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The expected signature of Brownian motion stopped on the boundary of a circle has finite radius of convergence

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Abstract

The expected signature is an analogue of the Laplace transform for probability measures on rough paths. A key question in the area has been to identify a general condition to ensure that the expected signature uniquely determines the measures. A sufficient condition has recently been given by Chevyrev and Lyons and requires a strong upper bound on the expected signature. While the upper bound was verified for many well-known processes up to a deterministic time, it was not known whether the required bound holds for random time. In fact, even the simplest case of Brownian motion up to the exit time of a planar disc was open. For this particular case we answer this question using a suitable hyperbolic projection of the expected signature. The projection satisfies a three-dimensional system of linear PDEs, which (surprisingly) can be solved explicitly, and which allows us to show that the upper bound on the expected signature is *not* satisfied.

Contents

1. *Introduction*

Let a probability measure on a subset of the real line have moments of all orders. Under which conditions do these moments pin down the probability measure uniquely? This is the well-studied *moment problem*. When the subset is compact, the answer is always affirmative. In the noncompact case uniqueness is more delicate (see [**14**]).

In stochastic analysis one is usually concerned with measures on some space of paths, the prime example being *Wiener measure* on the space of continuous functions. It turns out that for many purposes a good replacement for 'monomials' in this setting are the iterated integrals of paths. The collection of all of these integrals is called the *iterated-integrals signature*.

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For smooth paths $X : [0,T] \to \mathbb{R}^d$ (and with respect to a time horizon $T > 0$), it is defined, using Riemann–Stieltjes integration, as

$$\begin{aligned} S(X)_{0,T} &:= 1 + \int_0^T dX_s + \int_0^T \int_0^{r_2} dX_{r_1} \otimes dX_{r_2} \\ &\quad + \int_0^T \int_0^{r_3} \int_0^{r_2} dX_{r_1} \otimes dX_{r_2} \otimes dX_{r_3} + \cdots \in T((\mathbb{R}^d)) := \prod_{n=0}^{\infty} (\mathbb{R}^d)^{\otimes n}. \end{aligned} \tag{1}$$

It is well known (see [1] and references therein) that

- $S(X)_{0,T} \in G$, where $G \subset T((\mathbb{R}^d))$ is the group of *grouplike elements*;
- $S(X)_{0,T}$ completely characterizes the path X up to reparametrization and up to tree-likeness.

Let $X : \Omega \times [0,T] \to \mathbb{R}^d$ now be a stochastic process. For fixed $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is usually *not* smooth, so that we have to assume that the stochastic process possesses a 'reasonable' integration theory.

In particular assume that integrals of the form $\int g(X_s) dX_s$, exist for a large class of functions $g \in C(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^n))$ and that the fundamental theorem of calculus holds:

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \partial_{x_i} f(X_s) dX_s^i.$$

Examples include Brownian motion with Stratonovich integration and fractional Brownian motion with Hurst parameter strictly larger than 1/2 with Young integration. For further examples, see [4].

The iterated-integrals signature $S(X)_{0,T}$ defined by the expression (1) — now, using the given integration theory — is then a random variable. Let us assume that we can take its expectation level by level:

$$\mathbb{E}\left[\left\| \int_0^T \int_0^{r_n} \dots \int_0^{r_2} dX_{r_1} \otimes \cdots \otimes dX_{r_n} \right\| \right] < +\infty.$$

The choice of norm in fact does not matter here but we will equip $\mathbb{R}^d$ with the Euclidean norm and the projective norm $\|\cdot\|$ on $(\mathbb{R}^d)^{\otimes n}$ throughout this paper. We can then define expected signature level by level:

$$\begin{aligned} \mathsf{ExpSig}(X)_T &:= \mathbb{E}_{\mathsf{p}}[S(X)_{0,T}] \\ &:= \sum_{n=0}^{\infty} \mathbb{E}\left[\int_0^T \int_0^{r_n} \dots \int_0^{r_2} dX_{r_1} \otimes \cdots \otimes dX_{r_n} \right] \in T((\mathbb{R}^d)), \end{aligned} \tag{2}$$

where $\mathbb{E}_{\mathsf{p}}$ denotes the expectation level by level (p for "projective") of a G-valued random variable. The question arises:

$$\mathsf{ExpSig}(X) \text{ completely characterize the law of } X$$

The correspondence between X and $S(X)_{0,T}$ was studied in [1] and references therein. Here we will focus on

$$\mathsf{ExpSig}(X)_T \text{ completely characterize the law of} S(X)_{0,T} \text{ on } G$$

A sufficient condition for this to be the case is given in [2]: if $\mathsf{ExpSig}(X)_T$ has infinite radius of convergence, that is,

$$\sum_{n \geqslant 0} \|\mathrm{proj}_n \, \mathsf{ExpSig}(X)_T\| \lambda^n < +\infty, \tag{3}$$

for all $\lambda > 0$ then the law of $S(X)_{0,T}$ on G is the unique law with this (projective) expected value. Here $\mathrm{proj}_n : T((\mathbb{R}^d)) \to (\mathbb{R}^d)^{\otimes n}$ denotes projection onto tensors of length n.

Let us give two examples. Let μ be a probability measure on $\mathbb{R}$ having all moments and define

$$a_n := \int x^n \mu(dx).$$

Consider the stochastic process $X_t := tZ$, where Z is distributed according to μ. Since X is smooth, its signature is well defined and actually has the simple form

$$S(X)_{0,T} = 1 + TZ + \frac{T^2}{2!} Z^2 + \frac{T^3}{3!} Z^3 + \dots$$

Then

$$\mathsf{ExpSig}(X)_T = 1 + T a_1 + \frac{T^2}{2!} a_2 + \frac{T^3}{3!} a_3 + \dots,$$

and a sufficient condition for $\sum_n a_n T^n \lambda^n / n!$ to have infinite radius of convergence is $|a_n| \leqslant C^n$, for some $C > 0$.[†] Then [2, Proposition 6.1] applies, and the law of $S(X)_{0,T}$ on G is uniquely determined by these moments.

