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LOCAL EXISTENCE OF ANALYTIC SHARP FRONTS FOR SINGULAR SQG

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ABSTRACT. In this paper, we prove local existence and uniqueness of analytic sharp-front solutions to a generalised SQG equation by the use of an abstract Cauchy–Kowalevskaya theorem. Here, the velocity is determined by $u = |\nabla|^{-2\beta}\nabla^\perp\theta$ which (for $1 < \beta \leq 2$) is more singular than in SQG. This is achieved despite the appearance of pseudodifferential operators of order higher than one in our equation, by recasting our equation in a suitable integral form. We also provide a full proof of the abstract version of the Cauchy–Kowalevskaya theorem we use.

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1. INTRODUCTION

The β -generalised SQG equation, $\beta \in (0, 2]$ is the equation for the active scalar $\theta = \theta(x, t) \in \mathbb{R}$, $x \in \mathbb{R}^2$, $t \geq 0$,

$$\theta_t + (|\nabla|^{-2+\beta} \nabla^\perp \theta) \cdot \nabla \theta = 0.$$

Here, $|\nabla|^{-2+\beta}$ is defined as the Fourier multiplier $|\xi|^{-2+\beta}$. Choosing $\beta = 0$ corresponds to the two-dimensional incompressible Euler equations in vorticity form, while $\beta = 1$ corresponds to the Surface Quasi-Geostrophic (SQG) equation. In this paper, we consider special solutions to the family of equations for $\beta \in (1, 2)$. These solutions are of the form of an indicator of an evolving simply connected set with smooth boundary, termed sharp front solutions,

$$\theta(x, t) = \mathbf{1}_{x \in A(t)},$$

the evolution can be rephrased (by following the derivation in [20]) as a contour dynamics equation (CDE) for the boundary curve $z = \partial A(t)$, namely

$$z_t(s, t) \cdot N(s, t) = -C_\beta \int_{\mathbb{T}} \frac{(z_s(s_*, t) - z_s(s, t)) \cdot N(s, t)}{|z(s_*, t) - z(s, t)|^\beta} ds_*,$$

where N is the unit normal to z , and C_β should be the constant $\frac{\Gamma(\beta/2)}{\pi^{2-\beta} \Gamma((2-\beta)/2)}$, but for ease of presentation we will set $C_\beta := 1$. The torus \mathbb{T} is defined as $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, and we will take $[0, 1)$ as its fundamental domain in what follows. This integral term is parameterisation independent, and also finite so long as the parameterisation is regular and sufficiently smooth, and the curve does not self-intersect. By a suitable reparameterisation of $s \in \mathbb{T}$, this is equivalent to the contour dynamics equation

$$z_t(s, t) = - \int_{\mathbb{T}} \frac{z_s(s_*, t) - z_s(s, t)}{|z(s_*, t) - z(s, t)|^\beta} ds_* + \lambda(s, t) z_s(s, t) \quad (1.1)$$

$$=: \zeta(s, t) + \lambda(s, t) z_s(s, t), \quad (1.2)$$

for some function $\lambda = \lambda(s, t) \in \mathbb{R}$, satisfying $\lambda(0, t) = 0$ for every $t \geq 0$. The integral term $\zeta(s, t)$ is defined by (1.1).

The corresponding CDE for the Euler equation with $\beta = 0$, termed the vortex patch problem, was first derived in [24] and its systematic study can be found in the book [18]. The study of sharp fronts for the SQG equation was initiated in [21], where existence and uniqueness was proven in the class of smooth periodic graphs. Gancedo proved existence in [14] and later uniqueness [7] for Sobolev data and closed contours. The interpolated equations for $\beta \in (0, 1)$ was further studied in [14], [15], [16], and more recently the more singular sharp fronts

($\beta \in (1, 2)$) as in this paper, or other variants of SQG) have been studied in [4], [3], [19]. Non-periodic graph-type sharp fronts of (??) were studied in the recent papers, and of Hunter, Shu, and Zhang. They gave two different approaches in [27] and [29] to deriving the correct contour dynamics equation in this setting, and proved [26] global existence under a certain smallness condition. They also study two-front dynamics [25] and approximate equations for sharp fronts [28]. Finally, a second paper [30] by the authors has been prepared alongside this one that studies sharp fronts and almost-sharp fronts of the β -generalised SQG equation for a bounded domain in more detail.

In this paper, we are specifically interested in analytic sharp fronts, as in [11]. The motivation is as follows. There are striking similarities of the 3D Euler equation with SQG ($\beta = 1$) (see [5, 6] for example), that indicate that a sharp front is a good model of a vortex filament. Therefore, a key motivation for the study of the SQG sharp front is in developing methods that may transfer to the study of the vortex filaments.

It is also natural to study smooth approximations to the sharp fronts. These approximations, with a fixed time of existence despite arbitrarily large gradient (corresponding to the thickness of the transition region) have been called almost-sharp fronts, and have been studied for $\beta \in (0, 1)$ in [2], and for SQG in [8], [10], [12], [11], and [13]. These last three papers lay out a proof strategy that may be possible to extend to the range $\beta \in (1, 2)$. Therefore this paper represents the first step in proving the existence of almost-sharp fronts, in the sense of [13], for our more singular equation with $\beta \in (1, 2)$. We are interested in analytic solutions, as opposed to the Sobolev regularity found in [3], since this programme introduces Prandtl-like terms that require the analytic setting. To this end, the main theorem of this paper is

Theorem 1.1. *Let $z_0 = z_0(s) : \mathbb{T} \rightarrow \mathbb{R}^2$ be an analytic, simple, closed curve. Then there exists $T > 0$ and an analytic solution $z = z(s, t) : \mathbb{T} \times [0, T] \rightarrow \mathbb{R}^2$ to (1.1) such that $z(s, 0) = z_0(s)$.*

The remainder of this paper is organised as follows. In Section 2, we define the basic objects of study and some basic geometric notation. In Section 3, we establish some formulas for derivatives of ζ . In Section 4, we carefully rewrite the evolution equations in a form suitable for the application of the abstract Cauchy–Kowalevski Theorem. In section 5, we state and prove a version of the abstract Cauchy–Kowalevski Theorem, and then we apply it to prove the main theorem (Theorem 1.1) by using a suitable scale of Banach spaces.

2. PRELIMINARIES

To obtain a useful expression for λ , we will further reparameterise s so that

$$|z_s(s, t)|^2 = z_s(s, t) \cdot z_s(s, t) = L(t)^2, \quad s \in \mathbb{T}, t \geq 0.$$

That is to say, $z_s \cdot z_{ss}$ vanishes identically. In this parameterisation, s is not arc-length, but the vectors $z_s(s, t)$ have length $L(t)$ independent of s . The quantity $L(t)$ is the length of the curve z at time t . As in [14], the function λ can then be written explicitly in terms of z as follows. Taking the s derivative of z_t in equation (1.2) we find the equation

$$z_{st} = \zeta_s + \lambda z_{ss} + \lambda_s z_s. \quad (2.1)$$

Therefore

$$\frac{1}{2}(L^2)' = L'L = z_s \cdot z_{st} = z_s \cdot \zeta_s + \lambda_s L^2, \quad (2.2)$$

where we have substituted (2.1) in the last equality. Now (2.2) yields

$$\frac{L'}{L} = \frac{1}{L^2} z_s \cdot \zeta_s + \lambda_s. \quad (2.3)$$

Since λ is a periodic function in s , integrating (2.3) over \mathbb{T} , we obtain

$$\frac{L'}{L} = \int_{\mathbb{T}} \frac{z_s}{L^2} \cdot \zeta_s \, ds. \quad (2.4)$$

Hence, integrating (2.3) from 0 to s (using $\lambda(0, t) = 0$) and dividing by L^2 , we see that (using (2.4))

$$\begin{aligned} \lambda(s, t) &= s \frac{L'(t)}{L(t)} - \int_0^s \frac{z_s(s_1, t)}{L(t)^2} \cdot \zeta_s(s_1, t) \, ds_1 \\ &= -s \int_{\mathbb{T}} \frac{z_s(s_1)}{|z_s(s_1)|^2} \cdot \partial_{s_1} \int_{\mathbb{T}} \frac{z_s(s_0) - z_s(s_1)}{|z(s_0) - z(s_1)|^\beta} \, ds_0 \, ds_1 \\ &\quad + \int_0^s \frac{z_s(s_1)}{|z_s(s_1)|^2} \cdot \partial_{s_1} \int_{\mathbb{T}} \frac{z_s(s_0) - z_s(s_1)}{|z(s_0) - z(s_1)|^\beta} \, ds_0 \, ds_1, \end{aligned} \quad (2.5)$$

(hiding the t dependence for legibility reasons) which is an expression for λ completely determined by the values of z and z_s . Conversely, if λ is defined by (2.5), and z evolves via (1.1), then $\lambda(0, t) = 0$ for every t , and differentiating (1.1) readily yields

$$\frac{1}{2} \partial_t |z_s|^2 = \zeta_s \cdot z_s + \frac{\lambda}{2} \partial_s |z_s|^2 + \lambda_s |z_s|^2 = \frac{\lambda}{2} \partial_s |z_s|^2 + \mu(t) |z_s|^2,$$

where $\mu(t)$ is given by

$$\mu(t) = \lambda_s(s, t) + \frac{\zeta_s(s, t) \cdot z_s(s, t)}{|z_s(s, t)|^2}$$

$$= - \int_{\mathbb{T}} \frac{z_s(s_1)}{|z_s(s_1)|^2} \cdot \partial_{s_1} \int_{\mathbb{T}} \frac{z_s(s_0) - z_s(s_1)}{|z(s_0) - z(s_1)|^\beta} ds_0 ds_1.$$

One obtains by the method of characteristics for $F(s, t) = |z_s|^2(s, t)$, $F_0(s) = |z_s|^2(s, 0)$,

$$F(s, t) = F_0 \left(s + \int_0^t \lambda(s, \tau) d\tau \right) \exp \left(\int_0^t \mu(\tau) d\tau \right).$$

Hence, if $\frac{dF_0}{ds} = 0$, we have $\partial_s F = 0$ for all times, and (2.1) is satisfied. This proves the following proposition:

Proposition 2.1 (Parameterisation determines λ , [14]). *Suppose that $\lambda(0, t) = 0$, and that $z = z(s, t)$ is a smooth solution to the sharp front equation (1.1) with initial condition $z_0 = z_0(s)$ parameterised so that $\partial_s z_0 \cdot \partial_s^2 z_0 = 0$. Then $|z_s|^2$ is independent of s for all t , or equivalently $z_s \cdot z_{ss} = 0$, if and only if λ is given by (2.5).*

3. COMPUTING DERIVATIVES OF TERMS

When computing derivatives in s , it is useful to use the periodicity in s to rewrite the periodic integrands in terms of $s_* + s$ and s instead of s_* and s . This is so that the derivatives of difference quotients like $\frac{a(s+s_*)-a(s)}{|b(s+s_*)-b(s)|^\beta}$ retain their difference quotient structure.

