Birational non-rigidity of codimension 4 Fano 3-folds

by

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Declarations

I declare that, to the best of my knowledge, the material in this thesis is original and my own work, conducted under the supervision of Gavin Brown, unless otherwise indicated.

The material in this thesis has not been submitted for any other degree either at the University of Warwick or any other University.

Abstract

In this thesis we prove the birational non-rigidity of Picard rank 1 Fano 3-folds in codimension 4 having Fano index 1. This is done by explicitly constructing Sarkisov links for these varieties to other Mori fibre spaces.

We also consider those Fano 3-folds in codimension 4 and Fano index 1 having Picard rank 2, and we identify a Mori fibre space in its birational equivalence class. In a final short chapter, we begin this program for Fano 3-folds in codimension 4 having Fano index 2 by demonstrating a construction of them as quotients of index 1 Fano 3-folds.

Introduction

General overview

The construction of sequences of birational maps linking algebraic varieties to one another has been an active research topic since the development of Mori Theory and the Minimal Model Program, aimed at the birational classification of algebraic varieties, and indeed long before. This approach goes under the name of *Sarkisov Program*. In this context, and for certain algebraic varieties having Picard rank equal to 1, the notions of *birational rigidity* and *pliability* come into play. The pliability measures the number of different Mori fibre spaces that are birational to a given variety X. If this number is 1, the variety is said to be *birationally rigid*.

Different aspects of such birational transformations have been studied for several kinds of algebraic varieties. For instance, in the work [CPR00] by Corti, Pukhlikov, and Reid the authors examine the 95 Fano 3-fold weighted hypersurfaces of [Rei80a] and [IF00], proving their birational rigidity.

Our work, in contrast, focuses on proving the birational non-rigidity of certain Fano 3-folds in higher codimension.

The spirit of our approach follows the seminal work [CM04] of Corti and Mella for quartic Fano 3-folds, in which the authors show that quasi-smooth quartic Fano 3-folds having only one singularity of a certain type are not birationally rigid: in fact, their pliability is exactly 2. The result is achieved by studying certain sequences of birational maps called *Sarkisov links*.

In [BZ10], Brown and Zucconi study Sarkisov links for codimension 3 Fano 3-folds in index 1, proving the birational non-rigidity of the latter, provided the presence of a Type I centre. We obtain a similar result in our case. We largely use the techniques and the language developed in [BZ10], especially regarding the variation of GIT on toric varieties. The scenario in codimension 3 and index 1 is completed by Ahmadinezhad and Okada [AO18], where they prove the birational non-rigidity of the five remaining Hilbert series in codimension 3 and index 1 that do not have any Type I centre.

In this thesis we will only focus on codimension 4 Fano 3-folds having at least one Type I centre.

Fano 3-folds of Tom type

In the first part of this thesis we combine the strategies contained in [CPR00] and [CM04] together with the unprojection techniques developed in [Pap04] to tackle the birational geometry of the codimension 4 Fano 3-folds in index 1 having at least one Type I centre that are listed in the Graded Ring Database [BK$^+$15]. In particular we mainly focus on those deformation families arising from Type I unprojections of codimension 3 Fano 3-folds Z, and especially on those in the so-called *Tom format.* We call the outcomes of these unprojections *Fano 3-folds of Tom type.*

These varieties of Tom type constitute about a half of the known deformation families of codimension 4 Fano 3-folds; the other half is of Jerry type (see below).

Our main results proves that these varieties are not birationally rigid, and we give an explicit description of the Sarkisov links starting from them in terms of their ambient space and their basket of singularities. We summarise the results in Table 6.4. Along the way we encounter some interesting phenomena, highlighted explicitly in Chapter 3.

The construction we describe looks like this. In [BK$^+$15] we pick a codimension 4 Fano 3-fold $X \subset \mathbb{P}^7(a,b,c,d_1,d_2,d_3,d_4,r)$ with coordinates $x_1,x_2,x_3,y_1,y_2,y_3,y_4,s$, and we use the data in the Big Table of [BKR12b] to construct X explicitly via unprojection. Together with X we choose a Type I centre $p \in X$: the Kawamata blow-up of this point starts the link. In the notation above, we assume $p = P_s$. We use toric geometry to perform the blow-up, and we prove that

Proposition. In the notation above, the Kawamata blow-up Y_1 of X at the Tom centre $P_s \in X$ is contained in a rank 2 toric variety $\mathbb{F}_1$ having weights

$$\mathbb{F}_1 := \left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right)$$

(see Section 1.2.5 or Appendix of [BCZ04] for this notation).
In fact many varieties would fulfill the role of $\mathbb{F}_1$, but this variety in particular unfolds the birational geometry of X, as we explain below.

The weights of the rank 2 toric variety $\mathbb{F}_1$ describes a ray-chamber structure of its Mori cone. The mobile cone of $\mathbb{F}_1$ describes the behaviour of the Sarkisov link. The mobile cone of $\mathbb{F}_1$ and the mobile cone of Y_1 do not always coincide (by restriction of divisors to Y_1). For instance, the mobile cone of Y_1 can happen to be poorer (in some index 1 cases): when this occurs, the birational transformation associated to one of the rays of the mobile cone of $\mathbb{F}_1$ is an isomorphism when restricted to the variety Y_1. Note that the rank 2 toric variety $\mathbb{F}_1$ is built in such a way that it contains Y_1 and it reflects, at least partially, the birational geometry of Y_1. This is explained in Chapter 2.

The Sarkisov links for codimension 4 index 1 Fano 3-folds of Tom type proceed with a sequence of flops and flips. The endpoints of these sequences can be either divisorial contractions to a point or a line (a smooth rational curve) in another Fano 3-fold X' (of lower codimension), or del Pezzo fibrations, or conic bundles (see Tables 6.1 and 6.2).

For instance, consider X the Tom_1-type Fano 3-fold associated to the Hilbert series #11005, and $p \in X$ the Type I centre of type $\frac{1}{3}(1,1,2)$. Its Sarkisov link centred at p is

$$\begin{tikzcd}
 & Y_1 \arrow[dl, "\Phi"'] \arrow[dr] \arrow[rr, dashed, "\text{16 flops}", "\psi_1"'] & & Y_2 \arrow[r, dashed, "{(5,1,1,-1,-3;2)}", "\psi_2"'] & Y_3 \arrow[r, "\text{isomorphism}", "\psi_3"'] & Y_4 \arrow[dr, "\Phi'"] & \\
X & & Z \arrow[ll, dashed, "\text{unproj}"'] & & & & X'
\end{tikzcd}$$

where ψ_1 is constituted by 16 disjoint flopping $\mathbb{P}^1$, and ψ_2 is a hypersurface flip having weights $(5,1,1,-1,-3;2)$. After that, ψ_3 is a generalised flip of the toric ambient space whose exceptional locus do not intersect Y_3; therefore, the restriction of ψ_3 to Y_3 is an isomorphism. Lastly, Φ' is a divisorial contraction to a point on $X' = X_{4,4} \subset \mathbb{P}^5(1^4,2,3)$ (see Section 1.2.5 for notation, and Section 2.3.1 for link of this type in the proof of the Main Theorem 2.1.1).

The main result is therefore

Theorem. Picard rank 1 Fano 3-folds of Tom type having index 1, codimension 4, and at least one Type I centre are not birationally rigid.

This is Part **(B)** Theorem 2.1.1. The rest of the theorem contains the details of the geometry of the links including their flipping types and extremal contractions.

As a corollary of the above theorem, we construct a family with Hilbert series #5305 and general member having Picard rank 1.

On the Picard rank of Fano 3-folds of Tom type

An important observation is that whenever the Sarkisov link from X of Tom type terminates with another Fano 3-fold X', the latter has always lower codimension than X itself. Hence, since X and X' are birational, they ought to have the same Picard rank.

While, except for some computational results ([BF20]), very little is known regarding the Picard rank of codimension 4 Fano 3-folds, much more can be said for Fano 3-folds in lower codimension. Fano 3-folds in codimension up to 3 have Picard rank 1 whenever they are quasi-smooth. Therefore, if a Sarkisov link's endpoint is quasi-smooth, we can deduce straight away that the Picard rank of X must be 1. If there are no hypersurface flips in a link, and if the divisorial contraction Φ' contracts exactly a weighted

$\mathbb{P}^2$ (and not a surface in a weighted $\mathbb{P}^3$) to a point (or a line) in X', then X' is quasi-smooth. This situation occurs in 18 instances, in which we can therefore state that the corresponding Tom-type Fano 3-folds have Picard rank 1.

As it is clear from Table 6.4, the cases in which X' happens to be quasi-smooth are a very small minority. All the other endpoints X' have some extra (compound) singularities inherited either from the hypersurface flip(s) occurring in the link, or from the divisorial contraction Φ'. The treatment of this situation in which X' is not quasi-smooth is more delicate and it is not part of this thesis. However, we believe that by applying an appropriate Lefschetz-type theorem to X' we should be able to conclude that X', and therefore X, has Picard rank 1 even in the singular case.

Such result would ideally conclude the in-depth study of the geometry of (Tom-type) Fano 3-folds started in [BKR12a].

Fano 3-folds of Tom type and Picard rank 2

One of the hypotheses of the above theorem is that X must have Picard rank 1. This is surely needed to make sense of the notion of birational rigidity. Recall that, associated to each Hilbert series in [BKR12b], the deformation families corresponding to Tom-type formats might be either one or two. If there are two, we refer as *second Tom format* to the second one. All the second Tom formats of the varieties listed in the Table [BKR12b] fall into the description of [BKQ18], that is, they are in $\mathbb{P}^2 \times \mathbb{P}^2$ format. Therefore, their Picard rank is 2. In this case, even though it is not possible to talk about Sarkisov links anymore, a construction shaped on the one of Sarkisov links still leads to interesting conclusions, going beyond quasi-smooth Fano 3-folds.

Firstly, the birational links for these Fano varieties of second-Tom type present two divisorial contractions, simultaneous or consecutive, or a single divisorial contraction followed by a del Pezzo fibration, confirming that the Picard rank of X is 2. Note that they never give rise to conic bundles.

Theorem. Every Fano 3-fold in codimension 4 in second-Tom format presents a birational link terminating with either

- two divisorial contractions (when $d_1 > d_2 > d_3 > d_4$ and when $d_1 > d_2 = d_3 > d_4$);
- a divisorial contraction followed by a del Pezzo fibration (when $d_1 = d_2 > d_3 = d_4$).

In particular, they all terminate with a Mori fibre space.

Moreover, examining in detail the second-Tom format of the Hilbert series #10985 we discovered that its endpoint is a hypersurface $X' = X_5 \subset \mathbb{P}^4(1^4, 2)$. This does not fall into the description of [CPR00] because it has one compound singularity, and is quasi-smooth elsewhere.

Extracting from the other Type I centre of X leads to a birational link ending with another hypersurface $X'' = X_5 \subset \mathbb{P}^4(1^4, 2)$. However, X' and X'' are not isomorphic because they have non-isomorphic compound singularities. On the other hand, they have Picard rank 1, so they are Mori fibre spaces. Therefore,

Proposition. The pliability of $X' = X_5 \subset \mathbb{P}^4(1^4, 2)$ with exactly one compound singularity is $\mathcal{P}(X') \geq 2$.

Fano 3-folds of Jerry type

For the majority of this thesis we discuss Fano 3-folds of Tom type. However, for each Fano 3-fold of Tom type, there is at least one of *Jerry type*, that is, obtained by a Type I unprojection of a divisor $D \subset Z$ where Z is a codimension 3 Fano 3-fold in Jerry format.

Using an approach similar to the one above, it is possible to study these other varieties. Even though it is more articulated than for Tom, and although we do not completely resolve all the links from them, we do partially explain the behaviour of the Fano 3-folds of Jerry type.

The construction of the blow-up of X at p when X is of Jerry type depends on whether the following condition is satisfied or not.

Condition. Let P be the degree of the pivot entry of the Jerry format of Z. Consider the following statement:

One of the coordinates y_j of the weighted projective space $w\mathbb{P}^7$ is such that $\deg(y_j) = P$.

Thus, if X is of Jerry type, we prove the following proposition.

Proposition. Let X be a codimension 4 index 1 Fano 3-fold of Jerry type. If the above condition holds, then, the Kawamata blow-up Y_1 of X at p is contained in a rank 2 toric variety of the form

$$\mathbb{F}_1 = \left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \end{array}\right).$$

On the other hand, if the above condition does not hold, $\mathbb{F}_1$ has the same weights as in the Tom case.

Clearly the aspect of the variety $\mathbb{F}_1$ reflects the different nature of the birational geometry of Jerry-type Fano 3-folds. A case-by-case analysis shows that the phenomena occurring for Jerry-type Fano 3-folds range from a Tom-like behaviour to a much more unpredictable sequence of birational maps.

We included some examples in Chapter 3 and in Section 3.3.

Index 2 Fano 3-folds

While in index 1 the unprojection techniques give a concrete tool to construct Fano 3-folds in codimension 4, this does not happen in higher index. The last part of this thesis partially answers the question of explicitly constructing codimension 4 index 2 Fano 3-folds. The strategy is to combine the usual unprojection in index 1 with a quotient by $\mathbb{Z}/2\mathbb{Z}$ to view X as a double cover of another Fano $\tilde{X}$, where $\tilde{X}$ is a quasi-smooth Fano 3-fold having codimension 4 and index 2.

We achieve the following diagram.

$$\begin{tikzcd}
 & \text{codim } 4 & \text{codim } 3 \\
\text{index } 1 & X \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] & Z \arrow[l, "\text{unproj}"'] \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] \\
\text{index } 2 & \tilde{X} & \tilde{Z}
\end{tikzcd}$$

There are 34 Hilbert series that are candidate to have an index 1 Fano 3-fold with at least one Type I centre as their double cover. Our method applies to all of them, but not to all the deformation families in index 1. Here we consider a specific group action φ of $\mathbb{Z}/2\mathbb{Z}$ (see Chapter 5 for details).

We prove the following lemma.

Lemma. If a codimension 3 Fano 3-fold Z in Tom format is such that there exists a special member invariant under the group action φ, then the nodes on the divisor $D \subset Z$ are not fixed by the action.

This implies that we have a hope to construct a codimension 4 index 2 Fano 3-fold only if the index 1 codimension 3 counterpart has an even number of nodes. This helps discerning which formats could produce a double cover in codimension 4 by narrowing the range of possibilities to the only formats having even number of nodes.

This method explicitly constructs at least one deformation family for 32 different Hilbert series in index 2 in the Graded Ring Database [BK$^+$15].

In the last part of Chapter 5 we exhibit an explicit example of birational link starting from an index 2 Fano 3-fold of codimension 4. The upshot is that they are not birationally rigid. The description in this case is more challenging, and it is an ongoing joint work with Tiago Guerreiro.

Content of the chapters

In Chapter 1 we highlight the first definition necessary for the construction of Fano 3-folds in high codimension, together with the basics of the Sarkisov links.

In Chapter 2 we describe the construction of Sarkisov links for index 1 Fano 3-folds of Tom type in codimension 4 having Picard rank 1. Moreover, we prove a theorem outlining the behaviour of the links.

In Chapter 3 we provide explicit examples to the constructions explained in Chapter 2, showcasing the most relevant phenomena occurring. Section 3.3 is dedicated to comparing some of our results to the one of Takagi (cf. [Tak02]).

In Chapter 4.2 we examine birational links for index 1 Fano 3-folds of second-Tom type in codimension 4 having Picard rank 1. Here we draw conclusions regarding the pliability of a certain quintic Fano hypersurface having one compound singularity.

In Chapter 5 we construct codimension 4 Fano 3-folds of Tom type having index 2 as quotients of certain Fano 3-folds in index 1.

The Appendix 6 includes all the tables summarising the results of this thesis.

Chapter 1

Background theory

1.1 Setting and hypotheses

We work over the field of complex numbers $\mathbb{C}$.

In this chapter we summarise and make more precise the picture highlighted in the Introduction, setting the tone and the language for the rest of the chapters.

In this thesis X is a Fano 3-fold in codimension 4 with at most terminal singularities and Fano index 1, that is

Definition 1.1.1. A *Fano 3-fold* is a $\mathbb{Q}$-factorial normal complex projective 3-dimensional variety with terminal singularities whose anticanonical divisor $-K_X$ is ample.

In the literature it is also called $\mathbb{Q}$-Fano 3-fold.

The hypotheses $\mathbb{Q}$-factorial means that every Weil divisor on X is $\mathbb{Q}$-Cartier.

Definition 1.1.2. Consider a projective variety X and any resolution of singularities $\phi\colon \tilde{X} \to X$; call its exceptional divisors $E_1, \dots, E_n$. The canonical divisor of $\tilde{X}$ is $K_{\tilde{X}} = \phi^* K_X + \sum_{i=1}^{n} a_i E_i$ for some rational coefficients a_i. The variety X is said to have *terminal singularities* if for every $i \in \{1, \dots, n\}$ $a_i \in \mathbb{Q}_{>0}$.

Call *Fano index* the highest natural number q such that $-K_X = qA$ for $A \in \mathrm{Cl}(X)$.

In this setting the *codimension* of X is defined to be its codimension in its anticanonical embedding. More precisely, since X is Fano, we can define its (total) *anticanonical ring* $R(X, -K_X)$ as

$$R(X, -K_X)\colon = \bigoplus_{m \in \mathbb{N}} H^0(X, -mK_X) \ .$$

Any choice of the minimal generating set of the ring $R(X, -K_X)$ determines an embedding of X into a projective space $\mathbb{P}(a_0, \dots, a_n)$, where n and the weights $a_0, \dots, a_n$ are well defined by the ring. The codimension of X refers to this embedding.

A list of the possible candidates of codimension 4 Fano 3-folds satisfying the above definitions is contained in [BK$^+$15]. In order to classify them it is important to know what kind of equations define them.

In [IF00], [Rei80b], and [Alt98] the authors present a classification for codimension up to 3.

In codimension 1 all Fano 3-folds are hypersurfaces, and there are 95 distinguished families. In codimension 2 they are complete intersections, and there are 85 distinguished families. In codimension 3 there are 70 families: 1 family given by complete intersections, 69 given by pfaffians of 5×5 skew-symmetric matrices.

Regarding the codimension 4 case there are 145 numerical candidates listed in the [BK$^+$15], but there is no structure theorem for their equations as the ones above. This is a problem for our purposes, as we will need to know the equations (or at least some of the monomials that appear in them) of such codimension 4 varieties. In [BKR12a] the authors discovered that 115 of those 145 families can be realised as Type I unprojections of a codimension 3 Fano 3-fold. Their full list is contained in [BKR12b]. In particular they occur at least in two different ways, that is, one from a pfaffian variety defined on a matrix in Tom format and another one if the matrix is in Jerry format. These unprojections starting from two different formats lead to two topologically different deformation families. For a more detailed dissertation about Type I unprojections and Tom and Jerry formats refer to [Pap04].

This thesis examines the candidates among the 115 families arising from Type I unprojections. Therefore these Fano 3-folds in codimension 4 have at least one Type I centre as defined in [BZ10]. As explained in the following sections, the Tom and Jerry cases will present different issues and challenges when it comes to run the Sarkisov links, and also different geometric interpretations.

1.2 Strategy and notation

A *Sarkisov link* is a sequence of elementary birational maps, in a very precise sense. Let us recall first the definition of the birational maps that are building blocks for Sarkisov links.

1.2.1 Elementary birational maps

Definition 1.2.1. Consider $\varphi\colon X \to Z$ a birational map of projective varieties.

- φ is a *divisorial contraction* if it contracts a divisor in X.
- φ is a *small contraction* if it does not contract a divisor in X.

In the case where φ is a small contraction suppose that $K_X \cdot C < 0$ for each curve C contracted by φ. It is possible to define a *flip* to be a variety X^+ together with a birational morphism $\varphi^+ \colon X^+ \to W$ (another small contraction) such that

- X^+ is $\mathbb{Q}$-factorial;
- $K_{X^+} \cdot C^+ > 0$;
- K_{X^+} is φ^+-ample;
- $\psi \colon X \setminus C \xrightarrow{\cong} X^+ \setminus C^+$;
- the following diagram commutes

$$K_X \cdot C < 0 \qquad \begin{tikzcd} C \subset X \arrow[rr, dashed, "\psi"] \arrow[dr, "\varphi"'] & & X^+ \supset C^+ \arrow[dl, "\varphi^+"] \\ & Q \in W & \end{tikzcd} \qquad K_{X^+} \cdot C^+ > 0$$

A similar definition is for a *flop*, where both $K_X \cdot C$ and $K_{X^+} \cdot C^+$ are equal to 0 and ψ is an isomorphism in codimension 1.

Both flips and flops are isomorphisms in codimension 1.

Remark 1.2.1. Since a minimal model is defined to have nef canonical divisor, flips play a crucial role in the construction of the model itself, as they turn curves having negative intersection with the canonical divisor of a projective variety X into curves that have positive intersection. This means that K_{X^+} is actually closer to nefness than K_X.

In the 3-fold case it has been proven by Mori in [Mor88] that flips exist; their termination is proven by the work of Kawamata, Kollár, Mori, Reid, Shokurov and others. Thus the following definition is well-posed.

The formal definition of Sarkisov link stems from the one of *2-ray game*, as in [BZ10].

1.2.2 2-ray game

Definition 1.2.2. A *2-ray game* consists in the following sequence of birational transformations.

Consider a 3-fold X, and assume its Picard rank $\rho_X = 1$. Define Y as the blow up of X at a point $p \in X$; call $\phi \colon Y \to X$ the blow-up map. The Picard rank of Y is then $\rho_Y = 2$, so Y admits at most two possible contractions: one of them is ϕ itself. In the case where the second contraction does not exist we say that the 2-ray game "breaks". If the second contraction ϕ_0 exists, with target variety Z, there are the following occurring cases:

- Z is $\mathbb{Q}$-factorial: the 2-ray game stops, ϕ_0 is a divisorial contraction, and we say that it was "successful". We call the resulting sequence *Sarkisov link*;
- Z is not $\mathbb{Q}$-factorial: then ϕ_0 is a small contraction. We flip (or flop) to another Picard rank 2 variety Y_1. Consider Y_1 instead of Y and start again until the 2-ray game stops or breaks;

    the flip from Y does not exist: the 2-ray game breaks;

    Y_1 does not have the second contraction: the 2-ray game breaks.

The aim is to classify Sarkisov links run over $\mathbb{Q}$-Fano 3-folds as defined above. We first start from the easy case of weighted projective spaces. We afterwards move to the Fano cases.

A consequence of such analysis regards the notions of *birational rigidity* and *pliability.*

Definition 1.2.3. Let $X \to S$ be a Mori fibre space. Its *pliability* is the set of all Mori fibre spaces that are birational to X, up to a natural equivalence relation $\sim$ called *square birationality.* In symbols,

$$\mathcal{P}(X) = \{\text{Mfs} Y \to T | X \text{ is birational to } Y\}/\sim \ .$$

Definition 1.2.4. A birational map $f\colon X \dashrightarrow X'$ between two Mori fibre spaces $X \to S$ and $X' \to S'$ is said to be *square birational* if there exists a map $g\colon S \dashrightarrow S'$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

and the induced map on the generic fibres is biregular.

Definition 1.2.5. A Mori fibre space $X \to S$ is said to be *birationally rigid* if its pliability is 1, that is, $\mathcal{P}(X)$ contains only one element (X itself) up to square birationality.

1.2.3 Weighted projective spaces and rank 2 toric varieties

Running a Sarkisov link starting from a weighted projective space $w\mathbb{P}$ is straightforward: it suffices to choose a singularity of $w\mathbb{P}$ to blow up; this makes the Picard rank of $w\mathbb{P}$ increase by one. The blow up is a rank 2 toric variety, i.e. a scroll $\mathbb{F}$, defined by certain weights depending on those of $w\mathbb{P}$. Obviously, $\mathbb{F}$ comes with a certain initial polarisation.

Then, the Sarkisov link is performed by changing the GIT quotient on $\mathbb{F}$, i.e. by changing the irrelevant ideal in the definition of $\mathbb{F}$, that is, by changing its polarisation. Every step therefore consists in birational maps given by either flips or flops. This process will eventually stop after a finite number of steps, depending on how many variables the scroll has, ending with a map that makes the Picard rank of $\mathbb{F}$ drop by one. In particular, this last map is a divisorial contraction to another weighted projective space $w\mathbb{P}'$ or a fibration to a toric variety of lower dimension, as in Theorem 4.1 of [BZ10].

Remark 1.2.2. Note that changing the GIT quotient on $\mathbb{F}$ corresponds to considering alternatively stable and unstable loci in the Mori cone of the scroll.

As explained in [BZ10], it is possible to associate to any scroll a fan in a $\mathbb{Z}^2$-lattice, having a finite number of rays. It is the Mori cone of $\mathbb{F}_1$. Each ray is generated by the linear system corresponding to each bidegree in the scroll. They define maps given explicitly by monomials in those linear systems, that is, each of the maps going from the top row to the bottom row in (1.1) (in Subsection 1.2.5) is associated to a linear system of $\mathbb{F}_1$, and that each flip or flop (i.e. horizontal arrows in (1.1)) is based at one or more points in the Z_i. Changing the irrelevant ideal of $\mathbb{F}_1$, namely changing the GIT quotient on $\mathbb{F}_1$, performs isomorphisms in codimension 1, which could be either flips or flops, on the top row of 1.1. This produces a rank 2 birational link for $\mathbb{F}_1$.

More explicitly, the irrelevant ideal of $\mathbb{F}_1$ is $(t,s)\cap(x_1,x_2,x_3,y_1,y_2,y_3,y_4)$. We define $\mathbb{F}_2$ as the rank 2 toric variety having the same grading as $\mathbb{F}_1$ but having $(t,s,x_1,x_2,x_3)\cap(y_1,y_2,y_3,y_4)$ as irrelevant ideal. The definition of $\mathbb{F}_3$ and $\mathbb{F}_4$, if applicable, depends on which case of Theorem 2.1.1 we look at. For instance, if we consider case **(i)** of Theorem 2.1.1 we have that the irrelevant ideal of $\mathbb{F}_3$ is $(t,s,x_1,x_2,x_3,y_1)\cap(y_2,y_3,y_4)$, and the one of $\mathbb{F}_4$ is $(t,s,x_1,x_2,x_3,y_1,y_2)\cap(y_3,y_4)$. On the other hand, in case **(v)** of Theorem 2.1.1 the irrelevant ideal of $\mathbb{F}_3$ is $(t,s,x_1,x_2,x_3,y_1,y_2)\cap(y_3,y_4)$, while $\mathbb{F}_4$ is not defined: in this situation, the link is shorter. Explicit examples of these phenomena will be given in Chapter 3. This process is outlined explicitly in the examples of Section 3.

Remark 1.2.3. Such a link starting from a weighted projective space always exists and always terminates thanks to the finiteness of the number of rays generating the Mori cone of $\mathbb{F}$.

Each of such rank 2 toric varieties $\mathbb{F}$ is endowed with a 2×9 chart of weights, or bidegrees, representing the action of $\mathbb{C}^\times\times\mathbb{C}^\times$ on $\mathbb{F}$. It is possible to perform row operations on $\mathbb{F}$: this does not change the nature of the action, but only showcases the same action using a different basis of $\mathbb{C}^\times\times\mathbb{C}^\times$.

1.2.4 Fano varieties

The question is now how to translate this information from the weighted projective space case to Fano 3-folds.

Each of these Fano varieties X is embedded in a certain weighted projective space $w\mathbb{P}$. The ambient spaces of Fano varieties in any codimension are listed in the online database [BK$^+$15]. In particular, it is always possible to run explicitly a Sarkisov link for the ambient space of X. In order to see how X behaves along the link we need to find explicit equations for it. As outlined in the Introduction, there is no structure theorem for codimension 4 Fanos. But there is one for codimension 3 Fanos, that is,

Theorem 1.2.4 ([BE74])**.** *If a codimension 3 Fano 3-fold Z is Gorenstein, then it is realised as pfaffians of a 5×5 skew-symmetric matrix M.*

To set the notation, M is a weighted matrix with entries $\{a_{k,l}\}$

$$\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ & a_{2,3} & a_{2,4} & a_{2,5} \\ & & a_{3,4} & a_{3,5} \\ & & & a_{4,5} \end{pmatrix}$$

and weights $\{m_{k,l}\}$.

The unprojection technique described in [Pap04] allows to retrieve equations for codimension 4 varieties using the information in codimension 3. Morally, it consists in contracting to a point a divisor D in a variety and in seeing the said variety in a projective space having one dimension more. The way to force D to sit inside a variety described by pfaffians of a matrix M is to write M in either Tom or Jerry format. Picking a codimension 3 Fano Z_1, whose equations are the 5 pfaffians of M, if we unproject $D \subset Z_1$ we get a Fano in codimension 4 defined by the 5 pfaffian equations and the unprojection equations. These are the equations that we consider throughout the Sarkisov link just described. A more detailed notation is set in Section 1.2.5.

Recall the following definitions as in [BKR12a].

Definition 1.2.6. A 5×5 skew-symmetric matrix M is in Tom_k format if and only if each entry a_{ij} for $i, j \neq k$ is in the ideal I_D.

Definition 1.2.7. A 5×5 skew-symmetric matrix M is in Jerry_{kl} format if and only if each entry a_{ij} is in the ideal I_D whenever either i or j is in $\{k, l\}$.

Observe that M is a *graded matrix*, that is, each of its entries comes with a degree: so, each entry must be occupied by a polynomial in the given degree. A precise list of the grading for M in the case of codimension 3 Fano 3-folds is contained in [BKR12b].

In addition, if we consider either a Tom or Jerry format, the constraints of the formats need to be satisfied.

Remark 1.2.5. By considering Tom or Jerry formats we compromise on the quasi-smoothness of the codimension 3 Fano 3-folds. Putting M in such formats introduces some nodal singularities in the variety, which add up to the cyclic quotient singularities inherited from the ambient space.

Any polynomial in the prescribed degree satisfying the format constraints will do. However, the [BKR12b] gives an even more detailed information, listing also the minimum number of nodes of Z. In particular, these nodes can be concentrated only on the divisor $D \subset Z$: in this way Z is quasi-smooth off D. In order to gather the nodal singularities only on D, we need to choose a suitable member of the deformation family of Z by filling the entries of M with general polynomials of the right degree keeping the format unvaried, and twitching them to achieve the desired number of nodes as in [BKR12b].

More specifically, performing row/column operations on M allows to get rid of some terms in the entries of M.

Note that some variables have the same weight as certain entries of M. It is possible to place such variables in the compliant entries without loss of generality, as more extensively in Chapter 2. This eases the row/column operations, pivoting such modifications on the entries occupied by only one variable.

Suppose w is a coordinate of the ambient space $w\mathbb{P}^6$ of Z, and that its weight is the same as the weight of a certain entry of M. Call R the row of such entry. Then, the row operations we look at replace another row R' with the vector $R' - \xi w^{\zeta} R$, where $\xi \in \mathbb{C}^{\times}$ is a coefficient and ζ is the suitable power of w to cancel out the term in w in one entry of R'. To maintain the skew-symmetry of M we need to do the same for the corresponding columns C and C'.

Observe that not all row/column operations preserve the format. In the above notation, such operations are allowed if w is a generator of the ideal; more generally, if R is multiplied by an element of I_D.

The unprojection of the divisor $D \subset Z$ produces a quasi-smooth codimension 4 Fano 3-fold X defined by nine equations. Five of them are the five maximal pfaffians defining Z. The other four are called *unprojection equations*.

1.2.5 Notation

Let us introduce some notation. The first diagram is the Sarkisov link on the ambient spaces.

$$\begin{tikzcd}[column sep=small]
& \mathbb{F}_1 \arrow[dl, "\Phi"'] \arrow[dr, "\alpha_1"] \arrow[rr, dashed, "\Psi_1"] & & \mathbb{F}_2 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_2"] \arrow[rr, dashed, "\Psi_2"] & & \mathbb{F}_3 \arrow[dl, "\beta_2"'] \arrow[dr, "\alpha_3"] \arrow[rr, dashed, "\Psi_3"] & & \mathbb{F}_4 \arrow[dl, "\beta_3"'] \arrow[dr, "\Phi'"] & \\
w\mathbb{P} & & \mathbb{G}_1 & & \mathbb{G}_2 & & \mathbb{G}_3 & & w\mathbb{P}' \subset \mathbb{G}_4
\end{tikzcd} \tag{1.1}$$

Call

- $\mathbb{E}$ the exceptional locus of Φ;
- $\mathbb{E}'$ the exceptional locus of Φ';
- $\mathbb{A}_i$ the exceptional locus of α_i;
- $\mathbb{B}_i$ the exceptional locus of β_i.

Here $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_3$ are toric varieties. In our situation they are weighted projective spaces. The notation in the varieties setting is

$$\begin{tikzcd}[column sep=small]
& Y_1 \arrow[dl, "\phi"'] \arrow[dr, "\alpha_1"] \arrow[rr, dashed, "\psi_1"] & & Y_2 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_2"] \arrow[rr, dashed, "\psi_2"] & & Y_3 \arrow[dl, "\beta_2"'] \arrow[dr, "\alpha_3"] \arrow[rr, dashed, "\psi_3"] & & Y_4 \arrow[dl, "\beta_3"'] \arrow[dr, "\phi'"] & \\
X & & Z_1 \arrow[ll, dashed, "\text{unproj}"'] & & Z_2 & & Z_3 & & X'
\end{tikzcd} \tag{1.2}$$

The reason why the links have at most this number of steps will be clear in the following sections. We will refer to $\mathbb{F}$ as the scroll above when it is not necessary to specify its polarisation.

Since X is a codimension 4 3-fold, it sits inside a weighted $w\mathbb{P}^7$, while Z_1 sits inside a $w\mathbb{P}^6$. Therefore, the scroll $\mathbb{F}_1$ has 9 variables, called $t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4$ having respective weights

$$\begin{pmatrix} r_1 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ r_2 & r_3 & \alpha & \beta & \gamma & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}$$

Note that s is the unprojection variable.

Say that $Z_1 \subset \mathbb{P}^6(a, b, c, d_1, \ldots, d_4)$ with coordinates x_1, x_2, x_3, $y_1, \ldots, y_4$, and $X \subset \mathbb{P}^7(a, b, c, d_1, \ldots, d_4, r)$ with coordinates respectively x_1, x_2, x_3, $y_1, \ldots, y_4, s$.

Call $y_1, \ldots, y_4$ *relevant variables*, and their weights are $d_1 \geq d_2 \geq d_3 \geq d_4$.

Chapter 2

Tom and Jerry Sarkisov links

In this chapter we describe Sarkisov links for those index 1 Fano 3-folds X in codimension 4 that are obtained by Type I unprojection of codimension 3 $\mathbb{Q}$-Fano 3-folds Z in Tom or Jerry format, as explained later. We use the notation in Section 1.2.5 and of 1.2. In particular, we describe Sarkisov links starting from such varieties X. This is done step by step discussing the construction of Z, the blow-up Y_1 of X at the Type I centre arising from the unprojection, and studying the consequences that each possible variation of GIT quotient on the ambient space of the blow-up has on Y_1.

The results are summarised in Theorem 2.1.1 and Theorem 2.4.5. Some explicit examples are given in Chapter 3.

2.1 The Main Theorem

In this section we state the main theorem of the chapter, Theorem 2.1.1, which describes Sarkisov links starting from Fano 3-folds of Tom type. Its proof is contained in Section 2.3. This is also related to other works in the literature, such as Takagi's [Tak02], and a comparison with that can be found in Section 3.3.

Definition 2.1.1. Let X be a codimension 4 index 1 Fano 3-fold X listed in the table [BKR12b]. We say X is *of Tom Type* if it is obtained as Type I unprojection of the codimension 3 pair $Z \supset D$ in a Tom family (see Chapter 1 for background, Section 1.2 for notation, and Section 2.2.1 for details). The image of $D \subset Z$ in X is called *Tom centre*: it is a cyclic quotient singularity $p \in X$. In the unprojection setup $D \subset Z$, D is a complete intersection of four linear forms of weight $d_1, \dots, d_4$: we refer to $d_1, \dots, d_4$ as the *ideal weights*. Such X of Tom type is said to be *general* if $Z \supset D$ is general in its Tom family.

Theorem 2.1.1. *Let X be a general codimension 4 Fano 3-fold of Tom type and let $p \in X$ be a Tom centre. Suppose in addition that X has Picard rank $\rho_X = 1$. Then:*

(A) *X admits a Sarkisov link to a Mori fibre space $Y \to S$. The link is initiated by the Kawamata blow-up of $p \in X$.*

(B) *The Mori fibre space $Y \to S$ of **(A)** is not isomorphic to X. In particular, X is not birationally rigid.*

(C) *The geometry of each Sarkisov link in **(A)** is as follows. Let $d_1 \geq d_2 \geq d_3 \geq d_4$ be the four ideal weights for the Tom centre $p \in X$. In each case the Kawamata blow-up is followed by an algebraically irreducible flop of finitely many smooth rational curves, and proceeds as follows according to $d_1 \geq d_2 \geq d_3 \geq d_4$:*

- ***(i)*** $\underline{d_1 > d_2 > d_3 > d_4}$: *a flip followed by a second flip, followed by a divisorial contraction Φ' of $(2,0)$-type to another Fano 3-fold X';*
- ***(ii)*** $\underline{d_1 > d_2 = d_3 > d_4}$: *a flip (missed in cases #1218 and #1413) followed by a divisorial contraction Φ' of $(2,1)$-type to another Fano 3-fold X';*
- ***(iii)*** $\underline{d_1 = d_2 > d_3 > d_4}$: *two simultaneous flips, followed by a divisorial contraction Φ' of $(2,0)$-type to another Fano 3-fold X';*
- ***(iv)*** $\underline{d_1 > d_2 > d_3 = d_4}$: *a hypersurface flip, followed by a second hypersurface flip to a del Pezzo fibration: Φ' is of $(3,1)$-type;*
- ***(v)*** $\underline{d_1 = d_2 > d_3 = d_4}$: *two simultaneous flips followed by a del Pezzo fibration: Φ' of $(3,1)$-type;*
- ***(vi)*** $\underline{d_1 > d_2 = d_3 = d_4}$: *a toric flip (missed in case #6865) to a conic bundle: Φ' is of $(3,2)$-type;*
- ***(vii)*** $\underline{d_1 = d_2 = d_3 > d_4}$: *a divisorial contraction Φ' of $(2,1)$-type to another Fano 3-fold X';*
- ***(viii)*** $\underline{d_1 = d_2 = d_3 = d_4}$: *a conic bundle over a quadric surface in $\mathbb{P}^3$: Φ' is of $(3,2)$-type.*

The notation on fibrations and divisorial contractions in the above theorem is: (m, n) where m is the dimension of the exceptional locus of Φ' in Y_4 (where applicable) and n is the dimension of its image.

Remark 2.1.2. The flip in case **(v)** is of course algebraically irreducible (that is, its base is irreducible as an algebraic set), but it consists of two disjoint tubular neighbourhoods in the connected component of its exceptional locus. Such neighbourhoods are either both toric or both hypersurface. This means that the intersection between Y_2 and the contracted locus of the flip is not irreducible, and it is formed of two distinct connected components. In this situation, we say that we have two *simultaneous flips* of the variety Y_2.

In contrast, in case **(iv)** the link consists of two algebraically irreducible flips, one after the other.

Remark 2.1.3. In **(vii)** the exceptional divisor of Φ' is contracted to an irreducible (conic) curve $\Gamma \subset \mathbb{P}^2$.

Remark 2.1.4.

- This analysis does not involve the hypotheses $\rho_X = 1$ at all, although it is needed to state that the birational links constructed are Sarkisov links. See Chapter 4 for birational links that are not Sarkisov links ($\rho_X > 1$).
- In a few cases it is hard to determine who X' is. In these occasions, we need to suppose $\rho_X = 1$ to affirm that X' is a Fano 3-fold.
- We do not know which ones of these Fano 3-folds have Picard rank 1, but we do have some examples, provided by Takagi [Tak02] (see Section 3.3) and by [BKQ18] (computational). There is a belief that the first Tom format has Picard rank 1 (except for #12960). In addition, Chapter 4 gives a circumstantial evidence of this belief.
- In order to determine the Picard rank of X, it is crucial to observe that the endpoint X' of a link run from X, if it is another Fano 3-fold X', it always has codimension strictly lower than the codimension of X. Therefore, since $\rho_X = \rho_{X'}$, we can deduce the Picard rank of X from the study of the Picard rank of X'. The latter is surely 1 if X' is quasi-smooth (this happens in 18 cases); however, this should hold even when X' has singularities provided a suitable Lefschetz-type theorem, although this situation is not studied in this thesis. This would prove that at least one deformation family of X has $\rho_X = 1$ if a Sarkisov link from that deformation family terminates with another Fano 3-fold.

Remark 2.1.5. This theorem does not consider the Fano 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format listed in [BKQ18], as they have Picard rank 2. The Hilbert series #12960 is one of them, thus is not covered by the description in **(viii)** of Theorem 2.1.1. In particular, the ones having the "second Tom" will be examined in Chapter 4.

Theorem 2.1.1 could constitute a tool to prove that the Picard rank of some of these codimension 4 Fano 3-folds is 1.

Corollary 2.1.6. *The Fano 3-fold of Tom type X associated to the Hilbert series #5305 has Picard rank 1.*

Proof. Consider the Fano #5305 X of Tom_1 type and consider its Type I centre $p \sim \frac{1}{5}(1, 2, 3)$ in X. The Sarkisov links run on X from the centre p terminate with a divisorial contraction $\Phi' \colon Y_3 \to X'$, where X' is the Fano #5962 of codimension 3. In particular,

Φ' contracts the singular locus $\mathbb{E}'$ to a quasi-smooth point $p' \in X'$. Since Y_3 is also quasi-smooth (no hypersurface flips occur in this link), then X' is quasi-smooth as well. Therefore, by Theorem 3 of [BF20], X' has Picard rank 1. Therefore, this implies that X has Picard rank 1. $\square$

2.2 Construction of the birational links

Here and in the following subsections we explain how we construct the birational links described in Theorem 2.1.1.

2.2.1 The unprojection setup: construction of the pfaffian matrix M

The starting point is the following type of data, coming from [BKR12a] and [BK$^+$15].

