of ν^k.

As f^k is transverse to $\mathrm{diag}_k(M)$, the restriction

$$\tilde{N}_k = (f^k)^{-1}\ \mathrm{diag}_k(M) \longrightarrow \mathrm{diag}_k(M)$$

has the same normal bundle, i.e. the restriction of ν^k to $\tilde{N}_k$.

To finish the proof we only need to observe that $\tilde{f}_k$ is the product of $f^k\,|$ and a diffeomorphism.

□

5.17 PROPOSITION.- Let f be a self-transverse map. Then N_k is a smooth manifold, and f_k is a map of manifolds.

Proof.- The Σ_k-action on $\tilde{N}_k$ is smooth, free and properly discontinuous, so the space of orbits, N_k, is a smooth manifold and the induced map, f_k, is a smooth map ([6]).

□

5.18 THEOREM.- Let f be a self-transverse immersion and ν its normal bundle. Then, the map f_k is an immersion, with normal bundle isomorphic to the quotient of $\tilde{\nu}_k$ under the Σ_k-action given by permuting the factors.

Proof.- The Σ_k-actions on $F(N,k)$ and M^k are smooth, so they induce a Σ_k-action on the tangent bundles and they are given also by permutation of factors.

The map f^k is Σ_k-equivariant, so the same applies to the map $(d\,f)^k$ and the inclusion $TN^k \to f^*(TM)^k$ is compatible with the Σ_k-action.

Then, they induce a Σ_k-action on ν^k and it is given by permutation of factors, Thus the map

$$f_k = \tilde{f}_k/\Sigma_k : N_k \to \mathrm{diag}_k\, M$$

has normal bundle the quotient

$$\frac{\tilde{f}_k^*(TM)/\Sigma_k}{\widetilde{TN}_k/\Sigma_k} \cong \frac{f^*(TM)^k/\Sigma_k}{TN^k/\Sigma_k} \cong \frac{f^*(TM)^k}{(TN)^k}\Big/\Sigma_k \cong \tilde{\nu}_k/\Sigma_k \,.$$

□

To end this paragraph, we study the action of the map $\hat{h}_k$ on the geometric interpretation given by 5.6 .

5.19 DEFINITION.- We define the map $\theta^k : \mathcal{J}(M,\zeta) \to \mathcal{J}(M, \zeta^{(k)})$ as follows:

For any element of $\mathcal{J}(M,\zeta)$, we can choose a representative $(N,g,\bar{g})$ where $g=(f,e)$ and f is a self-transverse immersion. We also choose embeddings $e_k : N_k \to \mathbb{R}^\infty$. We define

$$\theta^k([\,(N,g,\bar{g})\,]) = [\,(N_k, g', \bar{g}')\,]$$

where,

i) $g' = (f_k, e_k)$, with f_k the immersion defined in 5.9 .

ii) $\bar{g}' = (\bar{f}', \bar{e}_k)$, where $\bar{e}_k$ is an extension of the embedding e_k to E_{ν_k}, and $\bar{f}'$ is the bundle map,

$$\begin{tikzcd} E_{\nu_k} \arrow[r, "\bar{f}'"] \arrow[d] & E\zeta^{(k)} \arrow[d] \\ N_k \arrow[r, "\bar{f}_1'"] & B\zeta^{(k)} \end{tikzcd} ,$$

given as a quotient by the Σ_k-action of the product map of the restriction of $\bar{g}^k$, and the map

$$E_{\tilde{\nu}_k} \to F(\mathbb{R}^\infty, k)$$

induced by e.

5.20 THEOREM.- [17] The diagram

$$\begin{tikzcd} \mathcal{J}(M,\zeta) \arrow[r, "\beta"] \arrow[d, "\theta_k"] & {[\,M, F(\mathbb{R}^\infty, T(\zeta))\,]} \arrow[d, "h_k^*"] \\ \mathcal{J}(M,\zeta^{(k)}) \arrow[r, "\beta"] & {[\,M, F(\mathbb{R}^\infty, T(\zeta^{(k)}))\,]} \end{tikzcd}$$

commutes. $\square$

§5.3 Pointed k-tuple points and good position.

Before defining good position, we need to study the manifold of pointed k-tuple points and its normal bundle.

5.21 DEFINITION.- Let $f: X \to Y$ be a map. The space $\tilde{X}_k$ has a Σ_{k-1} action induced by permuting the first $(k-1)$-factors. We define the <u>space of pointed k-tuple points</u> as $X'_k = \tilde{X}_k/\Sigma_{k-1}$.

The projection of X^k in the k^{th} factor induces a map

$$p : \tilde{X}_k \to X$$

that is Σ_{k-1}-invariant so it induces a map $f'_k : X'_k \to X$.

Notice that we can also define the space of i^{th}-pointed k-tuple point as

$$X_k^{(i)} = \tilde{X}_k/\Sigma_{k-i} .$$

Then, the identification map

$$\pi_k^{(i)} : X_k^{(i)} \to X_k^{(i-1)} ,$$

is a $(k-i)$- cover, and the diagram

$$\begin{tikzcd} X'_k \arrow[r, "\pi_k"] \arrow[d, "f'_k"'] & X_k \arrow[d, "f_k"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

5.22 PROPOSITION.- Let $f: N \to M$ be self transverse. Then, any $N_k^{(i)}$ is a smooth manifold and f'_k, $\pi_k^{(i)}$ are smooth maps.

<u>Proof</u>.- $\tilde{N}_k$ is a smooth manifold and, for any i, the Σ_{k-i}-action induced is smooth, free and properly discontinuous, so the quotient space $N_k^{(i)}$ is a smooth manifold and the projections $\pi_k^{(i)}$ are smooth. As p is a smooth map compatible with the Σ_{k-i}-action, the induced map f'_k is a smooth map.

5.23 PROPOSITION.- Let $f: N \to M$ be a self-transverse immersion. Then the map $p: \tilde{N}_k \to N$ is an immersion with normal bundle $\tilde{\nu}'_k$ isomorphic to the restriction of the product bundle $\nu^{k-1} \times \{0\}$.

<u>Proof</u>.- Using a riemannian metric, we can see the normal bundle as the orthogonal complement of TN in $f^* TM$ i.e.

$$f^*(TM) = TN \oplus \nu$$

so, using the product metric in TM^k, we have

$$\tilde{f}_k^*(TM) = T\tilde{N}_k \oplus \tilde{\nu}_k$$

By commutativity of

$$\begin{tikzcd}
\tilde{N}_k \arrow[r, "p"] \arrow[dr, "\tilde{f}_k"'] & N \arrow[d, "f"] \\
 & M
\end{tikzcd}$$

we have

$$f_k^*(TM) = p^* f^*(TM) = p^*(TN) \oplus p^*(\nu) = T\tilde{N}_k \oplus \nu(p) \oplus p^*(\nu)$$

so, as both are orthogonal complements

$$\tilde{\nu}_k = p^*(\nu) \oplus \nu(p)$$

But $\tilde{\nu}_k$ is the restriction of ν^k and $p^*(\nu)$ is the restriction of $\{0\} \times \nu$, so, $\nu(p) \cong \nu^{k-1} \times \{0\}$. $\square$

5.24 THEOREM.- Let $f: N \to M$ be a self-transverse immersion. Then, the map f'_k is an immersion whose normal bundle ν'_k is isomorphic to the quotient bundle $\tilde{\nu}'_k/\Sigma_{k-1}$ where the Σ_{k-1}-action is given by permutation of factors.

<u>Proof</u>.- As in 5.18, we see the Σ_{k-1}-action on $T\tilde{N}_k$ and TN induces one on $\tilde{\nu}'_k$ given by permutation of factors. Then, as in 5.18, the normal bundle of f'_k is the quotient bundle.

□

NOTE.- There is a unique map of bundles, over Π_k.

$$\tilde{\Pi}_k : \nu'_k \to \nu_k$$

closing the diagram,

$$\begin{tikzcd} TN \arrow[r] & TM \\ \nu'_k \arrow[u] \arrow[r] & \nu_k \arrow[u] \end{tikzcd}$$

and it is the quotient of the inclusion of bundles on $\tilde{N}_k$

$$\tilde{\nu}'_k \subset \tilde{\nu}_k \ .$$

its image is the bundle $\bigcup_{j=0}^{n-1} \nu^j \times \{0\} \times \nu^{k-j-1}/\Sigma_k$, that we call also ν'_k and it is the normal bundle of $N_k \subset \mathrm{Im}\, f$.

Inductively we have the bundle over $\tilde{N}_k$

$$\tilde{\nu}_k^{(i)} = \nu^{k-i} \times \{0\}$$

that gives a quotient bundle $\nu_k^{(i)}$ on $N_k^{(i)}$ and it is included in ν_k by the quotient of the map

$$\tilde{\nu}_k^{(i)} \subset \tilde{\nu}_k$$

having as image in $\tilde{\nu}_k$ the bundle

$$\bigcup_{j_1+j_2+\dots+j_i+i=k} \nu^{j_1} \times \{0\} \times \nu^{j_2} \times \{0\} \dots \times \nu^{j_i} / \Sigma_k$$

that we call also $\nu_k^{(i)}$ and it is the normal bundle of N_k in $\operatorname{Im} f_i$

Then the bundles over N_k

$$(\nu_k, \nu_k', \nu_k'', \dots, \nu_k^{(k-1)})$$

are the normal bundlesof N_k in

$$(M, \operatorname{Im} f_1, \operatorname{Im} f_2, \dots, \operatorname{Im} f_{k-1}).$$

Simmilarly, if we call also $\nu_k^{(i)}$ the bundle over N_k' induced by $\tilde{\nu}_k^{(i)}$, we have that the bundles over N_k'

$$(\nu_k', \nu_k'', \dots, \nu_k^{(k-1)})$$

are the normal bundles of N_k' in

$$(N, \operatorname{Im} f_1' \operatorname{Im} f_2', \dots, \operatorname{Im} f_{k-1}'')$$

and there is an obvious map from one set of bundles to another.

Now we turn to the description of good position.

