Type II Unprojections, Fano Threefolds and Codimension Four Constructions

by

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Declarations

I confirm that, to the best of my knowledge and except where otherwise stated, the content of this thesis is my own research. I confirm that this thesis has not been submitted to another university for any degree.

Abstract

The classification of singular Fano 3-folds remains an open problem in algebraic geometry. The purpose of this thesis is to prove the existence of new families of singular Fano 3-folds, specifically those whose anticanonical embedding is in codimension 4. This is achieved using unprojections. The projection of a scheme Y with coordinate ring $k[Y] := \mathbb{C}[x_0, \ldots, x_n]/I_Y$ is a scheme X defined by the coordinate ring $k[X] := \mathbb{C}[x_0, \ldots, x_m]/(I_Y \cap \mathbb{C}[x_0, \ldots, x_m])$ where $m < n$. Unprojection is a method of adjoining variables and equations to the ring of X to recover Y.

In Chapter 2, we define a new unprojection format allowing us to construct Gorenstein rings of codimension $n + 2$ from Gorenstein rings of codimension n. Following the naming conventions in the literature, these unprojections are type II and should be considered as type II_1 unprojections. We focus on the case where $n = 2$. This constructs codimension 4 Gorenstein rings which we define explicitly in Chapter 3.

In Chapter 4, we use codimension 4 Gorenstein rings to construct and prove the existence of 16 new families of codimension 4 Fano 3-folds. Demonstrably, each family corresponds to a distinct Hilbert series. By using type II_1 unprojections as in [33], we construct a second topologically distinct family of codimension 4 Fano 3-folds for these Hilbert series. The Hilbert scheme of these Fano 3-folds, therefore, contains at least 2 components that parametrize distinct Fano 3-folds.

In Chapter 5, we consider pre-existing families of codimension 4 Fano 3-folds which are described by [10] but also constructible using our methods.

Chapter 1

Introduction

Broadly speaking, a *Fano 3-fold* is a projective 3-dimensional variety defined over $\mathbb{C}$ with an ample anticanonical divisor. Fano 3-folds have classically been considered as smooth varieties, although we will not limit ourselves to this case in this thesis. Smooth Fano 3-folds are well understood and they are known to exist in exactly 105 families (see Chapter 12, [42]); Iskovskikh classified the cases where the second Betti number is 1 (see [22] and [23]) and Mori and Mukai, the second Betti number is greater than 1 (see [27]).

Fano 3-folds with singularities are significantly less well understood by comparison. In this thesis, we study Fano 3-folds with singularities. Our definition throughout will be:

Definition 1.0.1. A *Fano 3-fold* X is a complex normal projective 3-fold whose anticanonical divisor $-K_X$ is $\mathbb{Q}$-Cartier and ample; and whose singularities are $\mathbb{Q}$-factorial and terminal. If additionally the Picard rank of X is 1, we call X a *Mori-Fano 3-fold*.

In particular, we study index 1 Fano 3-folds: a Fano 3-fold X has *(Fano) index* r if r is the greatest integer such that $-K_X = rA$ for some ample Weil divisor A. Our choice of index is for narrative rather than mathematical purposes.

The classification of these Fano 3-folds remains an open problem and estimates suggest the existence of tens of thousands of families (see [8]). At present only a few hundred families are known explicitly. Many of the known families are of Fano 3-folds with codimension at most 3, where by codimension we refer to the standard anticanonical embedding (see Section 1.1). We are interested in codimension 4 Fano 3-folds.

1.1 The Graded Ring Database Project

Following the analysis of [2], we are able to predict Fano 3-folds. More accurately, we are able to predict their Hilbert series. There are several important ingredients: the relationships between Fano 3-folds, graded rings and Hilbert series; the sufficiency of weighted projective spaces; and the finiteness of singularities and genus.

The *anticanonical ring* of a Fano 3-fold X is the graded ring

$$R := \bigoplus_{m \in \mathbb{N}} H^0(X, -mK_X)$$

and it is equal to the homogeneous coordinate ring defining X. Immediately, we notice that it is sufficient to study only Fano 3-folds in weighted projective space: the generators of the anticanonical ring, say $x_0, \ldots, x_n$, describe an embedding of X as a projectively normal subvariety in $\mathbb{P}(a_0, \ldots, a_n)$ where $a_i := \mathrm{wt}(x_i)$ for $i = 0, \ldots, n$ and $n \in \mathbb{N}^+$. Therefore, in this thesis we consider only Fano 3-folds $X \subset \mathbb{P}(a_0, \ldots, a_n)$ which we denote by $X \subset w\mathbb{P}^n$ when the weights a_i are unknown.

Remark 1.1.1. In cases where the anticanonical divisor of a Fano 3-fold X is of the form $-K_X = rA$ for some ample Weil divisor A and some integer r, it can be useful to study rings such as $\bigoplus_{m \in \mathbb{N}} H^0(X, mA)$. We do not consider such rings because the Fano 3-folds in this thesis are index 1.

The *Hilbert series* of a Fano 3-fold X is the series

$$P_X(t) := \sum_{m \in \mathbb{N}} h^0(X, -mK_X) t^m.$$

By Riemann-Roch, the Hilbert series of X is known to be a rational function determined by the genus of X, $g := h^0(X, -K_X) - 2$, and a collection of singularities called a basket (see Section 4.1.3 or [37] for a more detailed discussion on baskets). The following theorem is commonly known as the plurigenus formula:

Theorem 1.1.1. (Theorem-Definition 4.6, [4]) Let X be a Fano 3-fold with genus g and denote $A := -K_X$. Then, the Hilbert series of X is such that

$$P_X(t) = \frac{1+t}{(1-t)^2} + \frac{t(1+t)A^3}{2(1-t)^4} - \sum_{\frac{1}{r}(1,a,r-a) \in \mathcal{B}} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r - \overline{bi})t^i}{2r}$$

where

1. the sum takes place over a basket $\mathcal{B}$ of finitely many terminal quotient singularities of the form $\frac{1}{r}(1, a, r-a)$ with $r, a \in \mathbb{N}^+$, $r > 1$ and $\mathrm{hcf}(r, a) = 1$;

2. b is an integer such that $ab = 1$ modulo r; and

3. $\overline{bi}$ denotes the minimal non-negative residue of bi modulo r.

Furthermore,

$$A^3 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r}.$$

It is clear that defining the Hilbert series of a Fano 3-fold X is equivalent to defining the genus g of X and a basket $\mathcal{B}$. We call $(g, \mathcal{B})$ the *numerical data* of X. A famous result is that the baskets and genera of Fano 3-folds are bounded (see Theorem 1.2 (3), [25]). For instance:

Theorem 1.1.2. (Theorem 5.1, [25]) Let X be a Mori-Fano 3-fold with genus g. Then,

$$2g - 2 \leq (-K_X)^3 \leq 6^3(24!)^2.$$

Theorem 1.1.3. (Proposition 1, [24]) Let X be a Mori-Fano 3-fold and $\mathcal{B}$ the associated basket of terminal cyclic quotient singularities. Then,

$$0 < \sum_{\frac{1}{r}(1,a,r-a)\mathcal{B}} \left(r - \frac{1}{r}\right) < 24.$$

The bounds imposed by these theorems result in a a finite set of data $(g, \mathcal{B})$. By substituting every possible pair $(g, \mathcal{B})$ into the formula for $P_X(t)$, we construct a finite list of rational functions inside which lies the Hilbert series of every existing Mori-Fano 3-fold. The rational functions obtained by this process are called *numerical candidates*. At this stage we do not know whether these numerical candidates are indeed the Hilbert series of a Fano 3-fold. Our task is to realise them in real life by constructing the appropriate Fano 3-fold. We would also like to make comments about the Hilbert scheme of a numerical candidate such as the number of Fano components.

The Graded Ring Database (shortened to GRDB) is an online resource which provides systematic predictions about polarised algebraic varieties, in particular Fano 3-folds, via their graded rings (see [8]). The GRDB is essentially a list of 52646 numerical candidates; however, each candidate is presented as a Fano 3-fold $X \subset \mathbb{P}(a_0, \ldots, a_n)$ with a Hilbert series equal to a particular numerical candidate. The process of obtaining a predicted Fano 3-fold from a numerical candidate $r(t)$ is subtle, but loosely speaking it involves presenting $r(t)$ in the form

$$r(t) = \frac{p(t)}{\prod_{i=0}^{n}(1 - t^{a_i})}$$

where the right hand side is the Hilbert series of some $X \subset \mathbb{P}(a_0, \ldots, a_n)$ with Hilbert numerator $p(t) \in \mathbb{Z}[t]$. This process is completed systematically:

Example 1.1.1. Suppose X is some Fano 3-fold with numerical data

$$g := -1 \text{ and } \mathcal{B} := \left\{ \frac{1}{2}(1,1,1), \frac{1}{8}(1,3,5), \frac{1}{11}(1,5,6) \right\}.$$

Then, by Theorem 1.1.1,

$$P_X(t) = 1 + t + t^2 + t^3 + t^4 + 2t^5 + 3t^6 + 3t^7 + 4t^8 + 4t^9 + \ldots.$$

To write P_X in a suitable format, we consecutively multiply P_X by $(1 - t^n)$ where n is the lowest non-zero power of t visible, i.e.

$$P_X(t)(1-t) = 1 + t^5 + t^6 + t^8 + \ldots$$

followed by

$$P_X(t)(1-t)(1-t^5) = 1 + t^6 + t^8 + \ldots.$$

Continuing in this manner, we eventually obtain

$$P_X(t)(1-t)(1-t^5)(1-t^6)(1-t^8)(1-t^{11}) = 1 - t^{30}$$

or equivalently

$$P_X(t) = \frac{1 - t^{30}}{(1-t)(1-t^5)(1-t^6)(1-t^8)(1-t^{11})}.$$

Our guess for X would be a hypersurface in $\mathbb{P}(1,5,6,8,11)$ defined by a degree 30 polynomial.

It is possible for a numerical candidate to be realised as several Fano 3-folds $X \subset w\mathbb{P}^n$ in different codimensions, $\operatorname{codim}(X) := n - 3$. For example:

Example 1.1.2. Let $X \subset \mathbb{P}(1,1,1,1,3)$ be a hypersurface defined by a degree 6 equation. Let $Y \subset \mathbb{P}(1,1,1,1,2,3)$ be a complete intersection defined by a degree 2 and a degree 6 equation. Then, the Hilbert series of X and Y are equal:

$$P_X(t) = \frac{1 - t^6}{(1-t)^4(1-t^3)} = \frac{1 - t^2 - t^6 + t^8}{(1-t)^4(1-t^2)(1-t^3)} = P_Y(t).$$

When $Y \subset \mathbb{P}(1,1,1,1,2,3)_{\langle x,y,z,u,v,w \rangle}$ is defined by $f_2 \in \mathbb{C}[x,y,z,u]$ and

$g_6 \in \mathbb{C}[x, y, z, u, v, w]$ of degree 2 and 6 respectively, Y has small deformations

$$Y_\lambda = \{f_2 - \lambda v = g_6 = 0\}$$

which allow us to eliminate v when $\lambda \in \mathbb{C} - \{0\}$. Thus Y_λ is isomorphic to a degree 6 hypersurface in $\mathbb{P}(1, 1, 1, 1, 3)$ when $\lambda \neq 0$ and $Y = Y_0$ lies on the boundary of the codimension 1 family $X \subset \mathbb{P}(1, 1, 1, 1, 3)$.

Another such example can be found in Section 4.1 of [9] where the Hilbert series

$$\frac{1 - 4t^3 + 4t^5 - t^8}{(1 - t)^5(1 - t^2)^2}$$

has a natural interpretation as a codimension 3 and a codimension 4 Fano 3-fold. If a numerical candidate could be realised as several Fano 3-folds in different codimensions, the GRDB will report a predicted Fano 3-fold $X \subset w\mathbb{P}^n$ where n is the least positive integer possible. We follow this convention.

Without confusion, the term numerical candidate will refer to rational functions and the predicted Fano 3-folds.

1.2 Realising Codimension 4 Candidates

The classification of Fano 3-folds $X \subset w\mathbb{P}^n$ in codimension $\mathrm{codim}(X) := n - 3$ at most 3 is well known. We have 95 families of weighted hypersurfaces, 85 families of codimension 2 complete intersections, 1 family of codimension 3 complete intersections and 69 families defined by the maximal Pfaffians of a 5×5 antisymmetric matrix (see [35], [2] and in particular Section 16 of [21]). In other words, we have realised the numerical candidates $X \subset w\mathbb{P}^n$ with $n \leq 6$.

In this thesis we are concerned with codimension 4 Fano 3-folds. There are 145 numerical candidates in codimension 4; that is, there are 145 pairs $(g, \mathcal{B})$ which produce rational functions that cannot be presented as the Hilbert series of a Fano 3-fold $X \subset w\mathbb{P}^{n+3}$ with $n < 4$.

Remark 1.2.1. There may be other codimension 4 Fano 3-folds which are not realised by these 145 numerical candidates; for example, the GRDB ignores degenerations (see Example 1.1.2). However, unlike the 145 numerical candidates we are concerned with, their Hilbert series have already been realised as lower codimension Fano 3-folds.

We do not expect the 145 codimension 4 numerical candidates to be realised as 145 topologically distinct codimension 4 Fano 3-folds. That is, we do not expect a fixed numerical candidate to build a unique family of Fano 3-folds. We expect more. A Hilbert series has finitely many families of associated Fano 3-folds and there may be multiple families for the same numerical candidate. In fact, many of the numerical candidates already realised have multiple families. For 116 of the codimension 4 candidates, Brown, Kerber and Reid successfully construct and prove the existence of at least 2 families of Fano 3-folds (see [10]). The candidates they work with are those which possess a particular cyclic quotient singularity:

Theorem 1.2.1. (Theorem 3.2, [10]) Let $X \subset \mathbb{P}(a_0, \ldots, a_7)$ be a numerical candidate for a codimension 4 Fano 3-fold. Up to reordering of a_i, there exist 116 numerical candidates such that the basket of X contains the cyclic quotient singularity $p = \frac{1}{a_7}(a_0, a_1, a_2)$. In these cases, the Hilbert scheme has at least 2 components containing quasismooth Fano 3-folds.

The techniques of Brown, Kerber and Reid cannot be extended to the remaining numerical candidates since a cyclic quotient singularity of the appropriate form does not exist. In this thesis we study a number of the remaining codimension 4 numerical candidates. We will construct and prove the existence of at least 2 families of Fano 3-folds for 16 numerical candidates containing a cyclic quotient singularity of a different shape. We prove the following result:

Theorem 1.2.2. Let $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_7)$ be a numerical candidate for a codimension 4 Fano 3-fold and suppose that X is not covered by Theorem 1.2.1. Up to relabelling of $a_1, \ldots, a_7$, there exist 16 numerical candidates such that the basket of X contains the cyclic quotient singularity $p = \frac{1}{a_7}(a_0, a_1, a_2)$. It is possible to realise these 16 numerical candidates as Fano 3-folds and for each candidate there exist 2 distinct families.

In other words, the Hilbert scheme for these 16 numerical candidates has at least 2 components containing Fano 3-folds.

Together Theorems 1.2.1 and 1.2.2 realise 132 of the 145 numerical candidates, i.e. 132 of the predicted Hilbert series occur as the Hilbert series of actual Fano 3-folds. For each of these 132 Hilbert series there exists at least 2 distinct families of Fano 3-folds. By counting distinct topological families of Fano 3-folds rather than distinct Hilbert series, we count over 500 families of codimension 4 Fano 3-folds $X \subset \mathbb{P}(a_0, \ldots, a_7)$.

Unprojections are used to prove Theorems 1.2.1 and 1.2.2 (see Section 1.3). For the 16 numerical candidates of Theorem 1.2.2, the first family will be constructed using a new unprojection method developed in Chapter 2, and the second family will be constructed using the unprojection method of [33]. Of the Fano 3-folds families in Theorem 1.2.2, 30 are newly realised as unprojections: Papadakis provides sketches for the constructions which realise 2 families of codimension 4 Fano 3-folds (see Sections 5.1 and 5.2 of [33]). Similarly, at least 10 families are new to the literature: using cluster algebras, [16] finds 1 family of Fano 3-folds for 10 candidates and 2 for the remaining 6. It is unknown whether the families of this thesis correspond to the families of [16].

1.3 Unprojections

Unprojections act as a substitute for the Gorenstein ring structure theory in high codimension. They provide a method of constructing and analysing high codimension Gorenstein rings in terms of lower codimension Gorenstein rings. Using unprojections to construct Fano 3-folds makes sense since their anticanonical rings are Gorenstein (see 5.1.9, [18]). There is a well established history of using unprojections to prove the existence of codimension 4 numerical candidates: the methods of [10] and [33] are in fact two distinct types of unprojection.

As previously mentioned, unprojections are intuitively the inverse of projection: the projection of the coordinate ring $k[Y] = \mathbb{C}[x_0, \dots, x_n]/I_Y$ is a subring of the form $k[X] = \mathbb{C}[x_0, \dots, x_m]/(I_Y \cap \mathbb{C}[x_0, \dots, x_m])$ where $n, m \in \mathbb{N}^+$ are such that $n < m$, so the unprojection is a method of recreating $k[Y]$ by adding variables and equations to $k[X]$.

Geometrically speaking, we wish to construct a birational map $\pi : X \dashrightarrow Y$ from a pair of schemes $D \subset X$ with D codimension 1 in X such that π contracts D and π is an isomorphism off D. There are many cases where this construction is possible. For example:

Example 1.3.1. Consider a codimension 2 complete intersection $Y_{2,2} \subset \mathbb{P}^4_{\langle x,y,z,u,v\rangle}$ defined by two degree 2 equations. Suppose that $Y_{2,2}$ contains the point $p_v := (0, 0, 0, 0, 1)$. Without loss of generality, $Y_{2,2}$ is defined by

$$vu + A_2 = vz + B_2 = 0$$

for some polynomials $A_2, B_2 \in \mathbb{C}[x, y, z, u]$ of degree 2. Define the hypersurface $X_3 \subset \mathbb{P}^3_{\langle x,y,z,u\rangle}$ by $A_2 z - B_2 u = 0$. The birational map

$$\pi : X_3 \dashrightarrow Y_{2,2}$$

$$\pi(x,y,z,u) = \left(x,y,z,u,-\frac{A_2}{u} = -\frac{B_2}{z}\right)$$

is intuitively the inverse of a projection from p_v on $Y_{3,3}$. We have that π is a contraction of $D := \{z = u = 0\} \subset X_3$ and an isomorphism off D. In particular, π is the Kawamata blow up of p_v and a contraction of finitely many lines

The crux is this: the homogeneous coordinate ring of Y is related to the ideals of X and D by some systematic calculation.

As far as this thesis is concerned, unprojections may be assumed to be a systematic method of constructing a Gorenstein ring $\mathcal{O}_Y$ from two smaller rings $\mathcal{O}_D$ and $\mathcal{O}_X$. We describe Example 1.3.1 in terms of rings as follows:

Example 1.3.2. Define $D, X = X_3$ and $Y = Y_{2,2}$ as in Example 1.3.1, and let $\mathcal{O}_X$ be the coordinate ring of X and I_D the ideal of D. Consider the $\mathcal{O}_X$-morphism $s : I_D \to \mathcal{O}_X$ where $s(z) = -B_2$ and $s(u) = -A_2$. The coordinate ring of Y is

$$\mathcal{O}_X[s] \cong \frac{\mathcal{O}_X[v]}{\langle vu + A_2, vz + B_2\rangle}.$$

That is, Y is the graph of s.

Unprojections were first used by Kustin and Miller in [26] as a method of describing codimension 4 Gorenstein ideals. Their systematic calculation was as follows:

Kustin-Miller Unprojection: Let $I_X \subset I_D$, be Gorenstein ideals of codimension $g-1$ and g inside the Gorenstein local ring R. Using the complex

$$\begin{tikzcd}
0 & R/I_X \arrow[l] \arrow[d] & M_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & M_{g-1} \arrow[l] \arrow[d, "(\alpha_1,\cdots,\alpha_g)^T"] & 0 \arrow[l] & \\
0 & R/I_D \arrow[l] & N_0 \arrow[l] & \ldots \arrow[l] & N_{g-1} \arrow[l] & N_g \arrow[l, "(\beta_1,\ldots,\beta_g)^T"'] & 0 \arrow[l]
\end{tikzcd}$$

obtained by the minimal resolutions of I_X and I_D, the ideal

$$I_X + \langle \beta_i v + \alpha_i : i = 1,\ldots,g\rangle$$

is a codimension g Gorenstein ideal in $R[v]$ for some indeterminate v. This ideal is the result of our unprojection. Note that $N_g \cong M_{g-1} \cong R$.

Since Kustin and Miller, the notion of unprojections has been expanded upon by Reid and Papadakis. There now exist many different types of unprojections where the Gorenstein assumptions are softened and multiple "unprojection indeterminates" are introduced (see [30], [40] and in particular Section 9 of [39],).

In Section 2.2, we describe a new unprojection method which we will later use to realise many codimension 4 numerical candidates.

1.4 Main Results and Structure of Thesis

We have two main results in this thesis. The first is the definition of a new unprojection; this result is more precisely stated as Theorem 2.2.1 and the proof is spread out across Chapters 2 and 3. We write a rough version of this result here:

Theorem 1.4.1. Define ideals $I_X \subset I_D$ in some positively graded ring $\mathcal{O}_{\mathrm{amb}}$ where I_D is as in Section 2.2, and I_X is codimension 1 inside I_D and such that $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal Gorenstein integral domain. Then:

1. The $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is generated by 1, s_0 and s_1 where 1 is the inclusion map and s_0 and s_1 are injective maps. We view s_0 and s_1 as rational functions having I_D as the ideal of denominators; that is, $s_0(f) = g \leftrightarrow s_0 = \frac{g}{f}$ (See Remark 2.2.3).

2. For some indeterminates T_0 and T_1, and ideal $I \subset \mathcal{O}_X[T_0, T_1]$, we have that $\mathcal{O}_X[s_0, s_1]$ is isomorphic to $\mathcal{O}_X[T_0, T_1]/I$, a Gorenstein ring with the same field of fractions as $\mathcal{O}_X$ (see Section 2.3).

3. The codimension of $\mathrm{Spec}(\mathcal{O}_X[T_0, T_1]/I) \subset \mathrm{Spec}(\mathcal{O}_{\mathrm{amb}}[T_0, T_1])$ is $\mathrm{codim}(I_X)+2$ (see Section 2.5).

4. The ring $\mathcal{O}_X[s_0, s_1]$ admits a presentation described in general terms. Moreover, we can describe this presentation precisely in the case where $\mathrm{codim}(I_X) = 2$ and codimension 4 rings are constructed (see Section 3.1.2).

Our new unprojection uses the rings $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ and $\mathcal{O}_D := \mathcal{O}_{\mathrm{amb}}/I_D$ to create a new ring $\mathcal{O}_X[T_0, T_1]/I$. The associated projection is the elimination of T_0 and T_1 from $\mathcal{O}_X[T_0, T_1]/I$.

The second main result of this thesis is Theorem 1.2.2 which is more accurately stated as Theorem 4.2.1. That is, we prove the existence of 16 new codimension 4 Fano 3-folds in weighted projective space. For each Fano 3-fold, we build one family by applying Theorem 2.2.1 and a second distinct family using the type II_1 unprojection of [33]. Loosely speaking, we define two varieties

$D \subset X \subset \mathbb{P}(a_0, \ldots, a_5)_{\langle x,y,z,u,v,w \rangle}$ by the ideals $I_X \subset I_D$ given in Theorem 1.4.1 (or Section 2.2). The ring created via the unprojection, $\mathcal{O}_X[T_0, T_1]/I$, will define a Fano 3-fold $Y \subset \mathbb{P}(a_0, \ldots, a_5, \mathrm{wt}(T_0), \mathrm{wt}(T_1))_{\langle x,y,z,u,v,w,T_0,T_1 \rangle}$. The projection in this case is the projection from the line $x = y = z = u = v = w = 0$. We have a weighted blow up followed by a flopping contraction (see Chapters 3 and 4).

In Chapter 2 we define our new unprojection. We argue that it is a type II unprojection and, moreover, is strongly related to type II_1 unprojections. To this end, we will expand the concept of type II_1 unprojections from the literature to include our format.

In Chapter 3, we explicitly calculate the rings defined by type II_1 unprojections. We also study various birational properties which will be helpful in proving Theorem 4.2.1.

In Chapter 4, we provide an extended example proving Theorem 4.2.1 for a single numerical candidate. In particular, we prove that there exist two distinct families of codimension 4 Fano 3-folds $Y \subset \mathbb{P}(2, 3, 4, 5, 6, 7, 8, 9)$ with genus $g_Y := -2$. The remaining numerical candidates are realised analogously; nevertheless, we sketch the proof.

The final chapter, Chapter 5, concerns several of the missing codimension 4 numerical candidates, the expansion of the type II unprojection definition and previously excluded codimension 4 numerical candidates. Recall that Theorem 4.2.1 considers numerical candidates of codimension 4 Fano 3-folds $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_7)$ with a singularity of the form $\frac{1}{a_7}(a_0, a_1, a_2)$ but excludes those already constructed by [10]. In this chapter we construct these excluded cases using type II_1 unprojections and predict a correspondence between the families of this thesis and the families of [10]. Although not the focus of this thesis, we also prove the existence of 7 codimension 4 Fano 3-fold families which cannot be constructed using the previously established methods of [10] or Chapter 4.

Chapter 2

A New Unprojection

Our first task is to construct Gorenstein rings.

2.1 Preliminaries

In the literature, there is some variation in the nomenclature used. We therefore provide the standard definitions used throughout this thesis.

Definition 2.1.1. (Definition 1.2.7 and Theorem 1.2.8, [12]) Let R be a Noetherian local ring with residue field $k := R/\mathfrak{m}$. The *depth* of a finite non-zero R-module M is

$$\mathrm{depth}(M) := \min\{i \geq 0 : \mathrm{Ext}^i_R(k, M) \neq 0\}.$$

Definition 2.1.2. (Definition 1.2.11, [12]) Let R be a Noetherian ring. The *grade* of a finite non-zero R-module M is

$$\mathrm{grade}(M) := \min\{i \geq 0 : \mathrm{Ext}^i_R(M, R) \neq 0\}.$$

For systematic reasons, the grade of $M = 0$ is infinity.

Definition 2.1.3. (Definition 1.4.15, [12]) Let R be a Noetherian ring and M a finite non-zero R-module. We say that M is *perfect* if

$$\mathrm{proj\ dim}(M) = \mathrm{grade}(M).$$

An ideal I of R is perfect if the R-module R/I is perfect.

Definition 2.1.4. (Definition 2.1.1, [12]) Let R be a Noetherian local ring. A finite non-zero R-module M is *Cohen-Macaulay* if $\mathrm{depth}(M) = \dim(M)$. The ring R is a *Cohen-Macaulay ring* if it is a Cohen-Macaulay R-module.

Definition 2.1.5. (Definition 2.1.1, [12]) Let R be an arbitrary Noetherian ring. A finite non-zero R-module M is said to be *Cohen-Macaulay* if $M_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \subset R$ in the sense of Definition 2.1.4. Similarly, R is said to be a *Cohen-Macaulay ring* if $R_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \subset R$.

Definition 2.1.6. (Chapter 9, [17]) Let R be a commutative ring with unity and let $P \subset R$ be a prime ideal. The *codimension* of P, $\operatorname{codim}(P)$, is the maximal $n \in \mathbb{N}$ such that

$$P_0 \subset P_1 \subset \cdots \subset P_n \subset P$$

is a chain of strictly increasing prime ideals. That is, $\operatorname{codim}(P) = \dim(R_P)$. For I, a proper ideal of R, we define

$$\operatorname{codim}(I) := \min\{\operatorname{codim}(P) : I \subset P \text{ and } P \text{ is a prime ideal of } R\}.$$

Definition 2.1.7. (Theorem 3.3.7, [12]) A local Noetherian ring R is *Gorenstein* if R is Cohen-Macaulay and the dualizing module of R, ω_R, exists and is such that $\omega_R \cong R$. If R is an arbitrary Noetherian ring, we say that R is Gorenstein if $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal $\mathfrak{m} \subset R$.

2.2 Format and Main Result

Unprojections provide a method of defining a large Gorenstein ring in terms of two smaller rings. Our first step towards constructing new Gorenstein rings will be to define the initial data of a new unprojection.

Fix $n, m, p \in \mathbb{N}^+$ such that $n \geq 2$. Define

$$\mathcal{O}_{\text{amb}} := \mathbb{Z}[x_j, y_j, w_i, z, v_l]$$

with $1 \leq j \leq n$, $1 \leq i \leq m$ and $1 \leq l \leq p$. Let $\mathcal{O}_{\text{amb}}$ be a positively graded ring such that the weight of z is even and

$$\operatorname{wt}(y_j) = \operatorname{wt}(x_j) + \frac{1}{2}\operatorname{wt}(z)$$

for all $1 \leq j \leq n$. Note that we have defined some number of extra indeterminates v_l to provide extra flexibility during the later sections (see Section 3.1.2); however, these indeterminates can be ignored for now.

Let $I_D \subset \mathcal{O}_{\text{amb}}$ be the ideal generated by $w_1 = \cdots = w_m = 0$ together with the 2×2 minors of the $2 \times 2n$ matrix

$$M := \begin{pmatrix} y_1 & \cdots & y_n & zx_1 & \cdots & zx_n \\ x_1 & \cdots & x_n & y_1 & \cdots & y_n \end{pmatrix}.$$

The ideal defined by only the 2×2 minors of M is prime and of codimension n in $\mathcal{O}_{\mathrm{amb}}$ (see the comment after Remark 2.1 in [31]); hence, the ideal I_D is prime and of codimension $n + m$.

Let $I_X \subset I_D$ be a homogeneous prime ideal of $\mathcal{O}_{\mathrm{amb}}$ such that $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal Gorenstein integral domain and I_X is codimension $n + m - 1$ in $\mathcal{O}_{\mathrm{amb}}$.

Remark 2.2.1. Without loss of generality, we assume that $w_i \notin I_X$ for $i = 1, \ldots, m$. If $I_X = \langle w_m \rangle + I'_X$ for some ideal $I_{X'}$, we work with $\mathcal{O}_{\mathrm{amb}}/\langle w_m \rangle$, the ideal I'_D defined by the 2×2 minors of M together with $w_1 = \cdots = w_{m-1} = 0$, and the ideal I'_X.

Our initial data for the unprojection consists of the rings $\mathcal{O}_X$ and $\mathcal{O}_D := \mathcal{O}_{\mathrm{amb}}/I_D$, where the latter is viewed as a quotient of $\mathcal{O}_X$. The initial data is equivalently the triple $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$.

The systematic method to construct a new Gorenstein ring is as follows:

Definition 2.2.1. Define $K(X)$ as the field of fractions of $\mathcal{O}_X$. The *unprojection ring of* $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is the $\mathcal{O}_X$-subalgebra

$$\mathcal{O}_X[I_D^{-1}] \subset K(X)$$

where $I_D^{-1} \subset K(X)$ is the $\mathcal{O}_X$-module

$$I_D^{-1} := \{f \in K(X) : fI_D \subset \mathcal{O}_X\}.$$

Remark 2.2.2. Note that $I_D^{-1} \subset K(X)$ is an $\mathcal{O}_X$-submodule and $\mathcal{O}_X[I_D^{-1}]$ is a ring: if $s \in I_D^{-1} - \mathcal{O}_X$, then $s^2 \notin I_D^{-1}$ but $s^2 \in \mathcal{O}_X[I_D^{-1}]$. In other words, I_D^{-1} and $\mathcal{O}_X[I_D^{-1}]$ are distinct as sets. This remark will be useful in Section 2.3 where we define a valuation on $\mathcal{O}_X[I_D^{-1}]$.

Remark 2.2.3. The modules I_D^{-1} and $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ are isomorphic. For each $x \in I_D$ and $f \in I_D^{-1}$, we may define $\widetilde{f} \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ by $\widetilde{f}(x) := fx$. Conversely, for $x \in I_D$ and $\widetilde{f} \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$, we may define $f := \frac{\widetilde{f}(x)}{x} \in I_D^{-1}$ which is well defined since $y\widetilde{f}(x) = x\widetilde{f}(y)$ for $x, y \in I_D$. Without confusion, we may use the notation of I_D^{-1} and $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ interchangeably.

It is not yet clear that the unprojection ring of $(I_X, I_D, \mathcal{O}_{\text{amb}})$ is Gorenstein. In the next few Sections we will prove the following theorem:

Theorem 2.2.1. There exists an isomorphism

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_X[T_0, T_1]}{\langle l_1, \dots, l_{2(n+m)}, q\rangle}$$

where for $1 \leq j \leq n$ and some $c_j, d_j \in \mathcal{O}_X$ we have

$$l_j := y_j T_1 + z x_j T_0 - c_j,$$

$$l_{n+m+j} := y_j T_0 + x_j T_1 - d_j;$$

for $1 \leq i \leq m$ and some $c_{n+i}, d_{n+i} \in \mathcal{O}_X$ we have

$$l_{n+i} := w_i T_1 - c_{n+i},$$

$$l_{2n+m+i} := w_i T_0 - d_{n+i};$$

and for some $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{O}_X$

$$q := T_1^2 - z T_0^2 + \alpha_0 T_0 + \alpha_1 T_1 + \alpha_2.$$

Furthermore, $\mathcal{O}_X[I_D^{-1}]$ is a Gorenstein ring, perfect as an $\mathcal{O}_{\text{amb}}[T_0, T_1]$-module and such that

$$\operatorname{codim}_{\mathcal{O}_{\text{amb}}[T_0,T_1]}(I_X + \langle l_1, \dots, l_{2(n+m)}, q\rangle) = n + m + 1.$$

Remark 2.2.4. We call T_0 and T_1 the unprojection indeterminates. We also refer to l_i as the linear equations of the unprojection ring and q the quadratic: this is a reference to their total degree with respect to T_0 and T_1.

Remark 2.2.5. In cases where I_X and I_D are not defined as specified, it is still possible for $\mathcal{O}_X[I_D^{-1}]$ to construct Gorenstein rings. We refer to the special way in which the data $(I_X, I_D, \mathcal{O}_{\text{amb}})$ is defined as an *unprojection format*.

2.3 Isomorphism

In this section, we prove the first statement of Theorem 2.2.1. We will prove that

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_X[T_0, T_1]}{\langle l_1, \dots, l_{2(n+m)}, q\rangle}$$

where l_i has total degree 1 and q has total degree 2 with respect to T_0 and T_1. The approach taken follows [31].

We start by noting some of the key properties of $\mathcal{O}_D$ and ω_D.

Remark 2.3.1. We use the standard notation of $D \subset X$ where

$$D := V(I_D) \subset \mathrm{Spec}(\mathcal{O}_{\mathrm{amb}}),$$

and

$$X := \mathrm{Spec}(\mathcal{O}_X) \subset \mathrm{Spec}(\mathcal{O}_{\mathrm{amb}}).$$

Lemma 2.3.1. The ring $\mathcal{O}_D$ and the variety D are non-normal. As varieties, the normalisation is defined by the morphism

$$\pi : \widetilde{D} := \mathrm{Spec}(\mathbb{Z}[a_1, \ldots, a_n, t, v_1, \ldots, v_p]) \longrightarrow D$$

where the corresponding morphism of rings

$$\pi_* : \mathcal{O}_D \longrightarrow \mathcal{O}_{\widetilde{D}} := \mathbb{Z}[a_1, \ldots, a_n, t, v_1, \ldots, v_p]$$

is defined by

$$y_j \mapsto a_j t, \qquad x_j \mapsto a_j, \qquad w_i \mapsto 0, \qquad v_j \mapsto v_j, \qquad z \mapsto t^2.$$

Remark 2.3.2. The ring $\mathcal{O}_{\widetilde{D}}$ is Gorenstein.

Remark 2.3.3. As π is an isomorphism in codimension 1, $\omega_D \cong \pi_* \omega_{\widetilde{D}} \cong \pi_* \mathcal{O}_{\widetilde{D}}$. We note that ω_D is Cohen-Macaulay.

Remark 2.3.4. As I_D is codimension 1 in $\mathcal{O}_X$ and $\mathcal{O}_X$ is a Gorenstein ring, we have that $\omega_D = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_D, \mathcal{O}_X)$ by the adjunction formula (see [38] Theorem 2.12 or [12] Theorem 3.3.7).

It is clear from Remark 2.3.3 and Lemma 2.3.1 that ω_D needs 2 generators as an $\mathcal{O}_D$-module. However:

Lemma 2.3.2. The dualizing module ω_D needs two generators as an $\mathcal{O}_X$-module, e_0 and e_1, which may be chosen so that

$$(e_0, e_1)M = 0$$

and $e_0 w_i = e_1 w_i = 0$ for all $i = 1, \ldots, m$.

Lemma 2.3.2 follows from Section 2.1 of [31] and Section 9.5 of [39]:

Proof. Let I'_D be defined by the 2×2 minors of M and let $\mathcal{O}'_{\text{amb}} := \mathcal{O}_{\text{amb}}/\langle w_1, \dots, w_m \rangle$. It is clear that $\mathcal{O}_D$ and $\mathcal{O}'_{\text{amb}}/I'_D$ are isomorphic as $\mathcal{O}_{\text{amb}}$-modules and as $\mathcal{O}_X$-modules. Consequently, ω_D and the dualizing module of $\mathcal{O}'_{\text{amb}}/I'_D$ are isomorphic. By [31] and [39], the dualizing module of $\mathcal{O}'_{\text{amb}}/I'_D$ has exactly two generators as an $\mathcal{O}_D$-module and the generators, say e'_0 and e'_1, may be chosen to ensure that $(e'_0, e'_1)M = 0$. We may therefore choose generators e_0 and e_1 for ω_D such that $(e_0, e_1)M = 0$ by virtue of our isomorphism. Furthermore, e_0 and e_1 are generators of ω_D as an $\mathcal{O}_X$-module since $\mathcal{O}_D$ is a quotient ring of $\mathcal{O}_X$.

All that remains to show is that $e_0 w_i = e_1 w_i = 0$ for all $i = 1, \dots, m$. However, this is clear from Remark 2.3.3 and Lemma 2.3.1. $\square$

We are interested in ω_D since we wish to calculate the generators of I_D^{-1}. This can be done using the following short exact sequence and recalling that I_D^{-1} is isomorphic to $\text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ (see Remark 2.2.3):

Lemma 2.3.3. There exists a short exact sequence

$$0 \to \mathcal{O}_X \to \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \to \omega_D \to 0. \tag{2.1}$$

Proof. This is standard. From the initial set up, we have the short exact sequence

$$0 \to I_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and the long exact sequence

$$\begin{aligned} 0 \to \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X) \to \text{Hom}_{\mathcal{O}_X}(I_D, \omega_X) \\ \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X) \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X) \to \cdots . \end{aligned}$$

By Remark 2.3.4 we know that $\omega_D \cong \text{Ext}_{\mathcal{O}_X}(\mathcal{O}_D, \mathcal{O}_X)$. As $\mathcal{O}_X$ is Gorenstein, $\omega_X \cong \mathcal{O}_X$ and we may replace all instances of ω_X with $\mathcal{O}_X$ in the long exact sequence. As $I_X \subset I_D$ is a strict inclusion, we have that $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_D, \mathcal{O}_X) = 0$. Since $\mathcal{O}_X$ is projective as a module over itself, $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$. Simplifying the long exact sequence accordingly provides our desired result. $\square$

Remark 2.3.5. The map $\text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \to \omega_D$ in sequence (2.1) is known as the Poincaré residue map and denoted by res_D.

Fix generators of ω_D, e_0 and e_1, as in Lemma 2.3.2 and let s_0 and s_1 be any lifting under res_D of e_0 and e_1 respectively. We may assume that s_0 and s_1 are injective (see Lemma 1.1 of [34]). Then:

Proposition 2.3.1. The $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$, equivalently I_D^{-1}, is generated by 1, s_0 and s_1.

Proof. Let $x \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. The $\mathcal{O}_X$-module ω_D is generated by e_0 and e_1 and therefore $\mathrm{res}_D(x) = \alpha e_0 + \beta e_1$ for some $\alpha, \beta \in \mathcal{O}_X$. Hence, $\mathrm{res}_D(x - \alpha s_0 - \beta s_1) = 0$. By the exact sequence (2.1), we have that $\ker(\mathrm{res}_D) \cong \mathcal{O}_X$ and $x = \alpha s_0 + \beta s_1 + \gamma$ with $\gamma \in \mathcal{O}_X$. $\square$

In particular:

Lemma 2.3.4. There exist $c_l, d_l \in \mathcal{O}_X$ where $l = 1, \dots, n+m$ such that

$$y_j s_1 + z x_j s_0 - c_j = 0,$$

$$w_i s_1 - c_{n+i} = 0$$

and

$$y_j s_0 + x_j s_1 - d_j = 0,$$

$$w_i s_0 - d_{n+i} = 0$$

for $1 \leq j \leq n$ and $1 \leq i \leq m$.