- A fixed projective plane $D := \mathbb{P}^2(a,b,c) \subset \mathbb{P}^6(a,b,c,d_1,\dots,d_4)$ with coordinates $x_1, x_2, x_3,\ y_1, \dots, y_4$ respectively and $d_1 \geq d_2 \geq d_3 \geq d_4$. So D is defined by the ideal $I_D := \langle y_1, y_2, y_3, y_4 \rangle$.
- A family $\mathcal{Z}_1$ of codimension 3 Fano 3-folds $Z \subset w\mathbb{P}^6$, each defined by maximal pfaffians of a skew-symmetric 5×5 syzygy matrix M whose entries have weights

$$\begin{pmatrix} m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ & m_{2,3} & m_{2,4} & m_{2,5} \\ & & m_{3,4} & m_{3,5} \\ & & & m_{4,5} \end{pmatrix}.$$

The plane is a divisor $D \cong \mathbb{P}^2_{x_1,x_2,x_3}(a,b,c)$ of $Z_1 \in \mathcal{Z}_1$ if the latter is written as pfaffians of a matrix M in either Tom or Jerry format. This subsection constructs M in this general setting by filling its entries by homogeneous polynomials in the x_i and y_j subject to the Tom and Jerry constraints (see Chapter 1).

The Big Table in [BKR12b] records exactly this data of $D \subset w\mathbb{P}^6$ and the weights of the syzygy matrix, together with the possible successful Tom and Jerry formats.

It is often possible to place each variable in a matrix position having the same degree, as long as all the Tom and Jerry format restrictions on M are satisfied. Since M has 10 entries and $\mathbb{P}^6$ only 7 coordinates, at least 3 entries have to be occupied by more general homogeneous general polynomials in the given degree.

Lemma 2.2.1. *Let $Z_1 \supset D$ be a general member of a Tom$_i$ family appearing in [BKR12b] where $i \in \{1,\dots,5\}$. Then we have the following.*

(i) *For each ideal generator y_j there is an entry $a_{k,l}$ of M with $k \neq i$, $l \neq i$ such that $d_j = m_{k,l}$, that is, in which y_j appears linearly.*

(ii) With the exception of the [BKR12b] entry #12960, there is an entry $a_{k,l}$ of M with $k = i$ or $l = i$ such that $m_{k,l}$ is equal to a, b, or c, that is, it is linear in at least one of the orbinates x_j.

Proof. This is the following observation about the weights m_{kl} of the syzygy matrix M of [BKR12b] and those of $w\mathbb{P}^6$. In each case (except for #12960), for any d_j there is an entry in the ideal part of M having weight d_j. Similarly it holds for the x_j. The fact that y_j and x_j appear linearly in such (suitable) entries is implied by the hypotheses of generality of Z_1. □

Later we analyse the entries of general M, and this Lemma guarantees certain monomials appearing in the pfaffian equations.

Remark 2.2.2. The only Hilbert series in [BK+15] that does not satisfy this condition is #12960, whose complementary variables x_j have weights $1, 1, 1$ respectively, while the weights of M are all 2. The successful Tom format in that Hilbert series results in X of Picard rank 2: it is in $\mathbb{P}^2 \times \mathbb{P}^2$-format, as listed in Table 1 of [BKQ18] although, as we see later, this is not related.

This phenomenon is probably not due to the fact that the Tom format #12960 has Picard rank 2: indeed, although the second Tom of #24078 is listed in [BKQ18] among the Fano 3-folds in codimension 4 having $\mathbb{P}^2 \times \mathbb{P}^2$-format, the complementary variable of weight 2 can be placed linearly in one of the complementary entries of M, which all have weight 2.

Following the notation in Section 1.2, the unprojection techniques described in [Pap01] give a birational map $Z_1 \to X \subset w\mathbb{P}^7$ that contracts D to a quotient singularity $P_s \in X$. It is this X whose Sarkisov links we study.

2.2.2 The Kawamata blow-up of a Fano: ambient space $\mathbb{F}_1$

We aim to make a Sarkisov link centered in P_s. Since $P_s \in X$ is a quotient singularity by construction, the first map of the Sarkisov link is a Kawamata blow-up of X at a cyclic quotient singularity $\frac{1}{r}(a, b, c)$ at P_s. Following a similar method as in [AZ17], we deduce the weights of a rank two toric variety $\mathbb{F}_1$ that is a blow-up of $w\mathbb{P}^7$ at P_s, which results in the Kawamata blow-up on X.

We consider $w\mathbb{P}^7$ as a toric variety with 1-skeleton given by primitive lattice vectors $\rho_s, \rho_{x_i}, \rho_{y_j}$, i.e. a weight lattice $N_{\mathbb{P}^7} \cong \mathbb{Z}^4$. These vectors satisfy the following relation

$$r\rho_s + a\rho_{x_1} + b\rho_{x_2} + c\rho_{x_3} + \sum_{j=1}^{4} d_j \rho_{y_j} = 0 \, .$$

To perform a blow-up of $w\mathbb{P}^7$ at P_s we add a new ray ρ_t to the fan inside the convex cone $\sigma_s := \langle \rho_{x_1}, \rho_{x_2}, \rho_{x_3}, \rho_{y_1}, \rho_{y_2}, \rho_{y_3}, \rho_{y_4} \rangle$; that is, an integer multiple of $\omega\rho_t$ of ρ_t is the integer positive sum of all rays other than ρ_s: there are many possible choices to choose the coefficients for this positive sum, and we will identify a particular one. The relation involving ρ_t is

$$- \omega\rho_t + \sum_{i=1}^{4} \omega_i \rho_{x_i} + \sum_{j=1}^{4} \delta_j \rho_{y_j} = 0 \ , \tag{2.1}$$

where ω, ω_i, $\delta_j > 0$ for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$.

In the language of the graded Cox rings, the bottom weights of the scroll $\mathbb{F}_1$ are the coefficient for the rays in the definition of ρ_t. Since ρ_s does not appear in the expression for ρ_t, its bottom weight is 0. Thus $\mathbb{F}_1$ looks like

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ -\omega & 0 & \omega_1 & \omega_2 & \omega_3 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{array} \right) . \tag{2.2}$$

Note that this is not yet well-formed: we connect to this later.

Recall the following theorem by Kawamata, [Kaw96]:

Theorem 2.2.3 (Kawamata)**.** *Let X be a 3-fold, and $p \in X$ a terminal cyclic quotient singularity $\frac{1}{r}(a,b,c)$. Suppose that $\phi\colon (E \subset Y) \to (\Gamma \subset X)$ is a divisorial contraction with $p \in \Gamma$ and Y terminal. Then, $\Gamma = \{p\}$ and ϕ is the weighted blow-up of p with weights (a,b,c) and therefore the exceptional divisor is $E \cong \mathbb{P}(a,b,c)$.*

Note that the Kawamata blow-up of a cyclic quotient singularity is unique, even though the bottom weights δ_j could be, in principle, chosen arbitrarily. In the following we give a recipe about how to choose the δ_j so that the 2-ray game described by $\mathbb{F}_1$ is a successful link for X.

The blow-up map is defined by the linear system $\left|\mathcal{O}\binom{1}{0}\right|$. Explicitly,

$$\begin{aligned} \Phi \,:\, \mathbb{F}_1 &\longrightarrow w\mathbb{P}^7 \\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto \left(t^{\frac{\omega_1}{\omega}} x_1, t^{\frac{\omega_2}{\omega}} x_2, t^{\frac{\omega_3}{\omega}} x_3, t^{\frac{\delta_1}{\omega}} y_1, t^{\frac{\delta_2}{\omega}} y_2, t^{\frac{\delta_3}{\omega}} y_3, t^{\frac{\delta_4}{\omega}} y_4, s \right) \ . \end{aligned}$$

On a local neighbourhood of P_s there is a weighted projective space $\mathbb{P}^6(\omega_1,\omega_2,\omega_3,\delta_1,\delta_2,\delta_3,\delta_4)$ contracted to the point P_s of index ω. Since P_s has index r, then ω must be equal to r.

On the other hand, we could assign many different values to $\omega_1, \omega_2, \omega_3$. However, we are interested in exhibiting a Kawamata blow-up, which is described in Theorem 2.2.3. From [BZ10] we know that the exceptional locus E of Φ is given by the vanishing of y_1, y_2, y_3, y_4. This means that $E \cong \mathbb{P}^2(\omega_1, \omega_2, \omega_3)$. Therefore, in order to achieve a Kawamata blow-up we choose the weights $\omega_1, \omega_2, \omega_3$ to be a, b, c respectively.

When restricted to its exceptional locus E, the map Φ becomes

$$\begin{aligned} \Phi :\ E &\longrightarrow \Gamma \\ (t, x_1, x_2, x_3) &\longmapsto \left(t^{\frac{a}{\omega}} x_1, t^{\frac{b}{\omega}} x_2, t^{\frac{c}{\omega}} x_3\right) \ . \end{aligned}$$

This achieves the construction of the Kawamata blow-up for X. The last thing that needs to be set is the value of the δ_j.

The equations of X come into play to determine the δ_j's. When pulling back the equations of X via Φ, each monomial will pick up extra t factors. Again, the choice of the δ_j's could be free, but we would like to cancel out the highest possible power of t: in other words, to get the equations of Y_1 we must saturate over t the total pullback of the equations of X. This is because we want the leading terms of the unprojection equations to be sy_j, as opposed to $sy_j t^\tau$, for τ a certain exponent greater than 1.

Localising at P_s allows to study each ideal variable via the unprojection equations.

To fix ideas, suppose we want to find δ_4, corresponding to y_4. We start with y_4 because it is the one with lowest weight $d_4 \leq d_3 \leq d_2 \leq d_1$. The unprojection equation of X involving y_4 is of the form $sy_4 = g_4(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$, where g_4 is a homogeneous polynomial of degree $r + d_4$. The pullback of the unprojection equation for y_4 is of the form

$$t^{\frac{\delta_4}{r}} sy_4 = g_4\left(t^{\frac{a}{r}} x_1, t^{\frac{b}{r}} x_2, t^{\frac{c}{r}} x_3, t^{\frac{\delta_1}{r}} y_1, t^{\frac{\delta_2}{r}} y_2, t^{\frac{\delta_3}{r}} y_3, t^{\frac{\delta_4}{r}} y_4\right) \ .$$

Separate from g_4 all its terms containing the variable y_4. Define h_4 the polynomial constituted by all the monomials of g_4 containing y_4, except for the term sy_4. The equality above becomes

$$t^{\frac{\delta_4}{r}}\left(sy_4 + t^{\frac{\kappa}{r}} h_4\right) = g_4'\left(t^{\frac{a}{r}} x_1, t^{\frac{b}{r}} x_2, t^{\frac{c}{r}} x_3, t^{\frac{\delta_1}{r}} y_1, t^{\frac{\delta_2}{r}} y_2, t^{\frac{\delta_3}{r}} y_3\right) \ , \tag{2.3}$$

where $g_4' := g_4 - h_4$ and κ is the minimum exponent that is possible to factorise from h_4.

Lemma 2.2.4. *It holds that $\delta_4 \geq d_4$.*

Proof. Every monomial in g_4 picks up a t factor because there is no pure monomial in s on the right hand side of the unprojection equation for y_4: this is by construction.

From the construction of ρ_t we know that each δ_j is greater than or equal to 1. We divide this proof in different cases according to the different types of monomials appearing in g_4. We indicate by $\underline{x}^l$ the multiplication of pure powers of x_1, x_2 and x_3, not necessarily all together, with different multiplicities, summarised by the multi-index l at the exponent. Similarly, we define $\underline{y}^{l'}$ as the multiplication of pure powers of y_1, y_2 and

y_3, not necessarily all together, with different multiplicities indicated by the multi-index l'. In the following description l and l' will vary from case to case.

- Monomials of the form $\underline{x}^l$, where $l = \deg(g_4) = r + d_4$. Since the top weights of x_1, x_2 and x_3 are the same as their bottom weights in the scroll $\mathbb{F}_1$ 2.2, then such monomials pick up a t factor with exponent $k = l = r + d_4$ in the pullback.

- Monomials of the form $\underline{x}^l \underline{y}^{l'}$, where $l + l' = \deg(g_4) = r + d_4$. Since $\delta_1, \delta_2, \delta_3 \geq 1$, the pullback of $\underline{x}^l \underline{y}^{l'}$ picks up a t factor with exponent k a least $l + l'$. So, $k \geq l + l' = r + d_4$.

- Monomials of the form $\underline{x}^l y_4^\lambda$, where $l + \lambda = \deg(g_4) = r + d_4$. They pick up a t factor with power $k \geq l + \lambda\delta_4 \geq r + d_4$.

- Monomials of the form $\underline{y}^{l'} y_4^\lambda$, where $l' + \lambda = \deg(g_4) = r + d_4$. They pick up a t factor with power $k \geq l' + \lambda\delta_4 \geq r + d_4$.

- Monomials of the form $\underline{x}^l \underline{y}^{l'} y_4^\lambda$, where $l + l' + \lambda = \deg(g_4) = r + d_4$. They pick up a t factor with power $k \geq l + l' + \lambda\delta_4 \geq r + d_4$.

Therefore the exponent for t relative to this kind of monomials is $r + d_4$, or higher. We choose δ_4 to be one of these values of k.

In conclusion, since every monomial in g_4 picks up a t factor with exponent at least $\frac{r+d_4}{r}$, we deduce that $\delta_4 \geq d_4$. □

The power of t gained by each y_j factor is greater or equal to $\frac{d_j}{\omega}$. This means that $\delta_j \geq d_j$. So the pullback of the unprojection equation for y_1 is of the form

$$t^{\frac{\delta_4}{r}} \left(s y_4 + t^{\frac{\kappa}{r}} h_4 \right) = t^{\frac{\tau_1}{r}} m_1 + \cdots + t^{\frac{\tau_{k_4}}{r}} m_{k_4} \ ,$$

for τ_l positive integers and m_l monomials of g_4'. So,

Definition 2.2.1. Define δ_4 as

$$\delta_4 := \min_{l \in \{1, \dots, k_4\}} \{\tau_l\} \ . \tag{2.4}$$

Note that since g_4' does not contain y_4, δ_4 is well-defined.

Remark 2.2.5. The scroll just obtained might not be well-formed. For a definition of well-formedness see Definition 3.1 in [Ahm17], which generalises to scrolls the notion of well-formedness for weighted projective spaces in [IF00].

The bottom weights of a well-formed scroll can be interpreted as the order of vanishing of the variables in the divisor D of the unprojection.

Definition 2.2.1 makes the distinction between the Tom and Jerry case come into play. The difference lies on detecting which one is the monomial in g that achieves the minimum of 2.2.1. The analysis in the Jerry case is contained in Section 2.4.

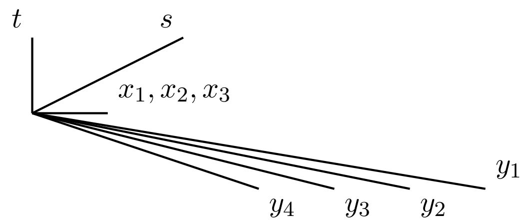
Proposition 2.2.6. *Let X be a codimension 4 index 1 Fano 3-fold of Fano type.*

Then the Kawamata blow-up of X at the Tom centre P_s is contained in a rank 2 toric variety of the form

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right). \tag{2.5}$$

Note that the scroll in Proposition 2.2.6 is well formed. Moreover, the blow-up map Φ is a morphism from $Y_1 \subset \mathbb{F}_1$ to $X \subset \mathbb{P}^7$.

Using the same notation introduced at the beginning of this subsection, we can view $\mathbb{F}_1$ as a toric variety whose 1-skeleton is spanned by the lattice vectors ρ_t, ρ_s, $\rho_{x_1}, \rho_{x_2}, \rho_{x_3}, \rho_{y_1}, \rho_{y_2}, \rho_{y_3}, \rho_{y_4}$. Each vector is defined by its bidegree in 2.5. The fan they generate looks like



(2.6)

where the y_j might generate the same rays depending on the value of the d_j. Underlying the above picture there is a ray-chamber structure that describes the 2-ray game for $w\mathbb{P}^7$. Each ray gives raise to a map of toric varieties. Suppose the bidegree of the chosen ray is $\binom{\iota_1}{\iota_2}$: its relative map is defined by the monomials having bidegree $\binom{\iota_1}{\iota_2}$ (or a natural multiple of it). In other words, these are the monomials in the linear system $\left|\mathcal{O}\binom{\iota_1}{\iota_2}\right|$. These are the maps α_i, β_i, Φ, Φ' introduced in Section 1.2.5. For instance, the map relative to ρ_s is defined by the monomials in $\left|\mathcal{O}\binom{r}{1}\right|$, and it is the blow-up map $\Phi\colon \mathbb{F}_1 \to w\mathbb{P}^7$. On the other hand, the map relative to ρ_{x_1}, ρ_{x_2}, ρ_{x_3} defined by the monomials in $\left|\mathcal{O}\binom{1}{0}\right|$ is $\alpha_1\colon \mathbb{F}_1 \to \mathbb{G}_1$. In conclusion, each ray corresponds to one of the toric varieties in the bottom row of the 2-ray game in 1.1, while each chamber corresponds to one of the $\mathbb{F}_i$ at the top row of 1.1. Passing from one chamber to another adjacent chamber means to perform the relative isomorphism in codimension 1 $\Psi_i\colon \mathbb{F}_i \to \mathbb{F}_{i+1}$, while approaching to the ray in between the two chambers from one side or another indicates the two maps $\alpha_i\colon \mathbb{F}_i \to \mathbb{G}_i$ and $\beta_i\colon \mathbb{F}_{i+1} \to \mathbb{G}_i$.

In the language of Geometric Invariant Theory, we are performing on $\mathbb{F}_1$ a variation of GIT to obtain the 2-ray game. This description of $\mathbb{F}_1$ will be useful in the examination of the explicit examples.

To prove Proposition 2.2.6 we need the following lemma.

Lemma 2.2.7. *Let Z be a codimension 3 $\mathbb{Q}$-Fano 3-fold defined by pfaffians of a 5×5 skew-symmetric matrix M in Tom format. Consider the Type I unprojection of Z at a divisor D. Then each unprojection equation contains at least one monomial purely in x_1, x_2, x_3.*

To prove Lemma 2.2.7 we partially refer to the notation in [Pap04]. We make use of the author's algorithm to compute unprojection equations, which we briefly summarise in the next paragraph.

Papadakis' algorithm for unprojection In [Pap04], Papadakis defines and explicitly constructs the Type I unprojection equations for Z in Tom format.

Suppose for simplicity that the matrix M is in format Tom_1. For $D \cong \mathbb{P}_{x_1,x_2,x_3}(a,b,c)$ the divisor in Z, and $I_D := \langle y_1, y_2, y_3, y_4 \rangle$, the graded matrix M is of the form

$$M = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ & a_{23} & a_{24} & a_{25} \\ & & a_{34} & a_{35} \\ & & & a_{45} \end{pmatrix} . \tag{2.7}$$

Here the a_{ij} are polynomials of the form

$$a_{ij} := \sum_{k=1}^{4} \alpha_{ij}^k y_k$$

for some polynomial coefficients α_{ij}^k. The a_{ij} are contained in the ideal I_D.

On the other hand, the p_j have to be polynomials not in I_D so that the Tom_1 constraints are satisfied.

For what concerns this specific paragraph, we follow Papadakis' notation, in which Pf_i is calculated by excluding the $(i+1)$-th row and the $(i+1)$-th column for $i \in \{0,1,2,3,4\}$.

Note that only $\mathrm{Pf}_1, \dots, \mathrm{Pf}_4$ are linear in the y_i; hence, there exists a unique matrix Q such that

$$\begin{pmatrix} \mathrm{Pf}_1(M) \\ \vdots \\ \mathrm{Pf}_4(M) \end{pmatrix} = Q \begin{pmatrix} y_1 \\ \vdots \\ y_4 \end{pmatrix} .$$

Explicitly,

$$Q = \begin{pmatrix} \mathrm{Pf}_1(N_1) & \mathrm{Pf}_1(N_2) & \mathrm{Pf}_1(N_3) & \mathrm{Pf}_1(N_4) \\ \mathrm{Pf}_2(N_1) & \ddots & & \vdots \\ \mathrm{Pf}_3(N_1) & & \ddots & \vdots \\ \mathrm{Pf}_4(N_1) & \cdots & \cdots & \mathrm{Pf}_4(N_4) \end{pmatrix}$$

where

$$N_i = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ & \alpha^i_{23} & \alpha^i_{24} & \alpha^i_{25} \\ & & \alpha^i_{34} & \alpha^i_{35} \\ & & & \alpha^i_{45} \end{pmatrix}. \tag{2.8}$$

and α^i_{kl} is the coefficient of y_i in a_{kl}.

Define H_i as the vector of length 4 whose i-th entry is $(-1)^{i+1}$ times the determinant of the submatrix of Q obtained by removing the i-th column and the i-th row. The vectors H_i satisfy the property that for all $i, j \in \{1, \dots, 4\}$

$$p_i H_j = p_j H_i \tag{2.9}$$

(Lemma 5.3 of [Pap04]). Therefore, the quotient $\frac{H_i}{p_i}$ is independent of i.

Papadakis defines the polynomials $g_1, \dots, g_4$ via the following equality of vectors of length 4

$$(g_1, g_2, g_3, g_4) = \frac{H_i}{p_i}.$$

For instance, g_1 is explicitly defined as the determinant of the matrix obtained deleting the first column and the first row of Q divided by p_1, i.e.

$$g_1 = \frac{1}{p_1} \det \begin{pmatrix} \mathrm{Pf}_2(N_2) & \mathrm{Pf}_2(N_3) & \mathrm{Pf}_2(N_4) \\ \mathrm{Pf}_3(N_2) & \mathrm{Pf}_3(N_3) & \mathrm{Pf}_3(N_4) \\ \mathrm{Pf}_4(N_2) & \mathrm{Pf}_4(N_3) & \mathrm{Pf}_4(N_4) \end{pmatrix}. \tag{2.10}$$

The g_j are the right hand sides of the unprojection equations, that is, the unprojection equations defining X are $sy_j = g_j$ for $j = 1, \dots, 4$.

This concludes our brief summary of Papadakis' algorithm to produce the unprojection equations of X. We use his techniques to deduce the statement of Lemmma 2.2.7 in our specific case.

Proof of Lemma 2.2.7. Recall that Z has index $i_Z = 1$, so the coordinate x_1 has weight 1. Hence, using the above notation, in every case p_j contains a monomial of the form $x_1^{\deg(p_j)}$.

On the other hand, there are different possibilities to fill the ideal entries a_{kl}. If the weight of an ideal entry is the same as the weight of one of the y_j, then it contains such ideal variable linearly, i.e α^j_{kl} is constant. Otherwise, it contains multiplications of y_j by the x_i, that is α^j_{kl} is a polynomial containing a term in the x_i. We can assume this without loss of generality.

Therefore, each N_j has at least one entry that is either a constant or a monomial in the x_i.

Recall that the vector of the g_j is independent on the choice of i in 2.9. This means that it is possible to consider only $\frac{H_1}{p_1}$. Therefore, we are excluding all $\mathrm{Pf}_1(N_j)$, that is, all $\mathrm{Pf}_i(N_j)$ involving the top row of the matrices N_j, which are the ones containing pure terms in x_1, x_2, x_3. Thus, each entry of Q in row 2, 3, and 4 contains a polynomial purely in x_1, x_2, x_3. The same holds for the g_i defined in 2.10. $\square$

Proof of Proposition 2.2.6. By Lemma 2.2.7, each unprojection equation contains at least one term depending only on the local coordinates of the Type I singularity at P_s, that is x_1, x_2, x_3. Therefore we do apply the procedure explained above to find δ_4. In particular, by the proof of Lemma 2.2.4, such pure monomial in the x_i realises the minimum value of Definition 2.2.1. Thus, by Lemma 2.2.7, we choose $\delta_4 = r + d_4$. Thus, δ_4 is equal to the degree of g_4.

In turn, we can apply this same strategy to δ_1, δ_2, δ_3, adapting the above construction, definitions and lemmas to the remaining δ_j. Note that the existence of a monomial in x_1, x_2, x_3 in each unprojection equation as stated in Lemma 2.2.7 implies that the order in which we determined the δ_j is unimportant, because $\delta_j = r + d_j$ for each $j \in \{1, 2, 3, 4\}$.

The weights in 2.5 follow by simple manipulation of the rows of the scroll we just defined. Summarising the observations made above about the bottom weights of 2.2, we have that

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ -r & 0 & a & b & c & d_1+r & d_2+r & d_3+r & d_4+r \end{array}\right).$$

If we subtract the second row to the third row of the above scroll we obtain an isomorphic rank 2 toric variety, whose Cox ring is given by

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ -r & -r & 0 & 0 & 0 & r & r & r & r \end{array}\right).$$

Finally, it only takes to divide the third row by $-r$ to get the final form of $\mathbb{F}_1$ presented in 2.5. $\square$

2.2.3 The Kawamata blow-up of a Fano: equations of the blow-up Y_1

We have just described a specific blow-up $\mathbb{F}_1$ of $w\mathbb{P}^7$, making choices for the bottom weights of $\mathbb{F}_1$ in order to keep track of the fact that X sits inside $w\mathbb{P}^7$. The following Proposition 2.2.10 and Lemma 2.2.11 are aimed to make sense of the choices made earlier to assign ω_i and δ_j.

Consider the pull-back $\Phi^*(X)$ of the nine equations of X. Referring to the 1-skeleton of $\mathbb{F}_1$ in 2.6, Φ is the map defined by the monomials having bidegree $\binom{r}{1}$.

Definition 2.2.2. Define the ideal of $Y_1 \subset \mathbb{F}_1$ as the saturation over t of the ideal of $\Phi^*(X)$.

However, the following statements will make Definition 2.2.2 more manoeuvrable and explicit.

Proposition 2.2.8. *The maps Φ and α_1 are proportional by a t factor (excluding s). In particular, $t^{-\frac{1}{r}}\Phi = \alpha_1$.*

Proof. Recall that Φ and α_1 are defined by monomials in the variables of $\mathbb{F}_1$ that are in $\left|\mathcal{O}\binom{r}{1}\right|$ and $\left|\mathcal{O}\binom{1}{0}\right|$ respectively.

As shown above, Φ is

$$\begin{aligned} \Phi \,:\, \mathbb{F}_1 &\longrightarrow w\mathbb{P}^7 \\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto \left(t^{\frac{a}{r}}x_1, t^{\frac{b}{r}}x_2, t^{\frac{c}{r}}x_3, t^{\frac{\delta_1}{r}}y_1, t^{\frac{\delta_2}{r}}y_2, t^{\frac{\delta_3}{r}}y_3, t^{\frac{\delta_4}{r}}y_4, s\right) \ , \end{aligned} \tag{2.11}$$

whereas α_1 is

$$\begin{aligned} \Phi \,:\, \mathbb{F}_1 &\longrightarrow w\mathbb{P}^6 \\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1,x_2,x_3,ty_1,ty_2,ty_3,ty_4) \ . \end{aligned}$$

Consider a variable w of $\mathbb{F}_1$ among $x_1,x_2,x_3,\ y_1,y_2,y_3,y_4$ with bidegree $\binom{\nu_1}{\nu_2}$. Call ζ the exponent of the t factor that w needs to pick up such that the bidegree of wt^ζ is proportional to $\binom{r}{1}$. In other words, we need to find ζ such that

$$\deg wt^\zeta = \begin{pmatrix}\nu_1\\ \nu_2+\zeta\end{pmatrix} = \lambda\begin{pmatrix} r\\ 1\end{pmatrix} \quad \text{for some } \lambda > 0\ .$$

Since $\lambda = \nu_2+\zeta$, we have that $\zeta = \frac{\nu_1}{r} - \nu_2$.

On the other hand, call ζ' the exponent of the t factor that w needs to pick up so that the bidegree of $wt^{\zeta'}$ is proportional to $\binom{1}{0}$. We need to have

$$\deg wt^{\zeta'} = \begin{pmatrix}\nu_1\\ \nu_2+\zeta'\end{pmatrix} = \mu\begin{pmatrix} 1\\ 0\end{pmatrix} \quad \text{for some } \mu > 0\ .$$

Here $\zeta' = -\nu_2$. Thus, $\zeta - \zeta' = \frac{\nu_1}{r} = \frac{1}{r}\deg_{w\mathbb{P}^7} w$. This means that on every variable $x_1, x_2, x_3, y_1, y_2, y_3, y_4$ of $\mathbb{F}_1$ the exponents ζ and ζ' differ only by $\frac{1}{r}$. $\square$

Proposition 2.2.8 obviously imples the following corollary.

Corollary 2.2.9. *The pull-backs $\Phi^*(\mathrm{Pf}(M))$ and $\alpha_1^*(\mathrm{Pf}(M))$ are equal up to a t factor.*

More precisely we mean that the evaluation of $\mathrm{Pf}(M)$ at the defining monomials of Φ is proportional to the evaluation of $\mathrm{Pf}(M)$ at the defining monomials of α_1.

Proposition 2.2.10. *If M is in Tom format, it is possible to cancel out from $\alpha_1^*(\mathrm{Pf}(M))$ a t factor with power at least 1.*

Proof. The ideal entries of M are occupied by polynomials in I_D: thus, they are formed by monomials either purely in the y_j or that are a multiplication of x_i and y_j.

This is true from what we said before: in other words, if we consider the pfaffian equation involving only ideal entries, it is divisible by t. $\square$

Let I_X be the ideal of X,

$$I_X := \langle f_1, \dots, f_5, f_6, \dots, f_9 \rangle$$

generated by polynomials $f_i := \mathrm{Pf}_i$ for $i \in \{1, \dots, 5\}$ and $f_i := sy_i - g_i$ for $i \in \{6, \dots, 9\}$.

Recall that Φ is expressed in 2.11 with fractional exponents for t. Since in the following we want the pull-back $\Phi^*(X)$ to have equation in a polynomial ring, we can write an equivalent expression for Φ by considering its multiplication by a $t^{\frac{r-a}{r}}$ factor. Thus,

$$\begin{aligned} & t^{\frac{r-a}{r}} \cdot \left(t^{\frac{a}{r}} x_1, t^{\frac{b}{r}} x_2, t^{\frac{c}{r}} x_3, t^{\frac{\delta_1}{r}} y_1, t^{\frac{\delta_2}{r}} y_2, t^{\frac{\delta_3}{r}} y_3, t^{\frac{\delta_4}{r}} y_4, s \right) \\ & \quad = \left(t x_1, t^{\frac{b(r-a)}{r} + \frac{b}{r}} x_2, t^{\frac{c(r-a)}{r} + \frac{c}{r}} x_3, t^{\frac{d_1(r-a)}{r} + \frac{\delta_1}{r}} y_1, t^{\frac{d_2(r-a)}{r} + \frac{\delta_2}{r}} y_2, t^{\frac{d_3(r-a)}{r} + \frac{\delta_3}{r}} y_3, t^{\frac{d_4(r-a)}{r} + \frac{\delta_4}{r}} y_4, t^{r-a} s \right) . \end{aligned} \tag{2.12}$$

The expression 2.12 has integer exponents.

Call $I_{\Phi^*X} := \langle \Phi^* f_1, \dots, \Phi^* f_5, \Phi^* f_6, \dots, \Phi^* f_9 \rangle$ using the above expression of Φ. Proposition 2.2.10 guarantees that, up to a t factor, Φ^* and α_1^* coincide on the pfaffian equations. Thus define the following polynomials

$$h_1 := \frac{\alpha_1^* \,\mathrm{Pf}_1(M)}{t^2} = \frac{\alpha_1^* f_1}{t^2} \; ; \tag{2.13}$$

$$h_i := \frac{\alpha_1^* \,\mathrm{Pf}_i(M)}{t} = \frac{\alpha_1^* f_i}{t^2} \qquad \text{for } i \in \{2, \dots, 5\} \; ; \tag{2.14}$$

$$h_i := \frac{\Phi^* f_i}{t^{\delta_{i-5} + r - a}} \qquad \text{for } i \in \{6, \dots, 9\} \; . \tag{2.15}$$

In addition, define the ideal $I_{Y_1} := (I_{\Phi^*X} : t^\infty)$ as the saturation of I_{Φ^*X} over t as in Definition 2.2.2.

Lemma 2.2.11. *We have that* $I_{Y_1} = \langle h_1, \ldots, h_5, h_6, \ldots, h_9 \rangle$.

Proof. For the saturation algorithm we refer to [CLO15]. Introducing a temporary variable z, define the ideal J as

$$J := \langle I_{\Phi^*X}, tz - 1 \rangle \subset S := R[z] \ ,$$

where $R := \mathbb{C}[t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4]$. Then, $(I_{\Phi^*X} : t^\infty) = J \cap R$ (see Chapter 4, §4 of [CLO15]). To write I_{Y_1} explicitly we study the Gröbner basis of J with respect to a complete monomial ordering $\succ$. This monomial ordering has to be such that the temporary variable z is the largest, and that s is the second largest. Then, we want it to be such that the monomials having the least number of y_j are larger. In other words, the monomial ordering $\succ$ is defined by the following matrix

$$\begin{pmatrix} z & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 & t \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d_1 - 1 & d_2 - 1 & d_3 - 1 & d_4 - 1 & 1 \\ 0 & 0 & a & b & c & d_1 - 1 & d_2 - 1 & d_3 - 1 & d_4 - 1 & 0 \\ 0 & 0 & a & b & c & d_1 - 1 & d_2 - 1 & d_3 - 1 & 0 & 0 \\ 0 & 0 & a & b & c & d_1 - 1 & d_2 - 1 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d_1 - 1 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.16}$$

Consider a polynomial k in which the variable z does not appear. Call $k_1 := LT(k)$ the leading term of k according to the monomial order 2.16: so $k = k_1 + k_2$ is the sum of the monomial k_1 and of the polynomial $k_2 := k - k_1$. Now compute the S-polynomials for $t^d k$ for some $d \geq 1$. The least common multiple between the respective leading terms of $t^d k$ and $tz - 1$ is $lcm\left(LT(t^d k), LT(tz-1)\right) = t^{d+1} k_1 z$. Then, following [CLO15],

$$\begin{aligned} S\left(t^d k, tz - 1\right) &= \frac{t^{d+1} k_1 z}{t^d k_1} \cdot t^d k - \frac{t^{d+1} k_1 z}{tz} \cdot (tz - 1) \\ &= t^{d+1} k z - t^{d+1} k_1 z + t^d k_1 \qquad (\text{since } k = k_1 + k_2) \\ &= t^{d+1} k_2 z + t^d k_1 = t^d \left(t k_2 z + k_1\right) \qquad (\text{since } tz = 1) \\ &= t^d k \ . \end{aligned}$$

Now focus on the polynomials $\Phi^* f_i$. If $i = 1$, the leading term of $\Phi^* f_1$ is of the form $LT(\Phi^* f_1) = y_{j_1} y_{j_2} t^2$. for certain $j_1, j_2 \in \{1, 2, 3, 4\}$. Similarly for $i \in \{2, \ldots, 5\}$, the leading term looks like $LT(\Phi^* f_i) = x_{j^i} y_{j_i} t$ for certain $j^i \in \{1, 2, 3\}$ and $j_i \in \{1, 2, 3, 4\}$.

For $i \in \{6, \ldots, 9\}$ instead, $LT(\Phi^* f_i) = sy_{i-5}t^{\delta_{i-5}+r-a}$. Note that the monomial ordering $\succ$ has been designed to identify as biggest the monomials having the lowest exponent of t. Therefore, for each $i \in \{1, \ldots, 9\}$ there is a suitable d such that $\Phi^* f_i = t^d h_i$. So, from the calculation shown above, we have that

$$S(\Phi^* f_i, tz - 1) + S\left(t^d h_i, tz - 1\right) = t^d h_1 \ .$$

Therefore, the Gröbner basis of J is

$$GB_\succ(\Phi^* f_1, \ldots, \Phi^* f_9, tz-1) = \left(t_1^h, th_2, th_3, th_4, th_5, t^{\delta_1+r-a}h_6, t^{\delta_2+r-a}h_7, t^{\delta+r-a}h_8, t^{\delta_4+r-a}h_9\right) \cup \{tz-1\} \ .$$

On the other hand, the highest common factor $hcf\,(LT(h_i), tz) = 1$ shows that $LT(h_i)$ and tz are coprime for all $i \in \{1, \ldots, 9\}$. Thus,

$$GB_\succ(h_1, \ldots, h_9, tz-1) = GB_\succ(h_1, \ldots, h_9) \cup \{tz-1\} \ .$$

In conclusion,

$$\begin{aligned} (\langle h_1, \ldots, h_9 \rangle : t^\infty) &= \langle GB_\succ(h_1, \ldots, h_9, tz-1) \cap R \rangle \\ &= \langle GB_\succ(h_1, \ldots, h_9) \rangle = \langle h_1, \ldots, h_9 \rangle \ . \end{aligned}$$

□

Remark 2.2.12. In conclusion, the choices of exponents of the t factors and the following elimination of them were made in a way such that the obtained ideal is precisely the saturation of the ideal of $\Phi^*(X)$.

2.3 Description of the link for Tom and proof of the Main Theorem

In this section we break down every step of the Sarkisov links described in Theorem 2.1.1. In doing so, we give a proof of Theorem 2.1.1.

Let $X \subset w\mathbb{P}^7$ be a general codimension 4 $\mathbb{Q}$-Fano 3-fold of Tom type and let $p \in X$ be a Tom centre. We first prove part **(B)** of Theorem 2.1.1.

*Proof of Theorem 2.1.1, **(B)**.* Consider a Sarkisov link for X that terminates with a divisorial contraction. Suppose that the endpoint Mori fibre space $Y \to S$ is a Fano 3-fold $X' \to S = \{pt\}$.

Let $\mathcal{B}_X$ be the basket of singularities of X. It is possible to track $\mathcal{B}_X$ throughout the link to retrieve the basket $\mathcal{B}_{Y_4}$ of Y_4. The basket $\mathcal{B}_{X'}$ of X' is a subset of $\mathcal{B}_{Y_4}$; that is, $\mathcal{B}_{X'}$ is $\mathcal{B}_{Y_4}$ minus the cyclic quotient singularities sitting inside the exceptional locus $E' := \mathbb{E}' \cap Y_4$. Moreover, if the determinant

$$\det \begin{pmatrix} d_3 & d_4 \\ -1 & -1 \end{pmatrix} = -1 \tag{2.17}$$

then E' is contracted to a Gorenstein point $p' \in X'$, which therefore does not contribute to the basket of X'.

If Φ blows up the cyclic quotient singularity of highest degree, neither the flops nor the flips will create a new cyclic quotient singularity of that degree. This means that the baskets $\mathcal{B}_X$ and $\mathcal{B}_{X'}$ are different, and therefore, $X \ncong X'$. On the other hand, if Φ blows up the cyclic quotient singularity of a lower degree, the flips will get rid of the one with higher degree, which will not be generated again. Thus, $X \ncong X'$.

If the absolute value of the above determinant is greater or equal than 2, the divisorial contraction Φ' might create a new orbifold singularity, but its order will not be higher than the absolute value of the determinant itself. Also, what we said for the basket of Y_4 still holds, that is, the cyclic quotient singularities of higher order are lost in the first blow-up and in the flips. Therefore, $\mathcal{B}_X \neq \mathcal{B}_{X'}$.

Now suppose that S is either a line or $\mathbb{P}^2$: thus, Y is Y_4. We conclude that X cannot be isomorphic to Y because their Picard ranks are different: 1 and 2 respectively.

$\square$

Remark 2.3.1. As an additional motivation to the proof of **(B)**, whereas we assume X quasi-smooth, X' is never quasi-smooth. Therefore, they cannot be isomorphic.

Moreover, in each case X' sits inside a weighted projective space having no more than seven coordinates. This is because the variable y_4 serves as the extra coordinate of the blow-up Φ', so it gets set as equal to one in X'; also, the unprojection equation $sy_4 = g_4$ globally eliminates the variable s.

The rest of this chapter is dedicated to proving part **(C)** of Theorem 2.1.1. Part **(C)** implies part **(A)** of Theorem 2.1.1.

The first step to construct the 2-ray game for X is blowing up the Tom centre $P_s \in X$: we obtain a Fano 3-fold $Y_1 \subset \mathbb{F}_1$ defined in Definition 2.2.2, where $\mathbb{F}_1$ is built as in Theorem 2.2.6. Then, by performing a variation of the GIT quotient of $\mathbb{F}_1$ we get a rank 2 birational link for $\mathbb{F}_1$. In other words, we change the irrelevant ideal of $\mathbb{F}_1$. This procedure is briefly explained in Section 1.2.3 and in the Appendix of [BCZ04].

We call Y_i the push-forward $\Psi_{i*}(Y_{i-1}) \subset \mathbb{F}_i$ of Y_{i-1} via Ψ_i. The Cox rings of the rank 2 toric varieties $\mathbb{F}_i$ can be naturally identified, as they are isomorphic in codimension 1. So similarly holds for the Cox rings of the varieties Y_i, for which we may choose the same generators of the quotient ideal. Throughout this thesis we identify these rings and these coordinates, for all $\mathbb{F}_i$ and Y_i.

We refer to the notation in Section 1.2.5 throughout the following chapters.

The first map of the birational link is Ψ_1.

Theorem 2.3.2. *The first step of the Sarkisov link starting with X, i.e. $\psi_1 : Y_1 \dashrightarrow Y_2$, consists of n simultaneous flops. The number n is equal to the number of nodes on $D \subset Z_1$.*

Proof. We divide the proof in a few claims.

Claim 1: α_1 contracts n lines. Following [BZ10], the locus $\mathbb{A}_1$ contracted by α_1 is defined by $\{y_1 = y_2 = y_3 = y_4 = 0\}$. Since Z_1 is in Tom format, Lemma 2.2.6 implies that α_1 is

$$\alpha_1 = \Phi_{\binom{1}{0}} \colon (t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) \mapsto (x_1, x_2, x_3, y_1 t, y_2 t, y_3 t, y_4 t) \ .$$

Thus, $Z_1 \cap Im(\alpha_1)$ restricted to $\mathbb{A}_1$ depends only on x_1, x_2, x_3, that is, it lies on D. Hence, over every node on D there is a $\mathbb{P}^1$ having coordinates t, s.