5.25 DEFINITION.- Let $f: N \to M$ be a self-transverse immersion with normal bundle ν. An immersion extending f.

$$\bar{f} : \nu \to M$$

is said to be in good position if, for any k, there are immersions

$$\bar{f}_k' : \nu_k' \to N \quad , \text{ extending } f_k' ,$$

$$\bar{f}_k : \nu_k \to M \quad , \text{ extending } f_k , \text{ and}$$

$$\bar{\bar{f}}_k : \pi_k^*(\nu_k) \to \nu, \text{ map of vector bundle over } f_k' ,$$

such that:

i) For any k, the diagram

$$\begin{tikzcd}
\nu \arrow[rrd, "\bar f"] & & \\
 & N \arrow[lu] \arrow[r, "f"] & M \\
 & \nu'_k \arrow[u, "\bar f'_k"'] \arrow[r, dashed, "\bar{\bar\pi}_k"] \arrow[ld] & \nu_k \arrow[u, "\bar f_k"'] \\
\pi^*_k(\nu_k) \arrow[uuu, "\bar{\bar f}_k"] \arrow[rru, dotted] & &
\end{tikzcd}$$

commutes, where the inclusion $\nu'_k \subset \pi'^*_k(\nu_k)$ is induced by the obvious isomorphism, $\pi^*_k(\nu_k) \cong \nu'_k \oplus f'^*_k(\nu)$.

ii) $\mathrm{Im}\,\bar f_k$ is the set of multiple points of $\bar f$ with multiplicity greater or equal to k .

In the next chapter we prove the existence and uniqueness of this extension, but now we are interested in seeing the importance of having one.

5.26 DEFINITION.- Let us assume that $[(N,g,\bar g)]$ lies in $\mathfrak{J}(M,\zeta)$ and $\bar f$ is an extension of f in good position. We define

$$\underline{M}_k = \mathrm{cl}(\mathrm{Im}\,\bar f_k - \mathrm{Im}\,\bar f_{k+1}) \subset \mathrm{Im}\,\bar f_k$$

$$\underline{N}_k = \mathrm{cl}(\mathrm{Im}\,f_k - \mathrm{Im}\,\bar f_{k+1}) \subset \mathrm{Im}\,\bar f_k$$

then, the obvious map $\underline{M}_k \to \underline{N}_k$ is a cube bundle classified by the map

$$\underline{N}_k \to F(\mathbb{R}^\infty, k) \times_{\Sigma_k} B\zeta^k .$$

Also, the restriction

$$f_k \big| : f_k^{-1}(\underline{M}_k) \to M$$

is an embedding, so we can consider $\underline{N}_k$ lying in M_k and $\underline{M}_k$ in ν_k , so we have,

5.27 PROPOSITION.- The map associated to the triple $(N,g,\tilde{g})$, h, restricts to the composite

$$\underline{M}_k \to F(\mathbb{R}^\infty, k)\times_{\Sigma_k} E\xi^k \to F_k(F(\mathbb{R}^\infty, T(\zeta))).$$

□

5.28 COROLLARY.- If h is as in 5.27, it restricts to the map

$$M - \mathrm{Im}\, \overline{F}_{k+1} \to F_k(F(\mathbb{R}^\infty, T(\zeta))) .$$

Proof.- We glue the restrictions to $\underline{M}_1, \underline{M}_2, \ldots, \underline{M}_k$ and all of them factor through F_k .

□

5.29 PROPOSITION.- The composite map

$$M - \mathrm{Im}\, \overline{F}_{k+1} \xrightarrow{h|} F_k(F(\mathbb{R}^\infty, T(\zeta))) \xrightarrow{h_k} T(\zeta^{(k)})$$

is the Thom Pontrjagin construction on the bundle $\underline{M}_k \to \underline{N}_k$.

Proof.- It is immediate, since the composition maps all the points in the cube bundle as the classifying map does and send the rest to the base point.

□

CHAPTER 6 : Extensions in good position.

In this chapter, we construct an extension of a self-transverse immersion

$$f : N \to M$$

to an immersion of its normal bundle ,ν , that is in good position.

It is done by glueing inductively a special type of chart described in the first paragraph.

§6.1 Some preliminaries .

Let us prove the existence of a special type of chart.

6.1 DEFINITION.- Let V be a real vector space. The set of subspaces $\{H_i\}_{i=1}^{r}$ is said to be in <u>general position</u> if, for any sequence $1 \leq i_1 < \ldots < i_s \leq r$, we have

$$\operatorname{cod}(H_{i_1} \cap \ldots \cap H_{i_s}) = \operatorname{cod} H_{i_1} + \ldots + \operatorname{cod} H_{i_s} \ .$$

Notice that, for any two different sets of subspaces of V in general position $\{H_i\}_{i=1}^{r}$ and $\{H_i'\}_{i=1}^{r}$, satisfying $\dim H_i = \dim H_i'$, for any i, there is a linear automorphsim of V

$$\psi : V \to V ,$$

such that, for any i, $\psi(H_i) = H_i'$.

6.2 DEFINITION._ Let M be a manifod . The set of submanifolds $\{M_i\}_{i=1}^{r}$ is said to be in <u>general position at $y \in$</u> $M_1 \cap \ldots \cap M_r$ if the set $\{T_y M_i\}$, of subspaces of $T_y M$ is in general position.

The set of submanifolds $\{M_i\}_{i=1}^{r}$ is said to be in <u>general position</u>

if for any $1 \le i_1 < \dots < i_s \le r$ and any $y \in M_{i_1} \cap \dots \cap M_{i_s}$, the set of submanifolds $\{M_{i_j}\}_{j=1}$ is in general position at y.

6.3 THEOREM ([12]).- Let $\{M_i\}_{i=1}^{r}$ be a set of submanifolds of M, in general position at $y \in M_1 \dots M_r$, and $\{H_i\}_{i=1}^{r}$ a set of subspaces of $\mathbb{R}^m$ in general position. Then, if $\dim H_i = \dim M_i$ and $m = \dim M$, there is a chart at y,

$$\phi : (W , y) \to (\mathbb{R}^m, 0)$$

such that, for any i, $\phi^{-1}(H_i) = M_i \cap W$.

□

6.4 THEOREM.- Let f be a self-transverse immersion and let y be a point of M such that $f^{-1}(y) = \{x_1,\dots,x_k\}$. Then, the set of subspaces of $T_y M$, $\{ df(Tx_i (N))\}_{i=1}^{r}$, is in general position.

Proof.- Let us define $H_i = df(Tx_i(N))$ and let $\{i_j\}$ be a subsequence $0 \le i_1 < \dots < i_s \le r$.

We define

$$\overline{x} = (x_{i_1},\dots,x_{i_s}) \in F(N,s)$$

$$\overline{y} = (y,\dots, y) \in \mathrm{diag}_s M.$$

Then, we have $f^s(F \bar{X}) = \bar{Y}$. As f is self-transverse,

$$Ty(M)^s = df^s(T_x (F(N,s))) + T_{\bar{y}} \mathrm{diag}_s M =$$

$$= (H_{i_1} \oplus \dots \oplus H_{i_s}) + T_{\bar{y}} \mathrm{diag}_s M ,$$

so

$$s \cdot \dim M = \dim H_{i_1} + \dots + \dim H_{i_s} + \dim \mathrm{diag}_s M - \dim(H_{i_1} \oplus \dots \oplus H_{i_s} \cap \mathrm{diag}_s M)$$

Thus,

$$\operatorname{cod} H_{i_1} + \ldots + \operatorname{cod} H_{i_s} = \dim M - \dim H_{i_1} \cap \ldots \cap H_{i_s} = \operatorname{cod} H_{i_1} \cap \ldots \cap H_{i_s}$$

and the set $\{H_i\}_{i=1}^{r}$ is in general position.

$\square$

6.5 THEOREM.- Let f be a self-transverse immersion and $a = \operatorname{cod} f = \dim M - \dim N$.
For any $y \in M$, if $f^{-1}(y) = \{x_1, \ldots, x_r\}$, there are

i) a chart for M at y, $\psi : (W, y) \to (\mathbb{R}^m, 0)$

ii) disjoint charts for N at x_i, $\phi_i : (U_i, x_i) \to (\mathbb{R}^n, 0)$

such that, if $t = \dim M - r\, a$, the composite map

$$(\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t) \xrightarrow{\phi_i^{-1}} (U_i, \operatorname{Im} f_r') \xrightarrow{f} (W, \operatorname{Im} f_r) \xrightarrow{\psi} (\mathbb{R}^t \times (\mathbb{R}^a)^r, \mathbb{R}^t)$$

is defined and it is the inclusion

$$\mathbb{R}^t \times (\mathbb{R}^a)^{r-1} \xrightarrow{\sim} \mathbb{R}^t \times \bar{H}_i \hookrightarrow \mathbb{R}^t \times (\mathbb{R}^a)^r ,$$

where $\bar{H}_i = (\mathbb{R}^a)^{i-1} \times \{0\} \times (\mathbb{R}^a)^{r-i}$.

Proof.- $\operatorname{Im} f_r'$ is a submanifold of N in a neighbourhood of x_i; so, for each i, there are submanifold charts of $\operatorname{Im} f_r'$ at x_i

$$\chi_i : (V_i, V_i \cap \operatorname{Im} f_r',) \to (\mathbb{R}^n, \mathbb{R}^t) ,$$

such that they are pairwise disjoint and $f|_{V_i}$ is an embedding.

We chose a submanifold chart of $\operatorname{Im} f_r$ at y

$$\psi : (W, W \cap \operatorname{Im} f_r) \to (\mathbb{R}^m, \mathbb{R}^t)$$

such that $f^{-1}(W) \subset \bigcup_{i=1}^{r} V_i$ and they sattisfy 6.3 with

$$H_i = \mathbb{R}^t \times \bar{H}_i \quad \text{and}$$
$$M_i = f(V_i)$$

If, now, we define U_i, $= V_i \cap f^{-1}(W)$, the composite,

$$\Phi : (\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t) \xrightarrow{x_i^{-1}} (U_i, \operatorname{Im} f'_r \cap U_i) \xrightarrow{f} (W, \operatorname{Im} f_r \cap W) \xrightarrow{\psi} (\mathbb{R}^t \times (\mathbb{R}^a)^r, \mathbb{R}^t)$$

is a diffeomorphism onto $(H_i, \mathbb{R}^t)$.