Proof. The $\mathcal{O}_X$-module ω_D is such that its module of linear relations is generated by

$$(e_0, e_1) \begin{pmatrix} y_1 & \dots & y_n & zx_1 & \dots & zx_n & 0 & \dots & 0 & w_1 & \dots & w_m \\ x_1 & \dots & x_n & y_1 & \dots & y_n & w_1 & \dots & w_m & 0 & \dots & 0 \end{pmatrix} = 0.$$

We have, for example, that

$$\mathrm{res}_D(y_1 s_0 + x_1 s_1) = y_1 e_0 + x_1 e_1 = 0$$

and $y_1 s_0 + x_1 s_1 \in \ker(\mathrm{res}_D)$. Since $\ker(\mathrm{res}_D) \cong \mathcal{O}_X$ by sequence (2.1), we have $y_1 s_0 + x_1 s_1 = d_1$ for some $d_1 \in \mathcal{O}_X$. The other relations are analogous. $\square$

Recall that X is normal and D is irreducible; hence, there exists a natural valuation

$$\mathrm{val}_D : K(X)^* \to \mathbb{Z} \cup \{\infty\}$$

(see [20], Chapter II, Section 6, Subsection Weil Divisors). The valuation is the order of vanishing along D and we have $\mathrm{val}_D(f) \geq -1$ for $f \in I_D^{-1}$.

Remark 2.3.6. As in Remark 2.4 of [31], any $f \in \mathcal{O}_X[I_D^{-1}]$ such that $\mathrm{val}_D(f) \geq 0$ is in fact an element of $\mathcal{O}_X$.

Remark 2.3.7. If $f \in K(X)$ is such that $\mathrm{val}_D(f) = -1$, then $f \in I_D^{-1}$: for any $g \in I_D$ we have $\mathrm{val}_D(fg) \geq 0$ and hence $fg \in \mathcal{O}_X$.

Let $\mathcal{O}_X[T_0, T_1]$ be the polynomial ring over $\mathcal{O}_X$ with indeterminates T_0 and T_1. Define

$$\phi : \mathcal{O}_X[T_0, T_1] \to \mathcal{O}_X[I_D^{-1}]$$

as the natural $\mathcal{O}_X$-algebra homomorphism extending $\phi(T_0) := s_0$ and $\phi(T_1) := s_1$. Define

$$l_j := y_j T_1 + z x_j T_0 - c_j,$$

$$l_{n+i} := w_i T_1 - c_{n+i},$$

$$l_{n+m+j} := y_j T_0 + x_j T_1 - d_j$$

and

$$l_{2n+m+i} := w_i T_0 - d_{n+i},$$

for $1 \leq j \leq n$ and $1 \leq i \leq m$. It is useful to restate Lemma 2.3.4 as follows:

Lemma 2.3.5. The homomorphism ϕ is surjective and such that

$$\left\langle l_1, \ldots, l_{2(n+m)} \right\rangle \subset \ker(\phi).$$

In addition to $\langle l_1, \ldots, l_{2(n+m)} \rangle$, the kernel of ϕ contains an equation of total degree 2 in T_0 and T_1:

Lemma 2.3.6. There exists

$$q := T_1^2 - z T_0^2 + \alpha_0 T_0 + \alpha_1 T_1 + \alpha_2 \in \ker(\phi)$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{O}_X$.

The general form of the quadratic and the proof of its existence follows as in Lemma 2.5, [31]. Namely, we calculate that

$$T_1 l_{n+m+1} - T_0 l_1 + d_1 T_1 - c_1 T_0 = x_1 (T_1^2 - z T_0^2),$$

and that

$$\begin{aligned}\mathrm{val}_D(x_1(s_1^2 - zs_0^2)) &\geq \min\{\mathrm{val}_D(s_1\phi(l_{n+m+1})), \mathrm{val}_D(s_0\phi(l_1)), \mathrm{val}_D(d_1s_1 - c_1s_0)\}\\ &\geq -1.\end{aligned}$$

As $\mathrm{val}_D(x_1) = 0$, we have that $\mathrm{val}_D(s_1^2 - zs_0^2) \geq -1$. Hence, by definition of I_D^{-1} and val_D, we have $s_1^2 - zs_0^2 \in I_D^{-1}$ and $s_1^2 - zs_0^2 = -\alpha_0 s_0 - \alpha_1 s_1 - \alpha_2$ for some $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{O}_X$.

In fact $l_1, \ldots, l_{2(n+m)}$ and q completely generate $\ker(\phi)$. We have that:

Proposition 2.3.2. $\ker(\phi) = \langle l_1, \ldots, l_{2(n+m)}, q\rangle$.

The proof follows that of Proposition 2.6, [31]. It is sufficient to consider $h \in \ker(\phi)$ which are linear in T_1 since q allows us to eliminate all even powers of T_1 from any element of $\ker(\phi)$. The result is then proven for h using induction on k, the total degree of h with respect to T_0 and T_1.

Suppose $h = \alpha T_0 + \beta T_1 + \gamma$ for some $\alpha, \beta, \gamma \in \mathcal{O}_X$ and $h \in \ker(\phi)$. The module of linear relations for ω_D is generated by

$$y_ie_1 + zx_ie_0 = y_ie_0 + x_ie_1 = w_je_0 = w_je_1 = 0$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. For simplicity of notation, we write these relations as

$$y_ie_1 + zx_ie_0 = y_ie_0 + x_ie_1 = 0$$

for $i = 1, \ldots, n + m$ where we define $y_{j+n} := w_j$ and $x_{j+n} := 0$ for $j = 1, \ldots, m$. Since

$$\alpha e_0 + \beta e_1 = \mathrm{res}_D(\alpha s_0 + \beta s_1 + \gamma) = \mathrm{res}_D(\phi(h)) = 0$$

is a linear relation in ω_D, we have that

$$\alpha e_0 + \beta e_1 = \sum_{i=1}^{n+m} \eta_i(y_ie_1 + zx_ie_0) + \zeta_i(y_ie_0 + x_ie_1)$$

for some $\eta_i, \zeta_i \in \mathcal{O}_X$. Consequently,

$$(\alpha s_0 + \beta s_1) - \sum_{i=1}^{n+m} \eta_i(x_izs_0 + y_is_1) + \zeta_i(y_is_0 + x_is_1) \in \ker(\mathrm{res}_D) = \mathcal{O}_X$$

and

$$h - \sum_{i=1}^{n+m} \eta_il_i + \zeta_il_{n+m+i} \in \mathcal{O}_X \cap \ker(\phi) = \{0\}.$$

For $k \geq 1$, we may write

$$h = \beta_0 T_0^{k+1} + \beta_1 T_1 T_0^k + L$$

for some $\beta_0, \beta_1 \in \mathcal{O}_X$ and $L \in \mathcal{O}_X[T_0, T_1]$ of total degree at most k with respect to T_0 and T_1. It must be the case that $\beta_0 s_0 + \beta_1 s_1 \in \mathcal{O}_X$, otherwise

$$\begin{aligned} \mathrm{val}_D(\phi(h)) &= \mathrm{val}_D(\beta_0 s_0^{k+1} + \beta_1 s_1 s_0^k + \phi(L)) \\ &= \mathrm{val}_D(\beta_0 s_0^{k+1} + \beta_1 s_1 s_0^k) \\ &\leq -k-1 \end{aligned}$$

and $\mathrm{val}_D(\phi(h)) = \mathrm{val}_D(0) = \infty$. That is, $\beta_0 s_0 + \beta_1 s_1 = \gamma \in \mathcal{O}_X$ and

$$h = (\beta_0 T_0 + \beta_1 T_1 - \gamma) T_0^k + \gamma T_0^k + L.$$

Clearly, $\beta_0 T_0 + \beta_1 T_1 - \gamma \in \ker(\phi)$ and hence $\gamma T_0^k + L \in \ker(\phi)$; therefore by our induction assumption, $\beta_0 T_0 + \beta_1 T_1 - \gamma$ and $\gamma T_0^k + L$ lie in $\langle l_1, \ldots, l_{2(n+m)}, q \rangle$. Our desired result follows immediately.

We can now prove the long awaited result of this section: by applying the isomorphism theorem to ϕ, we obtain

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_X[T_0, T_1]}{\langle l_1, \ldots, l_{2(n+m)}, q \rangle}.$$

2.4 Gorenstein

In this section, we prove that the unprojection ring is Gorenstein. For ease of notation, we define

$$I_Y := \langle l_1, \ldots, l_{2(m+n)}, q \rangle \subset \mathcal{O}_X[T_0, T_1]$$

and write

$$\mathcal{O}_Y := \mathcal{O}_X[T_0, T_1]/I_Y$$

where $l_1, \ldots, l_{2(n+m)}$ and q are defined as in Section 2.3.

We follow [31] (compare Lemmas 2.4.1, 2.4.2, 2.4.4 and 2.4.3 to Corollary 2.9, Proposition 2.10 and Theorem 2.15 of [31]). To show that $\mathcal{O}_Y$ is Gorenstein, we will instead show that $\mathcal{O}_Y/\langle T_0 \rangle$ is Gorenstein. This is sufficient since a local ring R is Gorenstein if and only if $R/\langle x_1, \ldots, x_n \rangle$ is a Gorenstein ring for an R-regular sequence $\langle x_1, \ldots, x_n \rangle$ (see Proposition 3.1.19, [12]). Note that T_0 is not a zero divisor since $\mathcal{O}_X[I_D^{-1}] \cong \mathcal{O}_Y$ is an integral domain.

Remark 2.4.1. Recall that for a positively graded ring $R := \bigoplus_{i\in\mathbb{N}} R_i$ over $\mathbb{Z}$, an ideal $\mathfrak{m} \subset R$ is maximal if and only if $\mathfrak{m} = \langle p \rangle \oplus R_1 \oplus R_2 \oplus \dots$ for some prime $p \in \mathbb{N}$. Throughout this section we work locally at a specific prime p but suppress the localisation notation.

Let $I_N := \mathrm{im}(s_0) \subset \mathcal{O}_X$ and define the ring $\mathcal{O}_N := \mathcal{O}_X/I_N$. By Proposition 2.8 of [31]:

Lemma 2.4.1. The codimension of I_N in $\mathcal{O}_X$ is 1.

Lemma 2.4.2. We have the following isomorphisms of $\mathcal{O}_X$-modules:

$$\mathcal{O}_Y/\langle T_0 \rangle \cong \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)/\langle s_0 \rangle \cong \mathrm{Hom}_{\mathcal{O}_X}(I_N, \mathcal{O}_X)/\langle i_N \rangle \cong \omega_N \tag{2.2}$$

where $i_N : I_N \to \mathcal{O}_X$ is the natural inclusion.

Proof. The result follows immediately from the proof of Proposition 2.10 of [31]. The first isomorphism of (2.2) follows by applying the fundamental theorem of homomorphisms to the $\mathcal{O}_X$-homomorphism $\sigma : \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \to \mathcal{O}_Y/\langle T_0 \rangle$ extending $\sigma(s_0) := T_0 = 0$ and $\sigma(s_1) := T_1$. Since $s_0 : I_D \to I_N$ is itself an isomorphism, the second isomorphism of (2.2) is constructed by defining the induced morphism $s_0^* : \mathrm{Hom}_{\mathcal{O}_X}(I_N, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ where $s_0^*(i_N) = s_0$. The codimension of I_N and the exact sequence

$$0 \to \mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_X}(I_N, \mathcal{O}_X) \to \omega_N \to 0$$

provide the final isomorphism of (2.2). ☐

To prove that $\mathcal{O}_Y/\langle T_0 \rangle$ is Gorenstein, we work with ω_N and will show that $\mathrm{depth}(\omega_N) = \mathcal{O}_X - 1$ and $\omega_N \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X)$.

Lemma 2.4.3. We have that $\mathrm{depth}(\omega_N) = \mathcal{O}_X - 1$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}_X$ and $k := \mathcal{O}_X/\mathfrak{m}$ the residue field. The long exact sequence of

$$0 \to \mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \to \omega_D \to 0$$

with respect to $\mathrm{Hom}_{\mathcal{O}_X}(k, -)$ is

$$\begin{aligned}\cdots \to \mathrm{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X) \to \mathrm{Ext}^i_{\mathcal{O}_X}(k, \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)) \to \mathrm{Ext}^i_{\mathcal{O}_X}(k, \omega_D) \to \\ \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k, \mathcal{O}_X) \to \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k, \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)) \to \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k, \omega_D) \to \cdots .\end{aligned}$$

As $\mathcal{O}_X$ and ω_D are Cohen-Macaulay, $\mathrm{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X)$ and $\mathrm{Ext}^j_{\mathcal{O}_X}(k, \omega_D)$ are 0 for $i = 0, \ldots, \dim(\mathcal{O}_X) - 1$ and $j = 0, \ldots, \dim(\mathcal{O}_X) - 2$. Note that the depth of ω_D is such that $\mathrm{depth}(\omega_D) = \dim(\mathcal{O}_X) - 1$ (see the start of Section 2.3 for the properties of I_D). Hence, by our long exact sequence $\mathrm{Ext}^i_{\mathcal{O}_X}(k, \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)) = 0$ for $i = 0, \ldots, \dim(\mathcal{O}_X) - 2$.

Define

$$h_2 : \mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \text{ where } (h_2(a))(b) = s_0(ab).$$

Note that $\ker(h_2) = 0$ by the injectivity of s_0 and $\mathrm{coker}(h_2) = \omega_N$ by Lemma 2.4.2. The associated long exact sequence of

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{h_2} \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \longrightarrow \omega_N \longrightarrow 0$$

with respect to $\mathrm{Hom}_{\mathcal{O}_X}(k, -)$ is

$$\cdots \to \mathrm{Ext}^i_{\mathcal{O}_X}(k, \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)) \to \mathrm{Ext}^i_{\mathcal{O}_X}(k, \omega_N) \to \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k, \mathcal{O}_X) \to \\ \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k, \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)) \to \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k, \omega_N) \to \mathrm{Ext}^{i+2}_{\mathcal{O}_X}(k, \mathcal{O}_X) \to \cdots .$$

We know that $\mathrm{Ext}^i_{\mathcal{O}_X}(k, \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)) = 0$ for $i = 0, \ldots, \dim(\mathcal{O}_X) - 2$ and $\mathrm{Ext}^j_{\mathcal{O}_X}(k, \mathcal{O}_X) = 0$ for $j = 0, \ldots, \dim(\mathcal{O}_X) - 1$. Hence, $\mathrm{Ext}^i_{\mathcal{O}_X}(k, \omega_N) = 0$ for $i = 0, \ldots, \dim(\mathcal{O}_X) - 2$ and our desired result follows. □

That is, the ring ω_N is Cohen-Macaulay.

Lemma 2.4.4. We have that $\omega_N \cong \mathrm{Ext}_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X)$.

Proof. As I_N is codimension 1 in $\mathcal{O}_X$ and $\mathcal{O}_X$ is a Gorenstein ring, we have that $\omega_N = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_N, \mathcal{O}_X)$ by the adjunction formula (see [38] Theorem 2.12). To prove our desired result, we will show that $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_N, \mathcal{O}_X) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X)$.

Define the following injective maps

$$h_1 : I_D \to \mathcal{O}_X \text{ where } h_1(a) = s_0(a),$$

$$h_2 : \mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \text{ where } (h_2(a))(b) = s_0(ab),$$

and

$$h_3 : \mathcal{O}_D \to \omega_D \text{ where } h_3(a) = a\,\mathrm{res}_D(s_0) \in \omega_D.$$

Applying the snake lemma to the complex

$$\begin{tikzcd}
 & I_D \arrow[r, hookrightarrow] \arrow[d, "h_1"'] & \mathcal{O}_X \arrow[r, twoheadrightarrow] \arrow[d, "h_2"'] & \mathcal{O}_D \arrow[r] \arrow[d, "h_3"'] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, hookrightarrow] & \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \arrow[r, twoheadrightarrow, "\operatorname{res}_D"'] & \omega_D &
\end{tikzcd}$$

gives the exact sequence

$$\ker(h_1) \to \ker(h_2) \to \ker(h_3) \to \operatorname{coker}(h_1) \to \operatorname{coker}(h_2) \to \operatorname{coker}(h_3).$$

As h_1, h_2 and h_3 are injective, we have that $\ker(h_1) = \ker(h_2) = \ker(h_3) = \{0\}$. Furthermore,

$$\operatorname{coker}(h_1) = \mathcal{O}_X / I_N = \mathcal{O}_N$$

by definition and $\operatorname{coker}(h_2) = \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)/\langle s_0 \rangle \cong \omega_N$ by Lemma 2.4.2. Therefore, the exact sequence simplifies to

$$0 \to \mathcal{O}_N \to \omega_N \to \operatorname{coker}(h_3) \to 0. \tag{2.3}$$

The associated long exact sequence of (2.3) with respect to $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ is

$$\begin{aligned} \cdots \to \operatorname{Ext}^1_{\mathcal{O}_X}(\operatorname{coker}(h_3), \mathcal{O}_X) &\to \operatorname{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X) \\ &\to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_N, \mathcal{O}_X) \to \operatorname{Ext}^2_{\mathcal{O}_X}(\operatorname{coker}(h_3), \mathcal{O}_X) \to \cdots . \end{aligned}$$

We claim that $\operatorname{Ext}^l_{\mathcal{O}_X}(\operatorname{coker}(h_3), \mathcal{O}_X) = 0$ for $l = 1, 2$ in which case we obtain the exact sequence

$$0 \to \operatorname{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_N, \mathcal{O}_X) \to 0$$

and the isomorphism $\operatorname{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_N, \mathcal{O}_X) \cong \omega_N$. To prove this claim, we will show that

$$\operatorname{coker}(h_3) \cong \mathcal{O}_X / \langle y_j, x_j, w_i : 1 \leq j \leq n \text{ and } 1 \leq i \leq m \rangle$$

and hence $\operatorname{Ext}^l_{\mathcal{O}_X}(\operatorname{coker}(h_3), \mathcal{O}_X) = 0$ for all $l < \dim(\mathcal{O}_X) - \dim(\operatorname{coker}(h_3))$ where $2 < \dim(\mathcal{O}_X) - \dim(\operatorname{coker}(h_3))$ (see Corollary 3.5.11, [12]).

Let $u = v + \operatorname{im}(h_3) \in \operatorname{coker}(h_3)$ where $v \in \omega_D$. By Lemma 2.3.2, the $\mathcal{O}_X$-module ω_D is generated by e_0 and e_1 and $v = v_0 e_0 + v_1 e_1$ for some $v_0, v_1 \in \mathcal{O}_X$.

In particular, we have that $u = v_1 e_1 + \mathrm{im}(h_3)$ where

$$v_1 \in \mathcal{O}_X - \langle x_j, y_j, w_i : 1 \leq j \leq n \text{ and } 1 \leq i \leq m \rangle.$$

This is because

$$y_j e_0 + x_j e_1 = z x_j e_0 + y_j e_1 = w_i e_1 = w_i e_0 = 0$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$ by our choice of generators and the fact that $\mathrm{im}(h_3) = \mathcal{O}_X e_0$. It is clear that $\mathrm{coker}(h_3) \cong \mathcal{O}_X / \langle x_j, y_j, w_i : 1 \leq j \leq n \text{ and } 1 \leq i \leq m \rangle$. $\square$

This proves that the unprojection ring is indeed Gorenstein.

2.5 Codimension

The proofs of Propositions 2.5.1 and 2.5.2 generalise that of Proposition 2.16, [31].

Proposition 2.5.1. As an $\mathcal{O}_{\mathrm{amb}}[T_0, T_1]$-module, $\mathcal{O}_Y$ is perfect.

Proposition 2.5.2. $\mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}[T_0,T_1]}(I_X + I_Y) = \mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_X) + 2 = n + m + 1.$

A finite $\mathbb{Z}[x_1, \ldots, x_n]$-module is Cohen-Macaulay if and only if it is perfect (see comment after Proposition 16.19 of [13]). Therefore, the perfectness of $\mathcal{O}_Y$ as an $\mathcal{O}_{\mathrm{amb}}[T_0, T_1]$-module is immediate from $\mathcal{O}_Y$ being a finitely generated Cohen-Macaulay $\mathcal{O}_{\mathrm{amb}}$-module .

To calculate the codimension of $I_X + I_Y$, we show that for $J := \langle l_1, \ldots, l_{2(n+m)}, q \rangle \cap \mathcal{O}_X[T_0]$

$$\dim(\mathcal{O}_X[I_D^{-1}]) = \dim(\mathcal{O}_X[T_0]/J) = \dim(\mathcal{O}_X),$$

in which case

$$\begin{aligned} \mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}[T_0,T_1]}(I_X + I_Y) &= \dim(\mathcal{O}_{\mathrm{amb}}[T_0, T_1]) - \dim(\mathcal{O}_X[I_D^{-1}]) \\ &= \dim(\mathcal{O}_{\mathrm{amb}}[T_0, T_1]) - \dim(\mathcal{O}_X) \\ &= \mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_X) + 2. \end{aligned}$$

As $\mathcal{O}_X[I_D^{-1}]$ is a finitely generated $(\mathcal{O}_X[T_0]/J)$-module and $\mathcal{O}_X[T_0]/J \subset \mathcal{O}_X[I_D^{-1}]$ is an integral extension, $\dim(\mathcal{O}_X[I_D^{-1}]) = \dim(\mathcal{O}_X[T_0]/J)$ is clear. The equality

$$\dim(\mathcal{O}_X) = \dim(\mathcal{O}_X[T_0]/J)$$

follows from T_0 being a regular element and $(\mathcal{O}_X[T_0]/J)/\langle T_0\rangle \cong \mathcal{O}_X/I_N$.

2.6 Remark on Gorensteinness

Recall that in Section 2.2 we defined I_D using the minors of

$$\begin{pmatrix} y_1 & \dots & y_n & zx_1 & \dots & zx_n \\ x_1 & \dots & x_n & y_1 & \dots & y_n \end{pmatrix}$$

where the entries x_j and y_j are indeterminates. It is desirable to perform unprojections in cases where the entries are polynomials (see Chapter 4 for many such examples). In this scenario, the unprojection ring is more delicate and the ring $\mathcal{O}_X[I_D^{-1}]$ may not Gorenstein. We recall the following example from [31] (see Remark 3.3, [31]):

Example 2.6.1. Define $\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[x, y, z]$, $I_D := \langle z, y\rangle$ and $I_X := \langle z^2x - y^3\rangle$. The $\mathcal{O}_X$-module I_D^{-1} is generated by $\mathcal{O}_X$ and

$$s := \frac{y^2}{z} = \frac{xz}{y}.$$

The unprojection ring of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathbb{Z}[x, y, z, s]}{\langle sz - y^2, sy - xz, s^2 - xy\rangle}$$

(compare with Example 1.3.2). The unprojection ring is the homogeneous coordinate ring of the twisted cubic and the projection is geometrically a blow up of its singular point; it is not Gorenstein.

In Example 2.6.1, the unprojection format used is not that of Section 2.2; however, the moral of this example remains. To achieve a Gorenstein ring in such a case, we can define the unprojection ring by tensoring over the standard unprojection as in Section 3 of [31].

Remark 2.6.1. For the most part, that is for the specific cases we study in this thesis, tensoring is not necessary and we simply replace indeterminates with polynomials in the setup of Section 2.2. Nevertheless, we record this extra definition for completeness.

"General Unprojection". Fix $m, n \in \mathbb{N}^+$ with $n \geq 2$. Let $\widehat{\mathcal{O}}_{\mathrm{amb}}$ be an equidimensional Gorenstein ring. Let $\widehat{I}_D \subset \widehat{\mathcal{O}}_{\mathrm{amb}}$ be a codimension $n + m$ ideal generated by the 2×2 minors of

$$\widehat{M} := \begin{pmatrix} \widehat{y}_1 & ... & \widehat{y}_n & \widehat{z}\widehat{x}_1 & ... & \widehat{z}\widehat{x}_n \\ \widehat{x}_1 & ... & \widehat{x}_n & \widehat{y}_1 & ... & \widehat{y}_n \end{pmatrix}$$

together with $\widehat{w}_1 = \cdots = \widehat{w}_m = 0$ for $\widehat{y}_i, \widehat{x}_i, \widehat{z}, \widehat{w}_k \in \widehat{\mathcal{O}}_{\mathrm{amb}}$. Let $\widehat{I}_X = \langle \widehat{f}_1, ..., \widehat{f}_r \rangle \subset \widehat{I}_D$ be a codimension $n+m-1$ ideal with $\widehat{f}_1, \ldots, \widehat{f}_r$ forming a regular sequence in $\widehat{\mathcal{O}}_{\mathrm{amb}}$. We write

$$\widehat{f}_j = \sum_{k=1}^{n^2+m} \widehat{v}_{jk} u_k$$

where u_k is a minimal basis of $\widehat{I}_D$ and $\widehat{v}_{jk} \in \widehat{\mathcal{O}}_X$. Let $(I_D, I_X, \mathcal{O}_{\mathrm{amb}})$ be defined as in Section 2.2 with $p = r(n^2 + m)$. Let $\mathcal{O}_Y$ be defined as in Section 2.4. The unprojection ring of $(\widehat{I}_X, \widehat{I}_D, \widehat{\mathcal{O}}_X)$ is the $\widehat{\mathcal{O}}_X$-algebra

$$\mathcal{O}_Y \otimes \widehat{\mathcal{O}}_{\mathrm{amb}}[T_0, T_1]$$

where the tensor product is over $\mathcal{O}_{\mathrm{amb}}[T_0, T_1]$. Note that there is a ring homomorphism $\phi : \mathcal{O}_{\mathrm{amb}}[T_0, T_1] \to \widehat{\mathcal{O}}_{\mathrm{amb}}[T_0, T_1]$.

2.7 Relations to Other Unprojections

2.7.1 Type II Unprojections

At present, the main unprojections in the literature are type I, type II, type III and type IV (see [34], [31], [30] and [40] respectively). We focus on type II unprojections.

The initial data for an unprojection consists of two ideals $I_X \subset I_D$ in some ring $\mathcal{O}_{\mathrm{amb}}$. The codimensions of I_X and I_D are such that

$$\mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_D) = \mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_X) + 1.$$

By fixing I_X such that $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal Gorenstein integral domain, the type I, II, III and IV conditions are conditions on I_D (see Section 9 of [39] for a nice description of these conditions). The conditions for I_D to be type II are as follows:

Reid's Type II Conditions. Let $I_D \subset \mathcal{O}_{\mathrm{amb}}$ be a homogeneous prime ideal such that $\mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_D) = \mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_X) + 1$. Suppose $I_X \subset I_D$. Then, I_D is type II if $\mathcal{O}_D$ is not normal but its normalisation $\mathcal{O}_{\widetilde{D}}$ is Gorenstein and needs two generators as an $\mathcal{O}_D$ module.

By Lemmas 2.3.1 and 2.3.2, it is clear that the unprojection format of Section 2.2 satisfies these conditions: it is therefore a type II unprojection.

2.7.2 Type II$_k$ Unprojections

There are many different unprojection formats $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ which fall under the umbrella of type II. A hint on how to construct some of these unprojections is provided in Section 9 of [39]:

"Slightly Non-normal Embeddings". Type II unprojections are a phenomena encountered when I_D is defined as the image of some map

$$\phi : \mathbb{P}(a_0, a_1, a_2) \to \mathbb{P}(ka_0, a_1, a_2, a_3, \dots, a_n)$$

where $k \in \mathbb{N}$ and $k > 1$.

Example 2.7.1. Define

$$\phi : \mathbb{P}(1, 3, 5) \to \mathbb{P}(2, 3, 4, 5, 6, 7)$$

$$\phi(a, b, c) := (x := a^2, y := b, z := 0, u := c, v := ac, w := a^7 + ab^2).$$

We claim that the image of ϕ is defined by the ideal I_D generated by the 2×2 minors of

$$\begin{pmatrix} v & w & xu & x^4 + xy^2 \\ u & x^3 + y^2 & v & w \end{pmatrix}$$

together with $z = 0$. Let $D \subset \mathbb{P}(2, 3, 4, 5, 6, 7)$ be defined by the equations of I_D. By evaluating the generating polynomials of I_D on $\phi(a, b, c)$, we see immediately that $\mathrm{im}(\phi) \subset D$. To prove that $D \subset \mathrm{im}(\phi)$, we note that any point $p := (x, y, z, u, v, w) \in D$ may be written as

$$p = \left(\frac{w^2}{(x^3 + y^2)^2}, y, 0, u, \frac{wu}{x^3 + y^2}, \frac{w(w^6 + y^2(x^3 + y^2)^6)}{(x^3 + y^2)^7} \right) = \phi \left(\frac{w}{x^3 + y^2}, y, u \right)$$

if $x^3 + y^2 \neq 0$,

$$p = \left(\frac{v^2}{u^2}, y, 0, u, v, \frac{v(v^6 + y^2u^6)}{u^7} \right) = \phi \left(\frac{v}{u}, y, u \right)$$

if $u \neq 0$ and

$$p = \phi(1, i, 0) = \phi(1, -i, 0)$$

if $x^3 + y^2 = u = 0$.

Example 2.7.2. Define

$$\phi : \mathbb{P}(1,3,5) \to \mathbb{P}(2,3,4,5,6,7)$$

$$\phi(a,b,c) := (x := a^2, y := b, z := ab, u := c, v := ac, w := a^7 + ab^2).$$

The image of ϕ is defined by the ideal I_D generated by the 2×2 minors of

$$\begin{pmatrix} z & v & w & xy & xu & x^4 + xy^2 \\ y & u & x^3 + y^2 & z & v & w \end{pmatrix}.$$

Remark 2.7.1. We briefly return to our ultimate goal of constructing Fano 3-folds. The aim is that D defined by I_D will define an exceptional divisor of a blow-up in the general picture

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

In Examples 2.7.1 and 2.7.2 we will have $X \subset \mathbb{P}(2,3,4,5,6,7)$ defined by a codimension 2 ideal $I_X \subset I_D$ and $Y \subset \mathbb{P}(2,3,4,5,6,7,\mathrm{wt}(T_0),\mathrm{wt}(T_1))$ a Fano 3-fold defined by the ring

$$\mathcal{O}_X[I_D^{-1}] \cong \mathcal{O}_X[T_0, T_1]/\langle l_1, \ldots, l_6, q\rangle.$$

We perform the $\mathbb{P}(1,3,5)$ weighted blow up of Y at a point and D will play the role of the exceptional divisor lying in X. (See Section 3.3 for a more detailed discussion of this diagram).

By viewing type II unprojections in terms of $\phi : \mathbb{P}(a_0, a_1, a_2) \mapsto \mathbb{P}(ka_0, a_1, \ldots, a_n)$, we obtain a natural concept of type II_k unprojections. Confusingly, the existing nomenclature dictates that ϕ defines a type II_{k-1} unprojection.

The ideal I_D defined in Example 2.7.1 is obviously a case of the unprojection format of Section 2.2. By Lemma 2.3.1, the ideal I_D of Section 2.2 can always be expressed as some map $\phi : \mathbb{P}(a_0, a_1, a_2) \to \mathbb{P}(2a_0, a_1, a_2, a_3, \ldots, a_n)$. It is therefore a type II_1 unprojection.

The ideal I_D generated in Example 2.7.2 is another case of type II_1 unprojections. The format of this ideal is generalised and expanded upon in [31] to define many type II_k unprojections:

The $\mathbf{2} \times (\mathbf{kn} + \mathbf{n})$ Format Fix $k, n, p \in \mathbb{N}^+$ such that $k, p \geq 1$ and $n \geq 2$. Define

$$\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[a_{i,j}, z, v_l]$$

with $1 \leq i \leq k+1$, $1 \leq j \leq n$ and $1 \leq l \leq p$. Let $\mathcal{O}_{\mathrm{amb}}$ be a positively graded ring. Define $I_D \subset \mathcal{O}_{\mathrm{amb}}$ as the homogeneous ideal generated by the 2×2 minors of the $2 \times (kn + n)$ matrix

$$\begin{pmatrix} a_{2,1} & \dots & a_{2,n} & a_{3,1} & \dots & a_{3,n} & \dots & a_{k+1,1} & \dots & a_{k+1,n} & za_{1,1} & \dots & za_{1,n} \\ a_{1,1} & \dots & a_{1,n} & a_{2,1} & \dots & a_{2,n} & \dots & a_{k,1} & \dots & a_{k,n} & a_{k+1,1} & \dots & a_{k+1,n} \end{pmatrix}.$$

Let $I_X \subset I_D$ be a homogeneous ideal of $\mathcal{O}_{\mathrm{amb}}$ such that $\operatorname{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_X) = nk - 1$ and $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal Gorenstein integral domain. The unprojection of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is the $\mathcal{O}_X$-subalgebra $\mathcal{O}_X[I_D^{-1}] \subset K(X)$ with I_D^{-1} and $K(X)$ defined as expected by Definition 2.2.1.

It is clear that such an unprojection is type II according to Reid's properties and type II_k according to the embedding definition. Information about the unprojection ring using the $2 \times (kn + n)$ format is discussed in [31]:

Theorem 2.7.1. There exists an isomorphism

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_X[T_0, T_1, \dots, T_k]}{I}$$

where I is the ideal defined by

$$l_{i,j,l} := a_{i+1,j}T_l + a_{i,j}T_{l+1} - c_{i,j,l}$$

and

$$l_{j,l} := za_{1,j}T_l + a_{k+1,j}T_{l+1} - d_{j,l}$$

for $1 \leq i \leq k$, $1 \leq j \leq n$ and $0 \leq l \leq k-1$ and some $c_{i,j,l}, d_{j,l} \in \mathcal{O}_X$;

$$q_{i,j} := T_iT_j - T_0T_{i+j} + \text{linear terms in } T_0, \dots, T_k$$

for $i + j \leq k$; and

$$r_{i,j} := T_iT_j - (-1)^{k+1}zT_0T_{i+j-k-1} + \text{linear terms in } T_0, \dots, T_k$$

for $i + j \geq k+1$. Moreover, $\mathcal{O}_X[I_D^{-1}]$ is a Gorenstein ring and codimension $nk + k$ as an $\mathcal{O}_{\mathrm{amb}}[T_0, \dots, T_k]$-module.

The key point is that type II_k unprojections introduce $k+1$ indeterminates and define an ideal of codimension $\operatorname{codim}(I_X)+k+1$.

2.7.3 Breaking the Type II_1 Definition

The unprojection format of Section 2.2 can be viewed as a close relative of the type $2\times(kn+n)$ unprojection format when $k=1$, although distinct from it. Both produce Gorenstein rings of codimension $n+1$ from Gorestein rings of codimension $n-1$, use matrices of a similar shape, and have presentations described by n equations of total degree 1 and a single equation of total degree 2 with respect to 2 new unprojection indeterminates (compare Sections 2.2 – 2.6 with [31]).

Moreover, the formats are also related isomorphically. Let $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ be defined in the format of Section 2.2 and let $I'_D \subset I_D$ be the ideal defined by only minors. Then,

$$\mathcal{O}_D := \mathcal{O}_{\mathrm{amb}}/I_D \cong \frac{\mathcal{O}_{\mathrm{amb}}}{I'_D + \langle w_i : 1 \leq i \leq m\rangle}$$

where $I'_D \subset \mathcal{O}_{\mathrm{amb}}/\langle w_i : 1 \leq i \leq m\rangle$ is in $2\times(kn+n)$ format.

In the literature, type II_k unprojections are "officially" defined as unprojections using the $2\times(kn+n)$ format (see [31] and [33]). We believe that this definition should be extended and therefore expand the definition of type II_1 unprojections to include the unprojection format of Section 2.2.

Fix $n, m, p \in \mathbb{N}$ such that $n \geq 2$, $m \geq 0$ and $p \geq 1$. If $m \geq 1$, let

$$\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[x_j, y_j, w_i, z, v_l]$$

be a positively graded ring such that $1 \leq j \leq n$, $1 \leq i \leq m$ and $1 \leq l \leq p$. If $m=0$, let

$$\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[x_j, y_j, z, v_l]$$

be a positively graded ring such that $1 \leq j \leq n$ and $1 \leq l \leq p$. In both cases, we suppose that the weight of z is even and

$$\mathrm{wt}(y_j) = \mathrm{wt}(x_j) + \frac{1}{2}\,\mathrm{wt}(z)$$

for $1 \leq j \leq n$. As before, we additionally define indeterminates v_l for extra flexibility. Define the $2\times 2n$ matrix,

$$M := \begin{pmatrix} y_1 & \cdots & y_n & zx_1 & \cdots & zx_n \\ x_1 & \cdots & x_n & y_1 & \cdots & y_n \end{pmatrix}.$$

If $m = 0$, let $I_D \subset \mathcal{O}_{\text{amb}}$ be the ideal generated by the 2×2 minors of M. Otherwise, let I_D be defined by the 2×2 minors of M together with the linear equations $w_1 = \cdots = w_m = 0$. Let $I_X \subset I_D$ be a homogeneous prime ideal of $\mathcal{O}_{\text{amb}}$ such that $\mathcal{O}_X := \mathcal{O}_{\text{amb}}/I_X$ is a normal Gorenstein integral domain and I_X is codimension $n + m - 1$. Then:

Definition 2.7.1. The *type* $\mathrm{II_1}^{(n,m)}$ *unprojection ring of* $(I_X, I_D, \mathcal{O}_{\text{amb}})$ is the $\mathcal{O}_X$-subalgebra

$$\mathcal{O}_X[I_D^{-1}] \subset K(X)$$

where $I_D^{-1} \subset K(X)$ is the $\mathcal{O}_X$-module $I_D^{-1} := \{f \in K(X) : fI_D \subset \mathcal{O}_X\}$. When the values of n and m are clear from context, we may refer to the unprojection ring as a type II_1 unprojection ring.

Remark 2.7.2. Note that we have naturally extended the unprojection format of Section 2.2 to include $m = 0$.

Remark 2.7.3. To reiterate: we think of type II_1 unprojections as in Section 2.7.2 and note that Definition 2.7.1 is unlikely to cover all type II_1 unprojections. We leave room for future extensions if and when new formats are discovered.

A corollary of Theorem 2.2.1 and Proposition 2.16 of [31] is then:

Corollary 2.7.1. Let $(I_X, I_D, \mathcal{O}_{\text{amb}})$ be in type $\mathrm{II_1}^{(n,m)}$ unprojection format. If $m \neq 0$ define $y_{i+n} := w_i$ and $x_{i+n} := 0$ for $1 \leq i \leq m$. Then, there exists an isomorphism

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_X[T_0, T_1]}{\langle l_1, \ldots, l_{2(n+m)}, q\rangle}$$

where

$$l_j := y_j T_1 + z x_j T_0 - c_j$$

and

$$l_{n+m+j} := y_j T_0 + x_j T_1 - d_j$$

for $j = 1, \ldots, n + m$, and

$$q := T_1^2 - zT_0^2 + \alpha_0 T_0 + \alpha_1 T_1 + \alpha_2$$

for some $c_j, d_j, \alpha_0, \alpha_1, \alpha_2 \in \mathcal{O}_X$. Furthermore, $\mathcal{O}_X[I_D^{-1}]$ is a Gorenstein ring, perfect as an $\mathcal{O}_{\text{amb}}[T_0, T_1]$-module and such that

$$\operatorname{codim}_{\mathcal{O}_{\text{amb}}[T_0,T_1]}(I_X + \langle l_1, \ldots, l_{2(n+m)}, q\rangle) = n + m + 1.$$

Chapter 3

Type II_1 Unprojection

3.1 Explicit Equations

Thus far we have described the general shape of the equations which define the type ${\mathrm{II}_1}^{(n,m)}$ unprojection ring. Recall that in Theorem 2.2.1 and Corollary 2.7.1 the type ${\mathrm{II}_1}^{(n,m)}$ unprojection ring of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is described by equations such as

$$l_1 := y_1T_1 + zx_1T_0 - c_1$$

and

$$q := T_1^2 - zT_0^2 + \alpha_0T_0 + \alpha_1T_1 + \alpha_2$$

where $c_1, \alpha_0, \alpha_1, \alpha_2$ are some unknowns in $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$. To define the equations of the unprojection ring completely, the unknowns must be known explicitly.

In the literature, only the explicit equations for the type ${\mathrm{II}_1}^{(2,0)}$ and type ${\mathrm{II}_1}^{(3,0)}$ unprojection ring are known (see Section 7.4, [15]; and Sections 3 and 4 of [33]).

