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ON STABLE COHOMOLOGY OF CENTRAL EXTENSIONS OF ELEMENTARY ABELIAN GROUPS

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ABSTRACT. We study when kernels of inflation maps associated to extraspecial p -groups in stable group cohomology are generated by their degree two components. This turns out to be true if the prime is large enough compared to the rank of the elementary abelian quotient, but false in general.

We feel honoured to dedicate this article to our friend and colleague Miles Reid on the occasion of his 70th birthday.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout below k will be an algebraically closed field of characteristic $l \geq 0$ and p will be a prime number assumed to be different from l if l is positive. Let G be a finite p -group. One defines the stable cohomology $H_{s,k}^*(G, \mathbb{Z}/p) = H_s^*(G, \mathbb{Z}/p)$ in the following way (this does depend on k , but we suppress it from the notation when there is no risk of confusion): for a finite-dimensional generically free linear G -representation V , let $V^L \subset V$ be the open subset where G acts freely. Then the ideal $I_{G,\text{unstable}}$ in the group cohomology ring $H^*(G, \mathbb{Z}/p)$ is defined to be, equivalently, the kernel of the natural homomorphism

$$(1) \quad H^*(G, \mathbb{Z}/p) \rightarrow H^*(\text{Gal}(k(V/G)), \mathbb{Z}/p)$$

or, more geometrically, the kernel of

$$(2) \quad H^*(G, \mathbb{Z}/p) \rightarrow \varinjlim_{U \subset V^L/G} H_{\text{ét}}^*(U, \mathbb{Z}/p)$$

where U runs over all nonempty Zariski open subsets of V^L/G .

Definition 1.1. We define $H_s^*(G, \mathbb{Z}/p)$ as

$$H_s^*(G, \mathbb{Z}/p) = H^*(G, \mathbb{Z}/p) / I_{G,\text{unstable}}.$$

A priori, this seems to depend on the choice of V , but really does not [Bo05, Thm. 6.8]. We often identify $H_s^*(G, \mathbb{Z}/p)$ with its image in $H^*(\text{Gal}(k(V/G)), \mathbb{Z}/p)$.

$H_s^*(G, \mathbb{Z}/p)$ is contravariant in the group G : if $\varphi: G' \rightarrow G$ is a group homomorphism, V' and V are generically free G' and G -representations with a dominant intertwining map $\Phi: V' \rightarrow V$ (meaning $\Phi(g'v') = \varphi(g')\Phi(v')$ for all $g' \in G', v' \in V'$), and $U' \subset V', U \subset V$ are nonempty G', G -invariant open subsets with $\Phi(U') \subset U$,

then the diagram

$$\begin{array}{ccc} BG' & \longrightarrow & BG \\ \uparrow & & \uparrow \\ U'/G' & \longrightarrow & U/G \end{array}$$

induces a homomorphism $H_s^*(G, \mathbb{Z}/p) \rightarrow H_s^*(G', \mathbb{Z}/p)$ independent of all choices. For a subgroup $H = G'$ of G and φ the inclusion, we call the induced homomorphism $H_s^*(G, \mathbb{Z}/p) \rightarrow H_s^*(H, \mathbb{Z}/p)$ restriction, or inflation. We also sometimes say that a class in $H_s^*(G', \mathbb{Z}/p)$ is induced from G when it is in the image of $H_s^*(G, \mathbb{Z}/p) \rightarrow H_s^*(G', \mathbb{Z}/p)$ and if this map is implied unambiguously by the context.

Definition 1.2. Put $L = k(V/G)$. The unramified group cohomology

$$H_{\text{nr}}^*(G, \mathbb{Z}/p) \subset H_s^*(G, \mathbb{Z}/p)$$

is defined as the intersection, inside $H^*(L, \mathbb{Z}/p)$, of $H_s^*(G, \mathbb{Z}/p)$ and $H_{\text{nr}}^*(L, \mathbb{Z}/p)$; here, as usual, $H_{\text{nr}}^*(L, \mathbb{Z}/p)$ are those classes that are in the kernel of all residue maps associated to divisorial valuations of L , i.e. those corresponding to a prime divisor on some normal model of L .

In this article we study a rather special class of groups.

Definition 1.3. For a prime p , an extraspecial p -group G is a p -group such that its center $Z(G)$ is cyclic of order p and $G/Z(G)$ is a nontrivial elementary abelian group.

This differs a bit from the arguably most common definition using the Frattini subgroup [Suz86, 4., §4, Def. 4.14], but it is equivalent to it by [Suz86, 4., §4, 4.16].