Claim 2: β_1 extracts n lines. Recall that if M is in Tom format then four of the five pfaffians are linear in the generators of the ideal I_D, whereas one is quadratic in those. To fix ideas, suppose without loss of generality that M is in Tom_1 format: under this convention, Pf_1 is quadratic and $\mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5$ are linear with respect to I_D.
From [BZ10] we know that the locus $\mathbb{B}_1 \in \mathbb{F}_2$ extracted by β_1 is defined by $\{t = s = 0\}$, which is isomorphic to a weighted $\mathbb{P}^3$. Therefore there is a weighted $\mathbb{P}^3$-bundle over the weighted $\mathbb{P}^2_{x_1,x_2,x_3} \cong D$.
Since $\mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5$ are linear on I_D, it is true that, restricting to $\{t = s = 0\}$,

$$\begin{pmatrix} \mathrm{Pf}_2 \\ \mathrm{Pf}_3 \\ \mathrm{Pf}_4 \\ \mathrm{Pf}_5 \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

where A is a 4×4 matrix defined as

$$A := \begin{pmatrix} \gamma_1(\mathrm{Pf}_2) & \gamma_2(\mathrm{Pf}_2) & \gamma_3(\mathrm{Pf}_2) & \gamma_4(\mathrm{Pf}_2) \\ \gamma_1(\mathrm{Pf}_3) & \gamma_2(\mathrm{Pf}_3) & \gamma_3(\mathrm{Pf}_3) & \gamma_4(\mathrm{Pf}_3) \\ \gamma_1(\mathrm{Pf}_4) & \gamma_2(\mathrm{Pf}_4) & \gamma_3(\mathrm{Pf}_4) & \gamma_4(\mathrm{Pf}_4) \\ \gamma_1(\mathrm{Pf}_5) & \gamma_2(\mathrm{Pf}_5) & \gamma_3(\mathrm{Pf}_5) & \gamma_4(\mathrm{Pf}_5) \end{pmatrix}$$

and $\gamma_i(\mathrm{Pf}_j) \in \mathbb{C}[x_1, x_2, x_3]$ is the coefficient of y_i in Pf_j.

Lemma 2.3.3. *For each point $p \in D$ the rank of $A_p := ev_p(A)$ is either 2 or 3.*

Proof of Lemma 2.3.3. Obviously the rank is at least 1.
Recall that there are six syzygies relating the five pfaffians of M: referring to the notation set in 2.7, one of them is

$$p_1 \operatorname{Pf}_2 + p_2 \operatorname{Pf}_3 + p_3 \operatorname{Pf}_4 + p_4 \operatorname{Pf}_5 = 0 \; .$$

which is a relation among $\operatorname{Pf}_2, \operatorname{Pf}_3, \operatorname{Pf}_4, \operatorname{Pf}_5$. Therefore, at any point $p \in D$ it is possible to express one of the last four pfaffians in terms of the other three. This means that we are left with only three equations that are linear on I_D. Thus, $rk(A_p) \leq 3$.
On the other hand, $rk(A_p) \geq 2$. To prove this note that the entries of A are all polynomials in $\mathbb{C}[x_1, x_2, x_3]$: this is because if we are considering the restriction to D, i.e. we impose the vanishing of all the y_i, we are actually killing all the monomials that come out from the non-linear (in I_D) terms of $\operatorname{Pf}_2, \operatorname{Pf}_3, \operatorname{Pf}_4, \operatorname{Pf}_5$. Since each of the Pf_j has at least one of the y_i appearing at least once, then there are at least two linearly independent column vectors in A. This concludes the proof of Lemma 2.3.3. □

Remark 2.3.4. As underlined before, the locus $\mathbb{B}_1$ is fibred over D with weighted $\mathbb{P}^3$ fibres. Therefore, for any point $p \in D$, if $rk(A_p) = 2$ the image of A is a 2-dimensional space in $\mathbb{P}^3$, which means that β_1 contracts a $\mathbb{P}^1 \subset \mathbb{B}_1 \cap Y_2$ to $p \in D$.
Analogously, if $rk(A_p) = 3$ the map β_1 is an isomorphism in a neighbourhood of a point $p' \subset \mathbb{P}^1 \subset \mathbb{B}_1$ to $p \in D$.

Remark 2.3.5. So far we used only four of the nine equations of X. This means that all the information about the flop is contained in the pfaffian equations. The last thing we need to check is that the unprojection equations do not play any role in the determination of the flop. Recall that the image of the maps of toric varieties α_1 and β_1 is $\mathbb{G}_1$, which is a rank 1 toric variety of dimension 10 which contains the weighted $\mathbb{P}^6$ that is the ambient space of Z_1. Its coordinates are $\xi_1 := x_1, \xi_2 := x_2, \xi_3 := x_3$, $\upsilon_1 := y_1 t, \upsilon_2 := y_2 t, \upsilon_3 := y_3 t, \upsilon_4 := y_4 t$, $\sigma_1 := s y_1, \sigma_2 := s y_2, \sigma_3 := s y_3, \sigma_4 := s y_4$. However, the variable s can be globally eliminated on D using the unprojection equations. So, even though Lemma 2.2.7 ensures that the restriction of the unprojection equations to $\mathbb{B}_1$ is non-trivial, we do not need to take those equations into account when studying the flop.

We could also observe that, on D, the Jacobian matrix of Z_1 is

$$J(Z_1)|_D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1(\mathrm{Pf}_2) & \gamma_2(\mathrm{Pf}_2) & \gamma_3(\mathrm{Pf}_2) & \gamma_4(\mathrm{Pf}_2) \\ 0 & 0 & 0 & \gamma_1(\mathrm{Pf}_3) & \gamma_2(\mathrm{Pf}_3) & \gamma_3(\mathrm{Pf}_3) & \gamma_4(\mathrm{Pf}_3) \\ 0 & 0 & 0 & \gamma_1(\mathrm{Pf}_4) & \gamma_2(\mathrm{Pf}_4) & \gamma_3(\mathrm{Pf}_4) & \gamma_4(\mathrm{Pf}_4) \\ 0 & 0 & 0 & \gamma_1(\mathrm{Pf}_5) & \gamma_2(\mathrm{Pf}_5) & \gamma_3(\mathrm{Pf}_5) & \gamma_4(\mathrm{Pf}_5) \end{pmatrix}$$

where the bottom right block is A. Therefore we deduce that

Lemma 2.3.6. *For each point $p \in D$, then $rk(J(Z_1)|_D)_p = rk(A_p)$.*

Claim 3: ψ_1 is a flop. From the previous two claims, ψ_1 is an isomorphism in codimension 1. We just need to check what is the intersection between $-K_{Y_i}$, for $i = 1, 2$, and the exceptional loci of α_1 and β_1 respectively. Both for i equal to 1 or 2, $-K_{Y_i}$ is of the form $\{x_1 = 0\}$. On the other hand, none of the points in $Sing(Z_1) \subset D$ satisfies the condition $x_1 = 0$. Therefore, $-K_{Y_i} \cdot \mathbb{P}^1_{t,s} = 0$ for $i = 1, 2$.

This completes the proof of 2.3.2. □

Remark 2.3.7. Note that this proof is completely independent from the form of the right-hand-side of the unprojection equations: the information about the flop is all encoded in the geometry of Z_1, as we would expect.

Now we want to show that, independently on the particular member in the family of Z_1, the nature of the birational maps at the rank 2 level is always the same throughout the deformation family of Z_1. In other words, given a general member of the deformation family of Z_1 in Tom format having only nodes on D prescribed by the [BKR12b], then the Sarkisov link of the associated X has the same behaviour, no matter the choice of the particular member, although the variables eliminated in the variables might change. For example, this is in contrast with the "starred monomials" of [CPR00].

For the purpose of the rest of this chapter, we introduce the following notation regarding the grading of the matrix M. These configurations arise many times. For simplicity, suppose that M is in Tom_1 format: the argument holds independently on the Tom format. For some suitable positive σ and τ, define

(A) The entries $a_{24}, a_{25}, a_{34}, a_{35}$ all have weight π. Hence, in order to have homogeneous pfaffians and positive weights, the other weights of M are

$$\begin{pmatrix} \sigma & \sigma & \pi+\sigma-\tau & \pi+\sigma-\tau \\ \hline & \tau & \pi & \pi \\ & & \pi & \pi \\ & & & 2\pi-\tau \end{pmatrix}. \tag{2.18}$$

(B) The entries a_{25}, a_{34} both have weight $d_1 = d_2$, while a_{24}, a_{35} are free. Hence, the other weights of M are

$$\left(\begin{array}{cccc} \sigma & \pi+\sigma-\upsilon & \pi+\sigma-\tau & 2\pi+\sigma-\tau-\upsilon \\ \hline & \tau & \upsilon & \pi \\ & & \pi & 2\pi-\upsilon \\ & & & 2\pi-\tau \end{array}\right). \tag{2.19}$$

2.3.1 Proof of (i)

Here we describe the flip that occurs when crossing the ray ρ_{y_1}. This proof is identical to the proof of the fact that crossing the ray ρ_{y_2} induces a second flip. The two proofs hold in case **(i)**.

Theorem 2.3.8. *Suppose $d_1 > d_2$ and that the point $P_{y_1} \in Z_2$. Then, the map $\psi_2 \colon Y_2 \dashrightarrow Y_3$ is a flip.*

Proof. Localise at the point $P_{y_1} \in Z_2$. So, after a row operation, $\mathbb{F}_2$ becomes

$$\left(\begin{array}{ccccc|cccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_1 & r+d_1 & a & b & c & 0 & d_2-d_1 & d_3-d_1 & d_4-d_1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

The exceptional locus of α_2 is $\mathbb{A}_2 = \{y_2 = y_3 = y_4 = 0\}$ (Lemma 4.5 of [BZ10]), that is,

$$\mathbb{A}_2 = \left(\begin{array}{ccccc|c} t & s & x_1 & x_2 & x_3 & y_1 \\ d_1 & r+d_1 & a & b & c & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \cong \mathbb{P}^4(d_1, r+d_1, a, b, c)$$

with coordinates t, s, x_1, x_2, x_3, y_1 respectively: it is such that $\alpha_2(\mathbb{A}_2) = P_{y_1}$.

In order to show that ψ_2 is a flip for the varieties, we need to look at the intersection $Y_2 \cap \mathbb{A}_2$ and show that it has codimension at least 3 in $\mathbb{P}^4(d_1, r+d_1, a, b, c)$. The unprojection equation $sy_1 = g_1$ allows to discard s locally above $P_{y_1} \in Z_2$. Therefore $Y_2 \cap \mathbb{A}_2$ has at least codimension 1. This is because $Y_2 \cap \mathbb{A}_2 \subset F \subset \mathbb{P}^4(d_1, r+d_1, a, b, c)$ where F is a hypersurface isomorphic to the weighted $\mathbb{P}^3(d_1, a, b, c)$ defined by the unprojection equation relative to y_1 in which y_1 has been set at 1.

From part **(i)** and **(ii)** of Lemma 2.2.1 we deduce that in one of the pfaffian equations there is a monomial of the form $x_i y_1$, which means that, locally at P_{y_1}, it is possible to discard x_i, i.e. x_i can be expressed as a function of the other variables: suppose that x_1 gets eliminated. Therefore, $Y_2 \cap \mathbb{A}_2$ has at least codimension 2 inside $\mathbb{P}^4(d_1, r+d_1, a, b, c)$.

From Lemma 2.2.7 we deduce that there is another unprojection equation that contains monomials in the x_i and t. Therefore, $Y_2 \cap \mathbb{A}_2 \subset S \subset F \subset \mathbb{P}^4(d_1, r+d_1, a, b, c)$ where $S \cong \mathbb{P}^4(d_1, b, c)$: $Y_2 \cap \mathbb{A}_2$ has at least codimension 3 in $\mathbb{P}^4(d_1, r+d_1, a, b, c)$. To prove that the codimension is exactly 3 we need to show that the remaining equations define a curve in S, so we need to exclude the case in which they define a single point or the empty set. The vanishing locus of the remaining equations cannot be the empty set because $P_{y_1} \in Z_2$, so there must be an intersection between Y_2 and $\mathbb{A}_2$. In addition, $Y_2 \cap \mathbb{A}_2$ cannot be a single point either for the following reason. Since X is quasi-smooth and $\mathbb{Q}$-factorial, the same holds for Y_1. Also Y_2 is quasi-smooth, but it is not isomorphic to Y_1 because $\beta_2 \colon Y_3 \to Z_2$ contracts the curve defined by the quadratic pfaffian equation (which is Pf_1 if M is in Tom_1 format). Thus, by $\mathbb{Q}$-factoriality, Y_2 must also contract a curve.

The last thing we need to check is that the intersection of $-K_{Y_2}$ with the exceptional locus of α_2 is positive and that the intersection of $-K_{Y_3}$ with β_2 is negative. This is true because $\{x_1 = 0\} \in |\mathcal{O}(-aK_{Y_2})|$ is relatively ample, so it meets every curve positively. $\square$

On the other hand,

Theorem 2.3.9. *If the point $P_{y_1} \notin Z_2$, the toric varieties flip $\Psi_2 \colon \mathbb{F}_2 \dashrightarrow \mathbb{F}_3$ restricted to Y_2 is an isomorphism $Y_2 \cong Y_3$.*

Proof. Recall that the equations of Z_2 are the same as Z_1. The fact that $P_{y_1} \notin Z_2$ means that there exists at least one pfaffian equation that is non-zero when evaluated at P_{y_1}. Moreover, $\alpha_2(\mathbb{A}_2) = P_{y_1}$; on the other hand, $\alpha_2(Y_2) = Z_2$. This means that the exceptional locus of the flip at the toric level does not intersect with Y_2, i.e. $\mathbb{A}_2 \cap Y_2 = \emptyset$. $\square$

The next Proposition is aimed at showing when the hypotheses of either Theorem 2.3.8 or Theorem 2.3.9 are verified. Everything depends on the nature of the weights of the matrix M.

Proposition 2.3.10. *Let M be in Tom format. If the weights of M fall in case **(B)**, then either the flip with base at $P_{y_1} \in Z_2$ or the flip with base at $P_{y_2} \in Z_3$ is an isomorphism.*

Proof. In case **(B)** two ideal entries with the same weight are positioned diagonally such that they get multiplied when considering $\mathrm{Pf}_1(M)$. Suppose that $\pi = d_1$. Thus, y_1 occupies both the entries a_{25} and a_{34}. From Theorem 2.2.11 and since y_1 appears linearly in those entries, we deduce that there is the monomial y_1^2 in the equations of Y_1. Therefore, repeating the proof of Theorem 2.3.9, we have that Ψ_2 is an isomorphism when restricted to Y_2. Analogously happens for $\pi = d_2$.

The weight π is never equal to d_3 or d_4. $\square$

Remark 2.3.11. There is only one Hilbert series lying in case **(i)**, #5870, whose matrix M is in the configuration **(A)**. The codimension 4 Fano 3-fold of Tom type X corresponding to #5870 lies in the weighted projective space $\mathbb{P}^7(1^2, 2^2, 3^2, 4, 5)$. The Tom centre considered is $\frac{1}{3}(1,1,2)$, therefore the generators y_1, y_2, y_3, y_4 of I_D have weight 5, 4, 3, 2 respectively. Here the second flip is skipped, namely the restriction of Ψ_3 to Y_3 is an isomorphism. In this case the weights of M (in Tom format) are

$$\begin{pmatrix} 2 & 2 & 3 & 3 \\ \hline & 3 & 4 & 4 \\ & & 4 & 4 \\ & & & 5 \end{pmatrix}.$$

In general, it is possible to fill the four entries with weight 4 with four different polynomials of degree 4 in I_D all containing y_2. However, performing row/column operations on M as described in 1.2.4 allows to get rid of the two copies of y_2 lying on the same diagonal: in this way, we can end up having y_2 appearing in either entries a_{24}, a_{35} or entries a_{25}, a_{34} only. Thus, $\mathrm{Pf}_1(M)$ contains the monomial y_2^2, which implies that the restriction of Ψ_3 to Y_3 is an isomorphism.

In conclusion, in this argument it is crucial that there is only one ideal generator having weight 4. The concurrent presence of configuration **(A)** and of two distinguished ideal generators having the same weight will lead to different consequences in **(iii)** and **(v)**.

Although the majority of the Hilbert series of case **(i)** falls in configuration **(B)**, it also happens that the weights of M are in configuration neither **(A)** nor **(B)**. In this situation, both ψ_1 and ψ_2 are flips. In particular, this means that the mobile cone of $\mathbb{F}_1$ coincides with the mobile cone of Y_1. In contrast, the skipping of a flip shows that the mobile cone of $\mathbb{F}_1$ is richer than the mobile cone of Y_1.

Theorem 2.3.8 and Proposition 2.3.10 can be also applied to the crossing of the wall adjacent to $d_2 > d_3$: in particular, this wall crossing is either a flip or an isomorphism.

Consider the rank 2 toric variety $\mathbb{F}_4$: in case **(i)**, $d_3 > d_4$. The end of the link is a divisorial contraction.

Lemma 2.3.12. *Suppose that $\rho_X = 1$. If $d_3 > d_4$, the map $\Phi' \colon \mathbb{F}_4 \to \mathbb{G}_4$ is a divisorial contraction of Y_4 to a Fano 3-fold $X' \subset \mathbb{P}' \subset \mathbb{G}_4$.*

Proof. Since $\rho_X = 1$, the exceptional divisor $\mathbb{E}'$ of Φ' is irreducible. Thus, $\rho_{X'} = 1$ as well. Moreover, X' is projective. In addition, $-K_{X'}$ is ample. Consider a curve Γ in X' that is not in the image of $\mathbb{E}'$ via Φ' and that is not in the image of the union of the right-hand-side contracted loci $\mathbb{B}_i$ of the flips. Such curve can be always found because the set of curves of X' lying in $\Phi'(\mathbb{E}')$ and the union of the proper transform of the $\mathbb{B}_i$ has codimension 2.

The curve Γ can be tracked back down to Y_1. The divisor $-K_{Y_1}$ is nef and big (that is, Y_1 is a so-called *weak Fano*): this is because $-K_{Y_1} = \alpha_1(-K_{Z_1})$, and the every curve in Y_1 is either strictly positive against $-K_{Y_1}$ and contracted to Z_1; or is a flopping curve. Therefore we have that $-K_{X'}\Gamma = -K_{Y_1}\tilde{\Gamma} > 0$, where $\tilde{\Gamma}$ is the proper transform of Γ, and is isomorphic to Γ. □

2.3.1.1 Identifying the end of the link

Lemma 2.3.12 shows that Φ' is a divisorial contraction to another Fano. When the determinant of the bidegrees of the right-hand-side irrelevant ideal of $\mathbb{F}_4$ is 1, it is possible to find the Hilbert series associated to the Fano X'.

Analogously to Section 2.2.2, the map $\Phi' \colon \mathbb{F}_4 \to \mathbb{G}_4$ is defined by all the monomials in the linear system $\left|\mathcal{O}\binom{d_3}{-1}\right|$. The variable y_4 will play the same role played by t for Φ. The restriction of Φ' to Y_4 shows that the equations of Y_4 constitute relations among the new coordinates of $\mathbb{G}_4$. This means that some of the equations of Y_4 eliminate (globally) some of the new coordinates of $\mathbb{G}_4$. The number, and the name, of such eliminated coordinates varies case by case. The global elimination of the variable $s' = sy_4^{\varsigma}$ of $\mathbb{G}_4$, for some exponent ς, always happens: this is due to the fourth unprojection equation $sy_4 = g_4$, that provides an expression of s' in terms of the other coordinates of $\mathbb{G}_4$.

This phenomenon might occur for other coordinates too, depending on each specific case. However, this shows that the weighted projective space $\mathbb{P}'$ that is the ambient space of X' is always strictly contained in $\mathbb{G}_4$. This calculates the ambient space of X'.

On the other hand, it is possible to track down the evolution of the basket of singularities of X along the link, in order to deduce the one for X'. Specifically, the basket $\mathcal{B}_{X'}$ is equal to $\mathcal{B}_{Y_4}$ minus the cyclic quotient singularities of $\mathcal{B}_{Y_4}$ contained in the exceptional locus $\mathbb{E}'$ of Φ'. Its basket and its ambient space determine the Hilbert series of X' univocally.

Remark 2.3.13. Studying the basket of singularities at each step of the link implies the investigation of which singularities get contracted and extracted each time. This is not always straightforward: we give the example of the Hilbert series #511 in Section 3.1.5, in which the basket $\mathcal{B}_{X'}$ is more complicated to find.

The equations of X' can be found by rewriting the equations of Y_4 in terms of the new coordinates of $\mathbb{G}_4$, and by excluding the ones used to perform the global elimination. Usually, the equations of X' retrieved in this way do not give the general member of the Hilbert series of X', but just a special member of the family.

This calculation is shown explicitly in the examples of Chapters 3 and 4.

Remark 2.3.14. Here we assumed that

$$\det \begin{vmatrix} d_3 & d_4 \\ -1 & -1 \end{vmatrix} = 1 \,.$$

In this case, we can still say that X' is a Fano 3-fold, because Lemma 2.3.12 still holds. In addition, by computing the exact evoultion of the basket of singularities along the link, we can identify X'.

2.3.2 Proof of (ii)

This case splits in two situations according to the weights of the matrix M.

The first is when M has weights as in **(B)**. Only two Hilbert series fall in this instance, namely #1218 and #1413. For both, the equations of Y_2 have a pure monomial in y_1 (similarly to the phenomenon described in 2.3.9). Therefore the following holds.

Theorem 2.3.15. *Consider the Hilbert series #1218, #1413 and the Fano 3-fold defined by* Tom_1 *for both. Then, their respective Sarkisov links evolve as follows:* ψ_1 *is a flop,* Ψ_2 *restricts to an isomorphism* ψ_2 *on* Y_2*,* ϕ' *is a divisorial contraction over* $\mathbb{P}^1_{y_2,y_3} \subset X'$*.*

Proof. By Theorem 2.3.2 we have that ψ_1 is a flop.

The weights of the matrix M of the two Hilbert series are as in **(B)**. Therefore, y_1, which is the only variable having degree d_1, occupies both the entries a_{25} and a_{34}, possibly added to a polynomial in I_D in degree d_1 involving the other variables: so $\mathrm{Pf}_1(M)$ is a polynomial containing y_1^2. Using the same proof strategy of 2.3.9 we see that ψ_2 is an isomorphism.

The last map is a divisorial contraction to another Fano 3-fold X', as shown in Lemma 2.3.12.

Note that $\mathbb{P}^1_{y_2,y_3} \subset X'$ in any case. So there aren't two distinct divisorial contractions, but only one polarised at $\mathbb{P}^1_{y_2,y_3}$. □

On the other hand, none of the other Hilbert series falling in $d_1 > d_2 = d_3 > d_4$ come from M with **(B)** weights. In this instance, the first flip ψ_2 is performed by the variety Y_2 too, and it is followed by a divisorial contraction to X'.

Theorem 2.3.16. *Let* Z_1 *be defined by a graded matrix* M *in Tom format having weights as in* ***(B)****. Then the Sarkisov link for* X *(except the Hilbert series #1218 and #1413) is constituted by: a flop, a flip, and a divisorial contraction to* $\mathbb{P}^1_{y_2:y_3} \subset X'$*.*

Proof. We connect this proof to the one for Theorem 2.3.15. As before, ψ_1 is a flop due to Theorem 2.3.2. Since the weights of M are not as in **(B)**, then the point P_{y_1} belongs to Z_2, which means that Y_2 is subject to the flip transformation ψ_2 that occurs on $\mathbb{F}_2$.

Lastly, the same proof for Theorem 2.3.15 holds with regards to the divisorial contraction Φ'. $\square$

2.3.3 Proof of (iii) and (v)

Now we need to study the behaviour of the link in the case where $d_1 = d_2$. Both **(iii)** and **(v)** share the same behaviour regarding the crossing of the ray ρ_{y_1,y_2} generated by y_1 and y_2.

Theorem 2.3.17. *Suppose $d_1 = d_2$. Then, there are two simultaneous flips based at two points in Z_2.*

Suppose that Z_1 is in Tom$_i$ format: the i-th pfaffian depends only on the six ideal entries of M. To fix ideas, let M be in Tom_1 format. Here we distinguish two different situations that are the specialisation to **(iii)** and **(v)** of **(A)** and **(B)**. We repeat the shape of the grading of M to stress the fact that in this case we have two different variables that fit the entries with weight $d_1 = d_2$.

(a) The entries $a_{24}, a_{25}, a_{34}, a_{35}$ all have the weight $d_1 = d_2$. So the weights of M are

$$M = \left(\begin{array}{cccc} \sigma & \sigma & d_1+\sigma-\tau & d_1+\sigma-\tau \\ \hline & \tau & d_1 & d_1 \\ & & d_1 & d_1 \\ & & & 2d_1-\tau \end{array} \right). \tag{2.20}$$

(b) The entries a_{25}, a_{34} both have the weight $d_1 = d_2$, while a_{24}, a_{35} are free. So the weights of M are

$$M = \left(\begin{array}{cccc} \sigma & d_1+\sigma-\upsilon & d_1+\sigma-\tau & 2d_1+\sigma-\tau-\upsilon \\ \hline & \tau & \upsilon & d_1 \\ & & d_1 & 2d_1-\upsilon \\ & & & 2d_1-\tau \end{array} \right). \tag{2.21}$$

Geometrically, α_2 contracts the locus $\mathbb{A}_2$ to a line $\mathbb{P}^1_{y_1:y_2} \subset \mathbb{G}_2$. So, the intersection $\mathbb{A}_2 \cap Y_2$ is mapped to $\mathbb{P}^1_{y_1:y_2} \cap Z_2$. In Lemma 2.3.19 and in Lemma 2.3.20 we discuss the nature of the intersection $\mathbb{P}^1_{y_1:y_2} \cap Z_2$ in cases **(a)** and **(b)** respectively. The idea is that $\mathbb{P}^1_{y_1:y_2}$ cuts out a rank 2 quadratic form in y_1, y_2, which determines two points in Z_2. Therefore, the variety Y_2 is subjected to two simultaneous flips.

Proposition 2.3.18. *There exists a rank 2 quadratic form in y_1, y_2 defined on $\mathbb{G}_2$ that determines two distinct points P_1, P_2 in Z_2.*

Proof. To fix ideas, let M be in Tom_1 format. Independently on **(a)** and **(b)**, without loss of generality we can assume that y_1 occupies the a_{25} entry and that y_2 occupies the a_{34} entry of M. Note that the equations of Z_2 are in terms of t as well, being the image of Y_2 via α_2. If any of y_1 or y_2 is in one of the entries in the top row of the matrix, it will surely pick up a t factor in the blow up of X, so it will vanish when restricted to $\mathbb{P}^1_{y_1:y_2}$. Moreover, if y_1 and y_2 appear in other entries of M they will need to be multiplied by some other variable.

Therefore, the quadratic form has to be found in the first pfaffian of M, i.e. it is the restriction of $\mathrm{Pf}_1(M)$ to $\mathbb{P}^1_{y_1:y_2}$. In particular, it is of the form $y_1^2 - y_1y_2 + y_2^2$ in case **(a)**, whereas it is $y_1^2 - y_1y_2$ in case **(b)**. Note that no other monomials, also coming from other equations, survive the restrictions for the reasons explained above. For both **(a)** and **(b)** the two quadratic forms describe two distinct points on Z_2. $\square$

Lemma 2.3.19. *Let Z_1 be defined by a graded matrix M in Tom format having weights as in* **(a)**. *Then, the following statements hold.*

- *If one of the two flips is toroidal, then the other one is also toroidal. Analogously, if one of the two flips is an hypersurface flip, then the other one is also an hypersurface flip.*
- *The two flips have exactly the same weights.*

Proof of Lemma 2.3.19. Let M have weights as in **(a)**. As in the proof of Proposition 2.3.18, it is possible to place y_1 and y_2 in the entries a_{25} and a_{34} respectively. Thus by looking at the pfaffians of M, locally at P_{y_1} we can eliminate a potential linear term in the entries a_{12} and a_{15}. Likewise, locally at P_{y_2} we can eliminate a potential linear term in the entries a_{13} and a_{14}. Since a_{12} and a_{13} have the same weights, y_1 and y_2 eliminate the same variable when localising at their respective coordinate points; or otherwise they do not eliminate any variable in those entries at all. The same happens for the entries a_{14} and a_{15}.

Note that the variables y_3 and y_4 cannot be eliminated, as they are always multiplied by a t factor on the top row, so they are not linear. Therefore, the birational transformations at P_1 and P_2 can only be flips.

This proves that α_2 contracts two loci of the same dimension: in fact, those loci are isomorphic. In conclusion, the flip phenomenon is completely symmetrical over y_1 and y_2 and the two points P_1 and P_2 in Z_2. $\square$

Remark 2.3.20. Let Z_1 be defined by a graded matrix M in Tom format having weights as in **(b)**. Then, if one of the flip is toroidal does not imply that the other one is toroidal. Analogously, if one of the flips is an hypersurface flip, then the other one is not necessarily an hypersurface flip.

In particular, the weights of each of the two flips could be different.

Proof of Remark 2.3.20. Let M have weights as in **(b)**: again, put y_1 and y_2 in the entries a_{25} and a_{34} respectively without loss of generality. Note that the weights in the top row of M are all different. This implies that y_1 and y_2 cannot eliminate the same variables, so the two flips at P_1 and P_2 cannot have the same weights.

Moreover, suppose that a certain linear variable w occupies the entry a_{14}. On the other hand, w can appear in the a_{15} entry only if multiplied by a polynomial $f_{d_1 - v}$ of degree $d_1 - v$. Thus, there is no hope for y_2 to eliminate w, and therefore the two flips can have different numbers of weights. In short, it is allowed to have a toric flip and an hypersurface flip simultaneously. □

The above statements prove Theorem 2.3.17.

Proof of Theorem 2.3.17. Proposition 2.3.18 shows that the image of α_2 determines two distinct points P_1, P_2 in Z_2. In a similar fashion to the proof of Theorem 2.3.8 it is possible to prove that $\Psi_2 \colon \mathbb{F}_2 \to \mathbb{F}_3$ is an algebraically irreducible flip. However, its restriction to the variety $Y_2 \subset \mathbb{F}_2$ is constituted of two distinct components, each one contracted to one of the two points $P_1, P_2 \in Z_2$.

Lemmas 2.3.19 and 2.3.20 clarify the nature of such components. □

Remark 2.3.21. Note that Theorem 2.3.17 holds both if $d_1 = d_2 > d_3 = d_4$ and $d_1 = d_2 > d_3 > d_4$. Essentially, it holds independently on how the link continues after crossing the ray ρ_{y_1, y_2}.

The continuation of the link is different for case $d_1 = d_2 > d_3 = d_4$ and $d_1 = d_2 > d_3 > d_4$. For the latter, item **(i)** holds by Lemma 2.3.12. For the former, we have that

Theorem 2.3.22. *If $d_2 > d_3 = d_4$, then Φ' is a del Pezzo fibration over $\mathbb{P}^1_{y_3, y_4}$.*

Proof. Consider the map of toric varieties $\Phi' \colon \mathbb{F}_4 \to \mathbb{P}^1_{y_3, y_4}$. In particular, $\mathbb{F}_4$ can be written as

$$\left(\begin{array}{ccccccc|cc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_3 & r + d_3 & a & b & c & d_2 - d_3 & d_2 - d_3 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

By definition, this is a weighted $\mathbb{P}^6$-bundle over $\mathbb{P}^1$. The intersection of Y_4 with the general fibre of this bundle clearly has dimension 2, given that the variables y_3 and y_4 now act as parameters. Moreover, the restriction of K_{Y_4} to such intersection is still ample. Therefore, Φ' is a del Pezzo fibration of Y_4 over $\mathbb{P}^1_{y_3, y_4}$. □

Lemma 2.3.23. *The intersection between Y_4 and the general fibre of the bundle defined by Φ' is smooth.*

Proof. Consider the generic fibre S of Φ': it is a surface in Y_4. Suppose S is singular. In particular, its closure inside the 3-fold Y_4 is a line. Therefore, Y_4 would contain a whole singular line, which is a contradiction with Y_4 being terminal. $\square$

In Table 6.1 we compute the degree of the general fibre of the del Pezzo fibration in each case.

2.3.4 Proof of (iv)

Similarly to what happens in case **(ii)**, the weights of the matrix M influence the behaviour of the link. Again, the distinction made in **(a)** and **(b)** plays a crucial role.

Proposition 2.3.24. *Suppose M has weights in configuration **(b)**. Then, either y_1 appears as a square in the equations of Y_2, or y_2 appears as a square in the equations of Y_3.*

Proof. Consider the configuration **(b)** of weights of M, assuming the format of M to be Tom_1 for the sake of simplicity. We have that $\mathrm{Pf}_1(M)$ involves the multiplication of two entries, a_{25} and a_{34}, having the same weight. In this instance, the entries a_{25} and a_{34} have weight either d_1 or d_2, depending on the specific Hilbert series considered. This time, in contrast to the proof of Proposition 2.3.18, by hypothesis we have only one variable for each d_1, d_2, namely, y_1 and y_2 respectively. Therefore, the quadratic form defined on $\mathbb{G}_2$ (or $\mathbb{G}_3$ respectively) is y_1^2 (or y_2^2 in turn). $\square$

The majority of Hilbert series that fall into case **(iv)** of Theorem 2.1.1 are such that the weights of M are in configuration **(b)**. Therefore,

Lemma 2.3.25. *If M has weights in configuration **(b)**, then either Ψ_2 or Ψ_3 is an isomorphism when restricted to Y_2 and Y_3 respectively.*

Proof. From the above Proposition we have that either y_1^2 appears in the equations of Y_2, or y_2^2 appears in the equations of Y_3. Therefore, analogously to the proof of case **(i)**, the point P_{y_1} does not belong to Z_2, or $P_{y_2} \notin Z_3$. So, the locus contracted by the ambient space flip does not intersect Y_2 (or Y_3). In conclusion, either Ψ_2 or Ψ_3 is an isomorphism. $\square$

Remark 2.3.26. Only the Hilbert series #20544 falling in case **(iv)** has a weight configuration of type **(a)**. Since the only variable with weight d_2 is y_2, it is possible to cancel out y_2 from the entries a_{25} and a_{34} via row/column operations. Therefore the equations of X have the monomial y_2^2. Nonetheless, no flip is missed. This is because, performing the blow-up of X and then saturating over t, we have that the term y_2^2 picks up a t factor.

Remark 2.3.27. The weights of the matrix M relative to the three Hilbert series #5516, #5867, #11437 are neither in configuration **(a)** nor **(b)**. Therefore, both Ψ_2 and Ψ_3 are flips for the varieties Y_2 and Y_3 respectively.

The last map Φ' of the link in case **(iv)** is a del Pezzo fibration, as proved in Theorem 2.3.22.

2.3.5 Proof of (vi)

There are six Hilbert series having ideal variables with weights $d_1 > d_2 = d_3 = d_4$.

Proposition 2.3.28. *The Sarkisov link starting from the Hilbert series #6865 is such that the restriction to Y_2 of the birational map Ψ_2 is an isomorphism.*

Proof. In the case of #6865, the weights of the matrix M are in configuration **(b)**. Therefore, in the same fashion as in the proof of **(iv)**, we deduce that the monomial y_1^2 appears in the equations of Y_2. This implies that Ψ_2 is an isomorphism on the variety Y_2. □

The other five Hilbert series falling in this case behave as expected.

Proposition 2.3.29. *Consider the Sarkisov link starting from X as in one of the five Hilbert series left in case* **(vi)**. *Then, the restriction to the variety Y_2 of the birational map Ψ_2 is a flip for Y_2.*

Proof. From [BKR12b] we see that the weights of M are neither in case **(a)** nor **(b)**. Thus, none of the ideal variables appears as a pure power in the equations of Y_2. The statement follows from the same reasoning contained in the proof of Theorem 2.3.8. □

The end of the link in this case is constituted by a conic bundle over a projective plane $\mathbb{P}^2$ defined by the coordinates y_2, y_3, y_4.

Proposition 2.3.30. *The map Φ' is a conic bundle over the projective plane $\mathbb{P}^2_{y_2,y_3,y_4}$.*

Proof. Localise $\mathbb{F}_3$ to the projective plane $\mathbb{P}^2(d_2, d_2, d_2)_{y_2,y_3,y_4}$. Recall that the variable s can be globally eliminated; this discards the four unprojection equations. We exclude s from the following expression of $\mathbb{F}_3$.

$$\left(\begin{array}{ccccc|ccc} t & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_2 & a & b & c & d_1 - d_2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

At the level of ambient spaces, $\mathbb{F}_3$ is a weighted $\mathbb{P}^4$-bundle over $\mathbb{P}^2$. Above each point of $\mathbb{P}^2(d_2, d_2, d_2)_{y_2,y_3,y_4}$ it is possible to locally eliminate two variables among t, x_1, x_2, x_3, y_1

via two of the pfaffian equations. The remaining three equations lie in the same principal ideal generated by one of them. Such equation is a conic in the three surviving variables of the fibre. The conic has coefficient in the base variables y_2, y_3, y_4. $\square$

2.3.6 Proof of (vii)

In this case there are no flips occurring in these Sarkisov links. They evolve as follows: ψ_1 is n simultaneous flops by Theorem 2.3.2, and it is followed by a divisorial contraction Φ' to a Fano 3-fold X' (as in Lemma 2.3.12). Localising $\mathbb{F}_2$ at $\mathbb{P}^3(d_1^3, d_4)$ having coordinates y_1, y_2, y_3, y_4 we have

$$\left(\begin{array}{ccccc|cccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_1 & r+d_1 & a & b & c & 0 & 0 & 0 & d_4 - d_1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

In particular, $d_4 - d_1$ is strictly negative. Practically speaking, this is the detail that makes Φ' a divisorial contraction and not a fibration.

2.3.7 Proof of (viii)

In case **(viii)** the first n flops are followed by a conic bundle over $\mathbb{P}^3(d_1, d_1, d_1, d_1)_{y_1,y_2,y_3,y_4}$. In this situation, a similar statement to the one of Proposition 2.3.30 holds, with an analogous proof.

All the links ending with conic bundles are summarised in Table 6.2.

2.4 Towards the analysis of Sarkisov links for Jerry

In this section we show how the techniques showed above for Tom change when M is in Jerry format. In particular, we discuss the shape of the toric variety $\mathbb{F}_1$ (see Proposition 2.4.1), and the behaviour of their Sarkisov links (see Theorem 2.4.5).

2.4.1 The blow-up for Jerry

The case in which the matrix M is in Jerry format does not always present the same phenomenon described in Proposition 2.2.6: this is because the unprojection equations do not always have a monomial only in the variables x_1, x_2, x_3.

Recall that a matrix in Jerry format, say $J_{i,j}$ to fix ideas, has a special entry μ_{ij} called *pivot*. Therefore we have a distinction into two sub-cases depending on whether the following condition is satisfied or not.

Condition 2.4.1. Let P be the degree of the pivot entry μ_{ij}. Consider the following statement:

There exists an ideal variable w of $\mathbb{F}$ such that $\deg(w) = P$.

Hence we have the following Proposition.

Proposition 2.4.1. *In the same hypothesis of Proposition 2.2.6, suppose the matrix M defining Z is in Jerry format.*

If Condition 2.4.1 holds, the bidegree of w is $\binom{P}{-2}$. Without loss of generality suppose w is y_4; then, the blow-up of X at P_s is contained in a scroll of the form

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \end{array}\right). \tag{2.22}$$

Note that the relevant weights are not necessarily in that order.

On the other hand, if Condition 2.4.1 does not hold, the scroll is of the form 2.5.

Analogously to the Tom case, to prove Proposition 2.4.1 we need the following lemma.

Lemma 2.4.2. *Let Z be a codimension 3 $\mathbb{Q}$-Fano 3-fold defined by pfaffians of a 5×5 skew-symmetric matrix M in Jerry format. Consider the Type I unprojection of Z at a divisor D. If Condition 2.4.1 holds, then there exists one unprojection equation that does not contain any monomial purely in x_1, x_2, x_3.*

On the other hand, if Condition 2.4.1 does not hold, then each unprojection equation has at least one monomial purely in x_1, x_2, x_3.

Proof. As in Section 3.8 of [Pap01], assume without loss of generality that M is of the form

$$M = \begin{pmatrix} e_1 & e_2 & b_3 & a_3 \\ & e_2 & b_2 & a_2 \\ & & b_1 & a_1 \\ & & & c \end{pmatrix}$$

where $e_i \notin I_D$, $a_i, b_i, c \in I_D$ are polynomials of degrees matching the gradings of M, and c occupies the pivot entry. Following Papadakis [Pap01], there exists a 3×4 matrix Q such that

$$\begin{pmatrix} \mathrm{Pf}_3(M) \\ \mathrm{Pf}_4(M) \\ \mathrm{Pf}_5(M) \end{pmatrix} = Q \begin{pmatrix} y_1 \\ \vdots \\ y_4 \end{pmatrix}$$

where the y_i are the generators of the ideal I_D and Q is defined as

$$\begin{aligned} Q_{1k} &:= \vartheta_{y_k}(c)e_1 - \vartheta_{y_k}(b_3)a_2 + \vartheta_{y_k}(b_2)a_3 \\ Q_{2k} &:= \vartheta_{y_k}(\mathrm{Pf}_4(M)) = \mathrm{Pf}_4(N_k) \\ Q_{3k} &:= \vartheta_{y_k}(\mathrm{Pf}_5(M)) = \mathrm{Pf}_5(N_k) \end{aligned} \tag{2.23}$$

where with $\vartheta_{y_k}(\cdot)$ we denote the coefficient of y_k in the polynomial in the argument and N_k are defined as in 2.8.

For $i = 1, \dots, 4$ call $h_i := \det \hat{Q}_i$ the four determinants of the 3×3 matrices obtained after deleting the i-th column of Q. Lemma 3.8.1 in [Pap01] shows that there exist polynomials K_i, L_i such that

$$h_i = e_1 K_i + (a_2 e_2 - e_3 a_3) L_i \qquad i = 1, \dots, 4 . \tag{2.24}$$

Define

$$g_i := K_i + a_1 L_i \qquad i = 1, \dots, 4 . \tag{2.25}$$

These are the right hand sides of the unprojection equations, that is $sy_i = g_i$.