In Section 3.1.2, we will define the explicit equations for the type ${\mathrm{II}_1}^{(2,1)}$ unprojection ring. This is done in the spirit of [26] and [39]. To calculate the linear equations of the unprojection ring, we construct a complex resolving $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ and $\mathcal{O}_{\widetilde{D}} \cong \mathbb{Z}[x_1, \ldots, x_n, t, v_1, \ldots, v_p]$. Recall that that $\widetilde{D}$ is the normalisation of D (see Lemma 2.3.1).

Remark 3.1.1. Note that this method does not calculate the quadratic equation, q, of the unprojection ring. A combination of luck and observation are required to find it (see Section 7.4, [15]; Section 4, [33]; and Theorem 3.1.1).

Remark 3.1.2. The method we use to define the linear equations of the unprojection ring can be expanded to type ${\mathrm{II}_1}^{(n,m)}$ unprojection rings. However, if

I_X were not a complete intersection, constructing the desired complex would be challenging.

Remark 3.1.3. An alternative method of obtaining the equations of the unprojection ring is to calculate the generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. Complexes are once again used and the equations of the unprojection can be calculated thanks to their known general form (see Appendix B.1).

3.1.1 Type $\mathrm{II}_1^{(2,0)}$ Unprojections

As a warm up, we discuss the type $\mathrm{II}_1{}^{(2,0)}$ unprojection ring. The results in this section can be found in Section 4.4 of [33], Example 9.8 of [39] and 4.10 – 4.12 and 7.3 of [15].

Let

$$\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[x_1, x_2, y_1, y_2, z, A_{12}, B_{11}, B_{12}, B_{22}]$$

be a positively graded ring such that the weight of z is even and $\mathrm{wt}(y_j) = \mathrm{wt}(x_j) + \frac{1}{2}\,\mathrm{wt}(z)$ for $j = 1, 2$.

Let $I_D \subset \mathcal{O}_{\mathrm{amb}}$ be the ideal generated by the 2×2 minors of

$$M := \begin{pmatrix} y_1 & y_2 & zx_1 & zx_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}.$$

Let $I_X \subset I_D$ be the homogeneous prime ideal of $\mathcal{O}_{\mathrm{amb}}$ defined by

$$f := A_{12}(x_2y_1 - x_1y_2) + B_{11}(y_1^2 - zx_1^2) + 2B_{12}(y_1y_2 - zx_1x_2) + B_{22}(y_2^2 - zx_2^2).$$

We assume that $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal and Gorenstein integral domain.

Theorem 3.1.1. The unprojection ring of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is

$$\frac{\mathcal{O}_X[T_0, T_1]}{\langle l_1, l_2, l_3, l_4, q\rangle}$$

where

$$l_1 := y_1T_1 + zx_1T_0 - y_2B_{22} + x_1A_{12},$$

$$l_2 := y_2T_1 + zx_2T_0 + y_1B_{11} + 2y_2B_{12} + x_2A_{12},$$

$$l_3 := y_1T_0 + x_1T_1 + 2x_1B_{12} + x_2B_{22},$$

$$l_4 := y_2T_0 + x_2T_1 - x_1B_{11}$$

and

$$q := T_1^2 - zT_0^2 - A_{12}T_0 + 2B_{12}T_1 + B_{11}B_{22}.$$

Remark 3.1.4. Note that the type ${\mathrm{II}_1}^{(2,0)}$ unprojection rings provided in the literature differ slightly from the one given in Theorem 3.1.1. Nevertheless, they are equal up to a change of coordinates.

We first construct the Koszul complex of $\mathcal{O}_X$ and the resolution of $\mathcal{O}_{\widetilde{D}}$. These complexes combine to form the complex

$$\begin{tikzcd}
0 & \mathcal{O}_X \arrow[l] \arrow[d] & \mathcal{O}_{\mathrm{amb}} \arrow[l] \arrow[d, "(1,0)^T"] & \mathcal{O}_{\mathrm{amb}} \arrow[l, "-f"'] \arrow[d, "\alpha"] & 0 \arrow[l] & \\
0 & \mathcal{O}_{\widetilde{D}} \arrow[l] & 2\mathcal{O}_{\mathrm{amb}} \arrow[l] & 4\mathcal{O}_{\mathrm{amb}} \arrow[l, "M"'] & 2\mathcal{O}_{\mathrm{amb}} \arrow[l, "N"'] & 0 \arrow[l]
\end{tikzcd}$$

where

$$N := \begin{pmatrix} y_2 & zx_2 \\ -y_1 & -zx_1 \\ x_2 & y_2 \\ -x_1 & -y_1 \end{pmatrix}$$

and

$$\alpha := \begin{pmatrix} -y_1B_{11} - 2y_2B_{12} - x_2A_{12} \\ -y_2B_{22} + x_1A_{12} \\ x_1B_{11} \\ 2x_1B_{12} + x_2B_{22} \end{pmatrix}.$$

The linear equations of the unprojection come from joining up the ends of the complexes:

$$(l_2, -l_1, l_4, -l_3)^T = N(T_1, T_0)^T - \alpha = 0.$$

That is, $l_1, \dots, l_4$ are 4 of the maximal Pfaffians of the 5×5 antisymmetric matrix

$$\begin{pmatrix} 0 & x_2 & x_1 & y_2 & y_1 \\ & 0 & T_0 & B_{11} & -T_1 - 2B_{12} \\ & & 0 & T_1 & B_{22} \\ & -\mathrm{Sym} & & 0 & zT_0 + A_{12} \\ & & & & 0 \end{pmatrix}.$$

The final Pfaffian,

$$T_1^2 - zT_0^2 - A_{12}T_0 + 2B_{12}T_1 + B_{11}B_{22},$$

is our quadratic.

Remark 3.1.5. We can verify that the fifth Pfaffian is the quadratic equation of the unprojection by replicating Lemma 3.1.1.

Remark 3.1.6. The polynomial f is often dropped when defining the unprojection ring since $f \in \langle l_1, \dots, l_4 \rangle$.

Remark 3.1.7. By viewing l_i as equations linear in T_0 and T_1, we can solve the linear system of equations to find $s_0, s_1 \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. For example,

$$l_1 x_1 - l_3 y_1 = n_1 - (y_1^2 - z x_1^2) T_0$$

and

$$z x_1 l_3 - l_1 y_1 = N_1 - (y_1^2 - z x_1^2) T_1$$

where $n_1, N_1 \in \mathcal{O}_X$. The rational functions

$$s_0 := \frac{n_1}{y_1^2 - z x_1^2} \text{ and } s_1 := \frac{N_1}{y_1^2 - z x_1^2}$$

can be shown to lie in I_D^{-1} and together with $\mathcal{O}_X$ generate I_D^{-1}. Every pair of linear equations gives a different definition for the rational functions s_0 and s_1; however, each definition is equal in the fraction field of $\mathcal{O}_X$.

3.1.2 Type $\mathrm{II}_1^{(2,1)}$ Unprojections

We now calculate the explicit equations of the type ${\mathrm{II}_1}^{(2,1)}$ unprojection ring.

Let

$$\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[x_1, x_2, y_1, y_2, w, z, A_{12}, B_{11}, B_{12}, B_{22}, C, \overline{A}_{12}, \overline{B}_{11}, \overline{B}_{12}, \overline{B}_{22}, \overline{C}]$$

be a positively graded ring with the weight of z even and $\mathrm{wt}(y_j) = \mathrm{wt}(x_j) + \frac{1}{2}\mathrm{wt}(z)$ for $j = 1, 2$. Let $I_D \subset \mathcal{O}_{\mathrm{amb}}$ be the homogeneous prime ideal defined by the 2×2 minors of

$$M := \begin{pmatrix} y_1 & y_2 & z x_1 & z x_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}$$

together with the linear equation $w = 0$. Let $I_X := \langle f, \overline{f} \rangle$ be a homogeneous codimension 2 ideal in $\mathcal{O}_{\mathrm{amb}}$ defined by

$$f := A_{12}(y_1 x_2 - x_1 y_2) + B_{11}(y_1^2 - z x_1^2) + 2B_{12}(y_1 y_2 - z x_1 x_2) + B_{22}(y_2^2 - z x_2^2) + Cw$$

and

$$\overline{f} := \overline{A}_{12}(y_1x_2 - x_1y_2) + \overline{B}_{11}(y_1^2 - zx_1^2) + 2\overline{B}_{12}(y_1y_2 - zx_1x_2) + \overline{B}_{22}(y_2^2 - zx_2^2) + \overline{C}w.$$

We assume that $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal Gorenstein integral domain.

By Theorem 2.2.1 and Corollary 2.7.1, we know that there exists an isomorphism

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_X[T_0, T_1]}{I_Y}$$

where I_Y is an ideal of $\mathcal{O}_X[T_0, T_1]$ defined by

$$l_1 := y_1T_1 + zx_1T_0 - c_1, \quad l_2 := y_2T_1 + zx_2T_0 - c_2, \quad l_3 := wT_1 - c_3,$$

$$l_4 := y_1T_0 + x_1T_1 - d_1, \quad l_5 := y_2T_0 + x_2T_1 - d_2, \quad l_6 := wT_0 - d_3$$

and

$$q := T_1^2 - zT_0^2 + \alpha_0T_0 + \alpha_1T_1 + \alpha_2$$

for some $c_1, c_2, c_3, d_1, d_2, d_3, \alpha_0, \alpha_1, \alpha_2 \in \mathcal{O}_X$. In particular:

Theorem 3.1.2. Define v_{ij} as the ij-th minor of

$$v := \begin{pmatrix} C & A_{12} & B_{11} & 2B_{12} & B_{22} \\ \overline{C} & \overline{A}_{12} & \overline{B}_{11} & 2\overline{B}_{12} & \overline{B}_{22} \end{pmatrix}$$

for $1 \le i < j \le 5$. Then,

$$l_1 := y_1T_1 + zx_1T_0 + x_1v_{12} - y_1v_{14} - y_2v_{15},$$

$$l_2 := y_2T_1 + zx_2T_0 + x_2v_{12} + y_1v_{13},$$

$$l_3 := wT_1 + y_1^2v_{34} + zx_2^2v_{45} + zx_1x_2v_{35} + x_2y_2v_{25} + x_2y_1v_{24} + x_1y_1v_{23} + y_2y_1v_{35},$$

$$l_4 := y_1T_0 + x_1T_1 + x_2v_{15},$$

$$l_5 := y_2T_0 + x_2T_1 - x_1v_{13} - x_2v_{14},$$

$$l_6 := wT_0 - x_2^2v_{25} - x_1x_2v_{24} - x_2y_2v_{45} - x_2y_1v_{35} - x_1^2v_{23} - x_1y_2v_{35} - x_1y_1v_{34}$$

and

$$q := T_1^2 - zT_0^2 - T_0v_{12} - T_1v_{14} + v_{15}v_{13}.$$

To calculate the linear equations of the unprojection we construct the Koszul complex of $\mathcal{O}_X$ and the minimal resolution of $\mathcal{O}_{\widetilde{D}}$. We join the complexes to obtain

$$\begin{tikzcd}
0 & \mathcal{O}_X \arrow[l] & \mathcal{O}_{\mathrm{amb}} \arrow[l] \arrow[d, "(1,0)^T"] & 2\mathcal{O}_{\mathrm{amb}} \arrow[l, "(f{,}\overline{f})"'] \arrow[d, "\beta"] & \mathcal{O}_{\mathrm{amb}} \arrow[l, "(\overline{f}{,}-f)^T"'] \arrow[d, "\alpha"] & 0 \arrow[l] & \\
0 & \mathcal{O}_{\widetilde{D}} \arrow[l] & 2\mathcal{O}_{\mathrm{amb}} \arrow[l] & 6\mathcal{O}_{\mathrm{amb}} \arrow[l, "M'"'] & 6\mathcal{O}_{\mathrm{amb}} \arrow[l, "A"'] & 2\mathcal{O}_{\mathrm{amb}} \arrow[l, "N"'] & 0 \arrow[l]
\end{tikzcd}$$

where

$$M' := \begin{pmatrix} y_1 & y_2 & w & zx_1 & zx_2 & 0 \\ -x_1 & -x_2 & 0 & -y_1 & -y_2 & -w \end{pmatrix},$$

$$A := \begin{pmatrix} y_2 & zx_2 & w & 0 & 0 & 0 \\ -y_1 & -zx_1 & 0 & 0 & w & 0 \\ 0 & 0 & -y_1 & -zx_1 & -y_2 & -zx_2 \\ x_2 & y_2 & 0 & w & 0 & 0 \\ -x_1 & -y_1 & 0 & 0 & 0 & w \\ 0 & 0 & -x_1 & -y_1 & -x_2 & -y_2 \end{pmatrix},$$

$$N := \begin{pmatrix} 0 & -w \\ -w & 0 \\ zx_2 & y_2 \\ y_2 & x_2 \\ -zx_1 & -y_1 \\ -y_1 & -x_1 \end{pmatrix},$$

$$\beta := \begin{pmatrix} x_2A_{12} + y_1B_{11} & x_2\overline{A}_{12} + y_1\overline{B}_{11} \\ -x_1A_{12} + 2y_1B_{12} + y_2B_{22} & -x_1\overline{A}_{12} + 2y_1\overline{B}_{12} + y_2\overline{B}_{22} \\ C & \overline{C} \\ -x_1B_{11} - 2x_2B_{12} & -x_1\overline{B}_{11} - 2x_2\overline{B}_{12} \\ -x_2B_{22} & -x_2\overline{B}_{22} \\ 0 & 0 \end{pmatrix}$$

and

$$\alpha := (\alpha_1, \ldots, \alpha_6)^T$$

with

$$\alpha_1 = y_1(y_1v_{34} + x_1v_{23} + y_2v_{35} + x_2v_{24}) + x_2(zx_2v_{45} + zx_1v_{35} + y_2v_{25}),$$

$$\alpha_2 = -x_2(x_2v_{25} + x_1v_{24} + y_2v_{45} + y_1v_{35}) - x_1(x_1v_{23} + y_2v_{35} + y_1v_{34}),$$

$$\alpha_3 = -x_2v_{12} - y_1v_{13},$$

$$\alpha_4 = x_1 v_{13} + x_2 v_{14},$$

$$\alpha_5 = x_1 v_{12} - y_1 v_{14} - y_2 v_{15},$$

and

$$\alpha_6 = x_2 v_{15}.$$

The equations $l_1, \ldots, l_6$ come from joining up the ends of the complexes, i.e. they are obtained by

$$(-l_3, -l_6, l_2, l_5, -l_1, -l_4)^T = N(T_0, T_1)^T - \alpha = 0.$$

With the linear equations of the unprojection defined as above, we may define the quadratic equation of the unprojection ring.

Lemma 3.1.1. Define

$$q' := T_1^2 - zT_0^2 - T_0 v_{12} - T_1 v_{14} + v_{15} v_{13}.$$

Then, $q' \in I_Y$ and $I_Y = \langle l_1, \ldots, l_6, q' \rangle$.

Proof. We follow Lemma 4.2 of [33], but identify a different candidate for q. By Theorem 2.2.1 and Corollary 2.7.1, we know that there exists an equation

$$q := T_1^2 - zT_0^2 + \alpha_0 T_0 + \alpha_1 T_1 + \alpha_2 \in I_Y$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{O}_X$. Define

$$q' := T_1^2 - zT_0^2 - T_0 v_{12} - T_1 v_{14} + v_{15} v_{13}.$$

Then,

$$q'(x_1 + x_2) = T_0(l_1 - l_2) + T_1(l_5 - l_4) + l_4(v_{14} - v_{13}) - v_{15} l_5 \in I_Y.$$

By the primality of I_Y and the fact that $(x_1 + x_2) \notin I_Y$, we must have $q' \in I_Y$. The term $q - q'$ has total degree at most 1 in T_0 and T_1, that is $q - q' \in \langle f, \overline{f}, l_1, ..., l_6 \rangle$. Therefore, $I_Y = \langle l_1, ..., l_6, q' \rangle$. $\square$

3.1.3 Remark on Type $\mathrm{II}_1^{(2,1)}$ and Type I Correspondence

Although for now we remain largely disinterested in type I unprojections, the explicit equations of Section 3.1.2 reveal that there exists a correspondence between type ${\mathrm{II}_1}^{(2,1)}$ unprojections and type I unprojections.

Despite already meeting type I unprojections (see Examples 1.3.1 and 1.3.2), we have yet to define them. A definition of type I unprojections can be achieved by modifying the type II_1 unprojection setup:

Definition 3.1.1. (See Section 1, [34]; Section 5, [32]; Lemma 2.1.1 and Definition 2.1.3 [29]) Let $I_X \subset I_D$ be ideals in some positively graded ring $\mathcal{O}_{\mathrm{amb}}$ such that $\mathcal{O}_X := \mathcal{O}_{\mathrm{amb}}/I_X$ and $\mathcal{O}_D := \mathcal{O}_{\mathrm{amb}}/I_D$ are Gorenstein rings with $\mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_D) = \mathrm{codim}_{\mathcal{O}_{\mathrm{amb}}}(I_X) + 1$. The $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is generated by two elements, i and s where i is a basis of $\mathcal{O}_X$ and $s \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is injective. The *type I unprojection ring* of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is the ring

$$\mathcal{O}_X[I_D^{-1}] = \mathcal{O}_X[s] \cong \frac{\mathcal{O}_X[S]}{\langle Sf - g : f \in I_D \rangle}$$

where S is some indeterminate and $s(f) = g$.

We are interested in type I unprojection rings which use the generic Tom ideal. We recall the following definition from [10] and [29]:

Definition 3.1.2. (Definition 2.2, [10]; Section 3.1.1, [29]) Let $R := \mathbb{Z}[x_k, z_k, b_{ij}^k,]$ with $2 \leq i < j \leq 5$ and $k = 1, 2, 3, 4$. The *generic Tom ideal* is generated by the maximal Pfaffians of the 5×5 antisymmetric matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ & 0 & b_{23} & b_{24} & b_{25} \\ & & 0 & b_{34} & b_{35} \\ & -\mathrm{Sym} & & 0 & b_{45} \\ & & & & 0 \end{pmatrix}$$

with $b_{ij} := \sum_{k=1} b_{ij}^k z_k$.

The generic Tom ideal I is a prime ideal of codimension 3, contained in $\langle z_1, \ldots, z_4 \rangle$ and such that the ring R/I is Gorenstein (see Theorem 3.1.1, [29]). Similarly, the ideal $\langle z_1, \ldots, z_k \rangle$ is a prime ideal of codimension 4 and such that $R/\langle z_1, \ldots, z_k \rangle$ is Gorenstein. We are in the perfect situation to apply type I unprojections.

The explicit equations of type I unprojections using the generic Tom ideal are known (see Section 3.3 of [29]). With these equations, it is straightforward to realise the explicit equations of the type ${\mathrm{II}_1}^{(2,1)}$ unprojection as a type I unprojection. We state this more concretely:

Proposition 3.1.1. Let

$$\mathcal{O}_{\mathrm{amb}} = \mathbb{Z}[x_1, x_2, y_1, y_2, z, A_{12}, B_{11}, B_{12}, B_{22}, C, \overline{A}_{12}, \overline{B}_{11}, \overline{B}_{12}, \overline{B}_{22}, \overline{C}]$$

be a positively graded ring with the weight of z even and $\mathrm{wt}(y_j) = \mathrm{wt}(x_j) + \frac{1}{2}\,\mathrm{wt}(z)$ for $j = 1, 2$. We define pairs of ideals $I_Y \subset I_D$ and $I_Z \subset I_E$ as follows:

- Let w be an indeterminate with positive weight. Let $I_D \subset \mathcal{O}_{\mathrm{amb}}[w]$ be the homogeneous ideal generated by the 2×2 minors of

  $$M := \begin{pmatrix} y_1 & y_2 & zx_1 & zx_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}$$

  together with the linear equation $w = 0$. Let $I_Y = \langle f, \overline{f} \rangle \subset \mathcal{O}_{\mathrm{amb}}[w]$ be a codimension 2 homogeneous prime ideal defined by the polynomials

  $$f := A_{12}(y_1x_2 - x_1y_2) + B_{11}(y_1^2 - zx_1^2) + 2B_{12}(y_1y_2 - zx_1x_2) + B_{22}(y_2^2 - zx_2^2) + Cw$$

  and

  $$\overline{f} := \overline{A}_{12}(y_1x_2 - x_1y_2) + \overline{B}_{11}(y_1^2 - zx_1^2) + 2\overline{B}_{12}(y_1y_2 - zx_1x_2) + \overline{B}_{22}(y_2^2 - zx_2^2) + \overline{C}w.$$

- Let T_0 and T_1 be indeterminates of positive weight and define $I_E = \langle C, \overline{C}, T_0, T_1 \rangle \subset \mathcal{O}_{\mathrm{amb}}[T_0, T_1]$. We define the homogeneous prime ideal $I_Z \subset \mathcal{O}_{\mathrm{amb}}[T_0, T_1]$ by the 4×4 Pfaffians of the antisymmetric 5×5 matrix

  $$\begin{pmatrix} 0 & x_1 & x_2 & y_1 & y_2 \\ & 0 & T_0 & B_{22}\overline{C} - C\overline{B}_{22} & -T_1 - 2B_{12}\overline{C} + 2C\overline{B}_{12} \\ & & 0 & T_1 & B_{11}\overline{C} - \overline{B}_{11}C \\ & -\mathrm{Sym} & & 0 & zT_0 - A_{12}\overline{C} + C\overline{A}_{12} \\ & & & & 0 \end{pmatrix}.$$

We assume that $\mathcal{O}_{\mathrm{amb}}[T_0, T_1]/I_Z$ and $\mathcal{O}_{\mathrm{amb}}[w]/I_Y$ are normal Gorenstein integral domains. Then, the type ${\mathrm{II}_1}^{(2,1)}$ unprojection of $(I_Y, I_D, \mathcal{O}_{\mathrm{amb}}[w])$ is equal to the type I unprojection of $(I_Z, I_E, \mathcal{O}_{\mathrm{amb}}[T_0, T_1])$.

The ideal I_Z in the statement of Proposition 3.1.1 is defined as $I_Z = \langle q, l_1, l_2, l_4, l_5 \rangle$ where the type ${\mathrm{II}_1}^{(2,1)}$ unprojection of $(I_Y, I_D, \mathcal{O}_{\mathrm{amb}}[w])$ is defined by $\langle f, \overline{f}, l_1, \ldots, l_6, q \rangle \subset \mathcal{O}_{\mathrm{amb}}[w, T_0, T_1]$ as in Section 3.1.2.

With some relabelling, we can see that the ideal I_Z in the statement of

Proposition 3.1.1 is a case of the generic Tom format where $I_Z \subset \langle T_0, T_1, C, \overline{C}\rangle$. Using the results of Section 3.3 [29] and setting w as the type I unprojection indeterminate, we verify that the type I unprojection of $(I_Z, I_E, \mathcal{O}_{\mathrm{amb}}[T_0, T_1])$ is equal to the type $\mathrm{II_1}^{(2,1)}$ unprojection of $(I_Y, I_D, \mathcal{O}_{\mathrm{amb}}[w])$.

For complete correctness, we should check that I_Z is codimension 3 and $\mathcal{O}_Z := \mathcal{O}_{\mathrm{amb}}[T_0, T_1]/I_Z$ is a Gorenstein ring. If I_Z were a codimension 3 ideal in $\mathcal{O}_{\mathrm{amb}}[T_0, T_1]$, the ring $\mathcal{O}_Z$ is a Gorenstein ring since it is defined by the $2n \times 2n$ Pfaffians of an antisymmetric $(2n+1)\times(2n+1)$ matrix, (see Buchsbaum-Eisenbud [14]). It is, therefore, sufficient to show that I_Z is codimension 3. The desired result follows immediately by Corollary 3.2.6 of [29].

Remark 3.1.8. Proposition 3.1.1 is a comment on the "general case" and cannot be applied immediately to every explicit example. Consider the case where C is not an indeterminate but a polynomial containing w. The ideal I_Z lies in $\mathcal{O}_{\mathrm{amb}}[T_0, T_1, w]$ not $\mathcal{O}_{\mathrm{amb}}[T_0, T_1]$ and hence w cannot be the type I unprojection indeterminate. Nevertheless Proposition 3.1.1 has many applications (see Section 5.1.2).

3.2 Free Resolutions

As an alternative method to proving that the type $\mathrm{II}_1^{(2,1)}$ unprojection ring is Gorenstein, we could construct a free resolution.

Define $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ as in Section 3.1.2. Let $R := \mathcal{O}_{\mathrm{amb}}[T_0, T_1]$ and define I_Y as the ideal defining the type II_1 unprojection ring in R. That is, $I_Y := I_X + \langle l_1, \ldots, l_6, q\rangle$ where $l_1, \ldots, l_6, q$ are defined in Section 3.1.2. The resolution of I_Y is

$$0 \longleftarrow R \xleftarrow[\alpha]{} R^9 \xleftarrow[\beta]{} R^{16} \xleftarrow[\gamma]{} R^9 \xleftarrow[\sigma]{} R \longleftarrow 0$$

with

$$\alpha := (l_2, -l_1, -l_5, l_4, q, l_6, -f, -\overline{f}, -l_3)$$

and

$$\sigma^T := (l_3 + B_{12}\overline{f} - \overline{B}_{12}f, \overline{f}, -f, l_6, -q, -l_4, l_5, l_1, -l_2).$$

The matrices β and γ are very large and we present them as the block matrices

$$\beta := \left(\begin{array}{c|c|c|c|c|c} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \end{array}\right) \quad \text{and} \quad \gamma := \left(\begin{array}{c|c|c} \gamma_1 & \gamma_2 & \gamma_3 \end{array}\right)$$

where

$$\beta_1 := \begin{pmatrix} \overline{C} & C\overline{B}_{12} - B_{12}\overline{C} - T_1 & 0 & -B_{22}x_2 \\ -x_2 & T_1 & -C\overline{B}_{11} + B_{11}\overline{C} & -T_0 \\ y_1 & 0 & -\overline{A}_{12}C + A_{12}\overline{C} - zT_0 & -C\overline{B}_{22} + B_{22}\overline{C} \\ y_2 & \overline{A}_{12}C - A_{12}\overline{C} + zT_0 & 0 & C\overline{B}_{12} - B_{12}\overline{C} - T_1 \\ 0 & y_1 & y_2 & x_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta_2 := \begin{pmatrix} T_0 & -\overline{B}_{12}x_1 - \overline{B}_{22}x_2 & y_1 & -B_{12}\overline{B}_{12}x_1 - B_{22}\overline{B}_{12}x_2 \\ B_{11}x_1 + B_{12}x_2 & 0 & \overline{B}_{11}x_1 & y_2 \\ -B_{12}y_1 - B_{22}y_2 + A_{12}x_1 & T_1 & -\overline{B}_{22}y_2 + \overline{A}_{12}x_1 & -x_1 z \\ B_{11}y_1 + A_{12}x_2 & -C\overline{B}_{11} + B_{11}\overline{C} & \overline{B}_{11}y_1 + \overline{B}_{12}y_2 + \overline{A}_{12}x_2 & -x_2 z \\ 0 & x_2 & 0 & 0 \\ C & 0 & \overline{C} & 0 \\ T_0 & 0 & 0 & -\overline{C} \\ 0 & 0 & T_0 & C \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta_3 := \begin{pmatrix} 0 \\ B_{12}\overline{B}_{11}x_1 + B_{22}\overline{B}_{11}x_2 - B_{11}\overline{B}_{22}x_2 \\ -B_{12}\overline{B}_{22}y_2 + \overline{A}_{12}B_{12}x_1 + \overline{A}_{12}B_{22}x_2 - A_{12}\overline{B}_{22}x_2 \\ B_{11}\overline{B}_{12}y_1 - B_{22}\overline{B}_{11}y_2 + B_{12}\overline{B}_{12}y_2 + B_{11}\overline{B}_{22}y_2 - \overline{A}_{12}B_{11}x_1 + A_{12}\overline{B}_{11}x_1 + A_{12}\overline{B}_{12}x_2 \\ 0 \\ B_{12}\overline{C} + T_1 \\ 0 \\ B_{12}T_0 \\ T_0 \end{pmatrix},$$

$$\beta_4 := \begin{pmatrix} -B_{12}y_1 - B_{22}y_2 + A_{12}x_1 & -\overline{B}_{12}y_1 - \overline{B}_{22}y_2 + \overline{A}_{12}x_1 & w \\ -w & B_{11}y_1 + A_{12}x_2 & \overline{B}_{11}y_1 + \overline{A}_{12}x_2 \\ 0 & -B_{22}x_2z & -\overline{B}_{22}x_2z \\ 0 & B_{11}x_1z + B_{12}x_2z & \overline{B}_{11}x_1z + \overline{B}_{12}x_2z \\ 0 & 0 & 0 \\ -x_1z & 0 & 0 \\ -\overline{B}_{12}y_1 - \overline{B}_{22}y_2 + \overline{A}_{12}x_1 & -T_1 & 0 \\ B_{12}y_1 + B_{22}y_2 - A_{12}x_1 & 0 & -T_1 \\ y_1 & C & \overline{C} \end{pmatrix},$$

$$
\beta_5 := \begin{pmatrix}
0 & 0 & B_{12}\overline{B}_{12}y_1 + B_{22}\overline{B}_{12}y_2 - \overline{A}_{12}B_{12}x_1 - A_{12}\overline{B}_{12}x_1 - A_{12}\overline{B}_{22}x_2 \\
0 & 0 & 0 \\
0 & w & 0 \\
0 & 0 & w \\
0 & 0 & 0 \\
-x_2 z & y_2 & -y_1 \\
\overline{B}_{11}y_1 + \overline{A}_{12}x_2 & \overline{B}_{11}x_1 + \overline{B}_{12}x_2 & \overline{B}_{22}x_2 \\
-B_{11}y_1 - A_{12}x_2 & -B_{11}x_1 - B_{12}x_2 & -B_{22}x_2 \\
y_2 & -x_2 & x_1
\end{pmatrix},
$$

$$
\beta_6 := \begin{pmatrix}
-B_{12}\overline{B}_{11}y_1 - B_{22}\overline{B}_{11}y_2 + B_{11}\overline{B}_{22}y_2 + A_{12}\overline{B}_{11}x_1 - \overline{A}_{12}B_{12}x_2 \\
B_{12}\overline{B}_{22}x_2 z - \overline{A}_{12}B_{22}y_2 + A_{12}\overline{A}_{12}x_1 \\
-B_{11}\overline{B}_{12}x_1 z + B_{22}\overline{B}_{11}x_2 z - B_{12}\overline{B}_{12}x_2 z - B_{11}\overline{B}_{22}x_2 z + \overline{A}_{12}B_{11}y_1 + \overline{A}_{12}B_{12}y_2 + A_{12}\overline{A}_{12}x_2 \\
-w \\
A_{12}\overline{C} - zT_0 \\
-C\overline{B}_{11}\overline{B}_{22} + B_{22}\overline{B}_{11}\overline{C} + \overline{A}_{12}T_0 + \overline{B}_{12}T_1 \\
B_{11}C\overline{B}_{22} - B_{11}B_{22}\overline{C} \\
-B_{12}\overline{C} - T_1
\end{pmatrix},
$$

$$\gamma_1 := \begin{pmatrix} -B_{12}\overline{C} - T_1 & B_{11}C\overline{B}_{22} + B_{12}^2\overline{C} - B_{11}B_{22}\overline{C} + B_{12}T_1 & C\overline{B}_{11}\overline{B}_{22} - B_{22}\overline{B}_{11}\overline{C} + B_{12}\overline{B}_{12}\overline{C} & \overline{A}_{12}C - A_{12}\overline{C} + zT_0 & -w \\ -x_2 & -B_{11}x_1 & -\overline{B}_{11}x_1 & -y_2 & 0 \\ x_1 & -B_{12}x_1 - B_{22}x_2 & -\overline{B}_{12}x_1 - \overline{B}_{22}x_2 & y_1 & 0 \\ -y_2 & B_{11}y_1 + B_{12}y_2 + A_{12}x_2 & \overline{B}_{11}y_1 + \overline{B}_{12}y_2 + \overline{A}_{12}x_2 & -x_2z & 0 \\ -\overline{C} & B_{12}\overline{C} + T_1 & \overline{B}_{12}\overline{C} & 0 & 0 \\ y_1 & B_{22}y_2 - A_{12}x_1 & \overline{B}_{22}y_2 - \overline{A}_{12}x_1 & x_1z & 0 \\ C & 0 & -C\overline{B}_{12} + B_{12}\overline{C} + T_1 & 0 & 0 \\ -T_0 & B_{12}T_0 & \overline{B}_{12}T_0 & T_1 & 0 \\ 0 & -C & -\overline{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & T_0 & 0 & -\overline{C} & 0 \\ 0 & 0 & T_0 & C & 0 \\ 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 := \begin{pmatrix}
-B_{11}\overline{B}_{12}x_1z + B_{22}\overline{B}_{11}x_2z - B_{12}\overline{B}_{12}x_2z - B_{11}\overline{B}_{22}x_2z - \overline{A}_{12}B_{12}y_2 & B_{12}\overline{B}_{22}x_2z - \overline{A}_{12}B_{12}y_1 \\
0 & w \\
w & 0 \\
0 & 0 \\
-\overline{B}_{11}x_1z - \overline{B}_{12}x_2z - \overline{A}_{12}y_2 & \overline{B}_{22}x_2z - \overline{A}_{12}y_1 \\
0 & 0 \\
B_{11}x_1z & -B_{12}x_1z - B_{22}x_2z \\
B_{12}\overline{B}_{11}y_1 - B_{11}\overline{B}_{12}y_1 + B_{22}\overline{B}_{11}y_2 - B_{11}\overline{B}_{22}y_2 + \overline{A}_{12}B_{11}x_1 - A_{12}\overline{B}_{11}x_1 + \overline{A}_{12}B_{12}x_2 - A_{12}\overline{B}_{12}x_2 & -\overline{A}_{12}B_{22}x_2 + A_{12}\overline{B}_{22}x_2 \\
x_2z & x_1z \\
-C\overline{B}_{11} + B_{11}\overline{C} & T_1 \\
\overline{B}_{11}y_1 + \overline{B}_{12}y_2 + \overline{A}_{12}x_2 & -\overline{B}_{22}y_2 + \overline{A}_{12}x_1 \\
-B_{11}y_1 - A_{12}x_2 & B_{12}y_1 + B_{22}y_2 - A_{12}x_1 \\
-C\overline{B}_{12} + B_{12}\overline{C} + T_1 & C\overline{B}_{22} - B_{22}\overline{C} \\
\overline{A}_{12}C - A_{12}\overline{C} + zT_0 & 0 \\
0 & -\overline{A}_{12}C + A_{12}\overline{C} - zT_0 \\
y_2 & y_1
\end{pmatrix}$$

and

$$
\gamma_3 := \begin{pmatrix}
B_{12}\overline{B}_{11}y_1 + B_{22}\overline{B}_{11}y_2 - B_{11}\overline{B}_{22}y_2 + \overline{A}_{12}B_{11}x_1 - A_{12}\overline{B}_{11}x_1 & B_{12}\overline{B}_{12}y_1 + B_{22}\overline{B}_{12}y_2 + \overline{A}_{12}B_{12}x_1 - A_{12}\overline{B}_{12}x_1 + \overline{A}_{12}B_{22}x_2 - A_{12}\overline{B}_{22}x_2 \\
0 & 0 \\
0 & 0 \\
0 & -w \\
\overline{B}_{11}y_1 & \overline{B}_{12}y_1 + \overline{B}_{22}y_2 \\
w & 0 \\
-B_{11}y_1 - B_{12}y_2 - A_{12}x_2 & -B_{22}y_2 + A_{12}x_1 \\
B_{22}\overline{B}_{11}x_2 - B_{11}\overline{B}_{22}x_2 & B_{22}\overline{B}_{12}x_2 - B_{12}\overline{B}_{22}x_2 \\
y_2 & -y_1 \\
0 & T_0 \\
-\overline{B}_{11}x_1 & -\overline{B}_{12}x_1 - \overline{B}_{22}x_2 \\
B_{11}x_1 + B_{12}x_2 & B_{22}x_2 \\
-T_0 & 0 \\
-T_1 & -C\overline{B}_{22} + B_{22}\overline{C} \\
C\overline{B}_{11} - B_{11}\overline{C} & C\overline{B}_{12} - B_{12}\overline{C} - T_1 \\
x_2 & -x_1
\end{pmatrix}.
$$

Remark 3.2.1. A free resolution is easy to construct using computer algebra software (we used Macaulay2 [19]).

To show that R is Gorenstein, we essentially need to check that R is Cohen-Macaulay and that the last entry of the resolution of R, equivalently the resolution of I_Y, is rank 1 (see [41]). In Section 2.4 we proved that R is Cohen-Macaulay and hence R is also Gorenstein using our free resolution.

The resolution of R that we have computed is in fact the minimal free resolution. The ring R is known to be a codimension 4 Gorenstein ring and it must therefore have a minimal free resolution of the form

$$0 \leftarrow R \leftarrow R^{k+1} \leftarrow R^{2k} \leftarrow R^{k+1} \leftarrow R \leftarrow 0$$

(see [41]). The minimal number of generators for I_Y is 9; hence

$$0 \leftarrow R \leftarrow R^9 \leftarrow ...$$

is minimal and our full free resolution is minimal also.

To compute the graded free resolution, we "twist" R so that homogeneous elements of degree i are mapped to homogeneous elements of degree i. For simplicity, we write

$$\begin{array}{ccc} \deg(f) = n, & \deg(\overline{f}) = \overline{n}, & \mathrm{wt}(z) = 2a, \\ \mathrm{wt}(y_1) = b, & \mathrm{wt}(y_2) = c, & \mathrm{wt}(w) = d, \\ \mathrm{wt}(T_0) = r & \text{and} & \mathrm{wt}(T_1) = r + a \end{array}$$

for some $n, \overline{n}, a, b, c, d, r \in \mathbb{N}$ and write

$$\begin{pmatrix} R(1) \\ R(2) \end{pmatrix} \leftrightarrow R(1) \oplus R(2).$$

Then, the graded free resolution is

$$
0 \leftarrow R \leftarrow \begin{pmatrix} R(-r-a-c) \\ R(-r-a-b) \\ R(-r-c) \\ R(-r-b) \\ R(-2r-2a) \\ R(-r-d) \\ R(-n) \\ R(-\overline{n}) \\ R(-r-a-d) \end{pmatrix} \leftarrow \begin{pmatrix} R(-r-a-c-d+\overline{n}) \\ R(-2r-2a-c) \\ R(-2r-2a-c) \\ R(-2r-a-c) \\ R(-2r-a-c) \\ R(-2r-a-c) \\ R(-r-a-c-b) \\ R(-r-a-b-c) \\ R(-2r-a-d) \\ R(-r-a-d-b) \\ R(-r-a-2d+n) \\ R(-r-a-2d-\overline{n}) \\ R(-r-a-d-c) \\ R(-r-d-c) \\ R(-r-d-b) \\ R(-2r-2a-d) \end{pmatrix} \leftarrow \begin{pmatrix} R(-2r-2a-c+\overline{n}-d) \\ R(-3r-2a-c) \\ R(-2r-2a-c-b) \\ R(-2r-2a-2c) \\ R(-r-a-c-2d+\overline{n}) \\ R(-2r-2a-d-c) \\ R(-2r-2a-d-b) \\ R(-2r-a-d-c) \\ R(-2r-2a-d-b) \end{pmatrix} \leftarrow R(-3r-2a-c-b-d) \leftarrow 0.
$$

3.3 Projective Geometry

Definition 3.3.1. Let $\mathbb{C}[x_0, \dots, x_k]$ be a positively graded polynomial ring with $a_i := \mathrm{wt}(x_i)$ for all $i = 0, \dots, k$. A subscheme $X \subset \mathbb{P}(a_0, \dots, a_k)_{\langle x_0, \dots, x_k \rangle}$ is *projectively Gorenstein* if the homogeneous coordinate ring of X, $\mathbb{C}[x_0, \dots, x_k]/I_X$, is itself Gorenstein.

