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# Dimensions, Embeddings and Iterated Function Systems

by

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# Abstract

In this thesis, we study the regularity of embeddings of finite dimensional subsets of Banach spaces into Euclidean spaces. We first consider subsets of Banach spaces with finite box–counting dimension and extend an embedding result due to Hunt & Kaloshin [15], which was previously known only for subsets of Hilbert spaces.

We then focus on almost homogeneous subsets of Banach spaces, which is a weaker notion of homogeneous sets, or sets with finite Assouad dimension. Olson & Robinson [25] showed that if $X$ is a subset of a Hilbert space and $X - X$ is almost homogeneous, then $X$ admits an almost bi–Lipschitz embedding into an Euclidean space. We extend this result for subsets of Banach spaces, using a weaker condition which requires that $X - X$ is almost homogeneous near the origin.

We also study the question of whether the set of differences is almost homogeneous at the origin, if the set itself is homogeneous. We answer the question negatively by considering a compact and homogeneous metric space $X$ such that the difference of isometric copies of $X$ into $L^{\infty}(X)$ is not almost homogeneous.

Finally, we find out that given an attractor $K$ of a Iterated function system that satisfies a weak separation condition, the Assouad dimension of $K-K$ is bounded above by twice the dimension of $K$. We then apply this result to a particular class of asymmetric Cantor sets.

# Chapter 1

# Introduction

Given a compact 'finite–dimensional' subset $X$ of a metric space $(M, d)$, it is natural to ask whether we can find a continuous map from $M$ onto some Euclidean space, such that if we restrict it to $X$, we obtain a homeomorphism from $X$ onto its image. The main question we want to address in this thesis is how we can understand the notion that an arbitrary subset $X$ of a Banach space is 'finite–dimensional'. There are many possible dimensions that we can consider and each of them provides different embedding properties into Euclidean spaces.

There have been embedding results concerning subsets with finite Hausdorff dimension by Mañé [22], finite box-counting dimension by Foias & Olson [9], finite Assouad dimension by Assouad [2], Olson & Robinson [25] and by Naor & Neiman [24]. In this thesis, we will concentrate on embeddings of subsets of Banach spaces with finite box-counting or Assouad dimension into Euclidean spaces.

In the second chapter, we concentrate on subsets of Banach spaces with finite box–counting dimension. The box-counting dimension $\mathrm{d}_B(X)$ of a compact set $X$ measures how does the minimum number of balls of radius $\epsilon$ that cover $X$ changes as $\epsilon \to 0$. In 1993, Ben Artzi et al [3] treated the case when $X$ is a subset of a Hilbert space with finite box–counting dimension and showed that there exists a projection into an Euclidean space which is injective on $X$ and has a Hölder continuous inverse. They also proved a sharp bound on the Hölder exponent. In 1996, Foias and Olson treated the case where $X$ is a subset of a Hilbert space $H$ with finite box–counting dimension and proved that there exists a dense set of linear maps $L\colon H \to \mathbb{R}^k$ which are injective on $X$ with Hölder inverses on the image of $X$. However, they did not obtain a bound on the Hölder exponent.

In 1999, Hunt and Kaloshin [15] introduced the thickness exponent $\tau$, an exponent which measures how well an arbitrary subset of a Banach space can be approximated by linear subspaces. They proved a breakthrough result that for any subset $X$ of a Hilbert space $H$ with finite box–counting dimension, and for every $0 < \theta < 1 + \tau/2$, there exists a $k \in \mathbb{N}$ and a dense set of linear maps $L\colon H \to \mathbb{R}^k$ that satisfies

$$\frac{1}{C_L}\|x-y\| \leq |Lx-Ly|^{\theta},$$

for any $x, y \in X$ and for some $C_L > 0$.

They actually proved that the set of linear maps that satisfy the above property is not only dense but also prevalent, a notion which can be viewed as an analogue of 'almost every' in the context of infinite dimensional spaces. They also showed that the bound on the Hölder exponent is sharp, i.e. there exists a set $X \subset H$ such that for any linear projection that is injective on $X$ and has a $\theta$-Hölder inverse, the exponent must be less or equal than $1 + \tau/2$. The sharpness result was also proved by Pinto De Moura and Robinson [6] by using the simpler example of an orthogonal sequence in $H$.

The authors attempted to extend the theorem for subsets of Banach spaces but their proof contained an error. Robinson [27] introduced the dual thickness and proved an embedding for subsets of Banach spaces with finite box–counting dimension. It has been an open question whether it is possible to prove the result for subsets of Banach spaces, using the thickness rather than the dual thickness. Motivated by this question, we prove an embedding result for subsets of Banach spaces with the restriction that the thickness exponent is less than 1. In particular, we prove the following theorem.

**Theorem 1.1.** *Let $X$ be a compact subset of a Banach space $\mathcal{B}$ with thickness exponent $\tau(X) < 1$ and box–counting dimension $\mathrm{d}_B(X) < \infty$. Then for any integer $k > 2d_B(X)$ and any given $\theta$ with*

$$0 < \theta < (1-\tau(X))\frac{k-2d_B(X)}{k\,(1+\tau(X))},$$

*a prevalent set of linear maps $L\colon \mathcal{B} \to \mathbb{R}^k$ satisfies:*

$$\|x-y\| \leq C_L|Lx-Ly|^{\theta}, \ \ \forall \ x, y \in X, \ \textit{for some } C_L > 0. \tag{1.1}$$

*In particular, $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

A second line of argument uses the Hahn–Banach theorem to construct linear embeddings from subsets of Banach spaces with finite box-counting dimension into Hilbert spaces. Using Hunt and Kaloshin's result for subsets of Hilbert spaces, we produce embedding theorems for subsets of Banach spaces into Euclidean spaces, even without the restriction for the thickness to be less than 1, with the cost of a factor of $1/1 + \mathrm{d}_B(X)$ in the bound of the Hölder exponent.

**Theorem 1.2.** *Let $X$ be a compact subset of a Banach space $\mathfrak{B}$ with thickness exponent $\tau(X)$ and box–counting dimension $\mathrm{d}_B(X)$. Then for any integer $k > 2\mathrm{d}_B(X)$ and any given $\theta$ with*

$$0 < \theta < \frac{k - 2d_B(X)}{k\,(1 + \mathrm{d}_B(X))\left(1 + \frac{\tau(X)}{2}\right)},$$

*there exists a linear map $L\colon \mathfrak{B} \to \mathbb{R}^k$ such that*

$$\|x - y\| \leq C_L|Lx - Ly|^{\theta}, \ \forall \ x, y \in X. \tag{1.2}$$

*In particular, $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

In the third chapter, we concentrate on the relation between the box-counting dimension of a compact subset of a Banach space and various thickness exponents. As mentioned above, these exponents play an important role in embedding theorems, especially when the embeddings are linear (as are the ones we consider). We study a particular class of orthogonal sets in $\ell_p$, for $p \in [1, \infty]$, which are used by De Moura & Robinson [6] to prove that Hunt and Kaloshin's theorem is sharp. We show that the thickness exponent of these sets equals the box–counting dimension when $p \in [1, 2]$ and satisfies a lower bound when $p > 2$.

In Chapter 4, we study doubling metric spaces $(X, d)$, which are spaces with the property that every ball in the set can be covered by a fixed number of balls with half the radius. In 1983, Assouad proved that for any $0 < \epsilon < 1$, doubling metric spaces admit bi–Hölder embeddings with Hölder exponent $\epsilon$, into an Euclidean space $\mathbb{R}^N$, for some $N$ depending on $\epsilon$. Assouad's pioneering work triggered a lot of research in the field of embeddings. Recent results by Naor & Neiman [24], based on work by Abraham, Bartal & Neiman [1], showed that we can actually choose $N$ in Assouad's theorem to be independent of $\epsilon$. The same result was also achieved independently by David & Snipes [5].

The doubling property remains invariant under bi–Lipschitz maps and we also know that any subset of an Euclidean space is doubling (see Chapter 9 in the book of Robinson [28]). Hence, a metric space needs to be doubling in order to admit a bi–Lipschitz embedding into an Euclidean space. However, the condition is not sufficient since there are examples of doubling spaces due to Lang & Plaut [21], Semmes [29] and Pansu [26] that cannot be embedded in a bi–Lipschitz way into any Hilbert space.

An equivalent definition of a doubling space is given which involves the

Assouad dimension $\mathrm{d}_A(X)$, a more local version of the box–counting dimension. The Assouad dimension is related to homogeneous sets, which are defined as follows.

**Definition 1.3.** *A subset $V$ of a metric space $(X,d)$ is said to be $(M,s)-$homogeneous if for every $r > \rho > 0$*

$$N_V(r,\rho) = \sup_{x\in V} N(V\cap B(x,r),\rho) \leq M\left(\frac{R}{r}\right)^s,$$

*where $N(V\cap B(x,r),\rho)$ denotes the minimum number of balls of radius $\rho$ required to cover $V\cap B(x,r)$.*

*The Assouad dimension of $V\subset (X,d)$, $\mathrm{d}_A(V)$ is defined as the infimum of all $s>0$ such that $V$ is $(M,s)$ homogeneous for some $M>0$.*

We are interested in the notion of a weaker class of embeddings which are called almost bi–Lipschitz. Given $\delta \geq 0$, we say that a map $L\colon (X,d_1)\to(Y,d_2)$, between two metric spaces is $\delta$-almost bi–Lipschitz if for $x\neq y\in X$, it satisfies

$$\frac{1}{C_L}\frac{d_1(x,y)}{\mathrm{slog}(d_1(x,y))^\delta} \leq d_2(L(x),L(y)) \leq C_L d_1(x,y),$$

where $\mathrm{slog}(x) = \log\left(x+\frac{1}{x}\right)$ is defined as the symmetric logarithm function, for $x>0$.

In 2010, Olson & Robinson [25] introduced a weaker notion of an $(\alpha,\beta)$–almost homogeneous metric space, which remains invariant under almost bi–Lipschitz maps.

**Definition 1.4.** *A metric space $(X,d)$ is $(\alpha,\beta)$-almost $(M,s)$- homogeneous if for any $0<\rho<r$*

$$N_X(r,\rho) \leq M\left(\frac{r}{\rho}\right)^s \mathrm{slog}(r)^\alpha \mathrm{slog}(\rho)^\beta. \tag{1.3}$$

The authors showed that when $X$ is almost homogeneous, it admits $\delta$-'almost' bi–Lipschitz embeddings into an infinite dimensional Hilbert space. In particular, the image in $H$ is almost homogeneous. They also obtained a bound on the exponent $\delta$.

The authors also study the case when $X$ is a subset of a Hilbert space such that the difference $X-X$ is almost homogeneous. They show that if such a condition is true then $X$ can be embedded into a sufficiently large Euclidean space using a linear map with an 'almost' Lipschitz inverse. In particular, they prove that when $X-X$ is homogeneous, $X$ admits $\delta$-almost bi–Lipschitz embeddings for all $\delta>3/2$. Unfortunately, the fact that $X$ is (almost) homogeneous does not necessarily imply that $X-X$ is also homogeneous (see examples of this kind of sets in chapter 9 in the book of Robinson [28]).

Sets of differences were also studied by Robinson [27], who proved that for any subset of a Banach space $X$ such that $X-X$ is homogeneous, $X$ admits $\delta$-almost bi–Lipschitz embeddings into Euclidean spaces, for all $\delta > 1$. The exponent $\delta$ was shown to be sharp, in this case.

The condition on the set of differences on the result of Olson & Robinson [25], was used to control covers of balls around the origin. In particular, their result can be restated to hold for sets $X$ such that $X - X$ is almost homogeneous at zero.

The interesting fact about this condition is that it remains invariant under linear almost bi–Lipschitz maps, as we show in section 4.2. In Section 4.2.1, we consider subsets of Banach spaces such that the set of differences is almost homogeneous at the origin, i.e. satisfy (1.3) only for balls around zero. We show that under this property, we can construct a linear embedding from a subset of a Banach space into a Hilbert space. Since the condition remains invariant, we then use Olson's & Robinson's result [25] to construct an embedding into an Euclidean space.

In Section 4.2.2, we extend the above result and we show that when $X - X$ is almost homogeneous at the origin, then there exists a prevalent set of linear almost bi–Lipschitz embeddings from $X$ into an Euclidean space. In particular, we show the following result.

**Theorem 1.5.** *Fix any $M \geq 1$, $s > 0$ and $\alpha, \beta \geq 0$. Suppose $X$ is a compact subset of a Banach space $\mathfrak{B}$ such that $X - X$ is $(\alpha, \beta)$-almost $(M, s)$-homogeneous at the origin. Then, given any $\delta > 1 + \frac{\alpha+\beta}{2}$, there exists a $N = N_\delta \in \mathbb{N}$ and a prevalent set of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^N$ that are injective on $X$ and bi–Lipschitz with $\delta$–logarithmic corrections. In particular, they satisfy*

$$\frac{1}{C_L} \frac{\|x - y\|}{\operatorname{slog}(\|x - y\|)^\delta} \leq |L(x) - L(y)| \leq C_L \|x - y\|, \tag{1.4}$$

*for some $C_L > 0$ and for all $x, y \in X$.*

We note that when $X - X$ is homogeneous, we obtain embeddings for any $\delta > 1$, exactly as in Robinson's result.

As mentioned above, Olson and Robinson showed [25] that any homogeneous metric space $(X, d)$ admits 'almost' bi–Lipschitz embeddings into a Hilbert space $H$ but it is not necessarily true that the image of $X$ in $H$ is homogeneous. However, they showed that the image can be 'well' approximated by finite–dimensional subspaces of $H$. It is natural to ask whether such a condition is sufficient for a subset of a Hilbert space to be embedded in an Euclidean space in an almost bi–Lipschitz way. Motivated by this question, we show that any subset of a Banach space that can be 'well approximated' by linear subspaces can be embedded into an infinite–dimensional Hilbert space in an almost bi–Lipschitz way.

In the next chapter, we study a metric space that was constructed by Laakso [20] and used by Lang and Plaut [21] as an example of a doubling space that cannot be embedded in a bi–Lipschitz way into any Hilbert space. Their construction was based on a sequence of graphs, equipped with the geodesic metric, which converges with respect to the Gromov–Hausdorff metric to a limiting metric space. We know by the result of Olson & Robinson [25], which was mentioned above that this set can be embedded in an almost bi–Lipschitz way into a Hilbert space. It is a natural question whether it also admits an almost bi–Lipschitz embedding into an Euclidean space.

Motivated by this question, we isometrically embed the metric space $X$ described above into $L^\infty(X)$ and look at the set of differences into $L^\infty(X)$. We show that $\Phi(X) - \Phi(X) \subset L^\infty(X)$ is not almost homogeneous at 0, for any choice of $(\alpha, \beta)$, thus giving an example of a set which is doubling but the set of differences is not $(\alpha, \beta)$-almost homogeneous at 0, for any choice of $(\alpha, \beta)$.

In Chapter 7, we study attractors of Iterated Function systems. In particular, we are interested in finite collections of functions $\{f_i \colon \mathbb{R}^s \to \mathbb{R}^s : i \in I\}$ that satisfy

$$|f_i(x) - f_i(y)| = c_i|x - y|,$$

for some $c_i < 1$, which holds for all $x, y \in \mathbb{R}^s$. These maps are called contracting similarities. It was proved by Hutchinson [17] that these systems determine a unique attractor $K$, i.e. a non-empty compact set such that

$$K = \bigcup_{i=1}^{|I|} f_i(K).$$

These attractors are called self similar fractals. The problem of bounding dimensions of self similar fractals has been extensively studied over the years by many authors. One of the most common dimensions that is considered, is the similarity dimension, which is denoted by $\mathrm{d_{sim}}$ and is defined as the positive number $D$ such that

$$\sum_{i=1}^{|I|} c_i^D = 1.$$

In general, for these sets we have that

$$\mathrm{d}_H(K) = \mathrm{d}_B(K) \leq \mathrm{d_{sim}}(K),$$

where $\mathrm{d}_H(K)$ is the Hausdorff dimension and $\mathrm{d}_B(K)$ is the box–counting dimension.

It has been shown by Hutchinson [17] that if there exists an non-empty open set $U$ such that all $f_i(U)$ are disjoint and $U$ contains the disjoint union of $f_i(U)$,

then

$$\mathrm{d}_H(K) = \mathrm{d}_B(K) = \mathrm{d}_{\mathrm{sim}}(K). \tag{1.5}$$

In case such an open sets exists, we say that the system satisfies the open set condition.

Zerner [30] introduced another condition, called the weak separation condition, which ensures that the images $f_i(K)$ do not overlap too much. In 2015, Fraser, Olson et al [11] used the notion of Ahlfors regularity and showed that if the weak separation condition is satisfied, then the Assouad dimension equals the Hausdorff dimension and in particular the box–counting dimension. In section 6.2, we give an independent proof of this result, without using Ahlfors regularity. The proof gives some insight for the more involved argument of bounding the Assouad dimension of differences of self similar fractals.

As discussed above, the Assouad dimension of sets of differences does not necessarily satisfy any bounds related with the Assouad dimension of the original set. Henderson [14] studied differences of self similar fractals and showed that there exist attractors of systems of contracting similarities on the real line that have arbitarily small Assouad dimension but the Assouad dimension of the set of differences is maximal. Our main purpose is to show that if the system satisfies a suitable separation condition, then we can achieve non–trivial bounds for the Assouad dimension of the set of differences. In particular, we show that under a suitable separation condition, we have $\mathrm{d}_A(K-K) \leq 2\mathrm{d}_A(K)$.

In the last chapter, we concentrate on Cantor sets, which are the most common examples of attractors of systems like the one we described above. Cantor sets are constructed by removing intervals from $[0,1]$ in an iterative process. The Cantor sets can be symmetric or non symmetric depending on whether the intervals that remain at each step of the iteration are of equal or non equal length.

A symmetric Cantor set, which is denoted by $C_\lambda$, is the attractor of a system of similarities

$$f_1(x) = \lambda x \qquad \text{and} \qquad f_2(x)\lambda x + (1-\lambda).$$

It is easy to see (see the book of Falconer [7]) that when $\lambda \leq 1/3$, the above maps satsify the open set condition for $U = (0,1)$, so all notions of dimensions coincide for these sets. Henderson [14] showed that differences of symmetric Cantor sets are actually attractors of another system of similarities on the unit interval that satisfies the open set condition. In particular, he showed that for all $\lambda < \frac{1}{2}$,

$$\mathrm{d}_A(C_\lambda - C_\lambda) < 2\mathrm{d}_A(C_\lambda).$$

We show that symmetric Cantor sets $C_\lambda$, for $\lambda < \frac{1}{3}$ satisfy the weak separation condition for differences, which immediately gives an example of a set that satisfies the weak separation condition for differences with a strict inequality for $\mathrm{d}_A(C_\lambda - C_\lambda)$.

In the second part of the chapter, we focus on non symmetric Cantor sets which are the attractors of the following system.

$$f_1(x) = c_1 x \qquad \text{and} \qquad f_2(x) = c_2 x + (1 - c_2).$$

We denote the Cantor set in this case by $K_{c_1 c_2}$. Henderson [14] showed that if $\frac{\log c_1}{\log c_2}$ is an irrational number then $\mathrm{d}_A(K_{c_1 c_2} - K_{c_1 c_2}) = 1$, which is maximal for this set. We show that we can prove a non trivial bound for $\mathrm{d}_A(K_{c_1 c_2} - K_{c_1 c_2})$ when $\frac{\log c_1}{\log c_2}$ is rational, $c_1 < 1/4$ and $c_2 < 1/4$. In particular, we prove the following theorem, which is the main result of this chapter

**Theorem 1.6.** *Suppose $c \in (0,1)$, $p_1 < p_2 \in \mathbb{N}$ such that $c^{p_1} < 1/4$ , $c^{p_2} < 1/4$. Let $K$ be the attractor of the system $\mathcal{F} = \{f_1, f_2\}$ where*

$$f_1(x) = c^{p_1} x, \qquad \textit{and} \qquad f_2(x) = c^{p_2} x + (1 - c^{p_2}).$$

*Then,*

$$\mathrm{d}_A(K - K) \leq 2\mathrm{d}_A(K).$$

# Chapter 2

# Embedding properties of sets with finite box-counting dimension

## 2.1 Background

In this chapter, we focus on subsets of Banach spaces with finite box–counting dimension, whose definition we now recall.

**Definition 2.1.** *Suppose that $(E, \|\cdot\|)$ is a normed space. Let $X$ a compact subset of $E$ and let $N(X,\epsilon)$ denote the minimum number of balls of radius $\epsilon$ with centres in $X$ required to cover $X$. The upper box-counting dimension of $X$ is*

$$\mathrm{d}_B(X) = \limsup_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon}. \tag{2.1}$$

It follows from the definition that if $d > \mathrm{d}_B(X)$, then there exists some positive constant $C = C_d$, such that

$$N(X,\epsilon) \leq C\epsilon^{-d}. \tag{2.2}$$

For the rest of the thesis, we will refer to $\mathrm{d}_B(X)$ as the box–counting dimension of $X$. We now want to recall some elementary but useful properties of the box–counting dimension, which we will be using in what follows. The proofs can be found in Robinson [28].

**Lemma 2.2.** *Suppose $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ are normed spaces.*

1. *If $f\colon (E_1, \|\cdot\|_1) \to (E_2, \|\cdot\|_2)$ is a Lipschitz function, i.e. there exists a constant $C > 0$, such that*

$$\|f(x_1) - f(x_2)\|_2 \leq C\|x_1 - x_2\|_1 \text{ for all } x_1, x_2 \in E_1,$$

*then for all compact subsets $X \subset E_1$ we have*

$$\mathrm{d}_B(f(X)) \leq \mathrm{d}_B(X).$$

2. *Let $E_1 \times E_2$ be the product space equipped with some product metric. Suppose also that $X \subseteq E_1$ and $Y \subseteq E_2$ are compact. Then,*

$$\mathrm{d}_B(X \times Y) \leq \mathrm{d}_B(X) + \mathrm{d}_B(Y).$$

In 1999, Hunt and Kaloshin [15] established the existence of a 'large' set of linear maps $L\colon H \to \mathbb{R}^k$ with Hölder continuous inverses on the image of $X$. In order to do so, they introduced a new quantity, called the thickness exponent of $X$, which measures how well an arbitrary subset of a Banach space can be approximated by finite–dimensional linear subspaces. We note that all Banach spaces we mention from now on are real.

**Definition 2.3.** *Let $X$ be a subset of a Banach space $\mathfrak{B}$. The thickness exponent of $X$ in $\mathfrak{B}$, $\tau(X, \mathfrak{B})$ is defined as:*

$$\tau(X, \mathfrak{B}) = \limsup_{\epsilon \to 0} \frac{\log d(X, \epsilon)}{-\log \epsilon},$$

*where $d(X, \epsilon)$ denotes the smallest dimension of those linear subspaces $V$ that satisfy*

$$\mathrm{dist}_{\mathfrak{B}}(x, V) \leq \epsilon \text{ for all } x \text{ in } X.$$

*If no such subspace exists, we set $d(X, \epsilon) = \infty$ and we adopt a similar convention throughout this thesis.*

Note that when $\tau > \tau(X)$, then there exists some positive constant $C$ such that

$$d(X, \epsilon) \leq C\epsilon^{-\tau}.$$

Therefore, when $X \subset \mathbb{R}^k$, for some $k \in \mathbb{N}$, then $\tau(X) = 0$.

It is easy to see that the thickness exponent is always bounded above by the box–counting dimension. Indeed, if we cover $X$ by $N(X, \epsilon)$ balls of radius $\epsilon$ and let

$V$ be the subspace of $\mathfrak{B}$ that is spanned by the centres of these balls, then every element of $X$ is within $\epsilon$ of $V$.

Moreover, Hunt & Kaloshin used the notion of prevalence, which was introduced by Hunt [16] in 1992 as a generalisation of the term 'almost every' in the context of infinite dimensional spaces. We only define what we need for the purpose of this thesis and without proving any of the statements. For a more detailed presentation, you can see Chapter 5 in the book of Robinson [28] or Chapter 6 in Benyamini & Lindenstrauss [4].

**Definition 2.4.** *Let $(V, \|\cdot\|)$ be a normed linear space. A Borel set $S \subset V$ is called shy if there exists a compactly supported probability measure $\mu$ on $V$ such that*

$$\mu(S+v) = 0,$$

*for every $v \in V$. In general, a set is shy if it is contained in a shy Borel set. A set is called prevalent if its complement in $V$ is shy.*

*Remark* 2.1. The set $\mathbb{E} = \mathrm{supp}(\mu)$, is called the 'probe space' and it easy to see that a set $S$ is prevalent if for every $v \in V$, $v + u \in S$, for $\mu-$almost every $u \in \mathbb{E}$. A prevalent set is dense with respect to the norm of $V$ and it is also straightforward (see Chapter 5 in Robinson) to show that the intersection of prevalent sets is prevalent.

Using the above structure, Hunt and Kaloshin [15] proved the following result.

**Theorem 2.5** (Hunt & Kaloshin, 1999)**.** *Let $X$ be a compact subset of a real Hilbert space $H$ with thickness exponent $\tau(X)$ and box–counting dimension $\mathrm{d}_B(X) < \infty$. Then for any integer $k > 2d_B(X)$ and any given $\theta$ with*

$$0 < \theta < \frac{k - 2d_B(X)}{k\left(1 + \frac{\tau(X)}{2}\right)},$$

*there exist a prevalent set of linear maps $L\colon H \to \mathbb{R}^k$ such that*

$$\|x - y\| \le C_L |Lx - Ly|^{\theta}, \ \ \forall \ x, y \in X, \ \text{for some } C_L > 0. \tag{2.3}$$

*In particular, every such $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

By using the fact that the thickness is bounded above by the box–counting dimension, the above theorem can be restated such that the range of the exponent $\theta$ depends solely on the box-counting dimension.

The authors attempted to extend the theorem for subsets of Banach spaces and their proof relies on the claim that there exists a linear isometry from the dual of any finite–dimensional subspace of $\mathfrak{B}$ to a linear subspace of the dual of $\mathfrak{B}$. However, Kakutani & Mackey[18] proved that this can only be true in the context of a Hilbert space. To circumvent this problem, Robinson [27] introduced a new exponent, the 'dual thickness' which was defined based on an approximation required in the course of Hunt and Kaloshin's argument.

**Definition 2.6.** *Suppose that $X$ is a subset of a Banach space $\mathcal{B}$ and for every $\epsilon, \theta > 0$ let $\mathrm{d}_\theta(X, \epsilon)$ denote the minimum dimension of all those subspaces $U$ of $\mathcal{B}^*$ with the property that for every $x, y \in X$ with $\|x - y\| \geq \epsilon$, there exists some $\phi \in U$, such that*

$$|\phi(x - y)| \geq \epsilon^{1+\theta}.$$

*Then, for every $\theta > 0$, we define*

$$\tau_\theta^*(X) = \limsup_{\epsilon \to 0} \frac{\log \mathrm{d}_\theta(X, \epsilon)}{-\log \epsilon},$$

*and then set*

$$\tau^*(X) = \lim_{\theta \to 0} \tau_\theta^*(X).$$

This admittedly unwieldy definition allows for the following result.

**Theorem 2.7** (Robinson, 2009)**.** *Let $X$ be a compact subset of a Banach space $\mathfrak{B}$ with dual thickness exponent $\tau^*(X) < \infty$ and box–counting dimension $\mathrm{d}_B(X) < \infty$. Then for any integer $k > 2\mathrm{d}_B(X)$ and any given $\theta$ with*

$$0 < \theta < \frac{k - 2\mathrm{d}_B(X)}{k\,(1 + \tau^*(X))},$$

*there exist a prevalent set of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^k$ such that*

$$\|x - y\| \leq C_L |Lx - Ly|^\theta, \;\; \forall\;\; x, y \in X, \;\; \textit{for some } C_L > 0. \tag{2.4}$$

*In particular, every such $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

In Chapter 3 we prove that the dual thickness is bounded above by twice the box–counting dimension of $X$ which gives a range of $\theta$ independent of the dual thickness. In particular, for any

$$0 < \theta < \frac{1}{1 + 2\mathrm{d}_B(X)}, \tag{2.5}$$

we can find an embedding into $\mathbb{R}^k$, for large enough $k$, such that the inverse is $\theta$-Hölder continuous.