For simplicity we introduce notation for a finite difference,

$$\mathcal{D}a(s, s_*) := a(s + s_*) - a(s).$$

The relevant quantities (suppressing the time variable) become

$$\zeta(s) = - \int_{\mathbb{T}} \frac{\mathcal{D}z_s(s, s_*)}{|\mathcal{D}z(s, s_*)|^\beta} ds_*, \quad (3.1)$$

$$\lambda(s) = s \int_{\mathbb{T}} \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) ds_1 - \int_0^s \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) ds_1. \quad (3.2)$$

Note that ζ is smooth if z is, by differentiating under the integral. Therefore, the first two derivatives of ζ are (writing $\mathcal{D}h$ for $\mathcal{D}h(s, s_*)$)

$$\begin{aligned} \zeta_s(s) &= - \int_{\mathbb{T}} \frac{\mathcal{D}z_{ss}}{|\mathcal{D}z|^\beta} - \beta \frac{\mathcal{D}z_s(\mathcal{D}z \cdot \mathcal{D}z_s)}{|\mathcal{D}z|^{\beta+2}} ds_*, \quad (3.3) \\ \zeta_{ss}(s) &= - \int_{\mathbb{T}} \left(\frac{\mathcal{D}z_{sss}}{|\mathcal{D}z|^\beta} - 2\beta \frac{\mathcal{D}z_{ss}(\mathcal{D}z \cdot \mathcal{D}z_s)}{|\mathcal{D}z|^{\beta+2}} \right. \\ &\quad \left. - \beta \frac{\mathcal{D}z_s(|\mathcal{D}z_s|^2 + \mathcal{D}z \cdot \mathcal{D}z_{ss})}{|\mathcal{D}z|^{\beta+2}} \right) \end{aligned}$$

$$+ \beta(\beta + 2) \frac{\mathcal{D}z_s(\mathcal{D}z \cdot \mathcal{D}z_s)^2}{|\mathcal{D}z|^{\beta+4}} \Big) ds_*, \quad (3.4)$$

which yields

$$\lambda_s(s) = \int_{\mathbb{T}} \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) ds_1 - \frac{z_s(s)}{L^2} \cdot \zeta_s(s), \quad (3.5)$$

$$\lambda_{ss}(s) = -\frac{z_s(s) \cdot \zeta_{ss}(s)}{L^2} - \frac{z_{ss}(s) \cdot \zeta_s(s)}{L^2}.$$

We have computed all the terms on the right hand side of the evolution equations

$$z_t = \zeta + \lambda z_s, \quad (3.6)$$

$$z_{st} = \zeta_s + \lambda_s z_s + \lambda z_{ss}, \quad (3.7)$$

$$z_{sst} = \zeta_{ss} + \lambda_{ss} z_s + 2\lambda_s z_{ss} + \lambda z_{sss}. \quad (3.8)$$

As mentioned earlier, we also need to assume that the initial curve z_0 does not self-intersect so that the integral $\zeta(s, t)$ makes sense. In analogy with [14], let us introduce the arc-chord condition via the function $\Gamma = \Gamma(z)$ below. It acts as a quantitative control on the length of the curve (since $\Gamma(0, 0) = \frac{1}{L}$) and self-intersection, and is a slight variant of the function F introduced in [14].

Definition 3.1 (Arc-chord condition). We say that a curve z satisfies the arc chord condition if the function $\Gamma(z) : \mathbb{T}^2 \rightarrow \mathbb{R}$ defined by

$$\Gamma(z)(s, s_*) := \begin{cases} \frac{|\sin(\pi s_*)|}{\pi |z(s + s_*) - z(s)|} & s_* \neq 0, \\ \frac{1}{|z_s(s)|} & s_* = 0, \end{cases} \quad (3.9)$$

belongs to $L^\infty(\mathbb{T}^2)$.

Lemma 3.2 (Analyticity of Γ). *Suppose z is an analytic curve with uniform speed parameterisation $s \in \mathbb{T}$, and $\Gamma(z) \in L^\infty(\mathbb{T}^2)$. Then z doesn't self-intersect, and $\Gamma(z)$ is analytic on \mathbb{T}^2 . In addition, $\Gamma(z)$ is bounded away from 0.*

Proof. Notice that $Z := \frac{z(s+s_*) - z(s)}{\sin(\pi s_*)/\pi}$ is analytic and does not vanish. The Euclidean norm of Z is therefore analytic, and the fact that this avoids zero since $\Gamma(z) \in L^\infty$ means that its reciprocal is also analytic. \square

To simplify the presentation slightly, we define $\text{Sin } s := \sin(\pi s)/\pi$.

The goal is to apply a Cauchy–Kowalevski type argument, as in [11]. The contour dynamics equation involves a term of order $1 + \beta$, which

means that a direct application of Cauchy-Kowaleski is not an option. We will need to carefully write the evolution equations for

$$z, z_s, z_{ss}, \text{ and } \Gamma,$$

in terms of analytic functions involving at most first order operators of those quantities plus one higher order term, that has a linear structure. To this end, we will use the following elementary version of the Fundamental Theorem of Calculus.

Lemma 3.3. *If $z \in C^1$ is a curve, $s, s_* \in \mathbb{T}$, then if we define for curves w ,*

$$\mathcal{I}(w)(s, s_*) := \int_0^1 w(s + (1 - \tau)s_*) d\tau, \quad (3.10)$$

then

$$\mathcal{D}z(s, s_*) = \mathcal{I}(z_s)(s, s_*)s_*.$$

It will be convenient to extend the definition of \mathcal{I} to two-variable functions $F = F(s, s_*)$ by the formula

$$\mathcal{I}(F)(s, s_*) := \int_0^1 F(s, (1 - \tau)s_*) d\tau. \quad (3.11)$$

This agrees with the previous definition (3.10) in the sense that if $F(s, s_*) = z(s + s_*)$, then $\mathcal{I}(F)(s, s_*) = \mathcal{I}(z)(s, s_*)$.

Proposition 3.4 (Expansion of Γ). *The function $\Gamma : \mathbb{T}^2 \rightarrow \mathbb{R}$ as defined in (3.9) satisfies the following first order expansion in s_* ,*

$$\Gamma(s, s_*, t) = \frac{1}{L(t)} + \mathcal{I}(\partial_{s_*}\Gamma)(s, s_*)s_*,$$

where explicitly, if $s_* \neq 0$ (ignoring the t variable),

$$\partial_{s_*}\Gamma(s, s_*) = \begin{cases} -\Gamma(s, s_*)^3 \frac{\mathcal{D}z}{\text{Sin } s_*} \cdot \left(\frac{z_s(s+s_*) \text{Sin } s_* - \mathcal{D}z(s, s_*) \cos(\pi s_*)}{(\text{Sin } s_*)^2} \right) & s_* \neq 0, \\ 0 & s_* = 0, \end{cases}$$

and using (3.11), we have $\mathcal{I}(\partial_{s_*}\Gamma)(s, s_*) := \int_0^1 (\partial_{s_*}\Gamma)(s, (1 - \tau)s_*) d\tau$.

Proof. The formula for $\mathcal{I}(\partial_{s_*}\Gamma)$ comes from applying the FTC to expand $\tilde{\Gamma}(s_*) := \Gamma(s, s_*)$ around $s_* = 0$ for fixed s ,

$$\tilde{\Gamma}(s_*) = \tilde{\Gamma}(0) + \int_0^{s_*} \tilde{\Gamma}'(s_* - \tilde{\tau}) d\tilde{\tau} = \tilde{\Gamma}(0) + \int_0^1 \tilde{\Gamma}'((1 - \tau)s_*) d\tau s_*.$$

Since $\tilde{\Gamma}(0) = \frac{1}{L}$, we just need to compute $\tilde{\Gamma}'$. We have

$$\begin{aligned}\tilde{\Gamma}'(s_*) &= \partial_{s_*} \Gamma(s, s_*) = \partial_{s_*} \left(\left| \frac{\mathcal{D}z(s, s_*)}{\text{Sin } s_*} \right|^{-1} \right) \\ &= - \left| \frac{\mathcal{D}z(s, s_*)}{\text{Sin } s_*} \right|^{-3} \frac{\mathcal{D}z(s, s_*)}{\text{Sin } s_*} \cdot \left(\partial_{s_*} \frac{\mathcal{D}z(s, s_*)}{\text{Sin } s_*} \right) \\ &= -\Gamma(s, s_*)^3 \frac{\mathcal{D}z(s, s_*)}{\text{Sin } s_*} \cdot \frac{(\text{Sin } s_*)z_s(s + s_*) - \cos(\pi s_*)\mathcal{D}z(s, s_*)}{(\text{Sin } s_*)^2},\end{aligned}$$

as claimed. For the behavior as $s_* \rightarrow 0$, writing $\text{Sin } s_* = s_* + O(s_*^3)$, and $\cos(\pi s_*) = 1 - \pi^2 s_*^2 + O(s_*^4)$,

$$\begin{aligned}\partial_{s_*} \Gamma(s, s_*) &= -\Gamma(s, 0)^3 z_s(s) \cdot \frac{z_s(s + s_*)s_* - (1 - \pi^2 s_*^2/2)\mathcal{D}z(s, s_*)}{s_*^2} + o(1) \quad (3.12)\end{aligned}$$

$$\begin{aligned}&= -\Gamma(s, 0)^3 z_s(s) \cdot \frac{z_s(s + s_*)s_* - (z_s(s)s_* + z_{ss}(s)s_*^2/2 + o(s_*^2))}{s_*^2} + o(1) \\ &\quad (3.13)\end{aligned}$$

$$\begin{aligned}&= -\Gamma(s, 0)^3 z_s(s) \cdot \frac{\mathcal{D}z_s(s, s_*)s_* - z_{ss}(s)s_*^2/2}{s_*^2} + o(1) \\ &= -\Gamma(s, 0)^3 z_s(s) \cdot \left(z_{ss}(s) - \frac{z_{ss}(s)}{2} + o(1) \right) + o(1) \quad (3.14)\end{aligned}$$

$$= o(1),$$

where in going from (3.12) to (3.13) we used a Taylor expansion of z to rewrite $\mathcal{D}z$, and at line (3.14) we used $z_s \cdot z_{ss} = 0$. \square

An analogous calculation gives the following.