From now on, $\mathcal{O}_{\mathrm{amb}}$ will be a polynomial ring over $\mathbb{C}$. Suppose that $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is defined in the usual type ${\mathrm{II}_1}^{(n,m)}$ unprojection format. Then, the type ${\mathrm{II}_1}^{(n,m)}$ unprojection ring of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is a Gorenstein ring (see Theorem 2.2.1 and Corollary 2.7.1) of codimension $n + m + 1$, (see Section 2.5). It is clear that we may define unprojections in the context of projective schemes:

Definition 3.3.2. Suppose that $X := \mathrm{Proj}(\mathcal{O}_{\mathrm{amb}}/I_X)$, $D := \mathrm{Proj}(V(I_D)) \subset X$ and $w\mathbb{P} := \mathrm{Proj}(\mathcal{O}_{\mathrm{amb}})$ are such that $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is in the usual type ${\mathrm{II}_1}^{(n,m)}$ unprojection format. Then, the *type* ${\mathrm{II}_1}^{(n,m)}$ *unprojection of* $(X, D, w\mathbb{P})$, or $D \subset X \subset w\mathbb{P}$, is

$$Y := \mathrm{Proj}(R) \subset \mathrm{Proj}(\mathcal{O}_{\mathrm{amb}}[T_0, T_1])$$

where $R = \mathcal{O}_X[T_0, T_1]/I$ is the type ${\mathrm{II}_1}^{(n,m)}$ unprojection ring of $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ as defined in Definitions 2.2.1 and 2.7.1. When the ambient space or unprojection format is clear from context, we call Y the unprojection of (X, D).

Example 3.3.1. Let $D \subset X \subset \mathbb{P}(1, 3, 4, 5, 6)_{\langle x,y,z,u,v \rangle}$ be such that D is defined by the 2×2 minors of

$$\begin{pmatrix} u & v & yz & z^2 + zx^4 \\ y & z + x^4 & u & v \end{pmatrix}$$

and X is defined by

$$f := (y^3 + zu)(u(z + x^4) - vy) + (z^2 + yu)(u^2 - zy^2) \\ + 2x^7(uv - zy(z + x^4)) + v(v^2 - z(z + x^4)^2).$$

Then, the type II_1 unprojection of (X, D) is the variety $Y \subset \mathbb{P}(1, 3, 4, 5, 5, 6, 7)_{\langle x,y,z,u,T_0,v,T_1 \rangle}$ defined by the maximal Pfaffians of the 5×5 antisymmetric matrix

$$\begin{pmatrix} 0 & z+x^4 & y & v & u \\ & 0 & T_0 & z^2+yu & -T_1-2x^7 \\ & & 0 & T_1 & v \\ & -\mathrm{Sym} & & 0 & zT_0+y^3+zu \\ & & & & 0 \end{pmatrix}.$$

Remark 3.3.1. The explicit equations of the unprojection provided earlier in Chapter 3 hold even though we are working over $\mathbb{C}$.

Remark 3.3.2. In the cases we encounter, the unprojection coordinates have positive degree and the unprojection ring is positively graded.

Proposition 3.3.1. Define $X := \mathrm{Proj}(\mathcal{O}_{\mathrm{amb}}/I_X)$, $D := \mathrm{Proj}(V(I_D)) \subset X$ and $w\mathbb{P} := \mathrm{Proj}(\mathcal{O}_{\mathrm{amb}})$ where $(I_X, I_D, \mathcal{O}_{\mathrm{amb}})$ is in the usual type ${\mathrm{II}_1}^{(n,m)}$ unprojection format. Then the type ${\mathrm{II}_1}^{(n,m)}$ unprojection of $(X, D, w\mathbb{P})$ is birational to X.

Proof. Suppose that $X \subset \mathbb{P}(a_0, \dots a_k)_{\langle x_0,\dots,x_k\rangle}$. Let Y be the type ${\mathrm{II}_1}^{(n,m)}$ unprojection of $(X, D, w\mathbb{P})$ and suppose without loss of generality that $Y \subset \mathbb{P}(a_0, \dots, a_k, b_0, b_1)_{\langle x_0,\dots,x_k,T_0,T_1\rangle}$. Define

$$\psi : X \dashrightarrow Y$$

$$\psi(x_0, \dots, x_k) := \left(x_0, \dots, x_k, \frac{s_0(d_i)}{d_i}, \frac{s_1(d_i)}{d_i}\right)$$

with d_i varying over a minimal basis for I_D and $s_0, s_1 \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ defined as in Section 2.3. Then, ψ is a birational map with inverse defined by the natural projection map, $(x_0, \dots, x_k, T_0, T_1) \mapsto (x_0, \dots, x_k)$. $\square$

The map ψ is known as the *unprojection map* of X to Y. For type ${\mathrm{II}_1}^{(n,m)}$ unprojections where $n + m \leq 3$, we can explicitly factorise the unprojection map as two blow-ups

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\psi"] & & Y \end{tikzcd}$$

where σ is the blow-up along $E := \{s_1(d_1) = s_0(d_1) = d_1 = 0\} \subset X$ and π is the blow down of $\sigma^{-1}(D)$. In particular, the fibers $\sigma^{-1}(p)$ is a unique point whenever $p \notin \{s_0(d_i) = 0\}$ and rational curves whenever $p \in \{s_0(d_i) = 0\}$. We will prove this in the case of type $\mathrm{II}_1^{(2,0)}$ unprojections. The type $\mathrm{II}_1^{(2,1)}$ and $\mathrm{II}_1^{(3,0)}$ unprojections behave similarly.

Remark 3.3.3. We do not comment on type ${\rm II_1}^{(n,m)}$ unprojections with $n+m>3$ since we do not know the explicit equations of the unprojection.

Remark 3.3.4. This factorisation is often used when X is a Fano 3-fold or a Calabi-Yau 3-fold. It will be very important for us in Sections 4.3.1 and 4.3.2.

3.3.1 Type $\mathrm{II}_1^{(2,0)}$ Unprojections

Let $D \subset w\mathbb{P}^4_{\langle x,y,z,u,v\rangle}$ be defined by the 2×2 minors of

$$M := \begin{pmatrix} u & v & zp_1 & zp_2 \\ p_1 & p_2 & u & v \end{pmatrix}$$

where $p_1, p_2 \in \mathbb{C}[x,y,z] - \{0\}$ and define

$$d_1 := up_2 - vp_1, \quad d_2 := u^2 - zp_1^2, \quad d_3 := uv - zp_1p_2 \quad \text{and} \quad d_4 := v^2 - zp_2^2$$

to form a basis of I_D. We will assume that D is codimension 2 in $w\mathbb{P}^4$.

Let $X \subset w\mathbb{P}^4$ be a hypersurface containing D and $Y \subset w\mathbb{P}^6_{\langle x,y,z,u,v,T_0,T_1\rangle}$ the type II_1 unprojection of (X,D) (see Section 3.1.1). By Proposition 3.3.1, we know that there exists a rational map

$$\psi : X \dashrightarrow Y$$

$$\psi(x,y,z,u,v) := \left(x,y,z,u,v,\frac{s_0(d_k)}{d_k},\frac{s_1(d_k)}{d_k}\right)$$

where $s_0, s_1 \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ and $\{1, s_0, s_1\}$ is the set of generators for $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ as an $\mathcal{O}_X$-module (see Section 2.3 for the definitions of s_0 and s_1). In particular, for X defined by

$$f := A_{12}d_1 + B_{11}d_2 + 2B_{12}d_3 + B_{22}d_4$$

where $A_{12}, B_{11}, B_{12}, B_{22} \in \mathbb{C}[x, y, z, u, v]$ we have that

$$\begin{aligned}
\frac{s_0(d_k)}{d_k} &= \frac{-p_1^2 B_{11} - 2p_1p_2B_{12} - p_2^2 B_{22}}{d_1} \\
&= \frac{-2p_1uB_{12} - p_2uB_{22} - p_1vB_{22} + p_1^2 A_{12}}{d_2} \\
&= \frac{p_1p_2A_{12} + p_1uB_{11} - p_2vB_{22}}{d_3} \\
&= \frac{p_1vB_{11} + p_2uB_{11} + 2p_2vB_{12} + p_2^2 A_{12}}{d_4}
\end{aligned}$$

and

$$\begin{aligned}
\frac{s_1(d_k)}{d_k} &= \frac{2p_1vB_{12} + p_2vB_{22} + p_1uB_{11}}{d_1} \\
&= \frac{vuB_{22} - p_1uA_{12} + 2p_1^2 zB_{12} + p_2p_1zB_{22}}{d_2} \\
&= \frac{-p_1^2 zB_{11} - p_1vA_{12} + v^2B_{22}}{d_3} \\
&= \frac{-uvB_{11} - 2v^2B_{12} - p_2vA_{12} - p_1p_2zB_{11}}{d_4}.
\end{aligned}$$

Remark 3.3.5. Recall Remark 3.1.7: these fractions could be obtained by viewing the linear equations of the unprojection as a linear system in T_0 and T_1. For example,

$$T_1d_1 - 2vp_1B_{12} - up_1B_{11} - vp_2B_{22} = l_1p_2 - l_2p_1 = 0.$$

We will now prove our claim that the unprojection map ψ factorises as

$$\begin{tikzcd}
 & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\
X \arrow[rr, dashed, "\psi"] & & Y
\end{tikzcd}$$

where $\sigma : Z \to X$ is the blow up of X along $E := \{s_0(d_1) = s_1(d_1) = d_1 = 0\}$ and $\pi : Z \to Y$ is the blow down of $\sigma^{-1}(D)$.

Remark 3.3.6. The idea is that the D and E are divisors of X such that $D + E$ is a Cartier divisor.

Let I be the ideal of E in X and define $e_1 := s_0(d_1)$, $e_2 := s_1(d_1)$ and

$e_3 := d_1$. The blow up of X along E, Z, is defined by the algebra

$$Bl_I(\mathcal{O}_X) := \bigcup_{i=1}^{3} \operatorname{Spec}\left(\mathcal{O}_X\left[\frac{I}{e_i}\right]\right) = \bigcup_{i=1}^{3} \operatorname{Spec}\left(\mathcal{O}_X\left[\frac{e_i}{e_j} : 1 \le j \le 3\right]\right).$$

We calculate one patch of the blow up Z:

Lemma 3.3.1. We have that

$$\mathcal{O}_X\left[\frac{s_0(d_1)}{d_1}, \frac{s_1(d_1)}{d_1}\right] \cong \mathcal{O}_X[t_0, t_1]/J$$

where t_0 and t_1 are indeterminates and

$$J := \langle f, d_i t_0 - s_0(d_i), d_i t_1 - s_1(d_i) : i = 1, 2, 3, 4\rangle.$$

Proof. The algebra $\mathcal{O}_X[I/d_1]$ has generators

$$x, y, z, u, v, t_0 := \frac{s_0(d_1)}{d_1}, t_1 := \frac{s_1(d_1)}{d_1}$$

and it is clear that $d_1 t_0 - s_0(d_1) = d_1 t_1 - s_1(d_1) = 0$ on $\mathcal{O}_X$ and $\mathcal{O}_X[I/d_1]$. The other equations of J are obtained by realising that $s_0(d_i)d_j = s_0(d_j)d_i$ on $\mathcal{O}_X$ for all $i, j = 1, 2, 3, 4$. A similar result holds for s_1. $\square$

With this in mind, we calculate the blow up Z:

Theorem 3.3.1. The blow up $Z \subset w\mathbb{P}^4 \times w\mathbb{P}^2_{\langle t_0, t_1, t_2\rangle}$ is defined by

$$u t_1 + z p_1 t_0 - v B_{22} t_2 + p_1 A_{12} t_2 = 0,$$

$$v t_1 + z p_2 t_0 + u B_{11} t_2 + 2 v B_{12} t_2 + p_2 A_{12} t_2 = 0,$$

$$u t_0 + p_1 t_1 + 2 p_1 B_{12} t_2 + p_2 B_{22} t_2 = 0,$$

$$v t_0 + p_2 t_1 - p_1 B_{11} t_2 = 0$$

and

$$t_1^2 - z t_0^2 - A_{12} t_0 t_2 + 2 B_{12} t_1 t_2 + B_{11} B_{22} t_2^2 = 0$$

together with f and the 2×2 minors of

$$m_k := \begin{pmatrix} t_0 & t_1 & t_2 \\ s_0(d_k) & s_1(d_k) & d_k \end{pmatrix}$$

for $k = 1, 2, 3, 4$.

It is clear from the definition of the blow-up of X along E that the minors of m_k must vanish on Z.

The remaining equations in t_0, t_1 and t_2 are obtained using the known relations of s_0 and s_1. For example,

$$\begin{aligned} d_i(vt_0 + p_2t_1 - p_1B_{11}t_2) &= vt_0d_i + p_2t_1d_i - p_1B_{11}t_2d_i \\ &= vt_2s_0(d_i) + p_2t_2s_1(d_i) - p_1B_{11}t_2d_i \\ &= (vs_0(d_i) + p_2s_1(d_i) - p_1B_{11}d_i)t_2 \\ &= 0. \end{aligned}$$

The final equality occurs since $vs_0 + p_2s_1 - p_1B_{11}$ is a linear relation on $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$.

Remark 3.3.7. Compare the equations of Z and the equations of Y. The equations of Z which are linear in t_0, t_1 and t_2 are

$$N(t_1, t_0)^T - \alpha t_2 = 0$$

and the linear equations of the unprojection are

$$N(T_1, T_0)^T - \alpha = 0$$

where N and α are defined as in Section 3.1.1.

Proposition 3.3.2. Let $S_0 := \{s_0(d_1) = \cdots = s_0(d_4) = 0\} \subset X$. If $p \notin D \cap S_0$, the fiber $\sigma^{-1}(p)$ consists of a unique point. Otherwise, $\sigma^{-1}(p)$ is a rational curve.

Proof. Suppose that $p \notin D$. Then, as $d_k \neq 0$ for some fixed k, the 2×2 minors of m_k set

$$t_0 = \tfrac{s_0(d_k)}{d_k} t_2 \text{ and } t_1 = \tfrac{s_1(d_k)}{d_k} t_2$$

and hence

$$\sigma^{-1}(p) = \left\{ p \times \left(\frac{s_0(d_k)}{d_k}, \frac{s_1(d_k)}{d_k}, 1 \right) \right\}.$$

Suppose that $p \in D - S_0$. As $p \notin S_0$, we have $s_0(d_k) \neq 0$ for some fixed k. However $d_k = 0$ since $p \in D$ and the minors of m_k therefore imply that $t_2 = 0$ and $t_0s_1(d_k) = t_1s_0(d_k)$. We note that since $D = \mathrm{im}(\phi)$ for

$$\phi : w\mathbb{P}^2 \to w\mathbb{P}^4$$

$$\phi(a, b, c) := (a, b, c^2, cp_1(a, b, c^2), cp_2(a, b, c^2)),$$

we may work with $p = (a, b, c^2, cp_1(a, b, c^2), cp_2(a, b, c^2))$. Considered as polynomials on $w\mathbb{P}^2$, we have that $s_1(d_k) = -cs_0(d_k)$. Hence,

$$\sigma^{-1}(p) = \{p \times (1, -c, 0)\}.$$

Suppose $p \in D \cap S_0$. If $p_1 = p_2 = 0$ on p, then $u = v = 0$ and the equations of Z which are linear in t_0, t_1 and t_2 all vanish on p. The fiber of p is therefore

$$\sigma^{-1}(p) = \{p \times (t_0, t_1, t_2) : t_1^2 - zt_0^2 - A_{12}t_0t_2 + 2B_{12}t_1t_2 + B_{11}B_{22}t_2^2 = 0\}.$$

Suppose instead that $p_2 \neq 0$ on p and let $p \times (t_0, t_1, t_2) \in \sigma^{-1}(p)$. By rearranging

$$vt_0 + p_2t_1 - p_1B_{11}t_2 = 0$$

we obtain the equality

$$t_1 = \frac{p_1B_{11}t_2 - vt_0}{p_2}.$$

By substituting this value for t_1 into the remaining equations of Z, we obtain

$$ut_1 + zp_1t_0 - vB_{22}t_2 + p_1A_{12}t_2 = \frac{-d_3t_0 + t_2s_0(d_3)}{p_2},$$

$$vt_1 + zp_2t_0 + uB_{11}t_2 + 2vB_{12}t_2 + p_2A_{12}t_2 = \frac{-d_4t_0 + t_2s_0(d_4)}{p_2},$$

$$ut_0 + p_1t_1 + 2p_1B_{12}t_2 + p_2B_{22}t_2 = \frac{d_1t_0 - t_2s_0(d_1)}{p_2}$$

and

$$t_1^2 - zt_0^2 - A_{12}t_0t_2 + 2B_{12}t_1t_2 + B_{11}B_{22}t_2^2 = \frac{B_{11}t_2(d_1t_0 - t_2s_0(d_1)) + t_0(t_0d_4 - t_2s_0(d_4))}{p_2^2}.$$

It is clear from the minors of m_j that on p the remaining equations are identically zero. Hence,

$$\phi^{-1}(p) = \left\{p \times \left(t_0, \frac{p_1B_{11}t_2 - vt_0}{p_2}, t_2\right)\right\}.$$

An analogous result holds when $p_1 \neq 0$. □

The blow down of the strict transform of D is defined by

$$\pi : Z \dashrightarrow Y$$

$$\pi((x, y, z, u, v) \times (t_0, t_1, t_2)) = \left(x, y, z, u, v, T_0 := \frac{t_0}{t_2}, T_1 := \frac{t_1}{t_2}\right).$$

In many cases $D \cap S_0$ consists of finitely many points. We have that σ blows up these points into rational curves and π maps these curves into a bouquet on p_{T_0}. We recall Example 3.3.1:

Example 3.3.2. Let $D \subset X \subset \mathbb{P}(1,3,4,5,6)_{\langle x,y,z,u,v\rangle}$ be such that D is defined by the 2×2 minors of

$$\begin{pmatrix} u & v & yz & z^2 + zx^4 \\ y & z + x^4 & u & v \end{pmatrix}$$

and X is defined by

$$f := (y^3 + zu)(u(z + x^4) - vy) + (z^2 + yu)(u^2 - zy^2) + \\ 2x^7(uv - zy(z + x^4)) + v(v^2 - z(z + x^4)^2).$$

Define the standard basis of I_D $\{d_1, \ldots, d_4\}$ where

$$d_1 := u(z + x^4) - vy,$$

$$d_2 := u^2 - zy^2,$$

$$d_3 := uv - zy(z + x^4)$$

and

$$d_4 := v^2 - z(z + x^4)^2.$$

We have that

$$s_0(d_2) = -2x^7yu - uv(z + x^4) - yv^2 + y^2(y^3 + vu)$$

and

$$s_0(d_3) = y(y^3 + zu)(z + x^4) + yu(z^2 + yu) - v^2(z + x^4).$$

If $p = (x, y, 0, u, v) \in D \cap S_0$, then it is clear from the equations of D, $s_0(d_2)$ and $s_0(d_3)$ that $p = (1, 0, 0, 0, 0)$. If $p = (x, y, 1, u, v) \in D \cap S_0$, then we must have $p \in \{(x, y, 1, y, 1 + x^4), (x, y, 1, -y, -1 - x^4)\}$ by definition of D. On such points $s_0(d_2)$ and $s_0(d_3)$ simplify to the polynomials

$$-2x^8y - 2x^7y^2 + x^4y^3 - 4x^4y + y^5 + y^3 - 2y$$

and

$$-x^{12} - 3x^8 + x^4y^4 + x^4y^2 - 3x^4 + 2y^4 + 2y^2 - 1$$

which have finitely many shared solutions. Hence, $D \cap S_0$ is a finite set of points.

3.3.2 Type $\mathrm{II}_1^{(2,1)}$ Unprojections

To convince ourselves that the birational geometry of the type ${\mathrm{II}_1}^{(2,0)}$ unprojection is typical, we now study the type $\mathrm{II}_1^{(2,1)}$ case.

Let $D \subset w\mathbb{P}^5_{\langle x,y,z,u,v,w\rangle}$ be a codimension 3 scheme defined by the 2×2 minors of the matrix

$$M := \begin{pmatrix} u & v & zp_1 & zp_2 \\ p_1 & p_2 & u & v \end{pmatrix}$$

together with $w = 0$ where $p_1, p_2 \in \mathbb{C}[x,y,z] - \{0\}$. We fix a basis of I_D

$$d_1 := w, \quad d_2 := up_2 - vp_1, \quad d_3 := u^2 - zp_1^2, \quad d_4 := uv - zp_1p_2, \quad d_5 := v^2 - zp_2^2.$$

Let X be a codimension 2 complete intersection which contains D. Without loss of generality we write $I_X = \langle f, \overline{f}\rangle$ where

$$f := A_{12}(up_2 - p_1v) + B_{11}(u^2 - zp_1^2) + 2B_{12}(uv - zp_1p_2) + B_{22}(v^2 - zp_2^2) + Cw$$

and

$$\overline{f} := \overline{A}_{12}(up_2 - p_1v) + \overline{B}_{11}(u^2 - zp_1^2) + 2\overline{B}_{12}(uv - zp_1p_2) + \overline{B}_{22}(v^2 - zp_2^2) + \overline{C}w$$

where $A_{12}, \overline{A}_{12}, B_{11}, \overline{B}_{11}, B_{12}, \overline{B}_{12}, B_{22}, \overline{B}_{22}, C, \overline{C} \in \mathbb{C}[x,y,z,u,v,w]$.

Let $Y \subset w\mathbb{P}^7_{\langle x,y,z,u,v,w,T_0,T_1\rangle}$ be the unprojection of (X, D). The unprojection map is

$$\psi : X \dashrightarrow Y$$

$$\psi(x,y,z,u,v,w) := \left(x,y,z,u,v,w,\frac{s_0(d_k)}{d_k},\frac{s_1(d_k)}{d_k}\right)$$

where $s_0, s_1 \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ and $\{1, s_0, s_1\}$ is the set of generators for $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ as an $\mathcal{O}_X$-module. In particular, if we define v_{ij} as the ij-th minor of

$$\begin{pmatrix} C & A_{12} & B_{11} & 2B_{12} & B_{22} \\ \overline{C} & \overline{A}_{12} & \overline{B}_{11} & 2\overline{B}_{12} & \overline{B}_{22} \end{pmatrix}$$

for $1 \le i < j \le 5$, the unprojection map can be described using

$$
\begin{aligned}
\frac{s_0(d_i)}{d_i} &= \frac{p_2^2v_{25} + p_2p_1v_{24} + p_2vv_{45} + p_2uv_{35} + p_1^2v_{23} + p_1vv_{35} + p_1uv_{34}}{d_1} \\
&= \frac{-p_1^2v_{13} - p_1p_2v_{14} - p_2^2v_{15}}{d_2} \\
&= \frac{p_1^2v_{12} - p_1uv_{14} - p_1vv_{15} - p_2uv_{15}}{d_3} \\
&= \frac{p_1p_2v_{12} - p_2vv_{15} + p_1uv_{13}}{d_4} \\
&= \frac{p_2^2v_{12} + p_2uv_{13} + p_1vv_{13} + p_2vv_{14}}{d_5}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{s_1(d_i)}{d_i} &= \frac{-u^2v_{34} - p_2^2zv_{45} - p_2p_1zv_{35} - p_2vv_{25} - p_2uv_{24} - p_1uv_{23} - vuv_{35}}{d_1} \\
&= \frac{p_1uv_{13} + p_2uv_{14} + p_2vv_{15}}{d_2} \\
&= \frac{p_1p_2zv_{15} - p_1uv_{12} + u^2v_{14} + uvv_{15}}{d_3} \\
&= \frac{-p_1^2zv_{13} - p_1p_2zv_{14} - p_1vv_{12} + uvv_{14} + v^2v_{15}}{d_4} \\
&= \frac{-p_1p_2zv_{13} - p_2^2zv_{14} - p_2vv_{12} - uvv_{13}}{d_5}.
\end{aligned}
$$

We claim that the unprojection map ψ can be factorised as

$$
\begin{tikzcd}
 & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\
X \arrow[rr, dashed, "\psi"] & & Y
\end{tikzcd}
$$

where σ is the blow up of X along $E = \{s_0(d_1) = s_1(d_1) = d_1 = 0\} \subset X$ and $\pi : Z \to Y$ is the blow down of $\sigma^{-1}(D)$. As in the type ${\mathrm{II}_1}^{(2,0)}$ case, the blow up of X along E is essentially defined by the equations of Y:

Theorem 3.3.2. The blow up $Z \subset w\mathbb{P}^5 \times w\mathbb{P}^2_{\langle t_0,t_1,t_2\rangle}$ is defined by $f, \overline{f}$ together with the 2×2 minors of the matrices

$$
m_k = \begin{pmatrix} t_0 & t_1 & t_2 \\ s_0(d_k) & s_1(d_k) & d_k \end{pmatrix}
$$

for $k = 1, \ldots, 5$ and the relations

$$ut_1 + zp_1t_0 + (p_1v_{12} - uv_{14} - vv_{15})t_2 = 0,$$

$$vt_1 + zp_2t_0 + (p_2v_{12} + uv_{13})t_2 = 0,$$

$$wt_1 + (u^2v_{34} + zp_2^2v_{45} + zp_2p_1v_{35} + p_2vv_{25} + p_2uv_{24} + p_1uv_{23} + vuv_{35})t_2 = 0,$$

$$ut_0 + p_1t_1 + p_2v_{15}t_2 = 0,$$

$$vt_0 + p_2t_1 - (p_1v_{13} + p_2v_{14})t_2 = 0,$$

$$wt_0 - (p_2^2v_{25} + p_2p_1v_{24} + p_2vv_{45} + p_2uv_{35} + p_1^2v_{23} + p_1vv_{35} + p_1uv_{34})t_2 = 0$$

and

$$t_1^2 - zt_0^2 - t_0t_2v_{12} - t_1t_2v_{14} + t_2^2v_{15}v_{13} = 0.$$

Remark 3.3.8. Compare the equations of Z with the equations of Y defined in Section 3.1.2.

Proposition 3.3.3. Let $S_0 := \{s_0(d_1) = \cdots = s_0(d_5) = 0\} \subset X$. If $p \notin D \cap S_0$, the fiber $\sigma^{-1}(p)$ consists of a unique point. If $p \in D \cap S_0$, then $\sigma^{-1}(p)$ is a rational curve.

The proof follows that of Proposition 3.3.2. For $p \notin D$, the minors m_k provide definitions of t_0 and t_1 in terms of t_2; therefore, the fiber of $p \notin D$ is a point. When $p \in D$, we use $D = \mathrm{im}(\phi)$ where $\phi : w\mathbb{P}^2 \to w\mathbb{P}^5$ is defined by $\phi(a, b, c) = (a, b, c^2, cp_1(a, b, c^2), cp_2(a, b, c^2), 0)$. If $p \in D - S_0$, we have that $s_0(d_k) = -cs_1(d_k)$ for $k = 1, \ldots, 5$ and the fiber of p is the point $p \times (1, -c, 0)$. Otherwise, a case by case analysis provides the rational curves we desire.

The blow down of the strict transform of D is defined by

$$\pi : Z \dashrightarrow Y$$

$$\pi((x, y, z, u, v, w) \times (t_0, t_1, t_2)) = \left(x, y, z, u, v, w, T_0 := \frac{t_0}{t_2}, T_1 := \frac{t_1}{t_2}\right).$$

Chapter 4

Application to Fano 3-folds

It is now the time to construct Fano 3-folds via type II_1 unprojections.

4.1 Preliminaries

We elaborate on Chapter 1 and recall the definitions of Fano 3-folds and numerical candidates. Many of the definitions and concepts are standard (we largely follow [4]).

4.1.1 Fano 3-folds

A Fano 3-fold X is a complex normal projective 3-fold whose anticanonical divisor $-K_X$ is $\mathbb{Q}$-Cartier and ample, and whose singularities are $\mathbb{Q}$-factorial and terminal (See Definition 1.0.1). We explain these terms below:

Definition 4.1.1. A variety X has *terminal singularities* if it is normal, rK_X is a Cartier divisor for some $r \in \mathbb{N}^+$, and any resolution of singularities $f : Y \to X$ with divisorial exceptional locus $\cup E_i$ satisfies $K_Y = f^*K_X + \sum a_i E_i$ with all $a_i > 0$.

Definition 4.1.2. A variety X is $\mathbb{Q}$*-factorial* if for every Weil divisor D on X there exists $r \in \mathbb{N}^+$ with rD a Cartier divisor.

For narrative purposes we focus on index 1 Fano 3-folds; however our methods can easily be extended to those with a higher index (see [11]):

Definition 4.1.3. Let X be a Fano 3-fold. The *(Fano) index* of X is the greatest integer r such that $-K_X = rA$ for some ample Weil divisor A.

Many of the Fano 3-folds we encounter will be quasismooth:

Definition 4.1.4. Let X be a closed subvariety of weighted projective space $w\mathbb{P}^n$. The *affine cone* C_X over X is the Spec of the homogeneous coordinate ring of X. We say that X is *quasismooth* if its affine cone C_X is smooth outside its vertex 0.

In essence, $X \subset w\mathbb{P}^n$ is quasismooth if its singularities come from the ambient space in which the variety lives rather than the equations of the variety itself. In practise, we check that a Fano 3-fold $X \subset w\mathbb{P}^n$ is quasismooth by checking that the Jacobian matrix of X at p is rank $n - \dim(X)$ for every $p \in X$.

4.1.2 Hilbert Series and Graded Rings

We recall the definition of the anticanonical ring of a variety X (see Section 1.1):

Definition 4.1.5. Let X be an irreducible projective variety over $\mathbb{C}$ and A an ample divisor on X. The *Riemann-Roch space* of mA for $m \in \mathbb{N}$, is

$$H^0(X, mA) := \{f \in \mathbb{C}(X) | \mathrm{div}(f) + mA \geq 0\}.$$

It is the finite dimensional vector space of rational functions $f \in \mathbb{C}(X)$ with divisor of poles $\leq mA$.

Definition 4.1.6. The *anticanonical ring* of a variety X is the graded ring

$$R(X, -K_X) := \bigoplus_{m \geq 0} H^0(X, -mK_X).$$

We will construct Fano 3-folds as $X = \mathrm{Proj}(R(X, -K_X))$. Recall that by using this description of Fano 3-folds it is sufficient to consider only varieties in weighted projective space: a choice of generators of $R(X, -K_X)$, say $x_0, \ldots, x_n$, allows us to embed X as a projectively normal variety in $\mathbb{P}(a_0, \ldots, a_n)$ with $a_i := \mathrm{wt}(x_i)$.

Remark 4.1.1. Since we are studying $X \subset w\mathbb{P}^n$, the anticanonical ring of X equals the homogeneous coordinate ring of X.

Definition 4.1.7. Let $X \subset w\mathbb{P}^n$ be a variety. The *Hilbert series* of X with respect to the ample divisor $A \subset X$ is

$$P_{X,A}(t) := \sum_{m \in \mathbb{N}} \dim(H^0(X, mA))t^m.$$

Unless otherwise specified, we will define the Hilbert series of X with respect to $A := -K_X$ and simplify our notation to $P_X(t)$.

4.1.3 Numerical Candidates

Theorem 1.1.1 states that the Hilbert series of a Fano 3-fold X can be defined as a rational function in terms of its genus, $g_X := h^0(X, -K_X) - 2$, and a basket of terminal cyclic quotient singularities, $\mathcal{B}_X = \{\frac{1}{r}(1, a, r-a)\}$.

Definition 4.1.8. (Definition 5.13, [21]) Let $r > 0$ and let $a_1, \ldots, a_n$ be integers. Suppose that $\mathbb{Z}_r$ acts on $\mathbb{A}^n_{\langle x_1,\ldots,x_n\rangle}$ with action

$$(x_1, \ldots, x_n) \mapsto (\epsilon^{a_1} x_1, \ldots, \epsilon^{a_n} x_n)$$

for ϵ a fixed primitive r-th root of unity. A singularity $Q \in X$ is a *quotient singularity* of type $\frac{1}{r}(a_1, \ldots, a_n)$ if (X, Q) is isomorphic to an analytic neighbourhood of $(\mathbb{A}^n, 0)/\mathbb{Z}_r$.

Definition 4.1.9. ((6.4), [37]) Let X be a Fano 3-fold and $p \in X$ a terminal singularity. There is a deformation $\{X_\lambda\}$ of $p \in X$ such that the general fibre has as its only singularities a number of terminal quotient singularities $\frac{1}{r}(a, -a, 1)$. The *basket* of $p \in X$ is this collection of terminal quotient singularities.

To obtain the *basket* of a Fano 3-fold X, we combine the baskets of p for all terminal singularities $p \in X$. For example,

$$\left\{\tfrac{1}{r_1}(a_1, -a_1, 1)\right\} \quad \text{and} \quad \left\{\tfrac{1}{r_1}(a_1, -a_1, 1), \{\tfrac{1}{r_2}(a_2, -a_2, 1)\right\}$$

combine to form

$$\left\{2 \times \frac{1}{r_1}(a_1, -a_1, 1), \frac{1}{r_2}(a_2, -a_2, 1)\right\}.$$

Remark 4.1.2. The singularities of a Fano 3-fold X may not exactly be the singularities of $\mathcal{B}_X$. However, the contribution of the actual singularities of X towards the Riemann-Roch formula is the same as the contribution of those in the basket (see Theorem 10.2, [37]). In practice, the Fano 3-folds constructed in this thesis have cyclic terminal quotient singularities, and in that case the basket is equal to the set of singularities of X.

For a Fano 3-fold X with genus g_X and basket $\mathcal{B}_X$, we call the pair $(g_X, \mathcal{B}_X)$ *numerical data of* X and note that it is equivalent to the Hilbert series of X by Theorem 1.1.1. As a result of the bounds imposed by Theorems 1.1.2 and 1.1.3, we create a finite set of numerical data for Fano 3-folds. Let S be the bounded set of pairs $\{g, \mathcal{B}\}$ defined by Theorems 1.1.2 and 1.1.3. We may predict the Hilbert series of Fano 3-folds by substituting $\{g, \mathcal{B}\} \in S$ into the Hilbert series definition provided by Theorem 1.1.1. We call such a rational function a *numerical candidate.*

Using the analysis of [8], [2], [3] and [4], it is possible to represent a numerical candidate as the Hilbert series of some projective variety $Y \subset w\mathbb{P}^n$. This can be done systematically (see Example 1.1.1). The predicted Fano 3-fold $Y \subset w\mathbb{P}^n$ for a numerical candidate is chosen such that the numerical candidate cannot be realised as a simpler Fano 3-fold where n is smaller (see Example 1.1.2). Without confusion, the term numerical candidate will refer to both rational functions and the predicted Fano 3-folds.

We are interested in the codimension 4 numerical candidates; that is, the numerical candidates P_Y which when analysed suggest codimension 4 Fano 3-folds $Y \subset \mathbb{P}(a_0, \ldots, a_7)$. There are 145 codimension 4 candidates.

Remark 4.1.3. For simplicity, one could read "numerical candidate" as "Fano 3-fold and Hilbert series predicted by the GRDB [8]". Alternatively, one could read "numerical candidate" as "a pair $\{g, \mathcal{B}\} \in S$" and the predicted Fano 3-fold as a suggestion on how to realise the candidate; a suggestion which always turns out to be correct.

4.2 Main Theorem

4.2.1 Statement of Result

Of the 145 codimension 4 numerical candidates, we are interested in those that are marked with a particular cyclic quotient singularity. We wish to identify those with a type II_1 centre:

Definition 4.2.1. (Definition 3.4, [6]) Let $X \subset \mathbb{P}(a_0, \ldots, a_m)$ be a Fano 3-fold with $p = \frac{1}{r}(b_0, b_1, b_2) \in X$ a cyclic quotient singularity such that $r \in \mathbb{N}^+$, $r > 1$ and b_i is a minimal non-negative residue modulo r for $i = 0, 1, 2$. We call p a *type* II_n *centre* if, up to relabelling, we have $r = a_m$, $b_i = a_i$ for $i = 0, 1$ and $n + 1$ is the smallest positive integer such that $(n+1)b_2 \in \{a_i : 2 \leq i < m\}$.

There are 50 codimension 4 numerical candidates marked with type II_1 centres. That is, there are 50 codimension 4 numerical candidates which lie in the correct ambient space and have a basket containing a suitable cyclic quotient singularity to satisfy Definition 4.2.1. By observation, we note that the ambient spaces and the type II_1 centres occur with a particular shape:

Proposition 4.2.1. If $Y \subset \mathbb{P}(a_0, a_1, \ldots, a_7)$ is a codimension 4 numerical candidate which is marked with a type II_1 centre $p = \frac{1}{r}(b_0, b_1, b_2)$, then up to relabelling we have $Y \subset \mathbb{P}(2b_0, b_1, b_2, b_3, b_4, b_5, r, r{+}b_0)$ and $p = \frac{1}{r}(b_0, b_1, b_2)$ for some $b_3, b_4, b_5 \in \mathbb{N}^+$.

Arguably, a number of the numerical candidates marked with type II_1 centres are not true candidates since they have already been realised as codimension 4 Fano 3-folds. Of these 50 numerical candidates, 34 possess a type I centre and are studied in [10].

Definition 4.2.2. (Section 3.2, [10]) Let X be a Fano 3-fold and $r \in \mathbb{N}^+$. A quotient singularity $p = \frac{1}{r}(1, a, r-a)$ of X with $1 < r$ is a *type I centre* if its orbinates are restrictions of global forms $x \in H^0(X, -K_X)$, $y \in H^0(X, -aK_X)$ and $z \in H^0(X, -(r-a)K_X)$ of the same weight.

Remark 4.2.1. Let $X \subset \mathbb{P}(a_0, \ldots, a_m)$ be a Fano 3-fold with $p = \frac{1}{r}(b_0, b_1, b_2) \in X$ a cyclic quotient singularity such that $r \in \mathbb{N}^+$, $r > 1$ and b_i is a minimal non-negative residue modulo r for $i = 0, 1, 2$. In this thesis, p is a type I centre if $r = a_m$ and $a_i = b_i$ for $i = 0, 1, 2$ up to reordering.

The codimension 4 numerical candidates marked with type I centres are identified in [10] and have been realised as Fano 3-folds. This includes the 34 numerical candidates marked with type I and type II_1 centres. In the hopes of constructing completely new Fano 3-folds, we ignore these cases until Chapter 5. We are now left with 16 numerical candidates which we list by their GRDB ID in Table 4.1.

Now we can state the main theorem of this thesis: each of these 16 numerical candidates can be realised as a quasismooth Fano 3-fold and moreover these Fano 3-folds occur in 2 distinct families.

Theorem 4.2.1. Let $Y \subset \mathbb{P}(2a_0, a_1, \ldots, a_7)$ be a codimension 4 numerical candidate marked with a type II_1 centre but no type I centre. The numerical candidate Y can be realised as a quasismooth codimension 4 Fano 3-fold and constructed as the type II_1 unprojection of (X, D) where $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_5)$ is a codimension 2 complete intersection. In particular, there exists a successful construction using type ${\mathrm{II}_1}^{(2,1)}$ unprojections and a second using type ${\mathrm{II}_1}^{(3,0)}$.

For a given codimension 4 candidate marked with a type II_1 and no type I centre, the type ${\mathrm{II}_1}^{(3,0)}$ and type ${\mathrm{II}_1}^{(2,1)}$ unprojections construct members of topologically distinct families. That is:

Corollary 4.2.1. Let $Y \subset w\mathbb{P}^7$ be a codimension 4 numerical candidate marked with a type II_1 centre but no type I centre. The Hilbert scheme of Y has at least 2 components containing quasismooth Fano 3-folds.