Thus each extraspecial p -group sits in an exact sequence

$$(3) \quad 1 \longrightarrow Z \longrightarrow G \xrightarrow{\pi} E \longrightarrow 1$$

where $Z \simeq \mathbb{Z}/p$ is the center of the group G and $E \simeq (\mathbb{Z}/p)^n$ is elementary abelian. Moreover, the skew-form given by taking the commutator of lifts of elements in E

$$(4) \quad \begin{aligned} \omega: E \times E &\rightarrow Z, \\ (x, y) &\mapsto [\tilde{x}, \tilde{y}] \end{aligned}$$

must be a symplectic form if G is extraspecial. Hence $n = 2m$ and the order of G is of the form p^{1+2m} for some positive integer m . One can be much more precise and prove that for each given order p^{1+2m} there are precisely two extraspecial p -groups of that given order, up to isomorphism [Hupp67, III, §13, 13.7 and 13.8] or [Gor07, Chapter 5, 5.]; but we do not need this detailed structure theory. We want to study the kernel of the ‘‘inflation map’’

$$(5) \quad K^G = \ker(\pi^*: H_s^*(E, \mathbb{Z}/p) \rightarrow H_s^*(G, \mathbb{Z}/p)).$$

This is a graded ideal in the graded ring $H_s^*(E, \mathbb{Z}/p)$ (graded by cohomological degree). It is natural to expect that this should be, in general, generated by its degree 2 component, or, even more precisely, by the class $\omega \in \text{Hom}(\Lambda^2 E, \mathbb{Z}/p) =$

$H_s^2(E, \mathbb{Z}/p)$ given by the extension; cf. also formula (12) in Section (3) for the description of the stable cohomology of abelian groups. In fact, Tezuka and Yagita in [TezYag11] study a very similar problem in §9, p. 4492 ff., see especially the problems they mention on p. 4494 top and bottom concerning what they cannot yet prove. Indeed, the expectation above is false in general (this is similar to the situation in ordinary group cohomology where conjectures that kernels of inflation maps associated to central extensions should always be the expected ones are false as well, see [Rus92, Prop. 9]). We show:

Theorem 1.4. *Let G be an extraspecial p -group of order p^{1+2m} as above. Then, provided $p > m$, the ideal K^G is generated by $\omega \in K_2^G$.*

Note that $\omega \in K_2^G$ always because it is the image, in $H_s^2(E, \mathbb{Z}/p)$, of the class $\tilde{\omega} \in H^2(E, \mathbb{Z}/p)$ giving the central extension G , which vanishes when pulled-back to $H^2(G, \mathbb{Z}/p)$ (the induced central extension of G has a section).

On the other hand:

Theorem 1.5. *Take $k = \mathbb{C}$. If G_0 is the extraspecial 2-group of order 2^{1+6} that is the preimage, under the natural covering map*

$$\mathrm{Spin}_7(k) \rightarrow \mathrm{SO}_7(k)$$

of the diagonal matrices $\mathrm{diag}(\pm 1, \dots, \pm 1)$ in $\mathrm{SO}_7(k)$, then K^{G_0} is not generated by its degree 2 piece $K_2^{G_0}$; here again, $K_2^{G_0} = \langle \omega \rangle$.

This does not seem to be related to the fact that $p = 2$ is a special prime; we believe similar examples could very likely be given for every other prime p as well, as will become apparent from the construction in the proof.

Remark 1.6. Theorems 1.4 and 1.5 should be seen in the following context, which provided us with motivation for this work.

- a) As pointed out in [BT11, Thm. 11], the Bloch-Kato conjecture (Voevodsky's theorem) implies that, letting $\Gamma = \mathrm{Gal}(k(V/G))$ as above, and denoting

$$\Gamma^a = \Gamma/[\Gamma, \Gamma], \quad \Gamma^c = \Gamma/[\Gamma, [\Gamma, \Gamma]],$$

the natural map $H^*(\Gamma^a, \mathbb{Z}/p) \rightarrow H^*(\Gamma, \mathbb{Z}/p)$ is surjective, its kernel K^{Γ^a} coincides with the kernel of $H^*(\Gamma^a, \mathbb{Z}/p) \rightarrow H^*(\Gamma^c, \mathbb{Z}/p)$, and is generated by its degree two component $K_2^{\Gamma^a}$ (note that since $[\Gamma, [\Gamma, \Gamma]] \subset [\Gamma, \Gamma]$, there is a natural homomorphism $\Gamma^c \rightarrow \Gamma^a$ giving $H^*(\Gamma^a, \mathbb{Z}/p) \rightarrow H^*(\Gamma^c, \mathbb{Z}/p)$); this follows not obviously from a spectral sequence argument, but in any case directly from the Bloch-Kato conjecture since for $L = k(V/G)$

$$H^n(\Gamma^a, \mathbb{Z}/p) \simeq (\Lambda^n L^*)/p, \quad H^n(\Gamma, \mathbb{Z}/p) \simeq K_n(L)/p$$

and the Milnor K-group $K_n(L)$ is a quotient of $L^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L^*$ by the n -th graded piece of the ideal generated by the Steinberg relations in degree two. Thus, whereas on the full profinite level, kernels of inflation maps are generated in degree two, this property is not inherited by finite quotients of the full Galois group, i.e. finite central extensions of finite abelian groups.