Lemma 3.5 (Expansion of Γ^β).

$$\Gamma^\beta(s, s_*) = \frac{1}{L(t)^\beta} + \mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s_*}\Gamma)(s, s_*).$$

In rewriting the evolution equations (3.6), (3.7), (3.8), the following dot products appear:

$$\mathcal{D}z \cdot \mathcal{D}z_s = \mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss})s_*^2, \quad (3.15)$$

$$|\mathcal{D}z_s|^2 = |\mathcal{I}(z_{ss})|^2 s_*^2, \quad (3.16)$$

$$\mathcal{D}z \cdot \mathcal{D}z_{ss} = \mathcal{I}(z_s) \cdot \mathcal{I}(z_{sss})s_*^2. \quad (3.17)$$

We will also need expressions for ζ , ζ_s and ζ_{ss} . Starting from (3.1), (3.3), and (3.4) we replace the finite differences $\mathcal{D}\partial_s^k z$ with the integral terms $\mathcal{I}(\partial_s^{k+1} z)_{s_*}$ via (3.15), (3.16), and (3.17). Then we rewrite $|\mathcal{D}z(s, s_*)|^{-1} = |\text{Sin } s_*|^{-1} \Gamma(s, s_*)$. We obtain the following expressions (already involving 3rd derivatives of z)

$$\zeta(s) = - \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(z_{ss})_{s_*}}{|\text{Sin } s_*|^\beta} ds_*, \quad (3.18)$$

$$\begin{aligned} \zeta_s(s) &= - \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(z_{sss})_{s_*}}{|\text{Sin } s_*|^\beta} ds_* , \\ &+ \beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss})(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))_{s_*}^3}{|\text{Sin } s_*|^{\beta+2}} ds_* . \end{aligned} \quad (3.19)$$

For the highest derivative ζ_{ss} , we additionally use the expansion of Γ^β in Lemma 3.5 to isolate a convolution operator for the most singular term.

$$\begin{aligned} \zeta_{ss}(s) &= - \frac{1}{L^\beta} \int_{\mathbb{T}} \frac{z_{sss}(s + s_*) - z_{sss}(s)}{|\text{Sin } s_*|^\beta} ds_* \\ &- \int_{\mathbb{T}} \mathcal{I}(\beta \Gamma^{\beta-1} \partial_{s_*} \Gamma)(s, s_*)_{s_*} \frac{z_{sss}(s + s_*) - z_{sss}(s)}{|\text{Sin } s_*|^\beta} ds_* \\ &+ 2\beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{sss})(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))_{s_*}^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \\ &+ \beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss})(|\mathcal{I}(z_{ss})|^2 + \mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))_{s_*}^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \\ &- \beta(\beta + 2) \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+4} \frac{\mathcal{I}(z_{ss})(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))_{s_*}^2 s_*^5}{|\text{Sin } s_*|^{\beta+4}} ds_* . \end{aligned} \quad (3.20)$$

To rewrite λ we start by considering $z_s \cdot \zeta_s$ (see (3.2)). Using (3.19) we find

$$\begin{aligned} z_s \cdot \zeta_s &= \int_{\mathbb{T}} -\Gamma(s, s_*)^\beta \frac{\mathcal{I}(z_{sss}) \cdot z_s s_*}{|\text{Sin } s_*|^\beta} \\ &+ \beta \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss}) \cdot z_s (\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))_{s_*}^3}{|\text{Sin } s_*|^{\beta+2}} ds_* . \end{aligned}$$

Notice that this already includes z_{sss} . However, notice that

$$\begin{aligned} \mathcal{I}(z_{sss}) \cdot z_s(s) s_* &= z_{ss}(s + s_*) \cdot z_s(s) \\ &= z_{ss}(s + s_*) \cdot (z_s(s + s_*) - \mathcal{I}(z_{ss}) s_*) \\ &= -z_{ss}(s + s_*) \cdot \mathcal{I}(z_{ss}) s_* , \end{aligned}$$

so we can remove completely the dependence on the third derivative:

$$\begin{aligned} z_s \cdot \zeta_s &= \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(z_{ss}) \cdot z_{ss}(s + s_*)s_*}{|\text{Sin } s_*|^\beta} \\ &\quad + \beta \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss}) \cdot z_s(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_*. \end{aligned} \quad (3.21)$$

Hence, we can write λ as (using $\frac{1}{L} = \Gamma(s, 0) = \Gamma(0, 0)$)

$$\begin{aligned} \lambda(s) &= s \int_{\mathbb{T}} \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) ds_1 - \int_0^s \frac{z_s(s_1)}{L^2} \cdot \zeta_s(s_1) ds_1 \\ &= \frac{s}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \left(\Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(z_{ss})(s_1, s_0) \cdot z_{ss}(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} + \beta \Gamma(s_1, s_0)^{\beta+2} \times \right. \\ &\quad \left. \frac{\mathcal{I}(z_{ss})(s_1, s_0) \cdot z_s(s_1)(\mathcal{I}(z_s)(s_1, s_0) \cdot \mathcal{I}(z_{ss})(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} \right) ds_0 ds_1 \\ &\quad - \frac{1}{L^2} \int_0^s \int_{\mathbb{T}} \left(\Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(z_{ss})(s_1, s_0) \cdot z_{ss}(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} + \beta \Gamma(s_1, s_0)^{\beta+2} \times \right. \\ &\quad \left. \frac{\mathcal{I}(z_{ss})(s_1, s_0) \cdot z_s(s_1)(\mathcal{I}(z_s)(s_1, s_0) \cdot \mathcal{I}(z_{ss})(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} \right) ds_0 ds_1. \end{aligned} \quad (3.22)$$

(Here \times denotes multiplication.) Similarly, we can rewrite (3.5) for λ_s as (using (3.21))

$$\begin{aligned} \lambda_s(s) &= \frac{1}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \left(\Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(z_{ss})(s_1, s_0) \cdot z_{ss}(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} + \beta \Gamma(s_1, s_0)^{\beta+2} \times \right. \\ &\quad \left. \frac{\mathcal{I}(z_{ss})(s_1, s_0) \cdot z_s(s_1)(\mathcal{I}(z_s)(s_1, s_0) \cdot \mathcal{I}(z_{ss})(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} \right) ds_0 ds_1 \\ &\quad - \frac{1}{L^2} \int_{\mathbb{T}} \left(\Gamma(s, s_*)^\beta \frac{\mathcal{I}(z_{ss})(s, s_*) \cdot z_{ss}(s + s_*)s_*}{|\text{Sin } s_*|^\beta} + \beta \Gamma(s, s_*)^{\beta+2} \times \right. \\ &\quad \left. \frac{\mathcal{I}(z_{ss})(s, s_*) \cdot z_s(s)(\mathcal{I}(z_s)(s, s_*) \cdot \mathcal{I}(z_{ss})(s, s_*))s_*^3}{|\text{Sin } s_*|^{\beta+2}} \right) ds_*. \end{aligned} \quad (3.23)$$

Finally for λ_{ss} recall that

$$\lambda_{ss}(s) = -\frac{1}{L^2} (\zeta_{ss} \cdot z_s + \zeta_s \cdot z_{ss}).$$

ζ_{ss} involves an operator of order higher than 3 on z . However we have $\zeta_{ss} \cdot z_s$; using the identities (which follow from $z_s \cdot z_{ss} = 0$),

$$z_{sss}(s) \cdot z_s(s) = -|z_{ss}(s)|^2,$$

$$z_{sss}(s + s_*) \cdot z_s(s) = -|z_{ss}(s + s_*)|^2 - z_{sss}(s + s_*) \cdot \mathcal{I}(z_{ss})(s, s_*)s_*,$$

we can rewrite

$$\begin{aligned} & \mathcal{D}z_{sss}(s, s_*) \cdot z_s(s) \\ &= -\mathcal{D}|z_{ss}|^2(s, s_*) - z_{sss}(s + s_*) \cdot \mathcal{I}(z_{ss})(s, s_*)s_* \\ &= \left(-2\mathcal{I}(z_{sss} \cdot z_{ss})(s, s_*) - z_{sss}(s + s_*) \cdot \mathcal{I}(z_{ss})(s, s_*) \right) s_*. \end{aligned}$$

Therefore, λ_{ss} depends on Γ and the first three derivatives of z . For reference, the full expansion of λ_{ss} is as follows, which comes from a similar derivation to that of (3.20) but the cancellations above are used to regularise the integral, in place of Lemma 3.5.

$$\begin{aligned} \lambda_{ss}(s) &= \frac{1}{L^2} \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{(-2\mathcal{I}(z_{sss} \cdot z_{ss}) - z_{sss}(s + s_*) \cdot \mathcal{I}(z_{ss}))s_*}{|\text{Sin } s_*|^\beta} \\ &\quad - 2\beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{sss}) \cdot z_s(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))s_*^3}{|\text{Sin } s_*|^{\beta+2}} \\ &\quad - \beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss}) \cdot z_s(|\mathcal{I}(z_{ss})|^2 + \mathcal{I}(z_s) \cdot \mathcal{I}(z_{sss}))s_*^3}{|\text{Sin } s_*|^{\beta+2}} \\ &\quad + \beta(\beta + 2)\Gamma(s, s_*)^{\beta+4} \frac{\mathcal{I}(z_{ss}) \cdot z_s(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))^2 s_*^5}{|\text{Sin } s_*|^{\beta+4}} \\ &\quad + \Gamma(s, s_*)^\beta \frac{\mathcal{I}(z_{sss}) \cdot z_{ss}s_*}{|\text{Sin } s_*|^\beta} \\ &\quad - \beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(z_{ss}) \cdot z_{ss}(\mathcal{I}(z_s) \cdot \mathcal{I}(z_{ss}))s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_*. \end{aligned} \quad (3.24)$$