Consider now the expected signature of a standard Brownian motion B calculated up to some *fixed* time $T > 0$. It is known (see for example [9, Proposition 4.10]) that

$$\mathsf{ExpSig}(B)_T = \exp\left(\frac{T}{2} \sum_{i=1}^{d} e_i \otimes e_i \right).$$

It follows that

$$\|\mathrm{proj}_{2n} \mathsf{ExpSig}(B)\| = \left\| \frac{T^n}{2^n n!} \left(\sum_{i=1}^{d} e_i \otimes e_i \right)^n \right\| \leqslant \frac{d^n T^n}{2^n n!},$$

and hence

$$\sum \| \mathrm{proj}_n \mathsf{ExpSig}(B)_T \| \lambda^n < +\infty, \tag{4}$$

for any $\lambda > 0$. Again, by [2, Proposition 6.1], the law of $S(B)_{0,T}$ is uniquely determined by $\mathsf{ExpSig}(B)_T$. The condition (4) in fact holds more generally when B is a Gaussian process or a Markov process; see [2, 3, 12] for further details.

In this paper we study properties of the expected signature, not up to deterministic time T, but up to a *stopping time* τ. Concretely, we consider the Brownian motion B^z in $\mathbb{R}^2$ started at some point z in the unit circle $\mathbb{D} := \{z \in \mathbb{R}^2 : |z|_2 \leqslant 1\}$, and stopped at hitting the boundary, that is,

$$\tau := \inf \{t \geqslant 0 : B_t^z \in \partial \mathbb{D}\}, \tag{5}$$

with $|\cdot|_2$ denoting the Euclidean norm on $\mathbb{R}^2$. In the notation introduced above, we are interested in

$$\Phi(z) := \mathsf{ExpSig}(X^z)_\infty,$$

where $X_t^z := B_{t \wedge \tau}^z$. In [8] it was shown that for every $n \in \mathbb{N}$ and $n \geqslant 2$, the nth term of Φ satisfies the following PDE:

$$\Delta(\mathrm{proj}_n (\Phi(z))) = -2 \sum_{i=1}^{d} e_i \otimes \frac{\partial \, \mathrm{proj}_{n-1} (\Phi(z))}{\partial z_i} - \left(\sum_{i=1}^{d} e_i \otimes e_i \right) \otimes \mathrm{proj}_{n-2} (\Phi(z)), \tag{6}$$

[†]The condition $|a_n| \leqslant C^n$ is of course more than enough in the classical moment problem to have uniqueness for the law μ on $\mathbb{R}$ [13, Example X.6.4].

with the boundary condition that for each $|z|_2 = 1$,

$$\mathrm{proj}_n\left(\Phi(z)\right) = \begin{cases} 0, & \text{if } n \geqslant 1 \\ 1, & \text{if } n = 0. \end{cases}$$

Additionally, one has $\mathrm{proj}_0(\Phi(z)) = 1$ and $\mathrm{proj}_1(\Phi(z)) = 0$ for all $z \in \mathbb{D}$. Using this, they were able to obtain the bound $\|\mathrm{proj}_n(\Phi(z))\| \leqslant C^n$ for some $C > 0$ [**8**, Theorem 3.6]. This bound is *not* enough to decide whether the radius of convergence for $\mathsf{ExpSig}(X^z)_\infty$ is infinite or not, but it is enough to deduce that $\mathsf{ExpSig}(X^z)_\infty$ has radius of converge strictly larger than 0. In this work we show that the radius of convergence is indeed finite.

Recall from [**2**, Proposition 6.1] that if A, B are G-valued random variables such that $\mathbb{E}_{\mathsf{p}}[A] = \mathbb{E}_{\mathsf{p}}[B]$ and $\mathbb{E}_{\mathsf{p}}[A]$ has an infinite radius of convergence, then $A \overset{\mathcal{D}}{=} B$. Our main theorem, proven in Section 6, is the following.

Theorem. *The expected signature* $\Phi(0) = \mathsf{ExpSig}(X^0)_\infty = \mathbb{E}_{\mathsf{p}}[S(X^0)_{0,\infty}]$ *of a two-dimensional Brownian motion stopped upon exiting the unit disk has a* finite *radius of convergence.*

Inspired by [**6, 10**], we consider the map $M \in L(\mathbb{R}^2, L(\mathbb{R}^3, \mathbb{R}^3))$ defined by

$$M : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & 0 \end{pmatrix}. \tag{7}$$

By [**6**, Lemma 3.1], one has

$$\|M\|_{L(\mathbb{R}^2, L(\mathbb{R}^3,\mathbb{R}^3))} = \sup_{|u|_2=1, |v|_2=1} |M(u)v|_2 = 1. \tag{8}$$

We may extend M as a continuous linear map on the k-times tensor product $(\mathbb{R}^2)^{\otimes k}$, by the following relation:

If $k = 0$, we take by convention that $(\mathbb{R}^d)^{\otimes 0} := \mathbb{R}$ and define

$$M(v) = 1 \quad \forall v \in \mathbb{R}. \tag{9}$$

For $k \geqslant 1$, if $v_1, \ldots, v_k \in \mathbb{R}^d$, then

$$M(v_1 \otimes \cdots \otimes v_k) = M(v_1) \cdots M(v_k). \tag{10}$$

Using (8) and the definition of projective norm, we see that

$$\|M(\mathrm{proj}_n(\Phi(z)))\|_{L(\mathbb{R}^3,\mathbb{R}^3)} \leqslant \|\mathrm{proj}_n(\Phi(z))\|.$$

Therefore, to show the finite radius of convergence theorem, it is sufficient to show that there exists $\lambda^* \in \mathbb{R}$ such that the matrix

$$\sum_{n=0}^{\infty} \lambda^n M(\mathrm{proj}_n(\Phi(z))) \tag{11}$$

at $z = 0$ diverges as λ tends to a finite number λ^*.

We proceed as follows. In Section 2, for $\lambda > 0$ and sufficiently small, we show that the matrix given in (11) acting on $(0, 0, 1) \in \mathbb{R}^3$ is smooth in z and solves a certain PDE. Using rotational invariance of Brownian motion, in Section 3 we rewrite said PDE solution in polar coordinates. In Sections 4 and 5, we obtain an explicit solution for the PDE (still, for λ small enough) in terms of Bessel functions. Finally, in Section 6 we show that the solution blows up as $\lambda \to \tilde{\lambda}$

for some $\tilde{\lambda} < +\infty$, proving our main theorem. The Appendix contains some auxiliary results on PDEs.