- b) The consequence of the Bloch-Kato conjecture in a) shows the importance to understand central extensions of abelian groups for the computation of stable and unramified cohomology. In [BT17] after formula 1.2 the authors mention that for a finite central extension $0 \rightarrow \mathbb{Z}/p \rightarrow G^c \rightarrow G^a \rightarrow 0$ of an abelian group G^a , the kernel of $H_s^*(G^a, \mathbb{Z}/p) \rightarrow H_s^*(G^c, \mathbb{Z}/p)$ contains the ideal generated by the kernel of $H_s^2(G^a, \mathbb{Z}/p) \rightarrow H_s^2(G^c, \mathbb{Z}/p)$; a preliminary version had *is equal to* instead of *contains*, and the present article shows that the containment can be strict contrary to what was expected.

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2. SOME LINEAR ALGEBRA

We establish some results concerning the exterior algebra of a symplectic vector space over a field of any characteristic. Most of this is contained in [Bour08, Ch. VIII, §13, 3., p. 203-210], but since the standing assumption in loc. cit., Ch. VIII, is to work over a field of characteristic 0 whereas we are interested in the case of a base field of finite characteristic, it is necessary to point out in detail which statements go through unchanged and which ones require adaptation.

Let \mathbb{F} be any field, and let V be a finite-dimensional \mathbb{F} -vector space of even dimension $n = 2m$. Suppose that V is symplectic, which means endowed with a non-degenerate alternating bilinear form Ψ . Let $\mathrm{Sp}_{2m}(V, \Psi) = \mathrm{Sp}_{2m}$ be the corresponding symplectic group. From e.g. [EKM08, Prop. 1.8] it follows that V is isometric to an orthogonal direct sum of m hyperbolic planes, in other words there exists a symplectic basis

$$(e_1, \dots, e_m, e_{-m}, \dots, e_{-1})$$

with $\Psi(e_i, e_j) = 0$ unless $i = -j$ when $\Psi(e_i, e_{-i}) = 1$. This is a statement entirely independent of the characteristic of \mathbb{F} , in particular, also holds in characteristic two (the form is then at the same time alternating and symmetric). Let V^* be the dual vector space to V , and (e_i^*) the basis dual to the basis (e_i) . We identify the alternating form Ψ with an element $\Gamma^* \in \Lambda^2 V^*$. Then

$$(6) \quad \Gamma^* = - \sum_{i=1}^m e_i^* \wedge e_{-i}^*.$$

Via the isomorphism $V \rightarrow V^*$ given by Ψ , the form Ψ induces a symplectic form Ψ^* on V^* . Identifying Ψ^* with an element Γ in $\Lambda^2 V$, one finds

$$(7) \quad \Gamma = \sum_{i=1}^m e_i \wedge e_{-i}.$$

One also denotes by $X_- : \Lambda^* V \rightarrow \Lambda^* V$ the endomorphism induced by left exterior product with Γ and by $X_+ : \Lambda^* V \rightarrow \Lambda^* V$ the endomorphism given by left interior product (contraction) with $-\Gamma^*$; more precisely,

$$(8) \quad X_+(v_1 \wedge \cdots \wedge v_r) = \sum_{1 \leq i < j \leq r} (-\Psi)(v_i, v_j) (-1)^{i+j} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_r.$$

Moreover, let $H : \Lambda^* V \rightarrow \Lambda^* V$ be the endomorphism that is multiplication by $(m-r)$ on $\Lambda^r V$ for $0 \leq r \leq 2m$. Then as in [Bour08, p. 207 and Ex. 19], it follows that

$$(9) \quad [X_+, X_-] = -H, \quad [H, X_+] = 2X_+, \quad [H, X_-] = -2X_-$$

so the vector subspace of $\text{End}(\Lambda^* V)$ generated by X_+, X_-, H is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{F})$. Moreover, for the action of $\mathfrak{sl}(2, \mathbb{F})$ on $\Lambda^* V$, the subspace $\Lambda^r V$ is the subspace of elements of weight $m - r$.

In the following Proposition and its proof, we make the conventions that for integers $i < 0$, $\Lambda^i V := 0$ and for binomial coefficients and positive integers n , $\binom{n}{i} := 0$.

Proposition 2.1. *Put $E_r = (\Lambda^r V) \cap \ker X_+$, the “primitive elements” in $\Lambda^r V$. If $p = \text{char } \mathbb{F} > \dim V/2 = m$ or $\text{char } \mathbb{F} = 0$, then*

- a) for $r \leq m - 1$, the restriction of X_- to $\Lambda^r V$ is injective;
- b) for $r \geq m - 1$, the restriction of X_- to $\Lambda^r V$ induces a surjection from $\Lambda^r V$ onto $\Lambda^{r+2} V$;
- c) for $r \leq m$,

$$\Lambda^r V = E_r \oplus X_-(\Lambda^{r-2} V).$$

Moreover, E_r coincides with the submodule $F_r \subset \Lambda^r V$ defined as the span of all “completely reducible” r -vectors $v_1 \wedge \cdots \wedge v_r$ such that $\langle v_1, \dots, v_r \rangle$ is a totally isotropic subspace of V . Here completely reducible means simply a pure wedge product of the above form $v_1 \wedge \cdots \wedge v_r$.

Proof. The proof is based on the following observations.