4. REWRITING THE EVOLUTION EQUATIONS

We will consider z , z_s and $z_s s$ as independent. We define

$$f(s, t) = z(s, t), \quad g(s, t) = z_s(s, t), \quad h(s, t) = z_{ss}(s, t).$$

4.1. **Evolution of f .** Rewriting (3.6) using (3.18) and (3.22), we obtain

$$\begin{aligned} & f_t(s) \\ &= - \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h)(s, s_*)s_*}{|\text{Sin } s_*|^\beta} ds_* \\ &\quad + g(s) \left(s \frac{1}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} \right. \\ &\quad \left. + \beta\Gamma(s_1, s_0)^{\beta+2} \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1)(\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \right) \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{L^2} \int_0^s \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1) s_0}{|\text{Sin } s_0|^\beta} \\
& + \beta \Gamma(s_1, s_0)^{\beta+2} \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1) (\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0)) s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \Big). \tag{4.1}
\end{aligned}$$

4.2. Evolution of g . Rewriting $z_{st} = \zeta_s + \lambda z_{ss} + \lambda_s z_s$ (3.7) using (3.19), (3.22), and (3.23), we obtain

$$\begin{aligned}
g_t(s) = & - \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h_s) s_*}{|\text{Sin } s_*|^\beta} ds_* \\
& + \beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h)(\mathcal{I}(g) \cdot \mathcal{I}(h)) s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \\
& + h(s) \left(s \frac{1}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1) s_0}{|\text{Sin } s_0|^\beta} \right. \\
& + \beta \Gamma(s_1, s_0)^{\beta+2} \\
& \times \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1) (\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0)) s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \\
& - \frac{1}{L^2} \int_0^s \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1) s_0}{|\text{Sin } s_0|^\beta} \\
& + \beta \Gamma(s_1, s_0)^{\beta+2} \\
& \times \left. \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1) (\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0)) s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \right) \\
& + g(s) \left(\frac{1}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1) s_0}{|\text{Sin } s_0|^\beta} \right. \\
& + \beta \Gamma(s_1, s_0)^{\beta+2} \\
& \times \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1) (\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0)) s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \\
& - \frac{1}{L^2} \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h)(s, s_*) \cdot h(s + s_*) s_*}{|\text{Sin } s_*|^\beta} \\
& + \beta \Gamma(s, s_*)^{\beta+2} \left. \frac{\mathcal{I}(h)(s, s_*) \cdot g(s) (\mathcal{I}(g)(s, s_*) \cdot \mathcal{I}(h)(s, s_*) s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \right). \tag{4.2}
\end{aligned}$$

4.3. **Evolution of h .** Rewriting $z_{sst} = \zeta_{ss} + \lambda_{ss}z_s + 2\lambda_s z_{ss} + \lambda z_{sss}$ (3.8) using (3.20), (3.22), (3.23), and (3.24), we obtain

$$\begin{aligned}
h_t &= \frac{-1}{L^\beta} \int_{\mathbb{T}} \frac{h_s(s+s_*) - h_s(s)}{|\text{Sin } s_*|^\beta} ds_* \\
&\quad - \int_{\mathbb{T}} \mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s_*}\Gamma)(s, s_*)s_* \frac{h_s(s+s_*) - h_s(s)}{|\text{Sin } s_*|^\beta} ds_* \\
&\quad + 2\beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h_s)(\mathcal{I}(g) \cdot \mathcal{I}(h))s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \\
&\quad + \beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h)(|\mathcal{I}(h)|^2 + \mathcal{I}(g) \cdot \mathcal{I}(h_s))s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \\
&\quad - \beta(\beta+2) \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+4} \frac{\mathcal{I}_{2,0}(\mathcal{I}(g) \cdot \mathcal{I}(h))^2 s_*^5}{|\text{Sin } s_*|^{\beta+4}} ds_* \\
&\quad + g(s) \left(\frac{1}{L^2} \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{(-2\mathcal{I}(h_s \cdot h) - h_s(s+s_*) \cdot \mathcal{I}(h))s_*}{|\text{Sin } s_*|^\beta} \right. \\
&\quad - 2\beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h) \cdot g(\mathcal{I}(g) \cdot \mathcal{I}(h))s_*^3}{|\text{Sin } s_*|^{\beta+2}} \\
&\quad - \beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h) \cdot g(|\mathcal{I}(h)|^2 + \mathcal{I}(g) \cdot \mathcal{I}(h_s))s_*^3}{|\text{Sin } s_*|^{\beta+2}} \\
&\quad + \beta(\beta+2)\Gamma(s, s_*)^{\beta+4} \frac{\mathcal{I}_{2,0} \cdot g(\mathcal{I}(g) \cdot \mathcal{I}(h))^2 s_*^5}{|\text{Sin } s_*|^{\beta+4}} \\
&\quad \left. + \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h_s) \cdot h s_*}{|\text{Sin } s_*|^\beta} - \beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h) \cdot h(\mathcal{I}(g) \cdot \mathcal{I}(h))s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \right) \\
&\quad + 2h(s) \left(\frac{1}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} \right. \\
&\quad - \beta\Gamma(s_1, s_0)^{\beta+2} \\
&\quad \times \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1)(\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \\
&\quad - \frac{1}{L^2} \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h)(s, s_*) \cdot h(s+s_*)s_*}{|\text{Sin } s_*|^\beta} \\
&\quad \left. - \beta\Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h)(s, s_*) \cdot g(s)(\mathcal{I}(g)(s, s_*) \cdot \mathcal{I}(h)(s, s_*))s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_* \right) \\
&\quad + h_s(s) \left(s \frac{1}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} \right.
\end{aligned}$$

$$\begin{aligned}
& -\beta\Gamma(s_1, s_0)^{\beta+2} \\
& \times \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1)(\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \\
& - \frac{1}{L^2} \int_0^s \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1)s_0}{|\text{Sin } s_0|^\beta} \\
& - \beta\Gamma(s_1, s_0)^{\beta+2} \\
& \times \frac{\mathcal{I}(h)(s_1, s_0) \cdot g(s_1)(\mathcal{I}(g)(s_1, s_0) \cdot \mathcal{I}(h)(s_1, s_0))s_0^3}{|\text{Sin } s_0|^{\beta+2}} ds_0 ds_1 \Big). \quad (4.3)
\end{aligned}$$

4.4. **Evolution of Γ .** The function Γ defined in (3.9) satisfies the evolution equation

$$\Gamma_t = \Gamma^3 \frac{\mathcal{D}z}{|\text{Sin } s_*|} \cdot \frac{\mathcal{D}z_t}{|\text{Sin } s_*|} = \frac{\Gamma^3 s_*^2}{(\text{Sin } s_*)^2} \mathcal{I}(g) \cdot \mathcal{I}(g_t). \quad (4.4)$$

4.5. **Summary of dependencies.** We now define the operators E_1 , E_2 , E_3 , and E_4 using the right-hand sides of the above equations. That is, E_1 , E_2 , and E_4 are defined as the right-hand sides of (4.1), (4.2) and (4.4) respectively. For E_3 in (4.3) we isolate the most singular term, involving a convolution, and use $g_t = E_2[g, h, h_s, \Gamma]$ to rewrite the term $\mathcal{I}(g_t)$. This gives

$$\begin{aligned}
f_t &= E_1[g, h, \Gamma], \\
g_t &= E_2[g, h, h_s, \Gamma], \\
h_t &= \frac{-1}{L^\beta} \int_{\mathbb{T}} \frac{h_s(s + s_*) - h_s(s_*)}{|\text{Sin } s_*|^\beta} ds_* + E_3[g, h, h_s, \Gamma], \quad (4.5)
\end{aligned}$$

$$\Gamma_t = E_4[\Gamma, g, F_2[g, h, h_s, \Gamma]]. \quad (4.6)$$

The convolution term in (4.5) is

$$\mathcal{H}_\beta(h) := - \int_{\mathbb{T}} \frac{h_s(s + s_*) - h_s(s_*)}{|\text{Sin } s_*|^\beta} ds_*.$$

Lemma 4.1. (*Skew-symmetry*) *The symbol of \mathcal{H}_β is purely imaginary.*

Proof. $-\mathcal{H}_\beta$ is the convolution of h_s with the renormalised distribution of $|\sin s|^{-\beta}$,

$$\langle \mathcal{R}_{|\text{Sin } s|^{-\beta}}, \phi \rangle := \int_{\mathbb{T}} \frac{\phi(s) - \phi(0)}{|\text{Sin } s|^\beta} ds.$$

As $|\text{Sin } s|^{-\beta}$ is even and real valued, so is the Fourier transform $F(k) = \mathcal{F}(\mathcal{R}_{|\text{Sin } s|^{-\beta}})(k)$. Hence, the symbol of \mathcal{H}_β is $i2\pi kF(k)$, which is purely imaginary. \square

Any operator $im(\partial_s)$ with a purely imaginary symbol $im(k)$ is skew-symmetric and so $(im(\partial_s)h, h)_{L^2} = 0$. We have $(im(\partial_s)g, h)_{L^2(\mathbb{T})} = (im(k)\hat{g}(k), \hat{h}(k))_{\ell^2(\mathbb{Z})} = -(\hat{g}(k), im(k)\hat{h}(k))_{\ell^2(\mathbb{Z})}$. More importantly the operator $\partial_t - im(\partial_s)$ is boundedly invertible on L^2 -based Sobolev spaces, with a solution defined via its Fourier coefficients,

$$\partial_t f - im(\partial_s)f = g \iff \hat{f}(k) = e^{-im(k)t} \hat{f}_0(k) + \int_0^t \hat{g}(k) e^{im(k)(\tilde{t}-t)} d\tilde{t}.$$

For the time dependent operator $\partial_t - L(t)^{-\beta} m(\partial_s)$, we can first perform the time rescaling $\partial_{t_0} f(s, t(t_0)) = L(t)^\beta \partial_t f$ with $t(t_0) = \int_0^{t_0} L(\tau)^\beta d\tau$. In these coordinates, the equation is $\partial_{t_0} f - im(\partial_s)f = g$. Writing $t_0(t)$ for the inverse of $t(t_0)$, applying the above formula gives

$$\begin{aligned} \partial_t f - iL(t)^{-\beta} m(\partial_s)f &= g \\ \iff \hat{f}(k, t) &= e^{-im(k)t_0(t)} \hat{f}_0(k) + \int_0^{t_0(t)} \hat{g}(k, t(\tilde{t}_0)) e^{im(k)(\tilde{t}_0 - t_0(t))} d\tilde{t}_0. \end{aligned}$$

A change of variables $\tilde{t}_0 = t_0(\tilde{t})$ gives

$$\hat{f}(k, t) = e^{-im(k)t_0(t)} \hat{f}_0(k) + \int_0^t L(\tilde{t})^{-\beta} \hat{g}(k, \tilde{t}) e^{im(k)(t_0(\tilde{t}) - t_0(t))} d\tilde{t}. \quad (4.7)$$

We will take advantage of this to allow the use of the abstract Cauchy–Kowalevski theorem, despite the fact that the original contour dynamics equation involves an operator of order higher than 1.