Defining,

$$\phi_i : (U_i, \operatorname{Im} f'_r \cap U_i) \xrightarrow{x_i} (\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t) \xrightarrow{\Phi} (H_i, \mathbb{R}^t) \xrightarrow{\bar{p}_i} (\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t)$$

where $\bar{p}_i$ is the projection into all factors but the i^{th}, we get the appropiate chart.

6.6 REMARK. If both M and N are compact manifolds, there are finite coverings by charts $\{(W, \psi_j)\}_{j=1}^{p}$ and $\{(U_{ji}, \phi_{ji})\}_{j=1\ i=1}^{p\ \ r_j}$ such that

i) W_j meets $\operatorname{Im} f_1, \operatorname{Im} f_2, \ldots, \operatorname{Im} f_{r_j}$ but does not meet $\operatorname{Im} f_{r_j+1}$.

ii) $f^{-1}(W_j) = \bigcup_{i=1}^{r_j} U_{ij}$.

iii) For any j, (W_j, ψ_j) and $\{(U_{ji}\ \phi_{ji})\}_{i=1}^{r_j}$ satisfy 6.5 .

Obviously then, these charts send the stratification

$$(M, \operatorname{Im} f_1, \operatorname{Im} f_2, \ldots, \operatorname{Im} f_r\ )$$

into the stratification of $\mathbb{R}^t \times (\mathbb{R}^a)^r$ given by $\{H_i\}$ and their finite intersections.

6.7 DEFINITION.- Let M' be a submanifold of M. A <u>partial tubular neighbourhood</u> of M' in M is a triple $T = (\xi, \epsilon, e)$ where

i) ξ is an inner product bundle over M' .

If, now, we define $U_i, = V_i \cap f^{-1}(W)$, the composite,

$$\phi : (\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t) \xrightarrow{x_i^{-1}} (U_i, \operatorname{Im} f_r' \cap U_i) \xrightarrow{f} (W, \operatorname{Im} f_r \cap W) \xrightarrow{\psi} (\mathbb{R}^t \times (\mathbb{R}^a)^r, \mathbb{R}^t)$$

is a diffeomorphism onto $(H_i, \mathbb{R}^t)$.

Defining,

$$\phi_i : (U_i, \operatorname{Im} f_r' \cap U_i) \xrightarrow{x_i} (\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t) \xrightarrow{\phi} (H_i, \mathbb{R}^t) \xrightarrow{\bar{p}_i} (\mathbb{R}^t \times (\mathbb{R}^a)^{r-1}, \mathbb{R}^t)$$

where $\bar{p}_i$ is the projection into all factors but the i^{th}, we get the appropiate chart.

6.6 REMARK. If both M and N are compact manifolds, there are finite coverings by charts $\{(W, \psi_j)\}_{j=1}^{p}$ and $\{(U_{ji}, \phi_{ji})\}_{j=1\ i=1}^{p\ \ r_j}$ such that

i) W_j meets $\operatorname{Im} f_1, \operatorname{Im} f_2, \ldots, \operatorname{Im} f_{r_j}$ but does not meet $\operatorname{Im} f_{r_j}+1$.

ii) $f^{-1}(W_j) = \bigcup_{i=1}^{r_j} U_{ij}$.

iii) For any j, (W_j, ψ_j) and $\{(U_{ji}\ \phi_{ji})\}_{i=1}^{r_j}$ satisfy 6.5 .

Obviously then, these charts send the stratification

$$(M, \operatorname{Im} f_1, \operatorname{Im} f_2, \ldots, \operatorname{Im} f_r\ )$$

into the stratification of $\mathbb{R}^t \times (\mathbb{R}^a)^r$ given by $\{H_i\}$ and their finite intersections.

6.7 DEFINITION.- Let M' be a submanifold of M. A <u>partial tubular neighbourhood</u> of M' in M is a triple $T = (\xi, \epsilon, e)$ where

i) ξ is an inner product bundle over M'.

ii) ϵ is a map $\epsilon : M' \to \mathbb{R}^+$

iii) e is an embedding into an open subset $e : D_\epsilon(\xi) \to M$.
where $D_\epsilon(\xi)$ is the open disc bundle of radius $\epsilon(y)$ over any $y \in M$.

In the case $\epsilon=1$ it agrees with the definition of tubular neighbourhood.

Let $T_1 = (\xi_1, \epsilon_1, e_1)$ and $T_2 = (\xi_2, \epsilon_2, e_2)$ be two partial tubular neighbourhoods, of M' in M, we say that the isomorphisms of inner product bundles

$$\psi : \xi_1 \longrightarrow \xi_2'$$

is an <u>isomorphism</u> <u>between</u> T_1 <u>and</u> T_2 if there is a map $\epsilon' : M' \to \mathbb{R}^+$ such that $\epsilon' \leq \inf(\epsilon_1, \epsilon_2)$, and the diagram

$$\begin{tikzcd} D_{\epsilon'}(\xi_1) \arrow[d, "\psi"'] \arrow[dr, "e_1|"] & \\ D_{\epsilon'}(\xi_2) \arrow[r, "e_2|"'] & M \end{tikzcd}$$

commutes. Notice that if $\epsilon_1 = \epsilon_2 = \epsilon' = 1$ we have the usual isomorphism between tubular neighbourhoods.

If $T = (\xi, \epsilon, e)$ is a tubular neighbourhood of M' in M and

$$h : (M, M') \xrightarrow{\sim} (N, N')$$

is a diffeomorphism of manifold pairs, we define

$$h_*(T) = ((h^{-1})^* \xi, \ \epsilon o h^{-1}, \ h \circ e \circ \bar{h}^{-1})$$

where $\bar{h}^{-1} : (h^{-1})^* \xi \to \xi$ is the isomorphism over h^{-1} .

6.8 LEMMA .- Let B be an open set in $\mathbb{R}^{m'} \times \{0\} \subset \mathbb{R}^m$, (ξ_1, e_1) and

(ξ_2, e_2) tubular neighbourhoods of B and $\mathbb{R}^{m'}$ in $\mathbb{R}^m$. For any compact set $V' \subset B$ there is an isotopy

$$H : \mathbb{R}^m \times I \to \mathbb{R}^m$$

such that $H_o = id$, and an isomorphism of vector bundles,

$$\psi : (H_1^{-1})^* \xi_1 \big|_V \longrightarrow \xi_2 \big|_V \quad ,$$

inducing an isomorphism of tubular neighbourhoods.

Proof.- Let $\beta : \xi_1 \big|_B \to \xi_2 \big|_B$ be the isomorphism of vector bundles given by the derivative of the map $e_1 \, e_2^{-1}$. Then, there exists an automorphism

$$\eta : \xi_2 \big|_B \to \xi_2 \big|_B$$

such that for any $x \in B$ η_x is self-adjoint and the composite $\psi = \eta o \beta$ is an isomorphism of inner product bundles.

Now, we want to define the isotopy. Let ψ_t be the isomorphism of vector bundle, $\psi_t = (1-t)\beta + t\psi$.

Now, we chose a neighbourhood of V in B, V_1, such that the composition

$$V_1 \times I \xrightarrow{e_1^{-1} \times 1} \xi_1 \times I \xrightarrow{\psi_t} \xi_2 \xrightarrow{e_2} V_1$$

is defined. Let $V_2 \subset V_1$ be a neighbourhood of V in B such that, for any $0 \le s, t \le 1$, we have $g_s(V_2) \subset g_t(V_1)$.

Let

$$\rho : \mathbb{R}^m \to [0,1]$$

be a smooth map with compact support in V_2 and $\rho = 1$ on a neighbourhood of V. Then, the maps

$$G_{s,t} : \mathbb{R}^m \to \mathbb{R}^m$$

defined by

$$G_{s,t}(x) = \begin{cases} (1-\rho(x))x + \rho(x)\, g_t\, g_s^{-1}(x) & x \in V_2 \\ x & x \notin V_2 \end{cases}$$

are smooth, and as $G_{t\,t} = 1$, there is a $\delta > 0$ such that $G_{s,t}$ is a diffeomorphism for any $|s-t| < \delta$.

Let n be such that $1/n < \delta$, then the maps,

$$H_t = G_{0,t/n} \,\ldots,\, G_{k-1/n\; t,\, t}$$

give the required isotopy.

$\square$

Notice that if the maps $e_1\, e_2^{-1}$ preserves the filtration of ξ_1 and ξ_2 given by the hyperplanes, all the construction can be done preserving it.

§6.2 Construction of extensions in good position.

In this paragraph f is a self-transverse immersion of N in M. With the notation

$$(\xi, \xi', \xi'', \ldots, \xi^{(n-1)}, P)$$

we mean an n-tuple of fiber bundles over the manifold P with fibre,

$$\left((\mathbb{R}^a)^n, \bigcup_{i=1}^{n} \bar{H}_i, \bigcup_{j,i=1}^{n} \bar{H}_i \cap \bar{H}_j, \ldots, \bigcup_{i=1}^{n} L_i, \{0\} \right)$$

when $L_i = \{0\} \times \mathbb{R}^a \times \{0\}$ and $\bar{H}_i$ are as in 6.5 .

6.9 THEOREM.- Let N_k be the manifold of the deepest multiple points, then, there are embeddings:

$$\bar{f}'_k : (\nu'_k, \nu''_k, \dots, N'_k) \to (N, \operatorname{Im} f'_2, \dots, \operatorname{Im} f'_k)$$

$$\bar{f}_k : (\nu_k, \nu'_k, \nu''_k, \dots, N_k) \to (M, \operatorname{Im} f_1, \operatorname{Im} f_2, \dots, \operatorname{Im} f_k)$$

such that the diagram

$$\begin{array}{ccc}
(\nu'_k, \nu''_k, \dots, N'_k) & \xrightarrow{\bar{f}_k^{-1}} & (N, \operatorname{Im} f'_2, \dots, \operatorname{Im} f'_k) \\
\downarrow & & \downarrow f \\
(\nu'_k, \nu''_k, \dots, N_k) & \xrightarrow{\bar{f}_k} & (\operatorname{Im} f_1, \operatorname{Im} f_2, \dots, \operatorname{Im} f_k)
\end{array}$$

commutes.