Table 4.1: Codimension 4 Candidates with a Type II_1 and No Type I Centre

ID	Numerical Candidate Y	g_Y	$\mathcal{B}_Y$
38	$Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$	-2	$\{7 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{8}(1,3,5)\}$
342	$Y \subset \mathbb{P}(1,4,6,7,7,8,9,10)$	-1	$\{2 \times \frac{1}{2}(1,1,1), \frac{1}{4}(1,1,3), \frac{1}{7}(1,1,6), \frac{1}{7}(1,3,4)\}$
360	$Y \subset \mathbb{P}(1,4,5,6,7,7,8,9)$	-1	$\{2 \times \frac{1}{4}(1,1,3), \frac{1}{6}(1,1,5), \frac{1}{7}(1,2,5)\}$
648	$Y \subset \mathbb{P}(1,3,4,4,5,5,6,7)$	-1	$\{\frac{1}{3}(1,1,2), 3 \times \frac{1}{4}(1,1,3), \frac{1}{5}(1,2,3)\}$
1069	$Y \subset \mathbb{P}(1,2,6,7,8,9,9,10)$	-1	$\{5 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{9}(1,1,8)\}$
1084	$Y \subset \mathbb{P}(1,2,5,6,7,8,8,9)$	-1	$\{4 \times \frac{1}{2}(1,1,1), \frac{1}{5}(1,2,3), \frac{1}{8}(1,1,7)\}$
1115	$Y \subset \mathbb{P}(1,2,4,5,6,7,7,8)$	-1	$\{5 \times \frac{1}{2}(1,1,1), \frac{1}{4}(1,1,3), \frac{1}{7}(1,1,6)\}$
1122	$Y \subset \mathbb{P}(1,2,4,5,5,6,6,7)$	-1	$\{5 \times \frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{6}(1,1,5)\}$
1172	$Y \subset \mathbb{P}(1,2,3,4,5,6,6,7)$	-1	$\{4 \times \frac{1}{2}(1,1,1), 2 \times \frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5)\}$
1256	$Y \subset \mathbb{P}(1,2,3,4,4,5,5,6)$	-1	$\{4 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,1,4)\}$
1350	$Y \subset \mathbb{P}(1,2,3,4,4,4,5,5)$	-1	$\{4 \times \frac{1}{2}(1,1,1), 3 \times \frac{1}{4}(1,1,3)\}$
2410	$Y \subset \mathbb{P}(1,2,2,3,4,5,5,6)$	-1	$\{7 \times \frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4)\}$
2438	$Y \subset \mathbb{P}(1,2,2,3,3,4,4,5)$	-1	$\{6 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3)\}$
2511	$Y \subset \mathbb{P}(1,2,2,3,3,3,4,4)$	-1	$\{5 \times \frac{1}{2}(1,1,1), 3 \times \frac{1}{3}(1,1,2)\}$
3509	$Y \subset \mathbb{P}(1,2,2,2,3,3,3,4)$	-1	$\{8 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)\}$
8051	$Y \subset \mathbb{P}(1,1,2,2,2,2,3,3)$	0	$\{7 \times \frac{1}{2}(1,1,1)\}$

Remark 4.2.2. In the statement of Theorem 4.2.1, $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_5)$ is not a Fano 3-fold. The variety X is not $\mathbb{Q}$-factorial since $Y \dashrightarrow X$ will be a projective morphism contracting finitely many rational curves (see Section 3.3). Nevertheless, it is a special member of a family whose general member is a Fano 3-fold in $\mathbb{P}(2a_0, a_1, \ldots, a_5)$.

In Section 4.3, we will provide the proof of Theorem 4.2.1 and Corollary 4.2.1 for a single numerical candidate. The method used is indicative of all remaining numerical candidates but we elaborate in Section 4.2.2.

4.2.2 Strategy

Let $Y \subset \mathbb{P}(2a_0, \ldots, a_5, r, r+a_0)$ be a codimension 4 numerical candidate marked with a type II_1 centre $\frac{1}{r}(a_0, a_1, a_2)$. We prove Theorem 4.2.1 by providing the appropriate and successful type II_1 unprojection constructions; we take our cue from the proof of Theorem 3.2 [10].

For each numerical candidate, Table A.1 provides two pairs of unprojection data (X, D) which realise Y. The initial data is chosen so that

$X_{i,j} \subset \mathbb{P}(2a_0, a_1, \ldots, a_5)_{\langle x,y,z,u,v,w \rangle}$ is a codimension 2 complete intersection where $i + j + 1 = 2a_0 + a_1 + \cdots + a_5$ and the Hilbert series of X and Y are related by

$$P_Y(t) = P_X(t) + \frac{t^r + t^{r+a_0}}{(1 - t^{2a_0})(1 - t^{a_1})(1 - t^{a_2})(1 - t^r)}.$$

Moreover, $D \subset X$ is chosen so that D is defined by

- either the 2×2 minors of

$$\begin{pmatrix} u & v & w & xp_1 & xp_2 & xp_3 \\ p_1 & p_2 & p_3 & u & v & w \end{pmatrix}$$

  where $p_i \in \mathbb{C}[x, y, z]$; or

- by $w = 0$ together with the 2×2 minors of

$$\begin{pmatrix} u & v & xp_1 & xp_2 \\ p_1 & p_2 & u & v \end{pmatrix}$$

  where $p_i \in \mathbb{C}[x, y, z]$.

Remark 4.2.3. The key idea is that $\mathbb{P}(a_0, a_1, a_2)$ maps to D (see Examples 2.7.1 and 2.7.2).

Let $Y' \subset \mathbb{P}(2a_0, a_1, \ldots, a_5, \deg(T_0), \deg(T_1))_{\langle x,y,z,u,v,w,T_0,T_1 \rangle}$ be the type II_1 unprojection of (X, D). We have that Y' lies in our desired ambient space by construction. We wish for $\frac{1}{r}(a_0, a_1, a_2)$ to be an isolated terminal cyclic quotient singularity in a Fano 3-fold. Hence, we wish for $a_0 + a_1 + a_2 - 1 = r$. The explicit equations of the unprojections give us the following Lemma:

Lemma 4.2.1. The unprojection indeterminates T_0 and T_1 are such that

$$\deg(T_0) = a_0 + a_1 + a_2 - 1 = r \text{ and } \deg(T_1) = 2a_0 + a_1 + a_2 - 1 = r + a_0.$$

It is also the case that the Hilbert series of Y' equals our desired numerical candidate:

Lemma 4.2.2. The Hilbert series of X and Y' are related by the equality

$$P_{Y',\mathcal{O}_{Y'}(1)}(t) = P_{X,\mathcal{O}_X(1)}(t) + \frac{t^r + t^{r+a_0}}{(1 - t^{2a_0})(1 - t^{a_1})(1 - t^{a_2})(1 - t^r)}.$$

Proof. We follow the proof of Theorem 3.1 from [6]. Let $R := \bigoplus_{k \in \mathbb{N}} R_k$ be the homogeneous coordinate ring of X where R_k is the additive group generated by degree k elements in $\mathbb{C}[x,y,z,u,v,w]$. Define $S := \bigoplus_{k\in\mathbb{N}} S_k$ to be the homogeneous coordinate ring of Y' where S_k is the additive group generated by degree k elements in $\mathbb{C}[x,y,z,u,v,w,T_0,T_1]$. The generators of S_k which occur in R_k contribute $P_{X,\mathcal{O}_X(1)}(t)$ to the Hilbert series of Y'.

Outside of R, we may ignore any monomials lying in $\langle T_0u, T_0v, T_0w, T_1u, T_1v, T_1w, T_1^2\rangle$ since the equations of Y' are of the form $T_0u+\dots$, $T_0v+\dots$, $T_0w+\dots$, $T_1u+\dots$, $T_1v+\dots$, $T_1w+\dots$ and $T_1^2+\dots$. That is, the remaining generators of S_k are of the form T_0h_1 or $T_0T_1h_2$ where $h_1,h_2\in\mathbb{C}[x,y,z,T_0]$. The generators of the form T_0h_1 contribute

$$\frac{t^r}{(1-t^{2a_0})(1-t^{a_1})(1-t^{a_2})(1-t^r)}$$

to the Hilbert series of Y' and the generators of the form $T_0T_1h_2$ contribute

$$\frac{t^{r+a_0}}{(1-t^{2a_0})(1-t^{a_1})(1-t^{a_2})(1-t^r)}.$$

$\square$

By choice of X and D, we also have:

Lemma 4.2.3. $\omega_{Y'} = \mathcal{O}_{Y'}(-1)$.

This statement is proven by calculating the adjunction number using the minimal free resolution of Y' or the Hilbert series $P_{Y',\mathcal{O}_{Y'}(1)}$. As $Y'\subset\mathbb{P}(2a_0,a_1,\dots,a_5,r,r+a_0)$ is well formed, we have

$$\omega_{Y'}\cong\mathcal{O}_{Y'}\left(k_{Y'}-3a_0-2r-\sum_{i=1}^{5}a_i\right)$$

for $k_{Y'}$ the adjunction number. Given our choice of X and D, we know that

$$k_{Y'}=\deg(T_0)-\deg(y)-\deg(z)+\frac{1}{2}\deg(x)=r-a_0-a_1-a_2=-1$$

(see Section 3.2 or Lemma 4.2.2).

To prove Theorem 4.2.1, we would choose a specific X with at worst terminal singularities which is quasismooth off D and such that the singular locus is a set of finitely many nodes. In this case, the unprojection map is a morphism between X and its type II_1 unprojection.

Let $\{1,s_0,s_1\}$ be the set of generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D,\mathcal{O}_X)$ as in Section 2.3 or Section 2.3 of [31] (depending on whether we're performing a type ${\mathrm{II}_1}^{(2,1)}$ or type

$\mathrm{II}_1{}^{(3,0)}$ unprojection). Let d_k be the standard minimal basis defining D. Then, the unprojection map

$$\psi : X \dashrightarrow Y'$$
$$(x,y,z,u,v,w) \mapsto \left(x,y,z,u,v,w,\frac{s_0(d_k)}{d_k},\frac{s_1(d_k)}{d_k}\right)$$

factorises as

$$\begin{tikzcd} & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\psi"] & & Y' \end{tikzcd}$$

where $\sigma : Z \to X$ is the blow up of X along $E := \{d_1 = s_0(d_1) = s_1(d_1) = 0\} \subset X$ and $\pi : Z \to Y'$ is the blow down of $\sigma^{-1}(D)$. In our constructions it will always be the case that σ blows up the singular locus of X to rational curves. As the singular locus consists of finitely many nodes, the nodes are blown up to finitely many rational curves. These curves are then mapped to a bouquet of rational curves on p_{T_0} by π.

Under the unprojection map, X and Y' are isomorphic away from D and a bouquet of rational curves through p_{T_0}. The existence of this isomorphism will allow us to prove that Y' is quasismooth and has only terminal $\mathbb{Q}$-factorial singularities by checking X (see Proposition 4.3.4, Corollary 4.3.2, Lemma 4.3.6 and Proposition 4.3.5 for an example). This completes our realisation of a numerical candidate as a Fano 3-fold.

Remark 4.2.4. Since Y' is now known to be a Fano 3-fold, the Hilbert series $P_{Y',\mathcal{O}_{Y'}(1)}$ is the Hilbert series $P_{Y',-K_{Y'}}$. Although perhaps unnecessary at this point, the genus $g_{Y'}$ and basket $\mathcal{B}_{Y'}$ can be read from the Hilbert series (see proof of Theorem 3.1, [6]).

Remark 4.2.5. Note that Table A.1 does not provide the specific defining equations of X since the general X containing D is sufficient. Theorem 4.2.1 and Corollary 4.2.1 can be stated in general terms. Let $Y \subset \mathbb{P}(2a_0, a_1, \ldots, a_5, r, r + a_0)$ be a numerical candidate for a codimension 4 Fano 3-fold marked with a type II_1 but no type I centre. Then:

1. The numerical candidate Y may be realised as a Fano 3-fold. This Fano 3-fold is constructed as the type II_1 unprojection of a codimension 2 complete intersection $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_5)$ containing D defined in the usual type II_1 manner.

2. There are at least 2 formats of (X, D) for which the general X containing D is quasismooth off D, the singular locus of X consists of finitely many nodes

and the unprojection is a quasismooth Fano 3-fold.

3. In different formats of (X, D) the initial X have different numbers of nodes on D; therefore, the unprojected varieties have different Betti numbers.

(Compare with statement of Theorem 3.2 [10]). Although this result is stated in general terms, it is most quickly proven using computer algebra. In Appendices B.2 and B.3, we provide the Magma code necessary to construct the general X containing D and the resulting unprojections.

To prove Corollary 4.2.1, we note that in different unprojection formats the initial X have different numbers of nodes on D (again, this follows the proof of Theorem 3.2 [10]). Following Section 2.3 of [7], the diagram which factorises the unprojection can be extended with a degeneration, $\widehat{X} \rightsquigarrow X$, from a Fano 3-fold $\widehat{X}$. We have the following arrangement of varieties

$$\begin{tikzcd} & & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ \widehat{X} \arrow[r, squiggly] & X \arrow[rr, dashed, "\psi"] & & Y' \end{tikzcd}$$

with the conifold transition $\widehat{X}$ to Z shrinking the vanishing cycles of $\hat{X}$ to nodes and then resolving the nodes as rational curves. In this scenario, the Euler characteristic of Y' can be calculated as

$$e(Y') = e(\widehat{X}) + 2N - 2$$

where N is the number of nodes on X (see Section 5 of [36] and Section 2.3 of [7]). If X is defined by a degree i and a degree j equation, $\widehat{X}$ is a Fano 3-fold in the same ambient space defined by equations of the same degree. Hence, for fixed numerical candidate the Euler characteristic of $\widehat{X}$ will be the same in the type ${\mathrm{II}_1}^{(2,1)}$ unprojection case as it will be in the type ${\mathrm{II}_1}^{(3,0)}$ unprojection case. The proof of Corollary 4.2.1 reduces to checking that the initial data for the type ${\mathrm{II}_1}^{(2,1)}$ and type ${\mathrm{II}_1}^{(3,0)}$ unprojections have different numbers of nodes.

4.3 Numerical Candidate # 38

Consider the numerical candidate

$$P_Y(t) := \frac{N_Y}{\prod_{i=2}^{9}(1 - t^i)}$$

where

$$N_Y := 1 - 2t^{12} - t^{13} - 2t^{14} - 2t^{15} - t^{16} + 2t^{19} + 2t^{20} + 3t^{21}$$
$$+ 3t^{22} + 2t^{23} + 2t^{24} - t^{27} - 2t^{28} - 2t^{29} - t^{30} - 2t^{31} + t^{43}.$$

This candidate is obtained from the data

$$g_Y := -2 \text{ and } \mathcal{B}_Y := \left\{7 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{8}(1,3,5)\right\}$$

(see numerical candidate # 38 on the GRDB [8]) Using the standard analysis, we expect to construct this numerical candidate as a codimension 4 Fano 3-fold $Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$ with numerical data $\{g_Y, \mathcal{B}_Y\}$. The singularity $\frac{1}{8}(1,3,5)$ is a type II_1 centre.

We prove Theorem 4.2.1 for this numerical candidate and, as a consequence, we prove the following corollary:

Corollary 4.3.1. There are at least 2 families of codimension 4 Fano 3-folds

$$Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$$

with $|-K_Y| = \emptyset$, equivalently with genus -2.

We indulge in proving Theorem 4.2.1 for this special numerical candidate rather than another candidate due to its empty anticanonical linear system; Y has no elephant. An *elephant* of a Fano 3-fold Y is a K3 surface $S \in |-K_Y|$ with at worst Du Val singularities. Many properties of Y are shared by S since the elephant is defined by the graded ring $R(Y, -K_Y)/\langle f \rangle$ where $f \in H^0(Y, -K_Y)$ is such that $S = \{f = 0\} \subset Y$. It is often easier to study the elephant than the Fano 3-fold itself and hence Fano 3-folds with empty anticanonical linear systems are in practice very hard to construct. They are also very rare: out of around 50000 numerical candidates, approximately 250 have empty anticanonical linear systems and very few cases are known explicitly (see Example 16.1, [21]; and Section 5.1, [33]).

4.3.1 Type $\mathrm{II}_1^{(2,1)}$ Construction

The first family of $Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$ we construct will be via type ${\mathrm{II}_1}^{(2,1)}$ unprojections.

Define $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)_{\langle x,y,z,u,v,w \rangle}$ by the degree 12 and degree 14 polynomials

$$f_{12} := (v(x^3+y^2) - uw) + (v^2 - xu^2) + z^3$$

and

$$f_{14} := x(v(x^3+y^2) - uw) + x(v^2 - xu^2) + (w^2 - x(x^3+y^2)^2) + z(x^5 + u^2 + zv).$$

Define $D \subset \mathbb{P}(2,3,4,5,6,7)$ by $z = 0$ together with the 2×2 minors of

$$\begin{pmatrix} v & w & xu & x^4 + xy^2 \\ u & x^3 + y^2 & v & w \end{pmatrix}.$$

It is clear that $D \subset X$ and (X, D) is in type ${\mathrm{II}_1}^{(2,1)}$ format. We claim that:

Proposition 4.3.1. The type II_1 unprojection of (X, D) is a quasismooth codimension 4 Fano 3-fold in $\mathbb{P}(2,3,4,5,6,7,8,9)$ with numerical data

$$g = -2 \text{ and } \mathcal{B} = \left\{ 7 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{8}(1,3,5) \right\}.$$

Before performing our type II_1 unprojection, we pointedly study X and D.

Lemma 4.3.1. The basket of X is

$$\mathcal{B}_X := \left\{ \frac{1}{5}(1,2,3), 2 \times \frac{1}{3}(1,1,2), 7 \times \frac{1}{2}(1,1,1) \right\}.$$

The quotient singularities of X are calculated as in Section 10.3 of [21]. In particular, the points p_u, p_y and $(0, i, 0, 0, 1, 0)$ together with

$$\{(1, 0, z, 0, v, 0) : z^3 + v^2 + v = z^2 v + v^2 + z + v - 1 = 0\}$$

are the cyclic quotient singularities and the variety X has at worst terminal singularities.

Lemma 4.3.2. We have that $\mathrm{Sing}(X) \subset \{z = 0\}$, or equivalently $\langle z \rangle \subset \sqrt{I_{\mathrm{Sing}(X)}}$. In particular, we have $z^{10} \in I_{\mathrm{Sing}(X)}$.

Lemma 4.3.3. We have that $\mathrm{Sing}(X) \subset D$.

Proof. We will study $p \in \mathrm{Sing}(X)$ through a series of case by case analyses and conclude that $p \in D$.

Suppose $p := (x, y, z, u, v, w) \in \mathrm{Sing}(X)$. By Lemma 4.3.2, we may assume that $p \in \{z = 0\}$. Let $J_X(p)$ be the Jacobian matrix of X evaluated at p. Then,

$$J_X(p)^T = \begin{pmatrix} 3x^2v - u^2 & -(7x^3 + y^2)(x^3 + y^2) + 4x^3v - 2xu^2 + y^2v + v^2 - uw \\ 2yv & -4x^4y - 4xy^3 + 2xyv \\ 0 & x^5 + u^2 \\ -2xu - w & -2x^2u - xw \\ x^3 + y^2 + 2v & x^4 + xy^2 + 2xv \\ -u & -xu + 2w \end{pmatrix}.$$

A singularity of X is defined by the 2×2 minors of $J_X(p)$. Equivalently, a singularity is defined by the 2×2 minors of

$$M := \begin{pmatrix} 3x^2v - u^2 & -7x^6 - 8x^3y^2 - y^4 + x^3v - xu^2 + y^2v + v^2 - uw \\ 2yv & -4x^4y - 4xy^3 \\ 0 & x^5 + u^2 \\ -2xu - w & 0 \\ x^3 + y^2 + 2v & 0 \\ -u & 2w \end{pmatrix}$$

where we have subtracted x copies of column 1 from column 2 in $J_X(p)^T$. Let M_{ij} be the ij-th 2×2 minor of M. Every 2×2 minor of M must vanish on p by definition of p being a singular point. In particular, we must have that the 2×2 minor

$$M_{36} = u(u^2 + x^5)$$

vanishes on p; hence $u = 0$ or $\pm u = -x = 1$.

- Suppose that $u = 0$. Then, $w = 0$ by the vanishing of $M_{46} = -2w^2$ on p. Additionally, $M_{23} = 2x^5yv$ must vanish on p and so $x = 0$, $y = 0$ or $v = 0$.
  - Suppose that $x = 0$. Then, on $p = (0, y, 0, 0, v, 0)$ we have that

    $$2y^5v = -2vy(-y^4 + f_{12}(p)) = -2vy(-y^4 + y^2v + v^2) = M_{12} = 0$$

    and either $v = 0$ or $y = 0$. When combined with the fact that $M_{15} = 0$ on p, we obtain the contradiction that $p = 0$.
  - Suppose that $x \neq 0$ and $y = 0$. Without loss of generality, let $x = 1$. Then, M_{35} evaluated at p implies that $v = -\frac{1}{2}$ whilst f_{12} evaluated at p implies that $v = 0$ or $v = -1$. We have a contradiction.

- Suppose that $x, y \neq 0$ and $v = 0$. Without loss of generality, let $x = -1$. Then, on $p = (-1, y, 0, 0, 0, 0)$ we have that $M_{35} = y^2 - 1 = 0$. Hence, $y = \pm 1$ and $p \in D$.

- Suppose that $u = -x = 1$. Evaluated on p, the minor $M_{46} = 2w(2 - w)$ must vanish and hence $w = 0$ or $w = 2$.
  - Suppose $w = 0$. Then, $M_{26} = 4y(y^2 - 1) = 0$ on p.
    - Suppose $y = 0$. Then, on p we have that $f_{12} = v^2 - v + 1 = 0$ and $M_{16} = v^2 - v - 6 = 0$. This system of polynomials is insoluble.
    - Suppose $y = \pm 1$. Then, $f_{12} = v^2 + 1$ on p and hence $p \in \{(-1, 1, 0, 1, \pm i, 0), (-1, -1, 0, 1, \pm i, 0)\} \subset D$.
  - Suppose $w = 2$. Then, on p we have $M_{56} = 4(2v - 1 + y^2) = 0$ and hence $v = \frac{1-y^2}{2}$. We obtain $y^4 - 2y^2 + 5 = -4f_{12}(p) = 0$ and $p \in \{(-1, \pm\sqrt{1 + 2i}, 0, 1, -i, 2), (-1, \pm\sqrt{1 - 2i}, 0, 1, i, 2)\} \subset D$.

- Suppose that $u = x = -1$. Evaluated on p, the minor $M_{46} = -2w(2 + w)$ must vanish and hence $w = 0$ or $w = -2$.
  - Suppose $w = 0$. Then, $M_{26} = -4y(y^2 - 1) = 0$ on p.
    - Suppose $y = 0$. Then, on p we have that $f_{12} = v^2 - v + 1 = 0$ and $M_{16} = -(v^2 - v - 6) = 0$. This system of polynomials is insoluble.
    - Suppose $y = \pm 1$. Then, $f_{12} = v^2 + 1$ on p and hence $p \in \{(-1, 1, 0, -1, \pm i, 0), (-1, -1, 0, -1, \pm i, 0)\} \subset D$.
  - Suppose $w = -2$. Then, on p we have $M_{56} = -4(2v - 1 + y^2) = 0$ and hence $v = \frac{1-y^2}{2}$. We obtain $y^4 - 2y^2 + 5 = -4f_{12}(p) = 0$ and $p \in \{(-1, \pm\sqrt{1 + 2i}, 0, -1, -i, -2), (-1, \pm\sqrt{1 - 2i}, 0, -1, i, -2)\} \subset D$.

□

That is, X is quasismooth off D. The fact that X is quasismooth off D is not surprising. When applying Bertini's theorem, we see that the general X containing D is quasismooth off

$$D \cup \{p_z\} \cup \{v = u = z = 0\} \cup \{z = x = w = 0\}.$$

By specifying that f_{12} and f_{14} contain certain terms, we maintain the generality of X containing D whilst adjusting quasismoothness to occur off D only. For example, the general X is quasismooth on p_z since we may choose $f_{12} = z^3 + \ldots$ and we specify that f_{12} contains a z^3 term.

Many of the points highlighted in the proof of Lemma 4.3.3 are equal when we take into account the group action defining $\mathbb{P}(2,3,4,5,6,7)$. The singular locus of X is

$$\{(-1,1,0,0,0,0),(-1,-1,0,1,\pm i,0),(-1,1,0,1,\pm i,0),\\ (-1,\pm\sqrt{1+2i},0,1,-i,2),(-1,\pm\sqrt{1-2i},0,1,i,2)\}$$

which consists of exactly 9 distinct points. Moreover:

Lemma 4.3.4. The singular locus of X consists of 9 nodes.

To prove this result we note that it is sufficient to work on D when studying $\mathrm{Sing}(X)$ since X is quasismooth off D. In particular, we will work on $\mathrm{im}(\phi)$ where

$$\phi : \mathbb{P}(1,3,5) \to \mathbb{P}(2,3,4,5,6,7)$$

$$\phi(a,b,c) := (a^2, b, 0, c, ac, a(a^6+b^2)).$$

Recall that in Example 2.7.1, we proved $D = \mathrm{im}(\phi)$.

Proof. Let J be the Jacobian ideal of X and $I \subset \mathbb{C}[a,b,c]$ its pullback using ϕ. By Lemma 4.3.3, there are exactly 9 singularities of X and $\mathrm{Sing}(X) \subset \{x \neq 0\}$. Therefore, the dimension of $(\mathbb{C}[x,y,z,u,v,w]/J)_{\langle x\rangle}$ as a $\mathbb{C}$-algebra is at least 9 with equality if and only if each singularity is a node. Since X is quasismooth off $D = \mathrm{im}(\phi)$, the singularities of X are nodes if and only if

$$(\mathbb{C}[a,b,c]/I)_{\langle a\rangle}$$

is 10-dimensional as a $\mathbb{C}$-algebra. The increase in dimension from 9 to 10 is because the fiber $\phi^{-1}(p)$ is a unique point everywhere except $p = (-1,1,0,0,0,0) \in \mathrm{Sing}(X)$ where p has two pre-images under ϕ.

The equations of $I_{\langle a\rangle}$ will be the 2×2 minors of the Jacobian matrix of X evaluated at $p := \phi(1,b,c)$, $J_X(p)$. We have that

$$J_X(p)^T = \begin{pmatrix} -c^2+3c & -b^4-8b^2-c^2+3c-7 \\ 2bc & -4b^3+2bc-4b \\ 0 & c^2+1 \\ -b^2-2c-1 & -b^2-2c-1 \\ b^2+2c+1 & b^2+2c+1 \\ -c & 2b^2-c+2 \end{pmatrix}.$$

The 2×2 minors of this matrix are clearly the 2×2 minors of

$$\begin{pmatrix} -c^2+3c & -b^4-8b^2-c^2+3c-7 \\ 0 & c^2+1 \\ b^2+2c+1 & b^2+2c+1 \\ -c & 2b^2-c+2 \end{pmatrix}$$

and hence $I_{\langle a \rangle}$ is defined by

$$\begin{aligned} 2b^4+4b^2c+4b^2+4c+2, c^3+c, -b^2c^2-2c^3-b^2-c^2 \\ -2c-1, -b^4c-2b^2c^2-2b^2c-2c^2-c, b^6+2b^4c+ \\ 9b^4+16b^2c+15b^2+14c+7, -c^4+3c^3-c^2+3c. \end{aligned}$$

After eliminating unnecessary equations, $I_{\langle a \rangle}$ is defined by

$$b^4+2b^2c+2b^2+2c+1, c^3+c, b^2c^2+c^2+b^2+1.$$

Hence

$$(\mathbb{C}[a,b,c]/I)_{\langle a \rangle} \cong R := \frac{\mathbb{C}[b,c]}{\langle b^4+2b^2c+2b^2+2c+1, c^3+c, b^2c^2+c^2+b^2+1 \rangle}.$$

We claim that R is 10-dimensional since $S := \{1, b, b^2, b^3, c, bc, b^2c, b^3c, c^2, bc^2\}$ is a basis of R as a $\mathbb{C}$-algebra. As linear independence is clear, we need only show that S generates R.

It is sufficient to check that S generates b^m, b^mc and b^mc^2 for $m \in \mathbb{N}$ since we have that $c^3 = -c$ on R and thus $c^k \in \{\pm c, \pm c^2\}$ for all $k \in \mathbb{N}^+$. The set S generates b^m, b^mc and b^mc^2 for $m = 1, 2, 3, 4$ because

$$\begin{aligned} b^2c^2 &= -(c^2+b^2+1), & b^3c^2 &= b(b^2c^2) = -(bc^2+b^3+b), \\ b^4 &= -(2b^2c+2b^2+2c+1), & b^4c &= -(2b^2c^2+2b^2c+2c^2+c), \end{aligned}$$

$$b^4c^2 = b^2(b^2c^2) = -b^2c^2-b^4-b^2$$

and in the remaining cases the monomial is itself an element of S.

Suppose that b^m, b^mc and b^mc^2 can be generated by S for a fixed $m \in \mathbb{N}_{\geq 4}$. Then, $b^{m+1}, b^{m+1}c, b^{m+1}c^2$ can be also be generated by S since

$$b^{m+1} = b^{m-3}b^4 = -(2b^{m-1}c+2b^{m-1}+2b^{m-3}c+b^{m-3}),$$

$$b^{m+1}c = b^{m-3}b^4c = -(2b^{m-1}c^2+2b^{m-1}c+2b^{m-3}c^2+b^{m-3}c)$$

and

$$b^{m+1}c^2 = b^{m-3}b^4c^2 = b^{m-3}(-b^2c^2 - b^4 - b^2) = -b^{m-1}c^2 - b^{m+1} - b^{m-2}.$$

By induction on $m \geq 4$, we have that $b^m, b^m c$ and $b^m c^2$ are generated by S for all $m \in \mathbb{N}^+$. Hence, the dimension of R is 10 as required. $\square$

We now study the unprojection. Let Y' be the type $\mathrm{II_1}^{(2,1)}$ unprojection of (X, D) as defined using the explicit equations of Section 3.1.2. We have that:

Proposition 4.3.2. $Y' \subset \mathbb{P}(2,3,4,5,6,7,8,9)_{\langle x,y,z,u,v,w,T_0,T_1\rangle}$.

Furthermore, Section 4.2.2 tells us that:

Proposition 4.3.3. The Hilbert series of Y', $P_{Y',\mathcal{O}_{Y'}(1)}(t)$, is equal to the numerical candidate.

It remains to check that Y' is a quasismooth Fano 3-fold; however, we may work on X instead of Y'. The unprojection map $\psi : X \dashrightarrow Y'$ is a birational map defined by

$$\psi : X \dashrightarrow Y'$$

$$\psi(x,y,z,u,v,w) = \left(x,y,z,u,v,w,\frac{s_0(d_k)}{d_k},\frac{s_1(d_k)}{d_k}\right)$$

for $k = 1,\dots,5$ where $\{1, s_0, s_1\}$ are generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ (see Section 2.3) and

$$d_1 := z, \quad d_2 := v(x^3+y^2) - uw, \quad d_3 := v^2 - xu^2,$$

$$d_4 := vw - xu(x^3+y^2), \quad d_5 := w^2 - x(x^3+y^2)^2$$

is a basis for I_D. In Section 3.2.2 the map ψ was shown to factorise as two blow ups

$$\begin{tikzcd} & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\psi"] & & Y' \end{tikzcd}$$

where $\sigma : Z \to X$ is the blow up of X along $E = \{z = s_0(z) = s_1(z) = 0\} \subset X$ and $\pi : Z \to Y'$ is the contraction of $\sigma^{-1}(D)$. Recall that σ blows up the points in $D \cap \{s_0(d_1) = \cdots = s_0(d_5) = 0\}$ to rational curves (see Proposition 3.3.3). In our scenario, only the nodes are blown up to rational curves and hence the map σ is a projective small resolution of nodes which resolves the nodes as rational curves.

Lemma 4.3.5. Let $S_0 := \{s_0(d_1) = \cdots = s_0(d_5) = 0\} \subset X$. Then, we have that $\mathrm{Sing}(X) = D \cap S_0$.

Proof. Using the expressions for $s_0(d_k)$ given in Section 3.3.2, we have that

$$s_0(d_1) = (x^3 + y^2)^2 + (x^3 + y^2)v + wu$$

$$s_0(d_2) = -u^2(z^2x - x^5 - u^2 - zv) - (x^3 + y^2)^2 z^2$$

$$s_0(d_3) = u^2(z^2x - x^5 - u^2 - zv) - z^2(uw + v(x^3 + y^2))$$

$$s_0(d_4) = (u(x^3 + y^2) + vu)(z^2x - x^5 - u^2 - zv) - z^2w(x^3 + y^2)$$

$$s_0(d_5) = ((x^3 + y^2)^2 + (x^3 + y^2)v + uw)(z^2x - x^5 - u^2 - zv).$$

Since $D = \operatorname{im}(\phi)$ and $\operatorname{Sing}(X) \subset D$, we work on $p := \phi(a, b, c)$ with $(a, b, c) \in \mathbb{P}(1, 3, 5)$. Suppose p is such that $c = 0$. Then, $s_0(d_2), s_0(d_3)$ and $s_0(d_4)$ vanish on p and $s_0(d_1)$ and $s_0(d_5)$ become

$$a^{12} + 2a^6b^2 + b^4 \quad \text{and} \quad -a^{10}(a^{12} + 2a^6b^2 + b^4)$$

on p respectively. By the explicit singular locus calculated in Lemma 4.3.3, it is clear that $p \in S_0 \cap D$ if and only if $p \in \operatorname{Sing}(X)$.

Suppose p is such that $c \neq 0$. To check that $p \in D \cap S_0$, it is sufficient to check that p vanishes on $s_0(d_1)$ and $s_0(d_2)$ because

$$s_0(d_2) = s_0(d_3),$$

$$cs_0(d_4) = -(ac + (a^6 + b^2))s_0(d_2)$$

and

$$s_0(d_5) = s_0(d_1)(-a^{10} - c^2).$$

If p vanishes on $s_0(d_1)$ and $s_0(d_2)$ we must have

$$(a^6 + b^2)(a^6 + b^2 + 2ac) = 0$$

and

$$c^2(a^{10} + c^2) = 0.$$

Clearly,

$$\begin{aligned}\phi(\{c \neq 0\}) \cap D \cap S_0 = \{&\phi(i, \pm 1, 1), \phi(i, \pm 1, -1),\\ &\phi(i, \pm\sqrt{(1 - 2i)}, 1), \phi(i, \pm\sqrt{(1 + 2i)}, -1)\} = \operatorname{Sing}(X) \cap \phi(\{c \neq 0\}).\end{aligned}$$

□

Quasismoothness of Y' can then be shown by the quasismoothness of X off D:

Proposition 4.3.4. The variety Y' is quasismooth.

Proof. Let $\Gamma := \bigcup_{p \in \mathrm{Sing}(X)} \sigma^{-1}(p)$. Then, since

$$Y' - \{p_{T_0}\} \cong Z - \sigma^{-1}(D) \subset Z - \Gamma \cong X - \mathrm{Sing(X)},$$

Y' is quasismooth off p_{T_0}. The variety Y' is also quasismooth at p_{T_0} since the Jacobian matrix of Y' at p_{T_0} has the non-zero 4×4 minor

$$\begin{vmatrix} \frac{\partial l_4}{\partial v} & \frac{\partial l_4}{\partial w} & \frac{\partial l_4}{\partial z} & \frac{\partial l_4}{\partial x} \\ \frac{\partial l_5}{\partial v} & \frac{\partial l_5}{\partial w} & \frac{\partial l_5}{\partial z} & \frac{\partial l_5}{\partial x} \\ \frac{\partial l_6}{\partial v} & \frac{\partial l_6}{\partial w} & \frac{\partial l_6}{\partial z} & \frac{\partial l_6}{\partial x} \\ \frac{\partial q}{\partial v} & \frac{\partial q}{\partial w} & \frac{\partial q}{\partial z} & \frac{\partial q}{\partial x} \end{vmatrix}.$$

□

Since Y' is quasismooth it immediately follows that:

Corollary 4.3.2. The singularities of Y' are $\mathbb{Q}$-factorial.

Similarly, Y' can be shown to have at worst terminal singularities because X has at worst terminal singularities:

Lemma 4.3.6. The variety Y' consists of only terminal singularities.

Proof. We have that Z consists of only terminal singularities since Γ is smooth and $X - \mathrm{Sing}(X) \cong Z - \Gamma$ has only terminal singularities. Therefore, $Y' - \{p_{T_0}\} \cong Z - \sigma^{-1}(D)$ has only terminal singularities. All that remains to check is that p_{T_0} is terminal, but this is clear since

$$(p_{T_0} \in Y') \cong \left(0 \in \frac{1}{8}(1,3,5)\right).$$

□

We can now prove that Y' is a Fano 3-fold and hence Y' realises our desired numerical candidate.

Proposition 4.3.5. The variety Y' is a Fano 3-fold.

Proof. As Y' has terminal $\mathbb{Q}$-factorial singularities, we need only show that the dualizing sheaf of Y' is such that $\omega_{Y'} = \mathcal{O}_{Y'}(-1)$. As Y' is well formed, the dualizing sheaf of Y' is

$$\omega_{Y'} = \omega_{\mathbb{P}^7} \otimes \mathcal{O}_{Y'}(k) = \mathcal{O}_{Y'}\left(k - \sum_{i=2}^{9} i\right)$$

where k is the adjunction number. By Section 3.2, $k = 43$ and we have our desired result. □

Remark 4.3.1. The numerical data of Y' matches the data of our numerical candidate. The Hilbert series $P_{Y',\mathcal{O}_{Y'}(1)}$ is also equal to $P_{Y',-K_{Y'}}$ since Y' is a Fano 3-fold.

4.3.2 Type $\mathrm{II}_1^{(3,0)}$ Construction

The second family of Fano 3-folds $Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$ is constructed using type $\mathrm{II_1}^{(3,0)}$ unprojections. A construction of this kind which uses different initial data can be found in Section 5.1 of [33] and Example 9.14 of [39]; however, we complete the unprojection and check that it is a Fano 3-fold.

Define $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)_{\langle x,y,z,u,v,w\rangle}$ by the degree 12 and 14 polynomials

$$f_{12} := y(yv - zu) + x(yw - z(x^3 + y^2)) + z(z^2 - xy^2) + (v^2 - xu^2)$$

and

$$\begin{aligned} f_{14} := u(yv - zu) &+ z(yw - z(x^3 + y^2)) \\ &+ (v + y^2)(z^2 - xy^2) + (w^2 - x(x^3 + y^2)^2) + x(uw - v(x^3 + y^2)). \end{aligned}$$

Define $D \subset X$ by the 2×2 minors of

$$M := \begin{pmatrix} z & v & w & xy & xu & x^4 + xy^2 \\ y & u & x^3 + y^2 & z & v & w \end{pmatrix}.$$

For ease of notation later on in this section, we define $\{d_1, \ldots, d_9\}$, a basis of I_D, where

$$\begin{aligned} &d_1 := zu - yv, \quad &&d_2 := z(x^3 + y^2) - yw, \quad &&d_3 := v(x^3 + y^2) - uw, \\ &d_4 := z^2 - xy^2, &&d_5 := zv - xyu, &&d_6 := zw - xy(x^3 + y^2), \\ &d_7 := v^2 - xu^2, &&d_8 := vw - xu(x^3 + y^2), &&d_9 := w^2 - x(x^3 + y^2)^2. \end{aligned}$$

We claim that:

Proposition 4.3.6. The type II_1 unprojection of (X, D) is a quasismooth codimension 4 Fano 3-fold in $\mathbb{P}(2,3,4,5,6,7,8,9)$ with numerical data $g = -2$ and $\mathcal{B} = \{7 \times \frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{8}(1,3,5)\}$.

As in Section 4.3.1:

Lemma 4.3.7. The variety X has genus $g_X = -2$ and basket of terminal cyclic quotient singularities $\mathcal{B}_X = \{\frac{1}{5}(1,2,3), 2 \times \frac{1}{3}(1,1,2), 7 \times \frac{1}{2}(1,1,1)\}$.

Lemma 4.3.8. We have that X is quasismooth off D

However:

Lemma 4.3.9. The singular locus of X consists of exactly 10 points.

To prove this result, we recall Example 2.7.2 where we showed that $D = \mathrm{im}(\phi)$ for

$$\phi : \mathbb{P}(1,3,5) \to \mathbb{P}(2,3,4,5,6,7)$$

$$\phi(a,b,c) := (x := a^2, y := b, z := ab, u := c, v := ac, w := a^7 + ab^2).$$

Proof. Since X is quasismooth off $D = \mathrm{im}(\phi)$, it is sufficient to work on $(a,b,c) \in \mathbb{P}(1,3,5)$ when investigating the singular locus of X.

Let $p = \phi(a,b,c)$ be a singularity of X. The Jacobian matrix of X evaluated at p, $J_X(p)$, is such that

$$J_X(p)^T = \begin{pmatrix} -3a^7b - ab^3 - c^2 & -7a^{12} - 11a^6b^2 - 2b^4 - 3a^7c - ab^2c \\ a^9 - 3a^3b^2 + abc & -3a^8b - 7a^2b^3 - 4a^3bc + ac^2 \\ -a^8 + a^2b^2 - bc & -a^7b + ab^3 + 2a^2bc - c^2 \\ -ab^2 - 2a^2c & a^9 + a^3b^2 - abc \\ b^2 + 2ac & -a^8 - a^2b^2 + bc \\ a^2b & 2a^7 + 3ab^2 + a^2c \end{pmatrix}.$$

If $p = \phi(0,b,c)$, we have that

$$J_X(p) = \begin{pmatrix} -c^2 & 0 & -bc & 0 & b^2 & 0 \\ -2b^4 & 0 & -c^2 & 0 & bc & 0 \end{pmatrix}.$$

It is clear from the 2×2 minors of $J_X(p)$ that p is a singularity of X if and only if $(0,b,c) = (0,1,\epsilon)$ for some $\epsilon^3 = 2$. Note that even though there are 3 distinct choices for ϵ, the group action of $\mathbb{P}(1,3,5)$ is such $\{(0,1,\epsilon) : \epsilon^3 = 2\}$ is a single point. That is, X has a single singularity on $\phi(\{a=0\})$.