5. CAUCHY–KOWALEVSKI THEOREM

In this section, we describe the abstract Cauchy–Kowalevski theorem that we use to prove existence and uniqueness of solutions. The version which we present below follows the notation Sammartino and Caffisch, [23] (see also [22]). We have changed assumption (CK3) below so that the Cauchy-type estimate remains valid for $\beta > \beta_0$.

Because of this difference, we give the full proof below. The strategy of the proof is based on the methods of [17] and [1]. After the proof, we also make some remarks about the differences with [17].

Definition 5.1. Let $\rho_0 > 0$. A Banach scale $\{X_\rho, 0 < \rho < \rho_0\}$ with norms $\|\cdot\|_\rho$ is a collection of Banach spaces such that $X_{\rho'} \subset X_{\rho''}$ with $\|\cdot\|_{\rho''} \leq \|\cdot\|_{\rho'}$ whenever $\rho'' \leq \rho' \leq \rho_0$.

Definition 5.2. Given a Banach scale X_ρ , $\tau > 0$ and $0 < \rho \leq \rho_0$ and $R > 0$, we define:

- (1) $X_{\rho,\tau}$ to be the set of all functions $u(t)$ from $[0, \tau]$ to X_ρ endowed with the norm

$$\|u\|_{\rho,\tau} = \sup_{0 \leq t \leq \tau} \|u(t)\|_\rho.$$

- (2) $Y_{\rho,\beta,\tau}$ is the set of functions u such that for $t \in [0, \tau]$, $u(t) \in X_{\rho-\beta t}$, with the norm

$$\|u(t)\|_{\rho,\beta,\tau} = \sup_{0 \leq t \leq \tau} \|u(t)\|_{\rho-\beta t}.$$

- (3) We will denote by $X_{\rho,\tau}(R)$ and $Y_{\rho,\beta,\tau}(R)$ the balls of radius R in $X_{\rho,\tau}$ and $Y_{\rho,\beta,\tau}$ respectively.

Theorem 5.3. *Suppose that there exist $\rho_0 > 0$, $R > 0$, $\beta_0 > 0$, and $0 < T < \rho_0/\beta_0$, such that the following holds:*

(CK1) *For every pair ρ, ρ' such that $0 < \rho' < \rho < \rho_0 - \beta_0 T$ and every $u \in X_{\rho,T}(R)$, the function $F(t, u) : [0, T] \rightarrow X_{\rho'}$ is continuous.*

(CK2) *For every ρ such that $0 < \rho \leq \rho_0 - \beta_0 T$, the function $F(t, 0) : [0, T] \rightarrow X_{\rho,T}(R)$ is continuous in t , and*

$$\|F(t, 0)\|_{\rho_0-\beta_0 t} \leq R_0 < R.$$

(CK3) *For any numbers $\beta \geq \beta_0$, $s < t < \min(T, \rho_0/\beta)$, $\rho' > 0$, function $\rho(s)$ such that $0 < \rho' < \rho(s) < \rho_0 - \beta s$, and functions $u_1, u_2 \in Y_{\rho_0,\beta,T^*}(R)$ we have for a constant C independent of β ,*

$$\|F(t, u_1) - F(t, u_2)\|_{\rho'} \leq C \int_0^t \frac{\|u_1(s) - u_2(s)\|_{\rho(s)}}{\rho(s) - \rho'} ds.$$

Then there exist $\beta > \beta_0$, and $T^ \leq T$ such that there is a unique u belonging to $Y_{\rho_0,\beta,T^*}(R)$ that solves the equation*

$$u = F(t, u).$$

Proof. Let β_0, ρ_0, T, R be fixed constants as in the theorem. We introduce the following weighted Banach space for $\gamma \in (0, 1)$ arbitrary but fixed, $\beta \gg 1$ to be chosen later, and $T^* = T^*(\beta) := \min(T, \rho_0/\beta) \leq T$,

$$\mathbb{S}^{\gamma,\beta} = \{u : [0, T^*] \rightarrow X_{\rho_0-\beta T^*} : \|u\|^{(\gamma,\beta)} < \infty\},$$

where the weighted norm $\|u\|^{(\gamma,\beta)}$ is defined by

$$\|u\|^{(\gamma,\beta)} = \sup_{\substack{t < T^* \\ 0 < \rho' < \rho_0 - \beta t}} \left(1 - \frac{\beta t}{\rho_0 - \rho'}\right)^\gamma \|u(t)\|_{\rho'}. \quad (5.1)$$

Note that $\|u\|^{(\gamma, \beta)} \leq \|u\|_{\rho_0, \beta, T^*}$. If $0 < \tilde{\beta} < \beta$, then $\rho_0 - \beta t < \rho_0 - \tilde{\beta} t$, so making the choice $\rho' = \rho_0 - \beta t$ we have

$$\left(1 - \frac{\tilde{\beta} t}{\rho_0 - \rho'}\right)^\gamma = \left(\frac{\beta - \tilde{\beta}}{\beta}\right)^\gamma.$$

This implies the following inequalities for $0 < \tilde{\beta} < \beta$,

$$\|u\|^{(\gamma, \beta)} \leq \|u\|_{\rho_0, \beta, T^*} \leq \left(\frac{\beta}{\beta - \tilde{\beta}}\right)^\gamma \|u\|^{(\gamma, \tilde{\beta})}. \quad (5.2)$$

Contraction-type inequality. Here, we prove that for any $\beta > \beta_0$, $\gamma \in (0, 1)$, and for any $u, v \in Y_{\rho_0, \beta, T^*}(R)$, we have (note that this implies $u \in \mathbb{S}^{\gamma, \beta}$)

$$\|F(t, u) - F(t, v)\|^{(\gamma, \beta)} \leq \frac{C2^{1+\gamma}\|u - v\|^{(\gamma, \beta)}}{\gamma\beta}. \quad (5.3)$$

In particular, if

$$\beta > \frac{C2^{1+\gamma}}{\gamma}, \quad (5.4)$$

then F is a contraction. Define for $0 < \rho' < \rho_0 - \beta s$, $s < T^*$,

$$\rho(s) := \frac{\rho' + \rho_0 - \beta s}{2}.$$

Since $T^* < \frac{\rho_0}{\beta}$, $\rho' < \rho(s) \leq \rho_0 - \beta_0 s$, so we can apply (CK3). If we define $\lambda(s)$ by $\rho(s) = \rho' + \frac{\lambda(s)}{2}$, i.e.

$$\lambda(s) := \rho_0 - \rho' - \beta s, \quad (5.5)$$

then

$$\rho(s) - \rho' = \frac{\lambda(s)}{2} = \rho_0 - \rho(s) - \beta s. \quad (5.6)$$

So from (CK3), we obtain for $t < T^*$,

$$\begin{aligned} & \|F(t, u) - F(t, v)\|_{\rho'} \\ & \leq C \int_0^t \frac{\|u - v\|_{\rho(s)}}{\rho(s) - \rho'} ds \\ & = C \int_0^t \frac{\|u - v\|_{\rho(s)}}{\rho(s) - \rho'} \cdot \underbrace{\frac{(\rho_0 - \rho(s) - \beta s)^\gamma}{(\rho_0 - \rho(s))^\gamma}}_{= \left(1 - \frac{\beta s}{\rho_0 - \rho(s)}\right)^\gamma} \frac{(\rho_0 - \rho(s))^\gamma}{(\rho_0 - \rho(s) - \beta s)^\gamma} ds \end{aligned} \quad (5.7)$$

$$\leq C(\rho_0 - \rho')^\gamma \|u - v\|^{(\gamma, \beta)} \int_0^t \frac{ds}{(\lambda(s)/2)^{1+\gamma}} \quad (5.8)$$

$$\begin{aligned}
&= C(\rho_0 - \rho')^\gamma 2^{1+\gamma} \|u - v\|^{(\gamma, \beta)} \int_0^t \frac{ds}{(\rho_0 - \rho' - \beta s)^{1+\gamma}} \quad (5.9) \\
&= \frac{C(\rho_0 - \rho')^\gamma 2^{1+\gamma} \|u - v\|^{(\gamma, \beta)}}{\gamma \beta} \cdot \frac{-1}{(\rho_0 - \rho' - \beta s)^\gamma} \Big|_{s=0}^t \\
&= \frac{C 2^{1+\gamma} \|u - v\|^{(\gamma, \beta)}}{\gamma \beta} \left(\underbrace{\frac{(\rho_0 - \rho')^\gamma}{(\rho_0 - \rho' - \beta t)^\gamma} - 1}_{= \left(1 - \frac{\beta t}{\rho_0 - \rho'}\right)^{-\gamma} > 1} \right) \\
&\leq \frac{C 2^{1+\gamma} \|u - v\|^{(\gamma, \beta)}}{\gamma \beta} \cdot \frac{1}{\left(1 - \frac{\beta t}{\rho_0 - \rho'}\right)^\gamma}.
\end{aligned}$$

In going from (5.7) to (5.8), we used the definition of $\|\cdot\|^{(\gamma, \beta)}$ in (5.1), $\rho(s) > \rho'$ and (5.6) and then (5.5) is used to obtain (5.9).