2. *Differentiability of the development of expected signature*

We first need two technical lemmas which assert that the development of the expected signature is twice differentiable, and satisfies the PDE we expect it to. In Lemma 1 we will adopt the multi-index notation

$$|(\alpha_1, \alpha_2)| = |\alpha_1| + |\alpha_2|, \qquad D^{(\alpha_1,\alpha_2)}u(z) = \frac{\partial^{\alpha_1+\alpha_2}u}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}}(z), \qquad (\alpha_1, \alpha_2) \in \mathbb{N}^2.$$

The convergence of the following series:

$$(\lambda M)\Phi(z) = \sum_{n=0}^{\infty} \lambda^n M \operatorname{proj}_n(\Phi(z))$$

for sufficiently small λ has been established in [8, Theorem 3.6].

LEMMA 1. *The function $z \mapsto \operatorname{proj}_n(\Phi(z))$ is twice continuously differentiable. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, all $z \in \mathbb{D}$ and all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \leqslant 2$, one has the bound*

$$\|D^\alpha \operatorname{proj}_n(\Phi(z))\| \leqslant C^n.$$

Moreover, there exists $\lambda^* > 0$ such that for all $\lambda < \lambda^*$

$$z \mapsto \sum_{n=0}^{\infty} \lambda^n M \operatorname{proj}_n(\Phi(z))$$

is twice differentiable in z and if $|\alpha| \leqslant 2$, then

$$D^\alpha \sum_{n=0}^{\infty} \lambda^n M \operatorname{proj}_n(\Phi(z)) = \sum_{n=0}^{\infty} \lambda^n D^\alpha M \operatorname{proj}_n(\Phi(z)).$$

Proof. Let $m \in \mathbb{N}$. By Theorem A.5, the function $z \mapsto \operatorname{proj}_n(\Phi(z))$ is twice continuously differentiable (it is in fact infinitely differentiable on $\mathbb{D}$). By Lemma A.4, there exists $C > 0$ such that for all $n \in \mathbb{N}$:

$$\|\operatorname{proj}_n(\Phi)\|_{W^{m,2}(\mathbb{D})} \leqslant C^n, \tag{12}$$

where the norm $\|\cdot\|_{W^{m,2}(\mathbb{D})}$ is the Sobolev norm on the unit disc $\mathbb{D}$ with respect to the variable z,

$$\|u\|_{W^{m,2}(\mathbb{D})} = \max_{|\alpha|=m} \|D^\alpha u\|_{L^2(\mathbb{D})}.$$

Note that as Φ is twice continuously differentiable in a strong sense by [8, Theorem 3.2], the derivative D^α below may be taken in the strong sense.

By [8, Theorem 2.2], which bounds the values of a function u in terms of the Sobolev norm of u, there is some constant $\tilde{C}(2)$ such that for all $z \in \mathbb{D}$ and $|\alpha| \leqslant 2$,

$$\begin{aligned} |D^\alpha \operatorname{proj}_n(\Phi(z))| &\leqslant \tilde{C}(2)\|D^\alpha \operatorname{proj}_n(\Phi)\|_{W^{2,2}(\mathbb{D})} \\ &\leqslant \tilde{C}(2)\|\operatorname{proj}_n(\Phi)\|_{W^{4,2}(\mathbb{D})} \\ &\leqslant \tilde{C}(2)\, C(4)^n. \end{aligned}$$

Since $M\operatorname{proj}_n(\Phi(z))$ is a linear image of $\operatorname{proj}_n(\Phi(z))$, the function $M\operatorname{proj}_n(\Phi(z))$ is twice continuously differentiable in z, and moreover, there exists $c>0$ such that for all $z\in\mathbb{D}$,

$$|D^\alpha M \operatorname{proj}_n(\Phi(z))| \leqslant c^n.$$

This bound also allows us to deduce that for $|\alpha|=2$, the series

$$\sum_{n=0}^{\infty}\lambda^n D^\alpha M \operatorname{proj}_n(\Phi(z))$$

converges uniformly and hence the series

$$\sum_{n=0}^{\infty}\lambda^n M \operatorname{proj}_n(\Phi(z))$$

is twice continuously differentiable and the derivatives can be taken inside the infinite summation. □

Lemma 2. *There exists $\lambda^*>0$ such that if $\lambda<\lambda^*$, the function F_λ defined by*

$$F_\lambda(z) = (\lambda M)\Phi(z)\begin{pmatrix}0\\0\\1\end{pmatrix} = \sum_{n=0}^{\infty}\lambda^n M\operatorname{proj}_n(\Phi(z))\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{13}$$

is twice continuously differentiable on $\mathbb{D}$, and satisfies

$$\Delta F_\lambda(z) = -2\lambda\sum_{i=1}^{2} Me_i\frac{\partial F_\lambda}{\partial z_i}(z) - \lambda^2\left(\sum_{i=1}^{2}(Me_i)^2\right)F_\lambda(z) \tag{14}$$

with $F_\lambda(z)=(0,0,1)$ for $z\in\partial\mathbb{D}$. Here (e_1,e_2) denotes the canonical basis of $\mathbb{R}^2$.

Proof. If we apply the linear map M to the PDE (6), then we have a matrix-valued PDE

$$\Delta(M\operatorname{proj}_n(\Phi(z))) = -2\sum_{i=1}^{2}Me_i\frac{\partial M\operatorname{proj}_{n-1}(\Phi(z))}{\partial z_i} - \left(\sum_{i=1}^{2}(Me_i)^2\right)M\operatorname{proj}_{n-2}(\Phi(z)), \tag{15}$$

together with the boundary condition

$$M\operatorname{proj}_0(\Phi(z)) = I_{3\times 3}$$
$$M\operatorname{proj}_1(\Phi(z)) = 0_{3\times 3}.$$

We may multiply both sides with λ^n, sum to infinity and apply to the vector (0,0,1) to get

$$\sum_{n=0}^{\infty}\lambda^n\Delta(M\operatorname{proj}_n(\Phi(z)))\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$= -2\lambda\sum_{i=1}^{2}Me_i\sum_{n=0}^{\infty}\lambda^n\frac{\partial}{\partial z_i}\operatorname{proj}_n(M\Phi(z))\begin{pmatrix}0\\0\\1\end{pmatrix} - \lambda^2\left(\sum_{i=1}^{2}(Me_i)^2\right)F_\lambda(z).$$

By Lemma 1, each infinite sum converges and we may take the derivatives outside the infinite sum. □

3. *A polar decomposition for the development*

Let $x = (x_1, x_2)^t \in \mathbb{R}^2$. Recall that

$$M(x) = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & 0 \end{pmatrix}.$$

We may consider $M(x)$ as a linear endomorphism of $\mathbb{R}^2 \oplus \mathbb{R}$ mapping (v, α) to $(\alpha x, \langle x, v\rangle)$.