- (I) Let E be any $\mathfrak{sl}(2, \mathbb{F})$ -module, and let ϵ be a primitive element, by which we mean, as usual, $X_+(\epsilon) = 0$ and ϵ is an eigenvector for some $\lambda \in \mathbb{F}$ for H . Then, as long as ν is an integer such that $1 \leq \nu < p$ it does make sense to define

$$\epsilon_\nu = \frac{(-1)^\nu}{\nu} X_-^\nu \epsilon, \quad \epsilon_0 = \epsilon, \quad \epsilon_{-1} = 0.$$

Then a straightforward computation with the commutation relations (9), done in [Bour08, Chapter VIII, §1, 2., Prop. 1], shows that

$$(10) \quad \begin{aligned} H\epsilon_\nu &= (\lambda - 2\bar{\nu})\epsilon_\nu \\ X_-\epsilon_\nu &= -(\bar{\nu} + 1)\epsilon_{\nu+1} \\ X_+\epsilon_\nu &= (\lambda - \bar{\nu} + 1)\epsilon_{\nu-1} \end{aligned}$$

as long as all indices of the occurring ϵ 's are $< p$. Here we put a bar on an integer to indicate that we consider it as an element of \mathbb{F} via the natural homomorphism $\mathbb{Z} \rightarrow \mathbb{F}$, which for us will however be usually not injective.

(II) If we define F_r as in the statement of the Proposition, then obviously $F_r \subset E_r$ and

$$(11) \quad \dim F_r = \binom{2m}{r} - \binom{2m}{r-2}, \quad 0 \leq r \leq m.$$

This is proven in [Bruyn09, Thm. 1.1] under no assumptions on $p = \text{char } \mathbb{F}$. For the module $E = \Lambda^*V$ we can thus display the action of the operators H, X_+, X_- schematically in the familiar way:

$$\begin{array}{ccc} \begin{array}{c} \Lambda^0 V \xrightarrow{X_-} \Lambda^2 V \\ \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \quad \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \\ \xleftarrow{X_+} \end{array} & \dots & \begin{array}{c} \Lambda^{2m-2} V \xrightarrow{X_-} \Lambda^{2m} V \\ \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \quad \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \\ \xleftarrow{X_+} \end{array} \\ \text{weight: } \quad \overline{\mathbf{m}} & & \overline{\mathbf{m}-2} \quad \dots \quad \overline{-(\mathbf{m}-2)} \quad \overline{-\mathbf{m}} \end{array}$$

and

$$\begin{array}{ccc} \begin{array}{c} V \xrightarrow{X_-} \Lambda^3 V \\ \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \quad \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \\ \xleftarrow{X_+} \end{array} & \dots & \begin{array}{c} \Lambda^{2m-3} V \xrightarrow{X_-} \Lambda^{2m-1} V \\ \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \quad \text{\scriptsize } \circlearrowleft \text{\scriptsize } H \\ \xleftarrow{X_+} \end{array} \\ \text{weight: } \quad \overline{\mathbf{m}-1} & & \overline{\mathbf{m}-3} \quad \dots \quad \overline{-(\mathbf{m}-3)} \quad \overline{-(\mathbf{m}-1)} \end{array}$$

Now we start to use the assumption that $p = \text{char } \mathbb{F} > m$.

If we start with a primitive element $\epsilon = \epsilon_0$ in one of the F_r , $0 \leq r \leq m$, of weight $\lambda = \overline{\mathbf{m}-r}$ in $\{0, \dots, \overline{\mathbf{m}}\}$, then the ϵ_ν as in item (I) above are all defined for $\nu = 0, \dots, m$. Moreover, if μ is the largest integer such that $\epsilon_\mu \neq 0$, then $\mu \leq m < p$ and μ can only possibly be equal to m if we start with ϵ_0 in F_0 ; excluding the latter case for a moment, we can use the third of formulas (10) to get

$$0 = X_+(\epsilon_{\mu+1}) = (\lambda - \bar{\mu})\epsilon_\mu$$

where now all indices are still $< p$, and one can only have that $\lambda - \bar{\mu} = 0$ in \mathbb{F} if μ is the unique lift of λ in $\{0, \dots, m\}$. If $\mu = m$ and $\epsilon_0 \in F_0$, the third of formulas (10) still shows that all of $\epsilon_0, \epsilon_1, \dots, \epsilon_m$ must be nonzero (since applying X_+ to any of them the appropriate number of times returns a nonzero multiple of ϵ_0 under the

standing assumptions), so putting all this together we can say that starting from a primitive $\epsilon_0 \in F_r$, $0 \leq r \leq m$, with weight $\lambda = \overline{m-r}$ we get a chain of nonzero vectors

$$\begin{array}{ccc} \epsilon_0 & \begin{array}{c} \xrightarrow{X_-} \\ \xleftarrow{X_+} \end{array} & \epsilon_1 & \dots & \epsilon_{m-r-1} & \begin{array}{c} \xrightarrow{X_-} \\ \xleftarrow{X_+} \end{array} & \epsilon_{m-r} \end{array}$$

$$\text{weight :} \quad \overline{m-r} \quad \overline{m-r-2} \quad \dots \quad \overline{-(m-r-2)} \quad \overline{-(m-r)}$$