We want to see in which cases the g_i have or not a monomial in the variables x_1, x_2, x_3. Definition 2.25 clearly shows that it is not possible to find it in the term $a_1 L_i$ for all i, since $a_i \in I_D$. On the other hand, there are hopes to find it in K_i. In order to do this, we need to look closer at the matrix Q.

Look first at the two bottom rows of Q.

From how we constructed M in Subsection 2.2.1, every ideal variable y_k occupies alone at least one entry of M. Hence, at least one entry of N_k is 1. Then, when computing the pfaffians defining the entries of the bottom rows of Q, we have that at least one monomial in each entry is in terms of x_1, x_2, x_3.

We now distinguish two cases, depending on whether condition 2.4.1 holds or not.

Suppose Condition 2.4.1 is satisfied. Therefore, the pivot entry is occupied by one of the ideal variables only; call it w to distinguish it. Explicitly, $c = w$. Thus, the last column of Q is the vector $(e_1, 0, 0)^T$. This implies that h_1, h_2, h_3 are divisible by e_1, so $K_i = \frac{h_i}{e_1}$ and $L_i = 0$ for $i = 1, 2, 3$.

Look now at the top row of Q. As already discussed, $Q_{14} = e_1 \notin I_D$. For $k = 1, 2, 3$, $\vartheta_{y_k}(c) = 0$, and at least one between $\vartheta_{y_k}(b_3)$ and $\vartheta_{y_k}(b_2)$ is equal to 1, because at least one entry of N_k is 1. Therefore, from the definition of Q_{1k} we deduce that $Q_{11}, Q_{12}, Q_{13} \in I_D$, so h_4 does not contain any monomial only in therms of x_1, x_2, x_3, thus neither does K_4.

Now suppose Condition 2.4.1 does not hold. The polynomial c in the pivot entry is now a general polynomial of degree P on which we perform row/column operations in order to simplify it by getting rid of some terms. Such operations must not break the *Jerry* format, namely, make monomials not in I_D appear in the ideal entries; in particular, this happens when using monomials not in I_D as coefficients of the row/column operations. This means it is not possible to get rid of the terms in c of the form $\mu \cdot \nu$ with $\mu \notin I_D$, $\nu \in I_D$. Therefore, when calculating the entries of the top row of Q, we have that they have at least one term in x_1, x_2, x_3, coming from the coefficients of c. $\square$

Proof of Proposition 2.4.1. Suppose that Condition 2.4.1 holds. Lemma 2.4.2 shows that the unprojection equation $sy_4 = g_4$ fails to have a monomial only in terms of x_1, x_2, x_3. Thus, the minimum in Definition 2.2.1 has to be achieved at a monomial containing at least one ideal variable, as g_4 only contains monomials of such sort. This means that the degree of g_4 is strictly less than δ_4, i.e. there exists an integer coefficient $\nu \in \mathbb{Z}$ such that

$$\delta_4 = \deg(g_4) + \nu r \ .$$

In fact, ν measures the least number of ideal variables (with multiplicity) appearing in the monomials of g_4. In order to prove that the bidegree of y_4 is $\binom{d_4}{-2}$ we need to show that $\nu = 1$. We need to look at the matrix Q. As in the proof of Lemma 2.4.2, the bottom rows of Q all contain at least one monomial in terms of x_1, x_2, x_3.

We want to show that there is one entry of the top row of Q having at least one monomial linear in the y_k. Surely, $\theta_{y_k}(c) = 0$ for $k = 1, 2, 3$. Moreover, for each $k = 1, 2, 3$ there exists $j \in \{1, 2, 3\}$ such that $\theta_{y_k}(b_j) = 1$. Each term of a_j contains at least one relevant variable. As proved before, the two bottom rows of the matrix Q contain at least one monomial in x_1, x_2, x_3 in each entry. Therefore, we want to show that there exists at least one term in one of the first three entries of the top row of Q having exactly one ideal variable with multiplicity 1. Such monomial certainly does not appear in the term $\vartheta_{y_k}(c)e_1$ of 2.23 since $e_1 \notin I_D$ and $\vartheta_{y_k}(c)$ is 1 if $k = 4$ and is 0 otherwise. On the other hand, there exist $k \in \{1, 2, 3\}$ and $z \in \{1, 2\}$ such that $\vartheta_{y_k}(b_j) = 1$. Moreover, up to a change of coordinates a_2 and a_3 contain a term that is exactly one of the ideal variables. $\square$

Remark 2.4.3. The pivot entry always vanishes twice on the divisor D. This means that whichever polynomial is in the pivot entry it has to vanish on D with order two. Thus, the -2 in the bidegree of w can be interpreted as the order of vanishing of w on D.

Remark 2.4.4. We can reformulate Proposition 2.4.1 stating the following.

The unprojection equation correspondent to w is

$$sw = g(x_1, x_2, x_3, y_1, y_2, y_3, y_4) \ .$$

If $g|_{\{y_j=0\}_1^4} = 0$, then the bottom degree of w is -2.
If $g|_{\{y_j=0\}_1^4} \neq 0$, then the bottom degree of w is -1.

2.4.2 Description of the link for Jerry

The classification of Sarkisov links in the Jerry case is determined also by the condition 2.4.1.

Theorem 2.4.5. *In the same hypotheses and notation of Theorem 2.1.1, suppose the matrix M is in Jerry format. Let X be a codimension 4 $\mathbb{Q}$-Fano 3-fold obtained as Type I unprojection of Z at a divisor D. Suppose X has Fano index 1 and Picard rank $\rho_X = 1$.*

The first step of the Sarkisov link run on X is a flop as in Lemma 2.3.2. Moreover, if condition 2.4.1 holds, the first flip of the link on $w\mathbb{P}^7$ does not affect the variety. In other words, the link for the Fano has an empty step.

Lastly, the Sarkisov link run on X does not break.

In the Tom case the shape of the scroll $\mathbb{F}$ suggests at first glance whether either fibrations or simultaneous flips could occur or not by looking at the relevant top weights. On the other hand in the Jerry case, if there exists a variable of $\mathbb{F}$ such that it generates the same linear system as w this could lead to fibrations or simultaneous flips even when the relevant top weights of $\mathbb{F}$ are all different. This makes the treatment of the Jerry case very difficult to systematise, as every specific example looks different from the others.

Theorems 2.1.1 and 2.4.5 both imply the following:

Theorem 2.4.6. *Let X be a Fano 3-fold in codimension 4 in either the hypotheses of theorem 2.1.1 or of Theorem 2.4.5. Then, X is not birationally rigid.*

Chapter 3

Examples of Tom and Jerry links

In this section we present several explicit examples of Sarkisov links for codimension 4 Fano 3-folds of Tom type, highlighting the main phenomena described in Theorem 2.1.1.

3.1 Tom examples

3.1.1 Example for (i): #10985, Tom_1

In this subsection we examine the Sarkisov link constructed from the pair (X, p) where X is the Tom type Fano 3-fold associated to the Hilbert series #10985 and $p \in X$ is the Tom centre $\frac{1}{2}(1,1,1)$. The Tom centre is chosen among the basket of singularities of X shown in the [BK$^+$15], which is $\mathcal{B}_X = \{\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)\}$. The ambient space of X is $\mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$, with coordinates x_1, x_2, x_3, s, y_4, y_3, y_2, y_1 respectively. The divisor D is $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1)$, defined by the ideal $I_D = \langle y_1, y_2, y_3, y_4 \rangle$. If the matrix M is in Tom_1 format, then $D \subset Z_1$.

In [BKR12b] we see that the nodes on D are 24, and that the weights of M are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \hline & 3 & 4 & 5 \\ & & 5 & 6 \\ & & & 7 \end{pmatrix}. \tag{3.1}$$

To summarise, we are looking at the following varieties:

#10985	X	$\subset \mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$	codimension 4	$\{\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)\}$
#10962	Z_1	$\subset \mathbb{P}^6(1^3, 3, 4, 5, 6)$	codimension 3	24 nodes on D

We fill the entries of M with linear terms as much as possible: the more detailed explanation of this process is in Subsection 2.2.1. This means that we aim to put ideal variables in an ideal entry having their same weight, and do analogously for the orbinates. The rest of the entries can be occupied by general polynomials in the given degrees, accordingly to the Tom_1 constraints. These polynomials can be eventually slimmed up by performing row/column operations as explained in 1.2.4. In this specific case, we end up with the following explicit matrix

$$M = \left(\begin{array}{cccc} x_1 & -x_2x_3 & -x_2^3 + y_4 & -x_3^4 + y_3 \\ \hline & y_4 & y_3 & y_2 \\ & & x_2^2 y_4 - y_2 & y_1 \\ & & & x_1^4 y_4 \end{array} \right). \tag{3.2}$$

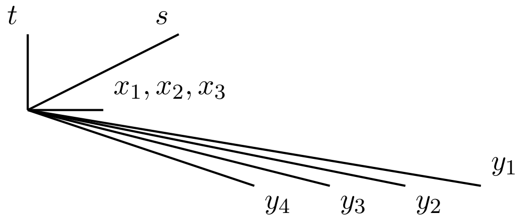
In particular, Z_1 has 24 nodes. The matrix M is built following what we explained in Section 2.2.1. The unprojection algorithm produces nine equations, defining X, as outlined in 1.2.4. Explicitly, the equations of X are

$$\begin{cases} x_1x_2^2y_4 - x_2^3y_4 + x_2x_3y_3 + y_4^2 - x_1y_2 = 0 \\ x_3^4y_4 - x_2x_3y_2 - y_4y_3 - x_1y_1 = 0 \\ x_1^5y_4 - x_3^4y_3 + x_2^3y_2 + y_3^2 - y_4y_2 = 0 \\ x_1^4x_2x_3y_4 + x_2^3x_3y_2 - x_3^4y_2 + x_1x_2^2y_1 - x_2^3y_1 + y_3y_2 + y_4y_1 = 0 \\ x_1^4y_4^2 + x_2^2y_4y_2 - y_2^2 - y_3y_1 = 0 \\ -x_2^4x_3 + x_1x_3^4 - x_1y_3 + y_4s = 0 \\ -x_1^6 - x_1x_2^5 + x_2^6 - x_2^3y_4 + x_1y_2 - y_3s = 0 \\ x_1^5x_2x_3 + x_1x_2^2x_3^4 - x_2^3x_3^4 - x_1x_2^2y_3 + x_2^3y_3 + x_1y_1 + y_2s = 0 \\ x_1^4x_2^2x_3^2 + x_2^3x_3^5 - x_3^8 + x_1^5y_4 - x_2^3x_3y_3 + x_3^4y_3 + x_2^3y_2 + x_2x_3y_1 - y_4y_2 - y_1s = 0 \end{cases} \tag{3.3}$$

Proposition 2.2.6 shows that the blow-up Y_1 of X at the Tom centre $p = P_s$ is contained in the rank 2 toric variety $\mathbb{F}_1$ with weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 2 & 1 & 1 & 1 & 6 & 5 & 4 & 3 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array} \right). \tag{3.4}$$

The Mori cone of $\mathbb{F}_1$ is given by the linear systems defined by the variables $t, s, x_1, x_2, x_3, y_4, y_3, y_2, y_1$, that is, $\mathbb{F}_1$ is associated to a fan generated by the lattice vectors ρ_t, ρ_s, ρ_{x_1}, ρ_{x_2}, $\rho_{x_3}, \rho_{y_1}, \rho_{y_2}$, ρ_{y_3}, ρ_{y_4} respectively. This defines a ray-chamber structure that will describe the link at the level of the rank 2 toric varieties $\mathbb{F}_i$.



The Kawamata blow-up of the Tom centre P_s is induced by the map Φ

$$
\begin{aligned}
\Phi \colon \mathbb{F}_1 &\longrightarrow \mathbb{P}^7(1^3,2,3,4,5,6) \\
(t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1t^{\frac{1}{2}}, x_2t^{\frac{1}{2}}, x_3t^{\frac{1}{2}}, y_4t^{\frac{5}{2}}, y_3t^{\frac{6}{2}}, y_2t^{\frac{7}{2}}, y_1t^{\frac{8}{2}}, s)
\end{aligned}. \tag{3.5}
$$

Consider the pull-back of the equations 3.3 of X. The ideal of Y_1 is defined as the saturation over t of the ideal of $\Phi^*(X)$, as in Definition 2.2.2.

Explicitly, after saturation we have the equations for Y_1

$$
\begin{cases}
-tx_1y_3 + sy_4 + x_1x_3^4 - x_2^4x_3 = 0 \\
ty_4^2 + x_1x_2^2y_4 - x_1y_2 - x_2^3y_4 + x_2x_3y_3 = 0 \\
tx_1y_2 - tx_2^3y_4 - sy_3 - x_1^6 - x_1x_2^5 + x_2^6 = 0 \\
-ty_4y_3 - x_1y_1 - x_2x_3y_2 + x_3^4y_4 = 0 \\
-tx_1x_2^2y_3 + tx_1y_1 + tx_2^3y_3 + sy_2 + x_1^5x_2x_3 + x_1x_2^2x_3^4 - x_2^3x_3^4 = 0 \\
-ty_4y_2 + ty_3^2 + x_1^5y_4 + x_2^3y_2 - tx_3^4y_3 = 0 \\
-t^2y_4y_2 + tx_1^5y_4 - tx_2^3x_3y_3 + tx_2^3y_2 + tx_2x_3y_1 + tx_3^4y_3 - sy_1 + x_1^4x_2^2x_3^2 + x_2^3x_3^5 - x_3^8 = 0 \\
ty_4y_1 + ty_3y_2 + x_1^4x_2x_3y_4 + x_1x_2^2y_1 + x_2^3x_3y_2 - x_2^3y_1 - x_3^4y_2 = 0 \\
x_1^4y_4^2 + x_2^2y_4y_2 - y_3y_1 - y_2^2 = 0
\end{cases} \tag{3.6}
$$

The birational link for $w\mathbb{P}^7$ is obtained performing a variation of the GIT quotient on $\mathbb{F}_1$, as outlined in Chapter 2.

From Theorem 2.3.2 we have that the map Ψ_1 is given by 24 simultaneous flops based at the 24 nodes of Z_1. Such flops arise when crossing the wall associated to the variables x_1, x_2, x_3, that is, they arise from transitioning from a chamber adjacent to the lattice vector ρ_{x_i} to the other adjacent chamber. This is obtained by changing the irrelevant ideal of $\mathbb{F}_1$.

Note that since the weights (3.1) of M are in configuration **(B)**, then either ψ_2 or ψ_3 is an isomorphism by Proposition 2.3.10. In particular, by looking at the equations 3.6 of Y_1 we notice that y_2 appears as a pure power: this implies that ψ_3 is an isomorphism.

In order to study ψ_2 we need to localise at $P_{y_1} \in Z_2$. This means that we look at the equations 3.6 locally analytically in a neighbourhood of the point $P_{y_1} \in Z_2$.

Practically, we treat y_1 as a local coordinate, so we perform row operations on $\mathbb{F}_2$ in order to write the weight of y_1 as either $\binom{\pm 1}{0}$ or $\binom{0}{\pm 1}$. To do so, we add six times the second row to the first row of 3.4: the grading of $\mathbb{F}_2$ becomes

$$\left(\begin{array}{ccccc|cccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 6 & 8 & 1 & 1 & 1 & 0 & -1 & -2 & -3 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

Recall from [BZ10] that the weights of the flip at the level of the rank 2 toric variety is $(6, 8, 1, 1, 1, -1, -2, -3)$. This notation stands for the contraction by α_2 of $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(6, 8, 1, 1, 1)$ to the point $P_{y_1} \in Z_2$, and the extraction by β_2 of $\mathbb{P}^2_{y_2,y_3,y_4}(1, 2, 3)$ from P_{y_1}. However, the intersection $\mathbb{P}^4_{t,s,x_2,x_3}(6, 8, 1, 1, 1) \cap Y_2$ can be a much smaller projective space than $\mathbb{P}^4$. Analogously, this might hold for $\mathbb{P}^1_{y_2,y_4}(1, 2, 3) \cap Y_3$. We can understand those intersections, and deduce the weights of the flip for Y_2, by using the following argument.

Every isomorphism in codimension 1 Ψ_i is based at a point (or a projective line) in $\mathbb{G}_i$. Localising at such a point (or at the points constituting the intersection of the projective line with Z_i), and using the equations of Y_i it is possible to write some of the variables as function of the others.

Examining the equations of Y_2 locally analytically at a neighbourhood of $P_{y_1} \in Z_2$ and considering y_1 as a local coordinate, we can set $y_1 = 1$ in the equations 3.6. Some linear monomials will emerge in the equations of Y_2 evaluated at $y_1 = 1$: those variables appearing linearly in $Y_2\big|_{y_1=1}$ can be expressed in terms of the other variables locally analytically. Thus, we can locally *eliminate* them. In this specific case, the evaluation of 3.6 at $y_1 = 1$ shows that s, x_1, y_3 appear linearly. Therefore, the weights of the flip for Y_2 are $(6, 1, 1, -1, -3)$, associated to the variables t, x_2, x_3, y_2, y_4 respectively.

Observe that it looks like that α_2 contracts a 2-dimensional locus inside Y_2 to the point P_{y_1}, thus α_2 does not seem like a small contraction, as required in flips. However, among the equations left after the local elimination process there is one involving t and y_4: that is $\mathrm{Pf}_2 = 0$. This means that there is an equation cutting out the contracted locus by one dimension.

In conclusion, ψ_2 is a flip having weights $(6, 1, 1, -1, -3; 3)$, where the last 3 in this notation tracks the degree of the equation involving the monomial ty_4. In other words, a weighted projective space $\mathbb{P}_{t,x_2,x_3}(6, 1, 1)$ containing a hypersurface of degree 3 with coefficients in $\mathbb{P}_{y_2,y_4}(1, 3)$ is flipped to $\mathbb{P}_{y_2,y_4}(1, 3)$. In particular, a $\frac{1}{6}(1, 1, 5)$ singularity in Y_2 is contracted to P_{y_1} via α_2, and a $\frac{1}{3}(1, 1, 2)$ is extracted in Y_3 from P_{y_1} via β_2.

This is a *hypersurface flip*. Despite the fact that there are three surviving equations after the elimination process, the equation cutting out $\mathbb{P}_{t,x_2,x_3}(6, 1, 1)$ is only one: the other two are multiples of it. This means that Pf_2 is the generator of the principal ideal of Y_2 on $\mathbb{P}_{t,x_2,x_3}(6,1,1)$.

As already mentioned, the map Ψ_3 based at P_{y_2} certainly defines a flip from $\mathbb{F}_3$ to $\mathbb{F}_4$, but one of the equations of Y_3 contains the monomial y_2^2, that is, P_{y_2} does not belong to Z_3. Thus, Y_3 is not affected by this flip. We call this phenomenon an *empty step* of the Sarkisov link.

The last map of the link is $\Phi' \colon \mathbb{F}_4 \to \mathbb{G}_4$. This is the map constituted by the basis of the linear system $\binom{4}{-1}$, which contracts the exceptional locus $\mathbb{E}' = \{y_4 = 0\}$ to the point $P_{y_3} \in \mathbb{G}_4$. Explicitly, it is

$$\begin{aligned} \Phi' \colon \mathbb{F}_4 &\longrightarrow \mathbb{G}_4 = \mathbb{P}^7(1,1,1,1,2,3,3,5) \\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1y_4, x_2y_4, x_3y_4, y_3, y_2y_4, y_1y_4^2, ty_4^3, sy_4^6)\ . \end{aligned} \tag{3.7}$$

The exceptional locus $\mathbb{E}'$ is isomorphic to $\mathbb{P}^7(4,6,1,1,1,2,1)$ with coordinates t,s,x_1,x_2,x_3,y_1,y_2 respectively: their weights are retrieved performing a localisation at P_{y_3}, in the same fashion as above. However, the intersection $\mathbb{E}' \cap Y_4$ is $\mathbb{P}^3(1,1,1,1)$, as we can eliminate the variables t,s,y_1 locally analytically in a neighbourhood of P_{y_3}.

We call X' the push-forward $\Phi'_*(Y_4)$ of Y_4 via Φ'. Practically, y_4 plays the role for Φ' that t played for Φ, being the extra variable needed to perform a blow-up: in this case, Φ' blows up the point $P_{y_3} \in X'$. The equations of X' are therefore given by evaluating the equations of Y_4 at $y_4 = 1$. Observe that this shows that the variables t and s can be algebraically expressed as functions of the other variables: two equations of $Y_4\big|_{y_4=1}$ are removed in order to perform this global elimination.

Call ς_i for $i \in \{1,\dots,8\}$ the coordinates of $\mathbb{G}_4$: the equations of X' are expressed in these coordinates. Since we globally eliminated two variables thanks to the equations of X', we deduce that $X' \subset w\mathbb{P}' \subset \mathbb{G}_4$, where $w\mathbb{P}' := \mathbb{P}^5(1,1,1,1,2,3)$ with coordinates $\varsigma_1,\dots,\varsigma_6$. So, Φ' restricts to $\phi' \colon Y_4 \to X' \subset \mathbb{P}^5(1,1,1,1,2,3)$.

If we consider the minimal basis of the ideal generated by the surviving equations of $Y_4\big|_{y_4=1}$ we have that the explicit equations of X' are

$$\begin{cases} \varsigma_1\varsigma_2^2\varsigma_4 - \varsigma_1\varsigma_4\varsigma_5 - \varsigma_1\varsigma_6 - \varsigma_2^3\varsigma_4 + \varsigma_2\varsigma_3\varsigma_4^2 - \varsigma_2\varsigma_3\varsigma_5 + \varsigma_3^4 = 0 \\ \varsigma_1^4 + \varsigma_2^2\varsigma_5 - \varsigma_4\varsigma_6 - \varsigma_5^2 = 0 \end{cases} \tag{3.8}$$

Note that the above equations both have degree 4 in $w\mathbb{P}'$.

In addition, it is possible to keep track of the singularities throughout the link. That is: X has $\frac{1}{2}(1,1,1)$ and $\frac{1}{6}(1,1,5)$ singularities. After the blowup Φ, Y_1 has only a singularity of type $\frac{1}{6}$: this holds for Y_2 too, as the basket does not change after the flops. The hypersurface flip Ψ_2 replaces $\frac{1}{6}(1,1,5)$ with $\frac{1}{3}(1,1,2)$, so Y_3 has one singularity of type $\frac{1}{3}$; same for Y_4, given that Y_3 and Y_4 are actually isomorphic. Lastly, ϕ' contracts a smooth locus, so the $\frac{1}{3}$ singularity of Y_4 is maintained in X'.

Now that we know the equations of X' and their degrees, the basket of X and its ambient space we deduce that X' is a representative of the family #16204 in [BK$^+$15], which is a Fano 3-fold in codimension 2.

Remark 3.1.1. Note that the Sarkisov link described above is of Type IV according to the notation in [HM13].

3.1.2 Example for (v): #20652, Tom_1, case (a)

Consider the pair (X, p) where X is the Tom type Fano 3-fold associated to the Hilbert series #20652 and $p \in X$ is the Tom centre $\frac{1}{2}(1,1,1)$.

The Tom centre is chosen among the basket of singularities of X shown in the [BK$^+$15], which is $\mathcal{B}_X = \{3 \times \frac{1}{2}(1,1,1)\}$. The ambient space of X is $\mathbb{P}^7(1^5, 2^3)$, with coordinates $y_1, y_2, x_1, x_2, x_3, y_3, y_4, s$ respectively. The divisor D is $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1)$, defined by the ideal $I_D = \langle y_1, y_2, y_3, y_4 \rangle$. The matrix M is in Tom_1 format, and $D \subset Z_1$.

The nodes on D are 7, and the weights of M are

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline & 2 & 2 & 2 \\ & & 2 & 2 \\ & & & 2 \end{pmatrix}. \tag{3.9}$$

Concisely, we are looking at the following varieties:

#20652	X	$\subset \mathbb{P}^7(1^5, 2^3)$	codimension 4	$\{3 \times \frac{1}{2}(1,1,1)\}$
#20543	Z_1	$\subset \mathbb{P}^6(1^5, 2^2)$	codimension 3	7 nodes on D

In a similar fashion to the previous example, we can construct the matrix M in Tom_1 format. For #20543 it is

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 \\ \hline & y_1 & y_2 & x_2y_4 - x_3y_3 + y_1 \\ & & x_1y_3 - y_2 & y_4^2 - y_2 \\ & & & x_1y_3 + y_1 \end{pmatrix}. \tag{3.10}$$

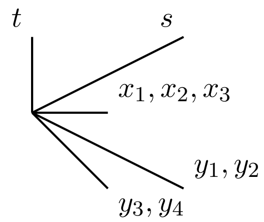
The nine unprojection equations defining X are

$$\begin{cases} x_3^2y_3 - x_2x_3y_4 + x_1y_1 - 2x_3y_1 + x_1y_2 + x_2y_2 + y_3y_2 = 0 \\ x_2x_3y_3 - x_2^2y_4 + x_1y_4^2 - x_2y_1 + y_3y_1 - x_1y_2 = 0 \\ x_1x_2y_3 + x_1y_3^2 - x_3y_4^2 + x_2y_1 + x_3y_2 - y_3y_2 = 0 \\ x_1^2y_3 + x_3y_1 - x_1y_2 - x_2y_2 = 0 \\ -x_1x_2^2 - x_2^3 - x_2x_3^2 + x_1^2y_4 + x_1x_2y_4 - x_1x_3y_4 - x_2x_3y_4 + y_3s = 0 \\ -x_1^3 + x_1^2x_2 + x_1x_2^2 + 2x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 \\ +x_3^3 + x_1^2y_3 + x_1x_2y_3 + x_1y_1 + x_2y_1 - x_1y_2 - x_2y_2 + x_3y_2 - y_4s = 0 \\ x_1x_2^2y_4 - x_1y_3^2y_4 - x_1^2y_4^2 - x_3^2y_4^2 + x_3y_4^3 + x_1x_2y_1 + x_2x_3y_1 + x_1y_3y_1 \\ -x_2y_4y_1 + x_1^2y_2 + x_3^2y2 - x_2y_4y_2 - x_3y_4y_2 + y_3y_4y_2 - y_4^2y_2 + y_1^2 - y_1y_2 + y_2^2 = 0 \\ x_1^2x_2^2 - x_1x_2^2x_3 + x_1^2x_2y_3 - x_1x_2x_3y_3 - x_1^3y_4 + 2x_1^2x_3y_4 + x_3^3y_4 \\ +x_2x_3y_1 - x_1x_2y_2 - x_2^2y_2 + x_2x_3y_2 + x_3y_4y_2 - y_2s = 0 \\ x_1^2x_2^2 + x_1x_2^3 + x_1^2x_2x_3 + x_1^2x_2y_3 + x_1x_2^2y_3 + x_1^2x_3y_3 - x_1^3y_4 - x_1^2x_2y_4 - x_1x_3^2y_4 \\ -x_2x_3^2y_4 + x_3^2y_1 - x_1x_2y_2 - x_2^2y_2 - x_1x_3y_2 - x_2x_3y_2 - x_1y_4y_2 - x_2y_4y_2 + y_1s = 0 \end{cases} \tag{3.11}$$

From Proposition 2.2.6 we have that Y_1 sits inside a rank 2 toric variety $\mathbb{F}_1$ having weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right). \tag{3.12}$$

This time, the Mori cone of $\mathbb{F}_1$ is given by the following fan



The Kawamata blow-up of the Tom centre P_s is the map Φ

$$\begin{aligned} \Phi\colon \mathbb{F}_1 &\longrightarrow \mathbb{P}^7(1^5, 2^3) \\ (t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) &\longmapsto (x_1t^{\frac{1}{2}}, x_2t^{\frac{1}{2}}, x_3t^{\frac{1}{2}}, y_4t^{\frac{3}{2}}, y_3t^{\frac{3}{2}}, y_2t^{\frac{4}{2}}, y_1t^{\frac{4}{2}}, s) \end{aligned}. \tag{3.13}$$

The expression of Φ having integer exponents of t is

$$\begin{array}{c}\Phi\colon \mathbb{F}_1 \longrightarrow \mathbb{P}^7(1^5,2^3)\\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) \longmapsto (x_1t,x_2t,x_3t,y_4t^2,y_3t^2,y_2t^3,y_1t^3,st)\end{array}. \tag{3.14}$$

Therefore, the equations of Y_1 are

$$\begin{cases} x_3^2y_3 - x_2x_3y_4 + x_1y_1 - 2x_3y_1 + x_1y_2 + x_2y_2 + ty_3y_2 = 0\\ x_2x_3y_3 - x_2^2y_4 + tx_1y_4^2 - x_2y_1 + ty_3y_1 - x_1y_2 = 0\\ x_1x_2y_3 + tx_1y_3^2 - tx_3y_4^2 + x_2y_1 + x_3y_2 - ty_3y_2 = 0\\ x_1^2y_3 + x_3y_1 - x_1y_2 - x_2y_2 = 0\\ -x_1x_2^2 - x_2^3 - x_2x_3^2 + tx_1^2y_4 + tx_1x_2y_4 - tx_1x_3y_4 - tx_2x_3y_4 + y_3s = 0\\ -x_1^3 + x_1^2x_2 + x_1x_2^2 + 2x_1^2x_3 + x_1x_2x_3 + x_2^2x_3+\\ x_3^3 + tx_1^2y_3 + tx_1x_2y_3 + tx_1y_1 + tx_2y_1 - tx_1y_2 - tx_2y_2 + tx_3y_2 - y_4s = 0\\ x_1x_2^2y_4 - t^2x_1y_3^2y_4 - tx_1^2y_4^2 - tx_3^2y_4^2 + t^2x_3y_4^3 + x_1x_2y_1 + x_2x_3y_1 + tx_1y_3y_1\\ -tx_2y_4y_1 + x_1^2y_2 + x_3^2y_2 - tx_2y_4y_2 - tx_3y_4y_2 + t^2y_3y_4y_2 - t^2y_4^2y_2 + ty_1^2 - ty_1y_2 + ty_2^2 = 0\\ x_1^2x_2^2 - x_1x_2^2x_3 + tx_1^2x_2y_3 - tx_1x_2x_3y_3 - tx_1^3y_4 + 2tx_1^2x_3y_4 + tx_3^3y_4+\\ tx_2x_3y_1 - tx_1x_2y_2 - tx_2^2y_2 + tx_2x_3y_2 + t^2x_3y_4y_2 - y_2s = 0\\ x_1^2x_2^2 + x_1x_2^3 + x_1^2x_2x_3 + tx_1^2x_2y_3 + tx_1x_2^2y_3 + tx_1^2x_3y_3 - tx_1^3y_4 - tx_1^2x_2y_4 - tx_1x_3^2y_4\\ -tx_2x_3^2y_4 + tx_3^2y_1 - tx_1x_2y_2 - tx_2^2y_2 - tx_1x_3y_2 - tx_2x_3y_2 - tx_1y_4y_2 - tx_2y_4y_2 + y_1s = 0 \end{cases} \tag{3.15}$$

The variation of GIT on $\mathbb{F}_1$ will give the 2-ray game.

Theorem 2.3.2 guarantees that Ψ_1 is given by 7 simultaneous flops based at the 7 nodes of Z_1. In terms of the ray-chamber structure of the fan of $\mathbb{F}_1$, we are crossing the first ray ρ_{x_i} for $i \in \{1,2,3\}$.

Observe that the weights 3.9 of M are in configuration **(a)**: from Proposition 2.3.18 we know that there is a quadratic form determining two points $P_1, P_2 \in Z_2$, constituting the intersection $Z_2 \cap \mathbb{P}^1_{y_1,y_2}$. Thus, Lemma 2.3.19 shows that the pencil of flips along the line $\mathbb{P}^1_{y_1,y_2} \subset \mathbb{G}_2$ restricts to two flips with base P_1 and P_2 respectively. So we look locally analytically in a neighbourhood of $P_1, P_2 \in Z_2$. Carrying out the same manipulation of $\mathbb{F}_2$ done in the previous example, we have that the grading of $\mathbb{F}_2$ is

$$\left(\begin{array}{ccccc|cccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4\\ 2 & 4 & 1 & 1 & 1 & 0 & 0 & -1 & -1\\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

The weights of the flip of rank 2 toric varieties based at $\mathbb{P}^1_{y_1,y_2}$ are $(2,4,1,1,1,-1,-1)$, where α_2 contracts $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(2,4,1,1,1)$ to $\mathbb{P}^1_{y_1,y_2}$, and β_2 extracts $\mathbb{P}^1_{y_3,y_4}$.

Now we look at the equations 3.15 of Y_1 locally analytically at a neighbourhood of P_1 and P_2 respectively, in order to understand the intersections $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(2,4,1,1,1) \cap Y_2$ and $\mathbb{P}^1_{y_3,y_4} \cap Y_2$.

We see that equation #9 and equation #8 of 3.15 make the variable s to be expressed in terms of the other variables at P_1 and P_2 respectively: therefore we say that s is eliminated algebraically at P_1 and P_2. Similarly happens for x_1 using equation #1 of 3.15. On the other hand, we can use either equation #2 to eliminate x_2 at P_1, or equation #3 to eliminate x_3 at P_2. We see that the intersection $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(2,4,1,1,1) \cap Y_2$ is formed by two disjoint loci, generated by t, x_2 and t, x_3 at P_1 and P_2 respectively. Nonetheless, they determine two projective lines $\mathbb{P}^1(2,1)$. The fact that this elimination process has not excluded y_3 nor y_4 shows that $\mathbb{P}^1_{y_3,y_4} \subset Y_2$.

Note that the variable t does not get eliminated. This is because in equation #7 of 3.15 the polynomial $t\left(y_1^2 - y_1y_2 + y_2^2\right)$ appears: the variable t could be eliminated only if $y_1^2 - y_1y_2 + y_2^2 \neq 0$, but P_1 and P_2 are exactly the two solutions of $y_1^2 - y_1y_2 + y_2^2 = 0$.

In conclusion, Ψ_2 restricts to two simultaneous Francia flips $(2,1,-1,-1)$ based at $P_1, P_2 \in Z_2$, as anticipated in Remark 2.1.2. In particular, two cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$ in Y_2 are contracted to P_1 and P_2 respectively via α_2, and β_2 extracts a smooth locus in Y_3. Therefore, Y_3 is a manifold having Picard rank 2.

The last map of the link is the fibration $\Phi' \colon \mathbb{F}_4 \to \mathbb{P}^1_{y_3,y_4}$. Recall that $-K_{Y_3} \sim \mathcal{O}\left(\binom{1}{0}\right)$. If F is a general fibre of Φ', then by adjunction we have that $K_F = (K_{Y_3} + F)\big|_F = K_{Y_3}\big|_F$. Thus, K_F is ample, F a del Pezzo and, as a consequence, Φ' a del Pezzo fibration.

Note that the unprojection variable s can be globally eliminated over each general point of $\mathbb{P}^1_{y_3,y_4}$. There is no other elimination that can be made. Therefore, the fibre F of the del Pezzo fibration sits inside a projective space $\mathbb{P}^6$ with coordinates $t, x_1, x_2, x_3, y_1, y_2$. The matrix M has become a matrix of linear forms in these variables. The equations of F are the five (quadratic) maximal pfaffians of M. Therefore, the degree of F, and of the del Pezzo fibration, is 5.

3.1.3 Example for (iv): #574, Tom$_1$, case (b)

Consider the pair (X, p) where X is the Tom type Fano 3-fold associated to the Hilbert series #574 and $p \in X$ is the Tom centre $\frac{1}{7}(1,3,4)$.

The basket of singularities of X shown in the [BK+15] is $\mathcal{B}_X = \{\frac{1}{3}(1,1,2), \frac{1}{5}(1,1,4), \frac{1}{5}(1,2,3), \frac{1}{7}(1,3,4)\}$. The ambient space of X is $\mathbb{P}^7(1,3,4,5^2,6,7^2)$, with coordinates $x_1, x_2, x_3,\ y_4, y_3, y_2, y_1, s$ respectively. The divisor D is $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,3,4)$, and the matrix M is in Tom_1 format, whose weights are

$$\begin{pmatrix} 3 & 4 & 5 & 6 \\ \hline & 5 & 6 & 7 \\ & & 7 & 8 \\ & & & 9 \end{pmatrix}. \tag{3.16}$$

There are 8 nodes on D. In short, we are looking at:

#574	X	$\subset \mathbb{P}^7(1,3,4,5^2,6,7^2)$	codimension 4	$\{\frac{1}{3}(1,1,2), \frac{1}{5}(1,1,4), \frac{1}{5}(1,2,3), \frac{1}{7}(1,3,4)\}$
#568	Z_1	$\subset \mathbb{P}^6(1,3,4,5^2,6,7)$	codimension 3	8 nodes on D.

Construct the matrix M in Tom_1 format as follows

$$M = \begin{pmatrix} x_2 & x_3 & -x_1^5 + y_4 & -x_2^5 - x_1 y_4 + y_2 \\ \hline & y_3 & y_2 & y_1 \\ & & -x_1 y_2 - y_1 & -x_2 y_3 - x_2 y_4 \\ & & & -x_1^3 y_2 + x_3 y_3 + x_3 y_4 - x_2 y_2 \end{pmatrix}. \tag{3.17}$$

Thus, the nine unprojection equations defining X are

$$\begin{cases} x_1^5 y_3 + x_1 x_2 y_2 - y_3 y_4 + x_3 y_2 + x_2 y_1 = 0 \\ x_2^2 y_3 + \frac{1}{2} x_2^2 y_4 + \frac{1}{2} x_1 y_3 y_4 - \frac{1}{2} y_3 y_2 + \frac{1}{2} x_3 y_1 = 0 \\ x_1^5 y_1 - x_1^3 x_2 y_2 + x_2 x_3 y_3 + x_2 x_3 y_4 - 2x_2^2 y_2 - x_1 y_4 y_2 + y_2^2 - y_4 y_1 = 0 \\ -x_1^6 x_2^2 - x_1^5 x_2 x_3 - x_1^3 x_2^3 - 2x_2^4 + x_1 x_2 x_3^2 + x_3^3 + x_2 x_3 y_4 + x_2^2 y_2 + y_3 s = 0 \\ x_1^8 x_3 - 2x_1^6 x_2^2 - x_1^7 y_4 + x_1^6 y_2 - 2x_1^3 x_2^3 - x_1^4 x_2 y_4 + x_1^5 y_1 \\ -x_1^3 x_3 y_4 - 4x_2^4 + x_1 x_2 x_3^2 - 2x_1 x_2^2 y_4 + x_3^3 + x_2 x_3 y_3 + x_2 x_3 y_4 + 2x_2^2 y_2 - y_4 y_1 - y_4 s = 0 \\ x_1^5 x_2 y_4 + x_1^3 x_3 y_2 - 2x_1 x_2^2 y_2 - x_1^2 y_4 y_2 - x_3^2 y_3 - x_3^2 y_4 - x_2 y_4^2 + x_1 y_2^2 - 2x_2^2 y_1 - x_1 y_4 y_1 + y_2 y_1 = 0 \\ x_1^{10} x_2 - x_1^5 x_3^2 - x_1^5 x_2 y_4 - x_1^6 y_1 + x_1^4 x_2 y_2 + x_1^3 x_3 y_2 + x_2^3 x_3 \\ -x_1 x_2 x_3 y_3 - x_3^2 y_3 - x_2 x_3 y_2 - 2x_2^2 y_1 + y_2 y_1 + y_2 s = 0 \\ x_1^3 y_3 y_2 - x_3 y_3^2 - x_3 y_3 y_4 - x_2 y_4 y_2 + x_1 y_2 y_1 + y_1^2 = 0 \\ -x_1^8 x_2^2 - 2x_1^5 x_2^3 - x_1^7 y_1 + 2x_1^5 x_2 y_2 + x_1^3 x_2^2 y_4 + x_1^4 x_3 y_2 + x_1 x_2^3 x_3 - x_1^2 x_2 x_3 y_3 + x_2^2 x_3^2 \\ -x_1 x_3^2 y_3 + 2x_2^3 y_4 - x_1 x_2 x_3 y_2 - 2x_1 x_2^2 y_1 - x_3^2 y_2 - x_2 y_4 y_2 + x_1 y_2 y_1 - y_1 s = 0 \end{cases} \tag{3.18}$$

The rank 2 toric variety $\mathbb{F}_1$ has weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 2 & 1 & 3 & 4 & 7 & 6 & 5 & 5 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right). \tag{3.19}$$

The Kawamata blow-up of the Tom centre P_s is the map Φ

$$\begin{aligned} \Phi\colon \mathbb{F}_1 &\longrightarrow \mathbb{P}^7(1^5,2^3) \\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1t^{\frac{1}{7}}, x_2t^{\frac{3}{7}}, x_3t^{\frac{4}{7}}, y_4t^{\frac{12}{7}}, y_3t^{\frac{12}{7}}, y_2t^{\frac{13}{7}}, y_1t^{\frac{14}{7}}, s) \end{aligned}, \tag{3.20}$$

which is equivalent to

$$\begin{aligned} \Phi\colon \mathbb{F}_1 &\longrightarrow \mathbb{P}^7(1^5,2^3) \\ (t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1t, x_2t^3, x_3t^4, y_4t^6, y_3t^6, y_2t^7, y_1t^8, st^6) \end{aligned}. \tag{3.21}$$

The equations of Y_1 are

$$\begin{cases} x_1^5y_3 + x_1x_2y_2 - ty_3y_4 + x_3y_2 + x_2y_1 = 0 \\ x_2^2y_3 + \frac{1}{2}x_2^2y_4 + \frac{1}{2}tx_1y_3y_4 - \frac{1}{2}ty_3y_2 + \frac{1}{2}x_3y_1 = 0 \\ x_1^5y_1 - x_1^3x_2y_2 + x_2x_3y_3 + x_2x_3y_4 - 2x_2^2y_2 - tx_1y_4y_2 + ty_2^2 - ty_4y_1 = 0 \\ -x_1^6x_2^2 - x_1^5x_2x_3 - x_1^3x_2^3 - 2x_2^4 + x_1x_2x_3^2 + x_3^3 + tx_2x_3y_4 + tx_2^2y_2 + y_3s = 0 \\ x_1^8x_3 - 2x_1^6x_2^2 - tx_1^7y_4 + tx_1^6y_2 - 2x_1^3x_2^3 - tx_1^4x_2y_4 + tx_1^5y_1 - tx_1^3x_3y_4 \\ -4x_2^4 + x_1x_2x_3^2 - 2tx_1x_2^2y_4 + x_3^3 + tx_2x_3y_3 + tx_2x_3y_4 + 2tx_2^2y_2 - t^2y_4y_1 - y_4s = 0 \\ x_1^5x_2y_4 + x_1^3x_3y_2 - 2x_1x_2^2y_2 - tx_1^2y_4y_2 - x_3^2y_3 - x_3^2y_4 - tx_2y_4^2 + tx_1y_2^2 - 2x_2^2y_1 - tx_1y_4y_1 + ty_2y_1 = 0 \\ x_1^{10}x_2 - x_1^5x_3^2 - tx_1^5x_2y_4 - tx_1^6y_1 + tx_1^4x_2y_2 + tx_1^3x_3y_2 + x_2^3x_3 - tx_1x_2x_3y_3 \\ -tx_3^2y_3 - tx_2x_3y_2 - 2tx_2^2y_1 + t^2y_2y_1 + y_2s = 0 \\ x_1^3y_3y_2 - x_3y_3^2 - x_3y_3y_4 - x_2y_4y_2 + x_1y_2y_1 + y_1^2 = 0 \\ -x_1^8x_2^2 - 2x_1^5x_2^3 - tx_1^7y_1 + 2tx_1^5x_2y_2 + tx_1^3x_2^2y_4 + tx_1^4x_3y_2 + x_1x_2^3x_3 - tx_1^2x_2x_3y_3 + x_2^2x_3^2 \\ -tx_1x_3^2y_3 + 2tx_2^3y_4 - x_1x_2x_3y_2 - 2tx_1x_2^2y_1 - tx_3^2y_2 - t^2x_2y_4y_2 + tx_1y_2y_1 - y_1s = 0 \end{cases} \tag{3.22}$$

From Theorem 2.3.2 we have that Ψ_1 is given by 7 simultaneous flops based at the 7 nodes of Z_1.