There is no known general relation between the thickness and the dual thickness in the context of a Banach space. However, Robinson [28] proved that zero thickness implies zero dual thickness, which immediately implies that subsets of Banach spaces with $\tau(X) = 0$ admit embeddings for any positive exponent $\theta < 1$. It is a natural question to ask whether we can prove an embedding theorem directly, i.e. without using the dual thickness when the thickness is small enough. In Section 2.4, we show that when $\tau(X) < 1$, such an embedding is true and we establish the same embedding result for $\tau(X) = 0$ directly, i.e. without having to use the dual thickness.

In Section 2.2, we consider $X$ as a compact subset of a Banach space with finite box-counting dimension and provide an embedding into a Hilbert space with a bound on the Hölder exponent of the inverse that depends on the box–counting dimension of $X$. As a corollary of this argument and Theorem 4.5 we immediately obtain an embedding into an Euclidean space that does not require the dual thickness. We also extend the result to any compact metric space using the Kuratowski embedding.

Then, in Section 2.4, we use the techniques introduced by Hunt and Kaloshin along with some key new arguments and prove an embedding theorem for subsets of Banach spaces with thickness exponent less than 1.

## 2.2 Embeddings from Banach into Hilbert spaces

In this section, we prove two embedding results into a Hilbert space. Both of them can be combined with Theorem 4.5 to provide an embedding theorem for compact subsets of Banach spaces into Euclidean spaces that does not require the arguments in Robinson's result [27].

### 2.2.1 Embedding when $\mathrm{d}_B(X)$ is finite

We first show that any compact subset of a Banach space with finite box–counting dimension embeds into a Hilbert space.

**Proposition 2.8.** *Suppose that $X$ is a compact subset of a Banach space $\mathfrak{B}$ with finite box-counting dimension. Then, for every $\alpha > 1 + \mathrm{d}_B(X)$ there exists a linear map $\Phi\colon \mathfrak{B} \to H$, where $H$ is a separable Hilbert space, such that for every $x, y \in X$*

$$C_\alpha^{-1}\|x - y\|^\alpha \le |\Phi(x) - \Phi(y)| \le C_\alpha\|x - y\|, \text{ for some } C_\alpha > 0.$$

We first prove that for every $n \in \mathbb{N}$ there exists a linear embedding $\phi_n$ into a Euclidean space $\mathbb{R}^{m_n}$ such that $\phi_n^{-1}$ satisfies a Lipschitz condition for all $x, y \in X$ with $\|x-y\| \geq 2^{-n}$.

**Lemma 2.9.** *Suppose that $X$ is as above. Then, given $d > \mathrm{d}_B(X)$ and $n \in \mathbb{N}$, there exist $\phi_n \in \mathcal{L}(\mathcal{B}; \mathbb{R}^{m_n})$, where $m_n \leq C 2^{2nd}$, such that $\|\phi_n\| \leq \sqrt{m_n}$ and*

$$|\phi_n(x-y)| \geq 2^{-(n+1)} \text{ whenever } \|x-y\| \geq 2^{-n}, \text{ for } x,y \text{ in } X.$$

*Proof.* Let $Z = X - X = \{x - y : x, y \in X\}$. Then, it is easy to see that $\mathrm{d}_B(Z) \leq 2\mathrm{d}_B(X)$. Indeed, $Z$ is the image of $X \times X$ under the Lipschitz map

$$(x, y) \mapsto x - y$$

and so by Lemma 2.2, we obtain

$$\mathrm{d}_B(f(X \times X)) = \mathrm{d}_B(X - X) \leq \mathrm{d}_B(X \times X) \leq 2\mathrm{d}_B(X).$$

Given $d$ as in the statement of the lemma, we can cover $Z$ by no more than $m_n = N(Z, 2^{-(n+2)}) \leq C_d\, 2^{2nd}$ balls of radius $2^{-(n+2)}$. Let the centres of these balls be $z_i$. By the Hahn–Banach Theorem, we can find $f_i \in \mathcal{B}^*$ such that $\|f_i\| = 1$ and $f_i(z_i) = \|z_i\|$. Now, define $\phi_n \colon \mathcal{B} \to \mathbb{R}^{m_n}$ as

$$\phi_n(x) = (f_1(x), ..., f_{m_n}(x)).$$

It is immediate that $\|\phi_n\| \leq \sqrt{m_n}$.

Suppose now that $z \in X - X$ such that $\|z\| \geq 2^{-n}$. Then, there exists some $i \leq m_n$ such that $\|z - z_i\| \leq 2^{-(n+2)}$, and therefore

$$\begin{aligned}|\phi_n(z)| &\geq |f_i(z)| = |f_i(z_i) + f_i(z - z_i)| \\ &\geq \|z_i\| - \|z - z_i\| \geq \|z\| - 2\|z - z_i\| \\ &\geq 2^{-n} - 2^{-(n+1)} \geq 2^{-(n+1)}. \square\end{aligned}$$

We now continue with the proof of Proposition 2.8.

*Proof of Proposition 2.8.* We first construct a new separable Hilbert space $H$ given an orthonormal basis $(e_i)_{i=1}^{\infty}$ of $\ell^2$ and a sequence $(m_i)_{i=1}^{\infty}$ of positive integers, by taking the collection

$$e_i \otimes w_j^{m_i} \qquad i \in \mathbb{N},\ j = 1, \ldots, m_i,$$

as an orthonormal basis for $H$, where $(w_j^N)_{j=1}^N$ denotes an orthonormal basis for $\mathbb{R}^N$. We define the inner product $\langle \cdot, \cdot \rangle$ on $H$ to ensure that this is indeed an orthonormal set, i.e. we set

$$\langle e_i \otimes w_j^{m_i}, e_{i'} \otimes w_{j'}^{m_{i'}} \rangle = \delta_{ii'}\delta_{jj'}.$$

In particular if $x_i \in \mathbb{R}^{m_i}$, $i \in \mathbb{N}$, then

$$\left\| \sum_{i=1}^{\infty} e_i \otimes x_i \right\|_H^2 = \sum_{i=1}^{\infty} \|x_i\|_{\mathbb{R}^{m_i}}^2.$$

Take $d = d_\alpha > 0$ such that $\alpha > 1 + d > 1 + \mathrm{d}_B(X)$. Then, for this $d > \mathrm{d}_B(X)$, we consider $\phi_n, m_n$ given by Lemma 2.9, and then from the above construction we consider $H$ based on the sequence $(m_n)_{n=1}^\infty$. Now, for $x \in \mathfrak{B}$ we set

$$\Phi(x) = \sum_{n=1}^{\infty} 2^{(1-\alpha)n} \phi_n(x) \otimes e_n \in H.$$

Clearly $\Phi$ is linear and

$$\|\Phi\|^2 \leq \sum_{n=1}^{\infty} 2^{2(1-\alpha)n} \|\phi_n\|^2 \leq \sum_{n=1}^{\infty} 2^{2(1+d-\alpha)n} < \infty,$$

since $1 + d - \alpha < 0$.

Now, take any $x, y \in X$ and suppose $x \neq y$ (the case $x = y$ is trivial). If $\|x - y\| \geq 1$, then it suffices to take $R > 0$ such that

$$X - X \subset B(0, R).$$

Therefore, using also that $\|\phi_1(x - y)\| \geq \frac{1}{4}$, we have that

$$\|\Phi(x - y)\| \geq 2^{1-\alpha} \|\phi_1(x - y)\| \geq \frac{2^{1-\alpha}}{4} \left( \frac{\|z\|}{R} \right)^\alpha = C_\alpha \|z\|^\alpha.$$

If $0 < \|x - y\| < 1$, consider $n$ such that

$$2^{-n} \leq \|x - y\| < 2^{-(n-1)}.$$

Thus, we obtain

$$\begin{aligned} \|\Phi(x - y)\| &\geq 2^{(1-\alpha)n} |\phi_n(x - y)| \\ &\geq 2^{(1-\alpha)n} 2^{-n-1} \geq C_\alpha \, 2^{-\alpha n + 1} \\ &\geq C_\alpha \|x - y\|^\alpha, \end{aligned}$$

which concludes the proof of the embedding into a Hilbert space. □

By combining Proposition 2.8 and Theorem 4.5, we can now obtain an embedding theorem for compact subsets of Banach spaces into finite–dimensional spaces. The difference here is that the range of the exponent depends on the thickness and the box–counting dimension rather than the dual thickness.

**Theorem 2.10** (**Theorem 1.2**). *Let $X$ be a compact subset of a Banach space $\mathfrak{B}$ with thickness exponent $\tau(X)$ and box–counting dimension $\mathrm{d}_B(X)$. Then for any integer $k > 2d_B(X)$ and any given $\theta$ with*

$$0 < \theta < \frac{k - 2\mathrm{d}_B(X)}{k\,(1 + \mathrm{d}_B(X))\left(1 + \frac{\tau(X)}{2}\right)},$$

*there exists a linear map $L\colon \mathfrak{B} \to \mathbb{R}^k$ such that*

$$\|x - y\| \le C_L |Lx - Ly|^{\theta}, \ \forall \ x, y \in X. \tag{2.6}$$

*In particular, $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

*Proof.* Take $\theta_1$ such that

$$\frac{\theta k\left(1 + \frac{\tau(X)}{2}\right)}{k - 2\mathrm{d}_B(X)} < \theta_1 < \frac{1}{1 + \mathrm{d}_B(X)},$$

and set

$$\theta_2 = \frac{\theta}{\theta_1} < \frac{k - 2\mathrm{d}_B(X)}{k\left(1 + \frac{\tau(X)}{2}\right)}.$$

By Proposition 2.8 (substituting $\alpha = \theta_1^{-1}$), there exists a separable Hilbert space $H$ and a linear map $\Phi\colon \mathfrak{B} \to H$ such that for every $x, y \in X$

$$C_{\theta_1}^{-1}\|x - y\| \le |\Phi(x) - \Phi(y)|^{\theta_1}, \text{ for some } C_{\theta_1} > 0.$$

We know from Lemma 2.2 that the box–counting dimension of $X$ does not increase under $\Phi$. We now check that the same holds for the thickness exponent of $X$. Take $\epsilon > 0$ and let $V$ be the linear subspace of $\mathfrak{B}$ with the smallest dimension among all those that satisfy

$$\mathrm{dist}(x, V) < \epsilon,$$

for all $x \in X$. Let $y = \Phi(x) \in \Phi(X)$ and if we let $v \in V$ such that

$$\|v - x\| < \epsilon,$$

then

$$\|y - \Phi(v)\| \leq \|\Phi\|\epsilon.$$

Since $\Phi(V)$ is a linear subspace of $H$ and $\dim(\Phi(V)) \leq \dim(V)$, we have

$$\begin{aligned}\tau(\Phi(x)) &= \limsup_{\epsilon\to 0} \frac{\log d_H\left(\Phi(X), \|\Phi\|\epsilon\right)}{-\log\|\Phi\|\epsilon} \\ &\leq \limsup_{\epsilon\to 0} \frac{\log d_{\mathfrak{B}}\left(X, \epsilon\right)}{-\log\|\Phi\|\epsilon} = \tau(X).\end{aligned}$$

Since

$$\theta_2 = \frac{\theta}{\theta_1} < \frac{k - 2d_B(X)}{k\left(1 + \frac{\tau(X)}{2}\right)} \leq \frac{k - 2d_B(X)}{k\left(1 + \frac{\tau(\Phi(X))}{2}\right)},$$

by Theorem 2.7 there exists a linear map $T\colon H \to \mathbb{R}^k$ and a positive constant $C_\theta$ such that

$$\|x - y\| \leq C_\theta |T(\Phi(x)) - T(\Phi(y))|^{\theta_1\theta_2},$$

for all $x, y \in X$. We conclude the proof by setting $L = T \circ \Phi$. □

Note that the above theorem gives an embedding for all compact subsets of Banach spaces with finite box–counting dimension, since $\tau(X) \leq \mathrm{d}_B(X)$. The result improves on the range of $\theta$ from Theorem 2.7 (see (2.5)) whenever

$$\tau(X) < \frac{2\mathrm{d}_B(X)}{1 + \mathrm{d}_B(X)}.$$

However, we note that when $\tau(X) = 0$, we obtain $\theta$-Hölder embeddings for any

$$0 < \theta < \frac{1}{1 + \mathrm{d}_B(X)},$$

which is not optimal.

The above embedding into a Banach space can also be used as a tool to prove an embedding theorem for compact metric spaces with finite box–counting dimension. We first recall the Kuratowski embedding theorem, which allows us to isometrically embed any compact metric space into a Banach space.

**Lemma 2.11** (Kuratowski Embedding)**.** *If $(X, d)$ is any compact metric space, then the map*

$$x \mapsto \Phi(x) = d(\cdot, x)$$

*is an isometry of $(X, d)$ onto a subset of $L^\infty(X)$.*

For a proof of the above result, see Heinonen [13]. Hence, we have the following corollary of Theorem 1.2 and Lemma 2.11, using the fact that

$$\tau(\Phi(X)) \leq \mathrm{d}_B(\Phi(X)) = \mathrm{d}_B(X),$$

where $\Phi$ is the isometry from Lemma 2.11 and $X$ is an arbitrary compact metric space.

**Corollary 2.12.** *Suppose $(X,d)$ is a compact metric space with finite box–counting dimension. Then, for any $k > 2\mathrm{d}_B(X)$ and any given $\theta$ with*

$$0 < \theta < \frac{k - 2\mathrm{d}_B(X)}{k\left(1 + \mathrm{d}_B(X)\right)\left(1 + \frac{\mathrm{d}_B(X)}{2}\right)},$$

*there exists a Lipschitz map $\psi\colon (X,d) \to \mathbb{R}^k$ such that*

$$\|x - y\| \leq C_\psi |\psi(x) - \psi(y)|^\theta, \ \forall \ x, y \in X.$$

The range of $\theta$ in the above Corollary improves on the respective range in the paper by Foias and Olson [9]. There, the authors use a direct argument to prove that if $d = \max\{1, \mathrm{d}_B(X)\}$, then for any given

$$\theta < \frac{1}{2d\left(1 + \frac{\mathrm{d}_B(X)}{2}\right)}$$

any compact metric space with finite box-counting dimension can be embedded into a sufficiently large Euclidean space such that the inverse is $\theta$-Hölder continuous.

### 2.2.2 Embedding when $\tau(X) < 1$

We now prove another embedding into a Hilbert space, in which the range of the Hölder exponent depends solely on the thickness exponent of $X$. This result also provides some motivation towards the next section.

**Proposition 2.13.** *Suppose that $X$ is a compact subset of a Banach space $\mathfrak{B}$ with thickness exponent $\tau(X) < 1$. Then, for every*

$$\alpha > \frac{1 + \tau(X)}{1 - \tau(X)}$$

*there exists a separable Hilbert space $H$ and a linear map $\Phi\colon \mathcal{B} \to H$, such that*

$$C_\alpha^{-1}\|x - y\|^\alpha \leq |\Phi(x) - \Phi(y)| \leq C_\alpha\|x - y\|, \textit{ for all } x, y \textit{ in } X.$$

Following the previous procedure, we first need the following Lemma.

**Lemma 2.14.** *Suppose that $\tau(X) < 1$. Then, for every $1 > \tau > \tau(X)$, there exists some $\beta_\tau > 1$ such that for every $n \in \mathbb{N}$, we can find $\phi_n \in \mathcal{L}(\mathcal{B}; \mathbb{R}^{m_n})$, $C_{\beta_\tau} > 0$, where $m_n \leq C_{\beta_\tau} 2^{\beta_\tau nk}$, with $\|\phi_n\| \leq \sqrt{m_n}$ and*

$$|\phi_n(x-y)| \geq 2^{-(\beta_\tau n+1)}, \text{ whenever } \|x-y\| \geq 2^{-n}.$$

Before we prove the above lemma, we recall the definition of an Auerbach basis for a finite–dimensional Banach space.

**Definition 2.15.** *Suppose that $U$ is a finite–dimensional Banach space. An Auerbach basis for $U$ is formed by a basis $\{e_1, ..., e_n\}$ of $U$ coupled with corresponding elements $\{f_1, ... f_n\}$ of $U^*$ that satisfy $\|f_i\| = \|e_i\| = 1$ and*

$$f_i(e_j) = \delta_{ij}.$$

For a proof of the existence of such a basis, see Exercise 7.3 in the book of Robinson [28], for example.

*Proof of Lemma 4.17.* Take any $\beta > 1$. Then, by the definition of the thickness exponent, there exists a subspace $V_n$ of $\mathcal{B}$ and some $C = C_\beta > 0$, such that $\dim(V_n) = m_n \leq C 2^{\beta n \tau}$ and

$$\operatorname{dist}(x, V_n) \leq 2^{-\beta n - 2}.$$

Suppose that $\{u_1^n, .., u_{m_n}^n\}$ is an Auerbach basis for $V_n$, and let $\{f_1^n, .., f_{m_n}^n\}$ be the corresponding elements of $V_n^*$ that satisfy $\|f_i^n\| = 1$, $\forall i$ and

$$f_i^n(u_j^n) = \delta_{ij}.$$

We now define a projection $P_n$ onto $V_n$ as

$$P_n(x) = \sum_{i=1}^{m_n} f_i^n(x) u_i^n$$

and define $\phi_n : \mathfrak{B} \to \mathbb{R}^{m_n}$ by setting

$$\phi_n(x) = (f_1^n(x), ..., f_{m_n}^n(x)).$$

Obviously $\|\phi_n\| \leq \sqrt{m_n} \leq 2^{\beta n \tau / 2}$. Moreover, let $z \in X - X$ be such that $\|z\| \geq 2^{-n}$ and choose $z_n \in V_n$ such that

$$\|z - z_n\| \leq 2^{-\beta n - 2}.$$

Then

$$\|z_n\| \geq 2^{-n} - 2^{-\beta n-2} \geq 2^{-n} - 2^{-n-2} \geq 2^{-n-1}.$$

Now, write $z_n = \sum_{i=1}^{m_n} z_n^i u_i^n$ and take $j \leq m_n$ such that $\|(z_n^1, ..., z_n^{m_n})\|_\infty = |z_n^j|$. Then,

$$\begin{aligned} |\phi_n(z)\|_2 &\geq |f_j^n(z)| \geq |f_j^n(z_n)| - |f_j^n(z - z_n)| \\ &\geq |z_n^j| - \|z - z_n\| \geq m_n^{-1}\|z_n\| - 2^{-\beta n-2} \\ &\geq C2^{-\beta n\tau}2^{-n} - 2^{-\beta n-2} = C2^{-n(1+\beta\tau)} - \frac{1}{4}2^{-\beta n}. \end{aligned}$$

Now, we choose $\beta = \beta_\tau$ such that

$$1 + \beta_\tau\tau = \beta_\tau \Leftrightarrow \beta_\tau = \frac{1}{1-\tau} > 1,$$

which concludes the proof. □

We now prove Proposition 4.16 .

*Proof of Proposition 4.16.* Take $1 > \tau > \tau(X)$ such that

$$\alpha > \frac{1+\tau}{1-\tau} = \beta_\tau + \tau\beta_\tau,$$

and let $\phi_n, m_n$ be as given in the previous lemma.

Now, let $(e_n)_{n=1}^\infty$ be the standard basis for $\ell_2$ and following the construction in the proof of Proposition 2.8 we define $H$ based on the sequence $(m_n)_{n=1}^\infty$. We now set

$$\Phi(x) = \sum_{n=1}^\infty 2^{(\beta_\tau-\alpha)n}\phi_n(x) \otimes e_n \in H.$$

Then,

$$\|\Phi\| \leq \sum_{n=1}^\infty 2^{(\beta_\tau-\alpha)n}2^{\tau\beta_\tau n} < \infty.$$

Now, take any $x, y \in X$ with $x \neq y$. If $\|x - y\| \geq 1$, we argue exactly as in the proof of Proposition 2.8. If $0 < \|x - y\| < 1$, let $n$ such that $2^{-n} \leq \|x - y\| < 2^{-n+1}$.

Therefore

$$\begin{aligned} \|\Phi(x-y)\| &\geq 2^{(\beta_\tau-\alpha)n}\|\phi_n(x-y)\|_2 \\ &\geq 2^{(\beta_\tau-\alpha)n}2^{-\beta_\tau n-1} \\ &\geq 2^{-\alpha n-1} \geq C_\alpha\|x-y\|^\alpha, \end{aligned}$$

which gives the desired result. □

Just as in the previous situation, we can now obtain a linear embedding from a compact subset of a Banach space with finite box-counting dimension into a finite–dimensional space such that the inverse is $\theta$-Hölder continuous for any

$$0 < \theta < \frac{1-\tau(X)}{(1+\tau(X))\left(1+\frac{\tau(X)}{2}\right)}.$$

However, in the next section, we give a more direct proof that not only improves this exponent, but also provides a prevalent set of embeddings.

## 2.3 A measure based on sequences of linear subspaces

Before we prove our main embedding result, we will recall, following Robinson [28], the construction of a compactly supported probability measure that is based on the ideas in Hunt and Kaloshin [15] and will play a key role in our proof. For a more detailed analysis, see also Appendix A.

Suppose that $\mathfrak{B}$ is a Banach space and $\mathcal{V} = \{V_n\}_{n=1}^{\infty}$ a sequence of finite–dimensional subspaces of $\mathfrak{B}^*$, the dual of $\mathfrak{B}$. Let us denote by $d_n$ the dimension of $V_n$ and by $B_n$ the unit ball in $V_n$.

Now, we fix a real number $\alpha > 1$ and define the space $\mathbb{E}_\alpha(\mathcal{V})$ as the collection of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^k$ given by

$$\mathbb{E} = \mathbb{E}_\alpha(\mathcal{V}) = \left\{ L = (L_1, L_2, ..., L_k) : L_i = \sum_{n=1}^{\infty} n^{-\alpha}\phi_{i,n},\ \phi_{i,n} \in B_n \right\}.$$

Let us also define

$$\mathbb{E}_0 = \left\{ \sum_{n=1}^{\infty} n^{-\alpha}\phi_{i,n},\ \phi_{i,n} \in B_n \right\}.$$

Clearly $\mathbb{E} = (\mathbb{E}_0)^k$.

To define a measure on $\mathbb{E}$, we first take a basis for $V_n$ so that we can identify $B_n$ with a symmetric convex set $U_n \subset \mathbb{R}^{d_n}$. Then, we construct each $L_i$ randomly by choosing each $\phi_{i,n}$ with respect to the normalised $d_n$–dimensional Lebesgue measure $\lambda_n$ on $U_n$. Finally, by taking $k$ copies of this measure we obtain a measure on $\mathbb{E}$. In particular we first consider $\mathbb{E}_0$ as a product space

$$\mathbb{E}_0 = \prod_{n=1}^{\infty} B_n,$$

and define a measure $\mu_0$ on $\mathbb{E}_0$ as

$$\mu_0 = \otimes_{n=1}^{\infty} \lambda_n.$$

Secondly, we consider $\mathbb{E} = \mathbb{E}_0^k$ and define $\mu$ on $\mathbb{E}$ as

$$\mu = \prod_{i=1}^{k} \mu_0.$$

For any map $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$, Hunt and Kaloshin [15] proved the following upper bound on

$$\mu\{L \in \mathbb{E} : |(f + L)x| \leq \epsilon\},$$

for $x \in \mathcal{B}$ and any $\epsilon > 0$. For a more detailed proof, see Robinson [28] or the Appendix in this thesis. (Appendix A)

**Lemma 2.16.** *Suppose that $x \in \mathcal{B}$, $\epsilon > 0$, $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$ and $\mathcal{V} = \{V_n\}$ as above. Then*

$$\mu\{L \in \mathbb{E} : |(f + L)(x)| < \epsilon\} \leq \left( n^{\alpha} d_n \frac{\epsilon}{|g(x)|} \right)^k,$$

*for any $g \in B_n$.*

## 2.4 Prevalent set of embeddings in $\mathbb{R}^k$

We are now ready to state and prove the main result of this section.

**Theorem 2.17 (Theorem 1.1).** *Let $X$ be a compact subset of a Banach space $\mathfrak{B}$ with thickness exponent $\tau(X) < 1$ and box–counting dimension $\mathrm{d}_B(X) < \infty$. Then for any integer $k > 2d_B(X)$ and any given $\theta$ with*

$$0 < \theta < (1 - \tau(X)) \frac{k - 2\mathrm{d}_B(X)}{k\,(1 + \tau(X))},$$

*there exists a prevalent set of linear maps $L \colon \mathcal{B} \to \mathbb{R}^k$ that satisfy*

$$\|x - y\| \leq C_L |Lx - Ly|^{\theta}, \;\; \forall \;\, x, y \in X. \tag{2.7}$$

*In particular, $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

The proof follows some of the techniques introduced in Hunt and Kaloshin's argument with some key differences. In particular, we first use the thickness exponent to construct a sequence of finite–dimensional subspaces of $\mathfrak{B}$, that 'approximate' $X$.

Then, we use an Auerbach basis to define a sequence of finite–dimensional subspaces of the dual of $\mathfrak{B}$ and define our probability measure based on this sequence.

*Proof.* Clearly (2.7) holds if and only if

$$\|z\| \leq C_L |Lz|^{\theta} \ \ \forall \ \ z \in X - X. \tag{2.8}$$

We want to bound the measure of linear maps that fail to satisfy (2.8) for some $z$ in a restricted subset of $X - X$. Take $1 > \tau > \tau(X)$ and $d > \mathrm{d}_B(X)$ such that

$$0 < \theta < (1-\tau)\frac{k-2d}{k\,(1+\tau)}. \tag{2.9}$$

Take some $\beta > 1$, which will be chosen later on and for every $n \in \mathbb{N}$, by definition of the thickness exponent, we can find a linear subspace $V_n \subset \mathcal{B}$ such that

$$\dim(V_n) \leq C_\beta 2^{\theta n \tau \beta} \tag{2.10}$$

and

$$\mathrm{d}(X, V_n) \leq \frac{2^{-\theta n \beta}}{3}. \tag{2.11}$$

Using an Auerbach basis for $V_n$, along with the Hahn–Banach theorem, we construct a subspace $G_n$ of $\mathcal{B}^*$, as follows. Suppose that

$$\{e_1^n, ..., e_{d_n}^n\}$$

is a basis for $V_n$ and

$$\{r_1^n, ..., r_{d_n}^n\}$$

is the corresponding basis for $V_n^*$, which satisfies:

$$\|r_i^n\| = 1, \ \ \forall \ \ i$$

and

$$r_i^n(e_j^n) = \delta_{ij}, \ \ \forall \ \ i \neq j.$$

Using the Hahn–Banach theorem, we extend the elements $r_1^n, ..., r_{d_n}^n \in V_n^*$ to maps $f_1^n, ..., f_{d_n}^n$ in $\mathfrak{B}^*$ and set

$$G_n = \langle f_1^n, .., f_{d_n}^n \rangle,$$

a subspace of $\mathcal{B}^*$, that is at most $d_n$–dimensional.

We now construct a measure based on the sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$, according to the definitions given in the beginning of this section.

In particular, if $S_n$ is the unit ball in $G_n$, we define

$$\mathbb{E} = \mathbb{E}_2(\mathcal{G}) = \{L = (L_1, .., L_k) : L_i = \sum_{n=1}^{\infty} n^{-2}\phi_{i,n}, \phi_{i,n} \in S_n\}.$$

We now fix a map $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$. Based on the above construction, we consider

$$Z_n = \{z \in X - X : \|z\| \geq 2^{-\theta n}\}$$

and

$$Q_n = \{L \in E : |(f + L)z| \leq 2^{-n}, \text{ for some } z \in Z_n.\}.$$

Our goal is to bound the measure of $Q_n$ by something summable over $n$ and use the Borel–Cantelli Lemma.