where according to the formulas in (10) and since $p > m$, the X_+ and X_- map each of the ϵ 's to a nonzero multiple of the subsequent one "up or down the ladder" as indicated in the previous diagram. In particular, X_+ and X_- induce isomorphisms between the vector subspaces indicated in the following diagram:

$$\begin{array}{ccc} F_r & \begin{array}{c} \xrightarrow{X_-} \\ \xleftarrow{X_+} \end{array} & X_-(F_r) & \dots & X_-^{m-r-1}(F_r) & \begin{array}{c} \xrightarrow{X_-} \\ \xleftarrow{X_+} \end{array} & X_-^{m-r}(F_r) \end{array}$$

for $0 \leq r \leq m$. In addition, the sum of $F_r, X_-(F_{r-2}), X_-^2(F_{r-4}), \dots$ inside $\Lambda^r V$ (for any $0 \leq r \leq 2m$, noting $F_s = 0$ for $s > m$) is direct: this can be seen by repeatedly applying X_+ and using the third formula of (10). Then a dimension count using item (II) at the beginning of the proof yields

$$\Lambda^r V = F_r \oplus X_-(F_{r-2}) \oplus X_-^2(F_{r-4}) \oplus \dots = F_r \oplus X_-(\Lambda^{r-2} V)$$

for any r , which proves $E_r = F_r$, c) in the statement of the Proposition as well as a) and b). \square

We will need one further piece of information concerning $E_r = F_r$, $0 \leq r \leq m$ later.

Theorem 2.2. *Let \mathbb{F} be a field of characteristic p . The $\mathrm{Sp}(2m, \mathbb{F})$ -module $E_r = F_r$, $1 \leq r \leq m$ is irreducible if*

$$p > m - \frac{r}{2} + 1.$$

More precisely, it is irreducible if and only if p does not divide

$$\prod_{0 \leq j \leq r, j \equiv r \pmod{2}} \binom{m - \frac{r+j}{2} + 1}{(r-j)/2}.$$

Proof. This is [PS83, p. 1313, Thm. 2]. \square

3. PROOFS OF MAIN RESULTS

We start by recalling that for an abelian p -group A

$$(12) \quad H_s^*(A, \mathbb{Z}/p) \simeq \Lambda^s A^\vee = \Lambda^s H^1(A, \mathbb{Z}/p) = \Lambda^s \mathrm{Hom}(A, \mathbb{Z}/p).$$

See e.g. [Bo05], Example after Remark 6.10. We now turn to extraspecial groups G sitting in an exact sequence (3), and retain the notation from Section 1.

Definition 3.1. A subgroup $A \subset E$ is called *totally isotropic* if it is a totally isotropic subspace of the \mathbb{F}_p -vector space E , i.e. the symplectic form ω vanishes identically on A .

Totally isotropic subgroups A can be characterised as the ones such that there exists an abelian subgroup $\tilde{A} \subset G$ of the same p -rank as A mapping onto A via π , where π is the natural surjection $\pi: G \rightarrow E$ (take for \tilde{A} the subgroup generated by the lifts to G of a minimal set of generators for A).

Proof of Theorem 1.4. For a totally isotropic subgroup A of E , consider the diagram

$$(13) \quad \begin{array}{ccc} H_s^*(E, \mathbb{Z}/p) & \xrightarrow{\pi^*} & H_s^*(G, \mathbb{Z}/p) \\ \downarrow r_A^E & & \downarrow r_{\tilde{A}}^G \\ H_s^*(A, \mathbb{Z}/p) & \xrightarrow{\pi^*} & H_s^*(\tilde{A}, \mathbb{Z}/p) \end{array}$$

where the vertical arrows are the restriction maps. From the description of the stable cohomology of abelian groups, one gets that the lower horizontal arrow is injective, and in fact an isomorphism. In other words, a class $\alpha \in H_s^r(E, \mathbb{Z}/p)$ that is nontrivial on a totally isotropic subgroup is *not* in the kernel of π^* . Applying Proposition 2.1 to the symplectic vector space $V = H^1(E, \mathbb{Z}/p) = E^*$, we see that in order to prove the Theorem it suffices to show that every nonzero class $\alpha \in E_r$, $0 \leq r \leq m$, is nontrivial on some totally isotropic subgroup. Since totally isotropic subgroups are invariant under the action of $\mathrm{Sp}_{2m}(\mathbb{F}_p)$, the classes in E_r that are trivial on all totally isotropic subgroups form a $\mathrm{Sp}_{2m}(\mathbb{F}_p)$ submodule; as E_r is irreducible by Theorem 2.2, this submodule is either reduced to zero or everything. Hence it suffices to prove that some class $\alpha \in E_r$, for every $0 \leq r \leq m$, is nontrivial on some totally isotropic subgroup. But this is clear: in the notation introduced at the beginning of Section 2, if we take the totally isotropic subgroup $A = \langle e_1^*, \dots, e_r^* \rangle$, then $\alpha = e_1 \wedge \dots \wedge e_r$ is nontrivial on it. \square

For the proof of Theorem 1.5 we need a few more auxiliary results. Assume $k = \mathbb{C}$ now. As in the statement of that Theorem, consider the preimage $G_0 \subset \mathrm{Spin}_7(k)$ of the diagonal matrices with entries ± 1 in $\mathrm{SO}_7(k)$.