Proof.- Without loss of generality, we can assume that N_k is connected, since otherwise we can repeat the construction for each component.

First, we reorder the coverings of 6.6, in such a way that $\{W_i\}_{j=1}^{p_0}$ are all that meet $\operatorname{Im} f_k$, and, for any ℓ, $\bigcup_{j=1}^{\ell} W_j$ is connected.

$\{U_{j\,i}\}_{j=1\ i=1}^{p_0\ \ k}$ is the associated covering of $\operatorname{Im} f'_k$ in N.

Now, we chose and open covering of $\operatorname{Im} f_k$, $\{W'_j\}_{j=1}^{p_0}$, satisfying

i) $W'_i \subset \operatorname{cl} W'_i \subset W_i$, and

ii) For any ℓ, $\bigcup_{j=1}^{\ell} W'_j$ is connected.

We define $U'_{j\,i} = U_{j\,i} \cap f^{-1}(W'_j)$

For any j, we define the n-tuple of fibre bundles:

$$(\xi_j, \xi'_j, \xi''_j, \dots, W_j \cap \operatorname{Im} f_k) = (W_j \cap \operatorname{Im} f_k) \times ((\mathbb{R}^a)^k, \bigcup_{i=1}^{k} H_i, \bigcup_{h,i=1}^{k} H_h \cap H_i, \dots, \{0\})$$

and the embedding

$$e_j : (W_j \cap \operatorname{Im} f_k) \times ((\mathbb{R}^a)^k, \bigcup_{i=1}^{k} H_i, \dots, \{0\}) \xrightarrow{\psi_j | \times 1} \mathbb{R}^t \times ((\mathbb{R}^a)^k, \bigcup_{i=1}^{k} H_i, \dots, \{0\})$$

$$\xrightarrow{\psi_j^{-1}} (M, \operatorname{Im} f_1, \operatorname{Im} f_2, \dots, \operatorname{Im} f_k) \cap W_j .$$

The couple (ξ_j, e_j) is a tubular neighbourhood of $W_j \cap \operatorname{Im} f_k$ in M.

Symmilarly, we define the bundles,

$$(\xi'_{ji}, \xi''_{ji}, \dots, U_{ji} \cap \operatorname{Im} f'_k) = (U_{ji} \cap \operatorname{Im} f'_k) \times ((\mathbb{R}^a)^{k-1}, \bigcup_{h=1}^{k-1} H'_h, \dots, \{0\})$$

and the embeddings

$$e_{ji} : (U_{ji} \cap \operatorname{Im} f'_k) \times ((\mathbb{R}^a)^{k-1}, \bigcup_{h=1}^{k-1} H'_h, \dots, \{0\}) \to (N, \operatorname{Im} f'_2, \dots, \operatorname{Im} f'_k) \cap U_{ji} .$$

the bundle maps

$$\Phi_j : \xi'_j = \coprod_{i=1}^{k} \xi'_{ji} \to \xi_j$$

induced by the inclusions,

$$(\mathbb{R}^a)^{k-1} \xrightarrow{\sim} H_i \hookrightarrow (\mathbb{R}^a)^k$$

makes commutative the square

$$\begin{tikzcd} \xi'_{ji} \arrow[r, "e_{ji}"] \arrow[d, "\Phi_j"'] & N \arrow[d, "f"] \\ \xi_j \arrow[r, "e_j"] & M \end{tikzcd} .$$

Now, we construct tubular neighbourhood of N'_k and N_k by glueing these bundles inductively after they have being changed by an isotopy, as follows:

Assume that we have already constructed a n-tuple of bundles $(\zeta_e, \zeta'_e, \dots, B_e)$ on an open neighbourhood of $\bigcup_{j=1}^{e} (\operatorname{cl} W'_j) \cap \operatorname{Im} f_k$ in $\operatorname{Im} f_k$ and an embedding,

$$g_e : (\zeta_e, \zeta'_e, \dots, B_e) \to (M, \operatorname{Im} f_1, \dots, B_e)$$

also n-tuples of bundles ($\zeta'_e,\ldots,f^{-1}(B_e)$) and an embedding,

$$g'_e : (\zeta'_e,\ldots,f^{-1}(B_e)) \to (N, \operatorname{Im} f'_2,\ldots,f^{-1}(B_e))$$

and a map of bundles

$$\Psi_e : (\zeta'_e,\ldots,f^{-1}(B_e)) \to (\zeta'_e,\ldots,B_e)$$

over f and such that $f \circ g'_e = g_e \circ \Psi_e$.

Let A_e be an open set in $\operatorname{Im} f_k$ such that

$$\bigcup_{j=1}^{e} (\operatorname{cl} W_j) \cap \operatorname{Im} f_k \subset A_e \subset \operatorname{cl} A_e \subset B_e .$$

Both (ξ_{e+1} , θ_{e+1}) and (ζ_e, g_e) are tubular neighbourhoods of $B_e \cap W_{e+1}$ so, by 6.8, there is an isotopy,

$$H_t : M \times I \to M$$

preserving the filtration of M by $\operatorname{Im} f_1, \operatorname{Im} f_2 \ldots$,

and an isomorphism,

$$\Psi = (H_1^{-1})^* \xi_{e+1}\big|_{A_e \cap W_{e+1}} \xrightarrow{\sim} \xi_e\big|_{A_e \cap W_{e+1}}$$

preserving the filtrations of both bundles.

Then, we define

$$\zeta_{e+1} = \zeta_e \cup_{\Psi} (H_1^{-1})^* \xi_{e+1}$$

$$g_{e+1} = g_e \cup_{\Psi} e_{e+1} \circ \bar{H}_1$$

and, obviously, it is a tubular neighbourhood over $(A_e \cup (W_{e+1} \cap \operatorname{Im} f_k))$ in M.

At both H and Ψ preserve the filtration. They lift to H' and Ψ' on N, ζ' and ζ', giving,

$$\zeta'_{e+1} = \zeta'_e \, U_\Psi \, , \, (M'^{-1})^* \xi'_{e+1}$$

$$g'_{e+1} = g'_e \, U_\Psi \, , \, e'_{e+1} \circ \bar{H}'_1$$

and a unique extension Ψ_{e+1} .

After finite induction, we have constructed a bundle ξ, and a embedding onto an open subset

$$g : \xi \to M$$

so for any $y \in \mathrm{Im}\, f_k$

$$\xi_y \cong T(\xi_y)_0 \xrightarrow{dg} T_y M \longrightarrow T_y M / T_y \,\mathrm{Im}\, f_k \cong (\nu_k)_y$$

is an isomorphism and, as g preserves filtration, it gives an isomorphism of bundles

$$\alpha : (\xi, \xi', \ldots, N_k) \xrightarrow{\sim} (\nu_k, \nu'_k, \ldots, N_k).$$

that lifts to an isomorphism

$$\alpha' : (\xi', \ldots, N'_k) \xrightarrow{\sim} (\nu'_k, \ldots, N'_k) .$$

Thus, if we define

$$f'_k = g' \circ \alpha'^{-1} \quad \text{and} \quad f_k = g \circ \alpha^{-1}$$

we get the maps we were looking for.

$\square$

6.10 NOTE.- If we choose an isomorphism

$$\theta : \Pi_k^* (\nu_k) \cong \bar{f}'^*_k(\nu) ,$$

we can define $\bar{f}^\sim_k$ as the composite of θ and the map

$$f'^*_k(\nu) \to \nu$$

Then, the map $\bar{f}\,|\mathrm{Im}\,\bar{f}'_k$ is defined as the only one that make the following diagram commute

$$\begin{tikzcd}
\nu \arrow[drr, "\bar f"] & & \\
& N \arrow[ul, bend left] \arrow[r, "f"] & M \\
& \nu'_k \arrow[u, "\bar f'_k"] \arrow[r] \arrow[dl, bend left] & \nu_k \arrow[u, "\bar f_k"'] \\
\Pi'_k(\nu_k) \arrow[uuu, "\bar f''_k"] \arrow[urr] & &
\end{tikzcd}$$

and, obviously, $\bar{f}$ is in good position.

6.11 THEOREM.- Let $f : N \to M$ be a self-transverse immersion. Then, there is an extension in good position

$$\bar{f} : \nu \to M$$

<u>Proof</u>.- It is done constructing, inductively, the maps

$$\bar{f}_\ell : \nu_\ell \to M$$

$$\bar{f}_\ell : \nu'_\ell \to N$$

and then $\bar{f}$ is defined as in 6.10.

If k is the deepest intersection, the result is 6.9.

We show how to construct $\bar{f}_{k-1}$ and $\bar{f}'_{k-1}$ from $\bar{f}_k$ and $\bar{f}'_k$

Let us define

$$(\bar{N}'_{k-1}, \bar{\bar{N}}'_{k-1}) = f'^{-1}_{k-1}\,(\mathrm{Im}\,\bar{f}'_k\,,\ \bar{f}'_k\,(D(\nu'_k)))\qquad \text{and}$$

$$(\bar{N}_{k-1}, \bar{\bar{N}}_{k-1}) = f^{-1}_{k-1}\,(\mathrm{Im}\,\bar{f}_k\,,\ \bar{f}_k\,(D(\nu_k)))\ .$$

Since f_{k-1} carries $\bar{N}_{k-1}$ to the image of $\bigcup_{i=0}^{R} \{0\} \times \nu \times \{0\}$, there are maps,

$$\nu_{k-1}|_{\bar{N}_{k-1}} \xrightarrow{\simeq} \nu_k \xrightarrow{\simeq} D(\nu_k) \qquad \text{and}$$

$$\nu'_{k-1}|_{\bar{N}'_{k-1}} \xrightarrow{\simeq} \nu'_k \xrightarrow{\simeq} D(\nu'_k) \ .$$

If we define g and q' as the composite of $\bar{f}_k$ and $\bar{f}'_k$ with the above diffeomorphism, the diagram,

$$\begin{tikzcd}
\nu'_{k-1}|_{\bar{W}'_{k-1}} \arrow[r] \arrow[d] & N \arrow[d, "f"] \\
\nu_{k-1}|_{\bar{W}_{k-1}} \arrow[r] & M
\end{tikzcd}$$

commutes.