Lemma 3. *For any linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$M(R(x)) = (R \oplus 1)M(x)(R^* \oplus 1),$$

where R^* is the transpose of R.

Proof. Note that

$$\begin{aligned}(R \oplus 1)M(x)(R^* \oplus 1)(v, \alpha) &= (R \oplus 1)M(x)(R^* v, \alpha) = (R \oplus 1)(\alpha x, \langle x, R^* v\rangle) \\ &= (\alpha R(x), \langle x, R^* v\rangle) = (\alpha R(x), \langle Rx, v\rangle) = M(R(x)).\end{aligned}$$ □

In what follows, we will use the notation

$$\triangle_n(0, t) = \{(t_1, \ldots, t_n) : 0 < t_1 < \cdots < t_n < t\}.$$

Corollary 4. *Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation map*

$$z \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} z.$$

Then there exists $\lambda^* > 0$ such that if $0 \leqslant \lambda < \lambda^*$, for all $z \in \mathbb{D}$,

$$(\lambda M)\Phi(R(z)) = (R \oplus 1)(\lambda M)\Phi(z)(R^* \oplus 1).$$

Proof. Brownian motion $B^{R(z)}$ starting at $R(z)$ has the same distribution as the rotated Brownian motion $R(B^z)$, where B^z starts from z. Let $\circ\mathrm{d}$ denote the Stratonovich differential. Then

$$\begin{aligned}(\lambda M)\Phi(R(z)) &= \sum_{n=0}^{\infty} \lambda^n \mathbb{E}^{R(z)}\left[\int_{\triangle_n(0,\tau_{\mathbb{D}})} M\big(\circ\mathrm{d}B^z_{t_1}\big) \cdots M\big(\circ\mathrm{d}B^z_{t_n}\big)\right] \\ &= \sum_{n=0}^{\infty} \lambda^n \mathbb{E}^{z}\left[\int_{\triangle_n(0,\tau_{\mathbb{D}})} M\big(R\big(\circ\mathrm{d}B^z_{t_1}\big)\big) \cdots M\big(R\big(\circ\mathrm{d}B^z_{t_n}\big)\big)\right].\end{aligned}$$

By Lemma 3

$$\begin{aligned}&\int_{\triangle_n(0,\tau_{\mathbb{D}})} M(R(\circ\mathrm{d}B_{t_1})) \cdots M(R(\circ\mathrm{d}B_{t_n})) \\ &\quad = \int_{\triangle_n(0,\tau_{\mathbb{D}})} (R \oplus 1)M(\circ\mathrm{d}B_{t_1})(R^* \oplus 1) \cdots (R \oplus 1)M(\circ\mathrm{d}B_{t_n})(R^* \oplus 1).\end{aligned}$$

As R is orthogonal, we have

$$\begin{aligned}&\int_{\triangle_n(0,\tau_{\mathbb{D}})} M(R(\circ\mathrm{d}B_{t_1})) \cdots M(R(\circ\mathrm{d}B_{t_n})) \\ &\quad = (R \oplus 1)\int_{\triangle_n(0,\tau_{\mathbb{D}})} M(\circ\mathrm{d}B_{t_1}) \cdots M(\circ\mathrm{d}B_{t_n})(R^* \oplus 1).\end{aligned}$$

Therefore,

$$
\begin{aligned}
&(\lambda M)\Phi(R(z)) \\
&= (R\oplus 1)\sum_{n=0}^{\infty}\lambda^n\mathbb{E}^z\int_{\triangle_n(0,\tau_{\mathbb{D}})} M\big(\circ \mathrm{d}B^z_{t_1}\big)\cdots M\big(\circ \mathrm{d}B^z_{t_n}\big)(R^*\oplus 1) \\
&= (R\oplus 1)(\lambda M)\Phi(z)(R^*\oplus 1).
\end{aligned}
$$

□

Corollary 5. *Define the functions* $A_\lambda, B_\lambda, C_\lambda : [0,1]\to\mathbb{R}$ *by*

$$
\begin{pmatrix} A_\lambda(r) \\ B_\lambda(r) \\ C_\lambda(r) \end{pmatrix} = F_\lambda(r,0), \tag{16}
$$

where F_λ *is the function defined by* (13)*. In polar coordinates, the expression of* F_λ *reads*

$$
F(r\cos\theta, r\sin\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_\lambda(r) \\ B_\lambda(r) \\ C_\lambda(r) \end{pmatrix}.
$$

Additionally, there exists $\lambda^* > 0$ *such that if* $\lambda < \lambda^*$*, then* $A_\lambda, B_\lambda, C_\lambda$ *are twice continuously differentiable functions in the variable* r *for all* $r \in [0,1]$*.*

Proof. The functions $A_\lambda, B_\lambda, C_\lambda$ are twice continuously differentiable because $F_\lambda(r,0)$ is twice continuously differentiable by Lemma 2. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of angle θ. For $z = (r\cos\theta, r\sin\theta) = R(r,0)$, the definition of F_λ gives

$$
F_\lambda(z) = (\lambda M)\Phi(z)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (\lambda M)\Phi(R(r,0))\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
$$

By Corollary 4, one has

$$
F_\lambda(z) = (R\oplus 1)(\lambda M)\Phi(R(r,0))(R^*\oplus 1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (R\oplus 1)F_\lambda(r,0),
$$

where

$$
R\oplus 1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

□

4. ODE for $A_\lambda, B_\lambda, C_\lambda$

Lemma 6. *There exists* $\lambda^* > 0$ *such that for* $\lambda < \lambda^*$*, the functions* $A_\lambda, B_\lambda, C_\lambda$ *defined in Lemma* 5 *satisfy*

$$
\begin{aligned}
r^2A''_\lambda(r) + rA'_\lambda(r) - A_\lambda(r) + \lambda^2r^2A_\lambda(r) + 2\lambda r^2C'_\lambda(r) &= 0 \\
r^2B''_\lambda(r) + B'_\lambda(r)r - B_\lambda(r) + r^2\lambda^2B_\lambda(r) &= 0 \\
C'_\lambda(r) + rC''_\lambda(r) + 2\lambda^2rC_\lambda(r) + 2\lambda rA'_\lambda(r) + 2\lambda A_\lambda(r) &= 0
\end{aligned} \tag{17}
$$

and

$$
A_\lambda(0) = 0,\ B_\lambda(0) = 0, A_\lambda(1) = 0
$$

$$
C'_\lambda(0) = 0,\ B_\lambda(1) = 0, C_\lambda(1) = 1.
$$

Proof. By Corollary 5,

$$F_\lambda \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta A_\lambda(r) - \sin\theta B_\lambda(r) \\ \sin\theta A_\lambda(r) + \cos\theta B_\lambda(r) \\ C_\lambda(r) \end{pmatrix}. \tag{18}$$