Lemma 3.2. G_0 is an extraspecial 2-group sitting in an exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow G_0 \rightarrow (\mathbb{Z}/2)^6 \rightarrow 0.$$

Proof. The existence of such an exact sequence is clear. The point is to verify that $\mathbb{Z}/2$ is the entire center of G_0 , and this is done in [Ad96, Chapter 4, Lemma on p. 22]; for this it is important that 7 coming from $\mathrm{Spin}_7(k)$ is odd: the center of the analogously defined groups for the even Spin-groups is larger (the claim that all of them are extraspecial in [Bak02, 5.5, p. 154] is erroneous). \square

Lemma 3.3. *Generically free linear $\mathrm{Spin}_7(k)$ -quotients $V/\mathrm{Spin}_7(k)$ and generically free linear G_0 -quotients W/G_0 are stably birationally equivalent. Here V resp. W are any generically free (finite-dimensional, complex) linear representations of $\mathrm{Spin}_7(k)$ resp. G_0 .*

Proof. This is proven in [Bo87, §3 ff.], but we include the easy argument for the sake of completeness and give a few more details. Consider the standard representation k^7 of $\mathrm{SO}_7(k)$; via the natural covering map it is a $\mathrm{Spin}_7(k)$ -representation. Inside $k^7 \oplus \cdots \oplus k^7$ (seven times), consider the subvariety R of tuples of vectors (v_1, \dots, v_7) that are mutually orthogonal. This is invariant under the group action, and has the structure of a tower of equivariant vector bundles over any of the summands k^7 . Let $P = ke_1 \times \cdots \times ke_7 \subset R$ be the Cartesian product of the lines through the standard basis vectors e_1, \dots, e_7 . Then P is a $(\mathrm{Spin}_7(k), G_0)$ -section of the action and [CTS07, Theorem 3.1] applies; in particular, given any generically free linear $\mathrm{Spin}_7(k)$ -representation V , then (a) $(V \oplus R)/\mathrm{Spin}_7(k)$ is stably equivalent to $V/\mathrm{Spin}_7(k)$ since R has the structure of a tower of equivariant vector bundles and one can apply the “no-name lemma” [CTS07, Thm. 3.6]; (b) in $V \oplus R$ the subvariety $V \times P$ is a $(\mathrm{Spin}_7(k), G_0)$ -section, whence $(V \oplus R)/\mathrm{Spin}_7(k)$ is birational to $(V \times P)/G_0$. This concludes the proof. \square

Theorem 3.4. *Generically free linear $\mathrm{Spin}_7(k)$ -quotients are stably rational; in particular, combining this with Lemma 3.3, G_0 has trivial unramified cohomology (with finite torsion coefficients).*

Proof. The fact that generically free $\mathrm{Spin}_7 \mathbb{C}$ -quotients are stably rational is proven in [Kor00] (despite the title only referring to Spin_{10}), see also [CTS07, §4.5]. The fact that unramified cohomology of stably rational varieties is trivial is proven, for example, in [CT95, Prop. 4.1.4]. \square

Proposition 3.5. *Let G be a finite group. Suppose that $\alpha \in H_s^*(G, \mathbb{Z}/p)$ is a class whose restriction, for any $g \in G$, to $H_s^*(Z(g), \mathbb{Z}/p)$ is induced from $H_s^*(Z(g)/\langle g \rangle, \mathbb{Z}/p)$; here $Z(g)$ is the centraliser of g in G . Then α is unramified.*

Proof. This follows from the way residue maps in Galois cohomology are defined, see [GMS, Chapter II, 7.] for the following: for $K = k(V)^G$, V a generically free G -representation, and v a geometric discrete valuation of K , one considers the completion K_v , and the decomposition group Dec_w where w is an extension of v to the separable closure K_s . Then $\mathrm{Gal}(K_v) \simeq \mathrm{Dec}_w \subset \mathrm{Gal}(K)$, and there is a split exact sequence

$$(14) \quad 1 \rightarrow I \rightarrow \mathrm{Gal}(K_v) \rightarrow \mathrm{Gal}(\kappa_v) \rightarrow 1$$

where κ_v is the residue field of v and $I \simeq \hat{\mathbb{Z}}$ is the topologically cyclic inertia subgroup. For a finite constant $\mathrm{Gal}(K)$ -module C of order not divisible by $\mathrm{char}(k)$, there is an exact sequence

$$0 \longrightarrow H^i(\mathrm{Gal}(\kappa_v), C) \longrightarrow H^i(\mathrm{Gal}(K_v), C) \xrightarrow{r} H^{i-1}(\mathrm{Gal}(\kappa_v), \mathrm{Hom}(I, C)) \longrightarrow 0$$