Let $a \neq 0$ and without loss of generality set $a = 1$. Then, p is defined by the 2×2 minors of

$$J_X(p)^T = \begin{pmatrix} -b^3 - c^2 - 3b & -2b^4 - b^2c - 11b^2 - 3c - 7 \\ -3b^2 + bc + 1 & -7b^3 - 4bc + c^2 - 3b \\ b^2 - bc - 1 & b^3 + 2bc - c^2 - b \\ -b^2 - 2c & b^2 - bc + 1 \\ b^2 + 2c & -b^2 + bc - 1 \\ b & 3b^2 + c + 2 \end{pmatrix}.$$

Equivalently p is defined by the 2×2 minors of

$$\begin{pmatrix} -c^2 & b^4 - 1 \\ 1 & 2b^3 - 3b^2c - bc + 3b - 2c \\ -1 & -2b^3 + 3b^2c + bc - 3b + 2c \\ -2c & 3b^3 + b^2 + 2b + 1 \\ 2c & -3b^3 - b^2 - 2b - 1 \\ b & 3b^2 + c + 2 \end{pmatrix}$$

where, in the previous matrix, we have used row 6 to eliminate all b terms in column 1. Using row 2 to eliminate as many terms as possible from column 1, we have that p is defined by the 2×2 minors of

$$\begin{pmatrix} 0 & 2b^3c^2 - 3b^2c^3 + b^4 - bc^3 + 3bc^2 - 2c^3 - 1 \\ 1 & 2b^3 - 3b^2c - bc + 3b - 2c \\ 0 & 0 \\ 0 & 4b^3c - 6b^2c^2 + 3b^3 - 2bc^2 + b^2 + 6bc - 4c^2 + 2b + 1 \\ 0 & -4b^3c + 6b^2c^2 - 3b^3 + 2bc^2 - b^2 - 6bc + 4c^2 - 2b - 1 \\ 0 & -2b^4 + 3b^3c + b^2c + 2bc + c + 2 \end{pmatrix}.$$

That is, the singular locus of X on $\{a \neq 0\}$ is defined by

$$g_1 := -4b^3c + 6b^2c^2 - 3b^3 + 2bc^2 - b^2 - 6bc + 4c^2 - 2b - 1$$

and

$$g_2 := 2b^4 - 3b^3c - b^2c - 2bc - c - 2;$$

we ignore the minor $2b^3c^2 - 3b^2c^3 + b^4 - bc^3 + 3bc^2 - 2c^3 - 1$ since it is generated by

g_1 and g_2.

As $g_1 = g_2 = bc = 0$ is insoluble, it is clear that p is a singularity if and only if $b \neq 0$ and $c \neq 0$. It is less clear that p is a singularity if and only if $(1 + 2b + b^2 + 3b^3) \neq 0$; however, $1 + 2b + b^2 + 3b^3 = g_2 = 0$ is an insoluble system of polynomials in $\mathbb{C}[b]$. By rearranging g_2 we have that

$$c = \frac{2b^4 - 2}{1 + 2b + b^2 + 3b^3}$$

and by substituting this value of c into g_1 we have that

$$-27b^9 - 63b^8 - 83b^7 - 112b^6 - 80b^5 - 31b^4 - 9b^3 + 33b^2 + 14b + 15 = 0. \quad (4.1)$$

As (4.1) has 9 distinct solutions, we count 9 distinct singularities on $\phi(\{a \neq 0\})$.

□

We claim that these 10 singularities are in fact 10 nodes. The singularity $p = (0, 1, 0, \epsilon, 0, 0)$ where $\epsilon^3 = 2$ can easily be seen to be a node since it is locally the intersection of 2 lines: working locally on U, the neighbourhood of p where $y \neq 0$, the lines $L_1, L_2 \subset U$ defined by $x = z = 0$ and $v = u - \epsilon = 0$ are such that $\{p\} = L_1 \cap L_2$.

The remaining singularities can be shown to be nodes in one fell swoop:

Lemma 4.3.10. The singular locus of X consists of 10 nodes.

Proof. By Lemma 4.3.9, there are exactly 10 singularities in X: a singularity on $\{x = 0\}$ and 9 singularities on $\{x \neq 0\}$. The singularity of X on $\{x = 0\}$ is already known to be a node and therefore it is sufficient to check the remaining 9 singularities of X.

Let J be the Jacobian ideal of X and I its pullback using ϕ. Then, the dimension of $(\mathbb{C}[x, y, z, u, v, w]/J)_{\langle x \rangle}$ as a $\mathbb{C}$-algebra is at least 9 with equality if and only if the singularities are all nodes. Let

$$g_1 := -4b^3c + 6b^2c^2 - 3b^3 + 2bc^2 - b^2 - 6bc + 4c^2 - 2b - 1$$

and

$$g_2 := 2b^4 - 3b^3c - b^2c - 2bc - c - 2.$$

Then, equivalently the singularities of X are nodes if and only

$$R := (\mathbb{C}[a, b, c]/I)_{\langle a \rangle} \cong \frac{\mathbb{C}[b, c]}{\langle g_1, g_2 \rangle}$$

is 9-dimensional (see the proof of Lemma 4.3.9).

We claim that $S := \{1, b, c, b^2, bc, c^2, b^3, b^2c, bc^2\}$ generates R and hence R is 9-dimensional. Since it is clear that S is linearly independent, we need only check that S is a generating set. We claim that any monomial $b^m c^n$ where $m, n \in \mathbb{N}$ and $m + n \leq 4$ is generated by S. The set S contains all elements $b^m c^n$ with $m + n \leq 3$ except c^3; however, c^3 is generated by S as

$$\begin{aligned} 18c^3 = -17b^3 - 12b^2c + 30bc^2 - 7b^2 - 30bc - 4c^2 - 2b - 8c - 9+ \\ (6b^2 - 9bc - 2b - 9)g_1 + (12bc - 18c^2 + 9b - 4c)g_2. \end{aligned}$$

Similarly, we have $b^n c^m$ generated by S for $m + n = 4$ since

$$\begin{aligned} 54b^2c^2 = 19b^3 - 60b^2c - 18bc^2 - 7b^2 + 30bc - 52c^2+ \\ 10b - 44c - 15 - (8b - 9)g_1 - (16c + 12)g_2, \end{aligned}$$

$$9b^3c = -2b^3 - 15b^2c - 4b^2 - 6bc - 4c^2 - 2b - 11c - 6 - 2bg_1 - (4c + 3)g_2,$$

$$3b^4 = -b^3 - 6b^2c - 2b^2 - 2c^2 - b - 4c - bg_1 - 2cg_2,$$

$$\begin{aligned} 162bc^3 = 107b^3 - 84b^2c - 126bc^2 + 97b^2 + 150bc - 110c^2+ \\ 44b + 116c - 3 + (54b^3 - 81b^2c - 18b^2 - 46b + 45)g_1+ \\ (108b^2c - 162bc^2 + 81b^2 - 36bc + 70c - 24)g_2 \end{aligned}$$

and

$$\begin{aligned} 486c^4 = 523b^3 + 516b^2c - 1332bc^2 - 108c^3 + 731b^2+ \\ 822bc - 250c^2 + 262b + 1162c + 381 + (270b^3 - 243b^2c- \\ 243bc^2 - 90b^2 - 54bc - 80b - 243c + 171)g_1 + (540b^2c- \\ 486bc^2 - 486c^3 + 405b^2 + 63bc - 108c^2 + 650c + 105)g_2; \end{aligned}$$

we used the computer algebra software Magma to obtain these expressions (see [5]). Fix $k \in \mathbb{N}$ such that $k \geq 4$ and suppose that S generates $b^m c^n$ for all $m + n < k$. Then, as

$$486c^k = 486c^{k-4}c^4 = (523b^3c^{k-4} + 516b^2c^{k-3} - 1332bc^{k-2} - 108c^{k-1} + \\ 731b^2c^{k-4} + 822bc^{k-3} - 250c^{k-2} + 262bc^{k-4} + 1162c^{k-3} + 381c^{k-4}),$$

$$9b^{k-1}c = 9b^{k-4}b^3c = -2b^{k-1} - 15b^{k-2}c - 4b^{k-2} \\ - 6b^{k-3}c - 4b^{k-4}c^2 - 2b^{k-3} - 11b^{k-4}c - 6b^{k-4},$$

$$54b^{k-2}c^2 = 54b^{k-4}b^2c^2 = 19b^{k-1} - 60b^{k-2}c - 18b^{k-3}c^2 \\ - 7b^{k-2} + 30b^{k-3}c - 52b^{k-4}c^2 + 10b^{k-3} - 44b^{k-4}c - 15b^{k-4}$$

and

$$162b^{k-3}c^3 = 162b^{k-4}bc^3 = 107b^{k-1} - 84b^{k-2}c - 126b^{k-3}c^2 + 97b^{k-2} + \\ 150b^{k-3}c - 110b^{k-4}c^2 + 44b^{k-3} + 116b^{k-4}c - 3b^{k-4}.$$

If $m \geq 4$, then

$$3b^mc^{k-m} = 3b^{m-4}c^{k-m}b^4 = -b^{m-1}c^{k-m} - 6b^{m-2}c^{k-m+1} - \\ 2b^{m-2}c^{k-m} - 2b^{m-4}c^{k-m+2} - b^{m-3}c^{k-m} - 4b^{m-4}c^{k-m+1}.$$

Hence, we have that S generates b^mc^n for $m + n = k$. Our desired result follows by induction.

□

Let Y' be the type $\mathrm{II_1}^{(3,0)}$ unprojection of (X, D) defined by the explicit equations of [33]. Then:

Proposition 4.3.7. The unprojection Y' is such that $Y' \subset \mathbb{P}(2,3,4,5,6,7,8,9)_{\langle x,y,z,u,v,w,T_0,T_1\rangle}$.

Proposition 4.3.8. Let $P_{Y',\mathcal{O}_{Y'}(1)}(t)$ be the Hilbert series of Y' with respect to $\mathcal{O}_{Y'}(1)$. Then, $P_{Y'}(t)$ is equal to our numerical candidate.

Proposition 4.3.7 is proven immediately by definition of the type $\mathrm{II_1}^{(3,0)}$ unprojection; Proposition 4.3.8 is proven immediately by applying Lemma 4.2.2.

Let $\{1, s_0, s_1\}$ be the generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ as in Section 2.2 of [31] or equivalently as in Remark 3.3.5. The unprojection map

$$\psi : X \dashrightarrow Y$$
$$(x, y, z, u, v, w) \mapsto \left(x, y, z, u, v, w, \frac{s_0(d_k)}{d_k}, \frac{s_1(d_k)}{d_k}\right)$$

factorises as

$$\begin{tikzcd} & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\psi"] & & Y \end{tikzcd}$$

where $\sigma : Z \to X$ is the blow up of X along $E := \{d_1 = s_0(d_1) = s_1(d_1) = 0\} \subset X$ and $\pi : Z \to Y$ is the blow down of $\sigma^{-1}(D)$ (see Section 3.3). Define the subvariety $S_0 := \{s_0(d_1) = \cdots = s_0(d_9) = 0\} \subset X$. Under σ, the fibers of $p \in D \cap S_0$ are rational curves and the fibers of $p \notin D \cap S_0$ are a point. We claim that $D \cap S_0 = \mathrm{Sing}(X)$, i.e. σ blows up only the nodes of X.

We start with the more palatable inclusion:

Lemma 4.3.11. We have that $D \cap S_0 \subset \mathrm{Sing}(X)$.

Proof. As X is quasismooth off $D = \mathrm{im}(\phi)$, it is sufficient to only consider points $p := \phi(a, b, c) \in D \cap S_0$.

Let $a = 0$. It is straightforward to show that the equations $s_0(d_k)$ vanish identically on p for $k = 1, 4, 5, 6, 7, 8, 9$. For $s_0(d_2)$ and $s_0(d_3)$ to vanish on p, we must have

$$2b^6 - bc^3 = 0$$

and

$$2b^5c - c^4 = 0.$$

That is, $p = (0, 1, 0, \epsilon, 0, 0)$ where $\epsilon^3 = 0$ and p is a known singular point of X.

Let $a \neq 0$ and without loss of generality set $a = 1$. If $b = 0$, then the vanishing of $s_0(d_2)$ and $s_0(d_3)$ at p means that $c^2 + 2c = -c^4 + 1 = 0$. This is insoluble so no such p exists. Similarly if $c = 0$, then the vanishing of $s_0(d_2)$ and $s_0(d_3)$ at p means that $2b^6 + b^5 + 2b^4 + b^3 + b^2 = -b^6 - b^4 + b^2 + 1 = 0$. This is another insoluble set of polynomials so no such p exists. Hence, p is such that $b, c \neq 0$. Recall that p is a singular point of X if and only if

$$g_1 := -4b^3c + 6b^2c^2 - 3b^3 + 2bc^2 - b^2 - 6bc + 4c^2 - 2b - 1,$$

$$g_2 := 2b^4 - 3b^3c - b^2c - 2bc - c - 2$$

vanish on p (see Lemma 4.3.9). As

$$b^2 g_1 + 2bcg_2 = -s_0(d_4)(p)$$

and

$$cg_2 = -s_0(d_7)(p),$$

we must have $p \in \mathrm{Sing}(X)$. $\square$

The proof that every singularity of X lies in $D \cap S_0$ follows similarly:

Lemma 4.3.12. We have that $\mathrm{Sing}(X) \subset D \cap S_0$.

Proof. As X is quasismooth off $D = \mathrm{im}(\phi)$, it is sufficient to only consider points $p := \phi(a, b, c) \in \mathrm{Sing}(X)$. Suppose that $a = 0$. Then,

$$D \cap S_0 \cap \phi(\{a = 0\}) = \mathrm{Sing}(X) \cap \phi(\{a = 0\})$$

(see proof of Lemma 4.3.11). Suppose that $a \neq 0$ and without loss of generality set $a = 1$. The singularities of X are points $\phi(1, b, c)$ which satisfy

$$g_1 := -4b^3c + 6b^2c^2 - 3b^3 + 2bc^2 - b^2 - 6bc + 4c^2 - 2b - 1$$

and

$$g_2 := 2b^4 - 3b^3c - b^2c - 2bc - c - 2.$$

When evaluated on p, $s_0(d_k)$ for $k = 1, \ldots, 9$ simplify to

$$-\frac{1}{2}bcg_1 - \left(c^2 - \frac{1}{2}b\right) g_2,$$

$$b\left(-b^2 + \frac{1}{2}bc\right) g_1 + \left(-2b^2c + bc^2 - \frac{1}{2}b^2 - c\right) g_2,$$

$$\frac{1}{2}\left(c(-b^2 + bc + 1)g_1 + (-2bc^2 + 2c^3 - b^2 - bc - 1)g_2\right),$$

$$-b^2g_1 - 2bcg_2,$$

$$-\frac{1}{2}bcg_1 - \left(c^2 + \frac{1}{2}b\right) g_2,$$

$$-\frac{1}{2}\left(b(bc + 2)g_1 + (2bc^2 - b^2 + 2c)g_2\right),$$

$$-cg_2,$$

$$\frac{1}{2}\left(c(b^2-bc-1)g_1+(2bc^2-2c^3-b^2+bc-1)g_2\right)$$

and

$$-\frac{1}{2}\left((b^3c+b^2c+c)g_1+(2b^2c^2-b^3+2bc^2-b^2-2bc-1)g_2\right)$$

respectively. Hence $\mathrm{Sing}(X) \subset D \cap S_0$. $\square$

The following results about the singularities of Y' now follow immediately (see their counterparts in Section 4.3.1 for a proof).

Proposition 4.3.9. The variety Y' is quasismooth.

Lemma 4.3.13. The singularities of Y' are $\mathbb{Q}$-factorial.

Lemma 4.3.14. The singularities of Y' are terminal cyclic quotient singularities.

Proposition 4.3.10. The variety Y' is a Fano 3-fold.

4.4 Proof of Corollary 4.2.1

Corollary 4.2.1 claims that the codimension 4 Fano 3-folds constructed in Sections 4.3.1 and 4.3.2 belong to topologically distinct families. To prove this, we recall our the sketch proof provided in Section 4.2.2.

Let $Y' \subset \mathbb{P}(2,3,4,5,6,7,8,9)$ be the type II_1 unprojection of (X, D) as in Section 4.3.1 or Section 4.3.2. In both cases, X has only ordinary nodes as singularities and it is possible to extend the diagram displaying the factorisation of the unprojection map to

$$\begin{tikzcd} & & Z \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ \widehat{X} \arrow[r, rightsquigarrow] & X \arrow[rr, dashed, "\psi"] & & Y' \end{tikzcd}$$

where $\widehat{X} \rightsquigarrow X$ is a degeneration from a Fano 3-fold to X. The Euler characteristic of Y' is such that

$$e(Y') = e(\widehat{X}) + 2N - 2$$

where N is the number of nodes on X (see Section 5 of [36] and Section 2.3 of [7]).

Regardless of whether Y' is defined as in Section 4.3.1 or Section 4.3.2, we have $e(\widehat{X}) = -32$: $\hat{X} \subset \mathbb{P}(2,3,4,5,6,7)$ is a Fano 3-fold defined by a degree 12 and a degree 14 polynomial. The Euler characteristic of $\widehat{X}$ is then provided by Appendix A.3 of [7].

To show that Sections 4.3.1 and 4.3.2 construct 2 distinct topological families of Fano 3-folds amounts to showing that our 2 sets of initial data have different numbers of nodes. This is immediate by Lemmas 4.3.4 and 4.3.10:

Proposition 4.4.1. The Euler characteristic of Y' defined as in Section 4.3.1 is -16. When Y' is defined as in Section 4.3.2, the Euler characteristic of Y' is -14.

Chapter 5

The Future

In this chapter we discuss topics of future research inspired by the content of this thesis. At present, the topics presented are at differing levels of completion.

5.1 Candidates with Type I Centres

There exist 34 codimension 4 numerical candidates marked with both a type I and a type II_1 centre. Thus far, and for the sake of realising new families of Fano 3-folds, we have ignored these candidates since they have already been realised using type I unprojections (see [10], Theorem 3.2). However, we may still desire to construct these Fano 3-folds by type II_1 unprojections. We have the following result:

Theorem 5.1.1. Let $Y \subset \mathbb{P}(2a_0, a_1, \ldots, a_7)$ be a codimension 4 numerical candidate marked with a type II_1 centre. The numerical candidate Y can be realised as a quasismooth codimension 4 Fano 3-fold and constructed as the type ${\mathrm{II}_1}^{(2,1)}$ unprojection of (X, D) where $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_5)$ is a codimension 2 variety.

The proof of this Theorem follows that of Theorem 4.2.1 and uses the constructions provided in Table A.1 of Appendix A. We elaborate on some of the subtleties hidden in this result.

5.1.1 Failure of Standard Type ${\mathrm{II}_1}^{(3,0)}$

Recall Theorem 4.2.1: if a codimension 4 numerical candidate Y is marked with a type II_1 centre but no type I centre, Y can be realised as a Fano 3-fold using type ${\mathrm{II}_1}^{(2,1)}$ and type ${\mathrm{II}_1}^{(3,0)}$ unprojections. Given the statement of Theorem 5.1.1, it is natural to wonder why we make no comment on the type ${\mathrm{II}_1}^{(3,0)}$ unprojection construction. For each codimension 4 numerical candidate marked with a type II_1 centre, it is always possible to construct a family using type ${\mathrm{II}_1}^{(2,1)}$ unprojections. We are currently unable to extend this statement to construct a family using type ${\mathrm{II}_1}^{(3,0)}$ unprojections. Indeed, a number of numerical candidates fail to be constructed using our standard type ${\mathrm{II}_1}^{(3,0)}$ unprojection model.

Consider a codimension 4 numerical candidate marked with both a type I and a type II_1 centre, $Y \subset \mathbb{P}(2a_0, a_1, \ldots, a_7)$. The standard model for constructing Y via a type ${\mathrm{II}_1}^{(3,0)}$ unprojection uses the initial data (X, D) with $X \subset \mathbb{P}(2a_0, a_1, \ldots, a_5)_{\langle x,y,z,u,v,w\rangle}$, a codimension 2 complete intersection, containing D where D is defined by the 2×2 minors of

$$M := \begin{pmatrix} u & v & w & xp_1 & xp_2 & xp_3 \\ p_1 & p_2 & p_3 & u & v & w \end{pmatrix}$$

with $p_1, p_2, p_3 \in \mathbb{C}[x, y, z]$ (see Section 4.2.2). Under this model, there exist 14 codimension 4 numerical candidates which fail to construct a quasismooth Fano 3-fold unprojection. In each case, X is not quasismooth off D and thus the unprojection is singular.

Table 5.1: Standard Type ${\mathrm{II}_1}^{(3,0)}$ Unprojection Model Failures

GRDB ID	Numerical Candidate	Centre	Standard X
1082	$Y \subset \mathbb{P}(1,2,5,6,7,9,11,13)$	$\frac{1}{6}(1,1,5)$	$X_{18,22} \subset \mathbb{P}(1,2,5,9,11,13)$
1167	$Y \subset \mathbb{P}(1,2,3,4,5,7,9,11)$	$\frac{1}{4}(1,1,3)$	$X_{14,18} \subset \mathbb{P}(1,2,3,7,9,11)$
1181	$Y \subset \mathbb{P}(1,2,3,4,5,5,7,12)$	$\frac{1}{4}(1,1,3)$	$X_{14,15} \subset \mathbb{P}(1,2,3,5,7,12)$
1182	$Y \subset \mathbb{P}(1,2,3,4,5,5,7,9)$	$\frac{1}{4}(1,1,3)$	$X_{12,14} \subset \mathbb{P}(1,2,3,5,7,9)$
1183	$Y \subset \mathbb{P}(1,2,3,4,5,5,7,7)$	$\frac{1}{4}(1,1,3)$	$X_{10,14} \subset \mathbb{P}(1,2,3,5,7,7)$
4938	$Y \subset \mathbb{P}(1,1,3,4,5,5,6,11)$	$\frac{1}{3}(1,1,2)$	$X_{12,15} \subset \mathbb{P}(1,1,4,5,6,11)$
5841	$Y \subset \mathbb{P}(1,1,2,2,3,5,7,9)$	$\frac{1}{2}(1,1,1)$	$X_{10,14} \subset \mathbb{P}(1,1,2,5,7,9)$
5845	$Y \subset \mathbb{P}(1,1,2,2,3,4,5,6)$	$\frac{1}{2}(1,1,1)$	$X_{8,10} \subset \mathbb{P}(1,1,2,4,5,6)$
5859	$Y \subset \mathbb{P}(1,1,2,2,3,3,5,8)$	$\frac{1}{2}(1,1,1)$	$X_{9,10} \subset \mathbb{P}(1,1,2,3,5,8)$
5860	$Y \subset \mathbb{P}(1,1,2,2,3,3,5,7)$	$\frac{1}{2}(1,1,1)$	$X_{8,10} \subset \mathbb{P}(1,1,2,3,5,7)$
5862	$Y \subset \mathbb{P}(1,1,2,2,3,3,5,5)$	$\frac{1}{2}(1,1,1)$	$X_{6,10} \subset \mathbb{P}(1,1,2,3,5,5)$
5866	$Y \subset \mathbb{P}(1,1,2,2,3,3,4,7)$	$\frac{1}{2}(1,1,1)$	$X_{8,9} \subset \mathbb{P}(1,1,2,3,4,7)$
5867	$Y \subset \mathbb{P}(1,1,2,2,3,3,4,5)$	$\frac{1}{2}(1,1,1)$	$X_{6,9} \subset \mathbb{P}(1,1,2,3,4,5)$
5963	$Y \subset \mathbb{P}(1,1,2,2,3,3,3,5)$	$\frac{1}{2}(1,1,1)$	$X_{6,8} \subset \mathbb{P}(1,1,2,3,3,5)$

Example 5.1.1. The numerical candidate $Y \subset \mathbb{P}(1,2,5,6,7,9,11,13)$ is produced by the data

$$g_Y := -1 \text{ and } \mathcal{B}_Y := \left\{ \frac{1}{6}(1,1,5), \frac{1}{13}(1,2,11) \right\}.$$

The singularities $\frac{1}{13}(1,2,11)$ and $\frac{1}{6}(1,1,5)$ are the type I and type II_1 centres respectively. To construct Y using the standard type ${\text{II}_1}^{(3,0)}$ unprojection model, our initial data (X, D) would be a codimension 2 complete intersection $X_{18,22} \subset \mathbb{P}(1,2,5,9,11,13)$ and an irreducible surface D where D is defined by the 2×2 minors of

$$M := \begin{pmatrix} u & v & w & yp_1 & yp_2 & yp_3 \\ p_1 & p_2 & p_3 & u & v & w \end{pmatrix}$$

with $p_1, p_2, p_3 \in \mathbb{C}[x, y, z]$ (see Section 4.2.2. We claim that the point p_w is a singularity off D for all X containing D. As w is degree 13 and $w^2 - yp_3^2 \in I_D$, it is clear that $p_w \in X - D$. Furthermore,

$$f_{18}, \frac{\partial f_{18}}{\partial x}, ..., \frac{\partial f_{18}}{\partial w} \in \mathbb{C}[x, y, z, u, v]$$

and therefore the rank of the Jacobian matrix of X evaluated at p_w is at most 1. That is, p_w lies in $\text{Sing}(X)$. The unprojection $Y' \subset \mathbb{P}(1,2,5,6,7,9,11,13)_{\langle x,y,z,u,T_0,T_1,v,w \rangle}$ of (X, D) is singular since X is isomorphic to Y' away from D and a bouquet of rational curves (see Section 3.3). In particular, we can check that $(0,0,0,0,0,0,0,1) \in \text{Sing}(Y')$ by calculating the Jacobian matrix of Y'.

Example 5.1.1 only proves that Y cannot be constructed by the standard type ${\text{II}_1}^{(3,0)}$ unprojection. It is possible that "non-standard" type ${\text{II}_1}^{(3,0)}$ unprojections or undiscovered type II_1 unprojections are successful in constructing a second family. We ask:

Question. For the 14 numerical candidates of Table 5.1, is there a constructible family of Fano 3-folds using type ${\text{II}_1}^{(3,0)}$ unprojections?

Question. For the 14 numerical candidates of Table 5.1, is there a second family of Fano 3-folds constructible by type II_1 unprojections?

5.1.2 Tom and Jerry Correspondence

Let Y be a codimension 4 numerical candidate marked with both a type I and type II_1 centre. It is known that Y can be realised as a Fano 3-fold using type I unprojections and type II_1 unprojections (see [10] and Theorem 5.1.1). However, type I and type II_1 unprojections provide two distinct methods of constructing Y as a Fano 3-fold. We ask the following question:

Question. Is there a correspondence between the Fano 3-folds constructed by type II_1 and the type I unprojections?

More generally, we could ask the related question:

Question. Is there a correspondence between the rings constructed by type II_1 and the type I unprojections?

Proposition 3.1.1 showed that, broadly speaking, type ${\mathrm{II}_1}^{(2,1)}$ unprojections correspond to type I unprojections in Tom format. That is, for

$$\mathcal{O}_{\mathrm{amb}} = \mathbb{Z}[x_1, x_2, y_1, y_2, z, A_{12}, B_{11}, B_{12}, B_{22}, C, \overline{A}_{12}, \overline{B}_{11}, \overline{B}_{12}, \overline{B}_{22}, \overline{C}, w],$$

$I_D \subset \mathcal{O}_{\mathrm{amb}}$ the ideal generated by the 2×2 minors of

$$M := \begin{pmatrix} y_1 & y_2 & zx_1 & zx_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}$$

together with $w = 0$, and $I_X = \langle f, \overline{f} \rangle \subset \mathcal{O}_{\mathrm{amb}}$ the codimension 2 ideal defined by

$$f := A_{12}(y_1x_2 - x_1y_2) + B_{11}(y_1^2 - zx_1^2) + 2B_{12}(y_1y_2 - zx_1x_2) + B_{22}(y_2^2 - zx_2^2) + Cw$$

and

$$\overline{f} := \overline{A}_{12}(y_1x_2 - x_1y_2) + \overline{B}_{11}(y_1^2 - zx_1^2) + 2\overline{B}_{12}(y_1y_2 - zx_1x_2) + \overline{B}_{22}(y_2^2 - zx_2^2) + \overline{C}w,$$

the type ${\mathrm{II}_1}^{(2,1)}$ unprojection ring $\mathcal{O}_X[I_D^{-1}]$ is a generic type I unprojection ring.

Remark 5.1.1. Note that we need various primality, homogeneity and positively graded properties but have dropped them here for simplicity of expression.

Unfortunately, we do not possess a similar statement for type ${\mathrm{II}_1}^{(3,0)}$ unprojection rings. Furthermore, Proposition 3.1.1 relies on everything in sight being an indeterminate and hence does not always translate directly to specific cases involving Fano 3-folds (see Section 3.1.3, Remark 3.1.8).

Nevertheless, a correspondence between Fano 3-folds constructed by type I and type II_1 unprojections can often be found in practice.

Consider the data

$$g_Y := -1 \text{ and } \mathcal{B}_Y := \left\{2 \times \frac{1}{2}(1,1,1), 2 \times \frac{1}{4}(1,1,3), \frac{1}{7}(1,2,5)\right\}$$

and the codimension 4 numerical candidate $Y \subset \mathbb{P}(1,2,3,4,4,5,5,7)$ (see GRDB ID # 1253). Since Y is marked with a type II_1 centre $\frac{1}{4}(1,1,3)$, Theorem 5.1.1 tells us that we are able realise the candidate as a quasismooth codimension 4 Fano 3-fold via type II_1 unprojections. In fact, we can construct three topologically distinct families as type II_1 unprojections from $X_{10,11} \subset \mathbb{P}(1,2,3,4,5,7)$ (see Table A.1). We will show that these families can also be constructed as type I unprojections of (Z, E) where $Z_{8,9,9,10,10} \subset \mathbb{P}(1,2,3,4,4,5,5)$ is defined by the Pfaffians of a 5×5 antisymmetric matrix in Tom_1, Tom_5 and Jer_{13} format matrix with respect to $E \cong \mathbb{P}(1,2,5)$. These are the families constructed by [10].

Remark 5.1.2. (Definition 2.2, [10]) Tom_i and Jer_{ij} are matrix formats that specify type I unprojection data, that is a codimension 3 ideal I_Z defined by the maximal Pfaffians of a 5×5 antisymmetric matrix and a codimension 4 complete intersection ideal $I_E \supset I_Z$. We have that Tom_i is defined by a matrix $P = (p_{jk})$ where $p_{jk} \in I_E$ for all $j, k \neq i$. A matrix $P = (p_{jk})$ is in Jer_{ij} format if $p_{kl} \in I_E$ whenever k or l equals i or j. The equations of the unprojection in these cases are described in [29].

Remark 5.1.3. Note that in the following examples we work with the general X containing D. However, for each example there exists a specific case with our desired properties:

1. $X_{10,11} \subset \mathbb{P}(1,2,3,4,5,7)$ is a codimension 2 complete intersection which is quasismooth off $D \subset X$ and such that $\mathrm{Sing}(X)$ is a set of finitely many nodes;
2. $Z_{8,9,9,10,10} \subset \mathbb{P}(1,2,3,4,4,5,5)$ is a codimension 3 variety which is quasismooth off some $E := \mathbb{P}(1,2,5) \subset Z$ and such that $\mathrm{Sing}(Z)$ is a set of finitely many nodes; and
3. in both cases the unprojection is a quasismooth Fano 3-fold that realises our desired numerical candidate.

Remark 5.1.4. We provide unprojection data (X, D) and (Z, E). It is straightforward to prove that the unprojections are equal by using the explicit equations of the unprojections.

The First "Tom" Family. This construction is one of many cases where we are able to apply Proposition 3.1.1 directly. Suppose that

$X_{10,11} \subset \mathbb{P}(1,2,3,4,5,7)_{\langle x,y,z,u,v,w\rangle}$ is a codimension 2 complete intersection containing the irreducible surface D where D is defined by $w = 0$ together with the 2×2 minors of

$$\begin{pmatrix} u & v & yz & y^3 + yx^4 \\ z & y^2 + x^4 & u & v \end{pmatrix}.$$

Since $D \subset X$, we write $I_X = \langle f, \overline{f} \rangle$ with

$$\begin{aligned} f := B_{11}(u^2 - yz^2) + 2B_{12}(uv - yz(y^2 + x^4)) \\ + B_{22}(v^2 - y(y^2 + x^4)^2) + A_{12}(u(y^2 + x^4) - vz) + Cw, \end{aligned}$$

of degree 10 and

$$\begin{aligned} \overline{f} := \overline{B}_{11}(u^2 - yz^2) + 2\overline{B}_{12}(uv - yz(y^2 + x^4)) \\ + \overline{B}_{22}(v^2 - y(y^2 + x^4)^2) + \overline{A}_{12}(u(y^2 + x^4) - vz) + \overline{C}w, \end{aligned}$$

of degree 11, where $A_{12}, \overline{A}_{12}, B_{ij}, \overline{B}_{ij}, C, \overline{C} \in \mathbb{C}[x, y, z, u, v]$ are some polynomials of the appropriate degree. Let $Y' \subset \mathbb{P}(1,2,3,4,4,5,5,7)$ be the type ${\mathrm{II}_1}^{(2,1)}$ unprojection of (X, D) defined as in Section 3.1.2. We may apply Proposition 3.1.1 since $A_{12}, \overline{A}_{12}, B_{ij}, \overline{B}_{ij}, C, \overline{C}$ do not contain any terms in $\langle w \rangle$; therefore, the equations q, l_1, l_2, l_4, l_5 are the Pfaffians of the 5×5 matrix in Proposition 3.1.1 which defines a variety $Z \subset \mathbb{P}(1,2,3,4,4,5,5)_{\langle x,y,z,u,T_0,v,T_1\rangle}$ containing $E := \{T_0 = T_1 = C = \overline{C} = 0\}$ in Tom_1 format. We verify that the unprojection of (X, D) equals the unprojection of (Z, E) by constructing the latter and comparing equations.

Remark 5.1.5. Proposition 3.1.1 can be applied to a number of the type ${\mathrm{II}_1}^{(2,1)}$ unprojection constructions of Table A.1. In these cases, the Fano 3-folds constructed by the type ${\mathrm{II}_1}^{(2,1)}$ unprojection can easily be written as "Tom" type I unprojections.

The Second "Tom" Family. In this construction we are unable to apply Proposition 3.1.1 since the linear equation of the type II_1 divisor cannot be the indeterminate introduced via the type I unprojection. Suppose that $X_{10,11} \subset \mathbb{P}(1,2,3,4,5,7)_{\langle x,y,z,u,v,w\rangle}$ is a codimension 2 complete intersection containing the irreducible surface D defined by $v = 0$ together with the 2×2 minors of the matrix

$$\begin{pmatrix} u & w & yz & y(x^6 + y^3 + z^2) \\ z & y^3 + x^6 + z^2 & u & w \end{pmatrix}.$$

Then, without loss of generality, we may assume that X is defined by the polynomials

$$f := B_{11}(u^2 - yz^2) + A_{12}(u(y^3 + z^2 + x^6) - wz) + Cv$$

of degree 10 and

$$\overline{f} := \overline{B}_{11}(u^2 - yz^2) + 2\overline{B}_{12}(uw - (y^3 + x^6 + z^2)yz) + \overline{A}_{12}(u(y^3 + z^2 + x^6) - wz) + \overline{C}v$$

of degree 11, where $A_{12}, \overline{A}_{12}, B_{11}, \overline{B}_{11}, \overline{B}_{12}, C, \overline{C} \in \mathbb{C}[x, y, z, u, v]$ are some polynomials of the appropriate degree. Let Y' be the type $\mathrm{II}_1{}^{(2,1)}$ unprojection of (X, D) defined by $\langle f, \overline{f}, l_1, \dots, l_6, q\rangle$ as in Section 3.1.2. Since

$$A_{12}, \overline{A}_{12}, B_{11}, \overline{B}_{11}, \overline{B}_{12}, C, \overline{C} \in \mathbb{C}[x, y, z, u, v],$$

we may check that $l_1, l_3, l_4, l_6, q \in \mathbb{C}[x, y, z, u, v, T_0, T_1] \cap \langle z, u, T_0, T_1\rangle$. Hence, we may define a codimension 3 variety $Z \subset \mathbb{P}(1, 2, 3, 4, 4, 5, 5)_{\langle x,y,z,u,T_0,v,T_1\rangle}$ by l_1, l_3, l_4, l_6, q which contains $E := \{z = u = T_0 = T_1 = 0\}$. More visually, Z is defined by the maximal Pfaffians of the 5×5 antisymmetric matrix

$$\begin{pmatrix} 0 & z & 0 & u & v \\ & 0 & T_0 & 0 & T_1 \\ & & 0 & T_1 & -zB_{11} \\ & -\mathrm{Sym} & & 0 & -yT_0 + A_{12}\overline{C} \\ & & & & 0 \end{pmatrix};$$

note that we have scaled f and $\overline{f}$ so that $2\overline{B}_{12} = A_{12} = 1$. This matrix is in Tom_5 format with respect to $\langle z, u, T_0, T_1\rangle$. We verify that the unprojection of (X, D) equals the unprojection of (Z, E) by constructing the latter and comparing equations.

The "Jerry" Family. Let $X_{10,11} \subset \mathbb{P}(1, 2, 3, 4, 5, 7)_{\langle x,y,z,u,v,w\rangle}$ be a codimension 2 complete intersection which contains D, the irreducible surface defined by the 2×2 minors of

$$\begin{pmatrix} u & v & w & yz & y^3 & yx^6 \\ z & y^2 & x^6 & u & v & w \end{pmatrix}.$$

Without loss of generality, we write $I_X = \langle f, \overline{f}\rangle$ where

$$f := B_{11}(u^2 - yz^2) + 2B_{12}(uv - y^3z) + B_{22}(v^2 - y^5) + A_{12}(vz - uy^2) + A_{13}(wz - ux^6)$$

is a polynomial of degree 10,

$$\begin{aligned}\overline{f} := \overline{B}_{11}(u^2 - yz^2) + 2\overline{B}_{12}(uv - y^3z) + \overline{B}_{22}(v^2 - y^5) + 2\overline{B}_{13}(uw - yzx^6) \\ + \overline{A}_{12}(vz - uy^2) + \overline{A}_{13}(wz - ux^6) + \overline{A}_{23}(y^2w - x^6v)\end{aligned}$$

is a polynomial of degree 11 and $A_{ij}, B_{ij}, \overline{A}_{ij}, \overline{B}_{ij} \in \mathbb{C}[x, y, z, u, v, w]$ are some polynomials of the appropriate degree. Note that for X to be quasismooth off D we require $B_{22}, \overline{B}_{13} \neq 0$; hence, we set $B_{22} = \overline{B}_{13} = 1$. Let Y' be the unprojection of (X, D) defined by $\langle f, \overline{f}, l_1, \ldots, l_6, q\rangle$ as in [33]. The polynomials

$$l_1, l_4, l_5, l_2 - f, q - 2B_{12}l_1 - l_2 + \frac{1}{2}f - 2A_{12}l_4 \in \mathbb{C}[x, y, z, u, v, T_0, T_1]$$

can be written as the maximal Pfaffians of the antisymmetric matrix

$$P := \begin{pmatrix} 0 & z & y^2 & u & v \\ & 0 & T_0 + zB_{12} & F & G \\ & & 0 & H & I \\ & -\mathrm{Sym} & & 0 & J \\ & & & & 0 \end{pmatrix},$$

where

$$F := u + y^2\overline{A}_{23} - z\overline{B}_{22}A_{13} + z\overline{A}_{13},$$

$$G := -T_1 - uB_{12} + zA_{12} + zA_{13}\overline{B}_{12} - z\overline{A}_{13}B_{12} - y^2\overline{A}_{23}B_{12},$$

$$H := T_1 - 3uB_{12} - v - zA_{12} + zA_{13}\overline{B}_{12} - z\overline{A}_{13}B_{12} - y^2\overline{A}_{23}B_{12},$$

$$I := 2uB_{11} - zA_{13}\overline{B}_{11} + z\overline{A}_{13}B_{11} + y^2\overline{A}_{23}B_{11} - 2x^6A_{13}$$

and

$$\begin{aligned}J := T_0y + 2uA_{12} + u\overline{A}_{23}B_{11} + zyB_{12} \\ + y^3 + zA_{12}\overline{A}_{13} - zA_{13}\overline{A}_{12} + y^2A_{12}\overline{A}_{23} + \\ \frac{1}{2}y^2\overline{A}_{23}^2B_{11} + A_{13}\overline{A}_{23}x^6 - \frac{1}{2}\overline{A}_{23}^2B_{11}y^2 - \overline{A}_{23}B_{11}u.\end{aligned}$$

To see that this matrix is in Jerry format we may simplify P using row and column operations. After adding B_{12} multiples of column 1 from column 3 (we symmetrically add B_{12} multiples of row 1 from row 2), a multiple of column 1 from column 2 (symmetrically row), $-\frac{1}{2}\overline{A}_{23}B_{12}$ multiples of column 1 from column 4 (symmetrically row) and $\frac{1}{2}\overline{A}_{23}$ multiples of column 3 from column 4 (symmetrically row), we have the 5×5 antisymmetric matrix

$$P' := \begin{pmatrix} 0 & z & y^2 & u + \frac{1}{2}y^2\overline{A}_{23} & v \\ & 0 & T_0 + y^2 & F & G \\ & & 0 & H & I \\ & -\mathrm{Sym} & & 0 & J \\ & & & & 0 \end{pmatrix},$$

where

$$F := -A_{13}\overline{B}_{22}z + \overline{A}_{13}z + \frac{1}{2}\overline{A}_{23}B_{12}z + \frac{1}{2}\overline{A}_{23}T_0 + \frac{3}{2}\overline{A}_{23}y^2 + 2u,$$

$$G := A_{12}z + A_{13}\overline{B}_{12}z - \overline{A}_{13}B_{12}z - \overline{A}_{23}B_{12}y^2 - B_{12}u - T_1 + v,$$

$$H := -A_{12}z + A_{13}\overline{B}_{12}z - \overline{A}_{13}B_{12}z - \frac{1}{2}\overline{A}_{23}B_{12}y^2 - 2B_{12}u + T_1 - v,$$

$$I := -A_{13}\overline{B}_{11}z - 2A_{13}x^6 + \overline{A}_{13}B_{11}z + \overline{A}_{23}B_{11}y^2 + 2B_{11}u + B_{12}v$$

and

$$\begin{aligned} J := A_{12}\overline{A}_{13}z + A_{12}\overline{A}_{23}y^2 + 2A_{12}u - A_{13}\overline{A}_{12}z - \frac{1}{2}A_{13}\overline{A}_{23}\overline{B}_{11}z \\ + \frac{1}{2}\overline{A}_{13}\overline{A}_{23}B_{11}z + \frac{1}{2}\overline{A}_{23}^2B_{11}y^2 + \overline{A}_{23}B_{11}u + B_{12}yz + T_0y + y^3. \end{aligned}$$

A final swap of rows and columns, shows that this matrix is in Jer_{13} format with respect to

$$I_E := \langle T_0 + y^2, z, u + \frac{1}{2}\overline{A}_{23}y^2, T_1 + \frac{1}{2}\overline{A}_{23}B_{12}y^2 - v\rangle.$$

The maximal Pfaffians of P' define a variety Z in $\mathbb{P}(1,2,3,4,4,5,5)_{\langle x,y,z,u,T_0,v,T_1\rangle}$ containing $E := V(I_E)$. Note that for our choice of row and column operations, the varieties defined by the Pfaffians of P' and P are equivalent.