As $A_\lambda, B_\lambda, C_\lambda$ are twice continuously differentiable for $\lambda < \lambda^*$, we may substitute (18) for F_λ into the equation in Lemma 2, which gives for $r > 0$

$$\Delta \begin{pmatrix} F_\lambda^{(1)} \\ F_\lambda^{(2)} \\ F_\lambda^{(3)} \end{pmatrix} = \begin{pmatrix} \partial_{rr}F_\lambda^{(1)} + \frac{1}{r}\partial_r F_\lambda^{(1)} + \frac{1}{r^2}\partial_{\theta\theta}F_\lambda^{(1)} \\ \partial_{rr}F_\lambda^{(2)} + \frac{1}{r}\partial_r F_\lambda^{(2)} + \frac{1}{r^2}\partial_{\theta\theta}F_\lambda^{(2)} \\ \partial_{rr}F_\lambda^{(3)} + \frac{1}{r}\partial_r F_\lambda^{(3)} + \frac{1}{r^2}\partial_{\theta\theta}F_\lambda^{(3)} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta A_\lambda''(r) - \sin\theta B_\lambda''(r) + \frac{1}{r}(\cos\theta A_\lambda'(r) - \sin\theta B_\lambda'(r)) \\ +\frac{1}{r^2}(-\cos\theta A_\lambda(r) + \sin\theta B_\lambda(r)) \\ \sin\theta A_\lambda''(r) + \cos\theta B_\lambda''(r) + \frac{1}{r}(\sin\theta A_\lambda'(r) + \cos\theta B_\lambda'(r)) \\ +\frac{1}{r^2}(-\sin\theta A_\lambda(r) - \cos\theta B_\lambda(r)) \\ C_\lambda''(r) + \frac{1}{r}C_\lambda'(r) \end{pmatrix}. \tag{19}$$

Using the identities

$$\frac{\partial}{\partial z_1} = \frac{\partial r}{\partial z_1}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z_1}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

$$\frac{\partial}{\partial z_2} = \frac{\partial r}{\partial z_2}\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial z_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta},$$

the right-hand side of (14) is

$$\begin{pmatrix} -\lambda^2\cos\theta A_\lambda(r) + \lambda^2\sin\theta B_\lambda(r) - 2\lambda\cos\theta C_\lambda'(r) \\ -\lambda^2\sin\theta A_\lambda(r) - \lambda^2\cos\theta B_\lambda(r) - 2\lambda\sin\theta C_\lambda'(r) \\ -2\lambda^2 C_\lambda(r) - 2\lambda\big[\cos^2\theta A_\lambda'(r) - \cos\theta\sin\theta B_\lambda'(r)\big] \\ \cdots - 2\lambda\big[\frac{\sin\theta}{r}(\sin\theta A_\lambda(r) + \cos\theta B_\lambda(r))\big] \\ \cdots - 2\lambda\big[\sin^2\theta A_\lambda'(r) + \sin\theta\cos\theta B_\lambda'(r)\big] \\ \cdots - 2\lambda\big[\frac{\cos\theta}{r}\cos\theta A_\lambda(r) - \frac{\cos\theta}{r}\sin\theta B_\lambda(r)\big] \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda^2\cos\theta A_\lambda(r) + \lambda^2\sin\theta B_\lambda(r) - 2\lambda\cos\theta C_\lambda'(r) \\ -\lambda^2\sin\theta A_\lambda(r) - \lambda^2\cos\theta B_\lambda(r) - 2\lambda\sin\theta C_\lambda'(r) \\ -2\lambda^2 C_\lambda(r) - 2\lambda A_\lambda'(r) - \frac{2\lambda}{r}A_\lambda(r) \end{pmatrix}. \tag{20}$$

Equating (19) and (20) gives the first equation as

$$\cos\theta A_\lambda''(r) - \sin\theta B_\lambda''(r) + \frac{1}{r}(\cos\theta A_\lambda'(r) - \sin\theta B_\lambda'(r))$$

$$+\frac{1}{r^2}(-\cos\theta A_\lambda(r) + \sin\theta B_\lambda(r))$$

$$= -\lambda^2\cos\theta A_\lambda(r) + \lambda^2\sin\theta B_\lambda(r) - 2\lambda\cos\theta C_\lambda'(r).$$

As this holds for all θ, we may equate the coefficients of $\sin\theta$ and $\cos\theta$ to obtain

$$A_\lambda''(r) + \frac{A_\lambda'(r)}{r} - \frac{1}{r^2}A_\lambda(r) = -\lambda^2 A_\lambda(r) - 2\lambda C_\lambda'(r)$$

$$-B_\lambda''(r) - \frac{1}{r}B_\lambda'(r) + \frac{1}{r^2}B_\lambda(r) = \lambda^2 B_\lambda(r) \tag{21}$$

and from the second equation,

$$\begin{aligned}\sin\theta A''_\lambda(r)+\cos\theta B''_\lambda(r)+\frac{1}{r}(\sin\theta A'_\lambda(r)+\cos\theta B'_\lambda(r))\\+\frac{1}{r^2}(-\sin\theta A_\lambda(r)-\cos\theta B_\lambda(r))\\=-\lambda^2\sin\theta A_\lambda(r)-\lambda^2\cos\theta B_\lambda(r)-2\lambda\sin\theta C'_\lambda(r)\end{aligned}$$

and therefore,

$$\begin{aligned}A''_\lambda(r)+\frac{1}{r}A'_\lambda(r)-\frac{1}{r^2}A_\lambda(r)&=-\lambda^2A_\lambda(r)-2\lambda C'_\lambda(r)\\B''_\lambda(r)+\frac{B'_\lambda(r)}{r}-\frac{B_\lambda(r)}{r^2}&=-\lambda^2B_\lambda(r).\end{aligned}\tag{22}$$