Multiplying both sides by $\left(1 - \frac{\beta t}{\rho_0 - \rho'}\right)^\gamma$ and taking a supremum over t and ρ with $t < T^*$, $\rho' < \rho_0 - \beta t$ yields the desired inequality (5.3).

Iteration scheme. Set $u_0 := 0$ and inductively define $u_n := F(t, u_{n-1})$. Then (CK2) implies that

$$\|u_1\|_{\rho_0, \beta, T^*} \leq \|u_1\|_{\rho_0, \beta_0, T} \leq R_0 < R. \quad (5.10)$$

The goal is to iteratively apply (5.3). For this, we need to show that $u_n \in Y_{\rho_0, \beta_0, T}(R)$ for every $n > 1$. This will introduce a second condition, depending on the difference $R - R_0 > 0$, requiring that β must be large enough.

Control of norm of u_n . Define the auxiliary sequence b_k ($k \geq 0$) by

$$b_k = \beta \left(1 - \frac{1}{2^k}\right). \quad (5.11)$$

Note that b_k is an increasing sequence with $b_k \rightarrow \beta$. Also

$$b_k \in (\beta/2, \beta). \quad (5.12)$$

Since we want to apply (5.3) (which is only valid for $\beta > \beta_0$) with b_k in place of β , our construction requires

$$\beta > 2\beta_0. \quad (5.13)$$

Also, note that $\left(\frac{\beta}{\beta - b_k}\right)^\gamma = 2^{\gamma k}$. Therefore, for $k \geq 0$, by choosing $u = u_{k+1} - u_k$ in the second inequality of (5.2) and then applying (5.3) k times,

$$\|u_{k+1} - u_k\|_{\rho_0, \beta, T^*} \leq 2^{\gamma k} \|u_{k+1} - u_k\|^{(\gamma, b_k)}$$

$$\begin{aligned}
 &\leq 2^{\gamma k} \left(\frac{C2^{1+\gamma}}{\gamma b_k} \right)^k \|u_1 - u_0\|^{(\gamma, b_k)} \\
 &\leq \left(\frac{C4^{1+\gamma}}{\gamma\beta} \right)^k \|u_1\|^{(\gamma, b_k)}, \tag{5.14}
 \end{aligned}$$

since $u_0 = 0$, and $b_k > \beta/2$ from (5.12). Then (5.10) and the first inequality of (5.2) imply

$$\|u_1\|^{(\gamma, b_k)} \leq \|u_1\|_{\rho_0, b_k, T^*} \leq R_0. \tag{5.15}$$

Applying (5.14), (5.15), and (5.11) we obtain

$$\begin{aligned}
 \|u_n\|_{\rho_0, \beta, T^*} &\leq \sum_{k=0}^{n-1} \|u_{k+1} - u_k\|_{\rho_0, \beta, T^*} \\
 &\leq \sum_{k=0}^{n-1} \left(\frac{C4^{1+\gamma}}{\gamma\beta} \right)^k \|u_1\|^{(\gamma, b_k)} \\
 &\leq R_0 \sum_{k=0}^{\infty} \left(\frac{C4^{1+\gamma}}{\gamma\beta} \right)^k \\
 &= \frac{R_0\gamma\beta}{\gamma\beta - C4^{1+\gamma}}.
 \end{aligned}$$

Therefore, in order to ensure that $u_n \in Y_{\rho_0, \beta, T^*}(R)$, we need

$$\beta > \frac{C4^{\gamma+1}}{\gamma(R - R_0)}. \tag{5.16}$$

Existence and uniqueness of solution. Let β be large enough so that

$$\beta > \max \left(\frac{C2^{1+\gamma}}{\gamma}, 2\beta_0, \frac{C4^{\gamma+1}}{\gamma(R - R_0)} \right).$$

Then (5.4), (5.13) and (5.16) are satisfied. Thus for some $R_0 < R_1 < R$, $u_n \in Y_{\rho_0, \beta, T^*}(R_1)$, and the contraction inequality (5.3) implies that there is a unique solution to $u = F(t, u)$ in $Y_{\rho_0, \beta, T^*}(R)$. \square

Remark 5.4. As mentioned earlier, our assumption (CK2) does not match the analogous assumption of [23]. Furthermore, while F indeed satisfies a contraction-type inequality in the weighted norm $\|u\|^{(\gamma)} := \sup_{\rho' < \rho_0 - \beta t} (\rho_0 - \rho' - \beta t)^\gamma \|u(t)\|_{\rho'}$ for $\beta \gg 1$ (which is what is proven in [17]), it does not seem possible to control the Y_{ρ_0, β, T^*} norm of the successive iterates u_n because the right inequality of (5.2) is not true for the norm $\|\cdot\|^{(\gamma)}$, since if one tries to similarly use $\rho' = \rho_0 - \beta t$

to bound $\|u\|_{\rho_0, \beta, T^*}$, one finds possible blow-up at $t = 0$. Instead, we normalise the weight $\frac{(\rho' - \rho_0 - \beta t)^\gamma}{(\rho' - \rho_0)^\gamma} = \left(1 - \frac{\tilde{\beta} t}{\rho_0 - \rho'}\right)^\gamma$.

Now, we define the Banach scale that we will use in the proof.

Definition 5.5. Given $l \in \mathbb{N}$ and $\rho > 0$, define the open set $U_\rho \subset \mathbb{C}$,

$$U_\rho := \{z \in \mathbb{C} : |\operatorname{Im} z| < \rho\}.$$

We say that a function $f : U_\rho \rightarrow \mathbb{R}^2$ is in $K^{l, \rho}$ if

- (1) $f(s + i\tilde{s})$ is analytic and 1-periodic (i.e. with period 1) in s .
- (2) For every $|\operatorname{Im} s| < \rho$, $\partial_s^\alpha f(\operatorname{Re} s + i \operatorname{Im} s) \in L^2_{\operatorname{Re} s}(\mathbb{T})$, that is, square integrable as a periodic function of the real part only.
- (3) The norm $\|f\|_{K^{l, \rho}}$ is finite, where

$$\|f\|_{K^{l, \rho}} := \sum_{\alpha \leq l} \sup_{|\tilde{s}| < \rho} \|\partial_x^\alpha f(s + i\tilde{s})\|_{L^2_s(\mathbb{T})}.$$

Definition 5.6. Given $l \in \mathbb{N}$ and $\rho > 0$, define U_ρ as in Definition 5.5.

We say that $f : U_\rho^2 \rightarrow \mathbb{R}$ is in $K_2^{l, \rho}$ if

- (1) $f(s + i\tilde{s}, s_* + i\tilde{s}_*)$ is 1-periodic in s and s_* , and analytic in U_ρ^2 .
- (2) For every $\alpha_1 + \alpha_2 \leq l$, $\max(|\tilde{s}|, |\tilde{s}_*|) < \rho$, $\partial_s^{\alpha_1} \partial_{s_*}^{\alpha_2} f \in L^2_{s, s_*}(\mathbb{T}^2)$.
- (3) The norm $\|f\|_{K_2^{l, \rho}}$ is finite, where

$$\|f\|_{K_2^{l, \rho}} := \sum_{\alpha_1 + \alpha_2 \leq l} \sup_{\substack{|\tilde{s}| < \rho \\ |\tilde{s}_*| < \rho}} \|\partial_s^{\alpha_1} \partial_{s_*}^{\alpha_2} f(s + i\tilde{s}, s_* + i\tilde{s}_*)\|_{L^2_s(\mathbb{T})} \| \cdot \|_{L^2_{s_*}(\mathbb{T})}.$$

The norm $\|f\|_{K_2^{l, \rho}}$ can also be written as

$$\|f\|_{K_2^{l, \rho}} = \sum_{\alpha_1 + \alpha_2 \leq l} \sup_{\substack{|\tilde{s}| < \rho \\ |\tilde{s}_*| < \rho}} \|\partial_s^{\alpha_1} \partial_{s_*}^{\alpha_2} f(s + i\tilde{s}, s_* + i\tilde{s}_*)\|_{L^2_{s, s_*}(\mathbb{T}^2)}.$$

If $\mathcal{F}f(k) := \int_{\mathbb{T}} f(s) e^{-2\pi i k s} ds$, $k \in \mathbb{Z}$ denotes the Fourier transform of f , then the $K^{l, \rho}$ norm is equivalent to the weighted Sobolev norm

$$\|f\|_{K^{l, \rho}} = \|e^{2\pi \rho |k|} (1 + |k|^l) \mathcal{F}f(k)\|_{\ell_k^2(\mathbb{Z})},$$

which can be seen by analytic continuation in s_0 of the well-known identity for the Fourier transform

$$\mathcal{F}_x[f(s - s_0)](k) = e^{-2\pi i s_0 k} \mathcal{F}f(k).$$

The following standard result follows from the Sobolev embedding theorem (see for instance [9]).

Proposition 5.7 (Banach Algebra). *(1) For $l \geq 1$, $K^{l, \rho}$ is a Banach algebra:*

$$\|uv\|_{K^{l, \rho}} \lesssim_{l, \rho} \|u\|_{K^{l, \rho}} \|v\|_{K^{l, \rho}}.$$

(2) For $l \geq 2$, $K_2^{l,\rho}$ is a Banach algebra:

$$\|uv\|_{K_2^{l,\rho}} \lesssim_{l,\rho} \|u\|_{K_2^{l,\rho}} \|v\|_{K_2^{l,\rho}}.$$

We now choose the Banach scale X_ρ to apply Theorem 5.3. The purpose of l is only to obtain the above Banach Algebra property, and is fixed to be any number $l \geq 2$. We now use the spaces $K^{l,\rho}$ and $K_2^{l,\rho}$ to define our Banach scale. The choice is made so that a representative element of X_ρ will be of the form (z, z_s, z_{ss}, Γ) .

Definition 5.8. Let $l \geq 2$ be arbitrary but fixed. The Banach scale X_ρ is

$$X_\rho := K^{l,\rho} \times K^{l,\rho} \times K^{l,\rho} \times K_2^{l,\rho}.$$

5.1. Adapting the equation for Theorem 5.3. In order to apply Theorem 5.3, we will need to:

Step 1 Rewrite the equations so that the evolution begins from zero initial data, which allows the iteration to begin using the estimate (CK2).

Step 2 Rewrite the evolutions in a suitable integral form that satisfies the continuity assumption (CK1) and the required Cauchy estimate (CK3).

We now implement these steps.