We point out that in the three examples above our correspondence was one directional. That is, we presented type II_1 unprojections as type I unprojections. We did not present type I unprojections as type II_1. We ask:

Question. For numerical candidates marked with both a type I and a type II_1 centre, can the families of Fano 3-folds constructed by type II_1 be built as families using type I projections?

The answer is expected to be yes. When we realise the numerical candidates using type II_1 constructions, the Fano 3-folds we construct contain the type I centre automatically. Since it is easy to perform "type I projections", we should be able to obtain the initial data of the type I unprojection: this is what we were doing in the three previous examples. It is beyond the scope of this thesis to check every codimension 4 Fano 3-fold construction of Table A.1 and [10] for such a correspondence. Nevertheless, we indicate a predicted or known correspondence for each numerical candidate in Table A.1.

Question. For numerical candidates marked with both a type I and a type II_1 centre, can the families of Fano 3-folds constructed by type I unprojections be built as families using type II_1 projections?

This question is more difficult since at present we are unable to perform "type II_1 projections".

Remark 5.1.6. Based on the families that have been constructed and checked at present, it appears that the type ${\mathrm{II}_1}^{(2,1)}$ families corresponds to the type I Tom families and the type ${\mathrm{II}_1}^{(3,0)}$ families correspond to the type I Jerry families. Brevity and the need for future comedic opportunities therefore suggest naming the type ${\mathrm{II}_1}^{(2,1)}$ unprojection format Thomasina and the ${\mathrm{II}_1}^{(3,0)}$ format Geraldine: some Thomasinas are known as Tom, some Geraldines are known as Jerry.

5.2 Missing Codimension 4 Candidates

A natural question now arises: what happens to the codimension 4 numerical candidates which are not marked by a type II_1 or a type I centre? Using the idea of type II_k unprojections established in Section 2.7.2, we are able to construct further Fano 3-folds. In particular, we are able to realise 7 more codimension 4 numerical candidates as Fano 3-folds by extending the existing notion of type II_2 unprojections.

Type II_2 unprojections have historically been defined as $\mathcal{O}_X[I_D^{-1}]$ with I_D defined by the 2×2 minors of matrices such as

$$M := \begin{pmatrix} u & w & v & s & xy & xz \\ y & z & u & w & v & s \end{pmatrix}$$

(see [31]). However, we believe that type II_2 unprojections occur using I_D defined as the image of

$$\phi : \mathbb{P}(a_0, a_1, a_2) \to \mathbb{P}(3a_0, a_1, \dots, a_n).$$

We therefore believe that there are more formats of I_D. Consider the following examples:

Example 5.2.1. The image of

$$\phi(a,b,c) := (x := a^3, y := b, z := c, u := ab, v := a^2b, w := ac, s := a^2c)$$

is defined as the 2×2 minors of M.

Example 5.2.2. Consider

$$\phi(a,b,c) = (x := a, y := c, z := b^3, u := bp_1 + b^2p_2, v := bp_3 + b^2p_4)$$

where $p_1, p_2, p_3, p_4 \in \mathbb{C}[a, b^3, c]$. Then $\mathrm{im}(\phi)$ is defined by the 3×3 minors of

$$\begin{pmatrix} u & v & -p_2z & -p_4z & -p_1z & -p_3z \\ -p_1 & -p_3 & u & v & -p_2z & -p_4z \\ -p_2 & -p_4 & -p_1 & -p_3 & u & v \end{pmatrix}$$

where we write $p_1, p_2, p_3, p_4 \in \mathbb{C}[x, y, z]$ in the obvious manner.

Example 5.2.2 is the key variety in this section. Unprojections with I_D defined as in Example 5.2.2 allow us to realise a further 7 codimension 4 numerical candidates which cannot be constructed by the results of [10] or Section 4.2; we provide realisations below.

GRDB # 501. The numerical candidate defined by

$$g = -1 \text{ and } \mathcal{B} = \left\{ \frac{1}{2}(1,1,1), 4 \times \frac{1}{3}(1,1,2), \frac{1}{8}(1,1,7) \right\}$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y \subset \mathbb{P}(1,3,6,7,8,8,9,10)_{\langle x,y,z,u,v,s,t,w \rangle}$ defined by

$$x^9y^3 - x^{10}v + 7x^6y^4 + x^6y^2z + x^5y^2u + 2x^5uz + y^6 + \\ 6x^2y^3u - x^4zv + y^2z^2 - 7xyzv + y^3t - z^3 + yuv + yus - vw,$$

$$-x^{10}y^2 - x^{10}z - 7x^7y^3 - x^5yv - x^5ys - xy^5 + y^3u - xy^2t - v^2 + zw,$$

$$x^{10}u - x^9v + x^5y^2z - 6x^6yv - x^5z^2 - x^2y^3z + y^3s - vt + uw,$$

$$-x^{15} - x^6y^3 - x^6yz - 6x^3y^4 - x^5w - y^5 - xy^2s - uv + zt,$$

$$-x^{10}z + x^6yu - x^5yv + xy^3z - 6x^2y^2v - xyz^2 + y^2w - vs + ut,$$

$$-x^{14} - 7x^{11}y - x^5y^3 - x^5t - x^2y^2z + xy^2u - y^2v - xyw - u^2 + zs,$$

$$\begin{aligned}&x^{10}s - x^9t + x^6y^2z - x^5y^2u - 7x^6yt - x^5uz + x^4y^2v + \\ &\qquad x^5yw - y^4z + 6xy^3v + 6x^2y^2w + y^2z^2 + xyzv - y^3t - t^2 + sw,\end{aligned}$$

$$\begin{aligned}&- x^{10}yz - x^9w + x^5y^2s - 6x^6yw + x^5u^2 - x^5zv + x^5zs + xy^4z \\ &\qquad - x^4uv + 6x^2y^3s + y^2uz - 6xyuv - xyzt - uz^2 + ys^2 + yut - tw\end{aligned}$$

and

$$\begin{aligned}&- x^{10}yu + x^9yv + x^{10}w + 6x^6y^2v - x^6y^2s + \\ &\qquad x^5yz^2 - x^5uv - x^5zt + x^4v^2 + xy^2uz - y^4s \\ &\qquad\qquad - y^2zv + 6xyv^2 + xyzw + z^2v - yst - yuw + w^2.\end{aligned}$$

GRDB # 512. The numerical candidate defined by

$$g = -1 \text{ and } \mathcal{B} = \left\{3 \times \frac{1}{3}(1,1,2), \frac{1}{5}(1,2,3), \frac{1}{7}(1,1,6)\right\}$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y \subset \mathbb{P}(1,3,5,6,7,7,8,9)_{\langle x,y,z,u,v,s,t,w\rangle}$ defined by

$$\begin{aligned}&9x^{17}+6x^{14}y+28x^{11}y^2+45x^{11}u+12x^8y^3+3x^{10}v+42x^8uy\\&\quad+19x^5y^4+4x^7yv+48x^5uy^2-3x^6yt+3x^6uz+9x^5u^2+3x^2y^5+7x^4y^2v\\&\quad+12x^2uy^3+4x^4uv-3x^3y^2t+x^3uyz+3x^4us+3x^2u^2y+4xy^3v-2x^3ut\\&\qquad+2xuyv-y^3t-x^2vt+3uy^2z+xuys-yv^2-uyt-xt^2+u^2z-tw,\end{aligned}$$

$$\begin{aligned}&-9x^{16}-6x^{13}y-19x^{10}y^2-13x^{10}u-9x^7y^3-14x^7uy\\&\qquad-2x^8t-x^4y^4+3x^4uy^2-x^5yt-3x^5uz+3x^4u^2+\\&\qquad\qquad 2xuy^3-x^2uyz-2x^3us+xu^2y-uys-t^2+vw,\end{aligned}$$

$$\begin{aligned}&9x^{15}+15x^{12}y+25x^9y^2+12x^9u+19x^6y^3+6x^6uy-3x^7t\\&\qquad+4x^3y^4-3x^5yv+3x^6w-2x^3uy^2-x^4yt+3x^4uz\\&\qquad\qquad-x^2y^2v+x^3yw-uy^3+xuyz+3y^2w-ts+uw,\end{aligned}$$

$$\begin{aligned}&-6x^{14}-14x^{11}y-19x^8y^2-27x^8u-9x^5y^3-21x^5uy-6x^6t-x^2y^4\\&\qquad-3x^5w-4x^2uy^2-2x^3yt-2x^3uz-x^2yw-3y^2t-uyz-ut+vs,\end{aligned}$$

$$\begin{aligned}&6x^{13}+2x^{10}y+12x^7y^2+8x^7u+14x^4y^3+3x^6s+8x^4uy-2x^5t\\&\quad+4xy^4-2x^3yv+x^3ys+3x^4w+2xuy^2-x^2yt-y^2v+3y^2s+xyw-tz+us,\end{aligned}$$

$$\begin{aligned}&-33x^{12}-29x^9y-40x^6y^2-13x^6u-14x^3y^3-3x^5s-7x^3uy-\\&\qquad 5x^4t-10y^4-x^2ys-2x^3w-7uy^2-2xyt-u^2+vz-yw,\end{aligned}$$

$$\begin{aligned}&-15x^{14}-31x^{11}y-45x^8y^2-20x^8u-43x^5y^3+3x^6yz-8x^7s\\&\quad-21x^5uy+5x^6t-11x^2y^4+5x^4yv+x^3y^2z-5x^4ys+2x^5w-5x^2uy^2\\&\qquad+4x^3yt+2xy^2v-4xy^2s+x^2yw+y^2t+xtz-xus-s^2+zw,\end{aligned}$$

$$
\begin{aligned}
18x^{16} + 24x^{13}y + 54x^{10}y^2 + 3x^{11}z + 27x^{10}u + 49x^7y^3 +&\\
x^8yz + 30x^7uy - 6x^8t + 29x^4y^4 - 6x^6yv + 6x^5y^2z - 3x^7w +&\\
28x^4uy^2 - 5x^5yt + 9x^5uz + 4x^4u^2 + 12xy^5 - 2x^3y^2v + x^2y^3z&\\
- 2x^3y^2s - x^4yw + 11xuy^3 - 4x^2y^2t + 3x^2uyz - 2x^3us + 2xu^2y&\\
- 3y^3v - x^2ut - y^3s - uyv + uz^2 - uys - xts - sw&
\end{aligned}
$$

and

$$
\begin{aligned}
- 9x^{15}y - 6x^{12}y^2 - x^9y^3 + 21x^9uy - 3x^{10}t + 3x^9w +&\\
16x^6uy^2 - 4x^7yt - 2x^7uz - 3x^5y^2s + 4x^6yw + 9x^3uy^3 -&\\
7x^4y^2t - 3x^4uyz - x^5tz - 3x^5us + 2x^3u^2y - 4x^4ut - x^2y^3s&\\
+ 3x^3y^2w + 3uy^4 - 4xy^3t - xuy^2z - x^2uys + 2x^3uw + u^2y^2&\\
- 2xuyt + x^2t^2 + y^3w + yvt - uzs + uyw + xtw + w^2.&
\end{aligned}
$$

GRDB # 550. The numerical candidate defined by

$$
g = -1 \text{ and } \mathcal{B} = \left\{ \frac{1}{2}(1,1,1), 3 \times \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{6}(1,1,5) \right\}
$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y \subset \mathbb{P}(1,3,4,5,6,6,7,8)_{\langle x,y,z,u,v,s,t,w \rangle}$ defined by

$$
\begin{aligned}
2x^{10}z + x^7yz - x^8v + x^6yu + 2x^4y^2z + x^2y^4 + 2x^4yt&\\
+ xy^3z + x^2yuz - y^2z^2 + xy^2t - z^2v + yus - vw,&
\end{aligned}
$$

$$
-x^6y^2 + 2x^4yu - x^2y^2z + xy^2u - x^2yt - y^2s - v^2 + zw,
$$

$$
2x^{10}y + x^7y^2 - x^4y^3 + 2x^4ys - y^3z + xy^2s - yzv - vt + uw,
$$

$$
-5x^8y - 4x^5y^2 - x^4yz - x^2y^3 - x^2ys - uv + zt - yw,
$$

$$
2x^{12} + x^9y - x^2y^2z + 2x^4w - y^4 - x^2zv + xyw - vs + ut,
$$

$$-x^{10}-2x^4v-y^2z-xyv-x^2w-u^2+zs-yt,$$

$$-4x^{14}-4x^{11}y-x^8y^2+2x^4y^2z+2x^4zv+xy^3z+x^2y^2v-y^3u+xyzv+x^2zw-yzt-t^2+sw,$$

$$2x^{10}u+2x^6y^3+x^7yu-x^8t+x^6ys+x^3y^4+x^2yzs-y^2uz-y^3v-uzv+ys^2-yzw-tw$$

and

$$-2x^{10}v-x^7yv+x^8w+2x^4y^4-x^6yt+xy^5+x^2y^3u+y^2zv+zv^2-yst+w^2.$$

GRDB # 577 The numerical candidate defined by

$$g=-1 \text{ and } \mathcal{B}=\left\{\frac{1}{2}(1,1,1), 3\times\frac{1}{3}(1,1,2), 2\times\frac{1}{5}(1,1,4)\right\}$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y\subset\mathbb{P}(1,3,4,5,5,6,6,7)_{\langle x,y,z,u,v,s,t,w\rangle}$ defined by

$$\begin{aligned}&-x^7t+x^4y^3+x^5z^2+x^5yu+4x^3y^2z+x^4zu-x^4yt+2x^2yz^2\\&\qquad+y^3z+xz^3+xyzu+xyzv+xy^2s+z^2u-yu^2-yzt-xt^2+yzs+tw,\end{aligned}$$

$$-x^6y^2-x^5yz-x^3y^3-x^4z^2-x^3ys+xyz^2-x^2zt-z^3-yzv-y^2s-t^2-uw,$$

$$-x^6y^2+x^5yz-2x^3y^3+x^4z^2-y^4+xyz^2+xy^2v+z^3-yzu+yzv-xzw-ts,$$

$$-x^7z-x^5y^2-2x^4yz-2x^2y^3-x^3z^2-x^3yv-3xy^2z-2yz^2-2xzt-y^2v+us+zw,$$

$$x^8y+x^7z+x^5y^2+2x^4yz+x^3z^2-x^3yu+xy^2z-y^2u+xzs-xyw-tv-zw,$$

$$-x^{10}-2x^7y-x^4y^2-2x^3yz-x^4t-3x^2z^2+x^3w-y^2z-2xyt-zt+uv-zs+yw,$$

$$\begin{aligned}&-x^9y-x^8z-2x^6y^2-3x^5yz-2x^4z^2+x^4yu-x^2y^2z+x^3yt\\&\qquad-x^3ys-4xyz^2+2xy^2u-2x^2zs-z^3+yzu+y^2t+xtv-s^2-vw,\end{aligned}$$

$$\begin{aligned}&2x^6yz-x^7s+x^4y^3+2x^5z^2+x^3y^2z+x^4zv-x^4ys+xy^4+x^2yz^2\\&\qquad+x^2y^2v+y^3z+2xz^3-2xyzu+xyzv-yzt+z^2v+yv^2-yzs-xts+sw\end{aligned}$$

and

$$\begin{aligned}&-x^7yz-x^6z^2-x^7w+2x^4y^2z-x^5yt-x^3yz^2-x^4zt\\&\qquad-x^3y^2v-x^4zs-x^4yw+3xy^3z-x^2z^3+x^2yzu-x^2yzv\\&\qquad\qquad+y^2z^2-xyzt-z^2t+yut-z^2s-yvs-yzw-xtw+w^2.\end{aligned}$$

GRDB # 872 The numerical candidate defined by

$$g=-1 \text{ and } \mathcal{B}=\left\{5\times\frac{1}{3}(1,1,2),\frac{1}{5}(1,1,4)\right\}$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y\subset\mathbb{P}(1,3,3,4,5,5,6,7)_{\langle x,y,z,u,v,s,t,w\rangle}$ defined by

$$\begin{aligned}&x^8u+x^6yz-x^7v+x^3y^3+x^4yv+2x^2y^2u+3x^2uz^2+xyu^2+y^3z\\&\qquad-y^2z^2+3yz^3+xy^2v-4xz^2v-2x^2v^2+u^3+yuv+uzv+yus+y^2t+vw,\end{aligned}$$

$$-x^{10}-3x^4z^2-x^3yu-x^2u^2-xy^2z-x^2yv-x^2zv-x^2ys+y^2u-xuv-xyt-v^2-zw,$$

$$x^8y-x^2y^3-x^2y^2z+3x^2yz^2-x^3yv-x^3zv+y^2s-vt-uw,$$

$$-x^9 - x^3y^2 - 3x^3z^2 - 2x^4v - y^3 - xu^2 - xys + x^2w - uv + zt,$$

$$x^7y - x^4y^2 - x^5v - xy^2z + 3xyz^2 - x^2zv - vs + ut - yw,$$

$$-x^8 - 2x^4u - x^2yz - 3x^2z^2 - 2x^3v + xuz - x^2t - u^2 - yv + zs + xw,$$

$$\begin{aligned} &- x^9y - x^6y^2 + 2x^3y^2z - 3x^3yz^2 + 2x^4zv + y^3z - 3y^2z^2 + xy^2v \\ &\qquad + xyzv + x^2v^2 + 2x^2vs - 2x^2ut - yus - yzt - xvt - xuw - t^2 - sw, \end{aligned}$$

$$\begin{aligned} &x^{10}y + x^8s + x^6yu - x^7t - 2x^4y^2z + 3x^4yz^2 - 2x^5zv \\ &\qquad - x^3y^2u - x^4uv + xy^4 - x^3v^2 + x^2y^2s + 3x^2z^2s - y^2uz + 3yuz^2 \\ &\qquad\qquad - xuzv + xyus - 3xz^2t - x^2vt + x^2uw + u^2s + ys^2 + uzt + tw \end{aligned}$$

and

$$\begin{aligned} &- x^8yz - x^8t + x^5y^3 - x^7w - x^4y^2u - x^5uv + x^2y^4 + x^2y^2z^2 \\ &\qquad - 3x^2yz^3 + x^3y^2v + x^3z^2v + xy^3u + x^2yuv - 3x^2z^2t + y^2zv - 3yz^2v \\ &\qquad\qquad + xzv^2 - y^2zs - xyvs - 3xz^2w - 2x^2vw - uvs - yst + uzw + w^2. \end{aligned}$$

GRDB # 878. The numerical candidate defined by

$$g = -1 \text{ and } \mathcal{B} = \left\{4 \times \frac{1}{3}(1,1,2), 2 \times \frac{1}{4}(1,1,3)\right\}$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y \subset \mathbb{P}(1,3,3,4,4,5,5,6)_{\langle x,y,z,u,v,s,t,w\rangle}$ defined by

$$\begin{aligned} &x^8z + x^5y^2 + 2x^6s + x^4yu + x^4zu + x^4yv + 3x^2y^3 \\ &\qquad + x^2yz^2 - x^2us + yu^2 - zu^2 - xs^2 + y^2t + sw, \end{aligned}$$

$$-2x^4y^2 - x^4z^2 - 3x^2zs - x^2yt - z^2u - yzv - s^2 - uw,$$

$$-x^4w + yzu - z^2u + y^2v - st,$$

$$-x^6z - x^3y^2 - x^4s - x^2zu - x^2yv - y^3 - xzs + ut + zw,$$

$$x^6y + x^4t + x^2yu - x^2zu - z^3 - sv - yw,$$

$$x^8 - x^5z - x^4u - 3x^2yz - x^3s + x^2w - ys + uv - zt,$$

$$\begin{aligned} &- x^7y - x^4y^2 + 2x^4z^2 - x^5t - x^3yu + x^3zu + \\ &\qquad x^2zs - 3x^2yt - y^2u + yzu + xsv - t^2 - vw, \end{aligned}$$

$$\begin{aligned} &x^8y + x^5yz + 2x^6t + x^4yu - x^4zu + x^4zv + x^2y^2z - 3x^2z^3 \\ &\qquad + xyzu - xz^2u + xy^2v - x^2ut - z^2s + zuv + yv^2 - xst + tw \end{aligned}$$

and

$$\begin{aligned} &- x^6yz + 2x^6w + x^3yz^2 - x^4ys - 2x^2yzu + 2x^2z^2u - 3x^2y^2v \\ &\qquad + y^2z^2 - z^4 - x^2uw - yus + zus - zsv - zut - yvt - yzw - xsw + w^2. \end{aligned}$$

GRDB # 1766. The numerical candidate defined by

$$g = -1 \text{ and } \mathcal{B} = \left\{ 2 \times \frac{1}{2}(1,1,1), 5 \times \frac{1}{3}(1,1,2) \right\}$$

is realised as the quasismooth codimension 4 Fano 3-fold $Y \subset \mathbb{P}(1,2,3,3,3,4,4,5)_{\langle x,y,z,u,v,s,t,w \rangle}$ defined by

$$\begin{aligned} &x^6u + 2x^4zy + x^4yu - x^5s + 2x^3z^2 + 3x^2zy^2 + x^2y^2u + 3xz^2y + 3zy^3 \\ &\qquad + xzyu + y^3u - xy^2s + x^2us + x^2zt + 2z^2u - u^3 - xs^2 + z^2v + zyt + sw, \end{aligned}$$

$$-x^4y^2 - x^3zy - 2xzy^2 - y^4 + z^2y - zyu - y^2s - xus - zyv - xzt - s^2 - uw,$$

$$x^6y + x^4y^2 + x^2y^3 + y^4 - x^2zu + x^2zv + z^2y - yu^2 - y^2s + zyv - st - zw,$$

$$-x^5y - 2x^4z - 4x^2zy - xy^3 + x^2yu - 2zy^2 - xzu + y^2u - xys - xzv - zs + ut + yw,$$

$$x^7 + x^5y + x^3y^2 + xy^3 + xz^2 - y^2u - xu^2 - xys - x^2w - sv + zt - yw,$$

$$-x^6 - 2x^2y^2 - xzy - 2y^3 - 2x^2s - z^2 - ys + uv - yt + xw,$$

$$\begin{aligned} &- 2x^8 - 3x^6y - 3x^4y^2 - 3x^2y^3 - 2x^2z^2 - y^4 + xy^2u \\ &\qquad + 3x^2u^2 + 2x^2ys - x^2uv - x^2yt - z^2y - zyu + yu^2 + \\ &\qquad\quad y^2s + xus + zyv - yuv + xsv - 2xzt - 2y^2t - xyw - t^2 - vw, \end{aligned}$$

$$\begin{aligned} &x^7y + x^6z + x^5y^2 + x^4zy + x^3y^3 + x^4yv - x^5t + x^2zy^2 + xy^4 \\ &\quad + x^3zu + x^3zv + xz^2y + zy^3 + xzyu - y^3u - 2xyu^2 - xy^2s + 2xzyv + y^3v \\ &\qquad - xy^2t + z^3 - zu^2 - zys - yus + zuv + zv^2 + zyt - xst + y^2w + tw \end{aligned}$$

and

$$\begin{aligned} &- x^6y^2 - x^4y^3 - x^6s - x^2y^4 - x^4ys - x^4yt - x^5w - y^5 + 2x^2zyu \\ &\qquad - x^2y^2s - x^2zyv - z^2y^2 + xz^2u + zy^2u - xz^2v - zy^2v - y^3t \\ &\qquad\qquad - xy^2w - z^2s + u^2s + ys^2 - usv - zvt - xsw + w^2. \end{aligned}$$

Given our experience of constructing codimension 4 Fano 3-folds using type II_1 and type I unprojections, we ask:

Question. Is it possible to find more than one family of Fano 3-folds for these 7 numerical candidates? Do these families correspond to distinct type II_2 unprojection constructions?

Question. In the literature, constructions using cluster algebras for these numerical candidates have been found (see [16]). Do the unprojection and cluster algebra methods construct different families of Fano 3-folds for these numerical candidates?

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Appendix A

Table of Type II_1 Unprojections

For each codimension 4 numerical candidate marked with a type II_1 centre, Table A.1 provides the information required to realise the candidate as a Fano 3-fold using type II_1 unprojections.

The codimension 4 candidates are listed by their GRDB ID and presented as $Y \subset w\mathbb{P}^7$. They are realised as the type ${\mathrm{II}_1}^{(n,m)}$ unprojection of (X, D). The choice of X is indicated by its GRDB ID and is always a codimension 2 variety $X \subset \mathbb{P}(a_0, \ldots, a_5)_{\langle x_0,\ldots,x_5\rangle}$ with $a_0 \leq \cdots \leq a_5$. We define $D \subset \mathbb{P}(a_0, \ldots, a_5)$ using the type ${\mathrm{II}_1}^{(n,m)}$ unprojection format and use the standard notation of Section 3.1.2 and [33]. In particular, we identify the vectors y and x with entries y_i and x_i respectively.

Remark A.0.1. We do not specify the equations of X since the general X containing D is sufficient. That is, for a given numerical candidate, the general X containing D constructed as in Table A.1 will be quasismooth off D and such that the singular locus is a set of finitely many nodes (see Section 4.2.2). The number of nodes is also described in Table A.1.

Remark A.0.2. In Table A.1 we also highlight the cases where the standard type ${\mathrm{II}_1}^{(3,0)}$ unprojection construction fails (see Section 5.1.1).

It is known by [10] that the codimension 4 numerical candidates marked with type I centres can be realised as Fano 3-folds using type I unprojections. When Table A.1 encounters such a numerical candidate, we predict the family of [10] to be constructed by the type II_1 unprojection (See Section 5.1.2). This prediction is based on calculating the Euler characteristic of the type II_1 unprojection and equating it to the Euler characteristic of the type I unprojection. We present this information as the GRDB ID and the Tom and Jerry formats of $Z \subset \mathbb{P}^6$ used by [10].

Remark A.0.3. In the cases where Proposition 3.1.1 is applicable and the family of [10] is known, we identify the family with an asterix: Tom_i^*. Note that Proposition 3.1.1 only identifies Tom families.

Remark A.0.4. In the case where a codimension 4 numerical candidate is not marked with a type I centre, there is no associated Tom and Jerry construction. We indicate this as "n/a".

Table A.1: Codimension 4 Fano 3-folds via Type II_1 Unprojections

Codimension 4 Candidate		Type II_1 Unprojection Initial Data				
ID	$Y \subset w\mathbb{P}^7$	ID	(n,m)	Data	Nodes	TJ
38	$Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$	37	$(3,0)$	$y = [x_2, x_4, x_5]$ $x = \left[x_1, x_3, x_0^3 + x_1^2\right]$ $z = x_0$	10	n/a
		37	$(2,1)$	$y = [x_4, x_5]$ $x = \left[x_3, x_0^3 + x_1^2\right]$ $z = x_0$ $w = x_2$	9	n/a
342	$Y \subset \mathbb{P}(1,4,6,7,7,8,9,10)$	338	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_1, x_0^5, x_2\right]$ $z = x_2$	17	n/a
		338	$(2,1)$	$y = [x_3, x_5]$ $x = [x_1, x_2]$ $z = x_2$ $w = x_4$	16	n/a

360	$Y \subset \mathbb{P}(1,4,5,6,7,7,8,9)$	359	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_1, x_2, x_0^6\right]$ $z = x_1$	17	n/a
		359	$(2,1)$	$y = [x_3, x_4]$ $x = [x_1, x_2]$ $z = x_1$ $w = x_5$	16	n/a
569	$Y \subset \mathbb{P}(1,3,4,5,5,6,7,9)$	546	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_1, x_2, x_0^7 + x_1x_2\right]$ $z = x_2$	20	Jer_{25}
		546	$(2,1)$	$y = [x_3, x_4]$ $x = [x_1, x_2]$ $z = x_2$ $w = x_5$	19	Tom_1^*
574	$Y \subset \mathbb{P}(1,3,4,5,5,6,7,7)$	547	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_1, x_2, x_0^5\right]$ $z = x_2$	18	Jer_{45}

		547	$(2,1)$	$y=[x_3,x_4]$ $x=[x_1,x_2]$ $z=x_2$ $w=x_5$	17	Tom_1^*
648	$Y\subset\mathbb{P}(1,3,4,4,5,5,6,7)$	640	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_0^2,x_1,x_2\right]$ $z=x_2$	15	n/a
		640	$(2,1)$	$y=[x_4,x_5]$ $x=[x_1,x_2]$ $z=x_2$ $w=x_3$	14	n/a
1069	$Y\subset\mathbb{P}(1,2,6,7,8,9,9,10)$	1068	$(3,0)$	$y=[x_2,x_3,x_5]$ $x=\left[x_0^5,x_1^3,x_4\right]$ $z=x_1$	27	n/a
		1068	$(2,1)$	$y=[x_3,x_5]$ $x=\left[x_0^6+x_1^3,x_4\right]$ $z=x_1$ $w=x_2$	24	n/a

1082	$Y \subset \mathbb{P}(1,2,5,6,7,9,11,13)$	1077	$(3,0)$	Standard Model Failure		
		1077	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^8 + x_1^4, x_0^{10} + x_1^5 + x_2^2\right]$ $z = x_1$ $w = x_5$	39	Tom_5^*
1084	$Y \subset \mathbb{P}(1,2,5,6,7,8,8,9)$	1068	$(3,0)$	$y = [x_4, x_3, x_5]$ $x = \left[x_0^5, x_1^2, x_2\right]$ $z = x_2$	27	n/a
		1068	$(2,1)$	$y = [x_3, x_5]$ $x = \left[x_0^4 + x_1^2, x_2\right]$ $z = x_2$ $w = x_4$	25	n/a
		1083	$(3,0)$	$y = [x_2, x_3, x_5]$ $x = \left[x_1^2, x_0^5, x_4\right]$ $z = x_1$	24	n/a

		1083	$(2,1)$	$y=[x_2,x_5]$ $x=\left[x_0^4+x_1^2,x_4\right]$ $z=x_1$ $w=x_3$	22	n/a
1091	$Y\subset\mathbb{P}(1,2,5,6,7,7,8,9)$	1080	$(3,0)$	$y=[x_2,x_4,x_5]$ $x=\left[x_0^4+x_1^2,x_3,x_0^8+x_1^4+x_1x_3\right]$ $z=x_1$	26	Jer_{34}
		1080	$(2,1)$	$y=[x_2,x_4]$ $x=\left[x_0^4+x_1^2,x_3\right]$ $z=x_1$ $w=x_5$	24	Tom_1^*
1115	$Y\subset\mathbb{P}(1,2,4,5,6,7,7,8)$	1114	$(3,0)$	$y=[x_2,x_3,x_5]$ $x=\left[x_0^3,x_1^2,x_4\right]$ $z=x_1$	21	n/a
		1114	$(2,1)$	$y=[x_3,x_5]$ $x=\left[x_0^4+x_1^2,x_4\right]$ $z=x_1$ $w=x_2$	19	n/a

1122	$Y \subset \mathbb{P}(1,2,4,5,5,6,6,7)$	1114	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_2, x_0^5 + x_0 x_1^2 + 2x_0 x_2, 2x_0^6 + x_1^3 + 3x_1 x_2\right]$ $z = x_1$	26	n/a
		1114	$(2,1)$	$y = [x_3, x_5]$ $x = \left[x_1^2 + x_2, x_0^6 + x_1^3\right]$ $z = x_1$ $w = x_4$	23	n/a
		1121	$(3,0)$	$y = [x_2, x_4, x_5]$ $x = \left[x_0^3, x_1^2, x_3\right]$ $z = x_1$	20	n/a
		1121	$(2,1)$	$y = [x_4, x_5]$ $x = \left[x_0^4 + x_1^2, x_3\right]$ $z = x_1$ $w = x_2$	18	n/a
1167	$Y \subset \mathbb{P}(1,2,3,4,5,7,9,11)$	1145	$(3,0)$	Standard Model Failure		

		1145	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_0^6+x_1^3+x_2^2, x_0^8+x_1^4\right]$ $z=x_1$ $w=x_5$	39	Tom_5^*
1172	$Y \subset \mathbb{P}(1,2,3,4,5,6,6,7)$	1171	$(3,0)$	$y=[x_2,x_3,x_5]$ $x=\left[x_1, x_0^3, x_4\right]$ $z=x_1$	18	n/a
		1171	$(2,1)$	$y=[x_2,x_5]$ $x=[x_1,x_4]$ $z=x_1$ $w=x_3$	17	n/a
1181	$Y \subset \mathbb{P}(1,2,3,4,5,5,7,12)$	1150	$(3,0)$	Standard Model Failure		
		1150	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_0^4+x_1^2, x_0^6+x_2^2+x_1^3\right]$ $z=x_1$ $w=x_5$	32	Tom_3^*

1182	$Y \subset \mathbb{P}(1,2,3,4,5,5,7,9)$	1151	$(3,0)$	Standard Model Failure		
		1151	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^4 + x_1^2, x_0^6 + x_1^3 + x_2^2\right]$ $z = x_1$ $w = x_5$	29	Tom_1 (1166) Tom_3 (1180)
1183	$Y \subset \mathbb{P}(1,2,3,4,5,5,7,7)$	1154	$(3,0)$	Standard Model Failure		
		1154	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^4 + x_1^2, x_0^6 + x_2^2 + x_1^3\right]$ $z = x_1$ $w = x_5$	27	Tom_3
1185	$Y \subset \mathbb{P}(1,2,3,4,5,5,6,8)$	1163	$(3,0)$	$y = [x_2, x_4, x_5]$ $x = \left[x_1, x_3, x_0^7\right]$ $z = x_1$	21	Jer_{24}

		1163	$(2,1)$	$y=[x_2,x_4]$ $x=[x_1,x_3]$ $z=x_1$ $w=x_5$	20	Tom_1^*
1186	$Y\subset\mathbb{P}(1,2,3,4,5,5,6,7)$	1165	$(3,0)$	$y=[x_2,x_4,x_5]$ $x=\left[x_1,x_3,x_0^6\right]$ $z=x_1$	20	Jer_{34}
		1165	$(2,1)$	$y=[x_2,x_4]$ $x=[x_1,x_3]$ $z=x_1$ $w=x_5$	19	Tom_1^*
1218	$Y\subset\mathbb{P}(1,2,3,4,5,5,5,6)$	1179	$(3,0)$	$y=[x_2,x_4,x_5]$ $x=\left[x_1,x_3,x_0^4\right]$ $z=x_1$	18	Jer_{45}
		1179	$(2,1)$	$y=[x_2,x_4]$ $x=[x_1,x_3]$ $z=x_1$ $w=x_5$	17	Tom_1

		1179	$(2,1)$	$y=[x_4,x_5]$ $x=\left[x_0^4+x_1^2,x_3\right]$ $z=x_1$ $w=x_2$	16	Tom_3
1253	$Y\subset\mathbb{P}(1,2,3,4,4,5,5,7)$	1165	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_2,x_1^2,x_0^6\right]$ $z=x_1$	24	Jer_{13}
		1165	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2,x_0^4+x_1^2\right]$ $z=x_1$ $w=x_5$	22	Tom_1^*
		1165	$(2,1)$	$y=[x_3,x_5]$ $x=\left[x_2,x_0^6+x_1^3+x_2^2\right]$ $z=x_1$ $w=x_4$	21	Tom_5
1256	$Y\subset\mathbb{P}(1,2,3,4,4,5,5,6)$	1171	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_2,x_0^4+x_1^2,x_0^5+x_0^2x_2+x_2x_1\right]$ $z=x_1$	24	n/a

		1171	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2,x_1^2+x_0^4\right]$ $z=x_1$ $w=x_5$	22	n/a
		1249	$(3,0)$	$y=[x_2,x_4,x_5]$ $x=\left[x_1,x_0^3,x_3\right]$ $z=x_1$	17	n/a
		1249	$(2,1)$	$y=[x_2,x_5]$ $x=[x_1,x_3]$ $z=x_1$ $w=x_4$	16	n/a
1350	$Y \subset \mathbb{P}(1,2,3,4,4,4,5,5)$	1249	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_2,x_0^3,x_1^2\right]$ $z=x_1$	20	n/a
		1249	$(2,1)$	$y=[x_3,x_5]$ $x=\left[x_2,x_0^4+x_1^2\right]$ $z=x_1$ $w=x_4$	18	n/a

1413	$Y \subset \mathbb{P}(1,2,3,3,4,4,5,5)$	1390	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_1, x_2, x_0^4\right]$ $z = x_1$	18	Jer_{35}
		1390	$(2,1)$	$y = [x_2, x_4]$ $x = [x_1, x_3]$ $z = x_1$ $w = x_5$	17	Tom_1^*
		1390	$(2,1)$	$y = [x_4, x_5]$ $x = \left[x_2, x_0^4 + x_1^2\right]$ $z = x_1$ $w = x_3$	16	Tom_3
2410	$Y \subset \mathbb{P}(1,2,2,3,4,5,5,6)$	2409	$(3,0)$	$y = [x_2, x_3, x_5]$ $x = [x_0, x_1, x_4]$ $z = x_1$	15	n/a
		2409	$(2,1)$	$y = [x_3, x_5]$ $x = \left[x_0^2 + x_1, x_4\right]$ $z = x_1$ $w = x_2$	14	n/a

2422	$Y \subset \mathbb{P}(1,2,2,3,3,4,5,7)$	2403	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_2, +x_0^4 + x_1^2 + x_2^2, x_0^6\right]$ $z = x_1$	26	Jer_{12}
		2403	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_1, x_0^4 + x_2^2\right]$ $z = x_1$ $w = x_5$	24	Tom_2^*
		2403	$(2,1)$	$y = [x_3, x_5]$ $x = \left[3x_0^2 + x_1 + x_2, 5x_0^6 + 3x_1^3 + x_2^3\right]$ $z = x_1$ $w = x_4$	23	Tom_5
2438	$Y \subset \mathbb{P}(1,2,2,3,3,4,4,5)$	2409	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_1 + 3x_2, x_0^3 + x_0x_1, 2x_1^2 + x_2^2\right]$ $z = x_1$	23	n/a
		2409	$(2,1)$	$y = [x_3, x_5]$ $x = \left[x_1 + x_2, x_0^4 + x_1^2\right]$ $z = x_1$ $w = x_4$	21	n/a