Now we proceed as in 6.9 constructing inductively ξ and ξ' , glueing charts on the bundles , leaving unchanged $\nu'_{k-1}|_{\bar{\bar{N}}'_{k-1}}$ and $\nu_{k-1}|_{\bar{\bar{N}}_{k-1}}$.

Then, we get the maps $\bar{f}'_{k-1}$ and $\bar{f}_{k-1}$ and, the map $\bar{f}|_{\nu|_{\mathrm{im}\,\bar{f}'_{k-1}}}$ is defined as in 6.10 . It is in good position since the maps

$\bar{f}'_{k-1}$ and $\bar{f}_{k-1}$ and the new maps

$$\nu_k \xrightarrow{\simeq} D(\nu_k) \xrightarrow{\bar{\bar{f}}_k} M \qquad \text{and}$$

$$\nu'_k \xrightarrow{\simeq} D(\nu'_k) \xrightarrow{\bar{\bar{f}}'_k} M$$

make the appropiate diagram commute.

Similarly, we proceed the induction downwards, shrinking at each step the width of the image of the normal bundle, and after a finite number of steps, we get the desired extension in good position

$$\bar{f} : \nu \to M .$$

□

6.11 REMARK.- By uniqueness of tubular neighbourhoods for immersions ([13]), any extension of f is regular homotopic to the one constructed, so any extension is regular homotopic to one in good position.

CHAPTER 7 Geometric Interpretation of the Classes $\hat{C}_k$.

To get this geometric interpretation, we have to use complex cobordism, so the first paragraph is a review of the theory, as sketched in ([25]), together with some results of [5] .

§7.1 Review of complex cobordism .

7.1 DEFINITION.- Let M be a manifold and $\xi : M \to BO$ be a map. A complex orientation of ξ, is a map , $\xi' : M \to BU$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\xi'"] \arrow[dr, "\xi"'] & BU \arrow[d] \\ & BO \end{tikzcd}$$

commutes up to a specified homotopy, where the map $BU \to BO$ is the limit of the inclusions $BU(n) \to BO(2n)$.

The couple (M, ν') is a weakly complex manifold if M is a compact manifold and ν' is a complex orientation of a map

$$\nu : M \to BO$$

classifying the stable normal bundle of M.

Similarly, we define weakly complex manifolds with boundary, making compatible the orientations of int M and ∂M.

7.2 DEFINITION.- For any pair of spaces, (X,A), we consider the set of triples (M, ν', f), where (M, ν') is an n-dimensional weakly complex compact manifold and f is a map of pairs,

$$f : (M, \partial M) \to (X,A) .$$

On this set, we define the following equivalence relation:
Two triples (M, ν', f) and $(\bar{M}, \bar{\nu}', \bar{f})$ are cobordant if there is an $(n+1)$-dimensional weakly complex compact manifold (W, ζ'), possibly with corners, and a map

$$F : W \to X$$

such that

i) ∂W splits in three parts $M, \bar{M}$ and δW, and M and $\bar{M}$ are disjoint.

ii) ζ' on M and $\bar{M}$ agrees with ν' and $\bar{\nu}'$.

iii) $F|_M = f$, $F|_{\bar{M}} = \bar{f}$ and $F(\delta W) \subset A$.

The set of equivalence classes, $U_n(X,A)$, is called the n^{th} complex bordism set and it is given a group structure with the operation induced by disjoint union of manifolds.

If g is a map of pairs

$$g : (X,A) \to (Y,B)$$

we define a homomorphism

$$g_n : U_n(X,A) \to U_n(Y,B)$$

by

$$g_n ([(M,\nu', f)]) = [(M,\nu', g \circ f)] .$$

Obviously, the map defined is functorial.

As always, we define

$$U_n(X) = U_n(X, \phi) \qquad \text{for any space } X$$

$$\tilde{U}_n((X,*)) = U_n(X, (*)) \qquad \text{for any based space } (X,*)$$

7.3 THEOREM.- The functors $\{U_n\}$ define a generalised homology theory on the category of pairs of topological spaces.

For a detailed proof see [5], we just recall that the boundary homomorphism

$$\partial_n : U_n(X, A) \to U_{n-1}(A),$$

is defined by

$$\partial_n([(M,\nu', f)]) = [(\partial M, \nu'|_{\partial M}, f|_{\partial M})],$$

and for any $U \subset X$ open, the inverse map of the excision

$$e : (X - U, A - U) \subset (X, A)$$

is given by

$$(e_*)^{-1}([(M,\nu', f)]) = [(f^{-1}(X - U), \nu'|, f|)].$$

Now, we define the associated cohomology theory.

7.4 DEFINITION.- Let $f: N \to M$ be a map of manifolds, we define $\operatorname{cod} f = \dim M - \dim N$. If $\operatorname{cod} f$ is even, we say that f has a complex orientation if there is a complex vector bundle over M, ζ, and an embedding $e : N \to E\zeta$ such that the classifying map of the normal bundle has a complex orientation.

If $\operatorname{cod} f$ is odd, we say that f has a complex orientation if the map

$$N \xrightarrow{f} M \xrightarrow{\simeq} M \times \{0\} \hookrightarrow M \times \mathbb{R}$$

has one.

Notice that a complex orientation of the map $M \to *$ is equivalent to a complex orientation of the map classifying the stable normal bundle of M.

7.5 DEFINITION.- Let M be a compact manifold. We consider the set of triples (N, f, α) where N is a compact manifold, $f : N \to M$ is a proper map of manifolds of codimension n and α is a complex orientation of f.

Two triples (N, f, α) and (N', f', α') are cobordant, if there is a triple (W, F, Λ) where

i) W is a compact manifold with $\partial W = N \cup N' \cup \delta W$.

ii) $F : W \to M \times I$ is a map of manifolds transverse to $M \times \{i\}$ for $i = 0,1$, and $F|_N : N \to M * \{0\}$ and $F|_{N'} : N' \to M \times \{1\}$ agree with f and f', and $F(\delta W) \subset \partial M \times I$.

iii) Λ is a complex orientation of F.

The set of equivalence classes, $U^n(X)$ is the n^{th} complex cobordism set and a group structure is given by the operation induced by disjoint union of manifolds.

Similarly, if (M, A) is a manifold pair, we can define $U^n(M, A)$ as cobordism classes of triples (N, f, α) where $\mathrm{Im}\, f \subset M - A$.

To define the action of a map of manifolds $g: M \to M'$ on the complex cobordism groups, we need to define transverse intersection, as follows:

Let $[(N, f, \alpha)]$ be lying in $U^n(M')$. Let $e: N \to E(\xi)$ be the embedding given by the complex orientation . We chose an embedding $e' : M \to \mathbb{R}^{\ell}$. By 5.12 we can assume that $E(\xi) \times M$ and $N \times \mathbb{R}^{\ell}$ have transverse images in $E_{\xi} \times \mathbb{R}^{\ell}$. Then we define, $N \pitchfork M$, the transverse intersection of N and M, as the intersection of those images.

The map

$$f' : N \pitchfork M \to M$$

is given by the second projection. Obviously, it has a complex orientation

α', associated to α, and the class $[(N \pitchfork M, f', \alpha')] \in U^n(M)$, is independent of all the choices in volved. We have defined a homomorphism

$$g^n : U^n(M') \to U^n(M) .$$

As before, we define $\tilde{U}^n(X,*) = U^n(X,\{*\})$ for any pointed space $(X, *)$.

7.6 THEOREM.- The functors $\{U^n\}$ define a generalised cohomology theory on the category of manifold pairs.

□

Now, we sketch a proof of the theorem stating that these theories agree with those associated to the spectrum MU.

7.7 THEOREM.- There are natural isomorphisms

$$\tilde{U}_*(X, *) \cong \widetilde{MU}_*(X, *) \quad \text{for any pointed space } (X,*)$$

$$\tilde{U}^*(M, *) \cong \widetilde{MU}^*(M, *) \quad \text{for any manifold } (M, *) .$$

<u>Proof</u>.- We prove it for $X = S^0 = M$. The general case essentially the same. Recall that the coefficients associated to MU are

$$\widetilde{MU}_n(S^0) = \widetilde{MU}^{-n}(S^0) = \lim_k [S^{2k-n}, MU(k)] .$$

Then we define the map

$$\alpha : \widetilde{MU}_n(S^0) \to \tilde{U}_n(S^0)$$

as follows:

Let $x \in \widetilde{MU}_n(S^0)$. We chose a representative $(f) \in [S^{2k-n}, MU(k)]$ where $f: S^{2k-n} \to MU(k)$ is transverse to $BU(k) \subset MU(k)$ (it can be done by 5.12) and then, we define $\alpha(x) = [f^{-1}(BU(k))]$. It is easy to prove that it is a well defined element of $\bar{U}_n(S^0)$, independent of the choices.

Also, we define the map $\beta : \tilde{U}_n(S^0) \to \widetilde{MU}_n (S^0)$ on an element $y = [(M, \nu')] \in \tilde{U}_n(S^0)$ by chosing an embedding $e : M \hookrightarrow \mathbb{R}^{2k-n}$. such that the associated normal bundle has a complex orientation. Now, we extend e to an embedding $\bar{e} : \nu (e) \to \mathbb{R}^{2k-n}$.

Let $(\bar{g}, g)$ be the classifying map of ν. Then, we define

$$f : S^{2k-n} \to MU(k)$$

by

$$f (t) = \begin{cases} \bar{g}(e^{-1}(t)) & \text{if } t \in \operatorname{Im} \bar{e} \\ * & \text{if } t \notin \operatorname{Im} \bar{e} \end{cases}$$

i.e. the Thom-Pontrjagin construction associated to ν (e). If we define $\beta (y) = [f]$, it is easy to see that β is well defined and it is the inverse map of α.

To end this paragraph, we state some results on duality of the U^* and U_* theories as proved in [5] .

7.8 DEFINITION.- Let ξ be an n-dimensional bundle over M and ξ' a complex orientation of it. We define the class $t(\xi) \in U^n(T (\xi))$ as

$$t(\xi) = [(M, i, \xi')]$$

where $i : M \to T(\xi)$ is induced by the zero-section.