Combining (21) and (22) and multiplying the equations throughout by r^2, we have

$$\begin{aligned}r^2A''_\lambda(r)+rA'_\lambda(r)-A_\lambda(r)+\lambda^2r^2A_\lambda(r)+2\lambda r^2C'_\lambda(r)&=0\\r^2B''_\lambda(r)+B'_\lambda(r)r-B_\lambda(r)+r^2\lambda^2B_\lambda(r)&=0\\C'_\lambda(r)+rC''_\lambda(r)+2\lambda^2rC_\lambda(r)+2\lambda rA'_\lambda(r)+2\lambda A_\lambda(r)&=0\end{aligned}\tag{23}$$

for all $r>0$. By continuity of the second derivatives of $A_\lambda,B_\lambda,C_\lambda$ (see Lemma 5), the equations hold for all $r\geqslant 0$. Again, using the continuity of the second derivatives of $A_\lambda,B_\lambda,C_\lambda$, we may substitute $r=0$ into (23) to get

$$A_\lambda(0)=B_\lambda(0)=C'_\lambda(0)=0.$$

Using the boundary conditions for $z\in\partial\mathbb{D}$ in Lemma 2, we have

$$\begin{pmatrix}A_\lambda(1)\\B_\lambda(1)\\C_\lambda(1)\end{pmatrix}=F_\lambda(1,0)=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$ □

5. *Solving the ODE for $A_\lambda,B_\lambda,C_\lambda$*

LEMMA 7. *Let*

$$\zeta=\sqrt{\frac{-1+i\sqrt{7}}{2}},\quad \alpha=\frac{1}{2}\zeta^3+\zeta,\quad d(\lambda)=\operatorname{Im}\big(\bar\alpha J_0(\lambda\zeta)J_1(\lambda\bar\zeta)\big),\tag{24}$$

where J_0,J_1 are the Bessel functions of the first kind. Fix $\lambda>0$ such that $d(\lambda)J_1(\lambda)\neq 0$. Then the real-valued functions defined for all $r>0$ by †

$$A_\lambda(r)=\frac{2\sqrt{2}}{d(\lambda)}\operatorname{Im}(J_1(\lambda\bar\zeta)J_1(\lambda\zeta r)),\quad B_\lambda(r)=0,\quad C_\lambda(r)=\frac{1}{d(\lambda)}\operatorname{Im}(\bar\alpha J_1(\lambda\bar\zeta)J_0(\lambda\zeta r))\tag{25}$$

are the unique solution of the differential system (17) satisfying the boundary conditions stated in Lemma 6.

Proof. Recall that, for $\nu=0,1$, Bessel's differential equation

$$x^2y''(x)+xy'(x)+(x^2-\nu^2)y(x)=0,\tag{26}$$

† Note that the two determinations of the square root in the definition of ζ yield the yield the same A_λ, B_λ and C_λ.

has a canonical basis of solutions consisting of the ordinary Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ [**11**, §10.2]. The function J_ν is entire, while Y_ν is analytic on $\mathbb{C}\setminus\mathbb{R}_-$, and $Y_\nu(x)$ diverges to $-\infty$ as $x\to 0$ along the positive reals.

Let $\mathcal{C}_\nu$ denote a cylinder function, that is, a linear combination $aJ_\nu + bY_\nu$ with coefficients a, b that do not depend on ν [**11**, §10.2(ii)]. One has [**11**, (10.6.2), (10.6.3)]

$$\mathcal{C}_0'(x) = -\mathcal{C}_1'(x), \qquad x\,\mathcal{C}_1'(x) + \mathcal{C}_1(x) = x\,\mathcal{C}_0(x). \tag{27}$$

The equation for B_λ is exactly Bessel's equation with $\nu = 1$ and $x = \lambda r$; its general solution for $r > 0$ is hence $B_\lambda(r) = \mathcal{C}_1(\lambda r)$. Since $Y_1(x)$ diverges as $x \to 0$ and λ is nonzero, the initial condition at 0 forces $b = 0$. Similarly, the initial condition at 1 implies $a = 0$ and therefore $B_\lambda = 0$, unless $J_1(\lambda) = 0$, in which case any $B_\lambda(r) = aJ_1(\lambda r)$ is a solution.

Let us turn to the coupled equations for A_λ and C_λ. Make the ansatz

$$C_\lambda(r) = f_0(r), \quad A_\lambda(r) = \alpha f_1(r), \qquad f_\nu(r) = \mathcal{C}_\nu(\lambda\zeta r), \tag{28}$$

where α, ζ are yet unspecified complex numbers. The equation involving C_λ'' becomes

$$r f_0''(r) + f_0'(r) + 2\lambda^2 r f_0(r) + 2\lambda\alpha(r f_1'(r) + f_1(r)) = 0. \tag{29}$$

The change of variable passing from $\mathcal{C}_0$ to f_0 transforms Bessel's equation into

$$r f_0''(r) + f_0'(r) + \lambda^2\zeta^2 r f_0(r) = 0,$$

and the relations (27) yield $r f_1'(r) + f_1(r) = \lambda\zeta r f_0(r)$, so (29) holds when

$$\lambda^2\zeta^2 r = 2\lambda^2 r + 2\lambda\alpha\cdot\lambda\zeta r,$$

that is, when $\zeta^2 = 2(1 + \alpha\zeta)$.

Similarly, the equation involving A_λ'' rewrites as

$$r^2\alpha f_1''(r) + r\alpha f_1'(r) + (\lambda^2 r^2 - 1)\alpha f_1(r) + 2\lambda r^2 f_0'(r) = 0, \tag{30}$$

and the last term on the left-hand side is equal to $2\lambda^2\zeta r^2 f_1(r)$ by (27). Thus, (30) reduces to Bessel's equation provided that $\zeta^2 = 1 - 2\alpha^{-1}\zeta$.

In summary, the functions (28) define a solution of (17) for any choice of a, b in the definition of $\mathcal{C}_\nu$ and α, ζ such that $\zeta^2 = 2(1+\alpha\zeta) = 1 - 2\alpha^{-1}\zeta$. The latter condition is equivalent to

$$\zeta^4 + \zeta^2 + 2 = 0, \qquad \alpha = \zeta^3/2 + \zeta.$$

Letting ζ now denote a fixed root of $\zeta^4 + \zeta^2 + 2$, say the one in (24), the choices

$$C_\lambda(r) = J_0(\lambda\zeta r), J_0(\lambda\bar\zeta r), Y_0(\lambda\zeta r), Y_0(\lambda\bar\zeta r) \tag{31}$$

provide us with four linearly independent[†] solutions, which hence form a basis of the solution space of the system of two linear differential equation of order two.