		2419	$(3,0)$	$y=[x_2,x_4,x_5]$ $x=[x_0,x_1,x_3]$ $z=x_1$	14	n/a
		2419	$(2,1)$	$y=[x_4,x_5]$ $x=[x_1,x_3]$ $z=x_1$ $w=x_2$	13	n/a
2511	$Y\subset\mathbb{P}(1,2,2,3,3,3,4,4)$	2419	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_2,x_1,x_0^3\right]$ $z=x_1$	18	n/a
		2419	$(2,1)$	$y=[x_3,x_4]$ $x=[x_2,x_1]$ $z=x_1$ $w=x_5$	17	n/a
3509	$Y\subset\mathbb{P}(1,2,2,2,3,3,3,4)$	3508	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=[x_0,x_1,x_2]$ $z=x_1$	14	n/a

		3508	$(2,1)$	$y=[x_4,x_5]$ $x=[x_1,x_2]$ $z=x_1$ $w=x_3$	13	n/a
4825	$Y \subset \mathbb{P}(1,1,4,6,7,8,9,10)$	4795	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_0^5+3x_1^5, x_0^6+x_1^6+x_2, x_0^7+x_1^7+x_1x_2\right]$ $z=x_2$	54	Jer_{24}
		4795	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_0^5+x_1^5, x_2\right]$ $z=x_2$ $w=x_5$	49	Tom_3^*
4915	$Y \subset \mathbb{P}(1,1,3,4,5,6,7,8)$	4823	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_0^4+x_1^4+x_2, x_0^5+x_1^5+2x_1x_2, 9x_1^6+x_0^2x_2\right]$ $z=x_2$	47	Jer_{35} (4895) Jer_{13} (4914)
		4823	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2, x_0^5+x_1^5\right]$ $z=x_2$ $w=x_5$	42	Tom_1 (4895) Tom_3^* (4914)

		4823	$(2,1)$	$y = [x_3, x_5]$ $a_1 = \left[x_1^4 + x_2, x_0^6 + x_0x_1x_2 + x_1^2x_2\right]$ $z = x_2$ $w = x_4$	41	Tom_2 (4895) Tom_5 (4914)
4938	$Y \subset \mathbb{P}(1,1,3,4,5,5,6,11)$	4836	$(3,0)$	Standard Model Failure		
		4836	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^3 + x_1^3, x_0^4 + x_1^4 + x_2\right]$ $z = x_2$ $w = x_5$	43	Tom_2^*
4939	$Y \subset \mathbb{P}(1,1,3,4,5,5,6,7)$	4837	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_0^3 + x_1^3, x_1^4 + x_2, x_0^5 + x_0x_2 + x_2x_1\right]$ $z = x_2$	38	Jer_{35} (4914) Jer_{24} (4937)
		4837	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^3 + x_1^3, x_2\right]$ $z = x_2$ $w = x_5$	35	Tom_1 (4914) Tom_2^* (4937)

4949	$Y \subset \mathbb{P}(1,1,3,4,5,5,6,6)$	4848	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_0^3 + x_1^3, x_2, x_0^4 + x_1^4 + 2x_2\right]$ $z = x_2$	36	Jer_{25}
		4848	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^3 + x_1^3, x_2\right]$ $z = x_2$ $w = x_5$	33	Tom_2^*
		4848	$(2,1)$	$y = [x_4, x_5]$ $x = \left[x_0^4 + x_1^4, x_2\right]$ $z = x_2$ $w = x_3$	32	Tom_1
5841	$Y \subset \mathbb{P}(1,1,2,2,3,5,7,9)$	5135	$(3,0)$	Standard Model Failure		
		5135	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^4 + x_1^4 + x_2^2, x_0^6 + x_1^6 + x_2^3\right]$ $z = x_2$ $w = x_5$	59	Tom_5^*

5845	$Y \subset \mathbb{P}(1,1,2,2,3,4,5,6)$	5138	$(3,0)$	Standard Model Failure		
		5138	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^3 + x_1^3, x_0^4 + x_1^4 + x_2^2\right]$ $z = x_2$ $w = x_5$	38	Tom_4^*
5859	$Y \subset \mathbb{P}(1,1,2,2,3,3,5,8)$	5154	$(3,0)$	Standard Model Failure		
		5154	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_2, x_0^4 + x_1^4\right]$ $z = x_2$ $w = x_5$	38	Tom_3^*
5860	$Y \subset \mathbb{P}(1,1,2,2,3,3,5,7)$	5155	$(3,0)$	Standard Model Failure		

		5155	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2,x_0^4+x_1^4\right]$ $z=x_2$ $w=x_5$	37	Tom_1 (5840) Tom_3^* (5858)
5862	$Y\subset\mathbb{P}(1,1,2,2,3,3,5,5)$	5156	$(3,0)$	Standard Model Failure		
		5156	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_0^2+x_1^2+x_2,x_0^4+x_1^4+x_2^2\right]$ $z=x_2$ $w=x_5$	35	Tom_3
5866	$Y\subset\mathbb{P}(1,1,2,2,3,3,4,7)$	5158	$(3,0)$	Standard Model Failure		
		5158	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2,x_0^3+x_1^3\right]$ $z=x_2$ $w=x_5$	33	Tom_2^*

5867	$Y \subset \mathbb{P}(1,1,2,2,3,3,4,5)$	5159	$(3,0)$	Standard Model Failure		
		5159	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_2, x_0^3 + x_1^3\right]$ $z = x_2$ $w = x_5$	31	Tom_2 (5858) Tom_2^* (5865)
5870	$Y \subset \mathbb{P}(1,1,2,2,3,3,4,5)$	5161	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_0x_1 + x_2, x_0x_2 + x_1^3, x_0^4 + x_2^2\right]$ $z = x_2$	32	Jer_{24} (5844) Jer_{12} (5865)
		5161	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_2, x_0^3 + +x_0x_2 + x_1^3 + x_1x_2\right]$ $z = x_2$ $w = x_5$	29	Tom_1 (5844) Tom_2^* (5865)
		5161	$(2,1)$	$y = [x_3, x_5]$ $x = \left[x_0^2 + x_1^2 + x_2, x_0^4 + x_2^2\right]$ $z = x_2$ $w = x_4$	28	Tom_2 (5844) Tom_5 (5865)

5914	$Y \subset \mathbb{P}(1,1,2,2,3,3,4,4)$	5200	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_0^2 + x_1^2 + x_2, x_0^3 + x_1x_2, x_0x_2 + x_1^3\right]$ $z = x_2$	30	Jer_{15}
		5200	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_0^2 + x_1^2 + x_2, x_0^3 + x_1^3 + 2x_0x_2\right]$ $z = x_2$ $w = x_5$	27	Tom_2
5963	$Y \subset \mathbb{P}(1,1,2,2,3,3,3,5)$	5258	$(3,0)$	Standard Model Failure		
		5258	$(2,1)$	$y = [x_3, x_4]$ $x = \left[x_2, x_0^2 + x_1^2\right]$ $z = x_2$ $w = x_5$	27	Tom_1 (5858) Tom_1^* (5962)
5970	$Y \subset \mathbb{P}(1,1,2,2,3,3,3,4)$	5261	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_2, x_0^2, x_1^3\right]$ $z = x_2$	26	Jer_{25} (5865) Jer_{14} (5962)

		5261	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2,x_0^2+x_1^2\right]$ $z=x_2$ $w=x_5$	24	Tom_1 (5865) Tom_1^* (5962)
		5261	$(2,1)$	$y=[x_3,x_5]$ $x=\left[x_2+x_0^2+x_1^2,x_0^3+x_1^3\right]$ $z=x_2$ $w=x_4$	23	Tom_2 (5865) Tom_4 (5962)
6217	$Y\subset\mathbb{P}(1,1,2,2,3,3,3,3)$	5514	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_2,x_0^2,x_1^2\right]$ $z=x_2$	23	Jer_{45}
		5514	$(2,1)$	$y=[x_3,x_4]$ $x=\left[x_2,x_0^2+x_1^2\right]$ $z=x_2$ $w=x_5$	21	Tom_1
6860	$Y\subset\mathbb{P}(1,1,2,2,2,3,3,5)$	5839	$(3,0)$	$y=[x_3,x_4,x_5]$ $x=\left[x_0,x_2,x_1^4\right]$ $z=x_2$	24	Jer_{13}

		5839	$(2,1)$	$y = [x_3, x_4]$ $x = [x_0, x_2]$ $z = x_2$ $w = x_5$	23	Tom_1^*
		5839	$(2,1)$	$y = [x_3, x_5]$ $x = \left[x_0, x_0^4 + x_1^4 + x_2^2\right]$ $z = x_2$ $w = x_4$	22	Tom_5
6865	$Y \subset \mathbb{P}(1,1,2,2,2,3,3,4)$	5843	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_0, x_2, x_1^3\right]$ $z = x_2$	22	Jer_{34}
		5843	$(2,1)$	$y = [x_3, x_4]$ $x = [x_0, x_2]$ $z = x_2$ $w = x_5$	21	Tom_1^*
6878	$Y \subset \mathbb{P}(1,1,2,2,2,3,3,3)$	5857	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = \left[x_0, x_2, x_1^2\right]$ $z = x_2$	20	Jer_{35}

		5857	$(2,1)$	$y = [x_3, x_5]$ $x = [x_0, x_2]$ $z = x_2$ $w = x_4$	19	Tom_1
		5857	$(2,1)$	$y = [x_4, x_5]$ $x = \left[x_0^2 + x_1^2, x_2\right]$ $z = x_2$ $w = x_3$	18	Tom_3
8051	$Y \subset \mathbb{P}(1,1,2,2,2,2,3,3)$	6858	$(3,0)$	$y = [x_3, x_4, x_5]$ $x = [x_1, x_0, x_2]$ $z = x_2$	18	n/a
		6858	$(2,1)$	$y = [x_4, x_5]$ $x = [x_1, x_2]$ $z = x_2$ $w = x_3$	17	n/a

Appendix B

Assorted Code

In this appendix we provide code pertinent to this thesis.

The defining equations of the type $\mathrm{II}_1^{(2,1)}$ unprojection ring can be calculated using the generators of the $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ (compare with the method of Section 3.1.1). In Section B.1 we calculate these generators using the computer algebra software Macaulay2 (see [19]).

In Section B.2 we recreate Section 4.3.1 using the computer algebra software Magma (see [5]). That is, we construct a codimension 4 Fano 3-fold $Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$ via a type $\mathrm{II}_1^{(2,1)}$ unprojection.

In Section B.3 we provide code which, when used in combination with the data of Table A.1, allows for the construction of random and successful initial data for type II_1 unprojections.

B.1 Calculating Hom

It is possible to calculate the explicit equations of the type $\mathrm{II}^{(2,1)}$ unprojection by computing the $\mathcal{O}_X$-module $I_D^{-1} \cong \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. In Section 2.3 the type II_1 unprojection was defined by choosing specific generators of ω_D, e_0 and e_1, and choosing any lift of these generators in $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ under the Poincaré residue map. To construct the explicit equations of the unprojection, we will choose specific generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ and show that the resulting images in ω_D satisfy the assumptions on e_0 and e_1. With the assumptions satisfied, the previously developed theory still holds and we are able to work explicitly.

Let

$$\mathcal{O}_{\mathrm{amb}} := \mathbb{Z}[a_{11}, a_{21}, a_{12}, a_{22}, w, z, v_{12}, v_{13}, v_{14}, v_{24}, u, \overline{v}_{12}, \overline{v}_{13}, \overline{v}_{14}, \overline{v}_{24}, \overline{u}]$$

be a positively graded ring such that the weight of z is even and

$$\mathrm{wt}(a_{2i}) = \mathrm{wt}(a_{1i}) + \frac{1}{2}\,\mathrm{wt}(z)$$

for $i = 1, 2$. Let $I_D \subset \mathcal{O}_{\mathrm{amb}}$ be the homogeneous ideal defined by the 2×2 minors of

$$M := \begin{pmatrix} a_{21} & a_{22} & za_{11} & za_{12} \\ a_{11} & a_{12} & a_{21} & a_{22} \end{pmatrix}$$

together with the linear equation $w = 0$. We define M_{ij} as the 2×2 minor of M generated by ordered columns $1 \leq i < j \leq 4$ and hence write $I_D = \langle M_{12}, M_{13}, M_{14}, M_{24}, w\rangle$. Let $I_X = \langle f, \overline{f}\rangle$ be the homogeneous ideal defined by

$$f := M_{13}v_{13} + M_{14}v_{14} + M_{12}v_{12} + M_{24}v_{24} + wu$$

and

$$\overline{f} := M_{13}\overline{v}_{13} + M_{14}\overline{v}_{14} + M_{12}\overline{v}_{12} + M_{24}\overline{v}_{24} + w\overline{u}.$$

We assume that $\mathcal{O}_X = \mathcal{O}_{\mathrm{amb}}/I_X$ is a normal Gorenstein integral domain. We restate Theorem 3.1.2 in our new notation:

Theorem B.1.1. There is an isomorphism

$$\mathcal{O}_X[I_D^{-1}] \cong \frac{\mathcal{O}_{\mathrm{amb}}[T_0, T_1]}{I_X + \langle l_1, \ldots, l_6, q\rangle}$$

where

$$l_1 := a_{21}T_1 + za_{11}T_0 + a_{11}B_{12} - a_{21}B_{14} - a_{22}B_{15},$$

$$l_2 := a_{22}T_1 + za_{12}T_0 + B_{12}a_{12} + B_{13}a_{21},$$

$$\begin{aligned} l_3 := wT_1 &+ (a_{21}a_{22} + a_{11}a_{12}z)B_{35} + a_{11}a_{21}B_{23} \\ &+ a_{12}^2 zB_{45} + a_{12}a_{21}B_{24} + a_{12}a_{22}B_{25} + a_{21}^2 B_{34}, \end{aligned}$$

$$l_4 := a_{21}T_0 + a_{11}T_1 + a_{12}B_{15},$$

$$l_5 := a_{22}T_0 + a_{12}T_1 - a_{11}B_{13} - a_{12}B_{14},$$

$$\begin{aligned} l_6 := wT_0 &- a_{11}^2 B_{23} - a_{11}a_{12}B_{24} - a_{11}a_{21}B_{34} \\ &- (a_{12}a_{21} + a_{11}a_{22})B_{35} - a_{12}^2 B_{25} - a_{12}a_{22}B_{45} \end{aligned}$$

and

$$q := T_1^2 - zT_0^2 - T_0B_{12} - B_{14}T_1 + B_{25}B_{13}$$

with B_{ij} defined to be the ij-th minor of

$$B := \begin{pmatrix} u & v_{12} & v_{13} & v_{14} & v_{24} \\ \overline{u} & \overline{v}_{12} & \overline{v}_{13} & \overline{v}_{14} & \overline{v}_{24} \end{pmatrix}$$

for $1 \leq i < j \leq 5$.

To prove this theorem we calculate the generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ as an $\mathcal{O}_X$-module.

Lemma B.1.1. The $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is generated by i, s_0, s_1 where i is the inclusion map, s_0 is the injective map defined by the natural extension of

$$s_0(w) := a_{11}(B_{35}a_{22} + B_{23}a_{11} + B_{24}a_{12} + B_{34}a_{21}) + a_{12}(B_{24}a_{12} + B_{35}a_{21} + B_{45}a_{22})$$

$$s_0(M_{12}) := -B_{13}a_{11}^2 - B_{14}a_{11}a_{12} - B_{15}a_{12}^2$$

$$s_0(M_{13}) := -B_{14}a_{11}a_{21} - B_{15}(a_{22}a_{11} + a_{21}a_{12}) + B_{12}a_{11}^2$$

$$s_0(M_{14}) := B_{13}a_{21}a_{11} - B_{15}a_{22}a_{12} + B_{12}a_{11}a_{12}$$

$$s_0(M_{24}) := B_{13}(a_{22}a_{11} + a_{21}a_{12}) + B_{14}a_{22}a_{12} + B_{12}a_{12}^2$$

and s_1 is the injective map defined by the natural extension of

$$s_1(w) := -a_{21}(B_{35}a_{22}+B_{24}a_{12}+B_{34}a_{21}+B_{23}a_{11})-a_{12}(B_{35}a_{11}z+B_{24}a_{22}+B_{45}a_{12}z)$$

$$s_1(M_{12}) := B_{13}a_{21}a_{11} + B_{14}a_{21}a_{12} + B_{15}a_{22}a_{12}$$

$$s_1(M_{13}) := B_{15}(a_{21}a_{22} + a_{11}a_{12}z) + B_{14}a_{21}^2 - B_{12}a_{21}a_{11}$$

$$s_1(M_{14}) := B_{15}a_{12}^2z - B_{13}a_{21}^2 - B_{12}a_{21}a_{12}$$

$$s_1(M_{24}) := -B_{13}a_{11}a_{12}z - B_{14}a_{12}^2z - B_{13}a_{21}a_{22} - B_{12}a_{22}a_{12}.$$

Proof. We may write the following presentation of I_D as an $\mathcal{O}_X$-module:

$$0 \longleftarrow I_D \xleftarrow[\alpha]{} \mathcal{O}_X^5 \xleftarrow[\beta]{} \mathcal{O}_X^8$$

where $\alpha^T = (w, -M_{12}, -M_{13} - M_{14}, -M_{24})$. Using the results of Chapter 3.9 [1], in particular Theorem 3.9.5, $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is isomorphic to the kernel of the map

$\mathcal{O}_X^5 \to \mathcal{O}_X^8$ defined by β^T. That is, $\mathrm{Hom}_{\mathcal{O}_{\mathrm{amb}}}(I_D, \mathcal{O}_X) \cong \langle U \rangle$ for some matrix U. We calculate U using Macaulay2 (see [19] for this computer algebra software).

We define $\mathcal{O}_{\mathrm{amb}}$, I_X and I_D.

```
i1: Oamb = QQ[v_12,vv_12,v_13,v_14,v_24,u,vv_13,vv_14,vv_24,uu,a_21,
    a_22,w,a_11,a_12,z];

i2 : f = (a_21^2-a_11*a_11*z)*v_13+(a_21*a_22-a_11*a_12*z)*v_14-
     (a_11*a_22-a_21*a_12)*v_12+(a_22^2-a_12*a_12*z)*v_24+w*u;

i3 : ff = (a_21^2-a_11*a_11*z)*vv_13+(a_21*a_22-a_11*a_12*z)*vv_14
     -(a_11*a_22-a_21*a_12)*vv_12+(a_22^2-a_12*a_12*z)*vv_24+w*uu;

i4 : I_X = ideal(f,ff);

o4 : Ideal of Oamb

i5 : I_D = ideal((a_21^2-a_11*a_11*z),(a_21*a_22- a_11*a_12*z),
     (a_11*a_22-a_21*a_12),(a_22^2- a_12*a_12*z),w);

o5 : Ideal of Oamb
```

The map β is computed using the resolution of I_D

```
i6 : CD = res I_D;

i7 : Beta = CD.dd_(2);

                 5          8
o7 : Matrix Oamb  <--- Oamb
```

To calculate U we desire β defined over $\mathcal{O}_X$ and must change rings:

```
i8 : Beta = sub(Beta,Oamb/I_X),;

i9 : BetaT = transpose(Beta),;
```

The kernel of β^T is then

```
i10 : U = kernel BetaT,;
```

```
i11 : U = generators U,;
```

The matrix U is a 5×47 matrix with entries in $\mathcal{O}_X$. We understand a column

$$U_j = (u_{1j}, u_{2j}, u_{3j}, u_{4j}, u_{5j})^T$$

of U as the map $\phi_{U_j} \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ defined by

$$-w \mapsto u_{1j}, \qquad M_{12} \mapsto u_{2j}, \qquad M_{13} \mapsto u_{3j}, \qquad M_{14} \mapsto u_{4j}, \qquad M_{24} \mapsto u_{5j}.$$

For us, the inclusion map i corresponds to zeroth column of U:

```
i12 : i = U_0;
```

To find the columns of U corresponding to s_0 and s_1, we discard any column U_j of U which is a multiple of `i`:

```
i13 : for j from 1 to 46 list
      isSubset(ideal(U_j_0),ideal(i_0)) and
      isSubset(ideal(U_j_1),ideal(i_1)) and
      isSubset(ideal(U_j_2),ideal(i_2)) and
      isSubset(ideal(U_j_3),ideal(i_3)) and
      isSubset(ideal(U_j_4),ideal(i_4))

o13 = {true, true, true, true, true, true, true, true, true,
      -----------------------------------------------------------
      true, false, true, true, true, true, true, true, true,
      -----------------------------------------------------------
      false, false, true, true, true, true, true, true, true,
      -----------------------------------------------------------
      true, true, true, true, true, true, true, true, true, true,
      -----------------------------------------------------------
      true, true, true, true, true, true, true, true, true}

i14 : l=oo;

i15 : for j from 0 to 45 do if l_j == false then print (j+1)
11
19
20
```

We define

```
i16 : s0 = U_11;


i17 : s1 = U_19;
```

By sight we can see that s0 and s1 correspond to the generators of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ defined in the statement of this Lemma. All that remains to show is that U_{20} is a combination of i, s0 and s1. This is immediate:

```
i18 : {U_20+s1+(v_14*uu-vv_14*u)*i}


o18 = {|0|}
        |0|
        |0|
        |0|
        |0|
```

We have now proven our result. $\square$

Remark B.1.1. We interpret $s_0 \in I_D^{-1}$ in the usual manner of $s_0(x) = y \leftrightarrow s_0 = \frac{y}{x}$. An analogous statement holds for s_1.

Remark B.1.2. We used Macaulay2 version 1.14 with packages FourTiTwo, Topcom, ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone and Truncations. Should a different Macaulay2 version be used, it is possible for different integers to be returned by the loop in the above. Nevertheless, we can proceed as expected by adjusting the columns chosen for s_0 and s_1.

Remark B.1.3. We note that B_{ij} is a homogeneous polynomial for all i and j, and the entries of the vectors s_0 and s_1 are homogeneous. Let (s_0/i) denote the 5×1 matrix where the j-th entry is the j-th entry of s0 divided by the j-th entry of i. Then every entry of (s_0/i) has the same degree and we may calculate the degree of s_0 as an unprojection indeterminate. An analogous statement holds for s_1.

With s_0 and s_1 now defined, we may find the linear relations of $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$.

Lemma B.1.2. For $i, s_0, s_1 \in \mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ defined as above, we have the following relations:

$$za_{11}s_0 + a_{21}s_1 + a_{11}B_{12} - a_{21}B_{14} - a_{22}B_{15} = 0,$$

$$za_{12}s_0 + a_{22}s_1 + B_{12}a_{12} + B_{13}a_{21} = 0,$$

$$\begin{aligned} ws_1 + (a_{21}a_{22} + a_{11}a_{12}z)B_{35} + a_{11}a_{21}B_{23} \\ + a_{12}^2 zB_{45} + a_{12}a_{21}B_{24} + a_{12}a_{22}B_{25} + a_{21}^2 B_{34} = 0, \end{aligned}$$

$$a_{21}s_0 + a_{11}s_1 + a_{12}B_{15} = 0,$$

$$a_{22}s_0 + a_{12}s_1 - a_{11}B_{13} - B_{14}a_{12} = 0$$

and

$$\begin{aligned} ws_0 - a_{11}^2 B_{23} - a_{11}a_{12}B_{24} - a_{11}a_{21}B_{34} \\ - (a_{12}a_{21} + a_{11}a_{22})B_{35} - a_{12}^2 B_{25} - a_{12}a_{22}B_{45} = 0. \end{aligned}$$

Proof. We check that the above relations hold by using the explicit definitions for i, s_0, s_1 provided in Lemma B.1.1. This can be done using the Macaulay2 code

```
i19 :  l1 = -a_11*z*s0-a_21*s1+(a_11*v_12*uu-a_11*u*vv_12-
       a_21*v_14*uu+a_21*u*vv_14-a_22*v_24*uu+a_22*u*vv_24)*i;

i20 : l2 = a_12*z*s0+a_22*s1+(-a_12*v_12*uu+a_12*u*vv_12-
      a_21*v_13*uu+a_21*u*vv_13)*i;

i21 : l3 = -w*s1+(-a_11*a_12*z*v_13*vv_24+a_11*a_12*z*v_24*vv_13-
      a_11*a_21*v_12*vv_13+a_11*a_21*v_13*vv_12-a_12*a_12*z*v_14*
      vv_24+a_12*a_12*z*v_24*vv_14-a_12*a_21*v_12*vv_14+a_12*a_21*
      v_14*vv_12-a_12*a_22*v_12*vv_24+a_12*a_22*v_24*vv_12-a_21^2*
      v_13*vv_14+a_21^2*v_14*vv_13-a_21*a_22*v_13*vv_24+a_21*a_22
      *v_24*vv_13)*i;

i22 : l4 = -a_21*s0-a_11*s1+(a_12*v_24*uu-a_12*u*vv_24)*i;

i23 : l5 = a_22*s0+a_12*s1+(a_11*v_13*uu-a_11*u*vv_13+a_12*v_14*uu
      -a_12*u*vv_14)*i;

i24 : l6 = w*s0+(v_14*vv_13*a_21*a_11-v_13*vv_14*a_21*a_11+
      v_24*vv_13*a_22*a_11-v_13*vv_24*a_22*a_11+vv_12*v_13*a_11^2
     -v_12*vv_13*a_11^2+ v_24*vv_13*a_21*a_12-v_13*vv_24*a_21*a_12+
```

```
v_24*vv_14*a_22*a_12-v_14*vv_24*a_22*a_12+vv_12*v_14*a_11*a_12-
v_12*vv_14*a_11*a_12+vv_12*v_24*a_12^2-v_12*vv_24*a_12^2)*i;
```

As `l1`, ..., `l6` are equal to 0, the relations hold for i, s_0 and s_1 as vectors and their counterparts in $\mathrm{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. $\square$

This provides the linear equations of the unprojection (see Section 2.3).

Remark B.1.4. Note that these linear equations can be predicted by recreating the Kustin-Miller style resolution of [39] (see Section 3.1.2).

Remark B.1.5. It may be the Macaulay2 may return different

The quadratic equation of the unprojection is obtained as in Lemma 3.1.1.

B.2 Example Unprojection Code

Codimension 4 Fano 3-folds can be constructed using computer algebra software. The Magma code below realises GRDB numerical candidate #38, $Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$, via a type $\mathrm{II}_1^{(2,1)}$ unprojection. With the help of the initial data provided by Table A.1, the code is easily edited to construct many numerical candidates.

First, we set up the initial data of the unprojection $D \subset X \subset w\mathbb{P}^5$ as in Section 4.3.1. We define the ambient space $w\mathbb{P}^5 = \mathbb{P}(2,3,4,5,6,7)$:

```
> P<x,y,z,u,v,w>:=ProjectiveSpace(Rationals(),[2,3,4,5,6,7]);
```

The divisor $D \subset \mathbb{P}(2,3,4,5,6,7)$ is defined by the 2×2 minors of a matrix M together with $z = 0$:

```
> M:=Matrix([[v,w,x*u,x*(x^3+y^2)],[u,x^3+y^2,v,w]]);
> D:=Scheme(P,Minors(M,2) cat [z]);
> M;
```

The output is:

```
[ v           w            x*u   x^4+x*y^2]
[ u    x^3+y^2               v           w]
```

The variety $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ is defined by the degree 12 polynomial and the degree 14 polynomial of Section 4.3.1:

```
> f12:=Minors(M,2)[6]+Minors(M,2)[5]+z^3;
> f14:=x*Minors(M,2)[6]+x*Minors(M,2)[5]+Minors(M,2)[2]+
  z*(x^5+z*v+u^2);
> X:=Scheme(P,[f12,f14]);
> f12;
> f14;
```

The outputs here are:

```
x^3*v + z^3 - x*u^2 + y^2*v + v^2 - u*w
```

and

```
-x^7 - 2*x^4*y^2 + x^5*z - x*y^4 + x^4*v - x^2*u^2 + x*y^2*v + z*u^2
+ z^2*v + x*v^2 - x*u*w + w^2
```

It is clear by construction that $D \subset X$; however, we use Magma to verify this:

```
> D subset X;
```

The output will be `true`. We check that X is quasismooth off D and the singular locus of X is a set of finitely many nodes. The singular locus of X is defined:

```
> SX:=JacobianSubrankScheme(X);
```

We can check that it is 0-dimensional, reduced and a subset of D using the code:

```
> Dimension(SX) eq 0;
> IsReduced(SX);
> SX subset D;
```

Again, the outputs are all `true`. In this case X has 9 nodes:

```
> Degree(SX);
```

This completes the definition of the initial data of the unprojection. Using the equations of Section 3.1.2, we know that the unprojection of (X, D) will lie in $\mathbb{P}(2, 3, 4, 5, 6, 7, 8, 9)$. We define this space:

```
> Q<x,y,z,u,v,w,T_0,T_1>:=ProjectiveSpace(Rationals(),[2,3,4,5,6,7,8,9]);
```

Since the unprojection equations will require the ideal of X inside this new space, we redefine X:

```
> f12:=x^3*v+z^3-x*u^2+y^2*v+v^2-u*w;
> f14:=-x^7-2*x^4*y^2+x^5*z-x*y^4+x^4*v-x^2*u^2+x*y^2*v+z*u^2+z^2*v
  +x*v^2-x*u*w+w^2;
```

The linear and quadratic equations of the unprojection will be defined using the explicit equations of Section 3.1.2. To avoid confusion, we identify

$$\begin{array}{ccc} z = \texttt{zz}, & w = \texttt{ww}, & A_{12} = \texttt{A\_12}, \\ \overline{A}_{12} = \texttt{AA\_12}, & B_{11} = \texttt{B\_11}, & \overline{B}_{11} = \texttt{BB\_11}, \\ B_{12} = \texttt{B\_12}, & \overline{B}_{12} = \texttt{BB\_12}, & B_{22} = \texttt{B\_22}, \\ \overline{B}_{22} = \texttt{BB\_22}, & C = \texttt{C}, & \overline{C} = \texttt{CC}, \end{array}$$

where the left-hand side is the notation used in Section 3.1.2 and the right-hand side is the notation used in our Magma code:

```
> x_1:=u;
> x_2:=x^3+y^2;
> y_1:=v;
> y_2:=w;
> zz:=x;
> ww:=z;
> A_12:=1;  B_11:=1;  B_22:=0;  B_12:=0;  C:=z^2;
> AA_12:=x; BB_11:=x; BB_22:=1; BB_12:=0; CC:=x^5+u^2+z*v;
```

The linear and quadratic equations of the unprojection are:

```
> l1:=x_1*zz*T_0+y_1*T_1+x_1*(C*AA_12-A_12*CC)-2*y_1*(C*BB_12-B_12*CC)
  -y_2*(C*BB_22-B_22*CC);
> l2:=x_2*zz*T_0+y_2*T_1+x_2*(C*AA_12-A_12*CC)+y_1*(C*BB_11-B_11*CC);
> l3:=ww*T_1+y_1^2*2*(B_11*BB_12-BB_11*B_12)+x_2^2*zz*2*(B_12*BB_22
  -BB_12*B_22)+x_2*x_1*zz*(B_11*BB_22-BB_11*B_22)+x_2*y_2*(A_12*BB_22
  -AA_12*B_22)+x_2*y_1*2*(A_12*BB_12-AA_12*B_12)+x_1*y_1*(A_12*BB_11-
  AA_12*B_11)+y_2*y_1*(B_11*BB_22-BB_11*B_22);
> l4:=y_1*T_0+x_1*T_1+x_2*(C*BB_22-B_22*CC);
> l5:=y_2*T_0+x_2*T_1-x_1*(C*BB_11-B_11*CC)-2*x_2*(C*BB_12-B_12*CC);
> l6:= ww*T_0-x_2^2*(A_12*BB_22-AA_12*B_22)-x_2*x_1*2*(A_12*BB_12-
  AA_12*B_12)-x_2*y_2*2*(B_12*BB_22-BB_12*B_22)-x_2*y_1*(B_11*BB_22-
  BB_11*B_22)-x_1^2*(A_12*BB_11-AA_12*B_11)-x_1*y_2*(B_11*BB_22-
  BB_11*B_22)-x_1*y_1*2*(B_11*BB_12-BB_11*B_12);
> q:=T_1^2-T_0^2*zz-T_0*(C*AA_12-A_12*CC)-2*T_1*(C*BB_12-B_12*CC)
```

```
  +(C*BB_22-B_22*CC)*(C*BB_11-B_11*CC);
```

The unprojection $Y \subset \mathbb{P}(2,3,4,5,6,7,8,9)$ of (X, D) is:

```
> Y:=Scheme(Q,[q,l1,l2,l3,l4,l5,l6,f12,f14]);
```

We may check that Y is a 3-fold of codimension 4:

```
> Dimension(Y) eq 3;
> Codimension(Y) eq 4;
```

and Y has an empty singular locus:

```
> SY:=JacobianSubrankScheme(Y);
> IsReduced(SY);
> Dimension(SY) eq -1;
```

The output for each line is `true`. For complete clarity, we compare the Hilbert numerator of Y to the Hilbert numerator of numerical candidate #38. The Hilbert series of Y is defined by

```
> PY:= HilbertSeries(Ideal(Y));
```

and its Hilbert numerator is

```
> HilbY:=PY*&*[1-Parent(PY).1^a : a in Gradings(Ambient(Y))[1]];
> HilbY;
```

The output is, as required:

```
t^43 - 2*t^31 - t^30 - 2*t^29 - 2*t^28 - t^27 + 2*t^24 + 2*t^23
+ 3*t^22 + 3*t^21 + 2*t^20 + 2*t^19 - t^16 - 2*t^15 - 2*t^14
- t^13 - 2*t^12 + 1
```

Remark B.2.1. The type $\mathrm{II}_1^{(3,0)}$ unprojection construction of Section 4.3.2 can be defined in a similar manner. Suitable Macaulay2 code is provided by [33] (see [28] in particular).

B.3 Initial Data Code

When using type II_1 unprojections to realise codimension 4 numerical candidates as Fano 3-folds, we are often more interested in the end result rather than the initial data of the unprojection. For our purposes, the general codimension 2 complete intersection $X \subset w\mathbb{P}^5$ containing D is sufficient. In this section we define code to construct "general" initial data.

Given the input of $(i, j, w\mathbb{P}^5, D)$, the code below constructs a random codimension 2 complete intersection $X_{i,j} \subset w\mathbb{P}^5$ which contains D, defined in the usual type II_1 manner. The constructed X is quasismooth off D and has a singular locus equal to finitely many nodes. In computer algebra, general loosely translates to random; therefore, the returned X is general in that it belongs to the Zariski open set of codimension 2 varieties $X_{i,j} \subset w\mathbb{P}^5$ which contain D, are quasismooth off D and have only finitely many nodes.

We begin by constructing random polynomials in Magma. For a given triple (`d`,`P`,`coeffs`), we define a function `randpoly` which constructs random polynomials of degree `d` on the projective space `P` where the coefficients of the polynomials are in the sequence `coeffs`:

```
> randpoly := func< P,d,coeffs | d ge 0 select &+[CoordinateRing(P)|
  Random(coeffs)*m:m in MonomialsOfWeightedDegree(CoordinateRing(P),
  d)] else CoordinateRing(P)!0 >;
```

Similarly, we define a function `randpolyD` which constructs a random polynomial of degree `d` in the ideal generated by `ID`:

```
> function randpolyD(P,ID,d,coeffs)
>   R := Universe(ID);
>   error if not R cmpeq CoordinateRing(P),"ID isn't a sequence of
    polys on P";
>   f := R!0;
>   for m in ID do
>       if WeightedDegree(m) le d then
>          f +:= m*randpoly(P,d-WeightedDegree(m),coeffs);
>       end if;
>   end for;
>   return f;
> end function;
```

We are now in a position to define our desired code. To construct $X_{i,j}$ in $w\mathbb{P}^5$ containing a type $\mathrm{II}_1^{(2,1)}$ divisor D, we define the function:

```
> function TypeII1_21(i,j,P,x,y,z,w : coeffs := [1..20])
>    error if not (#x eq 2 and #y eq 2),
     "Arguments 4 and 5 should have length 2";
```

```
>    error if not (i+j eq &+Gradings(P)[1]-1 and Dimension(P) eq 5),
     "Given equation degrees do not define an index 1 Fano 3-fold";
>    RP := CoordinateRing(P);
>    error if not (z in RP and w in RP and Universe(x) eq RP and
     Universe(y) eq RP),
     "Polynomial arguments must lie in the coordinate ring of P";
>    ZZXX:=[ x[i]*z: i in [1,2]];
>    M:=Matrix(CoordinateRing(P),2,4, y cat ZZXX cat x cat y);
>    ID:=Minors(M,2) cat [w];
>    D:=Scheme(P,ID);
>    fi:=randpolyD(P,ID,i,coeffs);
>    fj:=randpolyD(P,ID,j,coeffs);
>    X:=Scheme(P,[fi,fj]);
>    SX:=JacobianSubrankScheme(X);
>    is_ok := Dimension(SX) eq 0 and IsReduced(SX);
>    num_nodes := is_ok select Degree(SX) else -1;
>    is_ok and:= D subset X and SX subset D;
>    return is_ok, num_nodes, X, D;
> end function;
```

The input data of this function comprises of

1. `i` and `j`, the integers corresponding to the degrees of the equations defining our desired X;

2. `P`, the weighted projective space containing X;

3. `x` and `y` sequences of polynomials on `P` of length 2; and

4. `z` and `w` polynomials on `P`.

The type $\mathrm{II}_1^{(2,1)}$ divisor D is defined as in Section 3.1.2 by identifying `y` $= [y_1, y_2]$, `x` $= [x_1, x_2]$, `z` $= z$ and `w` $= w$. Note that the function `TypeII1_21` is defined with the extra optional sequence `coeffs` which specifies a range for the coefficients of the randomly chosen polynomials. If no information is provided here, the function runs with coefficients in $\{1, \ldots, 20\}$. The output of `TypeII1_21` will be:

1. `bool`, true or false depending on whether X is quasismooth off D, with a reduced singular locus consisting of finitely many nodes;

2. the number of nodes of X;

3. `X`, X itself; and

4. `D`, D itself.

Remark B.3.1. Note that the function has inbuilt error codes which occur if the input does not define a 2×4 matrix, if the data supplied for D does not lie in the appropriate coordinate space or if the provided $X_{i,j} \subset \mathbb{P}(a_0, \ldots, a_5)$ is such that $i + j - 1 \neq (a_0 + \cdots + a_5)$.

We may analogously define a function for the type $\mathrm{II}_1^{(3,0)}$ unprojection by altering the arguments and errors:

```
> function TypeII1_30(i,j,P,x,y,z : coeffs := [1..20] )
>     error if not (#x eq 3 and #y eq 3),
      "Arguments 4 and 5 should have length 3";
>     error if not (i+j eq &+Gradings(P)[1]-1 and Dimension(P) eq 5),
      "Given equation degrees do not define an index 1 Fano 3-fold";
>     RP := CoordinateRing(P);
>     error if not (z in RP  and Universe(x) eq RP
      and Universe(y) eq RP),
      "Polynomial arguments must lie in the coordinate ring of P";
>     ZZXX:=[ x[i]*z: i in [1,2,3]];
>     M:=Matrix(CoordinateRing(P),2,6, y cat ZZXX cat x cat y);
>     ID:=Minors(M,2);
>     D:=Scheme(P,ID);
>     fi:=randpolyD(P,ID,i,coeffs);
>     fj:=randpolyD(P,ID,j,coeffs);
>     X:=Scheme(P,[fi,fj]);
>     SX:=JacobianSubrankScheme(X);
>     is_ok := Dimension(SX) eq 0 and IsReduced(SX);
>     num_nodes := is_ok select Degree(SX) else -1;
>     is_ok and:= D subset X and SX subset D;
>     return is_ok, num_nodes, X, D;
> end function;
```

Example B.3.1. It is possible to construct a codimension 4 Fano 3-fold $Y \subset \mathbb{P}(2, 3, 4, 5, 6, 7, 8, 9)$ which is different to the one constructed in Section 4.3.1 by simply unprojecting using different initial data. We can set up initial data as follows:

```
> P<x,y,z,u,v,w>:=ProjectiveSpace(Rationals(),[2,3,4,5,6,7]);
> i:=12;
> j:=14;
> yy:=[w,v];
> xx:=[x^3+y^2,u];
> repeat
>    time bool,N,X,D := TypeII1_21(i,j,P,xx,yy,x,z);
> until bool;
```