7.9 PROPOSITION [5] .- $t(\xi)$ is a Thom class of ξ .

Proof.- It is immediate from the definition of Thom class.

7.10 THEOREM [5] .- i) Let M be an n-dimensional weakly complex closed manifold. Then, the Poincare duality isomorphism associated to $t(\nu_M)$,

$$PD : U_q(M) \xrightarrow{\simeq} U^{n-q}(M)$$

is given by

$$PD([(N, f, \xi')]) = [(N, f, \nu')]$$

when ν' is the complex orientation of ν_N given by ξ' and the complex orientation of ν_M.

ii) Let M be an n-dimensional weakly complex compact manifold. Then, the Lefschetz duality isomorphism associated to $t(\nu_M)$

$$LD : U_q(M, \partial M) \xrightarrow{\simeq} U^{n-q}(M)$$

is given as the one above.

□

7.11 COROLLARY.- Let M be an n-dimensional weakly complex manifold. Then, the fundamental class associated to $T(\nu_M)$ is given by $[(M, \nu'), 1] \in U^n(M)$.

Proof.- It is the image of $1 \in U_0(M)$ under the duality of 7.10 .

□

§7.2 Geometric interpretation of $c_k(\xi)$.

In all this paragraph M is an n-dimensional weakly complex closed manifold and the map $\xi : M \to BU$ classifies an isomorphism class of complex vector bundles.

7.12 LEMMA.- Let M' be a submanifold of M of codimension 0. Then, the diagram

$$\begin{tikzcd}
U_{n-q}(M) \arrow[r, "PD", "\sim"'] \arrow[d, "i_*"] & U^q(M) \arrow[dd, "j^*"] \\
U_{n-q}(M, M') & \\
U_{n-q}(M\text{-int } M', \partial M') \arrow[u, "e_*"', "\wr"] \arrow[r, "LD", "\sim"'] & U^q(M\text{-int } M')
\end{tikzcd}$$

commutes for every q, where the vertical maps are induced by the inclusion and e_* is an isomorphism by the excision property.

<u>Proof</u>.- Let $x = [(N, \xi', f)] \in U_{n-q}(M)$. As M' has codimension 0, the class $i_*(x) = [(N, \xi', i \circ f)] \in U_{n-q}(M, M')$ can be represented by $[(f^{-1}(cl(M-M')), \xi'|, i \circ f|)]$.

Following x clock wise, we get

$$e_* LD^{-1} j^* PD([(N, \xi', f)]) = e_* LD^{-1} j^*([(N, f, \xi')]) = (\text{by } 7.10) =$$

$$= e_* LD^{-1}([(f^{-1}(cl(M-M')), j \circ f, \xi'))]) =$$

$$(\text{since } f \text{ is transverse to } M') =$$

$$= e_*([(f^{-1}(cl(M-M')), \xi'|, j \circ f)]) = (\text{by } 7.10) =$$

$$= ([(f^{-1}(cl(M-M')), \xi'|, e \circ j \circ f)])$$

and this is $i_*(x)$ by the remark above.
□

Let $\hat{c}_k^{\vee}(\xi)$ be the element of $MU^{2k}(M)$ given by

$$\Sigma^{\infty} M \xrightarrow{\xi} \Sigma^{\infty} BU \xrightarrow{\tau} \Sigma^{\infty} QBU(1) \xrightarrow{\hat{h}_k} \Sigma^{\infty} T\gamma^{(k)} \xrightarrow{t_k} \Sigma^{2k} MU .$$

where t_k is the Thom class described in 7.8, so $\hat{c}_k^{\vee}(\xi) = \xi^* \circ \tau^* \circ \hat{h}_k^*(t_k)$

If $[(N,g,\bar{g})] \in (M,\gamma)$ is the element representing $\tau \circ \xi$, we choose an extension in good position $\bar{f} : \nu \to M$ and we define $M' = \operatorname{Im} \bar{f}_{k+1}$. Then we have the following result

7.13 THEOREM.

$$\iota_* (PD^{-1}(\hat{c}_k^{\vee} (\xi)) = \iota_*([(N_k, f_k \, \nu_k]) \in U_{n-2k}(M, \operatorname{Im} \bar{f}_{k+1})$$

<u>Proof</u>.- The diagram

$$\begin{tikzcd}
U^{2k}(M) \arrow[d, "j^*"] & U^{2k}(F(\mathbb{R}^{\infty}, BU(1))) \arrow[l, "(\tau\circ\xi|)^{2k}"'] \arrow[d, "i_k^*"] & \\
U^{2k}(M\text{-int } M') & U^{2k}(F_k(F(\mathbb{R}^{\infty}, BU(1)))) \arrow[l, "(\tau\circ\xi|)^{2k}"'] & U^{2k}(D_k(F(\mathbb{R}^{\infty}), BU(1))) \arrow[l, "p_k^*"] \arrow[ul, "\hat{h}_k^*"']
\end{tikzcd}$$

commutes, since the left hand square is induced by inclusions and restrictions and the right hand triangle commutes by 3.18 .

By definition, of $\hat{c}_k^{\vee}(\xi)$ above,

$$j^*(\hat{c}_j^{\vee}(\xi)) = j^* \circ (\tau\circ\xi \;)^* \circ \hat{h}_k^* (t_k) = (\tau\circ\xi| \;)^* \; p_k^* \; (t_k) .$$

As by 5.30, $\tau\circ\xi|$ is given by the Thom-Pontrjagin construction of M_k over N_k , $j^*(\hat{c}_k^{\vee}(M))$ is the Thom class of M_k , and $\iota_*(PD^{-1}(\hat{c}_k^{\vee}(\xi))) = e_* \, LD^{-1} \, j^* \, (\hat{c}_k^{\vee}(\xi))$ is given by the same element, $[(N_k, f_k), \nu_k']$.

By definition, this element also represents $i_*([(N_k, f_k), \nu_k])$.

$\square$

7.14 THEOREM.- In singular cohomology theory, the Chern class $c_k(\xi)$ is the Poincare dual of $f_{k\,*}([\,N_k\,])$, when $[\,N_k\,]$ is the standard fundamental class of N_k .

Proof.- Using the natural transformation t from complex bordism and cobordism to singular homology and cohomology, 7.13 is true when we replace U_* by H_* , so

$$\iota_* \circ PD^{-1}(\, c_k\,(\xi)) = \iota_*\,(t(\,[(N_k,\, f_k,\, \nu_k)])) = \iota_*\, f_{k*}([N_k]).$$

Since M' has the homotopy type of an $(n-2(k+1))$ dimensional complex, the map ι_* is a monomorphism in this dimension, so

$$PD^{-1}(c_k(\xi)) = f_{k*}\,([N_k]) \,.$$

$\square$

CHAPTER 8 Description of the bundle associated to an immersion.

In the preceeding chapters, we associated to any map

$$\xi : M \to BU$$

the composite map,

$$\xi' : M \xrightarrow{\xi} BU \xrightarrow{\tau} Q\,BU(1) ,$$

such that $\eta \circ \xi'$ is homotopic to ξ. Then, replacing $Q\,BU(1)$ by the weakly homotopic space $F(\mathbb{R}^\infty)(BU(1))$ and applying [17] we got a cobordism class $[(N,g,\tilde{g})]$ classifying the map ξ'.

Now, we want to describe the inverse procedure, i.e. given the triple $(N,g,\tilde{g})$ to get a description of the associated complex bundle.

§8.1 <u>More about infinite loop spaces</u>.

We want a closer study of the loop structure of BU.

8.1 DEFINITION.- Let $\mathcal{C}$ be an operad and $C : Top_* \to Top_*$ its associated functor. The functor

$$F : Top_* \to Top_*$$

is called a $\mathcal{C}$-functor iff there is a natural transformation

$$\lambda : F\,C \to F$$

such that the diagrams

i) $$\begin{tikzcd} F \arrow[r, "1_i"] \arrow[dr, equal] & F\,C \arrow[d, "\lambda"] \\ & F \end{tikzcd}$$

and

ii) $$\begin{tikzcd} F\,C\,C \arrow[r, "\lambda_C"] \arrow[d, "\lambda 1"'] & F\,C \arrow[d, "\lambda"] \\ F\,C \arrow[r, "\lambda"] & F \end{tikzcd}$$

commute.

8.2 NOTE [20] .- The functor $\Omega^j S^{i+j}$ is a $\mathcal{C}_{i+j}$-functor for any $j \geq 1$ and $i \geq 0$.

Let us recall the "double bar" construction of May ([20]).

8.3 DEFINITION.- Let $\mathcal{C}$ be an operad, F a $\mathcal{C}$-functor and X a $\mathcal{C}$-space. Then, we define the space $B(F, \mathcal{C}, X)$ as the geometric realization of the simplicial complex $B_*(F, \mathcal{C}, X)$ whose q-simplices are

$$B_q(F, \mathcal{C}, X) = F\,C^q\,(X),$$

the face maps are given as follows

$$\partial_0 \text{ by } \lambda 1_C^q \quad : \quad FC^{q+1}(X) \to F\,C^q(X)$$

$$\partial_i \text{ by } 1_F\, 1_C^{i-1}\, c\, 1^{q-i} \quad : \quad FC^{q+1}(X) \to F\,C^q(X)$$

$$\partial_q \text{ by } 1_F\, 1^q\, \theta \quad : \quad FC^{q+1}(X) \to F\,C^q(X)$$

where c is the collapsing map $C\,C \to C$ and θ is the $\mathcal{C}$-action on X.

The "degeneracy" maps are all given by the inclusion $X \to CX$.