Using the fact that $\mathrm{d}_B(X - X) \leq 2\mathrm{d}_B(X)$, we cover $Z_n$ by $C_1 2^{2nd}$ closed balls of radius $2^{-n}$, for some positive constant $C$. We observe that if $z$ is in the intersection of $Z_n$ with one of these balls, which we denote by $B(z_0, 2^{-n})$, then

$$|(f+L)z_0| \leq |(f+L)(z)|+|(f+L)(z-z_0)| \leq 2^{-n}+(\|f\|+\|L\|)2^{-n} = (1+\|f\|+\|L\|)2^{-n}.$$

But, $\|L\|$ is bounded uniformly for all $L \in \mathbb{E}$. Indeed

$$\|L\|^2 \leq \sum_{i=1}^{k} |L_i|^2$$

and

$$|L_i|^2 = \left|\sum_{n=1}^{\infty} n^{-2}\phi_{i,n}\right|^2 \leq \sum_{n=1}^{\infty} n^{-4} = C_2 < \infty.$$

Hence,

$$|(f + L)z_0| \leq M2^{-n},$$

for some positive constant $M$, which holds for all $L \in \mathbb{E}$ and depends on $f$, which is fixed.

Now, we wish to bound the measure of $L \in \mathbb{E}$ such that $(f + L)$ fails to satisfy (2.8), for some $z$ in $Y = Z_n \cap B(z_0, 2^{-n})$. From the above discussion, we have that

$$\mu\{L \in E : |(f + L)z| \leq 2^{-n} \text{ for some } z \in Y\} \leq \mu\{L \in E : |(f + L)z_0| \leq M2^{-n}\}.$$

Now, consider $z_n \in V_n$ such that $\|z_n - z_0\| \leq 2^{-\theta\beta n}/3$. Therefore,

$$\|z_n\| \geq 2^{-\theta n} - 2^{-\theta\beta n}/3 \geq 2^{-\theta n}.$$

We now write $z_n$ as

$$z_n = \sum_{i=1}^{d_n} z_n^i e_i^n,$$

and consider $j \leq d_n$ such that

$$z_n^j = \|(z_n^1, ..., z_n^{d_n})\|_\infty.$$

We now define

$$g_n = f_j^n,$$

which satisfies

$$\|g_n\| = 1 \ \text{ and } \ |g_n(z_n)| \geq d_n^{-1}\|z_n\|.$$

Hence,

$$\begin{aligned}|g_n(z_0)| &\geq d_n^{-1}\|z_n\| - \|z_n - z_0\| \geq C_\beta 2^{-n\theta\beta\tau} 2^{-\theta n} - 2^{-\theta\beta n}/3 \\ &= C_\beta 2^{-n\theta(\beta\tau+1)} - 2^{-n\theta\beta}/3.\end{aligned}$$

We now choose $\beta$ such that $\beta\tau + 1 = \beta \iff \beta = \frac{1}{1-\tau}$, which gives that

$$|g_n(z_0)| \geq C_3 2^{-n\theta\beta},$$

with $C_3 > 0$ independent of $n$. Using Lemma 2.16, we obtain:

$$\begin{aligned}\mu\{L \in E : |f(z_0) + L(z_0)| \leq M2^{-n}\} &\leq \left(n^2\, d_n \frac{M2^{-n}}{|g_n(z_0)|}\right)^k \\ &\leq C_3^k \left(n^2 2^{n\beta\theta\tau} 2^{-n} 2^{\theta\beta n}\right)^k.\end{aligned}$$

Thus,

$$\mu(Q_n) \leq C_1 C_3^k 2^{2nd} \left(n^2 2^{n\beta\theta\tau} 2^{-n} 2^{\theta\beta n}\right)^k,$$

so the sum $\sum_{n=1}^{\infty} \mu(Q_n)$ is finite iff

$$\theta < (1-\tau)\frac{k - 2\mathrm{d}_B(X)}{k\,(1+\tau)}.$$

Thus, by the Borel–Cantelli lemma $\mu(\limsup Q_n) = 0$, i.e. $\mu$–almost every $L$ lies in only a finite number of the $Q_n$. For such an $L$, there exists a $n_L$, such that for every $n \geq n_L$, $L$ does not belong to $Q_n$. In particular

$$\text{if } |z| \geq 2^{-n\theta} \text{ then } |(f+L)z| \geq 2^{-n}, \text{ for all } n \geq n_L.$$

To complete the argument, we use the fact that $X - X$ is compact and we claim the existence of an $R > 0$, such that $X - X \subseteq B(0, R)$.

Now, let $z \in X - X$ and consider the following cases

$$\text{if } |z| \geq 2^{-n_L\theta},$$

then

$$|(f+L)z| \geq 2^{-n_L} \geq \frac{2^{-n_L}}{R^{\frac{1}{\theta}}}|z|^{\frac{1}{\theta}}.$$

If

$$|z| \leq 2^{-n_L\theta},$$

then there exists $n \geq n_L$ such that

$$2^{-(n+1)\theta} \leq |z| < 2^{-n\theta}.$$

Thus,

$$|(f+L)z| \geq 2^{-(n+1)} > \frac{1}{2}|z|^{\frac{1}{\theta}}.$$

We now put these two cases together to conclude that

$$|(f+L)z| \geq C_L|z|^{\frac{1}{\theta}}, \tag{2.12}$$

where

$$C_L = \max\left\{\frac{2^{-n_L}}{R^{\frac{1}{\theta}}}, 2^{-1}\right\}. \tag{2.13}$$

All in all, for every map $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$, $f + L$ satisfies (2.12), for $\mu$-almost every $L \in \mathbb{E}$. Hence, there exists a prevalent set of maps $L \in \mathbb{E}$ that satisfy

$$\|z\| \leq C_L|Lz|^{\theta} \ \forall \ z \in X - X. \qquad \square$$

We note that when $\tau(X) = 0$, given any $0 < \theta < 1$, $X$ admits finite–dimensional embeddings with a $\theta$-Hölder continuous inverse, exactly as in the previous two theorems.

We also note that the above result provides a bigger range for $\theta$ comparing to (2.5), whenever

$$\frac{1-\tau(X)}{1+\tau(X)} > \frac{1}{1+2\mathrm{d}_B(X)} \Leftrightarrow \tau(X) < \frac{\mathrm{d}_B(X)}{1+\mathrm{d}_B(X)}.$$

The restriction on the thickness here is surprising and it is an interesting open problem whether there is a result that extends Theorem 1.1 for thickness $\tau \geq 1$.

# Chapter 3

# Thickness and Dual Thickness

In this chapter, we look closely at the thickness and dual thickness and how they relate to the box–counting dimension. We prove some general estimates and we also look at a particular class of sequences in $\ell_p$ which was used by Pinto de Moura and Robinson [6] to prove that Theorem 2.7 is asymptotically sharp as $k \to \infty$. We note that we already know that the thickness is bounded above by the box–counting dimension.

## 3.1 General estimates for $\tau$ and $\tau*$

We first give an immediate upper bound on the dual thickness based on yet another exponent.

**Definition 3.1.** *Suppose that $\mathfrak{B}$ is a Banach space and $X \subset \mathfrak{B}$. Then, given any $\alpha > 0$ and $\epsilon > 0$ we denote by $m_\alpha(X, \epsilon)$ the smallest dimension of all those finite–dimensional subspaces $V$ of $\mathfrak{B}^*$ such that whenever $x, y \in X$ with $\|x - y\| \geq \epsilon$ there exists some $\Phi \in V$ with $\|\Phi\| = 1$ that satisfies*

$$|\Phi(x - y)| \geq \alpha\epsilon.$$

*Then we define*

$$\sigma_\alpha(X) = \limsup_{\epsilon \to 0} \frac{\log m_\alpha(X, \epsilon)}{-\log \epsilon}.$$

Following Robinson [28], we have the following estimate.

**Lemma 3.2.** *Suppose that $\mathfrak{B}$ is a Banach space and $X \subset \mathfrak{B}$. Then*

$$\tau^*(X) \leq \sigma_\alpha(X), \ \ \forall\ \alpha > 0.$$

*Proof.* Let $\theta > 0$. Let $V$ be a finite–dimensional subspace of $\mathcal{B}^*$ such that for all $x, y \in X$ with $\|x - y\| \geq \epsilon$, there exists $\phi \in V$ with $\|\phi\| = 1$ and $|\phi(x-y)| \geq \alpha\epsilon$. If $\epsilon$ is small enough such that $\epsilon^\theta < \alpha$, then

$$|\phi(x-y)| \geq \epsilon^{1+\theta},$$

which gives

$$\tau_\theta^*(X) \leq \sigma_\alpha(X), \ \ \forall\, \theta > 0. \qquad \square$$

We now prove that in a Hilbert space the dual thickness is always bounded above by the thickness. The result was also proved by Robinson [28], using another exponent, called the Lipschitz deviation of $X$. We give an independent proof, which only depends on $\tau, \tau^*$.

**Lemma 3.3.** *Suppose $H$ is a Hilbert space and $X \subset H$, such that $\tau(X) < \infty$. Then*

$$\tau^*(X) \leq \tau(X).$$

*Proof.* Take $\epsilon > 0$ and let $U$ be a finite–dimensional subspace of $H$ such that

$$\dim(U) = d = d(X, \epsilon) \ \text{ and } \ \mathrm{dist}(X, U) < \epsilon.$$

Now we consider $P$, the orthonormal projection onto $U$. For all $x \in X$ we have

$$\|x - Px\| = \mathrm{dist}(x, U) < \epsilon.$$

Let

$$V = \{L \circ P : L \in U^*\} \subset H^*.$$

It is easy to see that $V$ is finite–dimensional and that $\dim(V) = d$.

Suppose that $x, y \in X$ satisfy $\|x - y\| \geq \epsilon$. Since $P(x-y) \equiv z \in U$, we define $L \colon U \to \mathbb{R}$ such that

$$L_z(u) = \frac{\langle u, z\rangle}{\|z\|},$$

for all $u \in U$.

Then, $\Phi = L_z \circ P \in V$ and $\|\Phi\| = 1$. Moreover,

$$|\Phi(x-y)| = |L_z(z)| = \|z\| = \|P(x) - P(y)\| = \|x - y\| \geq \epsilon.$$

Therefore,

$$\tau^*(X) \leq \sigma_1(X) \leq \tau(X). \qquad \square$$

In the context of a Banach space, there is no known relationship between the thickness and the dual thickness. In the paper of Robinson [27], it is claimed that the dual thickness is bounded above by the box–counting dimension, but there is an error in the proof given there. However, we can prove that the dual thickness is bounded above by the box dimension of the difference set $X - X$, which in particular is always bounded above by twice the box dimension of $X$. Hence, when the box–counting dimension is finite, the dual thickness is finite as well.

**Lemma 3.4.** *Suppose $\mathfrak{B}$ is a Banach space and $X \subset \mathfrak{B}$ compact. Then*

$$\tau^*(X) \leq \mathrm{d}_B(X - X) \leq 2\mathrm{d}_B(X).$$

*Proof.* Let $Z = X - X$. Given $\epsilon > 0$ and any $d > \mathrm{d}_B(Z)$, we find $N = N(Z, \epsilon) \lesssim \epsilon^{-d}$ balls of radius $\epsilon$ with centres $z_j$ that cover $Z$. By the Hahn–Banach theorem, for any $j \leq N$, we obtain linear functionals $\phi_j$ that satisfy

$$\|\phi_j\| = 1 \text{ and } |\phi_j(z_j)| = \|z_j\|.$$

We now define $V = \mathrm{span}(\phi_1, ..., \phi_N)$.

Suppose now that $z \in Z$ such that $\|z\| \geq 50\epsilon$ and let $j \leq N$ such that $\|z - z_j\| < \epsilon$. Thus,

$$\|z_j\| \geq 49\epsilon,$$

which gives

$$|\phi_j(z)| = |\phi_j(z - z_j) + \phi_j(z_j)| \geq \|z_j\| - \epsilon \geq 48\epsilon.$$

This shows that

$$\sigma_{48/100}(X) \leq \mathrm{d}_B(Z),$$

and the conclusion is immediate by Lemma 3.2. □

The above upper bound indicates that Theorem 4.5 is also true for subsets of Banach spaces $X$ such that $\mathrm{d}_B(X - X) < \infty$. Indeed, Robinson's argument [27] requires the sets of differences to have finite box–counting dimension, which in particular holds when the set itself has. Since $\tau^*(X) \leq \mathrm{d}_B(X - X)$, we have the following result

**Theorem 3.5.** *Suppose $\mathfrak{B}$ is a Banach space and let $X \subset \mathfrak{B}$ compact such that $\mathrm{d}_B(X - X) < \infty$. Then for any integer $k > \mathrm{d}_B(X - X)$ and any given $\theta$ with*

$$0 < \theta < \frac{k - \mathrm{d}_B(X - X)}{k\,(1 + \tau^*(X))},$$

*there exist a prevalent set of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^k$ such that*

$$\|x - y\| \le C_L |Lx - Ly|^\theta, \ \ \forall\ x, y \in X, \ \textit{for some}\ C_L > 0.$$

*In particular, every such $L$ is bijective from $X$ onto $L(X)$ with a Hölder continuous inverse.*

## 3.2 'Orthogonal' sequences in $\ell_p$

In the remainder of this chapter, we concentrate on a particular subset of $\ell_p$, for $p \in [1, \infty]$, and prove that some of the inequalities we know so far are sharp. These sets were first discussed by Ben Artzi et al [3] and have been used by Pinto De Moura & Robinson [6] as examples to show that the Hölder exponent of the inverses in Hunt and Kaloshin's Theorem 2.7 is asymptotically sharp.

Take $p \ge 1$ and let $(\alpha_n)_{n=1}^\infty$ be a decreasing sequence such that $\alpha_n \to 0$. Then, for all $n$ let $e_n = (0, 0, ..., 1, 0, ...)$, where the 1 is in the $n$th position and define

$$A = \{a_1, ..., a_n, ...\} = \{\alpha_1 e_1, ..., \alpha_n e_n, ...\}.$$

It is obvious that $a_i \in \ell_p$, for all $1 \le p < \infty$, hence $A \subset \ell_p$. We also have $a_i \in c_0$, where $c_0$ is the space of real sequences converging to zero equipped with the $\ell_\infty$ norm. Following Ben Artzi et al [3], we know that

$$\mathrm{d}_B(A; \ell_p) = \limsup_{n\to\infty} \frac{\log n}{-\log \|a_n\|} = \inf \left\{ \nu > 0 : \sum_{n=1}^{\infty} |a_n|^\nu < \infty \right\} \tag{3.1}$$

for all $p$. For an easier proof, see also Chapter 3 in the book of Robinson [28]. For the rest of this section, we implicitly understand the case $p = \infty$ as meaning $c_0$.

We first state without proof some additional properties of the box–counting dimension that we will need. The proofs can be found in Chapter 3 in the book of Robinson [28].

**Lemma 3.6.**

1. *Let $\mathfrak{B}$ a Banach space and $A \subset \mathfrak{B}$ compact. Let $M(A, \epsilon)$ be the maximum number of points in $A$ that are $\epsilon$–separated, meaning that $\|x - y\| \geq \epsilon$, for any $x, y$ in that collection. Then*

$$\mathrm{d}_B(A) = \limsup_{\epsilon \to 0} \frac{\log M(A, \epsilon)}{-\log \epsilon}.$$

2. *Suppose $(\mathfrak{B}_1, \|\cdot\|_1)$ and $(\mathfrak{B}_2, \|\cdot\|_2)$ are Banach spaces, $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ and*

$$\|u\|_2 \leq C\|u\|_1, \ \forall\ u \in \mathfrak{B}_1.$$

*Then, for all compact subsets $X \subset \mathfrak{B}_1$,*

$$\mathrm{d}_B(X; \mathfrak{B}_2) \leq \mathrm{d}_B(X; \mathfrak{B}_1).$$

We now prove that the inequality $\mathrm{d}_B(X - X) \leq 2\mathrm{d}_B(X)$ is sharp for this particular class of examples.

**Lemma 3.7.** *For all $p \geq 1$,*

$$\mathrm{d}_B(A - A; \ell_p) = 2\mathrm{d}_B(A). \tag{3.2}$$

*Proof.* We know that $\mathrm{d}_B(A - A; \ell_p) \leq 2\mathrm{d}_B(A)$. Hence, we need to show

$$\mathrm{d}_B(A - A; \ell_p) \geq 2\mathrm{d}_B(A).$$

We also have that $\ell_p \subset c_0$ and

$$\|u\|_\infty \leq \|u\|_p,$$

for any $p < \infty$. By part 2 of Lemma 3.6, we deduce that

$$\mathrm{d}_B(A - A; \ell_p) \geq \mathrm{d}_B(A - A; c_o).$$

Thus, it suffices to prove that

$$\mathrm{d}_B(A - A; c_0) \geq 2\mathrm{d}_B(A).$$

Let $\epsilon > 0$. Take $N \in \mathbb{N}$, such that $\|a_N\|_\infty < \epsilon \leq \|a_{N-1}\|_\infty$. Then

$$A \subseteq \bigcup_{i=0}^{N-1} B(x_i, \epsilon),$$

where $x_i = a_i, \forall\, 1 \leq i \leq N-1$ and $x_0 = 0$. It is also obvious that $N = N(A, \epsilon)$. Set $z_{ij} = x_i - x_j$, for all $i \neq j$. Then, we claim that

$$A - A \subset \bigcup_{i,j=0}^{N-1} B(z_{ij}, \epsilon) \bigcup B(0, \epsilon).$$

Take $z = a_m - a_n \in A - A$, such that $\|z\|_\infty \geq \epsilon$. Otherwise, $z \in B(0, \epsilon)$. Suppose, without loss of generality that $\|a_m\|_\infty \leq \|a_n\|_\infty$.Then,

$$\|z\|_\infty = \|a_n\|_\infty \geq \epsilon,$$

which implies that $n \leq N-1$ and $z_{n0} = x_n = a_n$. We now have two cases: if $\|a_m\|_\infty < \epsilon$, then

$$\|a_n - a_m - x_n\|_\infty = \|a_m\|_\infty < \epsilon,$$

while if $\|a_m\|_\infty \geq \epsilon$, then

$$\|a_n - a_m - z_{nm}\|_\infty = 0 < \epsilon.$$

Now, let $M = M(A - A, \epsilon)$ denote the maximum number of $\epsilon$–separated points in $A - A$, i.e.

$$\|y_k - y_l\|_\infty \geq \epsilon,$$

for all $y_k, y_l$ in that collection. Then, we claim that

$$\{z_{ij}\}_{i,j=0}^{N-1} \subseteq \{y_k\}_{k=1}^{M}.$$

Indeed suppose that for some $i, j, i \neq j$, $z_{ij} = x_i - x_j = a_i - a_j \neq y_k$, $\forall\, k$. Let $a_{n_k}, a_{m_k}$ such that $y_k = a_{n_k} - a_{m_k}$ and assume wlog that $i \neq n_k$. Then,

$$\begin{aligned}
|z_{ij} - y_k\|_\infty &= \|a_i - a_j - a_{n_k} + a_{m_k}\| \\
&= \|\alpha_i e_i - \alpha_j e_j - \alpha_{n_k} e_k + \alpha_{m_k} e_k\|_\infty \\
&\geq \|\alpha_i e_i\| \geq \epsilon,
\end{aligned}$$

contradicting the fact that $M(A - A, \epsilon)$ is the maximum number of $\epsilon$–separated points. All in all, we deduce that

$$M(A-A,\epsilon) \geq (N(A,\epsilon)-1)^2 - N(A,\epsilon) + 1 = N^2 - 3N + 2.$$

Using part 1 of Lemma 3.6 it follows that

$$\begin{aligned} \mathrm{d}_B(A-A;\infty) &= \limsup_{\epsilon\to 0} \frac{\log M(A-A,\epsilon)}{-\log\epsilon} \\ &\geq \limsup_{\epsilon\to 0} \frac{\log\left(N^2(A,\epsilon) - 3N(A,\epsilon) + 1\right)}{-\log\epsilon} \\ &\geq \limsup_{\epsilon\to 0} \frac{2\log N(A,\epsilon)}{-\log\epsilon} = 2\mathrm{d}_B(A), \end{aligned}$$

which finishes the proof. □

We now show that the thickness exponent and box–counting dimension of this 'orthogonal' sequence coincide whenever $p \leq 2$.

We first make the straightforward remark that whenever $(\mathcal{B}_1, \|\cdot\|_1) \subseteq (\mathcal{B}_2, \|\cdot\|_2)$, $X \subset \mathcal{B}_1$ and

$$\|u\|_2 \leq C\|u\|_1 \ \forall\ u \in \mathcal{B}_1,$$

then

$$\tau(X;\mathcal{B}_2) \leq \tau(X;\mathcal{B}_1).$$

In particular, since $\tau(X) \leq \mathrm{d}_B(X)$, we only need to show that the thickness exponent of this orthogonal sequence equals the box–counting dimension when $p = 2$, This has already been proven in Robinson [28]; here we give an alternative and slightly easier proof. We first need the following lemma.

**Lemma 3.8.** *Take $\{\alpha_n\}_{n=1}^{\infty}$ as usual and let $A_k = \{a_1, ..., a_k\} = \{\alpha_1 e_1, ..., \alpha_k e_k\} \subset \ell_2$. Then,*

$$d_{\ell_2}(A_k,\epsilon) \geq k\left(1 - \frac{\epsilon}{\|a_k\|}\right)^2.$$

*Proof.* We first remind ourselves that $d = d(A_k,\epsilon)$ denotes the smallest dimension among those finite-dimensional subspaces $V$ of $\ell_2$ that satisfy

$$\mathrm{dist}(x,V) \leq \epsilon,\ \forall\ x \in A_k.$$

For all $i \leq k$ take $v_i \in V$ such that $\|v_i - a_i\|_2 \leq \epsilon$. Then,

$$\dim(\mathrm{span}(v_1, ..., v_k)) = d.$$

Let $P$ denote the orthogonal projection onto $\operatorname{span}(v_1, ..., v_k)$. Thus, we have

$$\|a_i - Pa_i\|_2 = \operatorname{dist}(a_i, \operatorname{span}(v_1, ..., v_k)) \leq \|v_i - a_i\|_2 \leq \epsilon.$$

Moreover,

$$\|a_i - Pa_i\| = |\alpha_i|\|e_i - P(e_i)\| \geq |\alpha_k|(1 - \|Pe_i\|),$$

which gives

$$\|Pe_i\| \geq 1 - \frac{\epsilon}{\|a_k\|}.$$

We know that for every orthogonal projection in a Hilbert space,

$$\operatorname{rank} P = \sum_{i=1}^{\infty} \|Pe_i\|^2,$$

which proves that

$$d \geq k\left(1 - \frac{\epsilon}{\|a_k\|}\right)^2.$$

□

We can now show that the thickness exponent and the box–counting dimension are equal in this case. For the proof, we use the argument in Robinson [28] and the above lemma.

**Lemma 3.9.**

$$\tau(A; \ell_2) = \mathrm{d}_B(A) = \limsup_{n\to\infty} \frac{\log n}{-\log \|a_n\|}.$$

*Proof.* Take $n$ large enough such that $\|a_n\| < 1$ and take $n' \geq n$ such that

$$|\alpha_n| = |\alpha_{n+1}| = ... = |\alpha_{n'}| > |\alpha_{n'+1}|.$$

Let

$$\epsilon_n = \frac{\|a_n\| + \|a_{n'+1}\|}{4},$$

which implies

$$\frac{\|a_n\|}{4} < \epsilon_n < \frac{|a_n\|}{2}.$$

By the previous lemma, we have

$$d(A, \epsilon) \geq d(A_{n'}, \epsilon) \geq \frac{n'}{4} \geq \frac{n}{4}.$$

Therefore, we obtain

$$\begin{aligned}\tau(A) &\geq \limsup_{n\to\infty} \frac{d(A,\epsilon_n)}{-\log \epsilon_n} \geq \limsup_{n\to\infty} \frac{\log n - \log 4}{\log 4 - \log \|a_n\|} \\ &= \limsup_{n\to\infty} \left( \frac{\log n}{-\log \|a_n\|} \frac{1-\frac{\log 4}{\log n}}{1-\frac{\log 4}{\log \|a_n\|}} \right) \\ &= \limsup_{n\to\infty} \frac{\log n}{-\log \|a_n\|} = \mathrm{d}_B(A),\end{aligned}$$

by (3.1). □

It still remains open whether the thickness exponent and box–counting dimension of this particular set coincide, whenever $p > 2$. However, we give a lower bound for the thickness exponent of $A$ that depends on the box–counting dimension and the conjugate exponent of $p$.

**Lemma 3.10.** *Let $p \in [1, \infty]$ and $A \subset \ell_p$ as before. Then*

$$\tau(A) \geq \frac{q\,\mathrm{d}_B(A)}{q + \mathrm{d}_B(A)}, \tag{3.3}$$

*where $q$ is the conjugate exponent of $p$.*

We note that for $p = 1$ the right hand side of (3.3) becomes the box-counting dimension, giving a direct proof of what we proved in the previous lemma.

*Proof of Lemma 3.10.* Suppose $a_1, ..., a_k \in A$. Consider $\epsilon_k = \frac{1}{2}\|a_k\|k^{-1/q}$. Let $U$ be a subspace of $\ell_p$ such that $\dim(U) = d(\{a_1, ..., a_k\}, \epsilon_k) \leq d(A, \epsilon_k)$ and let $v_1, ..., v_k \in U$ be such that

$$\|v_i - a_i\| \leq \epsilon_k.$$

We claim that $v_1, , , , v_k$ are linearly independent and in particular that $k \leq \dim(U)$. Indeed, consider $\lambda_i \in \mathbb{R}$ such that

$$\sum_{i=1}^{k} \lambda_i v_i = 0.$$

Then,

$$\begin{aligned}
\epsilon_k \sum_{i=1}^{k} |\lambda_i| \geq \sum_{i=1}^{k} |\lambda_i| \, \|v_i - a_i\|_p &= \sum_{i=1}^{k_0} \|\lambda_i v_i - \lambda_i a_i\|_p \\
&\geq \| \sum_{i=1}^{k_0} \lambda_i v_i - \lambda_i a_i\|_p = \| \sum_{i=1}^{k_0} \lambda_i a_i\|_p \\
&= \left( \sum_{i=1}^{k_0} |\lambda_i \alpha_i|^p \right)^{1/p} \geq |\alpha_k| \left( \sum_{i=1}^{k_0} |\lambda_i|^p \right)^{1/p} \\
&\geq k^{-1/q} \sum_{i=1}^{k} |\lambda_i| \, .
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{k} |\lambda_i| \, (-\frac{1}{2} k^{-1/q}) \geq 0,$$

giving that $\lambda_i = 0$ for every $i$.

Thus,

$$\mathrm{d}(A, \epsilon_k) \geq k.$$

Now, we make the following computation

$$\begin{aligned}
\tau(A) &\geq \limsup_{k\to\infty} \frac{\log \mathrm{d}(A, \epsilon_k)}{-\log \epsilon_k} \geq \limsup_{k\to\infty} \frac{\log k}{-\log(\|a_k\| k^{-1/q})} \\
&= \limsup_{k\to\infty} \frac{\log k}{1/q \log k - \log \|a_k\|} \\
&= \frac{1}{\liminf \left( \frac{1}{q} - \frac{\log \|a_k\|}{\log k} \right)} = \frac{1}{\frac{1}{q} + \frac{1}{\mathrm{d}_B(A)}} \\
&= \frac{q \, \mathrm{d}_B(A)}{q + \mathrm{d}_B(A)},
\end{aligned}$$

which gives the desired lower bound. □

As mentioned in the beginning of the section, we do not know if the dual thickness is always bounded above by the box–counting dimension. However, Pinto de Moura and Robinson [6] relied on this fact to prove that the dual thickness and box–counting dimension of these orthogonal sequences coincide for every $p \in [1, \infty]$. In the next lemma, we show that this upper bound is true in this particular case.