The asymptotic behaviour of Y_1 at the origin, $Y_1(\lambda\zeta r) \sim -2(\pi\lambda\zeta r)^{-1}$ [**11**, (10.7.4)], shows that linear combinations involving any of the last two solutions (31) are incompatible with the conditions $A_\lambda(0) = C_\lambda'(0) = 0$. Therefore, one has

$$C_\lambda(r) = uJ_0(\lambda\zeta r) + vJ_0(\lambda\bar\zeta r),$$

$$A_\lambda(r) = u\alpha J_1(\lambda\zeta r) + v\bar\alpha J_1(\lambda\bar\zeta r)$$

for some $u, v \in \mathbb{C}$. The conditions $A_\lambda(1) = 0$, $C_\lambda(1) = 1$ translate into a linear system for u, v of determinant

$$\bar\alpha J_0(\lambda\zeta) J_1(\lambda\bar\zeta) - \alpha J_1(\lambda\zeta) J_0(\lambda\bar\zeta) = 2id(\lambda)$$

(where we have used the fact that $J_\nu(\bar z) = \overline{J_\nu(z)}$ [**11**, (10.11.9)]). When $d(\lambda) \neq 0$, the unique solution is $u = -\bar v = \bar\alpha J_1(\lambda\bar\zeta)$. Since $|\alpha|^2 = 2\sqrt{2}$, this leads to the expressions (25). □

[†]This follows, for instance, from the expressions [**11**, (10.2.2), (10.8.1)] and the fact that $(\lambda\zeta)^2 \neq 1$.

6. *Conclusion*

LEMMA 8. *In the notation of Lemma 7, there exists $\tilde{\lambda} > 0$ such that $C(0)$, viewed as a function of λ, has a pole at $\tilde{\lambda}$.*

Proof. Let us first show that $d(\lambda)$ has a zero lying in the interval $(2.5, 3)$. Consider the series expansions [**11**, (10.2.2)]

$$J_0(x) = \sum_{k=0}^{\infty}(-1)^k \frac{(x/2)^{2k}}{k!^2}, \qquad J_1(x) = \sum_{k=0}^{\infty}(-1)^k \frac{(x/2)^{2k+1}}{k!(k+1)!}. \tag{32}$$

For $x \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $|x| < 2(n+1)$, the remainders starting at index n of both series are bounded by

$$\sum_{k=n}^{\infty} \frac{|x/2|^{2k}}{k!^2} = \frac{|x/2|^{2n}}{n!^2} \sum_{k=n}^{\infty} \frac{|x/2|^{2(k-n)}}{((n+1)\cdots(n+k))^2} \leqslant \frac{1}{1 - |x|^2/(2n+2)^2} \frac{|x/2|^{2n}}{n!^2}. \tag{33}$$

In particular, for $x = \lambda\zeta$ or $x = \lambda\bar{\zeta}$ with $0 < \lambda \leqslant 3$, we have $|x/2| < 1.784$. For $n = 5$, the quantity (33) is bounded by 0.025. By replacing J_0 and J_1 by the first five terms of the series (32) in the expression of $d(\lambda)$ and propagating this bound by the triangle inequality, one can check that $d(2.5) < -0.06$. A similar calculation shows that $d(3) > 0.03$. Since $d(\lambda)$ is a continuous function of λ, it follows that $d(\tilde{\lambda})$ vanishes for some $\tilde{\lambda} \in (2.5, 3)$.

We still need to check that the numerator of $C_\lambda(0)$ in (25) does not vanish at $\tilde{\lambda}$. One has $J_0(0) = 1$. Taking $n = 3$ in (33) yields an expression of the form

$$\operatorname{Im}(\bar{\alpha} J_1(\lambda\bar{\zeta})) = \operatorname{Im}(c_0\lambda + c_1\lambda^3 + c_2\lambda^5 + \bar{\alpha}b), \qquad |b| \leqslant 1.12,$$

where one can check that $\operatorname{Im}(c_0) < -0.33$, $\operatorname{Im}(c_1) < -0.12$, $\operatorname{Im}(c_2) < -0.001$. For all $\lambda \geqslant 2.5$, this implies

$$\operatorname{Im}(\bar{\alpha} J_1(\lambda\zeta)) \leqslant \operatorname{Im}(c_0)\lambda + \operatorname{Im}(c_1)\lambda^3 + |\alpha||b| \leqslant -1.3.$$

The claim follows. □

REMARK 9. Instead of doing the calculation sketched in the proof manually, one can easily prove the result using a computer implementation of Bessel functions that provides rigorous error bounds. For example, using the interval arithmetic library Arb [**7**] via SageMath, the check that $d(\lambda)$ has a zero goes as follows. The quantities of the form `[x.xxx +/- eps]` appearing in the output are guaranteed to be rigorous enclosures of the corresponding real quantities. We check the presence of a zero in the interval $[2.82, 2.83]$ instead of $[2.5, 3.0]$ because having a tighter estimate simplifies the second step.

```
sage: zeta = CBF(sqrt((-1+I*sqrt(7))/2))
sage: alpha = zeta^3/2 + zeta
sage: lb, ub = CBF(282/100), CBF(283/100)
sage: (alpha.conjugate()*(lb*zeta).bessel_J(0)
....:                    *(lb*zeta.conjugate()).bessel_J(1))
[-13.208370024264 +/- 4.16e-13] + [-0.003639973760 +/- 4.63e-13]*I
sage: (alpha.conjugate()*(ub*zeta).bessel_J(0)
....:                    *(ub*zeta.conjugate()).bessel_J(1))
[-13.424373315124 +/- 4.75e-13] + [0.005782411521 +/- 4.38e-13]*I
```

One can then verify as follows that the image by the function $\lambda \mapsto \bar{\alpha} J_1(\lambda\bar{\zeta})$ of the interval $[2.82, 2.83]$ only contains elements of negative imaginary part.

```
sage: crit = lb.union(ub); crit # convex hull (real interval)
[2.8 +/- 0.0301]
sage: alpha.conjugate()*(crit*zeta.conjugate()).bessel_J(1)
[+/- 0.0707] + [-4e+0 +/- 0.303]*I
```

THEOREM 10. *The series expansion with respect to λ of $\Phi(0)$ has a finite radius of convergence.*

REMARK 11. Since the condition of [2] of uniqueness of laws is only *sufficient*, the question remains on whether there exists another law on G having the same moments as $S(X^0)_{0,T}$.