8.4 NOTE [20] .- This construction commutes (up to homotopy) with Ω, i.e.

$$B(\Omega F, \mathcal{C}, X) \simeq \Omega B(F, \mathcal{C}, X).$$

Then for any E_∞-operad, $\mathcal{C}$, if we define $\mathcal{D}_{i+j} = \mathcal{C}_{i+j} \times \mathcal{C}$ we have, for any $\mathcal{C}$-space X,

$$B(\Omega^j S^{i+j}, \mathcal{D}_{i+j}, X) \simeq \Omega^j B(S^{i+j}, \mathcal{D}_{i+j}, X).$$

8.5 DEFINITION.- Let $\mathcal{C}$ be an E_∞-operad and X a $\mathcal{C}$-space. We define

$$B_i(X) = \lim \Omega^j B(S^{i+j}, \mathcal{D}_{i+j}, X)$$

where the limit is taken with respect to the maps

$$\Omega^j B(S^{i+j}, \mathcal{D}_{i+j}, X) \to \Omega^j B(\Omega S^{i+j+1}, \mathcal{D}_{i+j+1}, X) \tilde{\to} \Omega^{j+1} B(S^{i+j+1}, \mathcal{D}_{i+j+1}, X).$$

It is obvious that $\{B_i(X)\}_{i \geq 0}$ is an infinite loop space.

8.6 THEOREM [20] .- Let $\mathcal{C}$ and X be as in 8.5. Then, the maps

$$X \leftarrow B(D_n, \mathcal{D}_n, X) \to B(\Omega^n S^n, \mathcal{D}_n, X) \to \Omega^n B(S^n, \mathcal{D}_n, X)$$

give in the limit the map

$$i : X \leftarrow B(D_\infty, \mathcal{D}_\infty, X) \to B(Q, \mathcal{D}_\infty, X) \to B_0 X$$

where the first map is a homotopy equivalence, the second one is a group completion and the last map is a weak equivalence.

So, if X is connected, i is a weak equivalence.

$\square$

8.7 PROPOSITION [26] .- Let, $\mathcal{C}$ be an E_∞-operad and $\Psi : \mathcal{C} \to \mathcal{C}_\infty$ a map of the associated coefficient systems.

Then, for any connected $\mathcal{C}$-space, X, the diagram

$$\begin{array}{ccc} C_\infty X & \xrightarrow{\sim} & C_\infty B_0 X \\ \uparrow \wr & & \\ C X & & \downarrow \\ \downarrow & & \\ X & \xrightarrow{\sim} & B_0 X \end{array}$$

commutes.

Proof.- The diagram

$$\begin{tikzcd}
C_\infty X \arrow[dd, bend right=60, leftarrow, "\psi"'] & C_\infty(B(D_\infty, \mathcal{D}_\infty, X)) \arrow[l, "\sim"'] \arrow[r] & C_\infty B_0 X \arrow[dd, bend left=60, "1"] \\
D_\infty X \arrow[u, "\wr"] \arrow[d, "\wr"] & D_\infty(B(D_\infty, \mathcal{D}_\infty, X)) \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\wr"] \arrow[d, equal] & D_\infty B_0 X \arrow[u, "\wr"] \arrow[d, "\wr"] \\
C\ X \arrow[d] & B(D_\infty D_\infty, \mathcal{D}_\infty, X)) \arrow[d] & C_\infty B_0 X \arrow[d] \\
X & B(D, \mathcal{D}_\infty, X) \arrow[l, "\sim"'] \arrow[r, "\sim"] & B_0 X
\end{tikzcd}$$

commutes, the bottom half by [20] and the upper half by naturality.

□

8.8 NOTE.- Then, the map $\eta : Q\,BU(1) \to BU$ is the composition

$$Q\,BU(1) \to Q\,BU \to Q\,B_0\,BU \to B_0\,BU \xleftarrow{\sim} BU ,$$

and using the commutative diagram

$$\begin{tikzcd}
 & Q(BU(1)) \arrow[r] & Q\,BU \arrow[r] & Q\,B_0\,BU \arrow[r] & B_0\,BU \\
F(\mathbb{R}^\infty)(BU(1)) & C_\infty(BU(1)) \arrow[l, "k"'] \arrow[r] \arrow[u, "\wr"] & C_\infty(BU) \arrow[r] \arrow[u, "\wr"] & C_\infty B_0\,BU \arrow[r] \arrow[u, "\wr"] & B_0\,BU \arrow[u, equal] \\
 & L_\infty BU(1) \arrow[ul, "\beta\circ\alpha", "\sim"'] \arrow[r] & L_\infty BU \arrow[u, "\wr"] \arrow[rr, "\theta"] & & BU \arrow[u, "\wr"]
\end{tikzcd}$$

the map η is homotopic to the composition

$$Q(BU(1)) \xleftarrow{\sim} C_\infty(BU(1)) \xrightarrow{k} F(\mathbb{R}^\infty)(BU(1)) \xleftarrow{\beta\circ\alpha} L_\infty(BU(1)) \to L_\infty(BU) \xrightarrow{\theta} BU.$$

§8.2 Description of the bundle.

We can now describe the complex vector bundle associated to a triple $(N, g, \bar{g})$.

8.9 THEOREM.- Let $h : M \to F(\mathbb{R}^\infty)(BU(1))$ be the map associated to the triple $(N, g, \bar{g})$. Then, there are triples , $(N, g, \bar{g}_t)$, such that the associated maps give a homotopy,

$$h_t : M \to F(\mathbb{R}^\infty)(BU(1)),$$

where $h_0 = h$ and h_1 factors as the composition

$$M \xrightarrow{\ell} L_\infty(BU(1)) \xrightarrow{\alpha} V_\infty(BU(1)) \xrightarrow{\beta} F(\mathbb{R}^\infty)(BU(1)).$$

<u>Proof.</u>- To lift h over β by a homotopy as described, is equivalent to change for any $m \in M$ the associated configuration in $\mathbb{R}^\infty$ to an orthonormal one. This is achieved by induction on $\{\mathrm{Im}\, \bar{f}_k\}_{k=0}^{n}$.

In the k^{th} step of the induction we have the map

$$h| : \underline{N}_k \to F(\mathbb{R}^\infty; k)$$

and a homotopy from $h|_{\partial\underline{\bar{N}}_k}$ to a map that factors as the composition

$$\partial\underline{N}_k \to V_\infty(k) \to F(\mathbb{R}^\infty; k).$$

We can extend it to a homotopy from $h|_{\underline{N}_k}$ to a map that factors

$$\underline{N}_k \xrightarrow{\bar{\ell}} V_\infty(k) \to F(\mathbb{R}^\infty, k).$$

This is extended to $\underline{M}_k$ composing with the projection.

Notice that if all the original configurations were orthonormal, the lifting can be achieved directly.

By a similar induction, we lift $\bar{\ell}$ to a composition

$$M \longrightarrow L_\infty(\ BU(1)\ ) \longrightarrow V_\infty\ (\ BU(1)\ )$$

but this time no homotopy is involved.

8.10 REMARK.- Notice that if for any $m \in M$, $h(m) = [(x_1,\dots,x_k),(L_1,\dots,L_k)]$ and $(x_1,\dots,x_k)$ are orthonormal in $\mathbb{R}^\infty$, the lifting is $\ell(m) = [e,(L_1,\dots,L_k)]$ where $e \in L_\infty(k)$ has to satisfy $e(e_{i1}) = x_i$ for any i.

8.11 THEOREM.- Let h be the composite

$$h : M \overset{\ell}{\to} L_\infty(\ BU(1)\ ) \overset{\alpha}{\to} V_\infty(BU(1)) \overset{\beta}{\to} F(\mathbb{R}^\infty)(\ BU(1)\ ).$$

The associated complex vector bundle, ξ, is given as follows:

Let $m \in M$ be such that $\ell(m) = [(e;(L_1,\dots,L_k)]$. Then, the fibre ξ_m is $A \oplus B \subset \mathbb{C}^\infty \times \mathbb{C}^\infty$, where A is the orthogonal complement in $\mathbb{C}^\infty \times \{0\}$ of the complex subspace generated by the configuration $\psi(e)$, and $B = \bigoplus_{i=1}^{k} \hat{L}_i$, where $\hat{L}_i$ is the image of L_i by the map $e_{\mathbb{C}}$.

<u>Proof</u>.- We chose automorphisms

$$g_i : \mathbb{C} \times \mathbb{C}^\infty \to \mathbb{C} \times \mathbb{C}^\infty$$

representing L_i, and then, we define the map

$$g : \mathbb{C}^\infty \times \mathbb{C}^\infty \to \mathbb{C}^\infty \times \mathbb{C}^\infty$$

on $\mathrm{Im}\,(e_{\mathbb{C}} \times e_{\mathbb{C}})$ as the only one filling the diagram

$$\begin{tikzcd}
(\mathbb{C} \times \mathbb{C}^\infty)^k \arrow[r] \arrow[d, "\times g_i"] & (\mathbb{C}^\infty \times \mathbb{C}^\infty)^k \arrow[r, "sh", "\sim"'] & (\mathbb{C}^\infty)^k \times (\mathbb{C}^\infty)^k \arrow[r, "e_{\mathbb{C}} \times e_{\mathbb{C}}"] & \mathbb{C}^\infty \times \mathbb{C}^\infty \arrow[d, dashed, "g"] \\
(\mathbb{C} \times \mathbb{C}^\infty)^k \arrow[r] & (\mathbb{C}^\infty \times \mathbb{C}^\infty)^k \arrow[r, "sh", "\sim"'] & (\mathbb{C}^\infty)^k \times (\mathbb{C}^\infty)^k \arrow[r, "e_{\mathbb{C}} \times e_{\mathbb{C}}"] & \mathbb{C}^\infty \times \mathbb{C}^\infty
\end{tikzcd} .$$

On the orthogonal complement, g is defined as the identity.

Then $\xi_m = g\,(\mathbb{C}^\infty \times \{0\})$.