and r is the residue map. The residue of an element $\beta \in H^i(K, C)$ is obtained by restricting to $H^i(K_v, C)$ and afterwards applying r . Now under the natural map

$$\mathrm{Gal}(K) \rightarrow G$$

the topologically cyclic inertia subgroup I will map to a cyclic subgroup of G generated by some element $g \in G$, and (14) being a central extension, $\mathrm{Gal}(K_v)$ will map into the centraliser $Z(g) \subset G$. Now if $C = \mathbb{Z}/p$ and $\alpha \in H_s^*(G, \mathbb{Z}/p)$ is a class whose restriction to $H_s^*(Z(g), \mathbb{Z}/p)$ is induced from $H_s^*(Z(g)/\langle g \rangle, \mathbb{Z}/p)$, then since there is a commutative diagram,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa_v) & \longleftarrow & \mathrm{Gal}(K_v) & \hookrightarrow & \mathrm{Gal}(K) \\ \downarrow & & \downarrow \psi & & \downarrow \tau \\ Z(g)/\langle g \rangle & \longleftarrow & Z(g) & \hookrightarrow & G \end{array}$$

and a factorisation

$$\begin{array}{ccc} H^i(Z(g), \mathbb{Z}/p) & \longrightarrow & H^i(\mathrm{Gal}(K_v), \mathbb{Z}/p) \\ & \searrow & \nearrow \\ & H_s^i(Z(g), \mathbb{Z}/p) & \end{array}$$

(the latter because $\mathrm{Gal}(K_v)$ sits inside $\mathrm{Gal}(k(V/(Z(g)))) \subset \mathrm{Gal}(K)$, $\mathrm{Gal}(k(V/(Z(g))))$ being the preimage of $Z(g)$ under τ), we get that the restriction of α to the decomposition group comes from $H^i(\mathrm{Gal}(\kappa_v), \mathbb{Z}/p)$, hence its residue is zero. \square

Proof of Theorem 1.5. Let $E = (\mathbb{Z}/2)^6$, $V = E^*$. On E we choose coordinates $x_1, x_2, x_3, y_1, y_2, y_3 \in E^*$ which form a symplectic basis and so that

$$\omega = \sum_{i=1}^3 x_i \wedge y_i.$$

To prove the Theorem we are going to proceed in the following Steps.

Step 1. We produce a class $\zeta \in H_s^4(E, \mathbb{Z}/2) = \Lambda^4 V$ that is not in the ideal generated by ω . More precisely, we will take

$$\zeta = x_2 \wedge x_3 \wedge y_2 \wedge y_3.$$

Step 2. We prove that $\pi^*(\zeta) \in H_s^4(G_0, \mathbb{Z}/2)$ is unramified using the criterion given in Proposition 3.5. By Theorem 3.4, we conclude $\pi^*(\zeta) = 0$ whence ζ is a class in the kernel of π^* not in the ideal generated by ω . Indeed, this argument works for any class in $H_s^4(E, \mathbb{Z}/2)$, not just the specific ζ .

Step 3. We check that the kernel of $\pi^*: H_s^2(E, \mathbb{Z}/2) \rightarrow H_s^2(G_0, \mathbb{Z}/2)$ is spanned by ω .

As kindly pointed out by the referee, for Step 1 it suffices to remark that $\omega^2 = 0$, but $\omega \wedge \zeta \neq 0$ so that ζ cannot be a multiple of ω .

Let us now prove Step 2. This uses another very nice simplification suggested by the referee, our initial proof was more complicated. We prove that $\pi^*(\vartheta) \in$

$H_s^4(G_0, \mathbb{Z}/2)$ is unramified for any class $\vartheta \in H_s^4(E, \mathbb{Z}/2)$, whence $H_s^4(E, \mathbb{Z}/2) \rightarrow H_s^4(G_0, \mathbb{Z}/2)$ is actually a zero map since G_0 has trivial unramified cohomology.

For an element g in E , we denote \tilde{g} any lift of g to G_0 . Denote the image of the centraliser $Z(\tilde{g}) \subset G_0$ in E by S_g . It has the following description:

$$S_g = \langle h \in E \mid \omega(g, h) = 0 \rangle,$$

in other words, it consists of all elements h in E whose preimages in G_0 commute with \tilde{g} . In order to show that $\pi^*(\vartheta)$ is unramified in the stable cohomology of G_0 , it is thus sufficient, by Proposition 3.5, to show:

(*) For any $\tilde{g} \in G_0$, the restriction of $\pi^*(\vartheta)$ to $H_s^*(Z(\tilde{g}), \mathbb{Z}/2)$ is induced from $H_s^*(S_g/\langle g \rangle, \mathbb{Z}/2)$ via the natural homomorphisms

$$Z(\tilde{g}) \rightarrow Z(\tilde{g})/\langle \tilde{g} \rangle \rightarrow S_g/\langle g \rangle.$$