Proof. Assume for contradiction that $\Phi(0)$ has an infinite radius of convergence. Then $F_\lambda(0)$ is an entire function in λ. We also know from Corollary 5 that there exists $\lambda^* > 0$ such that for real $\lambda < \lambda^*$

$$F_\lambda(0) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\lambda(0) \\ B_\lambda(0) \\ C_\lambda(0) \end{pmatrix}, \tag{34}$$

where $A_\lambda, B_\lambda, C_\lambda$ are defined by Lemma 7. By the Identity theorem, $A_\lambda, B_\lambda, C_\lambda$ are entire functions and (34) holds for all λ. This contradicts Lemma 8, and therefore $\Phi(0)$ has a finite radius of convergence. □

Appendix

Let Γ be a domain in $\mathbb{R}^d$.

DEFINITION A.1. Let u be a locally integrable function in Γ and α be a multi-index. Then a locally integrable function $r_\alpha u$ such that for every $g \in C_c^\infty(\Gamma)$,

$$\int_\Gamma g(x) r_\alpha(x) dx = (-1)^{|\alpha|} \int_\Gamma D^\alpha g(x) u(x) dx,$$

will be called *weak derivative* of u and r_α is denoted by $D^\alpha u$. By convention, $D^\alpha u = u$ if $|\alpha| = 0$.

DEFINITION A.2. Let $\tilde{d} \in \mathbb{N}$. The *Sobolev space* $W^{k,p}(\Gamma)$ for $p, k \in \mathbb{N}$ is defined to be the set of all $\mathbb{R}^{\tilde{d}}$-valued functions $u = (u^1, \dots, u^{\tilde{d}}) \in L^p(\Gamma)$ such that for every multi-index α with $|\alpha| \leqslant k$, the weak partial derivative $D^\alpha u$ belongs to $L^p(\Gamma)$, that is,

$$W^{k,p}(\Gamma) = \{u \in L^p(\Gamma) : D^\alpha u \in L^p(\Gamma)\ \forall |\alpha| \leqslant k\}.$$

It is endowed with the Sobolev norm defined as follows:

$$||u||_{W^{k,p}(\Gamma)} = \sum_{j=1}^{\tilde{d}} \left(\sum_{|\alpha| \leqslant k} \int_\Gamma |D^\alpha u^j(x)|^p dx \right)^{1/p}.$$

When $k = 0$, this norm coincides with the $L^p(\Gamma)$-norm, that is,

$$||u||_{W^{0,p}(\Gamma)} = ||u||_{L^p(\Gamma)}.$$

THEOREM A.3. *Let M be a second order differential operator with coefficients $\{a^{i,j}\}$. Let u be a weak solution of*

$$Mu = f(x),$$

$$u - g \in H_0^{1,2}(\Gamma).$$

Let Γ be a bounded domain of class C^{k+2} and let the coefficients of M be of class $C^{k+1}(\bar{\Gamma})$. Suppose that the following ellipticity condition holds: there exists $\lambda > 0$ such that for all $x \in \Gamma$ and all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$

$$\sum_{i,j=1}^{d} a^{i,j}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2.$$

Let $f \in W^{k,2}(\Gamma)$, $g \in W^{k+2,2}(\Gamma)$. Then

$$||u||_{W^{k+2,2}(\Gamma)} \leqslant c\big(||f||_{W^{k,2}(\Gamma)} + ||g||_{W^{k+2,2}(\Gamma)}\big),$$

with c depending on λ, d, Γ and on the C^{k+1}-norms for the $a^{i,j}$.

Proof. It is proved using [5, Theorem 8.13] and setting the boundary condition $\varphi = 0$. □

In the following we prove Lemma A.4 for $m \geqslant 2$, which is a generalization of Lemma 3.11 for the case $m = \lfloor \frac{d}{2} \rfloor$ in [8].

LEMMA A.4. *Let Γ be a bounded domain of class C^m in $\mathbb{R}^d$, where $m \geqslant 2$. Then there exists a constant C only depending on Γ and d, such that for every positive integer $n \geqslant 2$,*

$$||\operatorname{proj}_n(\Phi)||_{W^{m,2}(\Gamma)} \leqslant C\big(||\operatorname{proj}_{n-1}(\Phi)||_{W^{m,2}(\Gamma)} + ||\operatorname{proj}_{n-2}(\Phi)||_{W^{m,2}(\Gamma)}\big). \tag{A.1}$$

Proof. The proof of [8, Lemma 3.11] can be applied here directly, except for that we need to check that $\operatorname{proj}_n(\Phi) \in W^{m,2}$, which is proved in Theorem A.5. □

THEOREM A.5. *Suppose that Γ is a nonempty bounded domain in E. It follows Φ is infinitely differentiable in componentwise sense, that is, for all index I, $\operatorname{proj}_n \circ \Phi$ is infinitely differentiable for all n.*

Proof. Based on [8, Theorem 3.2], it shows that

$$\Phi(z) = \int_\Gamma G_\varepsilon(z-y) \otimes \Phi_\Gamma(y)dy = G_\varepsilon * \Phi(z),$$

where $K_\varepsilon(r)$ is a smooth distribution with compact support $[0, \frac{\varepsilon}{2}]$, $*$ is the convolution, and G_ε be a map from $\mathbb{R}^d$ to $T((\mathbb{R}^d))$ defined by

$$\Psi(z) = \Psi(z) = \mathbb{E}^0\Big[S(B_{[0,\tau_{\mathbb{D}(0,|z|)}]})|B_{\tau_{\mathbb{D}(0,|z|)}} = z\Big].$$

$$G_\varepsilon(z) = \Psi(-z)K_\varepsilon(|z|).$$

Since Ψ is smooth (in polynomial form) and K_ε is a smooth function with compact support, G_ε is a smooth function with compact support. It is easy to show that for any partial derivative $D^\alpha G_\varepsilon$ is L_1 integrable.

$$||D^\alpha G_\varepsilon||_{L^1} < +\infty.$$

On the other hand, $\Phi \in L^1$ as well, and so we have

$$||G_\varepsilon * D^\alpha \Phi||_{L^1} < +\infty.$$

Thus $G_\varepsilon * \Phi$ is infinitely differentiable, since $D^\alpha\Phi = (D^\alpha G_\varepsilon) * \Phi \in L^1$. □

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