The restriction of the map Ψ_2 to Y_2 is an isomorphism, by Theorem 2.3.9. The map Ψ_3 is instead an hypersurface flip, having weights $(1,3,1,-1,-1;5)$. Here a hypersurface (a curve) of degree 5 in $\mathbb{P}^2_{x_1,x_2,y_1}$ with coefficients in the variables y_3 and y_4 is flipped to $\mathbb{P}^1_{y_3,y_4}$; it has degree 5 because it contains the monomial $x_2^2y_3$ coming from Pf_3.

The final map Φ' is a del Pezzo fibration over $\mathbb{P}^1_{y_3,y_4}$ (Theorem 2.3.22).

The scroll $\mathbb{F}_4$ localised at $\mathbb{P}^1_{y_3,y_4}$ is

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 5 & 7 & 1 & 3 & 4 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right). \tag{3.23}$$

Remark 3.1.2. Note that x_3 appears only in the entries m_{13} and m_{45} of M. Moreover, by Proposition 2.2.10 we have that t does not appear in the entries m_{12}, m_{13}, and m_{ij} of $\alpha_1^*(M)$ for $i, j > 1$.

As a consequence, Pf_1 eliminates x_3, and Pf_5 eliminates t over the function field $k(y)$, for $y := \frac{y_3}{y_4}$. These two variables are both eliminated globally. In addition, s is also eliminated globally thanks to the unprojection equations.

Therefore, the general fibre of Φ' sits inside the $\mathbb{P}^3(1,3,2,1)$ having coordinates respectively x_1, x_2, y_1, y_2. In particular, the eliminated variables t and x_3 have weight 5 and 4 respectively in the general fibre of Φ'.

In this weighted projective space, the surviving pfaffian equations Pf_2, Pf_3, and Pf_4 have degree 8, 6, and 8. Since, from Theorem 2.3.22, the fibre of Φ' intersected with Y_4 is a smooth surface S, then S is defined by the degree 6 Pf_3: so $S = V_6 \subset \mathbb{P}^3(1,3,2,1)$. Therefore, it is a del Pezzo surface of degree 1 (cf [Isk77]).

3.1.4 Example for (vi): #16227, Tom_2

Let X be the Tom type Fano 3-fold associated to the Hilbert series #16227 and $p \in X$ be the Tom centre $\frac{1}{5}(1,2,3)$.

The basket of singularities of X shown in the [BK+15] is $\mathcal{B}_X = \{\frac{1}{5}(1,2,3)\}$. The ambient space of X is $\mathbb{P}^7(1^4,2^2,3,5)$, with coordinates $x_1, y_4, y_3, y_2, y_1, x_2, x_3, s$ respectively. The divisor D is $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,2,3)$, and the matrix M is in Tom_2 format, with weights

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ & 2 & 2 & 3 \\ & & 2 & 3 \\ & & & 3 \end{pmatrix}. \tag{3.24}$$

There are 4 nodes on D. We focus on the following varieties.

#16227	X	$\subset \mathbb{P}^7(1^4,2^2,3,5)$	codimension 4	$\{\frac{1}{5}(1,2,3)\}$
#16226	Z_1	$\subset \mathbb{P}^6(1^4,2^2,3)$	codimension 3	4 nodes on D

The matrix M in Tom_2 format is

$$M = \begin{pmatrix} x_1 & y_2 & y_3 & y_1 \\ & yx_2 & y_3^2 + x_2 & x_3 \\ & & y_1 & -x_1^2 y_3 - y_4^3 - x_2 y_4 \\ & & & -x_1^2 y_2 - y_2^3 + y_3^3 + y_4^3 \end{pmatrix}. \tag{3.25}$$

After performing the unprojection at $D \cong \mathbb{P}^2(1,2,3)$, we blow up X at the Type

I centre $P_s \in X$ of type $\frac{1}{5}(1,2,3)$. The equations for Y_1 are therefore

$$\begin{cases} t^2y_2y_3^2 - x_1y_1 + x_2y_2 - x_2y_3 = 0 \\ x_1^3y_3 + t^2x_1y_4^3 + x_1x_2y_4 - x_2y_1 + x_3y_2 = 0 \\ x_1^2y_2^2 + t^2y_2^4 - x_1^2y_3^2 - t^2y_2y_4^3 - t^2y_3y_4^3 - x_1y_1y_3 + x_2y_2y_3 - x_2y_3^2 - x_2y_3y_4 - y_1^2 = 0 \\ x_1^3y_2 + t^2x_1y_2^3 - t^2x_1y_3^3 - t^2x_1y_4^3 - t^2y_1y_3^2 - x_2y_1 + x_3y_3 = 0 \\ t^2x_1^2y_3^3 + t^4y_3^2y_4^3 - x_1^2x_2y_2 - t^2x_2y_2^3 + x_1^2x_2y_3 + t^2x_2y_3^3 + t^2x_2y_3^2y_4 + t^22x_2y_4^3 + x_2^2y_4 + x_3y_1 = 0 \\ -tx_1^4y_4^2 + t^3x_1^2y_3^2y_4^2 + tx_1^2x_2y_3^2 + t^3x_1y_1y_3y_4^2 + tx_1x_2y_1y_3 \\ -2tx_2^2y_4^2 - tx_1x_3y_4^2 - x_2^3 - x_1x_2x_3 - y_2s = 0 \\ -tx_1^4y_4^2 - t^3x_1^2y_2^2y_4^2 - x_1^4x_2 - tx_1^2x_2y_2^2 + 2t^3x_2y_3^2y_4^2 + tx_2^2y_3^2 \\ +2tx_2^2y_4^2 - tx_1x_3y_4^2 + x_2^3 + y_3s = 0 \\ -x_1^6 - x_1^4y_2^2 + x_1^2y_3^4 + y_3^3y_4^3 + 2x_1^2x_2y_3^2 + x_2y_3^4 + x_2y_3^3y_4 + x_2y_3y_4^3 \\ -x_2^2y_2^2 + x_2^2y_3^2 - x_1x_3y_3^2 + x_2^2y_3y_4 + x_3^2 - y_4s = 0 \\ -x_1^3y_2y_3y_4^2 - x_1y_2^3y_3y_4^2 + x_1^3y_3^2y_4^2 + x_1y_3y_4^5 - x_1^3x_2y_2y_3 - x_1x_2y_2^3y_3 + 2x_1^3x_2y_4^2 \\ +x_1x_2y_2^2y_4^2 - x_1x_2y_3^2y_4^2 + x_1x_2y_3y_4^3 + x_1^3x_2^2 + x_1x_2^2y_2^2 - x_1x_2^2y_3^2 + 2x_2x_3y_4^2 + x_2^2x_3 + y_1s = 0 \end{cases}$$

in the rank 2 toric variety $\mathbb{F}_1$ having weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 5 & 1 & 2 & 3 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

The map ψ_1 is formed of four simultaneous flops, whereas the map ψ_2 is an isomorphism on Y_2, as in Theorem 2.3.9.

At the level of toric varieties, the map ψ_3 is a fibre bundle $\psi_3 \colon \mathbb{F}_3 \to \mathbb{P}^2(1,1,1)_{y_2,y_3,y_4}$. We are interested in studying the fibres of such bundle.

We see that, locally at general points in $\mathbb{P}^2(1,1,1)_{y_2,y_3,y_4}$, it is possible to globally eliminate the following variables: s from the unprojection equations, x_2 from Pf_5, x_3 from Pf_3.

Over the general point in $\mathbb{P}^2(1,1,1)_{y_2,y_3,y_4}$ there is a conic in the remaining variables t, x_1, y_1 given by Pf_2. This is a quadratic form defined by the 3×3 matrix A in y_2, y_3, y_4

$$\begin{pmatrix} y_2^4 - y_2y_4^3 - y_3y_4^3 + y_2^2y_3^3 - y_2y_3^4 - y_2y_3^3y_4 & 0 & 0 \\ 0 & y_2^2 - y_3^2 & -\frac{1}{2}(y_3 + y_2y_3 + y_3^2 + y_3y_4) \\ 0 & -\frac{1}{2}(y_3 + y_2y_3 + y_3^2 + y_3y_4) & -1 \end{pmatrix}$$

and the explicit quadratic form is given by

$$\begin{pmatrix} t & x_1 & y_1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} t \\ x_1 \\ y_1 \end{pmatrix} = 0\,.$$

The conic bundle we obtain sits inside the $\mathbb{P}^2$-bundle over $\mathbb{P}^2(1,1,1)_{y_2,y_3,y_4}$

$$\left(\begin{array}{ccc|ccc} t & x_1 & y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 1 & 1 \end{array}\right).$$

We want to compute the discriminant Δ of this conic bundle. The fibre at a general point of $\mathbb{P}^2(1,1,1)_{y_2,y_3,y_4}$ contributes 6 to the discriminant. This is because the degree of the determinant of A is 6. Therefore, $\Delta \geq 6$.

The behaviour of the special fibres of ψ_3 determines the exact discriminant. Subdivide the base of the conic bundle in two affine patches: $\{y_2 = 0\}$ and $\{y_2 \neq 0\}$. At $\{y_2 \neq 0\}$ the contribution to the discriminant is 6, as we explained above. At $\{y_2 = 0\}$ instead, the fibre is singular, therefore this contributes by 1 to the discriminant. Therefore, $\Delta = 7$.

3.1.5 Example for (i): #511, Tom_4. The basket of X'

Consider the Tom type Fano 3-fold X associated to the Hilbert series #511 and $p \in X$ is the Tom centre $\frac{1}{14}(1,3,11)$.

The basket of singularities of X shown in the [BK+15] is $\mathcal{B}_X = \{\frac{1}{6}(1,1,5), \frac{1}{14}(1,3,11)\}$. The ambient space of X is $\mathbb{P}^7(1,3,5,6,7,8,11,14)$, with coordinates $x_1, x_2, y_4, y_3, y_2, y_1, x_3, s$ respectively. The divisor D is $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,3,11)$, and the matrix M is in Tom_4 format, with weights

$$\begin{pmatrix} 5 & 6 & 7 & 8 \\ & 7 & 8 & 9 \\ & & 9 & 10 \\ & & & 11 \end{pmatrix}.$$

There are 7 nodes on D. We focus on the following varieties.

#511	X	$\subset \mathbb{P}^7(1,3,5,6,7,8,11,14)$	codimension 4	$\{\frac{1}{6}(1,1,5), \frac{1}{14}(1,3,11)\}$
#510	Z_1	$\subset \mathbb{P}^6(1,3,5,6,7,8,11)$	codimension 3	7 nodes on D

The first 7 flops of ψ_1 are followed by the hypersurface flip with weights $(1,3,11,-2,-3;9)$ of ψ_2. Then, ψ_3 is the flip $(1,3,-1,-2)$, and $\Phi'\colon Y_4 \to X'$ is a divisorial contraction to a point in X'. In a similar fashion to the example for case **(i)** in Section 3.1.1, the ambient space of X' is $\mathbb{P}' = \mathbb{P}^4(1^2, 2, 3^2)$. Therefore, $X' = X_9 \subset \mathbb{P}' = \mathbb{P}^4(1^2, 2, 3^2)$ is the Fano hypersurface corresponding to the Graded Ring Database ID #5257. The basket of #5257 is $\{\frac{1}{2}(1,1,1), 3 \times \frac{1}{3}(1,1,2)\}$.

Let us now track how the basket of X changes along the link. The blow-up Φ gets rid of the $\frac{1}{14}$ singularity, and produces two new singularities, of index 3 and 11. Hence, the basket of Y_1 is $\mathcal{B}_{Y_1} = \{\frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5), \frac{1}{11}(1,3,8)\}$. The basket of Y_2 is identical to the one of Y_1, because the flops do not modify the basket.

The coordinates of the $(1,3,11,-2,-3;9)$ hypersurface flip are x_1, x_2, x_3, y_3, y_4 respectively, and there is an equation $f_9 = 0$ of degree 9 relating them to one another. A closer look to such equation reveals the behaviour of the singularities at this step. The polynomial f_9 we are after is $\mathrm{Pf}_2(M)$, which surely contains monomials such as x_3y_3 and x_2^3. In particular, the equation $f_9 = 0$ is of the form $x_3y_3 = x_2^3 + x_1^9$.

The presence of the x_2^3 monomial implies that the $\frac{1}{3}(1,1,2)$ at the point P_{x_2} of the locus contracted by α_2 is not being contracted in the variety, because P_{x_2} does not satisfy the equation $f_9 = 0$. Moreover, a qG-deformation of the $\frac{1}{3}(1,1,2)$ singularity at the point P_{y_4} shows that there are three $\frac{1}{3}(1,1,2)$ singularities instead of one, again because of the x_2^3 monomial in f_9.

By *qG-deformation* we mean a flat 1-parameter deformation $\mathcal{X} \to \Delta$ such that the total space $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein.

In conclusion, while the $\frac{1}{3}(1,1,2)$ singularity at P_{x_2} remains untouched in the ψ_2 flip, the $\frac{1}{11}(1,3,8)$ singularity at P_{x_3} is traded for a $\frac{1}{2}(1,1,1)$ singularity and $3\times\frac{1}{3}(1,1,2)$ singularities. Therefore, the basket of Y_3 is $\mathcal{B}_{Y_3} = \{\frac{1}{2}(1,1,1), (1{+}3)\times\frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5)\}$.

The flip given by ψ_3 is a toric flip $(1,3,-1,-2)$, so the singularities indicated in the right-hand side of the flip are the actual contracted singularities, and same for the left-hand side. Hence, $\mathcal{B}_{Y_4} = \{2 \times \frac{1}{2}(1,1,1), 3 \times \frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5)\}$.

Finally, Φ' contracts the divisor $\mathbb{E}' = \mathbb{P}^3(6,1,2,1)$ to a point in $X' \subset \mathbb{P}^4(1^2, 2, 3^2)$. Therefore, the basket of X' is $\mathcal{B}_{X'} = \{\frac{1}{2}(1,1,1), 3 \times \frac{1}{3}(1,1,2)\}$. This corresponds to the basket of singularities of #5257.

3.2 A Jerry example

We give here one detailed example of a Jerry construction. For more on this, we refer to the following Section 3.3.

In the same fashion as in the previous examples, consider the two Fanos

#10985	X	$\subset \mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$	codimension 4	$\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)$
#10986	Z_1	$\subset \mathbb{P}^6(1^3, 3, 4, 5, 6)$	codimension 3	26 nodes

where, as before, X is obtained by unprojecting Z_1 from a divisor $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1)$. Call the variables of $w\mathbb{P}^7$ as $x_1, x_2, x_3, y, z, u, v$, and the ideal $I_D = \langle y, z, u, v \rangle$.

This time Z_1 is defined by the pfaffians of a matrix M in Jerry$_{45}$ of weights

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & 3 & 4 & 5 \\ & & 5 & 6 \\ & & & 7 \end{pmatrix}.$$

The explicit matrix is

$$M = \begin{pmatrix} x_1 & -x_3^3 & y & z \\ & -x_2^3 + y & x_3 y + z & u \\ & & x_3^2 y + u & v \\ & & & x_3^4 y + x_1^3 z - x_2^2 u - x_3 v \end{pmatrix} \tag{3.26}$$

whose pivot entry is m_{45} and has degree 7. Therefore X is given by the five pfaffian equations of Z_1 and four unprojection equations, that is

$$\begin{cases} tx_1z + sy + x_1^5 - x_1x_2^3x_3 + x_1x_2^2x_3^2 - x_3^5 = 0 \\ -ty^2 + x_1x_3^2y - x_1u + x_2^3y - x_3^3y - x_3^2z = 0 \\ tx_1x_2^2y + 2tx_1x_3z - tx_1u - tx_3^3y - sz - x_1^2x_2^2x_3^2 + x_1^2x_3^4 - x_1x_2^5 + x_1x_2^2x_3^3 + x_1x_3^5 + x_2^3x_3^3 - x_3^6 = 0 \\ -tyz - x_1v + x_2^3z - x_3^2u = 0 \\ -tx_1^4y + tx_1x_3^2z + tx_1v + tx_2^3x_3y + tx_3^3z + su + x_1^5x_3^2 + x_1^4x_2^3 - x_1^4x_3^3 - x_1x_2^3x_3^3 + x_1x_3^6 - x_2^6x_3 + x_2^3x_3^4 = 0 \\ tx_3yz - tyu + tz^2 + x_1^4z - x_1x_2^2u + x_1x_3^4y - x_1x_3v = 0 \\ t^2yu - tx_1^3x_3^2y + tx_1x_2^2u - tx_1x_3^4y + 2tx_1x_3v + tx_2^5y - tx_2^3u + tx_2^2x_3^2z + tx_3^4z + tx_3^3u + tx_3^2v - sv + x_1^4x_3^4 \\ +x_1^3x_2^3x_3^2 - x_1^3x_3^5 - x_1x_2^5x_3^2 + x_1x_2^3x_3^4 - x_2^8 + x_2^5x_3^3 + x_3^8 = 0 \\ tx_3^2yz + tyv - tzu + x_1^3x_3^2z - x_2^2x_3^2u + x_3^6y - x_3^3v = 0 \\ tx_2^2yu + tx_3^2z^2 + tx_3yv + tx_3zu + tzv - tu^2 + x_1^4x_3^2z + x_1^4v - x_1^3x_3^3z + x_1^3x_3^2u - x_1x_2^2x_3^2u + x_1x_3^4u \\ -x_1x_3^3v - x_2^5u - x_2^3x_3v + x_2^2x_3^3u + x_3^6z + x_3^4v = 0 \end{cases} \tag{3.27}$$

Note that the condition 2.4.1 is not fulfilled in this case. Thus, by Theorem 2.4.1 we have that the blowup of X sits inside a toric variety of rank two having weights as in 2.5. Analogously to the Tom examples above, perform the variation of GIT quotient on $\mathbb{F}_1$ and the localisation process.

Again, Theorem 2.3.2 ensures that Ψ_1 is 26 flops.

The map Ψ_1 having base at $P_v \in Z_2$ is a $(6,1,1,-1)$ divisorial contraction, contracting a weighted $\mathbb{P}_{t,x_2,x_3}(6,1,1)$ to P_v. Localising first at P_u and then at P_z show that both Ψ_2 and Ψ_3 do not affect the varieties Y_2 and Y_3 respectively; this is because both u and z appear as pure squares in the equations, i.e. they do not belong to Z_2 and Z_3 respectively.

The map Φ' is

$$\begin{aligned} \Phi' \colon \mathbb{F}_4 &\longrightarrow \mathbb{P}^7(1^4, 2, 3) =: \mathbb{P}' \\ (t, s, x_1, x_2, x_3, v, u, z, y) &\longmapsto (x_1 y, x_2 y, x_3 y, z, uy, vy^2) \end{aligned} \tag{3.28}$$

Therefore the equation for X' are

$$\begin{cases} x_2^3 z - x_3^2 u - x_1 v - z(x_2^3 + x_1 x_3^2 - x_3^3 - x_3^2 z - x_1 u) = 0 \\ x_1 x_3^4 + x_1^4 z - x_1 x_2^2 u - x_1 x_3 v + (x_3 z + z^2 - u)(x_2^3 + x_1 x_3^2 - x_3^3 - x_3^2 z - x_1 u) = 0 \\ x_3^6 + x_1^3 x_3^2 z - x_2^2 x_3^2 u - x_3^3 v + (x_3^2 z - zu + v)(x_2^3 + x_1 x_3^2 - x_3^3 - x_3^2 z - x_1 u) = 0 \end{cases} \tag{3.29}$$

Note that they have degrees 4,5,6 in $\mathbb{P}'$.

Moreover, the blow up Y_1 of X at $\frac{1}{2}(1,1,1)$ has only a singularity of type $\frac{1}{6}(1,1,5)$; the same holds for Y_2. Therefore Φ' contracts $\frac{1}{6}(1,1,5)$ to a smooth point.

Hence, this proves that the endpoint of the link is X' #16204 sitting inside $\mathbb{P}^5(1^4, 2, 3)$.

3.3 Comparison with Takagi

In [Tak02], the author classifies all the possible extremal contractions Φ' appearing in sequences of flops and flips on $\mathbb{Q}$-factorial terminal Fano 3-folds Y of Picard rank $\rho_Y = 2$. We refer to the set-up in §3 of [Tak02]: what Takagi is explaining is a Sarkisov link starting from certain $\mathbb{Q}$-Fano 3-folds X with Picard rank 1 enjoying some additional properties (cf. "Main Assumption 0.1" of [Tak02]). In particular, these varieties are asked to have a singularity of type $\frac{1}{2}(1,1,1)$, that is blown up to initiate the sequence of birational transformations.

Six of the varieties falling in Takagi's assumption are in codimension 4 and have a Type I centre. In particular, three of them are of Tom-type, and follow the description of Theorem 2.1.1. They are: #24097 Tom_1 (number 4.4 in Takagi's paper) falling in case $d_1 = d_2 = d_3 < d_4$, #20652 Tom_1 (number 5.4) in case $d_1 = d_2 < d_3 = d_4$, and #16645 Tom_1 (number 2.2) in case $d_1 < d_2 = d_3 = d_4$.

We examine them here with our method, and show that the outcomes predicted by Theorem 2.1.1 match his results.

The remaining three Hilbert series indicated by Takagi are of Jerry type. We study them separately and compare them with Takagi's results.

#16645, Tom$_1$ Consider $X \subset \mathbb{P}(1^4, 2^4)$ with coordinates $x_1, x_2, x_3, y_4, y_1, y_2, y_3, s$ obtained unprojecting Z_1 #16338 in Tom$_1$ format at $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1)$. The basket of X is $\mathcal{B}_X = \{4 \times \frac{1}{2}(1,1,1)\}$. The matrix M defining Z_1 is

$$M = \begin{pmatrix} x_1 & x_2 & y_2 + y_3 & y_2 + x_3^2 + x_1 x_2 \\ & y_4 & y_1 & y_2 + y_3 \\ & & y_3 & y_1 + y_3 \\ & & & x_1 y_1 + x_2 y_3 + x_3 y_2 + y_4^3 \end{pmatrix}.$$

Start the Sarkisov link by blowing up one $\frac{1}{2}$ singularity; after 8 simultaneous flops we have a divisorial contraction $\Phi' \colon \mathbb{F}_2 \to \mathbb{G}_2 = \mathbb{P}^7(1^7, 3)$ with exceptional divisor $\mathbb{E}' := \{y_4 = 0\}$. On the other hand, $w\mathbb{P}' = \mathbb{P}^6$ is a smooth projective space. The intersection $\mathbb{E}' \cap Y_2$ is a conic $\Gamma := \{y_1^2 + y_1 y_3 + y_2 y_3 = 0\}$. In particular, Φ' contracts all the cyclic quotient singularities in the basket of Y_2. Therefore, Y_2 is contracted to a smooth $X' \subset \mathbb{P}^6$ #26988 in codimension 3.

This matches with what summarised by Takagi in Table 2 of [Tak02], No. 2.2 because the variety A_8 pinpointed by Takagi is exactly #26988.

#20652, Tom$_1$ As showed in Example 3.1.2, the end of the link is a del Pezzo fibration of degree 5. This complies with Table 5 of [Tak02], No. 5.4.

#24097, Tom$_1$ Consider the pair (X, p) where $X \subset \mathbb{P}^7(1^6, 2^2)$ is the Tom type Fano 3-fold associated to the Hilbert series #24097, and $p \in X$ is the Tom centre $\frac{1}{2}(1,1,1)$.

The coordinates of $\mathbb{P}^7(1^6, 2^2)$ are $x_1, x_2, x_3,\ y_2, y_3, y_4, y_1,\ s$ respectively. The unprojection of the divisor $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1) \subset Z_1$ in Tom$_1$ format produces X. Here Z_1 is #24077, and is defined by the five pfaffians of the matrix M

$$M = \left(\begin{array}{cccc} x_1 & x_2 & x_3 & -y_2^2 - x_3 y_3 \\ \hline & y_2 & y_3 & y_1 \\ & & y_4 & x_1 y_3 - y_4^2 \\ & & & -x_2 y_4 - x_3 y_4 + y_1 \end{array}\right).$$

There are 8 nodes on D.

The blow-up of $\mathbb{P}^7(1^6, 2^2)$ at P_s is the rank 2 toric variety $\mathbb{F}_1$ having weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right). \tag{3.30}$$

After the 8 simultaneous flops given by Ψ_1, the map Ψ_2 is a Francia flip $(2, 1, -1, -1)$.

The map Φ' is a weighted $\mathbb{P}^5$-bundle over the projective space $\mathbb{P}^2_{y_2,y_3,y_4}(1,1,1)$. We show that Y_3 is actually a conic bundle over that base. Note that Y_3 is smooth: therefore, referring to Section 3.1.4, we just need to compute the degree of the determinant of the matrix A in order to find the discriminant Δ.

We record here the five equations of Y_3 that originated from the pfaffians equations of Z_1. They are

$$\begin{cases} x_1y_3^2 + x_2y_2y_4 + x_2y_3y_4 - x_1y_4^2 - ty_3y_4^2 - y_2y_1 - y_4y_1 = 0 \\ x_1x_3y_3 + x_2^2y_4 + x_2x_3y_4 + t^2y_2^2y_4 + tx_3y_3y_4 - tx_3y_4^2 - x_2y_1 = 0 \\ t^2y_2^2y_3 + tx_3y_3^2 + x_1x_2y_4 + x_1x_3y_4 - x_1y_1 + x_3y_1 = 0 \\ t^2y_2^3 - x_1^2y_3 + tx_2y_3^2 - tx_1y_3y_4 + tx_1y_4^2 + x_2y_1 = 0 \\ x_3y_2 - x_2y_3 + x_1y_4 = 0 \end{cases}$$

At a general point in $\mathbb{P}^2_{y_2,y_3,y_4}(1,1,1)$, it is possible to globally eliminate the variables s thanks to the unprojection equations.

Now consider the line $\{y_4 = 0\}$ in the base $\mathbb{P}^2_{y_2,y_3,y_4}(1,1,1)$, and let us look at its two affine patches $\{y_2 \neq 0\}$ and $\{y_3 \neq 0\}$. We want to study the conic equations above each of these patches: in fact, they both contribute to the discriminant Δ.

Over the patch $\{y_2 \neq 0\}$, Pf_5 and Pf_1 globally eliminate the variables x_3 and y_1 respectively: hence they are $x_3 = x_2y_3$ and $y_1 = x_1y_3^2$. Replace their expressions in the remaining three pfaffian equations, obtaining

$$\begin{cases} t^2y_3 + tx_2y_3^3 - x_1^2y_3^2 + x_2x_1y_3^3 = 0 \\ x_1x_2y_3^2 - x_2x_1y_3^2 = 0 \\ t^2 - x_1^2y_3 + tx_2y_3^2 + x_2x_1y_3^2 = 0 \end{cases}$$

where Pf_2 is identically zero, and Pf_3 (above) is a multiple of Pf_4 by a y_3 factor. Therefore, the conic that Pf_4 describes is defined by the matrix

$$A_{y_2} = \begin{pmatrix} 1 & 0 & \frac{1}{2}y_3^2 \\ 0 & -y_3 & \frac{1}{2}y_3^2 \\ \frac{1}{2}y_3^2 & \frac{1}{2}y_3^2 & 0 \end{pmatrix}$$

as

$$\begin{pmatrix} t & x_1 & x_2 \end{pmatrix} \cdot A_{y_2} \cdot \begin{pmatrix} t \\ x_1 \\ x_2 \end{pmatrix} = 0\,.$$

The determinant $\det(A_{y_2}) = -\frac{1}{4}y_3^4(1+y_3)$.

On the other hand, over the patch $\{y_3 \neq 0\}$, Pf_1 and Pf_5 globally eliminate the variables x_1 and x_2 respectively: hence they are $x_1 = y_2 y_1$ and $x_2 = x_3 y_2$. Replace their expressions in the remaining three pfaffian equations: in a similar fashion to the other patch, the equation of the conic is $t^2 y_2^2 + t x_3 - y_2 y_1^2 + x_3 y_1 = 0$ given by Pf_3. It is defined by the matrix A_{y_3}

$$A_{y_3} = \begin{pmatrix} y_2^2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -y_2 \end{pmatrix}$$

and by the equation

$$\begin{pmatrix} t & x_3 & y_1 \end{pmatrix} \cdot A_{y_3} \cdot \begin{pmatrix} t \\ x_3 \\ y_1 \end{pmatrix} = 0\,.$$

The determinant $\det(A_{y_3}) = -\frac{1}{4} y_2(1 + y_2)$.

Even though the contribution of $\det(A_{y_2})$ and $\det(A_{y_3})$ to the discriminant might look like $5 + 2 = 7$, the solutions to $\det(A_{y_2}) = 0$ and $\det(A_{y_3}) = 0$ overlap at the point $(-1, -1, 0)$ which is counted twice. Therefore, $\Delta = 5 + 7 - 1 = 6$.

The map ϕ' is a conic bundle over the projective space $\mathbb{P}^2_{y_2,y_3,y_4}(1,1,1)$ discriminant $\Delta = 6$. This agrees with Table 4, No. 4.4 of [Tak02].

#16645, Jerry$_{45}$ Let (X, p) be the pair in which $X \subset \mathbb{P}^7(1^4, 2^4)$ is the Jerry type Fano 3-fold modelled on the Hilbert series #16645, and $p \in X$ is the Jerry centre $\frac{1}{2}(1,1,1)$. Name the coordinates of $\mathbb{P}^7(1^4, 2^4)$ $x_1, x_2, x_3, y_4, y_1, y_2, y_3, s$ respectively. The Fano 3-fold X is obtained via unprojection of the divisor $D \cong \mathbb{P}^2(1,1,1)_{x_1,x_2,x_3} \subset Z_1$ in Jerry$_{45}$. Here Z_1 is #16338, and is defined by the five pfaffians of the matrix M

$$M = \begin{pmatrix} x_1 & x_2 & y_1 & x_3 y_4 - y_3 \\ & x_3 & y_2 & x_3 y_4 + y_1 + y_2 \\ & & y_3 & y_2 + y_3 \\ & & & -y_4^3 + x_1 y_1 + x_3 y_2 - x_2 y_3 \end{pmatrix}$$

and there are 9 nodes on D.

The blow-up of X at the centre $p = P_s$ is contained in the rank 2 toric variety $\mathbb{F}_1$

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

In this case, the condition 2.4.1 is not satisfied, so $\mathbb{F}_1$ has the same shape that it has in the Tom case. After the 9 simultaneous flops given by ψ_1, the Sarkisov link presents a divisorial contraction Φ' to a point in the smooth projective space $\mathbb{P}^5$. In particular, Y_2 is contracted to the codimension 2 Fano 3-fold $X' = X_{2,3} \subset \mathbb{P}^5$ with Hilbert series #24076. Following Takagi's notation at the beginning of [Tak02], we have that X' is the smooth Fano 3-fold of type A_{10}.

This shows that X is No. 3.3 of Table 3, in [Tak02].

#20652, Jerry$_{23}$ Let $X \subset \mathbb{P}^7(1^5, 2^3)$ be the Jerry type Fano 3-fold associated to the Hilbert series #20652, and $p \in X$ be the centre $\frac{1}{2}(1,1,1)$. The coordinates of $\mathbb{P}^7(1^5, 2^3)$ are $x_1, x_2, x_3,\ y_3, y_4, y_1, y_2,\ s$ respectively. The unprojection of the divisor $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1) \subset Z_1$ in Jerry$_{2,3}$ format produces X. Here Z_1 is #20543, and is defined by the five pfaffians of the matrix M

$$M = \begin{pmatrix} y_3 & y_4 & x_1 & x_2 \\ & y_2 & y_1 & x_1y_4 \\ & & y_3^2 - x_1y_4 - y_1 & y_1 + y_2 \\ & & & -x_1^2 - x_2^2 + x_3^2 + y_4^2 \end{pmatrix}.$$

There are 8 nodes on D.

Note that the condition 2.4.1 is satisfied: without loss of generality, we assumed that the variable y_2 occupies the pivot entry m_{23} of M. Therefore, by Theorem 2.4.1, we have that the blow-up of $\mathbb{P}^7(1^5, 2^3)$ at P_s is the rank 2 toric variety $\mathbb{F}_1$ having weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 \end{array}\right).$$

The first map Ψ_1 of the birational link for X is given by 8 simultaneous flops. Since the hypotheses of Theorem 2.3.9 hold, then ψ_2 is an isomorphism of the variety Y_2.

The map Φ' is a conic bundle over the projective space $\mathbb{P}^2_{y_2,y_3,y_4}(2,1,1)$. We are interested in calculating its discriminant Δ.

Analogously to previous examples, we consider the line $\{y_3 = 0\}$ in the base space $\mathbb{P}^2_{y_2,y_3,y_4}(2,1,1)$ with coordinates y_2, y_4, and we look at its two affine patches $\{y_2 \neq 0\}$, $\{y_4 \neq 0\}$. On these affine patches we study the behaviour of the equations of $Y_2 \cong Y_3$. We report here only the five equations originated from the pfaffians equations of X. They are

$$\begin{cases} -tx_1y_3^2y_4 + x_1^2y_4^2 + x_1y_4y_1 - x_1^2y_2 - x_2^2y_2 + x_3^2y_2 + t^2y_4^2y_2 - y_1^2 - ty_1y_2 = 0 \\ tx_2y_3^2 - x_1^2y_4 - x_1x_2y_4 - x_2^2y_4 + x_3^2y_4 + ty_4^3 - x_1y_1 - x_2y_1 - tx_1y_2 = 0 \\ -x_1^2y_3 - x_2^2y_3 + x_3^2y_3 + x_1^2y_4 + t^2y_3y_4^2 + x_2y_1 = 0 \\ x_1y_4^2 + y_3y_1 + x_2y_2 + ty_3y_2 = 0 \\ ty_3^3 - x_1y_3y_4 - y_3y_1 - y_4y_1 + x_1y_2 = 0 \end{cases} .$$

We start looking at the affine patch $\{y_2 \neq 0\}$ of $\{y_3 = 0\}$. After the global elimination of the variables $x_1 = y_1y_4$ and $x_2 = -x_1y_4^2 = -y_1y_4^3$ (due to Pf_5 and Pf_4 respectively), and after the consequent substitution, the above equations become

$$\begin{cases} y_1^2y_4^4 + y_1^2y_4^2 - y_1^2y_4^2 - y_1^2y_4^6 + x_3^2 + t^2y_4^2 - y_1^2 - ty_1 = 0 \\ y_1^2y_4^3 + y_1^2y_4^5 + y_1^2y_4^7 + x_3^2y_4 + ty_4^3 - y_1^2y_4 + y_1^2y_4^3 - ty_1y_4 = 0 \\ y_1^2y_4^3 - y_1^2y_4^3 = 0 \end{cases}$$

Note that, after the substitution, Pf_3 is identically zero, and that $\mathrm{Pf}_2 = y_4 \,\mathrm{Pf}_1$. Therefore, the only surviving equation is Pf_1. It is a conic in the variables t, y_1, x_3 defined by the matrix

$$A = \begin{pmatrix} y_4^2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -1 + y_4^2 - y_4^6 \end{pmatrix} .$$

Its determinant has degree 8, therefore the discriminant $\Delta \geq 8$.

A similar calculation on the other patch $\{y_4 \neq 0\}$ shows that the fibre is not a conic. Therefore, the patch $\{y_4 \neq 0\}$ does not contribute to Δ.

This agrees with Table 4, No. 4.1 of [Tak02].

#24097, Jerry$_{15}$ Let $X \subset \mathbb{P}^7(1^6, 2^2)$ be the Jerry type Fano 3-fold relative to the Hilbert series #24097, where $p \in X$ is the centre $\frac{1}{2}(1,1,1)$. The coordinates of $\mathbb{P}^7(1^6, 2^2)$ are $x_1, x_2, x_3, y_4, y_3, y_2, y_1, s$ respectively. The unprojection of the divisor $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1) \subset Z_1$ in Jerry$_{1,5}$ format gives X. Here Z_1 is #24077: it is defined by the five pfaffians of the matrix M

$$M = \begin{pmatrix} y_4 & y_3 & y_2 & y_1 \\ & x_1 & x_2 & x_3y_3 + x_2y_4 \\ & & x_3 & y_2^2 - x_2y_3 - y_4^2 \\ & & & -x_1y_2 - 2y_3y_4 \end{pmatrix} .$$

There are 7 nodes on D. The five pfaffian equations of Y_1 are

$$\begin{cases} -x_1^2y_2 - tx_2y_2^2 + x_2^2y_3 + x_3^2y_3 + x_2x_3y_4 - 2tx_1y_3y_4 + tx_2y_4^2 = 0 \\ -ty_2^3 - x_1y_2y_3 + x_2y_2y_3 - 2ty_3^2y_4 + ty_2y_4^2 + x_3y_1 = 0 \\ -x_3y_2y_3 - x_1y_2y_4 - x_2y_2y_4 - 2ty_3y_4^2 + x_2y_1 = 0 \\ -x_3y_3^2 + ty_2^2y_4 - 2x_2y_3y_4 - ty_4^3 + x_1y_1 = 0 \\ x_1y_2 - x_2y_3 + x_3y_4 = 0 \end{cases}.$$

After 7 flops given by ψ_1, we have a divisorial contraction $\Phi'\colon Y_2 \to \mathbb{P}^3(2,1,1,1)$ of $(2,1)$-type, where the coordinates of $\mathbb{P}^3(2,1,1,1)$ are y_1, y_2, y_3, y_4 respectively. Recall that the variable s can be eliminated from each fibre of Φ'. Therefore, we just need to study the five pfaffian equations of Y_1.

Looking at the syzygies relating the five maximal pfaffians of M to one another, we see that, for each point in the base of Φ', Pf_1 can be written in terms of the other four pfaffians. We are left with four pfaffian equations, that are linear in the variables of the fibre t, x_1, x_2, x_3. Call L the 4×4 matrix recording the coefficients of $\mathrm{Pf}_2, \dots, \mathrm{Pf}_5$: the entries of L are in terms of the variables of the base only, i.e. y_1, y_2, y_3, y_4. In symbols,

$$\begin{pmatrix} \mathrm{Pf}_2 \\ \mathrm{Pf}_3 \\ \mathrm{Pf}_4 \\ \mathrm{Pf}_5 \end{pmatrix} = L \cdot \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Note also that the first syzygy relates such four linear pfaffians all together: therefore there are only three linearly independent pfaffians. Therefore, the determinant of L restricted on Y_2 is identically zero.

The map Φ' contracts its exceptional divisor $\mathbb{E}'$ to a curve $C \subset \mathbb{P}^3(2,1,1,1)$. The equations of C are given by the 3×3 minors of L. A simple computer algebra calculation on `Magma` shows that the degree of C is 7, and that its genus is $g(C) = 8$.

This coincides with what Takagi concluded in [Tak02]. Therefore, #24097, Jerry_{15} is No. 1.1 of Table 1 in [Tak02].

Chapter 4

Higher Picard rank Tom links

Looking at the table [BKR12b] we notice the presence, in 46 cases, of other deformation families in Tom format; following the common terminology also used in [BKQ18], we call these families *second Tom*. As shown in [BKQ18], these families contain quasi-smooth members whose equations are modelled on those of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ (cf Section 5 in [BKQ18]): they are in the so-called $\mathbb{P}^2 \times \mathbb{P}^2$-*format*. More precisely, the nine equations of X of second-Tom type can be retrieved from a 3×3 matrix as its nine 2×2 minors. The pfaffian matrix M in second Tom format is characterised by having a 0 in one or two of its entries. This can happen for instance when the polynomial occupying such an entry can be made 0 after row/column operations, or when the degree of that entry cannot be achieved by any polynomial in the variables x_i, y_j in the ideal I_D.

The most important feature of these Fano varieties X of second-Tom type is that they have Picard rank $\rho_X = 2$: see Proposition 2.1 of [BKQ18]. Other than the examples of Takagi [Tak02] and some computational cases, we know the Picard numbers of very few of the codimension 4 Fano 3-folds.

In this chapter we focus on birational links run on codimension 4 Fano 3-folds of second-Tom type. We will see that, even though we do not obtain Sarkisov links (because the starting variety X is not a Mori fibre space), a birational link construction is still licit, and can give interesting insights on the birational geometry of these higher Picard rank Fano varieties. In particular, we can find links to identify a Mori fibre space in the birational class of X, even though we do not know how to explicitly run the Minimal Model Program on X itself.