**Lemma 3.11.** *Suppose that $A = \{a_1, ..., a_k, \dots\} \subset \ell_p$ is as usual. Then,*

$$\tau^*(A) \leq \mathrm{d}_B(A), \ \textit{for any } p \in [1, \infty].$$

*Proof.* Take any $\epsilon > 0$. Then, there exists some $N = N(A, \epsilon)$, such that $|\alpha_N| < \epsilon$ and $|\alpha_{N-1}| \geq \epsilon$. Then, $\forall i \leq N$ let $x \in \ell_p$ and set $\phi_i(x) \in \mathbb{R}$ to be the $i$-th coordinate of $x$. Since we are just projecting on one direction we immediately obtain that $\|\phi_i\| = 1$ and

$$\phi_i(a_j) = \delta_{ij}\alpha_i.$$

Let

$$V = \text{span}(\phi_1, ..., \phi_N) \subset \ell_{p^*}.$$

Now, take any $a_n, a_m$ in $A$ such that $\|a_m\|_p \leq \|a_n\|_p$ and $\|a_n - a_m\|_p \geq 50\epsilon$. Let $i_n, i_m \leq N$ such that

$$\|a_n - a_{i_n}\|_p < \epsilon \text{ and } \|a_m - a_{i_m}\|_p < \epsilon.$$

Since $\|a_m\|_p \leq \|a_n\|_p$ and $\|a_n - a_m\|_p \geq 50\epsilon$, we obtain $\|a_n\|_p \geq 25\epsilon$ which gives $\|a_{i_n}\|_p \geq 24\epsilon$.

Set

$$\Phi = \frac{\phi_{i_n} - \phi_{i_m}}{\|\phi_{i_n} - \phi_{i_m}\|},$$

and we have $\Phi \in V$, $\|\Phi\| = 1$ and

$$\begin{aligned} |\Phi(a_n - a_m)| &\geq \frac{|(\phi_{i_n} - \phi_{i_m})(a_n - a_{i_n} + a_{i_n} - a_{i_m} + a_{i_m} - a_m)|}{2} \\ &\geq \frac{20\epsilon}{2} = 10\epsilon. \end{aligned}$$

Arguing as in Lemma (3.7), we obtain $\tau^*(A) \leq \sigma_{1/5}(A) \leq \text{d}_B(A)$. □

All in all, there is no known relation between the thickness and the dual thickness in the context of a Banach space. Moreover, it is still open whether or not the dual thickness is always bounded above by the box dimension.

# Chapter 4

# Embedding sets with finite Assouad dimension

## 4.1 Background

We say that a metric space $(X, d)$ is doubling, with doubling constant $K$, if for every $x \in X$ and $r > 0$, there exist $y_1, ..., y_K$ in $X$ such that

$$B(x,r) \subset \bigcup_{i=1}^{K} B(y_i, r/2).$$

We say that a metric space $(X, d)$ embeds into a normed space $(Y, \|\cdot\|)$ in a bi-Lipschitz way if there exists $f\colon X \to Y$ and some constant $L > 0$, such that for all $x, y \in X$

$$\frac{1}{L} d(x,y) \leq \|f(x) - f(y)\| \leq L\, d(x,y).$$

We also say that a map $\Phi\colon X \to Y$ is $\delta$-almost bi-Lipschitz if there exists an $L > 0$ such that for all $x, y \in X$

$$\frac{1}{L} \frac{d(x,y)}{\operatorname{slog}(d(x,y))^{\delta}} \leq \|\Phi(x) - \Phi(y)\| \leq L\, d(x,y).$$

We know that when a metric space embeds into an Euclidean space in a bi-Lipschitz way then it must be doubling, but there are examples due to Laakso [20], Lang & Plaut [21] and Semmes [29] that show that this condition is not sufficient. We discuss one of these examples in Chapter 5.

In 1983, Assouad [2] proved that any doubling metric space $(X, d)$ can be embedded into an Euclidean space in a bi–Hölder way, i.e. for any $0 < \alpha < 1$, there exists some $k \in \mathbb{N}$ and a $\phi\colon X \to \mathbb{R}^k$ such that

$$\frac{1}{L}d(x,y)^\alpha \leq |\phi(x) - \phi(y)| \leq Ld(x,y)^\alpha,$$

for all $x, y \in X$.

We also want to recall the notion of a homogeneous metric space (see also Definition 1.3). A subset $V$ of a metric space $(X,d)$ is said to be $(M,s)-$*homogeneous* if for every $x \in V$ and $r > \rho > 0$

$$N_V(r,\rho) = N(V \cap B(x,r),\rho) \leq M\left(\frac{R}{r}\right)^s,$$

where $N(V \cap B(x,r),\rho)$ denotes the minimum number of balls of radius $\rho$ required to cover $V \cap B(x,r)$.

The Assouad dimension of $K \subset (X,d)$, $\mathrm{d}_A(K)$ is the infimum of all $s > 0$ such that $K$ is $(M,s)$ homogeneous for some $M > 0$.

It can be easily shown that a metric space is homogeneous if and only if it is doubling (see Chapter 9 in the book of Robinson [28]). In 2010, Olson and Robinson [25] introduced a weaker notion of an $(\alpha,\beta)$-almost $(M,s)$- homogeneous metric space and they proved almost bi–Lipschitz embeddings into Euclidean spaces.

**Definition 4.1** (**Definition 1.4**). *A metric space* $(X,d)$ *is* $(\alpha,\beta)$*-almost* $(M,s)$*-homogeneous if for any* $0 < \rho < r$

$$N_X(r,\rho) \leq M\left(\frac{r}{\rho}\right)^s \mathrm{slog}(r)^\alpha \mathrm{slog}(\rho)^\beta,$$

*where*

$$\mathrm{slog}(x) = \log\left(x + \frac{1}{x}\right) > 0,$$

*is the symmetric logarithm of* $x$*, for* $x > 0$*.*

We give some useful properties of the symmetric logarithm, which we will be using frequently. For the proof, see the paper of Olson & Robinson [25].

**Proposition 4.2.** *Let* $C > 0$ *and* $\gamma \geq 0$*. There exist positive constants* $A_C, B_C, a_\gamma, b_\gamma, c$ *such that*

1. $|\log x| \leq \mathrm{slog}(x) \leq \log 2 + |\log x|$.

2. $A_C\,\mathrm{slog}(x) \leq \mathrm{slog}(Cx) \leq B_C\,\mathrm{slog}(Cx)$.

3. $a_\gamma\,\mathrm{slog}(x) \leq \mathrm{slog}(x\mathrm{slog}(x)^\gamma) \leq b_\gamma\,\mathrm{slog}(x)$.

4. If $2^{-(k+1)} \leq x \leq 2^{-k}$, then $\mathrm{slog}(x) \geq c\,\mathrm{slog}(2^{-k})$.

Using the above definition and Assouad's construction ([2]), Olson & Robinson [25] proved that any almost homogeneous metric space admits an almost bi–Lipschitz map in a Hilbert space. In particular they proved the following.

**Theorem 4.3.** *Suppose $(X, d)$ is a compact $(\alpha, \beta)$- almost homogeneous metric space and let $X$ be an infinite dimensional separable Hilbert space. Then, for any $\delta > \alpha + \beta + \frac{1}{2}$, there exists a $\delta$-almost bi-Lipschitz map $f\colon (X, d) \to H$.*

The authors also showed that almost bi–Lipschitz images of almost homogeneous sets remain almost homogeneous. They also proved an embedding theorem when $X$ is a subset of a Hilbert space $H$ such that $X - X$ is almost homogeneous. In particular, they proved the following

**Theorem 4.4.** *Suppose that $H$ is a Hilbert space and $X \subset H$ such that $X - X$ is $(\alpha, \beta)$-almost homogeneous, for some $\alpha, \beta \geq 0$. Then, for every*

$$\delta > \frac{3 + \alpha + \beta}{2},$$

*there exists a $N = N_\delta \in \mathbb{N}$ and a prevalent set of linear maps $L\colon H \to \mathbb{R}^N$ that are injective and $\delta$-almost bi-Lipschitz on $X$.*

Based on the above work, Robinson [27] proved an almost bi–Lipschitz embedding result for subsets of Banach spaces when $X - X$ is homogeneous.

**Theorem 4.5.** *Suppose $X$ is a compact subset of a real Banach space $\mathfrak{B}$ such that the set $X - X = \{x - y : x, y \in X\}$ is homogeneous. Then for any $\gamma > 1$, there exists a natural number $N$ and a prevalent set of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^N$, that are injective on $X$ and $\gamma$-almost bi–Lipschitz, i.e. for some constant $C_L$*

$$\frac{1}{C_L} \frac{\|x - y\|}{|\log \|x - y\||^\gamma} \leq |L(x) - L(y)| \leq C_L \|x - y\|,$$

*for all $x, y \in X$ such that $\|x - y\| \leq r_L < 1$.*

*Remark* 4.1. If $X$ is (almost) homogeneous, it is not necessary that $X - X$ will be (almost) homogeneous (see examples in Chapter 9 in the book of Robinson). The almost homogeneous condition on the set of differences is also not invariant under almost bi–Lipschitz maps. However, in Olson & Robinson's result [25], they only use the fact that $X - X$ is almost homogeneous to cover balls at the origin.

We show in the next section that if $X$ is a subset of a Banach space and $X - X$ satisfies the $(\alpha, \beta)$- almost homogeneous property at the origin, then $X$ admits an almost bi–Lipschitz embedding in some Euclidean space. We also show that this condition remains invariant under linear almost bi–Lipschitz maps.

## 4.2 Embeddings when $X - X$ is almost homogeneous at the origin

**Definition 4.6.** *Suppose $M, s > 0$ and let $(X, |\cdot|)$ be a normed space. Then, given any $\alpha, \beta \geq 0$, we say that $X - X$ is $(\alpha,\beta)$-almost $(M, s)$- homogeneous at the origin if given any $0 < \rho < r$, there exist $z_i \in X - X$ such that*

$$B_r(0) \cap X - X \subseteq \bigcup_{i=1}^{N} B_\rho(z_i)$$

*and*

$$N \leq \left(\frac{r}{\rho}\right)^s \mathrm{slog}(r)^\alpha \mathrm{slog}(\rho)^\beta.$$

It is trivial that when $X-X$ is $(\alpha, \beta)$-almost homogeneous, the above property is satisfied. We first show that if the above property is satisfied, then the box–counting dimension of $X - X$ is finite.

**Lemma 4.7.** *Take any $M \geq 1$ and $s \geq 0$. Suppose that $X$ is a compact subset of a Banach space $\mathfrak{B}$ such that $X - X$ is $(\alpha,\beta)$-almost homogeneous at the origin. Then,* $\mathrm{d}_B(X - X) < \infty$.

*Proof.* Take any $\epsilon > 0$. Let $R > 1$ be such that

$$X - X \subset B_R(0).$$

Suppose that $1 < \epsilon < R$. Since $X - X$ is almost homogeneous at 0, there exist $\{z_i\}_{i=1}^N \subset X - X$ such that

$$X - X \subset B_R(0) \cap X - X \subseteq \cup_{i=1}^N B_\epsilon(z_i).$$

Using properties of the slog function, we have

$$N \leq \left(\frac{R}{\epsilon}\right)^s \mathrm{slog}(R)^\alpha \mathrm{slog}(\epsilon)^\beta \leq R^s (\mathrm{slog} R)^\alpha (\log 2 + \log \epsilon)^\beta \epsilon^{-s} \leq C_R\, \epsilon^{-s}.$$

Suppose now that $\epsilon < 1 < R$. Since $X - X$ is almost homogeneous at 0, there exist $\{z_i\}_{i=1}^N \subset X - X$ such that

$$X - X \subset B_R(0) \cap X - X \subseteq \cup_{i=1}^N B_\epsilon(z_i),$$

and

$$N \leq \left(\frac{R}{\epsilon}\right)^{s} \mathrm{slog}(R)^{\alpha}\mathrm{slog}(\epsilon)^{\beta} \leq R^{s}(\mathrm{slog}R)^{\alpha}(\log\epsilon + \log\frac{1}{\epsilon})^{\beta}\epsilon^{-s}$$
$$\leq R^{s}(\mathrm{slog}R)^{\alpha}\left(\log\frac{1}{\epsilon}\right)^{\beta}\epsilon^{-s} \leq R^{s}(\mathrm{slog}R)^{\alpha}\epsilon^{-\beta-s},$$

which immediately implies that

$$\mathrm{d}_B(X - X) \leq \beta + s < \infty. \qquad \square$$

We now show that the condition on covers of balls around zero remains invariant for the sets of differences under linear almost bi–Lipschitz maps.

**Lemma 4.8.** *Let $M \geq 1$, $s \geq 0$ and suppose $\alpha, \beta \geq 0$. Suppose $X$ is a compact subset of a Banach space $\mathfrak{B}$ and suppose that $X - X$ is $(\alpha,\beta)$)-almost $(M, s)$- homogeneous at the origin. Let $\delta > 1$ and let $\mathbb{B}'$ be another Banach space. Suppose that $\Phi\colon \mathfrak{B} \to \mathfrak{B}$ is a bounded linear map such that*

$$\frac{1}{C}\frac{\|x-y\|}{\mathrm{slog}(\|x-y\|)^{\delta}} \leq \|\Phi(x) - \Phi(y)\| \leq C\|x-y\|, \tag{4.1}$$

*for some positive constant $C$ and for every $x, y \in X$. Then, $\Phi(X) - \Phi(X) \subset \mathbb{B}'$ is $(\alpha + \delta s, \beta)$-almost homogeneous at $0$.*

*Proof.* Let $0 < \rho < r$. We want to cover the ball centred at 0 in $\Phi(X) - \Phi(X)$. Let $x, y \in X$ be such that $\|\Phi(x) - \Phi(y)\| \leq r$.

Suppose first that $\|x - y\| \leq r$. Then, since $X - X$ is almost homogeneous at 0, there exist $z_i \in X - X$ such that

$$B_r(0) \cap X - X \subset \cup_{i=1}^{N} B_{\rho/C}(z_i),$$

and $N \leq \left(\frac{r}{\rho}\right)^{s}\mathrm{slog}(r)^{\alpha}\mathrm{slog}(\rho)^{\beta}$. Let $j \leq N$, such that $\|x - y - z_j\| \leq \rho/C$. Then, since $\Phi$ satisfies 4.1, we have

$$\|\Phi(x - y - z_j)\| \leq C\|x - y - z_j\| \leq \rho.$$

In particular we have that

$$B_r(0) \cap (\Phi(X) - \Phi(X)) \subset \cup_{i=1}^{N} B_{\rho}(\Phi(z_i))$$

and

$$N \leq \left(\frac{r}{\rho}\right)^{s}\mathrm{slog}(r)^{\alpha}\mathrm{slog}(\rho)^{\beta}.$$

Suppose now that $\|x-y\| > r$. Then, let $R > 4$ be such that $X-X \subset B_{R/4}(0)$. Assume without loss of generality that $r < R/4$. Then, by Lemma 4.2, we have

$$\frac{\|x-y\|}{R} \leq C\frac{1}{R}\|\Phi(x)-\Phi(y)\| \operatorname{slog}\left(\frac{\|x-y\|}{R}\right)^{\delta} \leq \frac{Cr}{R}\log\left(\frac{R}{\|x-y\|}\right)^{\delta}$$
$$\leq \frac{Cr}{R}\log\left(\frac{R}{r}\right)^{\delta} \leq \frac{Cr}{R}\operatorname{slog}\left(\frac{r}{R}\right)^{\delta} \leq C_R\, r\operatorname{slog}(r)^{\delta}.$$

Thus,

$$\|x-y\| \leq C_R\, r\operatorname{slog}(r)^{\delta},$$

We now again use the fact that $X-X$ is almost homogeneous at 0 to deduce a cover of the ball $B_{r\operatorname{slog}(r)^{\delta}}(0)$ in $X-X$ by at most $N$ balls of radius $\rho/C$. We now estimate $N$ using again properties of the symmetric logarithm (Lemma 4.2).

$$N \leq \left(\frac{r\operatorname{slog}(r)^{\delta}}{\rho}\right)^{s} \operatorname{slog}(r\operatorname{slog}(r)^{\delta})^{\alpha}\operatorname{slog}(\rho)^{\beta} \leq b_{\delta}\left(\frac{r}{\rho}\right)^{s}\operatorname{slog}(r)^{\delta s}\operatorname{slog}(r)^{\alpha}\operatorname{slog}(\rho)^{\beta}$$
$$\leq \left(\frac{r}{\rho}\right)^{s}\operatorname{slog}(r)^{\alpha+\delta s}\operatorname{slog}(\rho)^{\beta}.$$

Arguing as in the previous case, $\{\Phi(z_i)\}_{i=1}^{N} \subset \Phi(X)-\Phi(X)$ and we deduce that

$$B_r(0) \cap \Phi(X)-\Phi(X) \subset \bigcup_{i=1}^{N} B_{\rho}(\Phi(z_i)),$$

and

$$N \leq \left(\frac{r}{\rho}\right)^{s}\operatorname{slog}(r)^{\alpha+\delta s}\operatorname{slog}(\rho)^{\beta}.$$

In particular, $\Phi(X)-\Phi(X)$ is $(\alpha+\delta s, \beta)$- almost homogeneous. □

### 4.2.1 Embedding from a Banach into a Hilbert space

We now want to show that when $X$ is a subset of a Banach space such that $X-X$ is almost homogeneous at the origin, then it admits a linear almost bi–Lischitz embedding into a Hilbert space. In particular, by Lemma 4.8 the set of differences of the image of $X$ into $H$ will also satisfy the almost homogeneous property at 0. Following the techniques we introduced in the previous chapter, we use a Hahn-Banach argument to construct the embedding at a single scale, as in the following Lemma.

**Lemma 4.9.** *Let $M \geq 1$, $R \geq 1$ and $s > 0$ and suppose that $X$ is a compact subset of a Banach space $\mathfrak{B}$ such that $X - X$ is $(\alpha, \beta)$-almost $(M, s)$-homogeneous at $0$, for some $\alpha, \beta \geq 0$. Then, there exists a collection $(\phi_n)_{n=1}^{\infty}$ of elements of $\mathcal{L}(\mathfrak{B}; \mathbb{R}^{m_n})$ that satisfy $\|\phi_n\| \leq C_R \sqrt{m_n} \leq C_R n^{\frac{\alpha+\beta}{2}}$ and for every*

$$x, y \in X \text{ with } 2^{-(n+1)} R \leq \|x - y\| \leq R\, 2^{-n},$$

*we have that*

$$|\phi_n(x - y)| \geq \frac{1}{4}\|x - y\|.$$

*Proof.* Suppose $Z = X - X$. Since $Z$ is $(\alpha, \beta)$)-homogeneous we can cover

$$Z \cap B_{R2^{-n}}(0) = \{z \in X - X : \|z\| \leq R\, 2^{-n}\}$$

by no more than

$$m_n \leq M \left( \frac{R\, 2^{-n}}{R\, 2^{-(n+3)}} \right)^s \text{slog}(R\, 2^{-n})^{\alpha} \text{slog}(R 2^{-(n+2)})^{\beta} \leq C_R n^{\alpha+\beta} \log 2 = C_R\, n^{\alpha+\beta}$$

balls of radius $R2^{-(n+2)}$, for some positive constant $C$ that depends only on $M, s, R$. Let the centres of these balls be $z_j^n$, for $j \leq C_R\, n^{\alpha+\beta}$.

Using the Hahn–Banach Theorem, we can find $f_j^n \in \mathfrak{B}^*$ such that $\|f_j^n\| = 1$ and $f_j^n(z_j^n) = \|z_j^n\|$. Now, define $\phi_n \colon \mathfrak{B} \to \mathbb{R}^m$ by

$$\phi_n(x) = \left( f_1^n(x), ..., f_{m_n}^n(x) \right).$$

It is clear that $\|\phi_n\| \leq \sqrt{m_n} \leq C_R\, n^{\alpha+\beta/2}$ and if $z = x - y \in X - X$ is such that

$$2^{-(n+1)} R \leq \|z\| \leq R\, 2^{-n},$$

then for some $z_j^n$ we have that $\|z - z_j^n\| \leq R2^{-(n+2)}$ and so

$$\begin{aligned} |\phi_n| &\geq |f_j^n(z)| = |f_j^n(z - z_j^n + z_j^n)| \\ &\geq |f_j^n(z_j^n)| - |f_j^n(z - z_j^n)| \geq \|z_j^n\| - \|z - z_j^n\| \\ &\geq \|z\| - 2\|z - z_j^n\| \geq R2^{-(n+1)} - R2^{-(n+2)} \geq \frac{1}{4}\|z\|, \end{aligned}$$

which concludes the proof of the embedding at a single scale. □

The above Lemma allows for the following embedding.

**Theorem 4.10.** *Suppose that $X$ is a compact subset of a Banach space $\mathfrak{B}$ such that $X - X$ is $(\alpha, \beta)$-almost homogeneous at the origin. Then, given any $\delta > \frac{1+\alpha+\beta}{2}$, there exists a Hilbert space $H$ and a bounded linear map $\Phi\colon \mathfrak{B} \to H$, that satisfies*

$$\frac{1}{C_\Phi} \frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^\delta} \leq \|\Phi(x) - \Phi(y)\| \leq \|x-y\|,$$

*for some positive constant $C_\Phi$ and for every $x, y \in X$.*

*Proof.* Take $R > 6$ such that

$$X - X \subset B_{R/2}(0) \subset B_R(0).$$

Take $\delta$ such that

$$\delta > \frac{1+\alpha+\beta}{2}.$$

Let $m_n, \phi_n$ be from Lemma (4.9). Suppose $\{e_k\}_{k=1}^m$ is a basis for $\mathbb{R}^m$, which we cyclically extend to all $k \in \mathbb{N}$, as in the previous chapter. Then, we define $\Phi\colon \mathfrak{B} \to H$ by

$$\Phi(x) = \sum_{k=1}^{\infty} k^{-\delta} \phi_k(x) \otimes \hat{e_k}.$$

Then, $\Phi$ is obviously linear and for every $x \in \mathfrak{B}$, we have

$$\|\Phi(x)\|^2 \leq \sum_{n=1}^{\infty} |n^{-\delta} \phi_n(x)|^2 \leq \|x\|^2 \sum_{n=1}^{\infty} n^{-2\delta} n^{\alpha+\beta} = \|x\|^2 \sum_{n=1}^{\infty} n^{-2\delta+\alpha+\beta} < \infty.$$

Hence

$$\|\Phi\| \leq \sum_{n=1}^{\infty} n^{-2\delta+\alpha+\beta} < \infty,$$

since $\alpha + \beta - 2\delta < -1$. Then, for any $x, y \in X$, let $k \geq 1$ be such that

$$2^{-(k+1)} R \leq \|x-y\| \leq R 2^{-k}.$$

By definition of $\Phi$, we have

$$\|\Phi(x) - \Phi(y)\| = \|\Phi(x-y)\| \geq k^{-\delta} |\phi_k(x-y)| \geq k^{-\delta} \frac{1}{4} \|x-y\|.$$

Using properties of the symmetric logarithm (Lemma 4.2), we obtain

$$\begin{aligned} \operatorname{slog}(\|x-y\|)^\delta &\geq A_R \operatorname{slog}\left(\frac{\|x-y\|}{R}\right)^\delta \geq A_R\, b \operatorname{slog}(2^{-k})^\delta \\ &\geq A_R\, b\, k^\delta, \end{aligned}$$

for constants $A_R, b$ independent of $x, y$. Thus,

$$\|\Phi(x) - \Phi(y)\| \geq \frac{1}{C_\Phi} \frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^\delta}. \qquad \square$$

Hence, by Theorem 4.5 and Remark 4.1, we immediately obtain an almost bi–Lipschitz embedding into an Euclidean space. However, in the section we establish the existence of a prevalent set of almost bi–Lipschitz maps into Euclidean spaces directly.

We also note that the above theorem can be used to provide embeddings of compact metric spaces, using the isometric embedding $\Phi^* \colon (X, d) \to L^\infty(X)$, given by $x \mapsto d(x, \cdot)$, due to Kuratowski, which was also mentioned in the previous chapter (see Lemma 2.11). In particular, we can define '$X - X'$ in this context to mean

$$X - X \equiv \Phi^*(X) - \Phi^*(X) = \{f \in L^\infty(X) : f = d(x, \cdot) - d(y, \cdot), \text{ for } x, y \text{ in } X\}.$$

### 4.2.2 Embedding into an Euclidean space

We now extend the result of the previous subsection and prove the existence of a prevalent set of almost bi–Lipschitz embeddings into Euclidean spaces for a compact subset of a Banach space such that $X - X$ is almost homogeneous at the origin. We first want to recall Lemma 2.16, which will be used to bound the measure of maps that do not satisfy the almost bi–Lipschitz condition.

**Lemma 4.11** (Lemma 2.16)**.** *Suppose that $\mathfrak{B}$ is a Banach space and let $\mathbb{E} = \mathbb{E}_\gamma(\{V_n\})$ be a probe space defined as in section 2.3, based on a sequence $\{V_n\}_{n=1}^\infty \subset \mathfrak{B}^*$. Let $x \in \mathfrak{B}$, $\epsilon > 0$, $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$. Then*

$$\mu\{L \in \mathbb{E} : |(f + L)(x)| < \epsilon\} \leq \left(n^\gamma d_n \frac{\epsilon}{|g(x)|}\right)^k,$$

*for any $g \in B_n$, the unit ball in $V_n$.*

We also recall the result due to Robinson [27] which provides a prevalent set of injective and bi–Hölder embeddings from $X$ into an Euclidean space when $\mathrm{d}_B(X - X) < \infty$ (see Theorem 3.5).

We are now in position to state and prove the main result of this section. For the proof, we use techniques introduced by Olson & Robinson [25] and also used by Robinson [27], tailored to the weaker condition we now have at the origin.

**Theorem 4.12.** *Fix any $M \geq 1$, $s > 0$ and $\alpha, \beta \geq 0$. Suppose $X$ is a compact subset of a Banach space $\mathfrak{B}$ such that $X - X$ is $(\alpha, \beta)$-almost $(M, s)$-homogeneous at the origin. Then, given any $\delta > 1 + \frac{\alpha+\beta}{2}$, there exists a $N = N_\delta \in \mathbb{N}$ and a prevalent of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^N$ that are injective on $X$ and bi–Lipschitz with $\delta$–logarithmic corrections. In particular, they satisfy*

$$\frac{1}{C_L} \frac{\|x - y\|}{\operatorname{slog}(\|x - y\|)^\delta} \leq |L(x) - L(y)| \leq C_L \|x - y\|, \tag{4.2}$$

*for some $C_L > 0$ and for all $x, y \in X$.*

*Proof.* The proof consists of three parts. We first establish the existence of a prevalent set $T_1$ of linear maps $L$ that satisfy (4.2), for all $x, y \in X$ such that $\|x - y\| \leq r_L$, for some $r_L > 0$. We then use Theorem 3.5 to construct a prevalent set $T_2$ of linear maps that are injective on $X$ and have a Hölder continuous inverse. Finally, we show that all linear maps in $T_1 \cap T_2$, which in particular is a prevalent set, satisfy (4.2), for all $x, y \in X$.

Let $Z = X - X$. Let $R > 6$ be such that

$$Z \subset B_{R/2}(0) \subset B_R(0).$$

Let $\gamma > 1$ be such that

$$\delta > \frac{\alpha + \beta}{2} + \gamma > \frac{\alpha + \beta}{2} + 1.$$

By Lemma 4.9, for any given $n \in \mathbb{N}$, there exist a collection of functionals $\{f_i^n\}_{i=1}^{m_n} \subset \mathfrak{B}^*$ with $m_n \leq C_1 n^{\alpha + \beta/2}$, $\|f_i^n\| = 1$ and such that for any $z \in Z$ that satisfies $R\, 2^{-n+1} \leq \|z\| \leq R\, 2^{-n}$, there exists $f_j^n$ such that

$$|f_j^n(z)| \geq 2^{(-n+3)}.$$

Let

$$V_n = \operatorname{span}\{f_1^n, ..., f_{m_n}^n\}.$$

Let also $N \in \mathbb{N}$. Based on the sequence $\mathcal{V} = \{V_n\}_{n=1}^\infty$ and on $\gamma > 1$, we follow the construction in the previous chapter and we define a probe space $\mathbb{E}_\gamma = \mathbb{E}_\gamma(\mathcal{V}) \subset \mathcal{L}\left(\mathcal{B}, \mathbb{R}^N\right)$ with a measure $\mu$ compactly supported on $\mathbb{E}_\gamma(\mathcal{V})$.