Let $\bar{L}_i$ be the 1-dimensional subspace of $\mathbb{C}^\infty \times \{0\}$ given by

$$\mathbb{C} \cong \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^\infty \xrightarrow{j_i} \mathbb{C}^k \times \mathbb{C}^{\infty k}$$

where j_i is the inclusion in the i^{th} factor. Then,

$$g \circ (e_{\mathbb{C}} \times e_{\mathbb{C}})(\bar{L}_i) = (e_{\mathbb{C}} \times e_{\mathbb{C}})\ sh\ g_i(\bar{L}_i) = (e_{\mathbb{C}} \times e_{\mathbb{C}})\ sh\ (L_i) = \hat{L}_i .$$

On the other hand, the space A in the definition is orthogonal to $\mathrm{Im}\,(e_{\mathbb{C}} \times e_{\mathbb{C}})$, so g is the identity on it and, $g\,(A) = A$. Thus ξ_m splits as indicated

8.12 REMARK.

Recall that, if the map h is associated to the triple $(N, g, \bar{g})$, and $\bar{f} : \nu \to M$ is an extension in good position of f, for any $m \in M$ such that $\bar{f}^{-1}(m) = (\alpha_1, \dots, \alpha_k)$ with $\alpha_i \in \nu_{n_i}$, in the associated $h(m) = [\,(x_1, \dots, x_k)\ ;\ (L_1, \dots, L_k)\,]$ we have $L_i = \bar{f}\,(n_i)$.

To end this work, we get more detailed information in some particular cases

8.13 EXAMPLE.- Let us consider a map $h : M \to F(\mathbb{R}^\infty)\,(\,BU(1)\,)$ induced by a map $h : M \to F(\mathbb{R})(S^2)$,

As $S^2 = T(\epsilon^2)$, by 5.2, we can consider that it is represented by a triple $(N, g, \bar{g})$ where $f : N \to M$ is a codimension one immersion with trivial normal bundle. Then $\bar{f} : N \times D^2 \to M$ is an extension of f in good position.

Let $\alpha : S^2 \to \mathbb{C}P^1 = G_{1,2}$ be the one point extension of the map.

$$\alpha : D^2 \approx D_{\mathbb{C}} \to \mathbb{C}P'$$

defined by $\alpha(\lambda) = [\lambda, 1] = L_\lambda$ Then, $[\alpha]$ generates $\pi_2(\mathbb{C}P^\infty)$.

Let $q : N \to \mathbb{R}^\infty$ be a map such that, for any $(x,z),(x',z') \in N \times D^2$ such that $f(x,z) = f(x', z')$, $q(x)$ is orthogonal to $q(x')$. We can then lift the map h to $V_\infty(BU(1))$ using q and the lines $\hat{L}_i$ in 8.11 are given as follows:

Let be $u \in \mathbb{R}^\infty$. Then, we define the map

$$u : \mathbb{R}' \times \mathbb{R}' \to \mathbb{R}^\infty \times \mathbb{R}^\infty$$

by $u(s,t) = (su, tu)$. The subspaces $\hat{L}_i$ are then $q(x_i)_{\mathbb{C}}(L_{z_i})$.

8.14 EXAMPLE.- Let consider now the maps

$$h : M \to F(\mathbb{R}^\infty)(BU(1))$$

that factors through $BU(1)$.

By 5.26 a map that factors through $F_k(F(\mathbb{R}^\infty)(BU(1))$ can be represented by a triple $(N, g, \bar{g})$ such that the immersion

$$f : N \to M$$

has multiple points of multiplicity at most k. So, a map that factors through $BU(1)$ can be represented by a triple $(N,g,\bar{g})$ where f is an embedding of N in M, whose normal bundle is classified by a map in $BU(1)$.

Obviously, the bundle associated to any triple of this particular type is the Thom-Pontrjagin construction on the normal bundle of N in M.

It is easy to see that the restriction $\gamma|_{\mathbb{C}P^n}$ is represented by the inclusion $\mathbb{C}P^n \subset \mathbb{C}P^{n+1}$.

APPENDIX: On maps $f : BU \to BU$.

The goal of this appendix is to prove that a map $f : BU \longrightarrow BU$ that induces the identity in singular cohomology is itself homotopic to the identity. In order to do it, we have to recall the definition and some elementary properties of K-theory.

A.1 DEFINITION.- K-theory is the generalised cohomology theory $\tilde{K}^*$ whose functors are given by, for any pointed space X

$$\tilde{K}^{2n}(X) = [X, BU \times \mathbb{Z}] \quad \text{and}$$

$$\tilde{K}^{2n+1}(X) = [X, U] \quad , \text{ for any } n \geq 0 ,$$

where $U = \lim U(n)$.

The required natural equivalences at odd dimensions

$$, \tilde{K}^{2n+1}(SK) \simeq \tilde{K}^{2n+2}(X) ,$$

are given by the homotopy equivalence $\Omega U \simeq BU \times \mathbb{Z}$ proved in Bott periodicity theorem ([29]), and the equivalences at even dimensions

$$, \tilde{K}^{2n}(SX) \simeq \tilde{K}^{2n+1}(X) ,$$

are given by the equivalence $\Omega BU \simeq U$.

A.2 NOTE.- As $\tilde{K}^n$ and $\tilde{K}^{n+2}$ are naturally isomorphic, it is useful sometimes to consider $\tilde{K}^*$ as a $\mathbb{Z}_2$-graded cohomology theory.

Also, when necessary, we associate to the singular cohomology with coefficient in R, the $\mathbb{Z}_2$-graded cohomology theory $H^{**}(\ ; R)$ given by

$$\tilde{H}^{0*}(\ ;R) = \bigoplus_{n \geq 0} \tilde{H}^{2n}(\ ;R) \quad \text{and}$$

$$\tilde{H}^{1*}(\ ;R) = \bigoplus_{n \geq 0} \tilde{H}^{2n+1}(\ ;R) \ .$$

A.3 PROPOSITION [13] .- Let h^* and h'^* be two reduced cohomology theories and $h^*(S^0)$, $h'^*(S^0)$ their coefficient systems. Then

i) If h^* takes value in the category of $\mathbb{Q}$-vector spaces and

$$T, T' : h^* \to h'^* ,$$

are two natural transformations that agree on the coefficients, then $T = T'$.

ii) If both h^* and h'^* take values in the category of Q-vector spaces and

$$T : h^*(S^0) \to h'^*(S^0)$$

is a homomorphism, there is a natural transformation

$$T : h^* \to h'^*$$

extending it.

□

A.4 NOTE.- As an immediate consequence, any cohomology theory, h^*, taking values in the category of $\mathbb{Q}$-vector spaces, is naturally equivalent to the theory $\tilde{H}^*(\ ;\mathbb{Q}) \otimes h^*(S^0)$.

The rest of the appendix, we assume some knowledge of the Atiyah-Hirzebruch spectral sequence of a homology theory h^* , in particular that it is associated to an exact couple, $E\,C(X,h^*)$ and that the E^∞ terms are the quotients of the filtration of $h^*(X)$ given by

$$F_p(h^*(X)) = \ker (h^*(X) \to h^*(X_{p-1}))$$

where X_{p-1} is the (p-1)-skeleton of the complex X.

A.5 DEFINITION.- The character of K-theory (or Chern character) is the composite natural transformation of $\mathbb{Z}_2$-graded cohomology theories.

$$ch : \tilde{K}^* \to \tilde{K}^* \otimes \mathbb{Q} \cong \tilde{H}^{**}(\ ; \mathbb{Q})$$

where the second map is the natural equivalence of A.3.

A.6 THEOREM.- Let X be a complex such that $\tilde{H}^*(X,\mathbb{Z})$ is free. Then, the Chern character

$$ch (X) : \tilde{K}^*(X) \to \tilde{H}^{**}(\ ; \mathbb{Q})$$

is a monomorphism.

Proof.- The character induces a homomorphism of $\mathbb{Z}_2$-graded exact couples

$$ch : E\,C\,(X ; \tilde{K}^*) \to E\,C(X ; \tilde{H}_{\mathbb{Q}}^{**})$$

As $\tilde{K}^*$ is $\mathbb{Z}_2$-graded ordinary cohomology with coefficients in $\mathbb{Z}$ the E' term of its spectral sequence is $\tilde{H}^{**}(X ; \mathbb{Z})$. Obviously, the E' term of the second exact couple is $\tilde{H}^{**}(X; \mathbb{Q})$, so we get that the induced homomorphism,

$$ch_{(1)} : \tilde{H}^{**} (X ; \mathbb{Z}) \to \tilde{H}^{**}(X ; \mathbb{Q}) ,$$

is the coefficient homomorphism, so it is a monomorphism.

As ch is a map of exact couples, it commutes with the differentials, so in particular

$$\bar{d}_1\, ch_{(1)} = ch_{(1)}\, d_1 .$$

Since $\bar{d}_1 = 0$ and $ch_{(1)}$ is a monomorphism, $d_1 = 0$.

Inductively we get $d_2 = d_3 = \ldots = 0$ so the E^{∞}-term is the E'-therm.

The map

$$ch_{\infty} : \frac{F_*(\tilde{K}^*(X))}{F_{*-1}(\tilde{K}^*(X))} \longrightarrow H^{**}(X ; Q)$$

is then a monomorphism, so the map

$$ch : \tilde{K}^*(X) \to \tilde{H}^{**}(X ; Q)$$

is a monomorphism too.

$\square$

A.7 THEOREM.- Let $f: BU \to BU$ be a map such that the induced map in singular homology $f_* : H^*(BU ; \mathbb{Z}) \to H^*(BU ; \mathbb{Z})$ is the identity. Then, f is homotopic to 1_{BU}.

<u>Proof</u>.- The diagram

$$\begin{array}{ccc} K^*(BU) & \xrightarrow{f^*} & K^*(BU) \\ \downarrow ch & & \downarrow ch \\ H^{**}(BU;Q) & \xrightarrow{f^* = 1} & H^{**}(BU;Q) \end{array}$$

commutes, so

$$ch\, f^*(i_0) = f^*(ch(i_0)) = ch(i_0) = ch\, 1^*_{BU}(i_0),$$

where i_0 is the inclusion

$$BU \cong BU \times \{ 0 \} \hookrightarrow BU \times \mathbb{Z}$$

$H^*(BU ; \mathbb{Z})$ is free, so, by A.6, ch is injective and then

$$f^*(i_0) = 1_{BU}{}^* (i_0) .$$

So, there is a homotopy

$$F : BU \times I \to BU \times \mathbb{Z}$$

from i_0 of to $i_0 \circ 1_{BU}$.

As $BU \times I$ is connected, $\operatorname{Im} F \subset BU \times \{ 0 \}$, so we can lift to a homotopy

$$F : BU \times I \to BU$$

from f to 1_{BU} .

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