4.1 Mori fibre spaces arising from second Tom

Definition 4.1.1. From [BKR12a] we know that each codimension 3 Fano 3-fold Z admitting a Type I unprojection has at least two deformation families, one Tom and one Jerry. However, it could happen that it has one or two more Tom and Jerry families (one each at most). If this occurs, call *second Tom* the second Tom deformation family of Z, characterised by having a smaller number of nodes.

Remark 4.1.1. The Fano 3-folds of second-Tom type are in $\mathbb{P}^2 \times \mathbb{P}^2$ format (see the ones denoted by "subfamily of Tom" in Table 1 of [BKQ18]). We stress the fact that in this chapter we only consider the Fano 3-folds appearing in the table [BKR12b] that are of second-Tom type, together with the Hilbert series #12960. The latter does not have a second Tom, but its only Tom format is still in $\mathbb{P}^2 \times \mathbb{P}^2$ format, and our method applies to this as well.

We can summarise the result of this chapter with the following theorem.

Theorem 4.1.2. *Every Fano 3-fold in codimension 4 in second-Tom format and the unique Tom format of Fano #12960 present a birational link terminating with either*

- *two divisorial contractions (when $d_1 > d_2 > d_3 > d_4$ and when $d_1 > d_2 = d_3 > d_4$);*
- *a divisorial contraction followed by a del Pezzo fibration (when $d_1 = d_2 > d_3 = d_4$).*

Proof. We omit the detailed proof of this theorem because it is similar to the one contained in Chapter 2. We work out an in-depth example below.

The ones above are the only three configurations of the d_j in which Fano 3-folds of second-Tom type occur. □

Theorem 4.1.2 exhibits a Mori fibre space in the birational class of each X of second-Tom type. But in fact, more is true for #10985. The endpoints of its two links are not birationally rigid, even though we do not know a Sarkisov link that connects them.

We expect a similar behaviour for the other Fano 3-folds of second Tom type, as expressed in the following conjecture. If X is of second-Tom type and it has two Type I centres as in #10985, we expect it to be true. In addition, if X has only one Type I centre and X' has codimension 2, it is possible to run another extraction from X' in a similar fashion to [CM04]. Lastly, except for #4860 and #20652, if X has only one Type I centre whose endpoint X' has codimension 1, X does also have a Type II centre. Even though we do not know yet how to run this calculation from a Type II centre, we expect it would still lead to a new Mori fibre space.

Conjecture 4.1.1. *The birational-equivalence class of every Fano 3-fold in codimension 4 in second-Tom format contains at least two distinct Mori fibre spaces.*

We give an explicit example of the above construction in the following Section. In particular, we perform it from the two Type I centres of #10985. The endpoints X' and X'' of the two birational links are the hypersurface $X_5 \subset \mathbb{P}^4(1^4, 2)$. However, a more careful analysis shows that X' and X'' are not isomorphic, therefore the hypersurface $X_5 \subset \mathbb{P}^4(1^4, 2)$ has pliability at least 2.

Remark 4.1.3. Even though the varieties X of second-Tom type are not Mori fibre spaces, the birational links we obtain with this construction terminate with a Mori fibre space.

4.2 Hypersurface with high pliability: Fano #10985

Look again at the Hilbert series #10985, but this time let us analyse the second Tom, which is, in the notation of [BKR12b], a $\mathrm{Tom}_2 \bullet_{13,45}$ format. This means that the entries m_{13} and m_{45} of M are 0.

The basket of singularities of X is again $\{\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)\}$, but the deformation family of Z_1 is different from Example 3.1.1: this time M is in $\mathrm{Tom}_2 \bullet_{13,45}$ format.

In short, we are looking at the following Fano varieties,

#10985	X	$\subset \mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$	codimension 4	$\{\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)\}$
#10962	Z_1	$\subset \mathbb{P}^6(1^3, 3, 4, 5, 6)$	codimension 3	23 nodes

with the variables of $w\mathbb{P}^7$ being respectively $x_1, x_2, x_3, s,\ y_4, y_3, y_2, y_1$ and the divisor being $D \cong \mathbb{P}^2(1,1,1)_{x_1,x_2,x_3}$, on which Z_1 has 23 nodes.

The weights of the matrix M are the following

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & 3 & 4 & 5 \\ & & 5 & 6 \\ & & & 7 \end{pmatrix}.$$

We constructed explicitly the matrix M, that is

$$\begin{pmatrix} x_1 & 0 & y_4 & y_3 \\ & -x_2^3 - x_3^3 + y_4 & x_1^4 - x_3^4 + y_3 & x_1^5 - x_2^5 - y_2 \\ & & y_2 & y_1 \\ & & & 0 \end{pmatrix}. \tag{4.1}$$

Moreover, the equations of X are

$$
\begin{cases}
x_1^5 - x_1x_3^4 + x_1y_3 + y_4s = 0 \\
x_2^3y_4 + x_3^3y_4 - y_4^2 - x_1y_2 = 0 \\
-x_1^6 + x_1x_2^5 + x_1y_2 - y_3s = 0 \\
x_2^3y_3 + x_3^3y_3 - y_4y_3 - x_1y_1 = 0 \\
x_1^4x_2^3 + x_1^4x_3^3 - x_2^3x_3^4 - x_3^7 - x_1^4y_4 + x_3^4y_4 + x_1y_1 + y_2s = 0 \\
x_1^5y_4 + x_2^2x_3^3y_4 - x_1^4y_3 + x_3^4y_3 - x_2^2y_4^2 - x_1x_2^2y_2 - y_3^2 - y_4y_2 = 0 \\
-x_1^5x_2^3 + x_2^8 - x_1^5x_3^3 + x_2^5x_3^3 + x_1^5y_4 - x_2^5y_4 + x_2^3y_2 + x_3^3y_2 - y_4y_2 - y_1s = 0 \\
y_3y_2 - y_4y_1 = 0 \\
x_1^5y_2 - x_2^5y_2 - x_1^4y_1 + x_3^4y_1 - y_2^2 - y_3y_1 = 0
\end{cases}
\tag{4.2}
$$

According to Proposition 2.5, the blow up at P_s of $w\mathbb{P}^7$ is the scroll $\mathbb{F}_1$ given by

$$
\left(\begin{array}{cc|ccccccc}
t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\
0 & 2 & 1 & 1 & 1 & 6 & 5 & 4 & 3 \\
1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1
\end{array}\right). \tag{4.3}
$$

The Mori cone of $\mathbb{F}_1$ is identical to the one in Example 3.1.1.

The Kawamata blow-up of the Tom centre P_s is the map Φ

$$
\begin{aligned}
\Phi\colon \mathbb{F}_1 &\longrightarrow \mathbb{P}^7(1^3,2,3,4,5,6) \\
(t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1t^{\frac{1}{2}}, x_2t^{\frac{1}{2}}, x_3t^{\frac{1}{2}}, y_4t^{\frac{5}{2}}, y_3t^{\frac{6}{2}}, y_2t^{\frac{7}{2}}, y_1t^{\frac{8}{2}}, s)
\end{aligned}, \tag{4.4}
$$

while the expression of Φ having integer exponents of t is

$$
\begin{aligned}
\Phi\colon \mathbb{F}_1 &\longrightarrow \mathbb{P}^7(1^3,2,3,4,5,6) \\
(t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) &\longmapsto (x_1t, x_2t, x_3t, y_4t^4, y_3t^5, y_2t^6, y_1t^7, st)
\end{aligned}. \tag{4.5}
$$

Therefore, the equations of Y_1 are

$$\begin{cases} x_1^5 - x_1x_3^4 + tx_1y_3 + y_4s = 0 \\ x_2^3y_4 + x_3^3y_4 - ty_4^2 - x_1y_2 = 0 \\ -x_1^6 + x_1x_2^5 + tx_1y_2 - y_3s = 0 \\ x_2^3y_3 + x_3^3y_3 - ty_4y_3 - x_1y_1 = 0 \\ x_1^4x_2^3 + x_1^4x_3^3 - x_2^3x_3^4 - x_3^7 - tx_1^4y_4 + tx_3^4y_4 + tx_1y_1 + y_2s = 0 \\ x_1^5y_4 + x_2^2x_3^3y_4 - x_1^4y_3 + x_3^4y_3 - tx_2^2y_4^2 - x_1x_2^2y_2 - ty_3^2 - ty_4y_2 = 0 \\ -x_1^5x_2^3 + x_2^8 - x_1^5x_3^3 + x_2^5x_3^3 + tx_1^5y_4 - tx_2^5y_4 + tx_2^3y_2 + tx_3^3y_2 - t^2y_4y_2 - y_1s = 0 \\ y_3y_2 - y_4y_1 = 0 \\ x_1^5y_2 - x_2^5y_2 - x_1^4y_1 + x_3^4y_1 - ty_2^2 - ty_3y_1 = 0 \end{cases} \tag{4.6}$$

Theorem 2.3.2 shows the first step of the link are 23 simultaneous flops.

Crossing the wall corresponding to the variable y_1, we localise at the point $P_{y_1} \in \mathbb{G}_2$. Writing y_1 as a local coordinate we have that $\mathbb{F}_2$ becomes

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 6 & 8 & 1 & 1 & 1 & 0 & -1 & -2 & -3 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

Note that in the equations 4.6 of Y_2 there is no pure power of y_1, so the hypotheses of Theorem 2.3.9 are not satisfied and the flip is taking place. The variables that appear linearly locally analytically at a neighbourhood of $P_{y_1} \subset Z_2$ are s, y_4, and either x_1 or x_2; in particular, s and y_4 are globally eliminated. Therefore, Ψ_2 restricts to a hypersurface flip ψ_2 with weights $(6, 1, 1, -1, -2; 4)$, where α_2 contracts a hypersurface of degree 4 in $\mathbb{P}_{t,x_2,x_3}(6, 1, 1)$ and coefficients in $\mathbb{P}_{y_2,y_3}(1, 2)$ to P_{y_1}, and β_2 extracts $\mathbb{P}_{y_2,y_3}(1, 2) \subset Y_3$.

Analogously, we restrict the equations of Y_3 locally analytically at a neighbourhood of the point $P_{y_2} \in \mathbb{G}_3$. The weights of the rank 2 toric variety $\mathbb{F}_3$ become

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 5 & 7 & 1 & 1 & 1 & 1 & 0 & -1 & -2 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

This time around, the variables that are locally eliminated are t, x_1, and the ones globally eliminated are s, y_3. Therefore, the exceptional locus $\mathbb{A}_3$ restricted to Y_3 is $\mathbb{P}_{x_2,x_3,y_1}(1, 1, 1)$. On the other hand, the restriction of $\mathbb{B}_3$ to Y_4 is the $\frac{1}{2}$ quotient singularity at P_{y_4}. This shows that $Y_4 \cong Z_3$ via the map β_3 (which is actually a morphism), and that α_3 is the blow-up of the singularity $P_{y_4} \in Y_4$ of type $\frac{1}{2}(1, 1, 1)$.

Therefore, the Picard rank of Y_3 drops by one in the birational transformation determined by ψ_3. This happens when there is still another ray left to cross in the mobile cone of $\mathbb{F}_3$.

Performing again the elimination process at a neighbourhood of the point $P_{y_3} \in Y_4$ we have another divisorial contraction, the one we have usually called Φ'. The variables eliminated are s, y_2 (globally), and t (locally). Here, a surface $S_3 \subset \mathbb{P}_{x_1,x_2,x_3,y_1}(1,1,1,2)$ of degree 3 is contracted to the point $P_{y_4} \in X'$. In particular, the divisorial contraction Φ' is defined by the monomials in the linear system $\left|\mathcal{O}\binom{4}{-1}\right|$, that is,

$$\begin{aligned} \Phi' \colon \mathbb{F}_4 &\longrightarrow \mathbb{P}^7(1^4, 2, 3^2, 5) =: \mathbb{G}_4 \\ (t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) &\longmapsto (x_1y_4, x_2y_4, x_3y_4, y_3, y_2y_4, y_1y_4^2, ty_4^4, sy_4^6) \ . \end{aligned} \tag{4.7}$$

Call $x_1', x_2', x_3', y_4', y_3', y_2', y_1', t', s'$ the coordinates of $\mathbb{G}_4$. Looking at the equations 4.6 we notice that it is possible to ignore the coordinates y_1', t', s' because the terms sy_4, ty_4^2, and y_1y_4 appear in equations #1, #2, and #8 respectively (globally eliminated in this order). Hence, $X' \subset w\mathbb{P}' \subset \mathbb{G}_4$ where $w\mathbb{P}' = \mathbb{P}^4(1^4, 2)$.

The remaining equations after the elimination are (4.8). In order to find the explicit equations of X', let us write them in terms of the coordinates of $\mathbb{P}^4(1^4, 2)$ by multiplying them by a suitable power of y_4, which is t^3 for equation #4, t^4 for equation #6, and t^6 for equation #9. They become

$$\begin{cases} x_2'^3y_3' + x_3'^3y_3' - t'y_3' - x_1'y_1' = 0 \\ x_1'^5 + x_2'^2x_3'^3 - x_1'^4y_3' + x_3'^4y_3' - t'x_2'^2 - x_1'x_2'^2y_2' - t'y_3'^2 - t'y_2' = 0 \\ x_1'^5y_2' - x_2'^5y_2' - x_1'^4y_1' + x_3'^4y_1' - t'y_2'^2 - t'y_3'y_1' = 0 \end{cases} \ . \tag{4.8}$$

On the other hand, equations #1, #2, #8 express the variables s, t, y_1 in terms of the others, becoming

$$\begin{aligned} y_1' &= y_2'y_3' \\ t' &= x_2'^3 + x_3'^3 - x_1'y_2' \\ s' &= -x_1'^5 + x_1'x_3'^4 - t'x_1'y_3' \ . \end{aligned} \tag{4.9}$$

Replacing the above identities (4.9) in (4.8) we have the three equations

$$\begin{cases} x_2'^3 y_3' + x_3'^3 y_3' - (x_2'^3 + x_3'^3 - x_1' y_2') y_3' - x_1' y_2' y_3' = 0 \\ x_1'^5 + x_2'^2 x_3'^3 - x_1'^4 y_3' + x_3'^4 y_3' - (x_2'^3 + x_3'^3 - x_1' y_2') x_2'^2 - x_1' x_2'^2 y_2' \\ -(x_2'^3 + x_3'^3 - x_1' y_2') y_3'^2 - (x_2'^3 + x_3'^3 - x_1' y_2') y_2' = 0 \\ x_1'^5 y_2' - x_2'^5 y_2' - x_1'^4 y_2' y_3' + x_3'^4 y_2' y_3' - (x_2'^3 + x_3'^3 - x_1' y_2') y_2'^2 - (x_2'^3 + x_3'^3 - x_1' y_2') y_3' y_2' y_3' = 0\ . \end{cases} \tag{4.10}$$

We see that the third equation in 4.10 is a multiple of the second one by a y_2' factor, and the first equation is identically zero. Therefore it remains only the second equation:

$$x_1'^5 - x_1'^4 y_3' + x_3'^4 y_3' - x_2'^5 - (x_2'^3 + x_3'^3 - x_1' y_2')(y_3'^2 + y_2') = 0\ . \tag{4.11}$$

The one above is the equation of X', and it has degree 5 in the coordinates of $w\mathbb{P}'$. Thus, $X_5' \subset \mathbb{P}^4(1^4, 2)$. In addition, the basket of singularities of Y_1 is $\mathcal{B}_{Y_1} = \{\frac{1}{6}(1,1,5)\}$, which remains unvaried for Y_2. Then, the hypersurface flip ψ_2 replaces the $\frac{1}{6}$ singularity with one of type $\frac{1}{2}$. After that, ψ_3 contracts a singular locus to a $\frac{1}{2}$ singularity. Therefore, the basket of $Y_4 \cong Z_3$ is $\mathcal{B}_{Y_4} = \{2 \times \frac{1}{2}(1,1,1)\}$. Lastly, Φ' contracts a $\mathbb{P}(1,1,2)$ to a smooth point in X'; thus, $\mathcal{B}_{X'} = \{\frac{1}{2}(1,1,1)\}$.

The Fano 3-fold in codimension 1 sitting inside $\mathbb{P}^4(1^4, 2)$ defined by a degree 5 equation and having basket $\{\frac{1}{2}(1,1,1)\}$ is #16203: X' is a special member associated to that Hilbert series. Note that X' has a singularity at the point $P_{y_3'}$ like the ones described in [CM04].

Remark 4.2.1. According to [CPR00], X' should birationally rigid. Nonetheless, since X' has a singularity as in [CM04], actually it is birationally non-rigid.

For this calculation we could have also used the $\mathbb{P}^2 \times \mathbb{P}^2$ description of X, whose equations are given by the nine 2×2 minors of a 3×3 matrix N having weights

$$\begin{pmatrix} 1 & 3 & 4 \\ 3 & 5 & 6 \\ 2 & 4 & 5 \end{pmatrix}, \tag{4.12}$$

where the entry of degree 2 is occupied by the variable s only. The matrix N is therefore

$$N = \begin{pmatrix} x_1 & y_4 & y_3 \\ -x_2^3 - x_3^3 + y_4 & y_2 & y_1 \\ s & x_1^4 - x_3^4 + y_3 & x_1^5 - x_2^5 - y_2 \end{pmatrix}, \tag{4.13}$$

Remark 4.2.2. What we have just constructed is not a Sarkisov link, as the Picard rank drops by 2 because of the two consecutive divisorial contractions.

Therefore, the above proves the following theorem.

Theorem 4.2.3. *Define X as #10985 realised as a Tom_2 $\bullet_{13,45}$ unprojection. Then the Picard rank of X is $\rho_X \geq 2$.*

This shows that Sarkisov links are an effective tool to produce lower bounds for the Picard rank of a Fano 3-folds. In particular it means that X has a Mori cone of dimension at least 3. This observation lead to the idea that the Sarkisov link just computed could have been part of a larger link involving Fano 3-folds sitting inside rank 3 toric varieties.

4.2.1 Blow-up of #10985 from $\frac{1}{6}(1,1,5)$

The Fano 3-fold X associated to the Hilbert series #10985 also has another Type I centre, which is a $\frac{1}{6}(1,1,5)$ at the point $P_{y_1} \in X$. In particular, it also has a second-Tom format, that is a matrix M' in Tom_5, $\bullet_{14}$ format. The latter describes the same deformation family coming from the unprojection of the $\frac{1}{2}(1,1,1)$ centre at P_s, only obtained via a different unprojection.

This calculation retrieves the result of [CM04] because the endpoint of the 2-ray game starting with the blow-up of the $\frac{1}{6}(1,1,5)$ singularity of X is isomorphic to X'.

Using the matrix N in $\mathbb{P}^2 \times \mathbb{P}^2$ format in (4.13) it is possible to retrieve M' from M. The 3×3 matrix N' indicating the $\mathbb{P}^2 \times \mathbb{P}^2$ structure of the pair (X, P_{y_1}) is

$$\begin{pmatrix} x_1 & y_4 & y_3 \\ s & x_1^4 - x_3^4 + y_3 & x_1^5 - x_2^5 - y_2 \\ -x_2^3 - x_3^3 + y_4 & y_2 & y_1 \end{pmatrix}, \tag{4.14}$$

having weights

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

We can reconstruct the 5×5 matrix M' from N': so, M' is

$$\begin{pmatrix} x_1 & s & 0 & -x_2^3 - x_3^3 + y_4 \\ & 0 & y_4 & y_3 \\ & & x_1^4 - x_3^4 + y_3 & x_1^5 - x_2^5 - y_2 \\ & & & y_2 \end{pmatrix},$$

which is equal to the following matrix by performing a simple change of coordinates

$\bar{y}_3 := x_1^4 - x_3^4 + y_3$.

$$\begin{pmatrix} x_1 & s & 0 & -x_2^3 - x_3^3 + y_4 \\ & 0 & y_4 & \bar{y}_3 - x_1^4 + x_3^4 + \\ & & \bar{y}_3 & x_1^5 - x_2^5 - y_2 \\ & & & y_2 \end{pmatrix}.$$

Note that the unprojection variable relative to $\frac{1}{6}(1,1,5)$ is y_1, and that the unprojected divisor is $D' \subset Z_1' := \{\mathrm{Pf}_i(M') = 0\}_{i \in \{1,\ldots,5\}}$ defined by the ideal $I_{D'} := \langle y_3, y_4, s, x_1 \rangle$. Therefore, the matrix (4.2.1) is in Tom$_5$, $\bullet_{14}$ format.

The blow-up of $X \subset \mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$, having variables $x_1, x_2, x_3, s, y_4, \bar{y}_3, y_2, y_1$ respectively, at the point P_{y_1} is contained in the rank 2 toric variety $\mathbb{F}_1'$ having weights

$$\left(\begin{array}{cc|ccccccc} r & y_1 & x_2 & x_3 & y_2 & \bar{y}_3 & y_4 & s & x_1 \\ 0 & 6 & 1 & 1 & 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

Here we have an hypersurface flip with weights $(1,1,5,-1,-2;3)$ based at $P_{\bar{y}_3} \in Z_2'$, followed by one divisorial contraction to the $\frac{1}{2}(1,1,1)$ point P_{x_1} of weights $(1,1,1,-2)$, based at $P_{y_4} \in Z_3'$. The last divisorial contraction Φ'' has weights $(2,1,1,2,-1;4)$ contracting a degree 4 surface $S_4 \subset \mathbb{P}(2,1,1,2)$ to a point.

More explicitly, Φ'' is of the form

$$\begin{aligned} \Phi'' \colon \mathbb{F}_4' &\longrightarrow \mathbb{P}^7(1^4, 2, 3, 5, 7) =: \mathbb{G}_4' \\ (r, y_1, x_2, x_3, y_2, \bar{y}_3, y_4, s, x_1) &\longmapsto (x_1 r^2, s, x_1 x_2, x_1 x_3, x_1 y_4, x_1^2 \bar{y}_3, x_1^5 y_2, x_1^8 y_1). \end{aligned}$$

Define the coordinates of $\mathbb{P}^7(1^4, 2, 3, 5, 7)$ as $r', s', x_2', x_3', y_4', \bar{y}_3', y_2', y_1'$ respectively.

Analogously to the previous analysis of the $\frac{1}{2}$ weighted blow-up of X, we see that some of the equations of $Y_4' \subset \mathbb{F}_4'$ help expressing some of the coordinates of $\mathbb{P}^7(1^4, 2, 3, 5, 7)$ in terms of the others: this is the case of y_1', y_2', and $\bar{y}_3'$. The explicit expression of the two latter are

$$\begin{cases} y_2' &= y_4' \left(x_2'^3 + x_3'^3 - y_4' r' \right) \\ \bar{y}_3' &= s' y_4' \end{cases}, \tag{4.15}$$

and the three surviving pfaffians of M' are Pf_1, Pf_2, and Pf_4, that is

$$\begin{cases} \mathrm{Pf}_1 &= -y_4' \left(r'^5 - x_2'^5 - y_2' \right) + \bar{y}_3' \left(\bar{y}_3' r' - r'^4 + x_3'^4 \right) \\ \mathrm{Pf}_2 &= s' y_2' + \bar{y}_3' \left(-x_2'^3 - x_3'^3 + y_4' r' \right) \\ \mathrm{Pf}_4 &= \left(r'^5 - x_2'^5 - y_2' \right) - s' \left(\bar{y}_3' r' - r'^4 + x_3'^4 \right) \end{cases}.$$

After replacing equations (4.15) in the above equations, we have

$$\begin{cases} \mathrm{Pf}_1 & = -y'_4\left(r'^5 - x'^5_2 - y'_4\left(x'^3_2 + x'^3_3 - y'_4 r'\right)\right) + s'y'_4\left(s'y'_4 r' - r'^4 + x'^4_3\right) \\ \mathrm{Pf}_2 & = s'y'_4\left(x'^3_2 + x'^3_3 - y'_4 r'\right) + s'y'_4\left(-x'^3_2 - x'^3_3 + y'_4 r'\right) \equiv 0 \\ \mathrm{Pf}_4 & = \left(r'^5 - x'^5_2 - y'_4\left(x'^3_2 + x'^3_3 - y'_4 r'\right)\right) - s'\left(s'y'_4 r' - r'^4 + x'^4_3\right) \end{cases} , \qquad (4.16)$$

where Pf_1 is a multiple of Pf_4. In conclusion, the equation of $X'' = X_5 \subset \mathbb{P}^4(1^4, 2)$ is

$$\left(r'^5 - x'^5_2 - y'_4\left(x'^3_2 + x'^3_3 - y'_4 r'\right)\right) - s'\left(s'y'_4 r' - r'^4 + x'^4_3\right) = 0 \ . \qquad (4.17)$$

Proposition 4.2.4. *The pliability of X' is $\mathcal{P}(X') \geq 2$.*

Proof. Both X' and X'' sit inside the weighted projective space $\mathbb{P}^4(1^4, 2)$ and have Picard rank 1, so it makes sense to talk about their pliablity. Moreover, they each have a non-orbifold point inherited by the hypersurface flip happening in their respective 2-ray games. More explicitly, X' has a cA_2 singularity at the point P_{y_3} locally described by the equation $x_2^3 + x_3^3 = x_1 y_2$.

On the other hand, X'' has a cA_3 singularity at P_s which is, locally, $r^4 - x_3^4 = sy_4$. This means that their generic sections are not isomorphic. Therefore, $X' \ncong X''$, so $\mathcal{P}(X') \geq 2$. $\square$

Remark 4.2.5. Note that the sequence of birational transformations connecting X' and X'' is not a Sarkisov link.

Chapter 5

First steps towards Fano index 2

5.1 The existence of index 2 Fano varieties

The lack of a structure theorem for Fano 3-folds in codimension 4 forces to search for other ways to produce their equations. The (Type I) unprojection construction has supplied an efficient tool to deduce such equations from the ones of codimension 3 Fanos in either Tom or Jerry format. This works if these Fanos have Fano index 1.

If we look at the Fano index 2 case we see that the unprojection techniques are not applicable, as none of the 37 candidates Fano 3-folds in codimension 4 having index 2 admits a Type I centre, as in [BK$^+$15].

Despite this, it is still possible to use the unprojection to retrieve an explicit description for index 2 Fano 3-folds in codimension 4 from suitable families in index 1. The idea is to find an appropriate index 1 Fano X to be a double cover of each corresponding index 2 Fano $\tilde{X}$. This is suggested by an observation of the ambient spaces of these index 2 varieties, say $\tilde{X} \subset w\tilde{\mathbb{P}}$, that is, replacing a 2 with a 1 in the weights of the ambient space of $\tilde{X}$, there exists another candidate $X \subset w\mathbb{P}$ in the same codimension sitting inside such a manipulated weighted projective space.

We illustrate our approach by looking at a baby case in codimension 2, and describing explicitly the diagram 5.2. Although simpler than the examples we produce, it encodes some crucial phenomena encountered in the development of this chapter.

Example 5.1.1. Consider the codimension 2 index 1 Fano 3-fold $X = X_{4,4} \subset \mathbb{P}^5(1^4, 2, 3)$, #16204. Only in the span of this example we call the coordinates of $w\mathbb{P}^5$ according to their weight, that is, $x_1, \dots, x_4$ for the ones of weight one, y, z for the ones having weight 2 and 3 respectively. A projection from the Type I centre $P_z \in X$ of type $\frac{1}{3}(1,1,2)$ (with orbinates x_3, x_4, y) targets the index 1 codimension 1 Fano 3-fold $Z \subset \mathbb{P}^4(1^4, 2)$, #16203.

We consider the action γ of the cyclic group $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{P}^5(1^4, 2, 3)$ defined as the change of sign to the coordinate x_4. Suppose that we write the equations of X such that the coordinate x_4 appears only with even powers. They are of the form

$$\begin{cases} zx_1 = A_4(x_1, x_2, x_3, x_4^2, y) \\ zx_2 = B_4(x_1, x_2, x_3, x_4^2, y) \end{cases} ,$$

where A_4, B_4 are general homogeneous polynomials of degree 4. Therefore, the equation of Z is

$$x_2 A_4(x_1, x_2, x_3, x_4^2, y) = x_1 B_4(x_1, x_2, x_3, x_4^2, y) .$$

Define $\bar{x}_4 := x_4^2$ and consider the quotient $\tilde{X}$ of X by the group action that changes the sign of x_4. Thus, $\tilde{X} = \tilde{X}_{4,4}$ sits inside a new weighted projective space $\mathbb{P}^5(1^3, 2^2, 3)$. We can easily see that $-K_{\tilde{X}} \sim \mathcal{O}(-2)$. Therefore the index of $\tilde{X}$ is 2.

The fixed locus of the group action we considered is $Fix(\gamma) = \{x_4 = 0\} \cup \mathbb{P}^1(1,2)_{x_4,y}$. Note that the cyclic quotient singularity at $P_z \in X$ is fixed because it lies in the $\{x_4 = 0\}$ locus; it becomes of type $\frac{1}{3}(1,2,2)$ in $\tilde{X}$. This shows that the quotient does not produce new additional singularities: so $\tilde{X}$ is quasismooth. On the other hand, the intersection of the other component of the fixed locus of γ with X is empty: this is because the general polynomials A_4 and B_4 must contain monomials such as x_4^4 and y^2. Therefore $\tilde{X}$ is also terminal. We have just explicitly constructed equations for the index 2 Fano 3-fold with Hilbert series #40662, showing that the index 1 Fano 3-fold #16204 is its double cover.

Conversely, the same does not hold for Z. Here the intersection $Z \cap \mathbb{P}^1(1,2)_{x_4,y}$ is non-empty: actually, the whole line $\mathbb{P}^1(1,2)_{x_4,y}$ is contained in Z. Therefore, the quotient $\tilde{Z}$ of Z by the group action γ contains an entire line of cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$. This shows that $\tilde{Z}$ is not terminal, so does not appear in the Graded Ring Database [BK+15].

This specific construction can be summarised with the following diagram.

$$\begin{tikzcd}
\#16204 & \mathbb{P}^5(1^4,2,3) \supset X \arrow[d, "\gamma"'] & Z \subset \mathbb{P}^4(1^4,2) \arrow[l, "\text{unproj}"'] \arrow[d, "\gamma"] & \#16203 \\
\#40662 & \mathbb{P}^5(1^3,2^2,3) \supset \tilde{X} & \tilde{Z} \subset \mathbb{P}^4(1^3,2^2) &
\end{tikzcd}$$

This diagram is analogous to the one at 5.2.

The above example shows in a nutshell the achievements of the double-cover construction, and also the consequences it has at the codimension 3 level.

Here we resume the standard notation fixed in Section 1.2.5. Let us define the action γ of $\mathbb{Z}/2\mathbb{Z}$ on a weighted $\mathbb{P}^7$ as

$$w\mathbb{P}^7 \ni (t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) \longmapsto (t, s, -x_1, x_2, x_3, y_1, y_2, y_3, y_4) \tag{5.1}$$

that is, we change sign to the variable x_1. Recall that X has index 1, so we can assume the weight of x_1 to be 1.

The main goal of this section is to prove the following theorem.

Theorem 5.1.1. *There exist 32 Hilbert series of index 1 Fano 3-folds X in codimension 4 having at least a Type I centre such that the quotient $\tilde{X} := X/_{\mathbb{Z}/2\mathbb{Z}}$ via the group action γ (5.1) is an index 2 Fano 3-fold in codimension 4.*

We will later explain the reason for the number 32, and how this relates to what has been achieved so far in terms of explicit construction of index 2 Fano 3-folds. Theorem 5.1.1 implies the following corollary.

Corollary 5.1.2. *The index 1 Fano 3-fold X of Theorem 5.1.1 is a double cover for $\tilde{X}$.*

Here we describe our construction in the codimension 4 case, mimicking the one explained in the baby case of Example 5.1.1.

First we want to take X in codimension 4 and index 1 such that it is invariant under the action γ. To do so, we need to look at Z, the projection of X from a Type I centre, and write down a special member of Z that is invariant under γ. After that, we perform the unprojection to obtain a $\mathbb{Z}/2\mathbb{Z}$-invariant X. The last step is quotienting X by the group action γ and studying the quotient.

In the Graded Ring Database [BK$^+$15] there are 37 Hilbert series for Fano 3-folds in codimension 4 and index 2. The ones that our method does not construct are 5. One of them, #41028, lies in a non-weighted projective space, and was therefore already constructed by Iskovskih in [Isk77] and [Isk78].

Other two Hilbert series, #39569 and #39607, do have a double-cover candidate, but it does not have any Type I centre. Since in this thesis we consider only Type I unprojections, we will not examine these two examples.

Lastly, the two Hilbert series #40367 and #40378 do not have any index 1 double-cover candidate, so our method does not apply to them.

For the remaining 32 we therefore achieve the following diagram.

$$\begin{tikzcd} & \text{codim } 4 & & \text{codim } 3 \\ \text{index } 1 & X & & Z \arrow[ll, "\text{unproj}"'] \arrow[d, "\gamma"', "\mathbb{Z}/2\mathbb{Z}"] \\ \text{index } 2 & \tilde{X} \arrow[from=u, "\mathbb{Z}/2\mathbb{Z}"', "\gamma"] & & \tilde{Z} \end{tikzcd} \tag{5.2}$$

Table 6.3 summarises the pairs $(\tilde{X}, X)$ for each Hilbert series in index 2.

We break down the proof of Theorem 5.1.1 in some separated lemmas.

Practically speaking, the next lemma shows that, to perform the quotient and obtain $\tilde{X}$, we just need to replace x_1^2 with $\bar{x}$ in the equations of X, and that the ambient space of $\tilde{X}$ is the ambient space of X where a 1 has been replaced by a 2.

Lemma 5.1.3. *The Fano 3-fold $\tilde{X}$ sits inside the weighted projective space $\mathbb{P}^7(2, b, c, d_1, \ldots, d_4, r)$, with coordinates $\bar{x}, x_2, x_3, y_1, y_2, y_3, y_4, s$ respectively, and $\bar{x} := x_1^2$.*

Proof. Let us divide $\mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r)$ in affine patches. Pick, for instance, $\mathcal{U}_{x_2} := \{x_2 \neq 0\}$. In particular, $\mathcal{U}_{x_2}$ is given by the Spec of the degree-invariant fractions as

$$\operatorname{Spec}\mathbb{C}\left[\frac{x_1^b}{x_2}, \frac{x_3^{\frac{b}{c}}}{x_2}, \frac{y_1^{\frac{b}{d_1}}}{x_2}, \ldots, \frac{y_4^{\frac{b}{d_4}}}{x_2}, \frac{s^{\frac{b}{r}}}{x_2}, \ldots\right].$$

Similarly we can explicitly write all the other affine patches. Let the group action γ act on each patch. They are invariant if and only if the coordinate x_1 has even power. Such affine patches are defined by the ring of the invariants under the action, in which x_1 appears only with even powers. These same affine patches are exactly the affine patches of the weighted projective space $\mathbb{P}^7(2, b, c, d_1, \ldots, d_4, r)$.

This also proves that the quotient of X has the same equations as X, where x_1^2 has been replaced with the new coordinate $\bar{x}$. ☐

Lemma 5.1.4. *The Fano 3-fold $\tilde{X}$ has index 2.*

Proof. Consider the quotient map $f\colon X \to \tilde{X}$. The relation between the anticanonical bundles of X and $\tilde{X}$ is $-K_X = -f^*K_{\tilde{X}} - R$ where R is the ramification divisor. In our case, $-K_X = \{x_1 = 0\} \sim \mathcal{O}(1)$. Moreover, the ramification divisor is $R = \{x_1 = 0\}$. Therefore, $-f^*K_{\tilde{X}} = 2\{x_1 = 0\}$. This implies that $-K_{\tilde{X}} = \{\bar{x} = 0\} \sim \mathcal{O}(2)$: thus, $\tilde{X}$ has index 2. ☐

Lemma 5.1.5. *If X is quasi-smooth, then $\tilde{X}$ is quasi-smooth.*

Proof. Define the variety V as

$$V := \{p \in w\mathbb{P}^7 \ : \ rank\,(J|_p) < codim(X)\}\,.$$

The condition defining V is equivalent to looking at the vanishing locus of all 4×4 minors of the Jacobian matrix J_X of X (see [Har77]). By definition, if V is empty, then X is quasi-smooth. Suppose X quasi-smooth and compare J_X with $J_{\tilde{X}}$. The only difference between the two Jacobian matrices lies in the column relative to the derivative by x_1. Suppose $x_1 \neq 0$; then, the rank of J_X is equal to the rank of $J_{\tilde{X}}$.

On the other hand, if $x_1 = 0$ certain entries of the $\frac{\partial}{\partial x_1}$ column in J_X might vanish, while they would be just a constant in $J_{\tilde{X}}$. This is because, for each equation f_i of X,

$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x_1}$, and $\frac{\partial \bar{x}}{\partial x_1} = 2x_1$. So for $x_1 = 0$ we have that $\operatorname{rk} J_X \leq \operatorname{rk} J_{\tilde{X}}$. Therefore, $\tilde{X}$ is quasi-smooth if X is. $\square$

The last lemma is the last missing step to prove that $\tilde{X}$ can be found in the Graded Ring Database [BK+15].

Lemma 5.1.6. *The Fano 3-fold $\tilde{X}$ has terminal singularities.*

Proof. The fixed locus of the group action γ is $\operatorname{Fix}(\gamma) = \{x_1 = 0\} \cup \mathbb{P}_{\text{even}}$, where $\mathbb{P}_{\text{even}}$ is the weighted projective space defined by the vanishing of all the coordinates with odd weight, except for x_1. We want to study the intersection $X \cap \operatorname{Fix}(\gamma)$. Recall that the Type I centre of X is $P_s \in X$, having orbinates x_1, x_2, x_3; thus, it is (pointwise) fixed by γ, so $X \cap \{x_1 = 0\} \neq \emptyset$. In particular, all cyclic quotient singularities of X are fixed pointwise by γ.

On the other hand, X does not intersect the rest of the fixed locus, that is, $\mathbb{P}_{\text{even}}$. The reason for this lies in the shape of the unprojection equations of X. In Remark 5.1.7 we show that the order of the cyclic quotient singularities of an index 2 Fano 3-fold must be odd, and that therefore its orbinates must have weights $(2, b, c)$ where either b is odd and c even, or vice versa. Thus, in order for X to be a double-cover candidate for an index 2 Fano 3-fold $\tilde{X}$, the cyclic quotient singularities of X must have orbinates with weights $(1, b, c)$ and b, c as above. To fix ideas, suppose b odd and c even.

We want to prove that at least two unprojection equations of X contain a monomial of the form $x_1^\mu x_3^\nu$ for some μ, ν positive integers (μ and ν not the same for each of the unprojection equations). This is enough to prove that $X \cap \mathbb{P}_{\text{even}} = \emptyset$.

We study the unprojection equations of X via their algorithmic construction outlined in Section 2.2.2 (cf [Pap04]). Without loss of generality we can assume that the variable x_3 occupies one of the entries of M that are not in the ideal I_D. In addition, there are at least two entries not in I_D having even degree: therefore, we can always place a suitable even power of x_1 in the entry not occupied by x_3. Thus, all the matrices defined in (2.10) contain x_3 and even powers of x_1. They are eventually multiplied in the determinant of (2.10).

In conclusion, the only points of X fixed by the action γ are its orbifold points, and therefore $\tilde{X}$ has terminal singularities. $\square$

Looking more closely to the double covers created using diagram 5.2 we notice that they are drastically different depending on the codimension.

Remark 5.1.7. Note that our double-cover method does not produce index 2 Fano 3-folds in codimension 3 in the Graded Ring Database, that is, the quotient $\tilde{Z} := Z/_{\mathbb{Z}/2\mathbb{Z}}$ does not have terminal singularities. The reason lies in the proof of Lemma 5.1.6. While the intersection of X with the fixed locus of γ is just a finite number of points, the intersection $Z \cap \mathrm{Fix}(\gamma)$ contains much more. More precisely, Z and $\mathbb{P}_{\mathrm{even}}$ intersect along $D \subset Z$. This is because $D \cong \mathbb{P}^2(1, b, c)$ where either b or c is even, i.e. there is no configuration of $(1, b, c)$ such as $(1, 1, 1)$, $(1, 1, 3)$, $(1, 1, 5)$, $(1, 3, 5)$, etc. The reason for this is that terminal singularities on index 2 Fano 3-folds are of type $\frac{1}{3}(a, b, r-b)$ with $(a, r) = 1$ and $(b, r) = 1$. Since a is always 2 for index 2, this implies that r must be odd, and same for either one between b and $r - b$. From Lemma 5.1.3 we have that $\tilde{D} \cong \mathbb{P}^2(2, b, c)$. Therefore, the $\mathbb{Z}/2\mathbb{Z}$-quotient $\tilde{Z}$ of Z contains a line of $\frac{1}{2}$ singularities, sitting inside the divisor $\tilde{D} \subset \tilde{Z}$. Suppose b is even: such line is $\mathbb{P}^1(2, b)$.

This shows that $\tilde{Z}$ does not have terminal singularities, and thus is not listed in the Graded Ring Database [BK+15]. Recall that there are only two Hilbert series corresponding to terminal index 2 codimension 3 Fano 3-folds: one is smooth, so constructed by Iskovskih in [Isk77] and [Isk78]. The other one was constructed by Ducat in [Duc18]. The double-cover method described in this thesis does not construct them.

Proof of Theorem 5.1.1. The statement follows from the combination of Lemmas 5.1.3, 5.1.4, 5.1.5, and 5.1.6. □

5.1.1 Conjectural non-existence by computer algebra

Since the divisor D gets "folded in two" by the $\mathbb{Z}/2\mathbb{Z}$ action γ, it is interesting to study what happens to the nodes on $D \subset Z$.

Lemma 5.1.8. *Suppose there exists a special member of the deformation family of Z that is invariant under the $\mathbb{Z}/2\mathbb{Z}$ action γ.*

Then, the nodes on the divisor $D \subset Z$ are not fixed by γ. Moreover, they are pairwise-identified in the quotient $\tilde{Z}$.

Proof. From [BKR12a] we can assume that the nodes of Z only lie on the divisor D. Therefore, from Remark 5.1.7 we have that the nodes are not fixed by γ.