Following the argument of the previous chapter, we fix a map $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^N)$ and suppose that $K'$ is a Lipschitz constant that holds for all $L \in \mathbb{E}$. Then, we define

$$Z_n = \{z \in Z : 2^{(-n-1)}\, R \leq \|z\| \leq 2^{-n}\, R\}$$

and

$$Q_n = \{L \in \mathbb{E} : |(f+L)(z)| \le n^{-\delta}2^{-n}, \qquad \text{for some} \qquad z \in Z_n\}.$$

Since $X$ is $(\alpha, \beta)$- almost homogeneous at 0, given any $n \in \mathbb{N}$, there exist $\{z_i^n\}_{i=1}^{k_n} \subset Z$ such that

$$Z_n \subset B_{R\,2^{-n}}(0) \cap Z \subset \bigcup_{i=1}^{k_n} B_{n^{-\delta}2^{-n}}(z_i),$$

and

$$k_n \le M \left(\frac{R\,2^{-n}}{n^{-\delta}2^{-n}}\right)^s \mathrm{slog}(R\,2^{-n})^\alpha \mathrm{slog}(n^{-\delta}2^{-n})^\beta \le C_2 \left(n^\delta\right)^s n^{\alpha+\beta},$$

for some positive constant $C_2$ depending on $\alpha, \beta, M$. Now let $L \in Q_n$. Then there exists $z \in Z_n$ such that $|(f+L)(z)| \le n^{-\delta}2^{-n}$. Since $z \in Z_n$, there exists $z_i^n$ such that

$$\|z - z_i^n\| \le n^{-\delta}2^{-n},$$

which implies that

$$\begin{aligned} |(f+L)(z_i^n)| &\le |(f+L)(z_i^n) + (f+L)(z) - (f+L)(z)| \\ &\le n^{-\delta}2^{-n} + (\|f\| + \|L\|)n^{-\delta}2^{-n} \\ &\le (1 + \|f\| + K')n^{-\delta}2^{-n} = Kn^{-\delta}2^{-n}, \end{aligned}$$

where $K$ depends on $f$.

We now compute the measure of $Q_n$, based on Lemma 2.16. In particular, we have

$$\begin{aligned} \mu(Q_n) &\le \sum_{i=1}^{k_n} \mu\{L \in \mathbb{E} : |(f+L)(z_i^n)| \le (1+K)n^{-\delta}2^{-n}\} \\ &\le k_n \left(\dim(V_n)\, n^\gamma K n^{-\delta} 2^{-n} |\phi(z_i^n|^{-1}\right)^N, \end{aligned}$$

for any $\phi \in B_n$, the unit ball in $V_n$ Since $z_i^n \in Z_n$, there exists $f_i^n \in V_n$ such that $\|f_i^n\| = 1$ and $|f_i^n(z_i^n)| \ge 2^{-(n+3)}$. Therefore,

$$\mu(Q_n) \le Cn^{\delta s + \alpha + \beta + \frac{\alpha+\beta}{2}N + \gamma N - \delta N},$$

and $C > 0$ independent of $n$.

Since $\delta > \frac{\alpha+\beta}{2} + \gamma$, we can choose $N$ big enough such that

$$\frac{\left(\frac{\alpha+\beta}{2} + \gamma\right)N + 1}{N - s} < \delta,$$

which implies that

$$\delta s + \frac{\alpha+\beta}{2}N + \gamma N - \delta N < -1.$$

Therefore, $\sum_{n=1}^{\infty} Q_n < \infty$ and by the Borel-Cantelli Lemma, for $\mu$-almost every $L \in \mathbb{E}$, there exists an $n_L \geq 1$ such that for all $n \geq n_L$

$$2^{-(n+1)}\, R \leq |z| \leq 2^{-n}\, R \qquad \Rightarrow \qquad |(f+L)z| \geq n^{-\delta}2^{-n}.$$

Let $z \in Z$. If $\|z\| \leq R2^{-n_L}$, then there exists $n \geq n_L \geq 1$ such that

$$2^{-(n+1)}R \leq \|z\| \leq 2^{-n}\, R.$$

Therefore, arguing as in the end of Theorem 4.10, we obtain

$$|(f+L)(z)| \geq n^{-\delta}2^{-n} \geq n^{-\delta}2^{-n} \geq \frac{1}{A_R}\frac{\|z\|}{\operatorname{slog}(\|x-y\|)^{\delta}}.$$

Thus, we proved that there exists a prevalent set of bounded linear maps $L\colon \mathbb{B} \to \mathbb{R}^N$, denoted by $T_1$ such that all $L \in T_1$

$$\frac{1}{C'}\frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^{\delta}} \leq |L(x) - L(y)| \leq C'\|x-y\|,$$

for all $x, y \in X$ such that $\|x-y\| \leq R2^{-n_L}$, for some $n_L \geq 1$. By Lemma 4.7, we know that $\mathrm{d}_B(X-X) < \infty$. Hence, by Theorem 3.5, for a fixed $\theta < 1$, we establish the existence of a $N_1 \in \mathbb{N}$ and another prevalent set of linear maps $L\colon \mathbb{B} \to \mathbb{R}^{N_1}$, denoted by $T_2$ such that any $L \in T_2$ is $\theta$-bi-Hölder on $X$. Assume without loss of generality that $N_1 \leq N$ and let $T = T_1 \cap T_2$, which is still prevalent. Now suppose that $L \in T$ and $f \in \mathcal{L}(\mathfrak{B}, \mathbb{R}^N)$.

Let $z \in Z$. Let $m \geq 1$ such that

$$2^{-(m+1)}R \leq \|z\| \leq 2^{-m}R.$$

If $m \leq n_L$, we use that $f + L \in T \subset T_1$ and we fall in the previous case that we just proved. Suppose now that $m > n_L \geq 1$. Then, $f + L \in T \subset T_2$. In particular,

$$|(f+L)(z)| \geq \|z\|^{1/\theta} \geq R^{1/\theta}2^{-n_L/\theta}.$$

Hence,

$$\operatorname{slog}(\|x-y\|)^{\delta} \geq C_R \operatorname{slog}\left(\frac{\|x-y\|}{R}\right)^{\delta}$$
$$\geq C_R\,|\log 2^{-m}|^{\delta} = C_R\, m^{\delta} \log 2 > A_R\, n_L^{\delta}\, \log 2.$$

Thus,

$$|(f+L)(z)| \geq R^{1/\theta} 2^{-n_L/\theta} A_R\, n_L^{\delta}\, \log 2 \frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^{\delta}},$$

and the proof is now complete. □

Arguing as in the previous case, we can extend the above theorem for any compact metric space, using the Kuratowski embedding. In particular, the following theorem holds.

**Theorem 4.13.** *Suppose that $(X, d)$ is a metric space and let $\Phi\colon X \to L^{\infty}(X)$ be the Kuratowski embedding. Suppose that $\Phi(X) - \Phi(X)$ is $(\alpha, \beta)$-almost homogeneous at 0 Then, for any given $\delta > (\alpha + \beta)/2 + 1$, there exists a $N = N_{\delta} \in \mathbb{N}$ and a map $L\colon (X, d) \to \mathbb{R}^N$, which is bi–Lipschitz with $\delta$-logarithmic corrections and injective.*

## 4.3 Embedding when $X$ has 'better than zero' thickness

We know from the result of Olson & Robinson [25], that was mentioned before (Theorem 4.3) that any almost homogeneous metric space $(X, d)$ admits an almost bi–Lipschitz embedding into a Hilbert space. The authors prove that besides from being almost homogeneous, the image of $X$ can be very well approximated by linear subspaces of the Hilbert space $H$. We say it has 'better than zero' thickness. It is natural to ask whether we can embed subsets of Hilbert spaces that satisfy that property into Euclidean spaces.

Inspired by the above question, we show that a subset of a Banach space that can be well approximated by linear subspaces can be embedded into an infinite–dimensional Hilbert space in an almost bi–Lipschitz way. The techniques that we use are more similar to the ones we used in the previous chapter but we include the result here because it gives some indication towards proving an embedding result into an Euclidean space for almost homogeneous sets. We note that the condition here refers to $X$ rather than $X - X$.

The thickness exponent was defined in the previous chapter and measures how well can an arbitrary subset of a normed space can be approximated by linear subspaces. Before we proceed, we recall the definition.

**Definition 4.14.** *Let $X$ be a subset of a Banach space $\mathfrak{B}$. The thickness exponent of $X$ in $\mathfrak{B}$, $\tau(X, \mathfrak{B})$ is defined as:*

$$\tau(X, \mathfrak{B}) = \limsup_{\epsilon \to 0} \frac{\log d(X, \epsilon)}{-\log \epsilon},$$

*where $d(X, \epsilon)$ denotes the smallest dimension of those linear subspaces $V$ that satisfy*

$$\mathrm{dist}_{\mathfrak{B}}(x, V) \leq \epsilon \textit{ for all } x \in X.$$

*If no such subspace exists, we set $d(X, \epsilon) = \infty$.*

We now state the following result, which was proved by Olson & Robinson [25].

**Lemma 4.15.** *Suppose $X$ is a compact homogeneous metric space. Take $\delta > \frac{1}{2}$ and let $f = f_\delta \colon X \to H$ be the $\delta-$ almost bi–Lipschitz map from Theorem 4.3. Then, there exist $V_n \subseteq H$ and constants $C_1, C_2 \geq 1$ such that*

$$\mathrm{dist}_H(x, V_n) \leq C_1 \, 2^{-n} n^{-\delta}$$

*and*

$$dim(V_n) \leq C_2 \, n^{\delta}.$$

*In particular, $\tau(f(X)) = 0$.*

We observe that we can rephrase the above condition and say that $X$ has 'better than zero' thickness, if there exists a constant $C \geq 1$ such that for every $\epsilon > 0$

$$d(X, \epsilon) \leq C \, \mathrm{slog} \left( \frac{1}{\epsilon} \right)^s,$$

for some $s \geq 0$. We already know by the results in the previous chapter that compact subsets of Banach spaces with thickness exponent less than 1 admit bi–Hölder embeddings into Euclidean spaces. Motivated by the above result, we show that when a subset of a Banach space can be approximated by linear subspaces as in Lemma 4.15, then we can prove almost bi–Lipschitz embeddings into a Hilbert space.

**Proposition 4.16.** *Suppose that $X$ is a subset of a Banach space $\mathfrak{B}$ that has 'better than zero' thickness, for some $s \geq 0$. Then, for every*

$$\delta > \frac{3s}{2} + \frac{1}{2}$$

*there exists a separable Hilbert space $H$ and a bounded linear map $\Phi\colon \mathcal{B} \to H$, such that*

$$\frac{1}{C_\Phi} \frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^\delta} \leq |\Phi(x) - \Phi(y)| \leq C_\Phi \|x-y\|, \textit{ for all } x, y \textit{ in } X.$$

We first need the following Lemma.

**Lemma 4.17.** *Suppose that $X$ has better than zero thickness for some $s \geq 0$. Then, for every $n \in \mathbb{N}$, there exists $C > 1$, $\phi_n \in \mathcal{L}(\mathcal{B}; \mathbb{R}^{m_n})$, where $m_n \leq Cn^s$, $\|\phi_n\| \leq \sqrt{m_n}$ and*

$$|\phi_n(x-y)| \geq C\, 2^{-n-1} n^{-s}, \textit{ whenever } \|x-y\| \geq 2^{-n}.$$

*Proof.* Take any $n \in \mathbb{N}$. Then, by our hypothesis, there exists $V_n$ such that

$$\operatorname{dist}(x, V_n) \leq 2^{-n-1} n^{-s}$$

and

$$\dim(V_n) = m_n \leq C_1 \operatorname{slog}(2^{-n} n^{-s})^s \leq C_1 (\log 2 + s)^s n^s = C_s n^s$$

Suppose that $\{u_1^n, .., u_{m_n}^n\}$ is an Auerbach basis for $V_n$, and let $\{f_1^n, .., f_{m_n}^n\}$ be the corresponding elements of $V_n^*$ that satisfy $\|f_i^n\| = 1$, $\forall i$ and

$$f_i^n(u_j^n) = \delta_{ij}.$$

We now define a projection $P_n$ onto $V_n$ as

$$P_n(x) = \sum_{i=1}^{m_n} f_i^n(x) u_i^n$$

and define $\phi_n\colon \mathfrak{B} \to \mathbb{R}^{m_n}$ by setting

$$\phi_n(x) = (f_1^n(x), ..., f_{m_n}^n(x)).$$

Obviously $\|\phi_n\| \leq \sqrt{m_n} \leq Cn^{\frac{s}{2}}$. Moreover, let $z \in X - X$ be such that

$\|z\| \geq 2^{-n}$ and choose $z_n \in V_n$ such that

$$\|z - z_n\| \leq 2^{-n-1} n^{-s}.$$

Then

$$\|z_n\| \geq 2^{-n} - 2^{-n-1} n^{-s} \geq 2^{-n} - 2^{-n-1} = 2^{-n-1}.$$

Now, we write

$$z_n = \sum_{i=1}^{m_n} z_n^i u_i^n \in V_n$$

and suppose $j \leq m_n$ is such that $\|(z_n^1, ..., z_n^{m_n})\|_\infty = |z_n^j|$. Then,

$$\begin{aligned}\|\phi_n(z)\|_2 &\geq |f_j^n(z)| \geq |f_j^n(z_n)| - |f_j^n(z - z_n)| \\ &\geq |z_n^j| - \|z - z_n\| \geq m_n^{-1} \|z_n\| - 2^{-n} n^{-s} \\ &\geq C n^{-s} 2^{-n} - 2^{-n-1} n^{-s} \geq C\, 2^{-n-1} n^{-s},\end{aligned}$$

which finishes the proof of the embedding at a single scale. □

We now prove Proposition 4.16 .

*Proof of Proposition 4.16.* Take $p > 1$ such that

$$\delta > \frac{3s}{2} + p > \frac{3s}{2} + \frac{1}{2}$$

and let $\phi_n, m_n$ be as given in the previous lemma. We now set

$$\Phi(x) = \sum_{n=1}^{\infty} n^{-\left(\frac{s}{2}+p\right)} \phi_n(x) \otimes e_n \in H.$$

Then,

$$\|\Phi\| \leq \sum_{n=1}^{\infty} n^{-2\left(\frac{s}{2}+p\right)} n^{2\frac{s}{2}} = \sum_{n=1}^{\infty} n^{-2p} < \infty.$$

Now, take any $x, y \in X$ and suppose $x \neq y$ (the case $x = y$ is trivial). If $\|x - y\| \geq \frac{1}{2}$, then it suffices to take $R > 4$ such that

$$X - X \subset B(0, R/2).$$

Therefore, using also that $\|\phi_1(x - y)\| \geq \frac{C_1}{4}$, we have that

$$\|\Phi(x - y)\| \geq \|\phi_1(x - y)\| \geq \frac{C_1}{4} \geq \frac{C_1}{4} \frac{\|x - y\|}{R}.$$

Now, we also have

$$\operatorname{slog}(\|x-y\|)^{\delta} \geq A\,\frac{1}{4^{\delta}}\,\operatorname{slog}(4\|x-y\|)^{\delta} \geq \frac{A}{4^{\delta}}\,\delta\,\log 2.$$

Consequently,

$$\|\Phi(x-y)\| \geq C\,\frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^{\delta}}.$$

If $\|x-y\| < 1/2$, let $n \geq 1$ such that $2^{-n} \leq \|x-y\| < 2^{-n+1}$. Therefore

$$\begin{aligned}\|\Phi(x-y)\| &\geq n^{-\left(\frac{s}{2}+p\right)}\|\phi_n(x-y)\|_2\\ &\geq C_1\,n^{-\left(\frac{s}{2}+p\right)}2^{-n}n^{-s}\\ &\geq C_1 2^{-n}n^{-\frac{3s}{2}-p} \geq C_1 2^{-n}n^{-\delta}. \geq C\,\frac{\|x-y\|}{\operatorname{slog}(\|x-y\|)^{\delta}},\end{aligned}$$

which concludes the proof. □

It is an open question whether we can prove embeddings into Euclidean spaces for subsets of Hilbert spaces with 'better than zero' thickness. If such an embedding is true, then using Lemma 4.15 and Theorem 4.3, we can immediately embed in an almost bi–Lipschitz way any homogeneous metric space into an Euclidean space. Note that so far we only have this result when $X - X$ is almost homogeneous at 0 and as we show in the next chapter this is not sufficient to provide an embedding for any (almost) homogeneous space.

# Chapter 5

# The Laakso graphs

## 5.1 Construction of Laakso graphs

In this chapter, we consider a variation of the construction due to Laakso [20], which was used by Lang & Plaut [21] to construct a doubling metric space $X$ that cannot be embedded in a bi–Lipschitz way into any Hilbert space. We consider the Kuratowski embedding of $X$ into $L^\infty(X)$ and we show that $\Phi(X) - \Phi(X)$ is not almost homogeneous at 0 as a subset of $L^\infty(X)$. In particular, we do not inherit any control for covers of balls around 0 from the doubling property of $\Phi(X)$.

We first recall the Hausdorff distance for compact subsets of metric spaces.

**Definition 5.1.** *Suppose $(X, d)$ is a metric space and let $A, B$ be non-empty compact subsets of $X$. Then the Hausdorff distance is defined as*

$$d_H(A, B) = \max\{\operatorname{dist}(A, B), \operatorname{dist}(B, A)\}.$$

The Gromov–Hausdorff distance between compact metric spaces is defined as

**Definition 5.2.** *Suppose $X, Y$ are non-empty compact metric spaces. Then*

$$d_{GH}(X, Y) = \inf d_H(f(X), g(Y)),$$

*where the infimum is taken over all metric spaces $M$ and all possible isometric embeddings $f\colon X \to M$ and $g\colon Y \to M$.*

It is easy to check that $d_{GH}(X, Y) = 0$ if and only if $X$ is isometric to $Y$ (see the lecture notes from Heinonen [13] for a proof), proving that the set of all isometry classes of compact metric spaces equipped with $d_{GH}$ forms a metric space, which is complete (see Heinonen [13] again).

We now recall the construction due to Lang and Plaut [21] of a metric space that is homogeneous but does not embed in a bi–Lipschitz way into any Hilbert space. In order to study sets of differenes, we need to make the construction somewhat more concrete than that of Lang and Plaut [21]. We define the limiting metric space explicitly and then prove that it coincides with the one defined by Lang and Plaut.

Let $X_0$ be the unit interval $[0,1]$. To construct $X_{i+1}$ from $X_i$, we take six copies of $X_i$ and rescale them by the factor of $\frac{1}{4}$ as in the following figure (5.1).

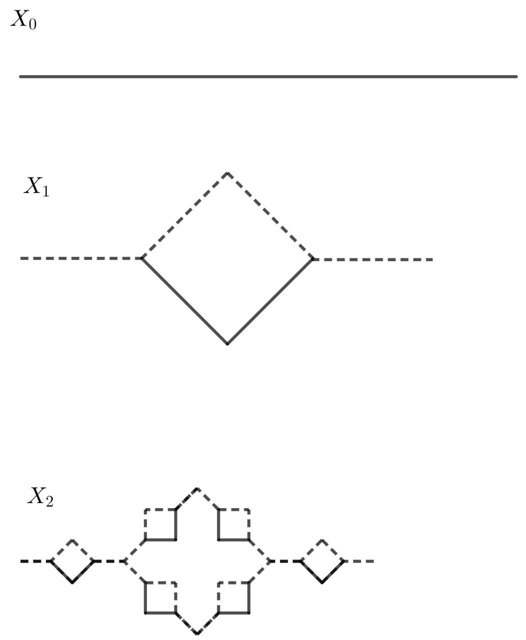


Figure 5.1: The first stages of the construction. At each step $i$ the dotted subset is isomorphic to $X_{i-1}$.

We note that each $X_i$ has diameter 1, has two endpoints, and comprises of $6^i$ edges of length $4^{-i}$ each. Every $X_i$, for $i > 0$ also includes $6^{i-1}$ 'squares', which we call 'edge cycles' for the rest of the chapter. We define a metric $\varrho_i(x, y)$ on each of the $X_i$ to be the geodesic distance, i.e. the shortest path that we need to travel on the graph to get from $x$ to $y$. For any $j > i$, we construct an isometric embedding of $X_i$ into $X_j$, by identifying vertices in $X_i$ with vertices in $X_j$ and endpoints with endpoints. The image of $X_i$ into $X_j$ is represented with the dotted lines in the above figure. It is also easy to see that $d_{GH}(X_i, X_j) < \left(\frac{1}{4}\right)^i$, and so $\{(X_i, \varrho_i)\}_{i=1}^{\infty}$ forms a Cauchy sequence in the Gromov–Hausdorff metric and it follows that it converges to a limiting metric space $X$, which is used by Lang and Plaut in their argument. We now construct this space $X$ explicitly using the following procedure

Let $(X_i, \varrho_i)$ be as above for any $j > i$ let $h_{i\to j}$ denote an isometric embedding of $X_i$ into $X_j$. Then, we take $X^* = \cup_{i=1}^{\infty} X_i$ and define a pseudometric on $X^*$, by setting

$$\varrho^*(x,y) = \begin{cases} \varrho_i(x,y) & \text{if } x, y \in X_i. \\ \varrho_j(h_{i\to j}(x), y) & \text{if } x \in X_i, y \in X_j \text{ and } i < j. \end{cases}$$

We now define a new metric space $X$, by identifying points in $X_i$ with their respective images in all $X_j$ for $j > i$. For all $x, y \in X^*$, we define the following equivalence relation

$$x \sim y \Leftrightarrow \varrho^*(x,y) = 0,$$

and we set $X = \cup_{i=1}^{\infty} [X_i]$. Then, for any $[x], [y] \in X$, we define

$$\varrho([x],[y]) = \varrho^*(x,y).$$

This definition of $X$ does not depend on the embedding we choose at each step, since if we consider another we end up with an isometric metric space.

Using the above construction, it is easy to check that

$$\mathrm{d}_{GH}(X, X_i) \to 0.$$

Indeed, let $\pi\colon X_i \to X$ be such that for any $x \in X_i$,

$$\pi(x) = [x].$$

It is immediate that $\pi$ is an isometry from $X_i$ onto $[X_i]$. Therefore,

$$\mathrm{d}_{GH}(X, X_i) = \mathrm{d}_{GH}(X, [X_i]) \leq \mathrm{d}_X(X, [X_i]).$$

Let $x \in X \setminus [X_i]$. Then, there exists $k > i$ such that $x \in [X_k]$. Then,

$$\mathrm{d}_X(x, [X_i]) = \mathrm{d}_{X_k}(x, h_{i\to k}(X_i)) \leq \left(\frac{1}{4}\right)^{i+1} \xrightarrow{i\to\infty} 0,$$

which proves that $X$ coincides with the metric space defined by Lang and Plaut. For the rest of the argument when we mention a point $x \in X_i$ we refer to the class $[x]$ with respect to the above equivalence relation.

## 5.2 Doubling property

We now recall Lang & Plaut's argument to show that $X$ is doubling. The proof is included for completeness and the above construction makes it more transparent.

**Theorem 5.3.** *The metric space $X$ defined above is doubling with doubling constant 6.*

*Proof.* Take $x \in X$ and $r$ with $0 < r \leq \frac{1}{2}$. Now, we choose $i$ such that

$$\frac{r}{2} \leq \left(\frac{1}{4}\right)^i < 2r,$$

and also let $k$ be the minimum natural number such that $x \in [X_k]$. We now have the following cases.

If $k \leq i$, then $x \in [X_i]$. Let $B_{X_i}(x, r)$ be the closed ball in $[X_i]$ and we set

$$Z = \partial B_{X_i}(x, r) \cup (B_{X_i}(x, r) \cap \{p, q\}),$$

where $p, q$ are the endpoints of $X_0$. Since $r \leq 2\left(\frac{1}{4}\right)^i$, $Z = \{t_1, ..., t_6\}$ contains no more than six points and as we can see from Figure 5.2 the closed balls in $X$ of radius $r$ centred at $Z$ cover $B_X(x, 2r) \cap [X_i] = B_{X_i}(x, 2r)$.

Now, we need to check that as we move forward into $[X_{i+1}], ...$ the new points that are added each time are still covered by the same balls. Let $y \in [X_j] \setminus [X_i]$, for $j > i$ and $\varrho(x, y) \leq 2r$. It is clear from the construction that $y$ must belong into some edge cycle $U$ in $[X_j]$. It is also clear that we can find an element $v$ of $[X_i]$ into the same edge cycle such that $\varrho(u, v) = \varrho(u, y)$ for all $u \in [X_j] \setminus U$ and $\varrho(x, v) \leq 2r$. (see also Figure 5.2).

Since $v \in B_X(x, 2r)$, there is some $t_s \in Z$ such that $\varrho(v, t_s) \leq r$. We now have two cases. If $t_s$ belongs to the same edge cycle as $v, y$, then

$$\varrho(y, t_s) \leq 2\left(\frac{1}{4}\right)^j < r$$

and if $t_s$ belongs outside the edge cycle, then

$$\varrho(y, t_s) = \varrho(v, t_s) < r,$$

as in the following Figure.

If $k > i$ and $x \in [X_k] \setminus [X_i]$, then $x$ belongs into some edge cycle $U$ in $[X_k]$. We claim that we can find $y \in [X_i]$ such that

$$B_X(x, 2r) \subseteq B_X(y, 2r).$$

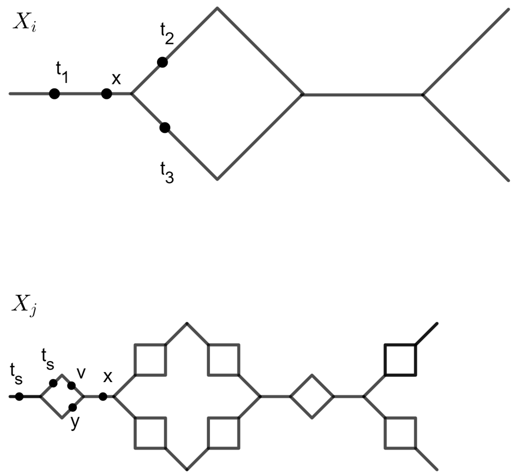


Figure 5.2: Covering the points that are added in $X_j$.

Indeed, arguing as before, let $y \in U \cap [X_i]$ be such that $\varrho(u, y) = \varrho(u, x)$, for all $u$ not in $U$. Now, let v $\in B_X(x, 2r)$.

If $v$ is not in $U$, then $\varrho(v, y) = \varrho(v, x) \leq 2r$, from the definition of $y$. If $v$ is in $U$, then

$$\varrho(v, y) \leq 2\left(\frac{1}{4}\right)^k < 2r.$$

Using what we already know for balls centred at $[X_i]$, the proof is complete. □

## 5.3 Differences of Laakso graphs

In order to define the set of differences, we recall the Kuratowski isometric embedding

$$\begin{aligned} \Phi\colon X &\to L^\infty(X) \\ x &\mapsto \varrho(\cdot, x). \end{aligned} \tag{5.1}$$

and we define

$$X - X = \Phi(X) - \Phi(X) = \{\varrho(\cdot, x) - \varrho(\cdot, y) : x, y \in X\}.$$

We now prove the main result of this chapter that gives a counterexample of a doubling set $X$ such that $X - X$ is not $(\alpha, \beta)$- almost homogeneous at 0 for any $\alpha, \beta \geq 0$.

**Theorem 5.4.** *If $X$ is the metric space defined above and $\Phi\colon X \to L^\infty(X)$ is the Kuratowski embedding defined in* (5.1) *then, $\Phi(X)-\Phi(X)$ is not $(\alpha,\beta)$-almost $(M,s)$-homogeneous for any choice of $\alpha,\beta,M,S\geq 0$.*

*Proof.* Suppose that $\Phi(X)-\Phi(X)$ is $(\alpha,\beta)$-almost $(M,s)$-homogeneous, for some $\alpha,\beta,M,S\geq 0$.

