Stable rationality and degenerations of conic bundles

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Thesis
submitted to the University of Warwick
for the degree of
Doctor of Philosophy

Department of Mathematics
March 2020

Contents

Acknowledgments.

I would like to express my sincere gratitude to my supervisor, Dr. Christian Bõhning, for the continuous support during my Ph.D. study, for his patience, motivation, and immense knowledge but especially for his mentorship, which helped me broadening my understanding of mathematics. I could not have imagined having a better advisor and mentor for my Ph.D.

I also want to thank all my family for believing in me and offering constant support: none of my achievements would have ever been realisable without their help.

Finally, I am indebted with my wife Claudia for her infinite patience during these years of intense study and work. No matter how far away from earth my mind was, she has never left my side.

Coventry, 21 − 03 − 2020

Addendum: I wish to extend my gratitude to my two examiners, who have carefully read my work and offered a much welcome external point of view on it. In particular, I am grateful to Professor Miles Reid for the really helpful notes on the usage of the English language in mathematics and to Professor Nicholas Shepherd-Barron for his deep, inquisitive questions that allowed me to demonstrate the cohesion between all chapters of this thesis.

Coventry, 21 − 6 − 2020

Declaration.

I hereby confirm that this thesis has been composed by myself and has not been submitted in any previous application for any degree. Each part of the thesis which is not product of my own work has been acknowledged with appropriate citations. Parts of this thesis are a result of collaborative research, whose results have been published in the papers [**ABBvB-19**, **ABBvB-18**].

Abstract.

The main task of the thesis is to illustrate a new techniue for establishing stable irrationality of certain conic bundles by degenerating to characteristic 2 and studying the unramified Brauer group of the degenerated conic bundle in terms of the geometry of its discriminant locus. The original material has been published in the collaborative research papers [**ABBvB-19**, **ABBvB-18**].

After introducing Voisin's degeneration method and its various refinements, we explain the envisaged application strategy, which relies on the interaction between Chow-theoretic properties (like existence of a decomposition of the diagonal and universal triviality of zero-cycles) and other invariants; particular regard is given to the Brauer group, whose prime-to-p torsion part is universally trivial for a smooth universally CH_0-trivial variety defined over fields of characteristic p. In Chapter 4 we improve this result, proving that this holds for the p-primary part as well (Theorem 4.1.1), thus extending the applicability of the Brauer group to degenerations where nontrivial classe have torsion order not coprime with the characteristic of the ground field..

We focus on studying applications of these techniques to conic bundles over fields of characteristic $\neq 2$ in Chapter 3: in particular, after recalling the construction of residue maps and unramified Brauer groups for low degrees, we give a geometric interpretation of these maps in terms of the discriminant locus of a conic bundle and finally prove a formula for the unramified Brauer group of a conic bundle (Theorem 3.4.15), attributed to Colliot-Thélène but not explicitly present in the literature. Furthermore, in Section 3.5 we perform a direct computation on a general conic bundle with quintic discriminant, showing that its unramified Brauer group is even trivial. This shows formally that one cannot prove stable irrationality of cubic threefold hypersurfaces with such strategy.

Finally, we extend these techniques to the case of conic bundles defined over fields of characteristic 2. After explaining how residue maps need to be re-defined in this case, we mirror the work done in Chapter 3, providing a geometric interpretation of residue maps in terms of the geometry of the discriminant locus and then establishing a formula for the unramified Brauer group in this case (Theorem 5.4.1). As an application, we run our full strategy on a particular conic bundle threefold, showing that it is not stably rational (Theorem 5.5.1) in the last Section.

Introduction.

This thesis investigates stable rationality of a class of rationally connected varieties that admit a structure of conic bundle through application of a degeneration-type argument introduced by Claire Voisin in 2013. In this respect, much of the effort is devoted to develop computable invariants that obstruct a Chow-theoretic property included in this degeneration method; this is achieved through a number of techniques stemming from disparate areas of algebra, arithmetic and K-theory; we also highlight our original contribution in the particular case of degenerations to conic bundles defined over fields of characteristic 2.

Some of these conic bundles might be not unirational either. The relationship between unirationality (and more generally rational connectedness), stable rationality and rationality has been studied since the second half of the XIX century, the first milestone of the problem dating back to the proof of the famous *Lüroth's Theorem*, which in modern language establishes the equivalence of rationality and unirationality for algebraic varieties of dimension 1. The efforts of the Italian school of algebraic geometry to replicate the result for varieties of dimension 2 were successful thanks to the work of Castelnuovo and Enriques but did not manage to give a satisfactory answer for higher dimensional varieties, whose geometry was too sophisticated for the heuristic arguments of the era: some incorrect attempts by Fano and Enriques are recorded in [**Roth-55**, Chapter V, Section 9]; nonetheless, Roth himself also proposed a new counter-example which was later proved to be invalid. It was only in the years 1971-72 that three indisputable classes of counter-examples finally appeared: these are described in the influential papers [**AM-72**, **CG-72**, **IM-71**] and were obtained with three different techniques. However, it was soon clear that each of these techniques was only effective if applied to very specific examples and either gave little information in more general settings (the Artin-Mumford invariant), or it was not generalisable to higher dimensions (the intermediate Jacobian by Clemens and Griffiths), or it was extremely hard to work with (the birational rigidity approach by Iskovskikh and Manin).