The equations of the nodes on $\tilde{Z}$ can be found by computing the 3×3 minors of the Jacobian matrix $\tilde{J}$ of $\tilde{Z}$ and then restricting it to $\tilde{D}$, i.e. $\bigwedge^3 \tilde{J}|_{\tilde{D}} = \underline{0}$. Such equations obviously depend on $\bar{x}$. These equations describe a finite number of points on $\tilde{D}$, that is, its nodes. Lifting these equations to Z, $\bar{x}$ is replaced by the new variable x_1^2. Therefore, the nodes found at the index 2 level are doubled in the index 1 level.

Therefore, the nodes on $D \subset Z$ are pairwise identified by γ in the quotient. □

We can therefore deduce the following corollary.

Corollary 5.1.9. *Consider Z as in Lemma 5.1.8. Then the number of nodes of Z is even.*

Now consider the codimension 4 candidates in index 1 to be double cover for one of the 32 Hilbert series in index 2. Such candidates have between two and four different deformation families, depending on the format of Z (cf [BKR12a]). Each of these formats has a certain number of nodes on $D \subset Z$. Corollary 5.1.9 show that, in order to be a double cover for $\tilde{X}$, the index 1 candidate X must be obtained from Z having even number of nodes. This excludes some of the deformation families of X, that is, dismisses the ones whose format has odd number of nodes.

Remark 5.1.10. Only the Hilbert series #24078 presents two possible Tom formats. Corollary 5.1.9 constitutes a criterion to exclude the first Tom format in the family #24078, which has 5 nodes. Therefore, the second Tom of the family #24078 cannot be invariant under the $\mathbb{Z}/2\mathbb{Z}$-action in 5.1.

We summarise in Table 6.3 the Tom formats that give rise to a double cover, together with the Jerry formats that could produce other deformation families for the same Hilbert series in index 2.

Although we have not investigated it thoroughly as the case of Fano 3-folds of Tom type, we do have some conjecture explaining the expected behaviour of the index 1 Fano 3-folds of Jerry type under the double-cover method. Through these conjectures we systematise the data collected via computer algebra.

Using the `tj` package for `Magma` that can be found in the Graded Ring Database website [BK+15] it is possible to produce a code checking whether the $\mathbb{Z}/2\mathbb{Z}$-invariance can be achieved with a certain Tom or Jerry format. The code shows that the formats giving rise to a $\mathbb{Z}/2\mathbb{Z}$-invariant Fano all share the features related to the number of nodes we explained and that, concerning the Jerry case, the condition 2.4.1 is involved.

For the Jerry case it has shown that there are 18 $\mathbb{Z}/2\mathbb{Z}$-invariant Jerry formats; 8 of them have some zero entries, the other 10 do not. In particular, it is possible to draw the following conclusions:

Conjecture 5.1.1. *If Z is defined by pfaffians of a $\mathbb{Z}/2\mathbb{Z}$-invariant matrix in Jerry format, then the numbers of nodes of Z is even.*

Conjecture 5.1.2. *If Z is defined by pfaffians of a $\mathbb{Z}/2\mathbb{Z}$-invariant matrix in Jerry format, then the condition 2.4.1 is satisfied, except for the format $Jerry_{12}$ of #11123.*

Note that the opposite implication in both Theorem 5.1.1 and 5.1.2 is false, although it seems true that

Conjecture 5.1.3. *Suppose Z is in Jerry format and has even number of nodes. If the condition 2.4.1 holds, then the deformation family of Z has a special member which is invariant under the $\mathbb{Z}/2\mathbb{Z}$-action 5.1.*

Remark 5.1.11. In [Duc18], Ducat constructs Fano 3-folds corresponding to the two Hilbert series #40663 and #40933. He constructs two deformation families for #40663, one in Tom_4 format and one in Jerry_{23} format. Regarding #40933, in [PR16] Prokhorov and Reid construct a deformation family in Jerry_{12} format. Here we construct the Tom_5 format of #40933.

In conclusion,

Theorem 5.1.12. *The double-cover method constructs at least one deformation family for 32 Hilbert series of index 2 Fano 3-folds in codimension 4.*

5.2 A birational link for an index 2 codimension 4 Fano 3-fold: the case of #39898

In the previous part of this chapter we constructed explicitly most of the codimension 4 index 2 Fano 3-folds. In this section we show a birational link starting from one of such Fano varieties, using similar techniques to the ones outlined in Chapter 2. In this case, the behaviour of the link is substantially different.

This is a work in progress joint with Tiago Guerreiro. This section is aimed to give a glimpse at this new development.

Consider the following Fanos:

#4896	X	$\subset \mathbb{P}^7(1^2,3,5,6,7,8,9)$	codimension 4	$2 \times \frac{1}{3}(1,1,2), \frac{1}{9}(1,1,8)$
#4895	Z	$\subset \mathbb{P}^6(1^2,3,5,6,7,8)$	codimension 3	14 nodes

The projection from the point $P_s \in X$ of type $\frac{1}{9}(1,1,8)$ gives the codimension 3 Fano 3-fold Z, containing the divisor $D \cong \mathbb{P}^2(1,1,8)_{x_1,x_2,x_3}$ with ideal $I_D := \langle y_1, y_2, y_3, y_4 \rangle$. Here y_1, y_2, y_3, y_4 have weights 7,6,5,3 respectively. In addition, Z is realised as pfaffians of a matrix M in Tom_3 format.

The Tom-type Fano 3-fold X obtained by a Type I unprojection of such $D \subset Z$ is a candidate to be a double cover of the codimension 4 Fano 3-fold in index 2 having Hilbert series #39898. This is because the general member of #39898 sits inside the weighted projective space $\mathbb{P}^7(1,2,3,5,6,7,8,9)$: Lemma 5.1.3 suggests that #39898 could be obtained as a $\mathbb{Z}/2\mathbb{Z}$-quotient of X #4896 via the group action γ defined in (5.1). Moreover, Z has even number of nodes, as in Corollary 5.1.9.

It is actually possible to write equations for X that are invariant (and not just equivariant) under the action γ, that is, in which the variable x_1 appears only with even powers. Therefore, the explicit equations for the $\mathbb{Z}/2\mathbb{Z}$-quotient $\tilde{X}$ are

$$
\begin{cases}
x_2^7y_4 + x_2^4y_2 + x_2^2y_4y_3 + \bar{x}^2y_2 + y_3^2 + y_4y_1 = 0\\
x_2^2y_4^3 + x_2^4y_1 + x_2^2y_4y_2 + y_4^2y_3 + \bar{x}^2y_1 + y_3y_2 + y_4x_3 = 0\\
\bar{x}^2y_4y_3 - y_4^2y_2 + \bar{x}y_4y_1 - y_2^2 + y_3y_1 = 0\\
x_2^8\bar{x}^2 + 2x_2^4\bar{x}^4 - x_2^7y_3 + \bar{x}^6 - x_2^4\bar{x}y_2 - x_2^4x_3 - \bar{x}^3y_2 - \bar{x}^2x_3 - y_3y_1 + y_4s = 0\\
x_2^7y_2 + x_2^4\bar{x}^2y_3 - x_2^4y_4y_2 + x_2^4\bar{x}y_1 + \bar{x}^4y_3 - x_2^2y_4^2y_3\\
-\bar{x}^2y_4y_2 + \bar{x}^3y_1 - y_4y_3^2 + y_2y_1 - y_3x_3 = 0\\
x_2^7y_1 - x_2^2y_4^2y_2 - \bar{x}^2y_3^2 - x_2^2y_2^2 + x_2^2y_3y_1 - \bar{x}y_3y_1 + y_1^2 - y_2x_3 = 0\\
-x_2^{14} - x_2^9y_3 - x_2^6\bar{x}y_4^2 - x_2^8y_2 - x_2^6y_4y_3 - x_2^6\bar{x}y_2 - 2x_2^7y_1 - x_2^2\bar{x}^3y_4^2 - x_2^4\bar{x}y_4y_3\\
-2x_2^4\bar{x}^2y_2 - x_2^2\bar{x}^2y_4y_3 - x_2^4y_3^2 - x_2^2\bar{x}^3y_2 - x_2^4\bar{x}x_3\\
-\bar{x}^3y_4y_3 - \bar{x}^4y_2 - \bar{x}^2y_3^2 - x_2^2y_3y_1 - \bar{x}^3x_3 - y_1^2 - y_3s = 0\\
-x_2^{11}\bar{x}^2 - x_2^7\bar{x}^4 + x_2^9y_4^2 + x_2^9y_2 + x_2^7y_4y_3 - x_2^4\bar{x}^3y_3 + x_2^6\bar{x}y_1 + x_2^7x_3 + x_2^4\bar{x}y_4y_2 - 2x_2^4\bar{x}^2y_1\\
-\bar{x}^5y_3 + x_2^2\bar{x}^3y_1 + \bar{x}^3y_4y_2 - 2\bar{x}^4y_1 + x_2^2y_4^2y_1 + x_2^2y_2y_1 + y_4y_3y_1 + y_1x_3 + y_2s = 0\\
x_2^6\bar{x}^2y_4^2 - x_2^7\bar{x}^2y_3 + x_2^6\bar{x}^2y_2 + x_2^7y_4y_2 - x_2^7\bar{x}y_1 + x_2^2\bar{x}^4y_4^2\\
+x_2^4\bar{x}^2y_4y_3 + x_2^2\bar{x}^4y_2 + x_2^4\bar{x}^2x_3 + \bar{x}^4y_4y_3 + x_2^4y_2^2 + x_2^2y_4y_3y_2 + \bar{x}^4x_3\\
-x_2^2y_4^2x_3 + \bar{x}^2y_2^2 - \bar{x}^2y_3y_1 - x_2^2y_2x_3 + y_3^2y_2 + y_4y_2y_1 - \bar{x}y_1^2 - y_4y_3x_3 - x_3^2 - y_1s = 0
\end{cases}
. \tag{5.3}
$$

Recall that the fixed locus of γ is $\mathrm{Fix}(\gamma) = \{x_1 = 0\} \cup \mathbb{P}^2(1,8,6)_{x_1,x_3,y_2}$. The projective space $\mathbb{P}^2(1,8,6)_{x_1,x_3,y_2} := \langle x_2, y_1, y_3, y_4, \rangle$ is the component of the fixed locus that we called $\mathbb{P}_{\text{even}}$ in Lemma 5.1.6. It is easy to see that D and $\mathbb{P}_{\text{even}}$ intersect along the projective line $\mathbb{P}^2(1,8)_{x_1,x_3}$.

The nodes on D can be found computing the 3×3 minors of the Jacobian matrix J of Z and then restricting it to D, i.e. $\bigwedge^3 J|_D = \underline{0}$. Their equations in the quotient $\tilde{Z}$ are

$$
\begin{cases}
x_2^{12}\bar{x}^2 + 3x_2^8\bar{x}^4 + 3x_2^4\bar{x}^6 - x_2^8x_3 + \bar{x}^8 - 2x_2^4\bar{x}^2x_3 - \bar{x}^4x_3 = 0\\
-x_2^{18} - x_2^14\bar{x}^2 - x_2^8\bar{x}x_3 - 2x_2^4\bar{x}^3x_3 - \bar{x}^5x_3 = 0\\
-x_2^{15}\bar{x}^2 - 2x_2^11\bar{x}^4 - x_2^7\bar{x}^6 + x_2^11x_3 + x_2^7\bar{x}^2x_3 = 0\\
x_2^8\bar{x}^2x_3 + 2x_2^4\bar{x}^4x_3 + \bar{x}^6x_3 - x_2^4x_3^2 - \bar{x}^2x_3^2 = 0\\
-x_2^{21} - x_2^11\bar{x}x_3 - x_2^7\bar{x}^3x_3 = 0\\
-x_2^{14}x_3 - x_2^4\bar{x}x_3^2 - \bar{x}^3x_3^2 = 0\\
-x_2^{11}\bar{x}^2x_3 - x_2^7\bar{x}^4x_3 + x_2^7x_3^2 = 0\\
x_2^4\bar{x}^2x_3^2 + \bar{x}^4x_3^2 - x_3^3 = 0
\end{cases}
$$

The above equations describe the 7 nodes on $\tilde{D} \subset \tilde{Z}$. Obviously they double in number when replacing x_1^2 back instead of $\bar{x}$.

Now that we explicitly constructed the index 2 Fano 3-fold #39898 we can use the techniques used in Chapter 2 to run a birational link starting from the pair $(\tilde{X}, P_s)$, where $P_s \in \tilde{X}$ is the cyclic quotient singularity of type $\frac{1}{9}(2,1,8)$, with orbinates $\bar{x}, x_2, x_3$ respectively. We use the same notation used for Sarkisov links introduced in Section 1.2.5. In Chapter 2 we started the link by performing a Kawamata blow up of the Type I centre at P_s; this step relied on the fact that Kawamata's theorem 2.2.3 held in our setting. This is no longer true in the index 2 context. In fact, Kawamata's theorem holds if the centre is of type $\frac{1}{r}(1,a,r-a)$, with a and r coprime. This condition on the centre is not fulfilled by $\frac{1}{9}(2,1,8)$. Therefore, a manipulation of the weight of the orbinates is needed. This means that we let $\mathbb{Z}/9\mathbb{Z}$ act on the orbinates until we get weights satisfying the hypotheses of Kawamata's theorem. What we get is an equivalent cyclic quotient singularity: in our case we have $\frac{1}{9}(2,1,8) \sim \frac{1}{9}(1,5,4)$.

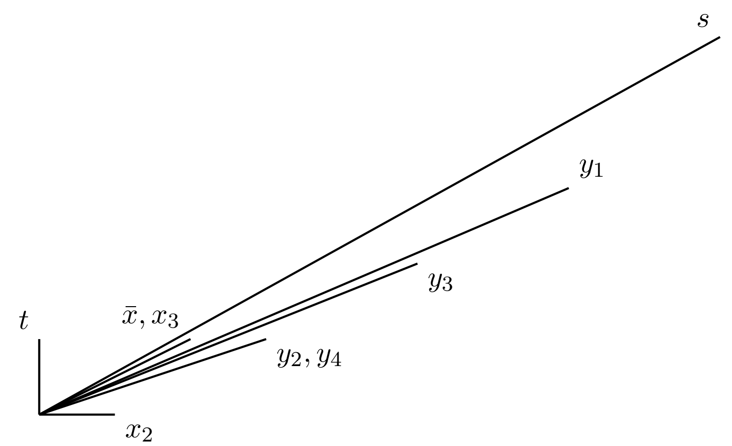
Applying Kawamata's theorem 2.2.3 and the same strategy to assign the bottom weights $\delta_1, \dots, \delta_4$ explained in Section 2.2.2, we obtain the following rank 2 toric variety.

$$\left(\begin{array}{cc|ccccccc} t & s & \bar{x} & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & 9 & 2 & 1 & 8 & 7 & 6 & 5 & 3 \\ -9 & 0 & 1 & 5 & 4 & 8 & 3 & 7 & 6 \end{array}\right).$$

Note that it is not well-formed, and that the lattice vectors $\rho_t, \dots, \rho_{y_4}$ are not in clockwise order as they were in the index 1 construction. After well-forming and reordering the above rank 2 toric variety we get

$$\mathbb{F}_1 := \left(\begin{array}{cc|ccccccc} t & s & \bar{x} & x_3 & y_1 & y_3 & y_4 & y_2 & x_2 \\ 0 & 9 & 2 & 8 & 7 & 5 & 3 & 6 & 1 \\ 1 & 5 & 1 & 4 & 3 & 2 & 1 & 2 & 0 \end{array}\right).$$

The toric variety $\mathbb{F}_1$ is associated to its Mori cone:



The reader can immediately notice that the first wall of the mobile cone of $\mathbb{F}_1$ is generated by the rays of only two of the orbinates. This plays an important role in determining the behaviour of Y_1 when crossing the $(\bar{x}, x_3)$-wall.

Fact 1. The first wall of the mobile cone is always generated by the following two vectors: $\rho_{\bar{x}}$, ρ_{x_i} for i equal to either 2 or 3, where the weight of x_i is even.

In contrast to the birational links in the index 1 case, here we have the following fact regarding the first step of the birational link.

Fact 2. If X is quasi-smooth, then the birational map $\psi_1 \colon Y_1 \to Y_2$ of the birational link for $\tilde{X}$ is an isomorphism.

The second and third maps in the birational link are both isomorphisms for the varieties Y_2 and Y_3 respectively.

The last map Φ' is a divisorial contraction to a Fano 3-fold X' in the weighted projective space $\mathbb{P}^7(1^3, 2^2, 3, 4, 5)$ correspondent to the Hilbert series #11106.

Note that X' has codimension 4. In fact, in the case of birational links for codimension 4 index 2 Fano 3-folds, the link does not always simplify the structure of X as it was happening in the index 1 context.

Chapter 6

Appendix: Tables

This table summarises the results for the Sarkisov links for index 1 codimension 4 Fano 3-folds X of Tom type having Picard rank 1 terminating with del Pezzo fibrations.

Table 6.1: Sarkisov links ending with del Pezzo fibrations

ID of codim 4	Centre	Format	Degree of dP
574	7	T_1	1
644	10	T_2	1
1395	9	T_5	1
1401	7	T_4	1
2421	8	T_5	1
5516	3	T_1	2
5519	3	T_1	2
5530	3	T_1	2
5845	6	T_4	2
5867	4	T_2	2
5870	5	T_2	3
5914	4	T_2	2
5970	4	T_1	3
6878	3	T_1	3
11004	7	T_2	2
11104	7	T_5	2
11123	5	T_4	2
11437	2	T_1	3
11437	5	T_3	3
11440	2	T_1	3

11455	2	T_1	4
16206	5	T_4	3
16228	4	T_2	3
16246	3	T_2	3
16339	3	T_1	4
20544	2	T_1	4
20652	2	T_1	5

This table summarises the results for the Sarkisov links terminating with conic bundles for index 1 Fano 3-folds X of Tom type in codimension 1 having Picard rank 1.

Table 6.2: Sarkisov links ending with conic bundles

ID of codim 4	Centre	Format
6865	4	T_1
12063	2	T_1
12960	2	T_1
16227	5	T_2
20524	4	T_4
20544	3	T_2
24078	3	T_1
24097	2	T_1

The following table collects all the 37 $\mathbb{Q}$-Fano 3-folds of index 2 in the Graded Ring Database [BK+15] together with their index 1 double cover, and the formats in codimension 3 index 1 that allow the construction described in Section 5.

Table 6.3: Index 2 Fano 3-folds in codimension 4

Index 2	Index 1	T&J
39557	327	$T_3,\ J_{24}$
39569	512	none
39576	569	T_1
39578	574	$T_1,\ J_{24\bullet 12}$
39605	869	$T_4,\ J_{13}$
39607	872	none
39660	1158	$T_5,\ J_{12}$
39675	1395	T_5
39676	1401	$\frac{1}{5}: T_2;\ \frac{1}{7}: T_4$

39678	1405	T_1
39890	4810	$T_3,\ J_{24}$
39898	4896	$T_3,\ J_{24}$
39906	4925	$\frac{1}{7}(1,1,6)\colon T_2;\ \frac{1}{7}(1,3,4)\colon T_1$
39912	4938	T_2
39913	4939	$\frac{1}{5}\colon T_1,\ J_{25\bullet 24};\ \frac{1}{7}\colon T_2,\ J_{14\bullet 13}$
39928	4987	T_5
39929	5000	$\frac{1}{5}\colon T_2;\ \frac{1}{9}\colon T_4$
39934	5052	$T_1,\ J_{23\bullet 13}$
39961	5176	$\frac{1}{5}\colon T_2;\ \frac{1}{7}\colon T_3$
39968	5260	$T_5,\ J_{13}$
39969	5266	$\frac{1}{5}\colon T_3,\ J_{24\bullet 25};\ \frac{1}{7}\colon T_4,\ J_{13\bullet 15}$
39970	5279	$\frac{1}{3}\colon T_1;\ \frac{1}{5}(1,1,4)\colon T_2;\ \frac{1}{5}(1,2,3)\colon T_1$
39991	5516	$\frac{1}{3}\colon T_1;\ \frac{1}{7}\colon T_3$
39993	5519	$\frac{1}{3}\colon T_1,\ J_{34};\ \frac{1}{5}\colon T_2,\ J_{12}$
40360	10963	$T_3,\ J_{24}$
40367	none	none
40370	11004	T_2
40371	11005	$\frac{1}{3}\colon T_1,\ J_{25\bullet 24};\ \frac{1}{5}\colon T_2,\ J_{14\bullet 13}$
40378	none	none
40399	11104	T_5
40400	11123	$\frac{1}{3}\colon T_3;\ \frac{1}{5}\colon T_4$
40407	11222	$T_1,\ J_{23\bullet 13}$
40663	16206	$T_4,\ J_{23}$
40671	16227	T_2
40672	16246	$T_2,\ J_{15\bullet 14}$
40933	24078	$T_5,\ J_{12}$
41028	none	none

Big Table of Birational links

Notation: The codimension 3 Fano 3-fold Z is embedded in a weighted $w\mathbb{P}^6 = \mathbb{P}^6(a,b,c,d_1,d_2,d_3,d_4)$ having coordinates $x_1,x_2,x_3,y_1,y_2,y_3,y_4$ respectively. It contains a divisor $D \subset Z$ defined by the vanishing of the variables y_1,y_2,y_3,y_4.

The codimension 4 Fano 3-fold X is embedded in a weighted $w\mathbb{P}^7 = \mathbb{P}^7(a,b,c,d_1,d_2,d_3,d_4,r)$ having coordinates $x_1,x_2,x_3,y_1,y_2,y_3,y_4,s$ respectively. It contains a Type I centre $P \in X$ produced by the Type I unprojection from Z.

The rank 2 toric variety $\mathbb{F}_1$ has 9 variables, called $t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4$ having respectively weights

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

The general notation we adopt for the birational links is:

$$\begin{tikzcd} & Y_1 \arrow[dl, "\phi"'] \arrow[r, dashed, "\psi_1"] & Y_2 \arrow[r, dashed, "\psi_2"] & Y_3 \arrow[r, dashed, "\psi_3"] & Y_4 \arrow[dr, "\phi'"] & \\ X & Z \arrow[l, dashed, "\text{unproj}"'] & & & & X' \end{tikzcd}.$$

Note that, depending on the d_j, the link might be shorter: in the case in which two of the d_j are equal, ψ_3 does not occur.

- The left-hand part of the table records the ID of the Hilbert series of index 1 and codimension 4 in the Graded Ring Database, together with the correspondent weights W of its ambient space $\mathbb{P}^7(W)$.
- The central part of the table contains:
  - The index r_P of the Type I centre $P \in X$, that is a cyclic quotient singularity $\frac{1}{r_P}(a,b,c)$. The projection of X from P gives the codimension 3 Fano 3-fold Z underlying the construction of X.

- The Tom_i format of Z (Definition 2.2 of [BKR12a]), together with the number of nodes of Z on the divisor D whose unprojection produces X. In particular, the number of nodes gives the number of flopping curves of ψ_1.
- The weights of the syzygy matrix M of Z.

• The right-hand side of the table records:

- The weights of $y_1, \dots, y_4$ in $\mathbb{P}(W)$, in the order in which they appear in the Mori cone of Y_1. In particular, $d_1 \geq d_2 \geq d_3 \geq d_4$.
- The nature of the map ψ_2: it can be either a toric flip (for instance, $(1, 5, -2, -3)$), or an hypersurface flip (e.g. $(1, 3, 10, -1, -3; 7)$), or an isomorphism (indicated by IM).
- The nature of the map ψ_3: it can have the same range of options of ψ_2. In addition, we write N/A when ψ_3 does not occur; this happens when two or more of the d_j are equal.
- The type of extremal contraction of Φ'. It can be either another Fano 3-fold, or a del Pezzo fibration, or a conic bundle. We use the following notation: (m, n) where m is the dimension of the exceptional locus of Φ' in Y_4 and n is the dimension of its image.
- The endpoint of the birational link. When the Φ' is a divisorial contraction to a point $Q \in X'$ it can happen that Q is an orbifold point: we record its index below Φ' using the notation r_Q.

Table 6.4: Birational links for codimension 4 Fano 3-folds

				Birational link				
ID: W of $\mathbb{P}(W) \supset X \ni P$	r_P	T_i, N	wtsM_Z	d_j	ψ_2	ψ_3	Φ'	End of link
327: $(1, 5, 5, 6, 7, 8, 9, 11)$	11	T_3 6	5 6 7 8 7 8 9 9 10 11	$9, 8, 7, 5$	$(1, 5, -1, -4)$	IM	$(2, 0)$ $r_Q = 2$	$X'_{9,10} \subset \mathbb{P}^5(1, 2, 3, 4, 5^2)$ is #1179

393: $(1,4,5,5,6,7,8,9)$	9	T_1 7	4 5 6 7 6 7 8 8 9 10	$8,7,6,5$	IM	$(1,4,1,-1,-2;2)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2,3,4,5^2)$ is #5175
455: $(1,4,4,5,6,7,9,13)$	13	T_5 5	4 5 6 7 6 7 8 8 9 10	$7,6,5,4$	$(1,4,-1,-3)$	IM	$(2,0)$	$X'_{10} \subset \mathbb{P}^4(1^2,2,3,4)$ is #5157
511: $(1,3,5,6,7,8,11,14)$	14	T_4 7	5 6 7 8 7 8 9 9 10 11	$8,7,6,5$	$(1,3,11,-2,-3;9)$	$(1,3,-1,-2)$	$(2,0)$	$X'_9 \subset \mathbb{P}^4(1^2,2,3^2)$ is #5257
549: $(1,3,4,5,6,7,10,13)$	13	T_5 6	4 5 6 7 6 7 8 8 9 10	$7,6,5,4$	$(1,3,10,-1,-3;7)$	IM	$(2,0)$	$X'_9 \subset \mathbb{P}^4(1^2,2,3^2)$ is #5257
569: $(1,3,4,5,5,6,7,9)$	9	T_1 6	3 4 5 6 5 6 7 7 8 9	$7,6,5,3$	IM	$(1,5,1,-1,-3;2)$	$(2,0)$ $r_Q=2$	$X' \subset \mathbb{P}^6(1,2,3^2,4^2,5)$ is #1409
570: $(1,3,4,5,5,6,7,8)$	8	T_2 7	3 4 5 6 5 6 7 7 8 9	$7,6,5,4$	$(1,5,-2,-3)$	$(1,3,-1,-2)$	$(2,0)$	$X'_{6,7} \subset \mathbb{P}^5(1^2,2,3^2,4)$ is #5261
574: $(1,3,4,5,5,6,7,7)$	7	T_1 8	3 4 5 6 5 6 7 7 8 9	$7,6,5,5$	IM	$(1,3,1,-1,-1;2)$	$(3,1)$	dP fibration of degree 1

642: $(1,3,4,4,5,6,7,11)$	11	T_2 5	$\begin{array}{cccc} 4&4&5&6\\ &5&6&7\\ &&6&7\\ &&&8\end{array}$	$6,5,4,3$	IM	$(1,4,1,-1,-2;2)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2,3^2,4,7)$ is #5262
644: $(1,3,4,4,5,6,7,10)$	10	T_2 6	$\begin{array}{cccc} 4&4&5&6\\ &5&6&7\\ &&6&7\\ &&&8\end{array}$	$6,5,4,4$	IM	$(1,3,1,-1,-1;2)$	$(2,1)$	dP fibration of degree 1
645: $(1,3,4,4,5,6,7,7)$	7	T_2 7	$\begin{array}{cccc} 4&4&5&6\\ &5&6&7\\ &&6&7\\ &&&8\end{array}$	$7,6,5,4$	$(7,1,3,-1,-3;4)$	IM	$(2,0)$	$X'_{6,7} \subset \mathbb{P}^5(1^2,2,3^2,4)$ is #5261
869: $(1,3,3,4,5,7,10,13)$	13	T_4 6	$\begin{array}{cccc} 3&4&5&7\\ &5&6&8\\ &&7&9\\ &&&10\end{array}$	$7,5,4,3$	$(1,3,10,-3,-4;7)$	$(1,3,10,-1,2)$	$(2,0)$	$X'_{6,7} \subset \mathbb{P}^5(1^2,2,3^2,4)$ is #5261
1082: $(1,2,5,6,7,9,11,13)$	13	T_5 9	$\begin{array}{cccc} 5&6&7&8\\ &7&8&9\\ &&9&10\\ &&&11\end{array}$	$9,7,6,5$	$(1,2,11,-2,-3;8)$	IM	$(2,0)$	$X'_{10} \subset \mathbb{P}^4(1^2,2^2,5)$ is #5837
1091: $(1,2,5,6,7,7,8,9)$	9	T_1 10	$\begin{array}{cccc} 4&5&6&7\\ &6&7&8\\ &&8&9\\ &&&10\end{array}$	$8,7,6,5$	IM	$(1,1,7,-1,-2;6)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2^2,3,5,7)$ is #5840
1158: $(1,2,3,5,5,7,12,17)$	17	T_5 4	$\begin{array}{cccc} 2&3&5&7\\ &5&7&9\\ &&8&10\\ &&&12\end{array}$	$7,5,3,2$	$(1,5,12,-2,-5;7)$	IM	$(2,0)$	$X'_{6,10} \subset \mathbb{P}^5(1^2,2,3,5^2)$ is #5156

1167: $(1,2,3,4,5,7,9,11)$	11	T_5 7	3 4 5 6 5 6 7 7 8 9	$7,5,4,3$	$(1,2,9,-2,-3;6)$	IM	$(2,0)$	$X'_9 \subset \mathbb{P}^4(1^2,2,3^2)$ is #5257
1169: $(1,2,3,4,5,7,7,9)$	7	T_1 7	3 4 5 6 5 6 7 7 8 9	$9,7,5,2$	$(9,1,-2,-7)$	IM	$(2,0)$ $r_Q=3$	$X' \subset \mathbb{P}^6(1,2,3^2,4,5,7)$ is #1394
	9	T_2 8	2 3 4 5 5 6 7 7 8 9	$7,5,4,3$	$(1,7,-3,-4)$	$(1,2,7,-1,-2;5)$	$(2,0)$	$X'_{6,6} \subset \mathbb{P}^5(1^2,2^2,3,4)$ is #5843
1181: $(1,2,3,4,5,5,7,12)$	12	T_3 4	3 4 4 5 5 5 6 6 7 7	$5,4,3,2$	IM	$(4,1,1,-1,-2;2)$	$(2,0)$	$X'_{8,10} \subset \mathbb{P}^5(1^2,2,3,5,7)$ is #5155
1182: $(1,2,3,4,5,5,7,9)$	5	T_1 11	3 4 5 6 5 6 7 7 8 9	$9,7,5,4$	$(9,1,2,-2,-5;4)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2,3^2,4,5)$ is #5264
	9	T_3 7	3 4 4 5 5 5 6 6 7 7	$5,5,4,3$	$(1,2,7,-1,-2;5)$, $(5,1,2,-1,-2;4)$	N/A	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^4(1^2,2^3,3)$ is #6858
1183: $(1,2,3,4,5,5,7,7)$	7	T_3 7	3 4 4 5 5 5 6 6 7 7	$7,5,4,3$	$(7,1,2,-2,-3;4)$	IM	$(2,0)$	$X'_{6,7} \subset \mathbb{P}^5(1^2,2^2,3,5)$ is #5839

1185: $(1,2,3,4,5,5,6,8)$	8	T_1 6	$\begin{smallmatrix}2&3&4&5\\ &4&5&6\\ &&6&7\\ &&&8\end{smallmatrix}$	$6,5,4,2$	IM	$(5,1,1,-1,-3;2)$	$(2,0)$ $r_Q=2$	$X'\subset\mathbb{P}^6(1,2^2,3^2,4,5)$ is #2420
1186: $(1,2,3,4,5,5,6,7)$	7	T_1 8	$\begin{smallmatrix}2&3&4&5\\ &4&5&6\\ &&6&7\\ &&&8\end{smallmatrix}$	$6,5,4,3$	IM	$(5,1,1,-1,-2;3)$	$(2,0)$	$X'\subset\mathbb{P}^6(1^2,2^2,3^2,5)$ is #5858
1218: $(1,2,3,4,5,5,5,6)$	5	T_1 10	$\begin{smallmatrix}2&3&4&5\\ &4&5&6\\ &&6&7\\ &&&8\end{smallmatrix}$	$6,5,5,4$	IM	N/A	$(2,1)$	$X'_{4,5}\subset\mathbb{P}^5(1^3,2^2,3)$ is #11102
		T_3 9 $\bullet_{12,45}$		$6,5,5,4$	$(1,2,-1,-1)$	$2\times(2,0)$	N/A	$X'_7\subset\mathbb{P}^4(1^3,2,3)$ is #10981
1251: $(1,2,3,4,4,5,7,11)$	11	T_5 4	$\begin{smallmatrix}2&3&4&5\\ &4&5&6\\ &&6&7\\ &&&8\end{smallmatrix}$	$5,4,3,2$	$(1,4,-1,-3)$	IM	$(2,0)$	$X'_8\subset\mathbb{P}^4(1^2,2^2,3)$ is #5838
1253: $(1,2,3,4,4,5,5,7)$	7	T_1 7	$\begin{smallmatrix}3&4&4&5\\ &4&4&5\\ &&5&6\\ &&&6\end{smallmatrix}$	$5,4,4,3$	IM	N/A	$(2,1)$	$X'\subset\mathbb{P}^6(1^3,2^2,3,5)$ is #11103
		T_5 6		$5,4,4,3$	$(1,2,-1,-1)$	N/A	$(2,1)$	$X'_{6,6}\subset\mathbb{P}^5(1^3,2,3,5)$ is #10982
1392: $(1,2,3,3,4,5,8,11)$	11	T_4 5	$\begin{smallmatrix}2&3&4&5\\ &4&5&6\\ &&6&7\\ &&&8\end{smallmatrix}$	$5,4,3,2$	$(1,3,8,-2,-3;5)$	$(1,3,-1,-2)$	$(2,0)$	$X'\subset\mathbb{P}^4(1^2,2^2,3^2)$ is #5857

1395: $(1,2,3,3,4,5,7,9)$	9	T_5 6	$\begin{matrix}3&3&4&5\\&4&5&6\\&&5&6\\&&&7\end{matrix}$	$5,4,3,3$	$(1,2,7,-1,-2;5)$	IM	$(3,1)$	dP fibration of degree 1
1397: $(1,2,3,3,4,5,5,8)$	5	T_1 9	$\begin{matrix}2&3&4&5\\&4&5&6\\&&6&7\\&&&8\end{matrix}$	$8,5,4,3$	$(8,1,-3,-5)$	$(1,3,-1,-2)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2^2,3^2,5)$ is #5858
	8	T_2 5	$\begin{matrix}3&3&4&4\\&4&5&5\\&&5&5\\&&&6\end{matrix}$	$5,4,3,2$	$(1,5,-2,-3)$	$(1,3,5,-1,-2;4)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2^2,3^2,5)$ is #5858
1401: $(1,2,3,3,4,5,5,7)$	5	T_2 8	$\begin{matrix}3&3&4&5\\&4&5&6\\&&5&6\\&&&7\end{matrix}$	$7,5,4,3$	$(7,1,2,-2,-3;4)$	IM	$(2,0)$	$X'_8 \subset \mathbb{P}^4(1^2,2^2,3)$ is #5838
	7	T_4 6	$\begin{matrix}3&3&4&4\\&4&5&5\\&&5&5\\&&&6\end{matrix}$	$5,4,3,3$	$(1,2,5,-1,-2;3)$	IM	$(3,1)$	dP fibration of degree 1
1405: $(1,2,3,3,4,5,5,5)$	5	T_1 8	$\begin{matrix}3&3&4&4\\&4&5&5\\&&5&5\\&&&6\end{matrix}$	$5,5,4,3$	$2\times(5,1,2,-1,-2;3)$	N/A	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2^3,3^2)$ is #6859
1410: $(1,2,3,3,4,4,5,7)$	7	T_3 5	$\begin{matrix}2&3&3&4\\&4&4&5\\&&5&6\\&&&6\end{matrix}$	$5,4,3,2$	$(1,4,-1,-3)$	IM	$(2,0)$	$X'_{5,6} \subset \mathbb{P}^5(1^2,2^2,3^2)$ is #5857

1413: $(1,2,3,3,4,4,5,5)$	5	T_1 8	2 3 3 4 4 4 5 5 6 6	$5,4,4,3$	IM	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^3,2^2,3^2)$ is #11122
		T_3 7 $\bullet_{12,45}$		$5,4,4,3$	$(1,2,-1,-1)$	$2\times(2,0)$	N/A	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
2405: $(1,2,2,3,5,7,9,11)$	11	T_4 7	2 3 4 5 5 6 7 7 8 9	$7,5,3,2$	$(1,2,9,-2-5;4)$	IM	$(2,0)$	$X' \subset \mathbb{P}^5(1^2,2^2,3,5)$ is #11102
2421: $(1,2,2,3,3,4,5,8)$	8	T_5 4	2 2 3 4 3 4 5 4 5 6	$4,3,2,2$	$(1,3,-1,-2)$	IM	$(3,1)$	dP fibration of degree 1
2422: $(1,2,2,3,3,4,5,7)$	7	T_2 6	2 2 3 4 3 4 5 4 5 6	$4,3,3,2$	IM	N/A	$(2,1)$	$X'_{4,4} \subset \mathbb{P}^5(1^3,2^3)$ is #11435
		T_5 5 $\bullet_{13}$		$4,3,3,2$	$(1,2,-1,-1)$	$2\times(2,0)$	N/A	$X'_6 \subset \mathbb{P}^4(1^3,2^2)$ is #11101
2427: $(1,2,2,3,3,4,5,5)$	5	T_1 7	2 2 3 4 3 4 5 4 5 6	$5,4,3,2$	$(5,1,-2,-3)$	$(1,3,-1,-2)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^2,2^3,3^2)$ is #6859
		T_4 6 $\bullet_{13}$		$5,4,3,2$	$(5,1,2,-1,-2;4)$	$(2,0)$	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^2,2^3,3)$ is #6858

4797: $(1,1,6,8,9,10,11,12)$	12	T_3 20	$\begin{smallmatrix}6&7&8&9\\&8&9&10\\&&10&11\\&&&12\end{smallmatrix}$	$10,9,8,6$	$(1,1,11,-1,-4;7)$	IM	$(2,0)$ $r_Q=2$	$X'_{10}\subset\mathbb{P}^4(1^2,2,3,4)$ is #5157
4810: $(1,1,5,7,8,9,10,11)$	11	T_3 18	$\begin{smallmatrix}5&6&7&8\\&7&8&9\\&&9&10\\&&&11\end{smallmatrix}$	$9,8,7,5$	$(1,1,10,-1,-4;6)$	IM	$(2,0)$ $r_Q=2$	$X'_{6,9}\subset\mathbb{P}^5(1^2,2,3,4,5)$ is #5159
4825: $(1,1,4,6,7,8,9,10)$	10	T_3 16	$\begin{smallmatrix}4&5&6&7\\&6&7&8\\&&8&9\\&&&10\end{smallmatrix}$	$8,7,6,4$	$(1,1,9,-1,-4;5)$	IM	$(2,0)$ $r_Q=2$	$X'_{6,8}\subset\mathbb{P}^5(1^2,2,3,4^2)$ is #5200
4839: $(1,1,4,5,6,7,8,9)$	5	T_1 21	$\begin{smallmatrix}4&5&6&7\\&6&7&8\\&&8&9\\&&&10\end{smallmatrix}$	$9,8,7,6$	$(9,1,1,-1,-3;3)$	IM	$(2,0)$	$X'_7\subset\mathbb{P}^4(1^3,2,3)$ is #10981
		T_2 20 $\bullet_{13,45}$		$9,8,7,6$	$(9,1,1,-1,-2;5)$	$(2,0)$	$(2,0)$	$X'_8\subset\mathbb{P}^4(1^3,2,4)$ is #10980
	9	T_3 14	$\begin{smallmatrix}4&5&5&6\\&6&6&7\\&&7&8\\&&&8\end{smallmatrix}$	$7,6,5,4$	$(1,1,8,-1,-3;5)$	IM	$(2,0)$	$X'_{5,6}\subset\mathbb{P}^5(1^3,2,3,4)$ is #10983
		T_5 13 $\bullet_{14}$		$7,6,5,4$	$(1,1,8,-1,-2;6)$	$(2,0)$	$(2,0)$	$X'_8\subset\mathbb{P}^4(1^3,2,4)$ is #10980
4850: $(1,1,4,5,6,6,7,13)$	13	T_2 4	$\begin{smallmatrix}4&4&5&6\\&5&6&7\\&&6&7\\&&&8\end{smallmatrix}$	$6,5,4,1$	IM	$(1,6,1,-1,-4;2)$	$(2,0)$ $r_Q=3$	$X'\subset\mathbb{P}^6(1^2,3,4,5,6,7)$ is #4914

4851: $(1,1,4,5,6,6,7,8)$	6	T_1 16	$\begin{array}{cccc}4&5&5&6\\&6&6&7\\&&7&8\\&&&8\end{array}$	$8,7,6,4$	$(8,1,1,-1,-4;4)$	IM	$(2,0)$ $r_Q=2$	$X' \subset \mathbb{P}^6(1^2,2,3,4^2,5)$ is #5201
	8	T_2 14	$\begin{array}{cccc}4&4&5&6\\&5&6&7\\&&6&7\\&&&8\end{array}$	$6,6,5,4$	$(1,1,7,-1,-2;5)$, $(6,1,1,-1,-2;4)$	N/A	$(2,0)$	$X'_6 \subset \mathbb{P}^4(1^3,2^2)$ is #11101
4860: $(1,1,4,5,6,6,7,7)$	7	T_2 14	$\begin{array}{cccc}4&4&5&6\\&5&6&7\\&&6&7\\&&&8\end{array}$	$7,6,5,4$	$(7,1,1,-1,-2;5)$	IM	$(2,0)$	$X'_7 \subset \mathbb{P}^4(1^3,2,3)$ is #10981
		T_4 13 $\bullet_{13}$		$7,6,5,4$	$(7,1,1,-1,-2;5)$	$(2,0)$	$(2,0)$	$X'_8 \subset \mathbb{P}^4(1^3,2,4)$ is #10980
4896: $(1,1,3,5,6,7,8,9)$	9	T_3 14	$\begin{array}{cccc}3&4&5&6\\&5&6&7\\&&7&8\\&&&9\end{array}$	$7,6,5,3$	$(1,1,8,-1,-4;4)$	IM	$(2,0)$ $r_Q=2$	$X'_{6,7} \subset \mathbb{P}^5(1^2,2,3^2,4)$ is #5261
4915: $(1,1,3,4,5,6,7,8)$	4	T_1 20	$\begin{array}{cccc}3&4&5&6\\&5&6&7\\&&7&8\\&&&9\end{array}$	$8,7,6,5$	$(8,1,1,-1,-3;5)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
		T_2 19 $\bullet_{13,45}$		$7,6,5,4$	$(8,1,1,-1,-2;6)$	$(2,0)$	$(2,0)$	$X'_7 \subset \mathbb{P}^4(1^3,2,3)$ is #10981
	8	T_3 12	$\begin{array}{cccc}3&4&4&5\\&5&5&6\\&&6&7\\&&&7\end{array}$	$6,5,4,3$	$(1,1,7,-1,-3;5)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002