Let $r=\left(\frac{1}{4}\right)^i$, for some $i\in\mathbb{N}$ and take the ball $B_{X-X}(0,2r)$. Then, there exist $\{g_j\}_{j=1}^N \subset \Phi(X)-\Phi(X)$ such that

$$B_{2r}(0)\cap X-X = B_{2r}(0)\cap\Phi(X)-\Phi(X)\subset \cup_{j=1}^N B_{X-X}(g_j,r),$$

and

$$N\leq M2^s \mathrm{slog}(2r)^\alpha \mathrm{slog}(r)^\beta \leq CM2^s i^{\alpha+\beta},$$

for some absolute positive constant $C$. Now, let $f\in B_{X-X}(0,2r)$. Then, there exist $x,y\in X$ such that

$$f(z)=\varrho(x,z)-\varrho(y,z),\ \forall z\in X.$$

We can easily check that $\|f\|_\infty=\varrho(x,y)<2r$. Similarly, let $t_j,s_j\in X$ such that

$$g_j(z)=\varrho(t_j,z)-\varrho(s_j,z),\ \forall z\in X.$$

Any time we choose $x,y\in X$ such that $\varrho(x,y)<2r$, we obtain an element of $B_{X-X}(0,2r)$. Let $[X_k]\subset X$ such that

$$t_j,s_j\in[X_k],$$

for all $j\leq N$. We now have two cases

If $k\leq i$, we show that for any edge cycle in $X_i$, there exist copies of some $t_j$ or $s_j$ that belong to this edge cycle. Suppose that there exist a cycle in $X_i$ that does not contain any images of $t_j,s_j$. Then, we choose $x,y\in X_i$ as in the Figure 5.3, where we zoom in at that specific cycle. Then, $x,y\in X$ satisfy

$$\begin{aligned}
&\varrho(t_j,x)>r,\ \forall j\leq N\\
&\varrho(s_j,x)>r,\ \forall j\leq N\\
&\varrho(s_j,y)>0,\ \forall j\leq N\\
&r<\varrho(x,y)<2r.
\end{aligned}$$

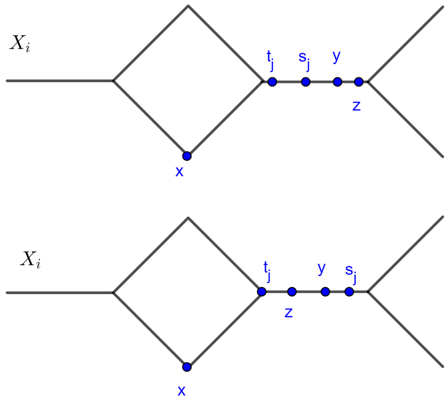


Figure 5.3: The edge cycle in $X_i$, which does not contain any $t_j, s_j$.

Since $f \in B_{X-X}(0, 2r)$, there exist $j \leq N$ such that

$$\|f - g_j\|_\infty < r \Leftrightarrow \|\varrho(x,z) - \varrho(y,z) - \varrho(t_j,z) + \varrho(s_j,z)\|_\infty < r,$$

for some $j \leq N$. Choosing $z$ as in the above figure, depending on the position of $t_j, s_j$ we have that

$$\|\varrho(x,z) - \varrho(y,z) - \varrho(t_j,z) + \varrho(s_j,z)\|_\infty = \varrho(t_j,x) + \varrho(s_j,y) > r$$

or

$$\|\varrho(x,z) - \varrho(y,z) - \varrho(t_j,z) + \varrho(s_j,z)\|_\infty = \varrho(x,y) + \varrho(s_j,t_j) > r,$$

a contradiction. We conclude that any edge cycle in $X_i$ contains one of the $t_i, s_i$ and since there are $6^{i-1}$ edge cycles contained in $X_i$, we deduce that

$$N \geq 6^{i-1},$$

a contradiction by choosing $i$ large enough.

If $k > i$, we consider the endpoints $v_{ij}, u_{ij}$ that enclose an edge cycle in $X_i$. We rescale the cycle by the appropriate factor to create an edge cycle in $X_k$, with the same endpoints in $X_k$ (with respect to the equivalence relation we have). Since distances are preserved, we only need to repeat the above argument for all these cycles in $X_k$ (see also the following figure).

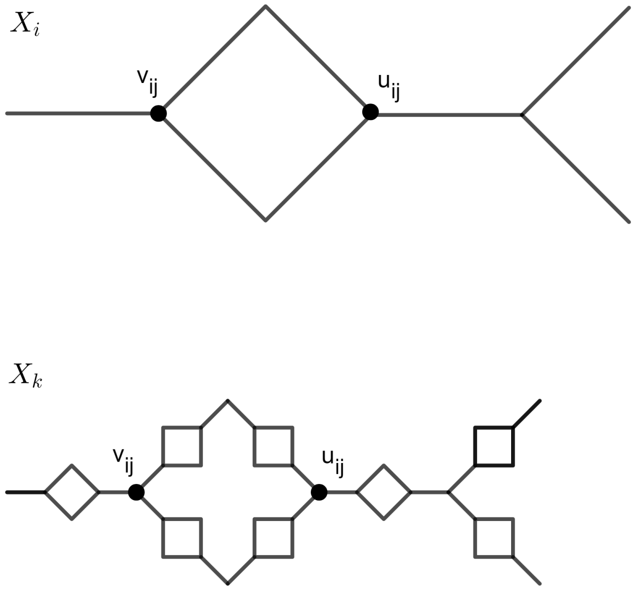


Figure 5.4: The case $k > i$.

$\square$

Since $\Phi$ is an isometry, the set $\Phi(X)$ is doubling but $\Phi(X) - \Phi(X)$ is not $(\alpha, \beta)$-homogeneous, for any $\alpha, \beta \geq 0$. Hence, the doubling property does not necessarily imply that we will have a control on covers of balls around 0 in $X - X$.

# Chapter 6

# Attractors of Iterated Function Systems in Euclidean spaces

## 6.1 Background

In this chapter, we study attractors of Iterated Function Systems in the context of a Euclidean space $\mathbb{R}^s$. Our main purpose is to establish non-trivial bounds on the Assouad dimension of differences of self-similar fractals, under some condition on the structure of the system.

We first want to set up our theory in the general context of any complete metric space $(X, d)$ and then concentrate on systems in $\mathbb{R}^s$. Suppose $(X, d)$ is a complete metric space and let $I = \{1, ..., |I|\}$ be a finite set of indices. We say that $\mathcal{F} = \{f_i \colon X \to X\}_{i \in I}$ is a system of contracting similarities, if for all $i \in I$, there exists $0 < c_i < 1$. such that

$$d(f_i(x), f_i(y)) = c_i \, d(x, y),$$

for all $x, y \in X$. Then, these maps are obviously contractions, so by the Banach fixed-point theorem they all have fixed points in $X$. We then say that a non-empty compact set $K \subset X$ is an attractor of the system if

$$K = \bigcup_{i=1}^{|I|} f_i(K).$$

It has been proven by Hutchinson [17] that every system $\mathcal{F}$ in a complete metric space $X$ defines a unique attractor $K$.

We now introduce some notation. Let $c_{min} = \min\{c_i : i \in I\}$ and $c_{max} = \max\{c_i : i \in I\}$ Let $\mathcal{I}^* = \cup_{k\geq 1} I^k$ be the set of all finite sequences with entries in $I$. For

$$\alpha = (i_1, ..., i_k) \in \mathcal{I}^*,$$

we write

$$f_\alpha = f_{i_1 i_2 \cdots i_k} = f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k},$$

and

$$c_\alpha = c_{i_1} ... c_{i_k}.$$

Let also

$$\bar{\alpha} = (i_1, ..., i_{k-1})$$

We also define for any $b < 1$,

$$I_b = \{\alpha \in \mathcal{I}^* : c_\alpha \leq b < c_{\bar{\alpha}}\}.$$

Finally, we define $C(I)$ to be the set all infinite sequences of integers $(i_p)_{p=1}^\infty$, with entries in $I$. We now state without proof some general properties of the attractor $K$, that we need. For the proofs, see the paper of Hutchinson [17].

**Proposition 6.1.** *Suppose that $(\mathcal{F}, K)$ is a system of contracting similarities with attractor $K$. Then we have the following*

1. *For any given $b < 1$, $K = \bigcup_{\alpha \in I_b} f_\alpha(K)$.*
2. *$K \supseteq f_{i_1}(K) \supseteq f_{i_1 i_2}(K) \supseteq \cdots \supseteq f_{i_1 \ldots i_p}(K) \supseteq \cdots$ and $\bigcap_{p=1}^\infty f_{i_1 \ldots i_p}(K)$ is a singleton, which is denoted by $k_v$, for $v = (i_1, i_2, \cdots, i_p, \cdots) \in C(I)$. $K$ is the union of all these singletons.*

## 6.2 The weak separation condition

Suppose now that $(\mathcal{F}, K)$ is a system of contracting similarities in $\mathbb{R}^s$ with an attractor $K$. One can show (see Hutchinson [17]) that a function $f \colon \mathbb{R}^s \to \mathbb{R}^s$ is a contracting similarity if and only if there exist $0 < c_f < 1$, $q_f \in \mathbb{R}$ such that

$$f(x) = c_f O_f(x) + q_f,$$

where $O_f \colon \mathbb{R}^s \to \mathbb{R}^s$ is an orthogonal transformation. The computation of dimensions of $K$ is of particular interest. One of the most common dimensions that we are interested in is the similarity dimension which is defined as follows.

**Definition 6.2.** *Suppose $(X, d)$ is a complete metric space and let $\mathcal{F} = \{f_i : X \to X\}_{i\in I}$ be a system of finitely many contracting similarities. The similarity dimension $d_{sim}$ is defined as the number $D$ such that*

$$\sum_{i\in I} c_i^D = 1.$$

In general, we know by Falconer [8] and McLaughlin [23] that the box–counting and Hausdorff dimensions of an attractor $K$ are equal and bounded above by the similarity dimension. This property does not hold in general for the Assouad dimension, as proven by Fraser [10]. However, if the system is defined on a Euclidean space and the images of the attractor under the maps $f_i$ do not overlap too much, then it can be shown that the box–counting dimension equals both the Hausdorff dimension and the Assouad dimension. An example of such a property is the weak separation condition, which was introduced by Zerner [30], in the context of an Euclidean space.

**Definition 6.3.** *Suppose that $\mathcal{F} = \{f_i : \mathbb{R}^s \to \mathbb{R}^s\}$ is an iterated function system, with $K$ as an attractor. We say that the IFS satisfies the weak separation property if there exists $\epsilon > 0$ such that for any given $0 < b < 1$ and any $\alpha, \beta \in I_b$, we have*

$$f_\alpha = f_\beta \qquad or \qquad \|f_\alpha^{-1} f_\beta - i_s\|_{L^\infty(F)} \geq \epsilon,$$

*where $i_s$ denotes the identity map $i_s : \mathbb{R}^s \to \mathbb{R}^s$.*

Fraser, Olson, Robinson and Henderson [11] used the notion of Ahlfors regularity and proved that the Assouad dimension coincides with the Hausdorff and box–counting dimensions, under the above condition. We give an independent proof of this result here, without using Ahlfors regularity, solely based on the definitions and the separation condition. Moreover, the proof provides us a useful mode for the analysis of sets of differences in the following section.

**Definition 6.4.** *We say that $\{x_j\}_{j=1}^k \subset \mathbb{R}^s$ are in general position if no $x_i$ lies in the affine space generated by any subcollection of the $\{x_j\}$ consisting of less or equal than $s$ points. Otherwise, no $m$ of them can lie in a $(m-2)$– dimensional hyperplane for $m \leq s$.*

We now state and prove the following general Lemma, which will give an equivalent property with the weak separation condition.

**Lemma 6.5.** *For every* $\{x_j\}_{j=0}^s \subset \mathbb{R}^s$ *in general position there exists an* $C > 0$ *such that for every affine map* $h\colon \mathbb{R}^s \to \mathbb{R}^s$ *of the form*

$$h(x) = Ax + b,$$

*where* $b$ *is a constant and* $A$ *is an* $s \times s$ *matrix, we have*

$$\|h - I\|_{L^\infty([0,1]^s)} \le C\,|h(x_{j_*}) - x_{j_*}|,$$

*for some* $j_* \in \{0, 1, \dots, s\}$ *that depends on* $h$.

*Proof.* Let $j_*$ be such that

$$|h(x_{j_*}) - x_{j_*}| = \max\{|h(x_j) - x_j| : j = 0, \dots, s\}$$

Since $x_j$ are in general position then there exist $\lambda_1, \dots, \lambda_s$ such that

$$x_0 = \sum_{j=1}^s \lambda_j x_j \qquad \text{and} \qquad \sum_{j=1}^s \lambda_j \ne 1.$$

Let $x \in [0,1]^s$ be such that

$$\|h - I\|_{L^\infty([0,1]^s)} = |h(x) - x|.$$

Choose $\{a_j\}_{j=1}^s$ such that

$$x = \sum_{j=1}^s a_j x_j.$$

Since $|x| \le \sqrt{s}$ and the $x_j$ are in general position, there is $C_1$ independent of $x$ such that

$$\sum_{j=1}^s |a_j| \le C_1.$$

Consequently,

$$\begin{aligned}
|h(x) - x| &= \left|\sum_{j=1}^s a_j(h(x_j) - x_j)\right| + \left|\left(\sum_{j=1}^s a_i - 1\right) b\right| \\
&\le C_1 \max\{|h(x_j) - x_j| : j = 1, ..., s\} + (C_1 + 1)\,|b| \\
&\le C_1 |h(x_{j_*}) - x_{j_*}| + (C_1 + 1)\,|b|.
\end{aligned}$$

It remains to estimate $b$ in terms of $h$.

To do this, we make the following computation:

$$2b = h(x_0) + h(-x_0) = h(x_0) - \sum_{j=1}^{s} \lambda_j h(x_j) + b\left(1 + \sum_{j=1}^{s} \lambda_j\right).$$

Hence,

$$\begin{aligned} b\left(1 - \sum_{j=1}^{s} \lambda_j\right) &= h(x_0) - \sum_{j=1}^{s} \lambda_j h(x_j) = h(x_0) - x_0 + x_0 - \sum_{j=1}^{s} \lambda_j h(x_j) \\ &= h(x_0) - x_0 + \sum_{j=1}^{s} \lambda_j x_j - \sum_{j=1}^{s} \lambda_j h(x_j) \\ &= h(x_0) - x_0 - \sum_{j=1}^{s} \lambda_j (h(x_j) - x_j) \end{aligned}$$

Therefore

$$\begin{aligned} |b| &= \frac{|h(x_0) - x_0 - \sum_{j=1}^{s} \lambda_j (h(x_j) - x_j)|}{|1 - \sum_{j=1}^{s} \lambda_j|} \\ &\leq \frac{|h(x_0) - x_0| + C_2 \max\{|h(x_j) - x_j| : j = 1, \ldots, s\}}{\delta} \\ &\leq \frac{C_2 + 1}{\delta} |h(x_{j_*}) - x_{j_*}| \end{aligned}$$

where $C_2 = \sum_{j=1}^{s} |\lambda_j|$ and $\delta = \left|1 - \sum_{j=1}^{s} \lambda_j\right|$.

All in all, we obtain

$$\|h - I\|_{L^\infty([0,1]^s)} \leq C|h(x_{j_*}) - x_{j_*}|,$$

where $C = C_1 + (C_1 + 1)(C_2 + 1)/\delta$. □

We now have the following Corollary.

**Corollary 6.6.** *Suppose that the IFS satisfies the weak separation condition. Then, for every $\{x_j\}_{j=0}^{s} \subset F$ in general position, there exists an $M > 0$ depending only on the $\{c_i\}_{i=1}^{|\mathcal{I}|}$ and $\{x_j\}_{j=0}^{s}$ such that*

$$|(f_\alpha - f_\beta)(x_j)| \geq \epsilon M r,$$

*for some $j \in \{0, \ldots, s\}$, which depends on $\alpha, \beta \in I_r$.*

*Proof.* Take $h = f_\alpha^{-1} f_\beta$ in Lemma 6.5. Then, for $f_\alpha, f_\beta$ such that $\alpha, \beta \in I_r$, let $j \leq s$

such that

$$|f_\alpha^{-1} f_\beta(x_j) - x_j| = |f_\alpha^{-1} f_\beta(x_j) - f_\alpha^{-1} f_\alpha(x_j)| \geq \epsilon,$$

which implies that

$$|(f_\alpha - f_\beta)(x_j)| \geq \epsilon M r,$$

for some $M > 0$. □

Before we proceed to the proof, we want to introduce some terminology from graph theory, which will be useful in what follows.

**Definition 6.7.** *We say that an undirected graph $G = (V, E)$ with $n$ vertices is complete if every two vertices are connected with a unique edge.*

**Definition 6.8.** *An $r$-colouring of the edges of a graph $(V, E)$ is a function $\mathfrak{g} \colon E \to \{1, 2, \cdots, r\}$.*

We now state the following version of Ramsey's theorem. For a more detailed analysis of Ramsey theory, see Chapter 1 in the book of Katz & Reimann [19]

**Theorem 6.9** (Ramsey's Theorem)**.** *Suppose that we have $r$ colours and let $n_1, \cdots, n_r$ be natural numbers. Then, there exist a number $R(r, n_1, n_2, \cdots, n_r)$ such that if $G$ is a complete graph with at least $R(r, n_1, n_2, \cdots, n_r)$ vertices, there exists an $1 \leq i \leq r$ and a complete subgraph $T$ of $G$ of order $n_i$ such that all the edges in $T$ are coloured with the colour $i$.*

An immediate corollary is the following.

**Corollary 6.10.** *Suppose that $G$ is a complete graph and suppose $N \in \mathbb{N}$. Suppose also that we have an $r$-colouring of the edges of $G$. If every monochromatic complete subgraph of $G$ has order at most $N$, then*

$$|G| < R(r, N+1, ..., N+1).$$

*Proof.* Suppose that $|G| \geq R(r, N+1, ..., N+1)$. Then, by Ramsey's theorem, there exists a complete monochromatic subgraph of order $N+1$, which violates the hypothesis. □

We now show directly that when the IFS satisfies the weak separation property then the Assouad dimension of the attractor equals its box–counting dimension. In particular, since the lower bound is always true, we only need to prove the upper bound. The proof provides a useful tool for the more involved analysis of sets of differences which follows in the next section.

**Theorem 6.11.** *Suppose that $\mathcal{F} = \{f_i \colon \mathbb{R}^s \to \mathbb{R}^s\}$ is an iterated function system that satisfies the weak separation property. Let $K$ be the attractor of the system and suppose that $K$ is not contained in a hyperplane. Then,*

$$\mathrm{d}_A(K) = \mathrm{d}_B(K).$$

*Proof.* Let $d > \mathrm{d}_B(K)$. Suppose, wlog that

$$K \subset B_1(0).$$

Then, we fix $y \in K$ and we have

$$K \subset B_2(y).$$

Now, suppose $x \in K$ and $r > 0$. Let

$$G_r(x) = \{f_\alpha : \alpha \in I_r, B_r(x) \cap f_\alpha(K) \neq \emptyset.\}$$

Then, we have

$$\begin{aligned} B(x,r) \cap F &= \cup_{\alpha \in I_r} B(x,r) \cap f_\alpha(K) \\ &= \cup_{f_\alpha \in G_r(x)} B(x,r) \cap f_\alpha(F) \\ &\subset \cup_{f_\alpha \in G_r(x)} B(x,r) \cap f_\alpha(B_2(y)) \\ &\subset \cup_{f_\alpha \in G_r(x)} B(x,r) \cap B_{2r}(f_\alpha(y)), \end{aligned}$$

for all $K \in F$. We claim that we can bound the cardinality of $G_r(x)$ independently of $r, x$. Since $K$ is not contained in a hyperplane, there exist $\{x_j\}_{j=0}^s \subset F$ in general position. By Lemma 6.6, for every choice of $f_\alpha, f_\beta \in G_r(x)$, there exists a $j \leq s$ such that

$$|f_\alpha(x_j) - f_\beta(x_j)| \geq \epsilon r. \tag{6.1}$$

Let

$$T_r(x) = \{0 \leq j \leq s : |f_\alpha(x_j) - f_\beta(x_j)| \geq \epsilon r, \qquad \text{for some} \qquad f_\alpha, f_\beta \in G_r(x).\}$$

Obviously, $|T_r(x)| \leq s+1$, for all $r, x$. We now consider $G_r(x)$ as a graph with vertices $f_\alpha$ and edges $E = \{\{f_\alpha, f_\beta\} : f_\alpha, f_\beta \in G_r(x)\}$. For any $j \in T_r(x)$, we say that the edge $\{f_\alpha, f_\beta\}$ is of color $j$ if

$$|f_\alpha(x_j) - f_\beta(x_j)| \geq \epsilon r.$$

Suppose that $P_r^j(x)$ is a complete monochromatic subgraph of $G_r(x)$, of color $j \leq s+1$. Then, for every $f_\alpha \in P_r^j(x)$, we have

$$\begin{aligned} B(x,r) \cap f_\alpha(F) \neq \emptyset &\Rightarrow B(x,r) \cap f_\alpha(B_2(x_j)) \neq \emptyset \\ &\Leftrightarrow B(x,r) \cap B_{2r}(f_\alpha(x_j)) \neq \emptyset, \end{aligned}$$

which implies that

$$|f_\alpha(x_j) - x| \leq 3r. \tag{6.2}$$

Moreover, for any $f_\alpha, f_\beta \in P_r^j(x)$, we have by definition

$$|f_\alpha(x_j) - f_\beta(x_j)| \geq \epsilon r. \tag{6.3}$$

In particular $f_\alpha(x_j) \neq f_\beta(x_j)$, for all $f_\alpha, f_\beta \in P_r^j(x)$. Consequently, in order to count the number of vertices in $P_r^j(x)$, it suffices to count the points $f_\alpha(x_j)$, for $f_\alpha \in P_r^j(x)$. By (6.3), the balls of radius $\epsilon r/2$, with centres $f_\alpha(x_j)$, for $f_\alpha \in P_r^j(x)$ are disjoint and by (6.2), all the centres lie in a ball of radius $3r$, centred at $x$. Thus,

$$\bigcup_{f_\alpha \in P_r^j(x)} B_{\frac{\epsilon r}{2}}(f_\alpha(x_j)) \subseteq B_{3r+\epsilon r}(x).$$

Therefore, if $\mu$ is the $s$-dimensional Lebesque measure, we have

$$|P_j^r(x)| \leq \frac{\mu\left(B_{3r+\epsilon r}(x)\right)}{\mu\left(B_{\frac{\epsilon r}{2}}\right)} = M',$$

which is independent of $r, x$. Since $G_r(x)$ is a complete graph and we bounded the order of any complete monochromatic subgraph independently of $r, x$, we have by corollary 6.10 That

$$|G_r(x)| \leq M,$$

independent of $r, x$.

We now enumerate $G_r(x)$ using the following parametrisation.

$$G_r(x) = \{f_{\alpha_k}\}_{k=1}^M.$$

Now, let $N = N(F, \rho/r)$ denote the number of balls of radius $\rho/r$ required to cover

$K$. Let the centres of those balls be $y_j$, for $j \leq N$. Then,

$$\begin{aligned} B(x,r) \cap K &\subseteq \cup_{k=1}^{M} f_{\alpha_k}(K) \\ &\subset \cup_{k=1}^{M} f_{\alpha_k}\left(\cup_{j=1}^{N} B_{\rho/r}(y_j)\right) \\ &= \cup_{k=1}^{M} \cup_{j=1}^{N} B_{\rho}(f_{\alpha_k}(y_j)) \end{aligned}$$

We know by definition of the box–counting dimension that there exists some constant $C > 0$ such that

$$N \leq C \left(\frac{r}{\rho}\right)^{d}.$$

Thus,

$$N_K(r,\rho) \leq MN \leq MC \left(\frac{r}{\rho}\right)^{d}.$$

Therefore, $d \geq \mathrm{d}_A(K)$ and since $d > \mathrm{d}_B(K)$ was arbitrary we have $\mathrm{d}_A(K) \leq \mathrm{d}_B(K)$. □

## 6.3 The weak separation condition for differences

In this section, we study differences of attractors of Iterated Function Systems in Euclidean spaces. We want to establish non trivial bounds for the Assouad dimension of the set of differences in terms of the Assouad dimension of the attractor. In particular, we show that under a suitable separation condition, the Assouad dimension of $K - K$ is bounded above by twice the Assouad dimension of $K$. Note that non-trivial bounds do not hold in general as there are examples on the real line due to Henderson [14] where $\mathrm{d}_A(K) < \epsilon$, for any $\epsilon > 0$ and $\mathrm{d}_A(K-K) = 1$.

**Definition 6.12.** *Suppose that $\mathcal{F} = \{f_i \colon \mathbb{R}^s \to \mathbb{R}^s\}$ is an system of contracting similarities. Suppose that $K$ is the attractor of the system. The IFS satisfies the weak separation condition for differences if there exist $M, \epsilon > 0$ and a collection of points $\{x_j\}_{j=0}^{M} \in K$ such that for every given $0 < b < 1$ and every $\alpha, \beta, \gamma, \delta \in I_b$, we have*

$$f_\alpha(K) - f_\beta(K) = f_\gamma(K) - f_\delta(K)$$

*or*

$$\|f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(x_i) + f_\delta(x_j)\| \geq \epsilon b,$$

*for some $i, j \leq M$ that depend on $\alpha, \beta, \gamma, \delta \in I_b$.*

We now recall the definition of Hausdorff distance, for compact subsets of metric spaces.

**Definition 6.13.** *Suppose $(X, d)$ is a metric space and let $A, B$ be non-empty compact subsets of $X$. Then the Hausdorff distance is defined as*

$$d_H(A, B) = \max\{\operatorname{dist}(A, B), \operatorname{dist}(B, A)\}.$$

We now prove that the weak separation for differences is also satisfied if for any scale $b < 1$, the sets $f_\alpha(K) - f_\beta(K)$, $f_\gamma(K) - f_\delta(K)$ for $\alpha, \beta, \gamma, \delta \in I_b$ are either equal or their Hausdorff distance is bounded away from zero.

**Lemma 6.14.** *Suppose that $\mathcal{F} = \{f_i : \mathbb{R}^s \to \mathbb{R}^s\}$ is a system of contracting similarities. Suppose that there exists a $\zeta > 0$ such that for any given $0 < b < 1$ we have that either*

$$f_\alpha(K) - f_\beta(K) = f_\gamma(K) - f_\delta(K)$$

*or*

$$d_H(f_\alpha(K) - f_\beta(K), f_\gamma(K) - f_\delta(K)) \geq \zeta b,$$

*for all $\alpha, \beta, \gamma, \delta \in I_b$. Then, the weak separation condition for differences is satisfied.*

*Proof.* Let $\alpha, \beta, \gamma, \delta \in I_b$. Suppose that

$$f_\alpha(K) - f_\beta(K) \neq f_\gamma(K) - f_\delta(K).$$

Let $\{x_j\}_{j=0}^M$ be an $\zeta/4$ net in $K$, i.e.

$$K \subset \bigcup_{j=0}^{M} B_{\zeta/4}(x_j) \qquad \text{and} \qquad |x_i - x_j| \geq \frac{\zeta}{4}.$$

Assume without loss of generality that

$$d_H(f_\alpha(K) - f_\beta(K), f_\gamma(K) - f_\delta(K)) = \operatorname{dist}(f_\alpha(K) - f_\beta(K), f_\gamma(K) - f_\delta(K)) \geq \zeta b.$$

Using the compactness of $K$, let $x, y \in K$ be such that

$$\operatorname{dist}(f_\alpha(K) - f_\beta(K), f_\gamma(K) - f_\delta(K)) = \operatorname{dist}(f_\alpha(x) - f_\beta(y), f_\gamma(K) - f_\delta(K)).$$

Let $i, j \leq M$ be such that

$$|x - x_i| \leq \frac{\zeta}{4} \qquad \text{and} \qquad |y - x_j| \leq \frac{\zeta}{4}.$$

Again by the compactness of $K$ suppose that $s, t \in K$ are such that

$$\operatorname{dist}(f_\alpha(x) - f_\beta(y), f_\gamma(K) - f_\delta(K)) = |f_\alpha(x) - f_\beta(y) - f_\gamma(s) + f_\delta(t)|.$$

Then, we deduce that

$$
\begin{aligned}
|f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(x_i) + f_\delta(x_j)| &\geq \operatorname{dist}(f_\alpha(x_i) - f_\beta(x_j), f_\gamma(K) - f_\delta(K)) \\
&= |f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(s) + f_\delta(t)|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(x_i) + f_\delta(x_j)| &\geq |f_\alpha(x) - f_\beta(y) - f_\gamma(s) + f_\delta(t)| - \\
&\quad - |f_\alpha(x_i) - f_\beta(x_j) - f_\alpha(x) + f_\beta(y)| \\
&\geq \zeta b - 2\frac{\zeta b}{4} = \frac{\zeta b}{2}.
\end{aligned}
$$

By taking $\epsilon = \zeta/2$, the proof is complete. □

We now state and prove the main result of this chapter.