Step 1. Define the initial conditions

$$\begin{aligned} f_0(s) &= f(s, 0), \\ g_0(s) &= g(s, 0), \\ h_0(s) &= h(s, 0), \\ \Gamma_0(s, s_*) &= \Gamma(s, s_*, 0). \end{aligned}$$

Then define the new variables $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\Gamma}$ by

$$\begin{aligned} \tilde{f}(s, t) &= f(s, t) - f(s, 0), \\ \tilde{g}(s, t) &= g(s, t) - g(s, 0), \\ \tilde{h}(s, t) &= h(s, t) - h(s, 0), \\ \tilde{\Gamma}(s, s_*, t) &= \Gamma(s, s_*, t) - \Gamma(s, s_*, 0). \end{aligned}$$

Then the evolution equations in terms of $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\Gamma}$ are

$$\tilde{f}_t = E_1[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{\Gamma} + \Gamma_0], \quad (5.17)$$

$$\tilde{g}_t = E_2[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{h}_s + h'_0, \tilde{\Gamma} + \Gamma_0], \quad (5.18)$$

$$\tilde{h}_t = (\tilde{\Gamma}(0, 0) + \Gamma_0(0, 0))^\beta \mathcal{H}_\beta(\tilde{h} + h_0),$$

$$+ E_3[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{h}_s + h'_0, \tilde{\Gamma} + \Gamma_0], \quad (5.19)$$

$$\tilde{\Gamma}_t = E_4[\tilde{\Gamma} + \Gamma_0, \tilde{g} + g_0, F_2[\tilde{g} + g_0, \tilde{h} + h_0, \tilde{h}_s + h'_0, \tilde{\Gamma} + \Gamma_0]]. \quad (5.20)$$

Step 2. The strategy is to integrate the equations for $\tilde{f}, \tilde{g}, \tilde{\Gamma}$ in time, and invert the operator $(\partial_t - (\tilde{\Gamma}(0, 0) + \Gamma_0(0, 0))\mathcal{H}_\beta)$ for the h equation. If we define the vector of functions u and initial conditions u_0 by

$$u(s, s_*, t) := \begin{pmatrix} \tilde{f}(s, t) \\ \tilde{g}(s, t) \\ \tilde{h}(s, t) \\ \tilde{\Gamma}(s, s_*, t) \end{pmatrix}, \quad u_0(s, s_*) = \begin{pmatrix} f_0(s) \\ g_0(s) \\ h_0(s) \\ \Gamma_0(s, s_*) \end{pmatrix},$$

then we can write the evolution equations as

$$u = F[u],$$

$$F[u] = \begin{pmatrix} F_1[u_0, u] \\ F_2[u_0, u] \\ F_3[u_0, u, \nabla u] \\ F_4[u_0, u] \end{pmatrix}, \quad (5.21)$$

where $\nabla u = \nabla_{s, s_*} u$ is the spatial gradient in s and s_* , and the component operators F_i of $F : X_\rho \rightarrow \cup_{\rho>0} X_\rho$ are

$$F_1 := f_0 + \int_0^t E_1 dt,$$

$$F_2 := g_0 + \int_0^t E_2 dt,$$

$$F_3 := \left((\partial_t - (\Gamma_0(0, 0, t) + \tilde{\Gamma}(0, 0))\mathcal{H}_\beta) \right)^{-1} E_3, \quad (5.22)$$

$$F_4 := \Gamma_0 + \int_0^t E_4 dt.$$

(The omitted inputs of F_i are as in (5.21), and the omitted inputs of E_i are as in (5.17), (5.18), (5.19), and (5.20).) The inverse operator in (5.22) is defined by (4.7). This completes the derivation of the equation to which Theorem 5.3 can be applied: it only remains to check that F satisfies the assumptions of Theorem 5.3.

5.2. Estimates. In this section, we give some estimates and explain how they are used to show that our system satisfies (CK3).

Many terms require a very similar approach; we will focus on a few representative terms that illustrate the main approach.

We write $U_1 = U_1(u), U_2 = U_2(u), \dots$ to denote the images of u under any of the following well behaved operators: $U_i(u) = u,$

$U_i(u)(s, s_*) = u(s + s_*, c)$, $U_i(u) = u(c, c_*)$, for constants c, c_* , or $U_i(u) = \mathcal{I}(u)$. (The c in $u(s + s_*, c)$ is arbitrary—the second input of u only affects $\tilde{\Gamma}$, and $\tilde{\Gamma}$ with first input $s + s_*$ doesn't appear.) For any collection of M such operators U_1, \dots, U_M ($M \geq 1$), we write $\mathbf{U}_M(u) = (U_1(u), \dots, U_M(u))$ for the function that takes values in $\mathbb{C}^{M'}$, where M' is an integer depending on M and the choices of U_i .

The operator $F_1[u_0, u]$ is a sum of time integrals of products of terms of the form

$$\Phi(\mathbf{U}_M(u)(s, s_*)),$$

where $\Phi : \mathbb{C}^{M'} \rightarrow \mathbb{C}^N$ is analytic, or terms of the form

$$\int_{\mathbb{T}} \Phi(\mathbf{U}_M(u)(s, s_*)) \, d\mu(s_*),$$

where μ is a finite measure in s_* , or terms of the form

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \Phi(\mathbf{U}_M(u)(s, s_*)) \, d\mu(s_*) \, ds.$$

For each one of these terms, we have the following elementary lemmas.

Lemma 5.9 (Triangle inequality for time integral). *For any function $u \in X_\rho$, $\|\int_0^t u \, dt\|_\rho \leq \int_0^t \|u\|_\rho \, dt$.*

Lemma 5.10 (local Lipschitz estimates). *Let $\Phi(x_1, \dots, x_{M'})$ be analytic on an open set containing the set $A = \{x : \sum_{i=1}^{M'} |x_i| \leq R\}$. Then Φ is locally Lipschitz on A ,*

$$|\Phi(x_1, \dots, x_{M'}) - \Phi(y_1, \dots, y_{M'})| \lesssim_{R, \Phi} \sum_{i=1}^{M'} |x_i - y_i|.$$

For $\|u\|_\rho \leq R$, we have the estimate

$$\|\Phi(\mathbf{U}_M(u_1)) - \Phi(\mathbf{U}_M(u_2))\|_{K_2^{l, \rho}} \lesssim_{R, \Phi, l} \|u_1 - u_2\|_\rho.$$

If in addition μ is a finite measure on \mathbb{T} , then we have the estimates

$$\left\| \int_{\mathbb{T}} \Phi(\mathbf{U}_M(u_1))(\cdot, s_*) - \Phi(\mathbf{U}_M(u_2))(\cdot, s_*) \, d\mu(s_*) \right\|_{K^{l, \rho}} \lesssim \|u_1 - u_2\|_\rho,$$

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}} \Phi(\mathbf{U}_M(u_1))(s, s_*) - \Phi(\mathbf{U}_M(u_2))(s, s_*) \, d\mu(s_*) \, ds \right| \lesssim \|u_1 - u_2\|_\rho.$$

In our application, we will use the finite measures $d\mu(s_*) = \frac{s_*^{k+1} ds_*}{|\sin s_*|^{\beta+k}}$, for some $k > 0$.

Example 5.11. In E_1 , the following terms appear (see (4.1)) :

$$\begin{aligned} \tilde{E}_1(u) &:= g(s) \frac{s}{L^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1) s_0}{|\text{Sin } s_0|^\beta} ds_0 ds_1 \\ &\quad - g(s) \frac{1}{L^2} \int_0^s \int_{\mathbb{T}} \Gamma(s_1, s_0)^\beta \frac{\mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1) s_0}{|\text{Sin } s_0|^\beta} ds_0 ds_1. \end{aligned}$$

The first term is (at least away from $s = 0, 1$ before periodising) a product of the analytic function of u and $u(0, 0)$, $g(s) \frac{s}{L^2}$ with the double integral against $\frac{s_0}{|\text{Sin } s_0|^\beta} ds_0 ds_1$ of the analytic function

$$\begin{aligned} &\Phi(u(s_1, s_0), \mathcal{I}(u)(s_1, s_0), u(s_0 + s_1, s_0 + s_1)) \\ &= \Gamma(s_1, s_0)^\beta \mathcal{I}(h)(s_1, s_0) \cdot h(s_0 + s_1). \end{aligned}$$

The second term is similar, and together with the first term, gives the analyticity at $s = 0, 1$ as well. Therefore, by Lemma 5.10, we have

$$\|\tilde{E}_1(u_1) - \tilde{E}_1(u_2)\|_{K^{l,\rho}} \lesssim \|u_1 - u_2\|_\rho,$$

which after integrating in time, is stronger than the required estimate (CK3). The other terms in E_1 are similar.

For F_2 , most of the terms are also treated in a different way, except one which involves h_s . For this term, we will use the following Cauchy-type estimate.