		T_5 11 $\bullet_{14}$		$6,5,4,3$	$(1,1,7,-1,-2;5)$	$(2,0)$	$(2,0)$	$X'_7 \subset \mathbb{P}^4(1^3,2,3)$ is #10981
4925: $(1,1,3,4,5,6,7,7)$	$[7,1]$	T_2 14	$\begin{smallmatrix}1&3&4&5\\&4&5&6\\&&7&8\\&&&9\end{smallmatrix}$	$7,5,4,3$	$(7,1,-3,-4)$	$(1,1,6,-1,-2;4)$	$(2,0)$	$X'_{5,6} \subset \mathbb{P}^5(1^3,2,3,4)$ is #10983
	$[7,3]$	T_1 6	$\begin{smallmatrix}3&4&4&5\\&5&5&6\\&&6&7\\&&&7\end{smallmatrix}$	$7,6,5,1$	$(7,1,-1,-6)$	IM	$(2,0)$ $r_Q=4$	$X' \subset \mathbb{P}^6(1^2,3,4^2,5,6)$ is #4988
4938: $(1,1,3,4,5,5,6,11)$	11	T_2 4	$\begin{smallmatrix}3&3&4&5\\&4&5&6\\&&5&6\\&&&7\end{smallmatrix}$	$5,4,3,1$	IM	$(1,5,1,-1,-3;2)$	$(2,0)$ $r_Q=2$	$X' \subset \mathbb{P}^6(1^2,2,3,4,5,6)$ is #5162
4939: $(1,1,3,4,5,5,6,7)$	5	T_1 14	$\begin{smallmatrix}3&4&4&5\\&5&5&6\\&&6&7\\&&&7\end{smallmatrix}$	$7,6,5,3$	$(7,1,1,-1,-4;3)$	IM	$(2,0)$ $r_Q=2$	$X' \subset \mathbb{P}^6(1^2,2,3^2,4^2)$ is #5302
	7	T_2 12	$\begin{smallmatrix}3&3&4&5\\&4&5&6\\&&5&6\\&&&7\end{smallmatrix}$	$5,5,4,3$	$(5,1,1,-1,-2;3)$, $(1,1,6,-1,-2;4)$	N/A	$(2,0)$	$X'_{4,5} \subset \mathbb{P}^5(1^3,2^2,3)$ is #11102
4949: $(1,1,3,4,5,5,6,6)$	6	T_2 12	$\begin{smallmatrix}3&3&4&5\\&4&5&6\\&&5&6\\&&&7\end{smallmatrix}$	$6,5,4,3$	$(6,1,1,-1,-3;3)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
		T_4 11 $\bullet_{13}$		$6,5,4,3$	$(6,1,1,-1,-2;4)$	$(2,0)$	$(2,0)$	$X'_7 \subset \mathbb{P}^4(1^3,2,3)$ is #10981

4987: $(1,1,3,4,4,5,9,13)$	13	T_5 4	1 3 4 5 4 5 6 7 8 9	$5,4,3,1$	$(1,4,9,-1,-4;5)$	IM	$(2,0)$ $r_Q=2$	$X' \subset \mathbb{P}^6(1^2,2,3,4^2)$ is #5200
4989: $(1,1,3,4,4,5,6,7)$	4	T_1 16	3 4 4 5 5 5 6 6 7 7	$7,6,5,4$	$(7,1,1,-1,-3;4)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2,3^2,4)$ is #11003
		T_2 15 $\bullet_{14}$		$7,6,5,4$	$(7,1,1,-1,-2;5)$	$(2,0)$	$(2,0)$	$X'_{5,6} \subset \mathbb{P}^5(1^3,2,3,4)$ is #10983
	7	T_3 11	3 4 4 5 4 4 5 5 6 6	$5,4,4,3$	$(1,1,6,-1,-2;4)$	N/A	$(2,1)$	$X'_5 \subset \mathbb{P}^4(1^4,2)$ is #16203
		T_5 10		$5,4,4,3$	$(1,1,6,-1,-2;4)$	N/A	$2\times(2,0)$	$X'_6 \subset \mathbb{P}^4(1^4,2)$ is #16202
5000: $(1,1,3,4,4,5,5,9)$	5	T_2 14	1 3 4 5 4 5 6 7 8 9	$9,5,4,3$	$(9,1,-4,-5)$	$(1,1,4,-1,-2;2)$	$(2,0)$	$X'_{5,6} \subset \mathbb{P}^5(1^3,2,3,4)$ is #10983
	9	T_4 4	3 3 4 4 4 5 5 5 5 6	$5,4,3,1$	$(1,5,-1,-4)$	IM	$(2,0)$ $r_Q=2$	$X' \subset \mathbb{P}^6(1^2,2,3,4^2,5)$ is #5201
5002: $(1,1,3,4,4,5,5,6)$	4	T_1 15	3 3 4 5 4 5 6 5 6 7	$6,5,5,4$	$(6,1,1,-1,-2;4)$	N/A	$(2,1)$	$X'_5 \subset \mathbb{P}^4(1^4,2)$ is #16203

		T_2 14 $\bullet_{13,45}$		6, 5, 5, 4	$(6,1,1,-1,-1;5)$	N/A	$2\times(2,0)$	$X_6'\subset\mathbb{P}^4(1^4,3)$ is #16202
	5	T_1 12	$\begin{matrix}3&4&4&5\\&4&4&5\\&&5&6\\&&&6\end{matrix}$	6, 5, 4, 3	$(6,1,1,-1,-3;3)$	IM	$(2,0)$	$X'\subset\mathbb{P}^6(1^3,2,3^2,4)$ is #11003
		T_3 11 $\bullet_{24}$		6, 5, 4, 3	$(6,1,1,-1,-2;4)$	N/A	$2\times(2,0)$	$X_{5,6}'\subset\mathbb{P}^5(1^3,2,3,4)$ is #10983
	6	T_2 11	$\begin{matrix}3&3&4&4\\&4&5&5\\&&5&5\\&&&6\end{matrix}$	5, 4, 4, 3	$(1,1,5,-1,-2;3)$	N/A	$(2,1)$	$X_5'\subset\mathbb{P}^4(1^4,2)$ is #16203
		T_4 10 $\bullet_{13}$		5, 4, 4, 3	$(1,1,5,-1,-1;4)$	N/A	$2\times(2,0)$	$X_6'\subset\mathbb{P}^4(1^4,3)$ is #16202
5052: $(1,1,3,4,4,5,5,5)$	5	T_1 12	$\begin{matrix}3&3&4&4\\&4&5&5\\&&5&5\\&&&6\end{matrix}$	5, 5, 4, 3	$2\times(5,1,1,-1,-2;3)$	N/A	$(2,0)$	$X'\subset\mathbb{P}^6(1^3,2^2,3,4)$ is #11105
5140: $(1,1,2,4,5,6,7,8)$	8	T_3 12	$\begin{matrix}2&3&4&5\\&4&5&6\\&&6&7\\&&&8\end{matrix}$	6, 5, 4, 2	$(1,1,7,-1,-4;3)$	IM	$(2,0)$ $r_Q=2$	$X_{6,6}'\subset\mathbb{P}^5(1^2,2^2,3,4)$ is #5843
5163: $(1,1,2,3,4,5,6,7)$	3	T_1 20	$\begin{matrix}2&3&4&5\\&4&5&6\\&&6&7\\&&&8\end{matrix}$	7, 6, 5, 4	$(7,1,1,-1,-3;4)$	IM	$(2,0)$	$X_{4,5}'\subset\mathbb{P}^5(1^3,2^2,3)$ is #11102
		T_2 19 $\bullet_{13,45}$		7, 6, 5, 4	$(7,1,1,-1,-2;4)$	N/A	$2\times(2,0)$	$X_6'\subset\mathbb{P}^4(1^3,2^2)$ is #11101

	7	T_3 10	$\begin{smallmatrix}2&3&3&4\\&4&4&5\\&&5&6\\&&&6\end{smallmatrix}$	$5,4,3,2$	$(1,1,6,-1,-3;3)$	IM	$(2,0)$	$X'_{5,6}\subset\mathbb{P}^5(1^3,2^2,3)$ is #11102
		T_5 9 $\bullet_{14}$		$5,4,3,2$	$(1,1,6,-1,-2;4)$	N/A	$2\times(2,0)$	$X'_6\subset\mathbb{P}^4(1^3,2^2)$ is #11001
5176: $(1,1,2,3,4,5,5,7)$	5	T_2 12	$\begin{smallmatrix}1&2&3&5\\&3&4&6\\&&5&7\\&&&8\end{smallmatrix}$	$7,5,3,2$	$(7,1,-2,-5)$	IM	$(2,0)$	$X'_{6,6}\subset\mathbb{P}^5(1^3,2,3,5)$ is #10982
	7	T_3 6	$\begin{smallmatrix}1&2&3&4\\&3&4&5\\&&5&6\\&&&7\end{smallmatrix}$	$5,4,3,1$	$(5,1,-1,-4)$	IM	$(2,0)$ $r_Q=2$	$X'_{6,6}\subset\mathbb{P}^5(1^2,2^2,3,4)$ is #5843
5177: $(1,1,2,3,4,5,5,6)$	5	T_1 7	$\begin{smallmatrix}2&3&3&4\\&4&4&5\\&&5&6\\&&&6\end{smallmatrix}$	$6,5,4,1$	$(6,1,-1,-5)$	IM	$(2,0)$ $r_Q=3$	$X'\subset\mathbb{P}^6(1^2,2,3^2,4,5)$ is #5267
	6	T_2 11	$\begin{smallmatrix}1&2&3&4\\&3&4&5\\&&5&6\\&&&7\end{smallmatrix}$	$5,4,3,2$	$(5,1,-2,-3)$	$(1,1,5,-1,-2;3)$	$(2,0)$	$X'_{4,6}\subset\mathbb{P}^5(1^2,2^3,3)$ is #6858
5202: $(1,1,2,3,4,4,5,9)$	9	T_2 4	$\begin{smallmatrix}2&2&3&4\\&3&4&5\\&&4&5\\&&&6\end{smallmatrix}$	$4,3,2,1$	IM	$(1,4,1,-1,-2;2)$	$(2,0)$	$X'\subset\mathbb{P}^6(1^3,2,3,4,5)$ is #10984
5203: $(1,1,2,3,4,4,5,6)$	4	T_1 12	$\begin{smallmatrix}2&3&3&4\\&4&4&5\\&&5&6\\&&&6\end{smallmatrix}$	$6,5,4,2$	$(6,1,1,-1,-4;2)$	IM	$(2,0)$ $r_Q=2$	$X'\subset\mathbb{P}^6(1^2,2^2,3^2,4)$ is #5865

	6	T_2 10	$\begin{array}{cccc} 2&2&3&4\\ &3&4&5\\ &&4&5\\ &&&6 \end{array}$	$4,4,3,2$	$(4,1,1,-1,-2;2)$, $(1,1,5,-1,-2;3)$	N/A	$(2,0)$	$X'_{4,4} \subset \mathbb{P}^5(1^3,2^3)$ is #11435
5215: $(1,1,2,3,4,4,5,5)$	5	T_2 10	$\begin{array}{cccc} 2&2&3&4\\ &3&4&5\\ &&4&5\\ &&&6 \end{array}$	$5,4,3,2$	$(5,1,1,-1,-3;2)$	IM	$(2,0)$	$X'_{4,5} \subset \mathbb{P}^5(1^3,2^2,3)$ is #11102
		T_4 9 $\bullet_{13}$		$5,4,3,2$	$(5,1,1,-1,-2;3)$	N/A	$2\times(2,0)$	$X'_6 \subset \mathbb{P}^4(1^3,2^2)$ is #11001
5260: $(1,1,2,3,3,5,8,11)$	11	T_5 4	$\begin{array}{cccc} 1&2&3&5\\ &3&4&6\\ &&5&7\\ &&&8 \end{array}$	$5,3,2,1$	$(1,3,8,-2,-3;5)$	IM	$(2,0)$	$X'_7 \subset \mathbb{P}^4(1^3,2,3)$ is #10981
5263: $(1,1,2,3,3,4,7,10)$	10	T_5 4	$\begin{array}{cccc} 1&2&3&4\\ &3&4&5\\ &&5&6\\ &&&7 \end{array}$	$4,3,2,1$	$(1,3,7,-1,-3;4)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
5265: $(1,1,2,3,3,4,5,8)$	4	T_2 12	$\begin{array}{cccc} 1&2&3&5\\ &3&4&6\\ &&5&7\\ &&&8 \end{array}$	$8,5,3,2$	$(8,1,1,-3-5;5)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
	8	T_4 4	$\begin{array}{cccc} 1&2&3&3\\ &3&4&4\\ &&5&5\\ &&&6 \end{array}$	$4,3,2,1$	$(4,1,-1,-3)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
5266: $(1,1,2,3,3,4,5,7)$	5	T_3 6	$\begin{array}{cccc} 1&2&3&4\\ &3&4&5\\ &&5&6\\ &&&7 \end{array}$	$7,4,3,1$	$(7,1,-3,-4)$	IM	$(2,0)$ $r_Q=2$	$X'_{5,6} \subset \mathbb{P}^5(1^2,2^2,3^2)$ is #5857

	7	T_4 4	1 2 3 3 3 4 4 5 5 6	$5,3,2,1$	$(5,1,-2,-3)$	IM	$(2,0)$	$X'_{5,6} \subset \mathbb{P}^5(1^3,2,3,4)$ is #10983
5268: $(1,1,2,3,3,4,5,6)$	3	T_1 15	2 3 3 4 4 4 5 5 6 6	$6,5,4,3$	$(6,1,1,-1,-3;3)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2^2,3^2)$ is #11122
		T_2 14 $\bullet_{14}$		$6,5,4,3$	$(6,1,1,-1,-2;4)$	$(2,0)$	$(2,0)$	$X'_{4,5} \subset \mathbb{P}^5(1^3,2^2,3)$ is #11101
	6	T_3 9	2 3 3 4 3 3 4 4 5 5	$4,3,3,2$	$(1,1,5,-1,-2;3)$	IM	$(2,1)$	$X'_{3,4} \subset \mathbb{P}^5(1^4,2^2)$ is #16225
		T_5 8		$4,3,3,2$	$(1,1,5,-1,-1;4)$	N/A	$2\times(2,0)$	$X'_5 \subset \mathbb{P}^4(1^3,2)$ is #16203
5279: $(1,1,2,3,3,4,5,5)$	3	T_1 16	1 2 3 4 3 4 5 5 6 7	$5,5,4,3$	$(5,1,2,-1,-2;3)$, $(5,1,1,-1,-2;3)$	N/A	$(2,0)$	$X'_{4,4} \subset \mathbb{P}^5(1^3,2^3)$ is #11435
	[5, 1]	T_2 10	1 2 3 3 3 4 4 5 5 6	$5,3,3,2$	$(5,1,4,-2,-3;4)$	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^4,2,3)$ is #16226
	[5, 2]	T_1 6	2 3 3 4 3 3 4 4 5 5	$5,4,3,1$	$(5,1,-1,-4)$	IM	$(2,0)$ $r_Q = 2$	$X' \subset \mathbb{P}^6(1^2,2^2,3^2,4)$ is #5865

5303: $(1,1,2,3,3,4,4,7)$	4	T_2 11	1 2 3 4 3 4 5 5 6 7	$7,4,3,2$	$(7,1,-3,-4)$	$(1,3,-1,-2)$	$(2,0)$	$X'_{4,5} \subset \mathbb{P}^5(1^3,2^2,3)$ is #11102
	7	T_4 4	2 2 3 3 3 4 4 4 4 5	$4,3,2,1$	$(1,4,-1,-3)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2,3^2,4)$ is #11003
5305: $(1,1,2,3,3,4,4,5)$	4	T_1 11	1 2 3 3 3 4 4 5 5 6	$5,4,3,2$	$(5,1,-2,-3)$	$(4,1,1,-1,-2;2)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2^2,3^2)$ is #11122
	5	T_1 6	2 2 3 3 3 4 4 4 4 5	$4,4,3,1$	$2 \times (4,1,-1,-3)$	N/A	$(2,0)$ $r_Q = 2$	$X' \subset \mathbb{P}^6(1^2,2^2,3^3)$ is #5962
5306: $(1,1,2,3,3,4,4,5)$	3	T_1 14	2 2 3 4 3 4 5 4 5 6	$5,4,4,3$	$(5,1,1,-1,-2;3)$	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^3,2^2,3^2)$ is #11122
		T_2 13 $\bullet_{13,45}$		$5,4,4,3$	$(5,1,1,-1,-1;4)$	N/A	$2 \times (2,0)$	$X'_5 \subset \mathbb{P}^4(1^3,2)$ is #16203
	4	T_1 10	2 3 3 4 3 3 4 4 5 5	$5,4,3,2$	$(5,1,1,-1,-2;3)$	IM	$(2,0)$	$X'_7 \subset \mathbb{P}^4(1^3,2,3)$ is #10981
		T_3 9 $\bullet_{24}$		$5,4,3,2$	$(5,1,1,-1,-2;3)$	$(2,0)$	$(2,0)$	$X'_{4,5} \subset \mathbb{P}^4(1^3,2^2,3)$ is #11102

	5	T_2 9	2 2 3 3 3 4 4 4 4 5	4, 3, 3, 2	$(4, 1, 1, -1, -2; 2)$	N/A	$(2, 1)$	$X'_5 \subset \mathbb{P}^4(1^4, 2)$ is #16203
		T_4 8 $\bullet_{13}$		4, 3, 3, 2	$(4, 1, 1, -1, -1; 3)$	N/A	$2 \times (2, 0)$	$X'_5 \subset \mathbb{P}^4(1^3, 2)$ is #16203
5410: $(1, 1, 2, 3, 3, 4, 4, 4)$	4	T_1 10	2 2 3 3 3 4 4 4 4 5	4, 4, 3, 2	$2 \times (4, 1, 1, -1, -2; 2)$	N/A	$(2, 0)$	$X' \subset \mathbb{P}^6(1^3, 2^3, 3)$ is #11436
5516: $(1, 1, 2, 3, 3, 3, 4, 7)$	3	T_1 14	1 2 3 4 3 4 5 5 6 7	7, 4, 3, 3	$(7, 1, -3, -4)$	$(1, 1, 3-1, -1; 2)$	$(3, 1)$	dP fibration of degree 2
	7	T_3 4	2 3 3 3 3 3 3 4 4 4	3, 3, 2, 1	$2 \times (1, 3, -1, -2)$	N/A	$(2, 0)$	$X' \subset \mathbb{P}^6(1^3, 2^2, 3, 4)$ is #11105
5519: $(1, 1, 2, 3, 3, 3, 4, 5)$	3	T_1 12	2 3 3 4 3 3 4 4 5 5	5, 4, 3, 3	$(5, 1, 1, -1, -2; 3)$	IM	$(3, 1)$	dP fibration of degree 2
	5	T_3 8	2 3 3 3 3 3 3 4 4 4	3, 3, 3, 2	N/A	N/A	$(2, 1)$	$X'_4 \subset \mathbb{P}^4(1^5)$ is #20521
5530: $(1, 1, 2, 3, 3, 3, 4, 4)$	3	T_1 12	2 2 3 3 3 4 4 4 4 5	4, 4, 3, 3	$2 \times (4, 1, 1, -1, -1; 3)$	N/A	$(3, 1)$	dP fibration of degree 2

		T_2 11 $\bullet_{13,45}$		4, 4, 3, 3	(2, 1)	N/A	(3, 1)	dP fibration of degree 2
	4	T_1 9	2 3 3 3 3 3 3 4 4 4	4, 3, 3, 2	$(4,1,1,-1,-2;2)$	N/A	(2, 1)	$X' \subset \mathbb{P}^6(1^5,2,3)$ is #20523
		T_3 8		4, 3, 3, 2	$(4,1,1,-1,-1;3)$	N/A	$2 \times (2,0)$	$X' \subset \mathbb{P}^6(1^5,2,3)$ is #20523
5841: $(1,1,2,2,3,5,7,9)$	9	T_5 5	1 2 3 4 3 4 5 5 6 7	5, 3, 2, 1	$(1,2,7,-2,-3;4)$	IM	(2, 0)	$X'_6 \subset \mathbb{P}^4(1^3,2^2)$ is #11101
5845: $(1,1,2,2,3,4,5,6)$	6	T_4 8	2 2 3 3 3 4 4 4 4 5	4, 3, 2, 2	$(1,1,5,-1,-2;3)$	IM	(3, 1)	dP fibration of degree 2
5859: $(1,1,2,2,3,3,5,8)$	8	T_3 4	1 2 2 3 3 3 4 4 5 5	3, 2, 2, 1	IM	N/A	(2, 1)	$X'_{4,4} \subset \mathbb{P}^5(1^4,2,3)$ is #16204
5860: $(1,1,2,2,3,3,5,7)$	3	T_1 13	1 2 3 4 3 4 5 5 6 7	7, 5, 3, 2	$(7,1,-2,-5)$	IM	(2, 0)	$X' \subset \mathbb{P}^5(1^3,2^2,3,5)$ is #11103
	7	T_3 5	1 2 2 3 3 3 4 4 5 5	3, 3, 2, 1	$(1,2,5,-1,-2;3)$ $(3,1,-1,-2)$	N/A	(2, 0)	$X'_{4,4} \subset \mathbb{P}^5(1^3,2^3)$ is #11435

5862: $(1,1,2,2,3,3,5,5)$	5	T_3 5	1 2 2 3 3 3 4 4 5 5	$5,3,2,1$	$(5,1,-2,-3)$	IM	$(2,0)$	$X'_{4,6} \subset \mathbb{P}^5(1^3,2,3^2)$ is #11002
5866: $(1,1,2,2,3,3,4,7)$	7	T_2 4	2 2 2 3 3 3 4 3 4 4	$3,2,2,1$	IM	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^4,2,3,4)$ is #16205
5867: $(1,1,2,2,3,3,4,5)$	4	T_2 9	1 2 2 3 3 3 4 4 5 5	$5,3,2,2$	$(5,1,1,-2,-3;2)$	$(1,1,2,-1,-1;1)$	$(3,1)$	dP fibration of degree 2
	5	T_2 5	2 2 2 3 3 3 4 3 4 4	$4,3,2,1$	$(4,1,-1,-3)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2^2,3^2)$ is #11122
5870: $(1,1,2,2,3,3,4,5)$	3	T_1 11	2 2 3 3 3 4 4 4 4 5	$5,4,3,2$	$(5,1,1,-1,-3;2)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2^3,3)$ is #11436
		T_2 10 $\bullet_{15}$		$5,4,3,2$	$(5,1,1,-1,-2;3)$	$(2,0)$	$(2,0)$	$X'_{4,4} \subset \mathbb{P}^6(1^3,2^3)$ is #11435
	5	T_2 8	2 2 2 3 3 3 4 3 4 4	$3,3,2,2$	$(1,1,4,-1,-1;3)$ $(3,1,1,-1,-1;2)$	N/A	$(3,1)$	dP fibration of degree 3
		T_5 7 $\bullet_{14}$		$4,3,3,2$	$(2,1)$	N/A	$(3,1)$	dP fibration of degree 2

5914: $(1,1,2,2,3,3,4,4)$	4	T_2 8	$\begin{matrix} 2&2&2&3 \\ &3&3&4 \\ &&3&4 \\ &&&4 \end{matrix}$	$4,3,2,2$	$(4,1,1,-1,-2;2)$	IM	$(3,1)$	dP fibration of degree 2
5963: $(1,1,2,2,3,3,3,5)$	3	T_1 11	$\begin{matrix} 1&2&2&3 \\ &3&3&4 \\ &&4&5 \\ &&&5 \end{matrix}$	$5,3,3,2$	$(5,1,-2,-3)$	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^4,2^2,3)$ is #16226
	5	T_1 5	$\begin{matrix} 2&2&3&3 \\ &2&3&3 \\ &&3&3 \\ &&&4 \end{matrix}$	$3,3,2,1$	$(1,3,-1,-2)$ $(3,1,-1,-2)$	N/A	$(2,0)$	$X' \subset \mathbb{P}^6(1^3,2^3,3)$ is #11436
5970: $(1,1,2,2,3,3,3,4)$	3	T_1 10	$\begin{matrix} 2&2&2&3 \\ &3&3&4 \\ &&3&4 \\ &&&4 \end{matrix}$	$4,3,3,2$	$(4,1,1,-1,-2;2)$	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^4,2^3)$ is #16338
		T_2 9 $\bullet_{14}$		$4,3,3,2$	$(4,1,1,-1,-1;1)$	N/A	$2\times(2,0)$	$X'_{3,4} \subset \mathbb{P}^5(1^4,2^2)$ is #16225
	4	T_1 8	$\begin{matrix} 2&2&3&3 \\ &2&3&3 \\ &&3&3 \\ &&&4 \end{matrix}$	$3,3,2,2$	$2\times(1,1,3,-1,-1;2)$	N/A	$(3,1)$	dP fibration of degree 3
		T_4 7 $\bullet_{23}$		$4,3,3,2$	$(2,1)$	N/A	$(3,1)$	dP fibration of degree 2
6217: $(1,1,2,2,3,3,3,3)$	3	T_1 9	$\begin{matrix} 2&2&3&3 \\ &2&3&3 \\ &&3&3 \\ &&&4 \end{matrix}$	$3,3,3,2$	N/A	N/A	$(2,1)$	$X'_{3,3} \subset \mathbb{P}^5(1^5,2)$ is #20552

6860: $(1,1,2,2,2,3,3,5)$	5	T_1 5	$\begin{smallmatrix} 1&2&2&3\\ &2&2&3\\ &&3&4\\ &&&4 \end{smallmatrix}$	$3,2,2,1$	IM	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^4,2^2,3)$ is #16226
		T_5 4 $\bullet_{14}$		$3,2,2,1$	$(1,2,-1,-1)$	N/A	$2\times(2,1)$	$X'_{4,4} \subset \mathbb{P}^5(1^4,2,3)$ is #16204
6865: $(1,1,2,2,2,3,3,4)$	4	T_1 8	$\begin{smallmatrix} 1&2&2&3\\ &2&2&3\\ &&3&4\\ &&&4 \end{smallmatrix}$	$3,2,2,2$	IM	N/A	$(3,2)$	Conic bundle
6878: $(1,1,2,2,2,3,3,3)$	3	T_1 9	$\begin{smallmatrix} 1&2&2&3\\ &2&2&3\\ &&3&4\\ &&&4 \end{smallmatrix}$	$3,3,2,2$	$(3,1,1,-1,-1;2)$ $(1,2,-1,-1)$	N/A	$(3,1)$	dP fibration of degree 3
		T_3 8 $\bullet_{12,45}$		$3,3,2,2$	$(2,1)$	N/A	$(3,1)$	dP fibration of degree 2
10963: $(1,1,1,3,4,5,6,7)$	7	T_3 10	$\begin{smallmatrix} 1&2&3&4\\ &3&4&5\\ &&5&6\\ &&&7 \end{smallmatrix}$	$5,4,3,1$	$(1,1,6,-1,-4;2)$	IM	$(2,0)$ $r_Q=2$	$X'_{5,6} \subset \mathbb{P}^5(1^3,2,3,4)$ is #10983
10985: $(1,1,1,2,3,4,5,6)$	2	T_1 24	$\begin{smallmatrix} 1&2&3&4\\ &3&4&5\\ &&5&6\\ &&&7 \end{smallmatrix}$	$6,5,4,3$	$(6,1,1,-1,-3;3)$	IM	$(2,0)$	$X'_{4,4} \subset \mathbb{P}^5(1^4,2,3)$ is #16204
		T_2 23 $\bullet_{13,45}$		$6,5,4,3$	$(6,1,1,-1,-2;4)$	$(2,0)$	$(2,0)$	$X'_5 \subset \mathbb{P}^4(1^4,2)$ is #16203

	6	T_3 8	1 2 2 3 3 3 4 4 5 5	$4,3,2,1$	$(1,1,5,-1,-2;4)$	IM	$(2,0)$	$X'_{4,4} \subset \mathbb{P}^4(1^4,2,3)$ is #16204
		T_5 7 $\bullet_{14}$		$4,3,2,1$	$(1,1,5,-1,-2;3)$	$(2,0)$	$(2,0)$	$X'_5 \subset \mathbb{P}^4(1^4,2)$ is #16203
11004: $(1,1,1,2,3,3,4,7)$	7	T_2 4	1 1 2 3 2 3 4 3 4 5	$3,2,1,1$	IM	$(1,3,1,-1,-1;2)$	$(3,1)$	dP fibration of degree 2
11005: $(1,1,1,2,3,3,4,5)$	3	T_1 10	1 2 2 3 3 3 4 4 5 5	$5,4,3,1$	$(5,1,-1,-4)$	IM	$(2,0)$ $r_Q = 2$	$X' \subset \mathbb{P}^6(1^3,2^2,3,4)$ is #11105
	5	T_2 8	1 1 2 3 2 3 4 3 4 5	$3,3,2,1$	$(3,1,-1,-2)$, $(1,1,4,-1,-2;2)$	N/A	$(2,0)$	$X'_{3,4} \subset \mathbb{P}^5(1^4,2^2)$ is #16225
11021: $(1,1,1,2,3,3,4,4)$	4	T_2 8	1 1 2 3 2 3 4 3 4 5	$4,3,2,1$	$(4,1,-1,-3)$	IM	$(2,0)$	$X'_{4,4} \subset \mathbb{P}^5(1^4,2,3)$ is #16204
		T_4 7 $\bullet_{13}$		$4,3,2,1$	$(4,1,1,-1,-2;2)$	$(2,0)$	$(2,0)$	$X'_5 \subset \mathbb{P}^4(1^4,2)$ is #16203
11104: $(1,1,1,2,2,3,5,7)$	7	T_5 4	1 1 2 3 2 3 4 3 4 5	$3,2,1,1$	$(1,2,5,-1,-2;3)$	$(2,0)$	$(3,1)$	dP fibration of degree 2

11106: $(1,1,1,2,2,3,4,5)$	2	T_1 16	$\begin{matrix}1&2&2&3\\ &3&3&4\\ &&4&5\\ &&&5\end{matrix}$	$5,4,3,2$	$(5,1,1,-1,-3;2)$	$(2,0)$	$(2,0)$	$X' \subset \mathbb{P}^6(1^4,2^2,3)$ is #16226
		T_2 15 $\bullet_{14}$		$5,4,3,2$	$(5,1,1,-1,-2;3)$	$(2,0)$	$(2,0)$	$X'_{3,4} \subset \mathbb{P}^5(1^4,2^2)$ is #16225
	5	T_3 7	$\begin{matrix}1&2&2&3\\ &2&2&3\\ &&3&4\\ &&&4\end{matrix}$	$3,2,2,1$	$(1,1,4,-1,-2;2)$	N/A	$(2,1)$	$X'_{3,3} \subset \mathbb{P}^5(1^5,2)$ is #20522
		T_5 6		$4,3,2,1$	$(1,1,4,-1,-1;3)$	N/A	$2\times(2,0)$	$X'_4 \subset \mathbb{P}^4(1^5,1)$ is #20521
11123: $(1,1,1,2,2,3,3,5)$	3	T_2 8	$\begin{matrix}1&1&2&3\\ &2&3&4\\ &&3&4\\ &&&5\end{matrix}$	$5,3,2,1$	$(5,1,-2,-3)$	IM	$(2,0)$	$X'_{3,4} \subset \mathbb{P}^5(1^4,2^2)$ is #16225
	5	T_4 4	$\begin{matrix}1&1&2&2\\ &2&3&3\\ &&3&3\\ &&&4\end{matrix}$	$3,2,1,1$	$(1,3,-1,-2)$	IM	$(3,1)$	dP fibration of degree 2
11125: $(1,1,1,2,2,3,3,4)$	2	T_1 15	$\begin{matrix}1&1&2&3\\ &2&3&4\\ &&3&4\\ &&&5\end{matrix}$	$4,3,3,2$	$(4,1,-1,-2)$	IM	$(2,1)$	$X'_{3,3} \subset \mathbb{P}^5(1^5,2)$ is #20522
		T_2 14 $\bullet_{13,45}$		$4,3,3,2$	$(4,1,1,-1,-1;3)$	N/A	$2\times(2,1)$	$X'_4 \subset \mathbb{P}^4(1^5)$ is #20521

	3	T_1 8	$\begin{smallmatrix}1&2&2&3\\&2&2&3\\&&3&4\\&&&4\end{smallmatrix}$	$4,3,2,1$	$(4,1,-1,-3)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^4,2,3)$ is #16226
		T_3 7 $\bullet_{24}$		$4,3,2,1$	$(4,1,1,-1,-2;2)$	$(2,0)$	$(2,0)$	$X'_{3,4} \subset \mathbb{P}^5(1^4,2^2)$ is #16225
	4	T_2 7	$\begin{smallmatrix}1&1&2&2\\&2&3&3\\&&3&3\\&&&4\end{smallmatrix}$	$3,2,2,1$	$(1,3,-1,-2)$	N/A	$(2,1)$	$X'_{3,3} \subset \mathbb{P}^5(1^5,2)$ is #20522
		T_4 6 $\bullet_{13}$		$3,2,2,1$	$(3,1,1,-1,-1;2)$	N/A	$2\times(2,1)$	$X'_4 \subset \mathbb{P}^4(1^5)$ is #20521
11222: $(1,1,1,2,2,3,3,3)$	3	T_1 8	$\begin{smallmatrix}1&1&2&2\\&2&3&3\\&&3&3\\&&&4\end{smallmatrix}$	$3,3,2,1$	$2\times(3,1,-1,-2)$	N/A	$(2,0)$	$X' \subset \mathbb{P}^6(1^4,2^3)$ is #16338
11437: $(1,1,1,2,2,2,3,5)$	2	T_1 13	$\begin{smallmatrix}1&1&2&3\\&2&3&4\\&&3&4\\&&&5\end{smallmatrix}$	$5,3,2,2$	$(5,1,-2,-3)$	$(1,1,2,-1,-1;1)$	$(3,1)$	dP fibration of degree 3
	5	T_3 4	$\begin{smallmatrix}1&2&2&2\\&2&2&2\\&&3&3\\&&&3\end{smallmatrix}$	$2,2,1,1$	$2\times(2,1,-1,-1)$	N/A	$(3,1)$	dP fibration of degree 3
11440: $(1,1,1,2,2,2,3,4)$	2	T_1 12	$\begin{smallmatrix}1&2&2&3\\&2&2&3\\&&3&4\\&&&4\end{smallmatrix}$	$4,3,2,2$	$(4,1,1,-1,-2;2)$	IM	$(3,1)$	dP fibration of degree 3

	4	T_3 6	1 2 2 2 2 2 2 3 3 3	2, 2, 2, 1	N/A	N/A	(2, 1)	$X'_{2,3} \subset \mathbb{P}^5(1^6)$ is #24076
11455: (1, 1, 1, 2, 2, 2, 3, 3)	2	T_1 12	1 1 2 2 2 3 3 3 3 4	3, 3, 2, 2	2 × (3, 1, 1, −1, −1; 2)	N/A	(3, 1)	dP fibration of degree 4
		T_2 11 $\bullet_{13,45}$		3, 3, 2, 2	(2, 1)	N/A	(3, 1)	dP fibration of degree 2
	3	T_1 7	1 2 2 2 2 2 2 3 3 3	3, 2, 2, 1	(3, 1, −1, −2)	N/A	(2, 1)	$X' \subset \mathbb{P}^6(1^5, 2^2)$ is #20543
		T_3 6		3, 2, 2, 1	(3, 1, 1, −1, −1; 2)	N/A	2 × (2, 1)	$X'_{3,3} \subset \mathbb{P}^5(1^5, 2)$ is #20552
12063: (1, 1, 1, 2, 2, 2, 2, 3)	2	T_1 10	1 2 2 2 2 2 2 3 3 3	3, 2, 2, 2	(3, 1, 1, −1, −1; 2)	N/A	(3, 2)	Conic bundle
	3	T_1 6	2 2 2 2 2 2 2 2 2 2	2, 2, 2, 1	N/A	N/A	(2, 1)	$X' \subset \mathbb{P}^6(1^6, 7)$ is #24077
12960: (1, 1, 1, 2, 2, 2, 2, 2)	2	T_1 8	2 2 2 2 2 2 2 2 2 2	2, 2, 2, 2	N/A	N/A	(3, 2)	Conic bundle

16206: $(1,1,1,1,2,3,4,5)$	5	T_4 6	$\begin{smallmatrix}1&1&2&2\\ &2&3&3\\ &&3&3\\ &&&4\end{smallmatrix}$	$3,2,1,1$	$(1,1,4,-1,-2;2)$	IM	$(3,1)$	dP fibration of degree 3
16227: $(1,1,1,1,2,2,3,5)$	5	T_2 4	$\begin{smallmatrix}1&1&1&2\\ &2&2&3\\ &&2&3\\ &&&3\end{smallmatrix}$	$2,2,1,1$	IM	N/A	$(3,2)$	Conic bundle
16228: $(1,1,1,1,2,2,3,4)$	2	T_1 10	$\begin{smallmatrix}1&1&2&2\\ &2&3&3\\ &&3&3\\ &&&4\end{smallmatrix}$	$4,3,2,1$	$(4,1,-1,-3)$	IM	$(2,0)$	$X' \subset \mathbb{P}^6(1^5,2,3)$ is #20523
		T_2 9 $\bullet_{15}$		$4,3,2,1$	$(4,1,1,-1,-2;2)$	$(2,0)$	$(2,0)$	$X'_{3,3} \subset \mathbb{P}^5(1^5,2)$ is #20522
	4	T_2 6	$\begin{smallmatrix}1&1&1&2\\ &2&2&3\\ &&2&3\\ &&&3\end{smallmatrix}$	$2,2,1,1$	$(1,1,3,-1,-1;2)$ $(2,1,-1,-1)$	N/A	$(3,1)$	dP fibration of degree 3
		T_5 5 $\bullet_{14}$		$3,2,2,1$	$(2,1)$	N/A	$(3,1)$	dP fibration of degree 3
16246: $(1,1,1,1,2,2,3,3)$	3	T_2 6	$\begin{smallmatrix}1&1&1&2\\ &2&2&3\\ &&2&3\\ &&&3\end{smallmatrix}$	$3,2,1,1$	$(3,1,-1,-2)$	IM	$(3,1)$	dP fibration of degree 3
16339: $(1,1,1,1,2,2,2,3)$	2	T_1 9	$\begin{smallmatrix}1&1&1&2\\ &2&2&3\\ &&2&3\\ &&&3\end{smallmatrix}$	$3,2,2,1$	$(3,1,-1,-2)$	N/A	$(2,1)$	$X' \subset \mathbb{P}^6(1^6,2)$ is #24077

		T_2 8 $\bullet_{14}$		3, 2, 2, 1	(3, 1, 1, −1, −1; 2)	N/A	2 × (2, 1)	$X'_{2,3} \subset \mathbb{P}^5(1^6)$ is #24076
	3	T_1 6	1 1 2 2 1 2 2 2 2 3	2, 2, 1, 1	(1, 2, −1, −1) (2, 1, −1, −1)	N/A	(3, 1)	dP fibration of degree 4
		T_4 5 $\bullet_{23}$		2, 2, 1, 1	(2, 1)	N/A	(3, 1)	dP fibration of degree 3
16645: (1, 1, 1, 1, 2, 2, 2, 2)	2	T_1 8	1 1 2 2 1 2 2 2 2 3	3, 2, 2, 1	(3, 1, −1, −2)	N/A	(2, 1)	$X' \subset \mathbb{P}^6$ is #26988
20524: (1, 1, 1, 1, 1, 2, 3, 4)	4	T_4 5	1 1 2 2 1 2 2 2 2 3	2, 1, 1, 1	(1, 1, 3, −1, −1; 2)	N/A	(3, 2)	Conic bundle
20544: (1, 1, 1, 1, 1, 2, 2, 3)	2	T_1 7	1 1 2 2 1 2 2 2 2 3	3, 2, 1, 1	(3, 1, −1, −2)	N/A	(3, 1)	dP fibration of degree 4
	3	T_2 5	1 1 1 1 2 2 2 2 2 2	2, 1, 1, 1	(1, 2, −1, −1)	N/A	(3, 2)	Conic bundle
20652: (1, 1, 1, 1, 1, 2, 2, 2)	2	T_1 7	1 1 1 1 2 2 2 2 2 2	2, 2, 1, 1	2 × (2, 1, −1, −1)	N/A	(3, 1)	dP fibration of degree 5

		T_2 6 $\bullet_{15}$		2, 2, 1, 1	(2, 1)	N/A	(3, 1)	dP fibration of degree 4
24078: (1, 1, 1, 1, 1, 1, 2, 3)	3	T_1 5	$\begin{matrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{matrix}$	1, 1, 1, 1	N/A	N/A	(3, 2)	Conic bundle
		T_5 4		1, 1, 1, 1	N/A	N/A	(3, 2)	Conic bundle
24097: (1, 1, 1, 1, 1, 1, 2, 2)	2	T_1 6	$\begin{matrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{matrix}$	2, 1, 1, 1	(2, 1, −1, −1)	N/A	(3, 2)	Conic bundle

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