**Theorem 6.15.** *Suppose that $\mathcal{F} = \{f_i \colon \mathbb{R}^s \to \mathbb{R}^s\}$ is a system of contracting similarities and let $K$ be the attractor of the system. If the IFS satisfies the weak separation for differences then*

$$
\mathrm{d}_A(K - K) \leq 2\mathrm{d}_A(K). \tag{6.4}
$$

*Proof.* The argument is similar with the argument of the previous section. We use a Ramsey theory argument to prove that given any $0 < r < 1$ and $z \in K - K$, the cardinality of set of maps $(f_\alpha, f_\beta)$ such that

$$
B_r(z) \cap f_\alpha(K) - f_\beta(K) \neq \emptyset
$$

is independent of $r, z$.

Assume without loss of generality that

$$
K - K \subseteq B_1(0).
$$

Let $d = \mathrm{d}_A(K)$ and let also $r, \rho$ such that $0 < \rho < r < 1$. Now, suppose $z \in K - K$.

Note that for any $x \in K$, we have

$$
K \subseteq B_1(x).
$$

Let

$$
\begin{aligned}
G_r(z) = \{(f_\alpha, f_\beta) : \alpha, \beta \in I_r, B_r(z) \cap f_\alpha(K) - f_\beta(K) \neq \emptyset \quad &\text{and} \\
f_\alpha(K) - f_\beta(K) \neq f_\gamma(K) - f_\delta(K), \text{ for all } (f_\gamma, f_\delta) \neq (f_\alpha, f_\beta)\}.
\end{aligned}
$$

We now observe by properties of $K$ (see Proposition 6.1) that

$$\begin{aligned} B_r(z)\cap(K-K) &\subseteq B_r(z)\cap\left(\bigcup_{(\alpha,\beta)\in I_r\otimes I_r} f_\alpha(K)-f_\beta(K)\right) \\ &= \bigcup_{(f_\alpha,f_\beta)\in G_r(z)} B_r(z)\cap(f_\alpha(K)-f_\beta(K)) \\ &\subseteq \bigcup_{(f_\alpha,f_\beta)\in G_r(z)} B_r(z)\cap(B_r(f_\alpha(x))\cap K - B_r(f_\beta(y))\cap K) \\ &= \bigcup_{(f_\alpha,f_\beta)\in G_r(z)} B_r(z)\cap B_{2r}(f_\alpha(x)-f_\beta(y)), \end{aligned}$$

for all $x, y \in K$.

We now claim that we can bound the cardinality of $G_r(z)$ independently of $r, z$. Indeed, by the weak separation property, we can find $\{x_j\}_{j=0}^M \subset F$ such that for each choice of $(f_\alpha, f_\beta), (f_\gamma, f_\delta) \in G_r(z)$, we can find $i, j \le M$ such that

$$|f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(x_i) + f_\delta(x_j)| \ge \epsilon r. \tag{6.5}$$

Based on the above, we interpret $G_r(z)$ as a graph we say that an edge $\{(f_\alpha, f_\beta), (f_\gamma, f_\delta)\}$ is of color $(i, j)$, if

$$|f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(x_i) + f_\delta(x_j)| \ge \epsilon r > 0. \tag{6.6}$$

We claim that there exists $N$ independent of $r, z$ such that

$$|G_r(z)| \le N.$$

Let $T_{ij}$ be any complete monochromatic subgraph of $G_r(z)$ of color $(i, j)$. Therefore, for all $(f_\alpha, f_\beta), (f_\gamma, f_\delta) \in T_{ij}$, (6.6) is satisfied for the same $x_i, x_j$. In particular for each $(f_\alpha, f_\beta), (f_\gamma, f_\delta) \in T_{ij}$ we have

$$f_\alpha(x_i) - f_\beta(x_j) \ne f_\gamma(x_i) - f_\delta(x_j). \tag{6.7}$$

Hence, the number of vertices in $T_{ij}$ equals the number of points $\{f_\alpha(x_i) - f_\beta(x_j) : (f_\alpha, f_\beta) \in T_{ij}\}$. For $(f_\alpha, f_\beta) \in T_{ij} \subset G_r(z)$, we also have

$$\begin{aligned} B_r(z)\cap(f_\alpha(F)-f_\beta(F)) \ne \emptyset &\Rightarrow B_r(z)\cap(B_r(f_\alpha(x_i)) - B_r(f_\beta(x_j))) \ne \emptyset \\ &\Leftrightarrow B_r(z)\cap(B_{2r}(f_\alpha(x_i)-f_\beta(x_j))) \ne \emptyset. \end{aligned}$$

Therefore, we deduce

$$|f_\alpha(x_i) - f_\beta(x_j) - z| \leq 3r,$$

and we also know that

$$|f_\alpha(x_i) - f_\beta(x_j) - f_\gamma(x_i) + f_\delta(x_j)| \geq \epsilon r.$$

Therefore, all the balls of radius $\epsilon r/2$ and centres $f_\alpha(x_i) - f_\beta(x_j)$, for $(f_\alpha, f_\beta) \in T_{ij}$ are disjoint and all the centres lie in a ball of radius $3r$ around $z$. It is immediate from (6.7) that

$$|T_j| \leq \frac{\mu(B_{3r+\epsilon r}(z))}{\mu(B_{\frac{\epsilon r}{2}}(f_\alpha(x_i) - f_\beta(x_j)))} \leq N_1,$$

independent of $r, z, (i,j)$. Hence, by Ramsey's Theorem, we have that

$$|G_r(z)| \leq N,$$

independent of $r, z$.

Now, we enumerate $G_r(z)$ using the following parametrisation

$$G_r(z) = \{(f_{\alpha_k}, f_{\beta_k})\}_{k=1}^N$$

Take any $x \in K$. Then, we have

$$\begin{aligned} B_r(z) \cap (K - K) &\subseteq \bigcup_{k=1}^N B_r(z) \cap (f_{\alpha_k}(K) - f_{\beta_k}(K)) \\ &\subseteq \bigcup_{k=1}^N B_r(z) \cap (B_r(f_{\alpha_k}(x)) \cap K - B_r(f_{\beta_k}(x)) \cap K). \end{aligned}$$

Since $f_{\alpha_k}(x), f_{\beta_k}(x) \in K$, we can cover each of these balls centred at those points by $N' = N_K(r, \rho/2)$ balls of radius $\rho/2$ centred at $K$. Let the centres of those balls be $z_i^k$. Then,

$$\begin{aligned} B_r(z) \cap (K - K) &\subseteq \bigcup_{k=1}^N \left( \bigcup_{i=1}^{N'} B_{\frac{\rho}{2}}(z_i^k) - \bigcup_{j=1}^{N'} B_{\frac{\rho}{2}}(z_j^k) \right) \\ &\subseteq \bigcup_{k=1}^N \bigcup_{i,j=1}^{N'} B_\rho(z_i^k - z_j^k) \end{aligned}$$

Thus,

$$N_{F-F}(r, \rho) \leq N(N')^2 \leq NC \left(\frac{r}{\rho}\right)^{2d}. \qquad \square$$

# Chapter 7

# Cantor sets

Cantor sets are in general some of the most common examples of self similar fractals. They are constructed by an iterated process of removing intervals from the unit interval $[0, 1]$. We first focus on symmetric Cantor sets, where at each stage of the iteration the intervals that remain are of the same length. We will show that symmetric Cantor sets satisfy the weak separation condition for differences. In particular, the Assouad dimension of differences of symmetric Cantor sets obeys bounds in terms of the Assouad dimension of the Cantor set itself.

## 7.1 Symmetric Cantor sets

Symmetric Cantor sets are constructed by removing intervals of the same length from $[0, 1]$ repeatedly. In particular, let $\lambda < 1/2$ and suppose that $C_0$ is the interval $[0, 1]$. We define $C_{k+1}$ by removing intervals of length $c_k\lambda$, from $C_k$, where $c_k$ is the length of the intervals in $C_k$ (see also Figure 7.1). Then, the middle -$\lambda$ Cantor set is defined as

$$C = \cap_{k=0}^{\infty} C_k.$$


Figure 7.1: The first stages of the iteration for the middle–1/3 Cantor set.

A symmetric Cantor set can also be defined as the attractor of an Iterated Function system. For any $\lambda < 1/2$, the middle- $\lambda$ Cantor set $\mathcal{C}_\lambda$, is the attractor of the following iterated function system.

$$f_1(x) = \lambda x \qquad \text{and} \qquad f_2(x) = \lambda x + (1-\lambda).$$

We recall the open set condition, which holds if there exists an non-empty open set $U$ such that

$$U \supseteq \bigcup_{i=1}^{|I|} f_i(U) \qquad \text{and} \qquad f_i(U) \cap f_j(U) = \emptyset.$$

It is easy to see (see Chapter 13 in the book of Falconer) that the Cantor set satisfies the open set condition for $U = (0,1)$ which in particular implies that

$$\mathrm{d}_A(C_\lambda) = \mathrm{d}_B(C_\lambda) = \mathrm{d}_{sim}(C_\lambda) = \frac{\log 2}{\log \frac{1}{\lambda}}.$$

Henderson [14] studied the Assouad dimension of the set of differences $C_\lambda - C_\lambda$ and showed that it is strictly bounded above by twice the Assouad dimension of $C_\lambda$. This is trivial when $\lambda \geq 1/3$ since

$$2\mathrm{d}_A(C_\lambda) = 2\frac{\log 2}{\log \frac{1}{\lambda}} > 1 = \mathrm{d}_A((-1,1)) \geq \mathrm{d}_A(C_\lambda - C_\lambda).$$

If $\lambda < 1/3$, Henderson [14] showed that $C_\lambda - C_\lambda$ is an attractor of another system of similarities, which satisfies the open set condition. In particular, he showed that

$$\mathrm{d}_A(C_\lambda - C_\lambda) = \frac{\log 3}{\log \frac{1}{\lambda}} > 2\mathrm{d}_A(C_\lambda).$$

The above formula can be also obtained by taking the product of the Cantor set with itself and projecting onto the span of $(-1,1)$.

We show that the Cantor set $C_\lambda$, for $\lambda < 1/3$ satisfies the weak separation property for differences, which immediately gives an example of a set which satisfies that property and

$$\mathrm{d}_A(C_\lambda - C_\lambda) < 2\,\mathrm{d}_A(C_\lambda).$$

The proof also provides a useful technique for the more involved analysis of asymmetric Cantor sets.

**Proposition 7.1.** *The Cantor $C_\lambda$, for $\lambda < \frac{1}{3}$ satisfies the weak separation property for differences.*

*Proof.* Fix $\lambda < 1/3$. Then, $C_\lambda$ is the attractor of the Iterated Functions system

$$f_1(x) = \lambda x \qquad \text{and} \qquad f_2(x) = \lambda x + (1-\lambda).$$

Take any $0 < b < 1$. We claim that there exists a $\delta > 0$, such that for any $\alpha, \beta, \gamma, \delta \in I_b$ we have

$$f_\alpha(C_\lambda) - f_\beta(C_\lambda) = f_\gamma(C_\lambda) - f_\delta(C_\lambda).$$

or

$$|f_\alpha(x) - f_\beta(y) - f_\gamma(x) + f_\delta(y)| \geq \delta b,$$

for every $x, y \in C_\lambda$. In particular, this obviously implies the weak separation for differences by choosing any single point in the Cantor set.

Now, we fix $0 < b < 1$. Then, it is easy to see for any $\alpha = (i_1, \cdots, i_k)$, $\beta = (i_1, \cdots, i_m) \in I_b$, we have that $k = m$ and

$$c_\alpha = c_\beta = \lambda^k \leq b \leq \lambda^{k-1} = c_{\bar{\alpha}} = c_{\bar{\beta}}. \tag{7.1}$$

We also have that for any $\alpha = (i_1, \cdots, i_k) \in I_b$, there exists some translation $q_\alpha$ such that for any $x \in C_\lambda$

$$f_\alpha(x) = \lambda^k x + q_\alpha$$

and

$$q_\alpha = \sum_{i=0}^{k-1} t_i \lambda^i (1-\lambda) = (1-\lambda) \sum_{i=0}^{k-1} t_i \lambda^i,$$

for $t_i \in \{0, 1\}$.

Therefore, for any $\alpha, \beta, \gamma, \delta \in I_b$, there exists some $k \in \mathbb{N}$ such that $|\alpha| = |\beta| = |\gamma| = |\delta| = k$ and for any $x, y \in C_\lambda$

$$|f_\alpha(x) - f_\beta(y) - f_\gamma(x) + f_\delta(y)| = (1-\lambda) \left| \sum_{i=0}^{k-1} a_i \lambda^i \right|,$$

where $a_i \in \{-1, -2, 0, 1, 2\}$.

Suppose now that

$$f_\alpha(C_\lambda) - f_\beta(C_\lambda) \neq f_\gamma(C_\lambda) - f_\delta(C_\lambda).$$

We claim that $\left| \sum_{i=0}^{k-1} a_i \lambda^i \right| \neq 0$.

Indeed, suppose without loss of generality that

$$d_H(f_\alpha(C_\lambda) - f_\beta(C_\lambda), f_\gamma(C_\lambda) - f_\delta(C_\lambda)) = \text{dist}(f_\alpha(C_\lambda) - f_\beta(C_\lambda), f_\gamma(C_\lambda) - f_\delta(C_\lambda))$$

and let $x_0, y_0 \in C_\lambda$ be such that

$$d_H(f_\alpha(C_\lambda) - f_\beta(C_\lambda), f_\gamma(C_\lambda) - f_\delta(C_\lambda)) = \text{dist}(f_\alpha(x_0) - f_\beta(y_0), f_\gamma(K) - f_\delta(K)) > 0.$$

Then,

$$\begin{aligned}\left|\sum_{i=0}^{k-1} a_i\lambda^i\right| &= |f_\alpha(x_0) - f_\beta(y_0) - f_\gamma(x_0) + f_\delta(y_0)| \\ &\geq \text{dist}(f_\alpha(x_0) - f_\beta(y_0), f_\gamma(K) - f_\delta(K)) > 0.\end{aligned}$$

Suppose that $\sum_{i=0}^{k-1} a_i\lambda^i > 0$. We claim that there exist $\widehat{a_i} > 0$ such that

$$\sum_{i=0}^{k-1} a_i\lambda^i = \sum_{i=0}^{k-1} \widehat{a_i}\lambda^i.$$

We construct $\widehat{a_i}$ by the following process. If $a_{k-1} \geq 0$, we set $\widehat{a_{k-1}} = a_{k-1}$. If $a_{k-1} < 0$, we write

$$a_{k-1} = \left(\frac{1}{\lambda} + a_{k-1}\right)\lambda^{(k-1)} - \lambda^{(k-2)}$$

and we set

$$\widehat{a_{k-1}} = \left(\frac{1}{\lambda} + a_{k-1}\right).$$

Then, $\widehat{a_{k-1}} > 0$, since $\lambda < \frac{1}{3}$.

Now, if $a_{k-2}\lambda^{(k-2)} - \lambda^{(k-2)} = (a_{k-2} - 1)\lambda^{k-2} < 0$, then we again write

$$(a_{k-2} - 1)\lambda^{(k-2)} = \left(\frac{1}{\lambda} + a_{k-2} - 1\right)\lambda^{(k-2)} - \lambda^{(k-3)}$$

and we set

$$\widehat{a_{(k-2)}} = \frac{1}{\lambda} + a_{k-2} - 1.$$

Then, $\frac{1}{\lambda} + a_{k-2} - 1 > 0$, since $\lambda < 1/3$ and we carry on this procedure until we construct $\widehat{a_0}$. Now, for all $1 \leq i \leq k-1$, we have that

$$\widehat{a_i} = a_i \text{ or } (\frac{1}{\lambda} + a_i) \text{ or } (\frac{1}{\lambda} + a_i - 1),$$

which are all non negative. We claim that

$$A = \sum_{i=1}^{k-1} \widehat{a_i} \lambda^i < 1.$$

We observe that for all $1 \le i \le k-1$, $\widehat{a_i} \le 2$. Therefore

$$A \le 2 \sum_{i=1}^{k-1} \lambda^i < 2 \sum_{i=1}^{k-1} \left(\frac{1}{3}\right)^i \le 2(\frac{3}{2} - 1) = 1.$$

Hence, $\widehat{a_0} > -1$. If $\widehat{a_0} < 0$, then

$$\widehat{a_0} \le -1,$$

a contradiction.

By a symmetric argument, i.e. by subtracting $c$ where necessary we have that if the sum is negative then it can be written such that all the coefficients are non–positive. Assume that $\widehat{a_i} \ge 0$, for all $i$. In particular, by the construction above, we observe that if $\widehat{a_i} > 0$, then

$$\widehat{a_i} \ge \left(\frac{1}{\lambda} - 3\right) > 0. \tag{7.2}$$

Let $0 \le m \le k-1$ such that $\widehat{a_m} > 0$. Then, by (7.1), (7.2), we have

$$\begin{aligned} |f_\alpha(x) - f_\beta(y) - f_\gamma(x) + f_\delta(y)| &= (1-\lambda) \left| \sum_{i=0}^{k-1} a_i \lambda^i \right| = (1-\lambda) \sum_{i=0}^{k-1} \widehat{a_i} \lambda^i \\ &\ge (1-\lambda) a_m \lambda^m \ge (1-\lambda) \left(\frac{1}{\lambda} - 3\right) \lambda^{k-1} \\ &\ge (1-\lambda) \left(\frac{1}{\lambda} - 3\right) b, \end{aligned}$$

which concludes the proof. □

We note that we can actually make all the coefficients in the above construction either negative or positive, depending on whether the sum is negative or positive. We will need this in the following section, where we prove that a particular class of Asymmetric Cantor sets satisfies the weak separation condition for differences.

## 7.2 Non-symmetric Cantor sets

Non-symmetric Cantor sets are constructed by an iterative process of removing intervals of different lengths from the unit interval. In particular let $c_1, c_2 \in (0, 1)$

such that $c_1 + c_2 < 1$. Suppose that $C_0 = [0,1]$. We construct $C_1$ by removing an interval of length $1 - c_1 - c_2$ from $C_0$ and we set $C_1$ to be the remaining two intervals. We carry on by removing intervals of proportionate length from each of the intervals in $C_k$ (see also Figure 7.2).


Figure 7.2: The first stages of the iteration for the asymmetric Cantor set.

Asymmetric Cantor sets are also the attractors of the following Iterated function system

$$f_1(x) = c_1 x \qquad \text{and} \qquad f_2(x) = c_2 x + (1 - c_2),$$

for $c_1, c_2 \in (0,1)$, such that $c_1 + c_2 < 1$. We denote the non symmetric Cantor set by $K_{c_1 c_2}$. It has been proven by Henderson that if $\frac{\log c_1}{\log c_2}$ is an irrational number, then $\mathrm{d}_A(K_{c_1c_2} - K_{c_1c_2}) = 1$, which is maximal for this set.

It is an open question whether we can show that the Assouad dimension of $K - K$ is bounded by twice the Assouad dimension of $K$ when $\frac{\log c_1}{\log c_2}$ is any rational number.

In this section, we show that if $\frac{\log c_1}{\log c_2}$ is a rational number and $c_1 < c_2 < \frac{1}{4}$, then the weak separation for differences is satisfied. In particular, we prove the following theorem, which is the main result of this chapter.

**Theorem 7.2 (Theorem 1.6).** *Suppose $c \in (0,1)$, $p_2 < p_1 \in \mathbb{N}$ such that $c^{p_1} < c^{p_2} < 1/4$. Let $K$ be the attractor of the system $\mathcal{F} = \{f_1, f_2\}$ where*

$$f_1(x) = c^{p_1} x, \qquad \textit{and} \qquad f_2(x) = c^{p_2} x + (1 - c^{p_2}).$$

*Then, $K$ satisfies the weak separation condition for differences. In particular,*

$$\mathrm{d}_A(K - K) \le 2\mathrm{d}_A(K) \le 2\mathrm{d}_{\mathrm{sim}}(K).$$

The proof follows a similar procedure with the one for symmetric Cantor sets, but is significantly more involved.

*Proof.* Fix any $0 < b < 1$. Let $\alpha, \beta, \gamma, \delta \in I_b$. Then, for any $x, y \in K$, we have

$$f_\alpha(x) - f_\beta(y) - f_\gamma(x) + f_\delta(y) = (c_\alpha - c_\gamma)x + (c_\delta - c_\beta)y + q_\alpha - q_\beta - q_\gamma + q_\delta, \quad (7.3)$$

for some translations $q_\alpha, q_\beta, q_\gamma, q_\delta$. By definition of $I_b$, we have that for any $\alpha \in I_b$

$$c_\alpha \geq c_{\bar{\alpha}} c^{p_1} \geq b c^{p_1} \quad (7.4)$$

Assume that

$$f_\alpha(K) - f_\beta(K) \neq f_\gamma(K) - f_\delta(K).$$

Assume first that $q_\alpha - q_\beta - q_\gamma + q_\delta = 0$. By compactness of $K$, let $x_0, y_0 \in K$ such that

$$d_H(f_\alpha(K) - f_\beta(K), f_\gamma(K) - f_\delta(K)) = \operatorname{dist}(f_\alpha(x_0) - f_\beta(y_0), f_\gamma(K) - f_\delta(K)) > 0.$$

Therefore,

$$|f_\alpha(x_0) - f_\alpha(y_0) - f_\gamma(x_0) + f_\delta(y_0)| \geq \operatorname{dist}(f_\alpha(x_0) - f_\beta(y_0), f_\gamma(K) - f_\delta(K)) > 0.$$

It is immediate from (7.3) that either $c_\alpha \neq c_\gamma$ or $c_\beta \neq c_\delta$. Assume without loss of generality that $c_\alpha < c_\gamma$. Then, since $0, 1 \in K$ we have by (7.4)

$$|f_\alpha(1) - f_\beta(0) - f_\gamma(1) + f_\delta(0)| = |c_\alpha - c_\gamma| = c_\gamma \left(1 - \frac{c_\alpha}{c_\gamma}\right) \geq b c^{p_2} \left(1 - \frac{c_\alpha}{c_\gamma}\right).$$

We claim that there exists $M > 0$ such that

$$\left(1 - \frac{c_\alpha}{c_\gamma}\right) \geq M.$$

Indeed, let $n_1, n_2, m_1, m_2$ be such that

$$c_\alpha = c^{p_1 n_1 + p_2 n_2} \qquad \text{and} \qquad c_\gamma = c^{p_1 m_1 + p_2 m_2},$$

with $p_1 m_1 + p_2 m_2 < p_1 n_1 + p_2 n_2$. Then,

$$\frac{c^{p_1 n_1 + p_2 n_2}}{c^{p_1 m_1 + p_2 m_2}} \leq \frac{c^{p_1 m_1 + p_2 m_2 + 1}}{c^{p_1 m_1 + p_2 m_2}} = c,$$

which implies that

$$\left(1 - \frac{c_\alpha}{c_\gamma}\right) \geq (1 - c).$$

Thus, the weak separation property is satisfied when $q_\alpha - q_\beta - q_\gamma + q_\delta = 0$. Suppose now that $q_\alpha - q_\beta - q_\gamma + q_\delta \neq 0$. By (7.3), we have that

$$|f_\alpha(0) - f_\beta(0) - f_\gamma(0) + f_\delta(0)| = |q_\alpha - q_\beta - q_\gamma + q_\delta|.$$

We claim that there exists $M_1 > 0$ such that

$$|q_\alpha - q_\beta - q_\gamma + q_\delta| \geq M_1 b.$$

We want to write $q_\alpha - q_\beta - q_\gamma + q_\delta$ in terms of powers of $c^{p_1}$ and $c^{p_2}$. For $\alpha \in I_b$, let $n_\alpha, m_\alpha \in \mathbb{N}$ be such that

$$c_\alpha = c^{n_\alpha p_1 + m_\alpha p_2}.$$

Then, we have

$$q_\alpha = (1 - c^{p_2}) \left( c^0 \left( \sum_{j=0}^{m_\alpha - 1} t_{o\,j} c^{p_2\,j} \right) + \cdots + c^{p_1(n_\alpha - 1)} \left( \sum_{j=1}^{m_\alpha - 1} t_{n_\alpha - 1\,j} c^{p_2\,j} \right) \right)$$

for some $t_{ij} \in \{0, 1\}$, where $0 \leq i \leq n_\alpha - 1$ and $0 \leq j \leq m_\alpha - 1$.