Lemma 5.12 (Cauchy Estimate). *For any $l \geq 0, \rho \geq 0, \rho' \in (0, \rho)$,*

$$\|\nabla u\|_{\rho'} \leq \frac{C}{\rho - \rho'} \|u\|_\rho.$$

Proof. It is enough to prove this for the component spaces $K^{l,\rho}$ and $K_2^{l,\rho}$. We give the proof for $K^{l,\rho}$, since $K_2^{l,\rho}$ can be treated in exactly the same way. That is, we shall prove for $u \in K^{l,\rho}$,

$$\|\partial_s u\|_{K^{l,\rho'}} \leq \frac{C}{\rho - \rho'} \|u\|_{K^{l,\rho}}.$$

From the definition of the $K^{l,\rho}$ norm in Definition 5.5, it suffices to prove that for every $v = \partial_s^r u$, $r = 0, 1, \dots, l$,

$$\sup_{|\tilde{s}| < \rho'} \|\partial_s v(s + i\tilde{s})\|_{L_s^2} \leq \frac{C}{\rho - \rho'} \sup_{|\tilde{s}| < \rho} \|v(s + i\tilde{s})\|_{L_s^2}. \quad (5.23)$$

Set $0 < \delta < \rho - \rho'$. Then the well-known Cauchy Integral Formula for a derivative gives for $z = s + i\tilde{s} \in \mathbb{C}$, $|\tilde{s}| < \rho'$,

$$\partial_s v(s + i\tilde{s}) = \frac{1}{2\pi i} \int_{|z-w|=\delta} \frac{v(w)}{(z-w)^2} dw = \frac{1}{2\pi i} \int_{|w|=\delta} \frac{v(s + i\tilde{s} + w)}{w^2} dw.$$

Taking the L_s^2 norm and using the periodicity of v in the real part to obtain (5.24),

$$\begin{aligned}
 \|\partial_s v(s + i\tilde{s})\|_{L_s^2} &\leq \frac{1}{2\pi} \int_{|w|=\delta} \frac{\|v(s + i\tilde{s} + w)\|_{L_s^2}}{|w|^2} dl(w) \\
 &= \frac{1}{2\pi} \int_{|w|=\delta} \frac{\|v(s + i(\tilde{s} + \text{Im } w))\|_{L_s^2}}{|w|^2} dl(w) \quad (5.24) \\
 &\leq \frac{1}{2\pi\delta^2} \int_{|w|=\delta} dl(w) \sup_{|\tilde{s}| < \rho' + \delta} \|v(s + i\tilde{s})\|_{L_s^2} \\
 &\leq \frac{1}{\delta} \sup_{|\tilde{s}| < \rho} \|v(s + i\tilde{s})\|_{L_s^2}.
 \end{aligned}$$

(dl is the arc-length measure on $|w| = \delta$.) Taking a limit $\delta \rightarrow \rho - \rho'$, and then a supremum over all \tilde{s} with $|\tilde{s}| < \rho'$ leads to (5.23). By the earlier discussion, we have finished the proof of Lemma 5.12. \square

Example 5.13. The first term of (4.2) is

$$\tilde{E}_2(u) = - \int_{\mathbb{T}} \Gamma(s, s_*)^\beta \frac{\mathcal{I}(h_s) s_*}{|\text{Sin } s_*|^\beta} ds_*.$$

To show that this term satisfies the assumption (CK3), first use the local Lipschitz estimates of Lemma 5.10 but treating the integrand as an analytic function of u and $\mathcal{I}(\nabla u)$. This yields a local Lipschitz estimate

$$\|\tilde{E}_2(u_1) - \tilde{E}_2(u_2)\|_{K^{l,\rho}} \lesssim \|u_1 - u_2\|_\rho + \|\nabla(u_1 - u_2)\|_\rho.$$

Now apply the Cauchy estimate for the second term; this shows that (CK3) is satisfied.

In a similar way, F_4 can be controlled by using the bounds on F_2 , since F_2 appears in F_4 .

The term E_3 , which uses the auxillary operator used in (4.7) to define F_3 is defined by (4.5). It involves the following terms where ∇u appears,

$$\begin{aligned}
 E_{31} &= \int_{\mathbb{T}} \mathcal{I}(\beta\Gamma^{\beta-1}\partial_{s_*}\Gamma)(s, s_*) s_* \frac{h_s(s + s_*) - h_s(s)}{|\text{Sin } s_*|^\beta} ds_*, \\
 E_{32} &= -2\beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h_s)(\mathcal{I}(g) \cdot \mathcal{I}(h)) s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_*, \\
 E_{33} &= -\beta \int_{\mathbb{T}} \Gamma(s, s_*)^{\beta+2} \frac{\mathcal{I}(h)(|\mathcal{I}(h)|^2 + \mathcal{I}(g) \cdot \mathcal{I}(h_s)) s_*^3}{|\text{Sin } s_*|^{\beta+2}} ds_*, \\
 E_{34} &= h_s G,
 \end{aligned}$$

where G is a collection of terms involving only u and not ∇u , defined by the last four lines of (4.6). These terms are controlled by combining the above lemmas with the skew-symmetry (Lemma 4.1) and the Cauchy-type estimate of Lemma 5.12.

Theorem 5.14. *Let $z_0 : \mathbb{T} \rightarrow \mathbb{R}^2$ be an analytic curve with $\Gamma_0 = \Gamma(z_0) \in L^\infty(\mathbb{T}^2)$. Then there exists $T^* > 0, \rho_0 > 0$ and $\beta > 0$ such that a unique solution to (1.1) exists in the space $u \in Y_{\rho_0, \beta, T^*}$.*

Proof. Since z_0 is analytic on \mathbb{T} , there exists $\rho > 0$ such that z_0 admits an analytic continuation to a complex neighbourhood $\mathbb{T} + i(-\rho, \rho)$ of \mathbb{T} , that belongs to the space $K^{l, \rho}$ (recall that we have already fixed some $l \geq 2$). From z_0 , we define the initial data to (5.21) as

$$u_0 = (z_0, \partial_s z_0, \partial_s^2 z_0, \Gamma(z_0)) \in X_{\rho_0}.$$

The operator F in (5.21) is continuous, satisfying (CK1) for Theorem 5.3, and there exists β_0, T such that (CK2) is satisfied. With the estimates and earlier discussion in this subsection, (CK3) is satisfied, so Theorem 5.3 applies, proving the result. \square

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REFERENCES

- [1] K Asano. On the abstract Cauchy–Kowalewski theorem. In *Functional analytic methods in complex analysis and applications to partial differential equations (Trieste, 1993)*, pages 185–197. World Sci. Publ., River Edge, NJ, 1995.
- [2] Z. Atkins. Almost sharp fronts: limit equations for a two-dimensional model with fractional derivatives, 2013.
- [3] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, and J. Wu. Generalized Surface Quasi-Geostrophic equations with singular velocities. *Comm. Pure Appl. Math.*, 65(8):1037–1066, 2012.
- [4] D. Chae, P. Constantin, & J. Wu. Inviscid models generalizing the two-dimensional Euler and the Surface Quasi-Geostrophic equations. *Arc. for Rat. Mech. and Analysis*, 202(1):35–62, Oct 2011.
- [5] P. Constantin, A. Majda & E. Tabak. Singular front formation in a model for quasigeostrophic flow”. *Phys. Fluids* 6(1), 9-11,(1994).
- [6] P. Constantin, A. Majda & E. Tabak. Formation of strong fronts in the 2 – D quasigeostrophic thermal active scala. *Nonlinearity* 7(6) 1495–1533, (1994).
- [7] A. Córdoba, D. Córdoba, & F. Gancedo. Uniqueness for SQG patch solutions. *ArXiv e-prints*, May 2016.

- [8] D. Córdoba, C. Fefferman, & J.L. Rodrigo. Almost sharp fronts for the Surface Quasi-Geostrophicequation. *Proceedings of the National Academy of Sciences of the United States of America*, 101(9):2687–2691, 2004.
- [9] L.C. Evans. Partial Differential Equations. *Graduate Studies in Mathematics*, 19, 1998.
- [10] C. Fefferman, G. Luli, and J.L. Rodrigo. The spine of an SQG almost-sharp front. *Nonlinearity*, 25(2):329, 2012.
- [11] C. Fefferman & J.L. Rodrigo. Analytic sharp fronts for the Surface Quasi-Geostrophic equation. *Comm. in Math. Phys.*, 303(1):261–288, 2011.
- [12] C. Fefferman & J.L. Rodrigo. Almost sharp fronts for SQG: the limit equations. *Comm. in Math. Phys.*, 313(1):131–153, 2012.
- [13] C.L. Fefferman & J.L. Rodrigo. Construction of Almost-Sharp Fronts for the Surface Quasi-Geostrophic equation. *Arch. for Rat. Mech. and Analysis*, 218(1):123 – 162, 2015.
- [14] F. Gancedo. Existence for the α -patch model and the QG sharp front in Sobolev spaces. *Advances in Mathematics*, 217(6):2569 – 2598, 2008.
- [15] F. Gancedo & N. Patel. On the local existence and blow-up for generalized SQG patches. *arXiv e-prints*, page arXiv:1811.00530, November 2018.
- [16] A. Kiselev, Y. Yao, & A. Zlatoš. Local regularity for the modified sqg patch equation. *Comm. Pure Appl. Math.*, 70(7):1253–1315, 2017.
- [17] M. C. Lombardo, M. Cannone, & M. Sammartino. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.*, 35(4):987–1004, 2003.
- [18] A. Majda & A. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge Texts in Applied Mathematics, 2002.
- [19] K. Ohkitani. Asymptotics and numerics of a family of two-dimensional generalized surface quasi-geostrophic equations. *Phys. of Fluids*, 24(9):095101, 2012.
- [20] J.L. Rodrigo. The Vortex Patch problem for the Surface Quasi-Geostrophic equation. *Proc. of the National Academy of Sciences*, 101(9):2684–2686, 2004.
- [21] J.L. Rodrigo. On the evolution of Sharp Fronts for the Quasi-Geostrophic equation. *Comm. Pure Appl. Math.*, 58(6):821–866, 2005.
- [22] M.V. Safonov. The abstract Cauchy–Kovalevskaya theorem in a weighted Banach space. *Comm. Pure Appl. Math.*, 48(6):629–637, 1995.
- [23] M. Sammartino & R.E. Caffisch. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space I. existence for Euler and Prandtl equations. *Comm. in Math. Phys.*, 192(2):433–461, 1998.
- [24] N.J. Zabusky, M.H. Hughes, & K.V. Roberts. Contour dynamics for the euler equations in two dimensions. *J. of Comp. Phy.*, 30(1):96 – 106, 1979.
- [25] J. K. Hunter, J. Shu, and Q. Zhang. Two-front solutions of the SQG equation and its generalizations. *arXiv preprint*, 2019.
- [26] J. K. Hunter, J. Shu, and Q. Zhang. Global Solutions of a Surface Quasi-Geostrophic Front Equation. *arXiv preprint*, 2018.
- [27] J. K. Hunter and J. Shu. Regularized and approximate equations for sharp fronts in the surface quasi-geostrophic equation and its generalizations. *Nonlinearity*, 31(6):2480–2517, 2018.
- [28] J. K. Hunter, J. Shu, and Q. Zhang. Local well-posedness of an approximate equation for SQG fronts, *J. Math. Fluid Mech.*, 20(4):1967–1984, 2018.
- [29] J. K. Hunter, J. Shu, and Q. Zhang. Contour dynamics for Surface Quasi-Geostrophic fronts. *arXiv preprint*, 2019.

- [30] C. Khor and J. L. Rodrigo. On sharp fronts and almost-sharp fronts for singular SQG. In preparation.