If $g = 0$, this is obvious since ϑ comes from $H_s^*(E, \mathbb{Z}/2)$. Hence we will assume in the sequel that $g \neq 0$. Without loss of generality, we can also assume that $\dim S_g \geq 4$ since otherwise ϑ restricts to zero on S_g , hence will also be zero on $Z(\tilde{g})$ whence (*) is trivially verified. Since g is nonzero and ω is nondegenerate, S_g is a hyperplane, hence has dimension 5. Furthermore, we can assume that $g = e_1$ is the standard basis element whence

$$S_g = \langle e_1, e_2, e_{-2}, e_3, e_{-3} \rangle.$$

Then a direct computation shows

$$\Lambda^4(S_g^*) = (\omega|_{S_g} \wedge \Lambda^2(S_g^*)) \oplus \langle x_2 \wedge y_2 \wedge x_3 \wedge y_3 \rangle$$

and the first summand is in the kernel of $H_s^4(S_g, \mathbb{Z}/2) \rightarrow H_s^4(Z(\tilde{g}), \mathbb{Z}/2)$, and the second summand is induced from $H_s^4(S_g/\langle g \rangle, \mathbb{Z}/2)$.

Finally, for Step 3, note that there are generically free E - and G_0 -representations V_E and V_{G_0} such that, denoting by a superscript L the loci where the actions are free, $(V_{G_0}^L)/G_0$ maps dominantly to $(V_E^L)/E$, and the induced field extension $k(V_{G_0}^L)^{G_0} \supset k(V_E)^E$ factors as

$$\begin{array}{c} k(V_{G_0}^L)^{G_0} \xrightarrow{\cong} k(\mathcal{S})(t) \\ \uparrow \\ k(\mathcal{S}) \\ \uparrow \\ k(V_E)^E \end{array}$$

where \mathcal{S} is a Severi-Brauer scheme over $k(V_E)^E$ and t is an indeterminate: indeed, one can take for V_E any generically free E -representation, which is at the same time a G_0 -representation via the homomorphism $G_0 \rightarrow E$, and for V_{G_0} one takes $W \oplus V_E$, where W is a generically free G_0 -representation in which the center $\mathbb{Z}/2$ of G_0 acts nontrivially via multiplication by scalars. Then the sought-for Severi-Brauer scheme \mathcal{S} is $(\mathbb{P}(W) \times V_E^L)/G_0 \rightarrow (V_E^L)/E$, over which $(W \setminus \{0\} \times V_E^L)/G_0$

is a k^* -principal bundle (Zariski locally trivial). Hence the tower of fields above. By Amitsur's theorem [GS06, 4.5.1], the kernel of $\mathrm{Br}(k(V_E)^E) \rightarrow \mathrm{Br}(k(\mathcal{S}))$ is cyclic generated by the class of \mathcal{S} , and $\mathrm{Br}(k(\mathcal{S})) \rightarrow \mathrm{Br}(k(\mathcal{S})(t))$ is injective. Since the two-torsion in the Brauer group of the fields Λ involved here is precisely $H^2(\Lambda, \mathbb{Z}/2)$, the definition of stable cohomology given in formula (1) shows that the kernel of $\pi^*: H_s^2(E, \mathbb{Z}/2) \rightarrow H_s^2(G_0, \mathbb{Z}/2)$ is cyclic. Since it contains the nontrivial class ω , it is generated by ω . \square

Remark 3.6. As kindly pointed out by the referee, Step 3 of the above proof can be given a simpler proof based on ideas of multilinear algebra similar to the ones used in Theorem 1.4, which imply more generally:

Lemma 3.7. *For any extraspecial p -group G the degree 2 component*

$$K_2^G = \ker(H_s^2(E, \mathbb{Z}/p) \rightarrow H_s^2(G, \mathbb{Z}/p))$$

is generated by ω .

Proof. It suffices to show that if $\eta \in H_s^2(E, \mathbb{Z}/p) = \Lambda^2(E^*)$ is such that it vanishes on each $u \wedge v \in \Lambda^2 E$ for u, v generating an isotropic subspace, then this η is a multiple of ω . Indeed, this is true in any characteristic since the subspace of $\Lambda^2 E$ generated by isotropic planes $u \wedge v$ has codimension 1 (hence it has orthogonal of dimension 1 generated by ω): indeed, the isotropic planes

$$e_i \wedge e_j, \quad j \neq -i$$

and

$$(e_i + e_{i+1}) \wedge (e_{-i} - e_{-i-1}), \quad 1 \leq i \leq m-1$$

generate a codimension 1 subspace. (This is also implied by [Bruyn09, Thm. 1.1]).

The assertion of this Lemma also follows from Amitsur's Theorem in a way analogous to the method used in Step 3 of the proof of Theorem 1.5, but the proof here is clearly much easier. \square

Remark 3.8. The phenomenon, on which the proof of Theorem 1.5 is based to a large extent, that X_- as in Proposition 2.1 can fail to be surjective on some $\Lambda^r V$ with $r \geq m-1$ in cases where $p \leq m$, is not related to $p=2$, but reoccurs for other primes: it is only to do with the fact that $(X_-)^p = 0$ in characteristic p . We therefore strongly suspect that examples of the type given in Theorem 1.5 where K_2^G fails to generate the ideal K^G , for some extraspecial group G , exist for all primes p .

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