Assume that $N_1 = \max\{n_\alpha, n_\beta, n_\gamma, n_\delta\}$ and $N_2 = \max\{m_\alpha, m_\beta, m_\gamma, m_\delta\}$. Then,

$$q_\alpha - q_\beta - q_\gamma + q_\delta = (1 - c^{p_2}) \left( c^0 \left( \sum_{j=0}^{N_2 - 1} a_{o\,j} c^{p_2\,j} \right) + \cdots + c^{p_1(N_1 - 1)} \left( \sum_{j=1}^{N_2 - 1} a_{(N_1 - 1)j} c^{p_2\,j} \right) \right)$$

where $a_{ij} \in \{-2, -1, 0, 1, 2.\}$, for all $i \leq N_1 - 1$ and $j \leq N_2 - 1$. Let

$$A_i = \sum_{j=0}^{N_2 - 1} a_{ij} c^{p_2 j}.$$

Since $c^{p_2} < 1/4 < 1/3$, by the argument in the previous section (see proof of Proposition 7.1), we can rewrite all negative $A_i$ such that all the coefficients $a_{ij}$ are negative and all positive $A_i$, such that all $a_{ij}$ are positive. In this case we also note by the previous argument that if $a_{ij} < 0$, for some $i, j$ then

$$-2 \leq a_{ij} \leq 3 - \frac{1}{c^{p_2}} < -1$$

and if $a_{ij} > 0$, for some $i, j$, then

$$1 \leq \frac{1}{c^{p_2}} - 3 \leq a_{ij} \leq 2.$$

Consequently, if $A_i < 0$ then

$$-2\sum_{j=1}^{\infty} c^{p_2 j} \leq -2\sum_{j=1}^{N_2-1} c^{p_2 j} \leq A_i < -\sum_{j=1}^{N_2-1} c^{p_2 j}, \tag{7.5}$$

and if $A_i > 0$, we have

$$1 \leq \sum_{j=1}^{N_2-1} c^{p_2 j} \leq A_i \leq 2\sum_{j=1}^{N_2-1} c^{p_2 j} \leq 2\sum_{j=1}^{\infty} c^{p_2 j}. \tag{7.6}$$

Assume that $q_\alpha - q_\beta - q_\gamma + q_\delta > 0$. We want to rewrite the above sum such that all $A_i$ are non–negative. If $A_{N_1-1} \geq 0$, we set $\widehat{A_{N_1-1}} = A_{N_1-1}$. If $A_{N_1-1} < 0$, we set

$$\widehat{A_{N_1-1}} = \left(\frac{1}{c^{p_1}} + A_{N_1-1}\right) c^{p_1(N_1-1)} - c^{p_1(N_1-2)}.$$

We claim that $\left(\frac{1}{c^{p_1}} + A_{N_1-1}\right) > 0$. Indeed, since $A_{N_1-1} < 0$, we have that

$$A_{N_1-1} \geq -2\sum_{i=0}^{N_2-1} c^{p_2 i} \geq \frac{-2}{1-c^{p_2}} \geq -3, \tag{7.7}$$

since $c^{p_2} < \frac{1}{3}$. Thus, $\widehat{A_{N_1-1}} > 0$, since $c^{p_1} < \frac{1}{4}$. Now, arguing is in the symmetric Cantor set case, if $A_{N_1-2} - 1 \geq 0$, we set $\widehat{A_{N_1-2}} = A_{N_1-2} - 1$, while if $A_{N_1-2} - 1 < 0$, we set

$$\widehat{A_{N_1-2}} = \left(\frac{1}{c^{p_1}} + A_{N_1-2} - 1\right),$$

which is positive since since $c^{p_1} < 1/4$ and $A_{N_1-2} \geq -3$. We then write

$$\begin{aligned} A_{N_1-2}c^{p_1(N_1-2)} &= \left(\frac{1}{c^{p_1}} + A_{N_1-2} - 1\right) c^{p_1(N_1-2)} - c^{p_1(N_1-3)} \\ &= \widehat{A_{N_1-2}} c^{p_1(N_1-2)} - c^{p_1(N_1-3)}. \end{aligned}$$

We continue the process until we have defined $\widehat{A_0}$. We note that for all $1 \leq i \leq n-1$, by (7.6) and (7.7), we have

$$\frac{1}{c^{p_1}} - 4 \leq \widehat{A_i} \leq 2\sum_{j=1}^{N_2-1} c^{p_2 j}.$$

Hence,

$$\sum_{i=1}^{N_1-1} \widehat{A_i} c^{ip_1} \leq 2B_j \sum_{i=1}^{\infty} c^{p_1 i},$$

where

$$B_j = \sum_{j=1}^{N_2-1} c^{p_2 j}.$$

Therefore,

$$\sum_{i=1}^{N_1-1} \widehat{A_i} c^{ip_1} \leq B_j \left( \frac{2c^{p_2}}{1-c^{p_2}} \right) < B_j,$$

since $c^{p_2} < 1/3$. Since we have assumed $q_\alpha - q_\beta - q_\gamma + q_\delta > 0$, we need

$$\widehat{A_0} > -B_j = -\sum_{j=1}^{N_2-1} c^{p_2 j}.$$

But, from (7.5), (7.6), we deduce that if $\widehat{A_0} < 0$, then it must satisfy

$$\widehat{A_0} \leq -\sum_{j=1}^{N_2-1} c^{p_2 j}.$$

Consequently, $\widehat{A_0} \geq 0$. By a symmetric argument, if $q_\alpha - q_\beta - q_\gamma + q_\delta < 0$, we can rewrite the sum such that all the $A_i$ are non–positive, for $0 \leq i \leq N-1$. Assume without loss of generality that $q_\alpha - q_\beta - q_\gamma + q_\delta > 0$. Then, we have

$$q_\alpha - q_\beta - q_\gamma + q_\delta = (1-c^{p_2}) \sum_{i=0}^{N_1-1} \widehat{A_i} c^{ip_1}.$$

By the above construction, we deduce that if $\widehat{A_i} > 0$, then $\widehat{A_i} \geq \frac{1}{c^{p_1}} - 4 > 0$, for $0 \leq i \leq N_1 - 1$. Moreover, for every $i$, we have that

$$\widehat{A_i} = \sum_{j=0}^{N_2-1} \widehat{a_{ij}} c^{jp_2}.$$

Similarly, if $\widehat{a_{ij}} > 0$, for some $0 \leq j \leq N_2 - 1$, then $\widehat{a_{ij}} \geq \frac{1}{c^{p_2}} - 3 > 0$.

Assume that $N_1 = n_\alpha = \max\{n_\alpha, n_\beta, n_\gamma, n_\delta\}$ and $N_2 = m_\beta$. Assume without loss of generality that $m_\beta \leq n_\alpha$. Thus, $m_\alpha \leq m_\beta \leq n_\alpha$. Let $0 \leq m \leq n_\alpha - 1$ such that $\widehat{A_m} > 0$. Let also $0 \leq n \leq m_\alpha - 1 \leq m_\beta - 1$ such that $\widehat{a_{mn}} > 0$. Thus,

$$\begin{aligned} |q_\alpha - q_\beta - q_\gamma + q_\delta| &\geq (1-c^{p_2})\widehat{A_m} c^{p_1 m} \geq (1-c^{p_2})\widehat{a_{mn}} c^{p_2 n} c^{p_1 m} \\ &\geq (1-c^{p_2}) \left( \frac{1}{c^{p_2}} - 3 \right) c^{(m_\alpha - 1)p_2} c^{(n_\alpha - 1)p_1} \\ &\geq (1-c^{p_2}) \left( \frac{1}{c^{p_2}} - 3 \right) c_{\bar{\alpha}} \geq (1-c^{p_2}) \left( \frac{1}{c^{p_2}} - 3 \right) b, \end{aligned}$$

which concludes the proof that the system satisfies the weak separation condition for differences. Since the weak separation condition for differences trivially implies the standard weak separation condition we deduce that $\mathrm{d}_{\mathrm{sim}}(K) \geq \mathrm{d}_B(K) = \mathrm{d}_A(K)$, which implies the desired result by Theorem 6.15. □

Consequently, we have the following result

**Theorem 7.3.** *Suppose $c_1 < c_2 < 1/4$ such that $\frac{\log c_1}{\log c_2}$ is rational. Suppose that $K_{c_1c_2}$ is the attractor of the system $\mathcal{F} = \{f_1, f_2\}$ such that*

$$f_1(x) = c_1 x \qquad \textit{and} \qquad f_2(x) = c_2 x + (1 - c_2).$$

*Then,*

$$\mathrm{d}_A(K_{c_1c_2} - K_{c_1c_2}) \leq 2\mathrm{d}_A(K_{c_1c_2}) \leq 2\mathrm{d}_{\mathrm{sim}}(K_{c_1c_2}).$$

*Proof.* Suppose that

$$\frac{\log c_1}{\log c_2} = \frac{p_1}{p_2}.$$

Then, $c_1 = c_2^{p_1/p_2}$. Let $c = c_2^{1/p_2}$. Then, $c_1 = c^{p_1}$ and $c_2 = c^{p_2}$. Moreover, $c^{p_1} < c^{p_2} < 1/4$. Thus, by the above Theorem, the result follows immediately. □

We note that the only case that needs to be covered is when the above theorem can be a useful tool for computing explicit bounds for the Assouad dimension of differences of non symmetric Cantor sets. Let $c \in (0, 1)$, $c_1 = c^p, c_2 = c^{2p}$ such that $c^{2p} < c^p < \frac{1}{4}$. Then, we can explicitly compute the similarity dimension $\mathrm{d}_{\mathrm{sim}}(A_{c_1c_2})$. In particular, let $D$ such that

$$c^{pD} + c^{2pD} = 1.$$

By solving the quadratic equation for $c^p$, we find that

$$D = \frac{\log \phi}{p \log \left(\frac{1}{c}\right)},$$

where

$$\phi = \frac{2}{\sqrt{5} - 1}.$$

Thus,

$$\mathrm{d}_A(A_{c_1c_2} - A_{c_1c_2}) \leq \frac{2 \log \phi}{p \log \left(\frac{1}{c}\right)}.$$

We note that in the above argument we only require one of the exponents to be less than 1/4 and the other one to be less than 1/3. Moreover, it can be easily shown that if both of the exponents are bigger than 1/3, then the asymmetric Cantor set contains an interval, which trivially implies the bound on the Assouad dimension of the set of differences.

# Chapter 8

# Conclusions and Future Work

## 8.1 Conclusion

The results presented in this thesis extend several embedding theorems, previously known only for subsets of Hilbert spaces, into subsets of Banach spaces. In particular, using a combination of methods introduced by Hunt and Kaloshin, with some key new ingredients, we established several new embedding theorems for subsets of Banach spaces with finite box–counting dimension that depend solely on the box–counting dimension or the thickness exponent.

We prove an embedding theorem for subsets of Banach spaces such that the set of differences is almost homogeneous at 0. The theorem is an extension of the respective result for subsets of Hilbert spaces, such that $X - X$ is almost homogeneous, that was proved by Olson & Robinson [25]. In particular, we prove the theorem under a weaker condition which only deals with balls around 0 rather than any point in $X - X$.

Our embedding theorems rely on a property that holds for the set of differences. It is natural to ask whether the doubling property is enough to obtain local covers for balls around 0. We give an example of a metric space which is doubling but not $(\alpha, \beta)$-almost homogeneous at 0, for any $\alpha, \beta \geq 0$, which answers the question negatively.

However, in the context of Iteration Function Systems, we show that if the system satisfies a suitable separation condition, then the Assouad dimension of differences of attractors obeys non trivial bounds related to the Assouad dimension of the attractor itself. In particular, we show that particular examples of symmetric and asymmetric Cantor sets fall in the above class.

## 8.2 Future Work

There are a number of questions that arise naturally from the results presented in this thesis and we would like to list some of them.

**Question 8.1.** In Section 2.4, we consider a set $X$ with finite box counting dimension and thickness exponent less than 1. Is it possible to extent the theorem without the restriction of the thickness exponent being less than 1, in such a way that it improves on Theorem 1.2?

**Question 8.2.** In Section 3.2, we show that the thickness exponent of any 'orthogonal' sequence in $\ell_p$, for $p \in [1, 2]$ equals the box dimension. We also show a lower bound for $p > 1$. De Moura & Robinson showed that if a orthogonal sequence $A$ in $\ell_\infty$ can be linearly $\theta$-bi–Hölder embedded in some Euclidean space, then

$$\theta \leq \frac{1}{1 + \mathrm{d}_B(A)}.$$

Can we find an orthogonal sequence $A$ in $\ell_\infty$, such that $\tau(A) < \mathrm{d}_B(A)$? A positive answer to this question would yield a negative answer to question 8.1.

**Question 8.3.** In Section 4.2.2, we construct a prevalent set of linear almost bi–Lipschitz embeddings from a subset of Banach space $X$ into some Euclidean space, when $X - X$ is almost homogeneous at 0. Is it possible to extend the theorem for doubling subsets of Banach spaces?

**Question 8.4.** Suppose $X$ is a doubling subset of a Banach space. Can we find an almost bi–Lipschitz embedding $\psi$ from $X$ into another Banach space $Y$ such that the set of differences $\psi(X) - \psi(X)$ into $Y$ is almost homogeneous at 0? This would yield a positive answer to the question above.

**Question 8.5.** In Section 4.2, we show that if $X - X$ is almost homogeneous and can be embedded into a Hilbert space, using a linear almost bi–Lipschitz map $\Phi$, then $\Phi(X) - \Phi(X)$ is almost homogeneous at 0. This leads to the following question. Is the condition that $X - X$ is almost homogeneous at 0 necessary in order to have linear bi–Lipschitz embeddings into some Euclidean space? (we already know by Theorem 4.12 that it is sufficient).

**Question 8.6.** In Section 4.3, we show that subsets of Banach spaces that can be well approximated by linear subspaces can be linearly almost bi–Lipschitz embedded into some Hilbert space $H$. Olson & Robinson show that any doubling metric space admits an almost bi–Lipschitz embedding in a Hilbert space $H$ and the image satisfies the above property. Can we show that subsets of a Hilbert space with this 'better than zero thickness' property embed in an almost bi–Lipschitz way into some $\mathbb{R}^k$? This would yield a positive answer to question 8.3.

**Question 8.7.** In Section 5.3, we give an example of a doubling set $X$ such that $X - X$ is not almost-homogeneous at 0. We know that $X$ cannot be embedded into any Hilbert space, with a bi–Lipschitz map. It is also true from theorem 4.3 that $X$ can be almost bi-Lipschitz embedded into a Hilbert space. Can we construct an almost bi–Lipschitz map from $X$ into an Euclidean space?

**Question 8.8.** In Section 6.3, we establish a weak separation condition that allows us to bound the Assouad dimension of differences of self similar fractals in a non–trivial way. In Lemma 6.14, we give a sufficient condition for the weak separation to hold. Is it true that the two conditions are equivalent?

**Question 8.9.** In Section 7.2, we prove that a particular class of non symmetric Cantors sets satisfies the weak separation property for differences. In particular, we obtain non-trivial bounds for the Assouad dimension of differences of asymmetric Cantor sets when $c_1 < c_2 < 1/4$, and

$$\frac{\log c_1}{\log c_2} = \frac{p_1}{p_2},$$

for $p_1, p_2 \in \mathbb{N}$. Henderson [14] showed that if $\frac{\log c_1}{\log c_2}$ is irrational then the Assouad dimension of the set of differences is maximal. Is it true that when $c_1 < 1/4$, $c_2 \geq 1/4$ and $\frac{\log c_1}{\log c_2}$ is rational, then the weak separation property for differences is always satisfied?

**Question 8.10.** Suppose $H$ is a Hilbert space. Let $f\colon H \to H$ be a contracting similarity , i.e. it satisfies

$$\|f(x) - f(y)\| = c\|x - y\|,$$

for all $x, y \in H$ and for some $c < 1$. We know by Hutchinson [17], there exists a unitary operator $U\colon H \to H$ and a point $q \in H$ such that

$$f(x) = c\,U(x) + q.$$

In particular, $f$ is bijective and the inverse $f^{-1}$ is a similarity that satisfies

$$\|f^{-1}(x) - f^{-1}(y)\| = \frac{1}{c}\|x - y\|,$$

for all $x, y \in H$. Suppose $\mathcal{F} = \{f_i\colon H \to H\}$ is a system of similarities like the one described above with an attractor $K$. What we can we say about the Assouad dimension of $K$? Can $K$ be embedded into an Euclidean space in an almost bi–Lipschitz way? Can we formulate a separation condition, similar to the one that Zerner introduced [30] to show that the set of differences is almost homogeneous at 0 under that condition?

# Appendix A

# Measures on infinite dimensional Banach spaces

In this appendix, we present several useful results from the book of Robinson [28] about measures on infinite dimensional Banach spaces, that we are using throughout the thesis. We first want to recall the construction from section 2.3 of a compactly supported probability measure that is based on a construction from Hunt and Kaloshin [15].

Suppose that $\mathfrak{B}$ is a Banach space and $\mathcal{V} = \{V_n\}_{n=1}^{\infty}$ a sequence of finite–dimensional subspaces of $\mathfrak{B}^*$, the dual of $\mathfrak{B}$. Let us denote by $d_n$ the dimension of $V_n$ and by $B_n$ the unit ball in $V_n$.

Now, we fix a real number $\alpha > 1$ and define the space $\mathbb{E}_\alpha(\mathcal{V})$ as the collection of linear maps $L\colon \mathfrak{B} \to \mathbb{R}^k$ given by

$$\mathbb{E} = \mathbb{E}_\alpha(\mathcal{V}) = \left\{ L = (L_1, L_2, ..., L_k) : L_i = \sum_{n=1}^{\infty} n^{-\alpha} \phi_{i,n},\ \phi_{i,n} \in B_n \right\}.$$

Let us also define

$$\mathbb{E}_0 = \left\{ \sum_{n=1}^{\infty} n^{-\alpha} \phi_{i,n},\ \phi_{i,n} \in B_n \right\}.$$

Clearly $\mathbb{E} = (\mathbb{E}_0)^k$.

In order to define a measure on $\mathbb{E}$, we first take a basis for $V_n$ so that we can identify $B_n$ with a symmetric convex set $U_n \subset \mathbb{R}^{d_n}$. Then, we construct each $L_i$ randomly by choosing each $\phi_{i,n}$ with respect to the normalised $d_n$–dimensional Lebesgue measure $\lambda_{d_n}$ on $U_n$. Finally, by taking $k$ copies of this measure we obtain a measure on $\mathbb{E}$. In particular we first consider $\mathbb{E}_0$ as a product space

$$\mathbb{E}_0 = \prod_{n=1}^{\infty} B_n,$$

and define a measure $\mu_0$ on $\mathbb{E}_0$ as

$$\mu_0 = \otimes_{n=1}^{\infty} \lambda_n.$$

Secondly, we consider $\mathbb{E} = \mathbb{E}_0^k$ and define $\mu$ on $\mathbb{E}$ as

$$\mu = \prod_{i=1}^{k} \mu_0.$$

For any map $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$, Hunt and Kaloshin [15] proved an upper bound on

$$\mu\{L \in \mathbb{E} : |(f + L)x| \leq \epsilon\},$$

for $x \in \mathcal{B}$ and any $\epsilon > 0$. In order to prove such an estimate, we need the following lemma, which can be found in the book of Robinson [28].

**Lemma A.1.** *Suppose $\mathcal{L}^n$ is the Lebesque measure on $\mathbb{R}^n$ and $\lambda_n$ is the uniform $n$-dimensional Lebesque measure. If $a \in \mathbb{R}$ and $x \in \mathfrak{B}$ then*

$$\lambda_{d_n}\{\phi \in B_n : |a + \phi(x)| < \epsilon\} \leq d_n \left(\frac{\epsilon}{|g(x)|}\right), \tag{A.1}$$

*for any $g \in B_n$, the unit ball in $V_n$.*

*Proof.* For the proof, we recall the Brunn-Minkowski inequality (see Gardner [12] for a detailed proof) which says that if $T_1, T_2$ are two convex subsets of $\mathbb{R}^n$, then

$$\mathcal{L}^n((1-t)T_1 + tT_2)^{1/n} \geq (1-t)\mathcal{L}^n(T_1)^{1/n} + t\mathcal{L}^n(T_2)^{1/n},$$

for $t \in [0, 1]$.

Let $x \in \mathfrak{B}$. Suppose that $g(x) \neq 0$, otherwise the inequality is trivial. Let

$$P = \{p \in V_n : p(x) = 0\}.$$

It is easy to see that $P$ is a subspace of $\mathfrak{B}^*$ and

$$\dim(P) = \dim(V_n) - 1 = d_n - 1.$$

Let $\phi \in B_n$ such that $|\phi(x) + a| < \epsilon$. We have

$$[g(x)\phi - \phi(x)g](x) = 0.$$

Let $p' \in P$ such that $g(x)\phi = \phi(x)g + p'$. Thus,

$$\phi = \frac{\phi(x)}{g(x)}g + \frac{1}{g(x)}p',$$

which implies that there exists $p \in P$ such that

$$\phi = \frac{\phi(x)}{g(x)}g + p.$$

Therefore,

$$\left\| \phi + \frac{a}{g(x)}g - p \right\| = \frac{|\phi(x) + a|}{|g(x)|}\|g\| < \frac{\epsilon}{|g(x)|}\|g\|.$$

Suppose that $P$ is represented by a hyperplane $\Pi$ in $\mathbb{R}^{d_n}$ and $g$ is identified with a vector $v \in U_n$, which is a symmetric convex set in $\mathbb{R}^{d_n}$. Let also

$$b = \frac{a}{g(x)}.$$

Then, the probability on the left hand side of (A.1) is bounded above by the probability that an element $u \in U_n$ lies between

$$\left(-b - \frac{\epsilon}{|g(x)|}\right) v + \Pi \qquad \text{and} \qquad \left(-b + \frac{\epsilon}{|g(x)|}\right) v + \Pi.$$

For any $s \in \mathbb{R}$, let

$$K_s = U_n \cap (\Pi + sv),$$

the intersection of the symmetric convex set $U_n$ with translations of the hyperplane $\Pi$. By the Brunn-Minkowski inequality, we have that the function

$$s \to \mathcal{L}^{d_n - 1}(K_s)$$

is a concave function. In particular, it attains its maximum value at 0. If $\theta$ denotes the smallest angle that $\Pi$ makes with the vector $v$ in $U_n$, we have

$$\lambda_{d_n}\{\phi \in B_n : |\phi(x) + a| < \epsilon\} \leq \frac{\mathcal{L}^{d_n-1}(K_0)|v|\frac{2\epsilon}{|g(x)|}\sin\theta}{\mathcal{L}^{d_n}(U_n)}.$$

Since $U_n$ is a symmetric convex set, it contains the cone with base $K_0 = U_n \cap \Pi$ and vertex $v$ and the cone with base $K_0$ and vertex $-v$. Consequently,

$$\mathcal{L}^{d_n}(U_n) \geq \frac{2\mathcal{L}^{d_n-1}(K_0)|v|\sin\theta}{d_n},$$

which implies that

$$\lambda_{d_n}\{\phi \in B_n : |\phi(x) + a| < \epsilon\} \leq d_n \frac{\epsilon}{|g(x)|}. \qquad \square$$

We can now show the following estimate.

**Lemma A.2** (Lemma 2.16). *Suppose that* $x \in \mathcal{B}$, $\epsilon > 0$, $f \in \mathcal{L}(\mathfrak{B}; \mathbb{R}^k)$ *and* $\mathcal{V} = \{V_n\}$ *as above. Then*

$$\mu\{L \in \mathbb{E}_\alpha : |(f + L)(x)| < \epsilon\} \leq \left(n^\alpha d_n \frac{\epsilon}{|g(x)|}\right)^k,$$

*for any* $g \in B_n$.

*Proof.* By definition of the measure $\mu$, we have

$$\begin{aligned}\mu\{L \in \mathbb{E} : |(f + L)(x)| < \epsilon\} &\leq \mu\{L \in \mathbb{E} : |(f_i + L_i)(x)| < \epsilon \text{ for all } i \leq k.\} \\ &= \prod_{i=1}^{k} \mu_0\{L \in \mathbb{E}_0 : |(f_i + L)(x)| < \epsilon\}\end{aligned}$$

Take any $f_0 \in \mathfrak{B}^*$. Then we estimate

$$\begin{aligned}&\mu_0\{L \in \mathbb{E}_0 : |(f_0 + L)(x)| < \epsilon\} \\ &= \bigotimes_{m=1}^{\infty} \lambda_{d_m} \left\{ \{\phi_m\}_{m=1}^{\infty} \in \mathbb{E}_0 : \left| f_0(x) + \sum_{m=1}^{\infty} m^{-\alpha}\phi_m(x) \right| < \epsilon \right\} \\ &= \bigotimes_{m=1}^{\infty} \lambda_{d_m} \left\{ \{\phi_m\}_{m=1}^{\infty} \in \mathbb{E}_0 : \left| \left( f_0(x) + \sum_{m \neq n}^{\infty} m^{-\alpha}\phi_m(x) \right) + n^{-\alpha}\phi_n(x) \right| < \epsilon \right\}.\end{aligned}$$

For

$$a = f_0(x) + \sum_{m \neq n}^{\infty} m^{-\alpha},$$

Lemma (A.1) implies that

$$\begin{aligned}\lambda_{d_n}\{\phi \in B_n : |a + n^{-\alpha}\phi(x)| < \epsilon\} &= \lambda_{d_n}\{\phi \in B_n : |n^{-\alpha}\phi(x)| < \epsilon\} \\ &= \lambda_{d_n}\{\phi \in B_n : |\phi(x)| < n^\alpha \epsilon\} \\ &\leq d_n \frac{\epsilon\, n^\alpha}{|g(x)|},\end{aligned}$$

for any $g \in B_n$.

By definition of the product measure $\bigotimes_{m=1}^{\infty} \lambda_{d_m}$, we obtain

$$\mu_0\{L \in \mathbb{E}_0 : |(f_0 + L)(x)| < \epsilon\} \leq d_n \frac{\epsilon\, n^\alpha}{|g(x)|},$$

which implies that

$$\mu\{L \in \mathbb{E} : |(f + L)(x)| < \epsilon\} \leq \left( n^\alpha d_n \frac{\epsilon}{|g(x)|} \right)^k,$$

for any $g \in B_n$. □

# Bibliography


[1] Ittai Abraham, Yair Bartal, and Ofer Neiman. Embedding metric spaces in their intrinsic dimension. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 363–372. Society for Industrial and Applied Mathematics, 2008.

[2] P. Assouad. Plongements lipschitziens dans $\mathbb{R}^n$. *Bulletin de la Société Mathématique de France*, 111:429–448, 1983.

[3] A. Benartzi, A. Eden, C. Foias, and B. Nicolaenko. Hölder continuity for the inverse of mañé' s projection. *Journal of Mathematical Analysis and Applications*, 178(1):22–29, 1993.

[4] Yoav Benyamini and Joram Lindenstrauss. *Geometric nonlinear functional analysis*, volume 48. American Mathematical Soc., 1998.

[5] Guy David and Marie Snipes. A non-probabilistic proof of the assouad embedding theorem with bounds on the dimension. *Analysis and Geometry in Metric Spaces*, 1:36–41, 2013.

[6] E Pinto De Moura and J. C. Robinson. Orthogonal sequences and regularity of embeddings into finite-dimensional spaces. *Journal of Mathematical Analysis and Applications*, 368(1):254–262, 2010.

[7] Kenneth J Falconer. *The geometry of fractal sets*, volume 85. Cambridge university press, 1986.

[8] Kenneth J Falconer. Dimensions and measures of quasi self-similar sets. *Proceedings of the American mathematical society*, 106(2):543–554, 1989.

[9] C. Foias and E. Olson. Finite fractal dimension and hölder-lipschitz parametrization. *Indiana University Mathematics Journal*, pages 603–616, 1996.

[10] Jonathan Fraser. Assouad type dimensions and homogeneity of fractals. *Transactions of the American Mathematical Society*, 366(12):6687–6733, 2014.

[11] Jonathan M Fraser, Alexander M Henderson, Eric J Olson, and James C Robinson. On the assouad dimension of self-similar sets with overlaps. *Advances in Mathematics*, 273:188–214, 2015.

[12] Richard Gardner. The brunn-minkowski inequality. *Bulletin of the American Mathematical Society*, 39(3):355–405, 2002.

[13] Juha Heinonen. *Geometric embeddings of metric spaces.* Number 90. University of Jyväskylä, 2003.

[14] Alexander M Henderson. *Assouad dimension and the Open set condition.* Dissertation for Msc Degree, University of Nevada, Reno, 2013.

[15] Brian R Hunt and Vadim Yu Kaloshin. Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces. *Nonlinearity*, 12(5):1263, 1999.

[16] Brian R Hunt, Tim Sauer, and James A Yorke. Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. *Bulletin of the American mathematical society*, 27(2):217–238, 1992.

[17] John E Hutchinson. *Fractals and self similarity.* University of Melbourne.[Department of Mathematics], 1979.

[18] S. Kakutani and G. W. Mackey. Two characterizations of real hilbert space. *Annals of Mathematics*, pages 50–58, 1944.

[19] Matthew Katz and Jan Reimann. *An Introduction to Ramsey Theory*, volume 87. American Mathematical Soc., 2018.

[20] Tomi J Laakso. Plane with $a_\infty$-weighted metric not bilipschitz embeddable to $\mathbb{R}^n$. *Bulletin of the London Mathematical Society*, 34(6):667–676, 2002.

[21] Urs Lang and Conrad Plaut. Bilipschitz embeddings of metric spaces into space forms. *Geometriae Dedicata*, 87(1-3):285–307, 2001.

[22] R. Mañé. *On the dimension of the compact invariant sets of certain non-linear maps.* Springer, 1981.

[23] John McLaughlin. A note on hausdorff measures of quasi-self-similar sets. *Proceedings of the American Mathematical Society*, 100(1):183–186, 1987.

[24] A. Naor and O. Neiman. Assouad's theorem with dimension independent of the snowflaking. *arXiv preprint arXiv:1012.2307*, 2010.

[25] E. Olson and J. C. Robinson. Almost bi-lipschitz embeddings and almost homogeneous sets. *Transactions of the American Mathematical Society*, 362(1):145–168, 2010.

[26] Pierre Pansu. Métriques de carnot-carathéodory et quasiisométries des espaces symétriques de rang un. *Annals of Mathematics*, pages 1–60, 1989.

[27] J. C. Robinson. Linear embeddings of finite-dimensional subsets of banach spaces into euclidean spaces. *Nonlinearity*, 22(4):711, 2009.

[28] J. C. Robinson. *Dimensions, embeddings, and attractors*, volume 186. Cambridge University Press, 2010.

[29] Stephen Semmes. On the nonexistence of bilipschitz parameterizations and geometric problems about $a_\infty$-weights. *Revista Matemática Iberoamericana*, 12(2):337–410, 1996.

[30] Martin Zerner. Weak separation properties for self-similar sets. *Proceedings of the American Mathematical Society*, 124(11):3529–3539, 1996.