The mathematical community began therefore to examine more closely the intermediate notion of *stable rationality*, giving rise to the so-called *stable Lüroth problem* – that is to say, distinguishing unirational, stably rational varieties from those which are unirational but not stably rational. The motivation came from the promising Artin and Mumford example of unirational threefold which is not only irrational, but even stably irrational; the invariant used to show stable irrationality of a (smooth) variety X was the torsion subgroup in the third singular cohomology, which has to vanish for any stably rational variety. It was clear at the time that this subgroup could be identified with the Brauer group of X itself, which in turn embeds into the Brauer group of the function field $k(X)$; this latter object elicited much interest for various reasons: first of all, it was much easier to manipulate, since it could be identified with some Galois cohomology group − an abelian group. And, on a second instance, if it were possible to characterise the image of $\mathrm{Br}(X)$ inside $\mathrm{Br}\, k(X)$ in function-field theoretic terms, such characterisation would have been already invariant up to birational isomorphism.

This characterisation resulted from the theory of unramified elements and residue maps developed by Serre and Merkurjev: it is, essentially, a valuation-theoretic characterisation which, in some special instances, is able to single out elements in $\mathrm{Br}\, X$ as those classes in $\mathrm{Br}\, k(X)$, called *unramified classes*, that belong to the image of all the natural maps $\mathrm{Br}\, A_v \to \mathrm{Br}\, k(X)$, where A_v is the valuation ring of a geometric discrete, rank 1 valuation v on $k(X)$. Such description of $\mathrm{Br}(X)$ as subgroup of unramified classes holds for any smooth variety, as shown later in the thesis. Unramified Brauer classes were furthermore characterised, for the m-torsion part prime to $\mathrm{char}\, k$, as those classes in $\mathrm{Br}\, k(X)[m]$ that are in the kernel of certain "residue maps" $\partial_v \colon \mathrm{Br}\, k(X)[m] \to H^1(k(v), \mathbb{Z}/m)$ for every discrete rank 1 valuation v on $k(X)$; following Merkurjev, it is also possible to give a description of unramified classes in $\mathrm{Br}(X)[p]$ for $p = \mathrm{char}\, k$ via certain (different) residue maps, which although are defined only conditionally. Moreover, thanks to "purity" properties established by Grothendieck and improved recently by Gabber, in many situations this condition is actually computable since the family of valuations that one needs to check can be reduced to valuations corresponding to prime divisors on a fixed model of X.

Particularly fruitful applications have emerged for the class of conic bundles, that is to say, varieties X admitting a fibration $X \to B$ over a smooth, projective variety B whose fibres are all isomorphic to plane conics. A series of favourable circumstances make the study of residue maps on conic bundles more enticing: a conic bundle $X \to B$ (with smooth generic fibre) corresponds to a 2-torsion class α in the group $\mathrm{Br}\, k(B)$ and the residues $\partial^2_{v_D}(\alpha)$ where D varies through divisors on B can be described geometrically in terms of the discriminant locus of the conic bundle $X \to B$. The idea of Colliot-Thélène and Ojanguren is then to produce the non-zero Brauer classes on X as pull-backs of Brauer classes represented by certain other conic bundles on B, whose residue profiles are a proper subset of the residue profile of the given conic bundle $X \to B$. Hence, one also has to understand the geometric meaning of residues because in the course of this approach it becomes necessary to decide when the residues of two conic bundles along the same divisor are equal.

By employing these techniques, Colliot-Thélène gave a formula for the subgroup of unramified elements in $\operatorname{Br} k(X)$, where X is a conic bundle over a rational smooth surface B, with certain restrictions on the discriminant locus. For such formula, which allegedly had been already known in some forms to the previous generation of algebraic geometers, we give a full proof, since it does not seem to be stated explicitly anywhere in the literature.

All of these techniques are however suitable to give results *for specific examples* only: the reason is practically given by the fact that the Brauer group is not invariant under deformation. The inability of this invariant to establish results for families of varieties, which is shared by many classical invariants for stable rationality, led to a substantial scarceness of counter-examples, until the introduction of Voisin's degeneration method.

The key intuition is to "fill" the gap between stable irrationality and non-vanishing of the (unramified) Brauer group − neither of which is preserved under degeneration, especially if singular fibres are allowed − with an abstract notions which acts a a "bridge" between them and that, in addition, behaves well under deformation. Voisin has introduced two of these new notions, which are Chow-theoretic in nature: decomposition of the diagonal (**CDD**) and universal triviality of zero-cycles (**UCT**). While the first one has a more geometric significance, they are both very difficult to check and not particularly insightful in the direct study of "nearly rational" varieties. Nevertheless, they are related both to stable rationality and to many classical invariants that obstruct stable rationality.

$$X \text{ is stably rational } \Rightarrow \begin{array}{c}\text{all smooth models } \widetilde{X} \simeq X \\ \text{admit a } \mathbf{CDD}/\text{are } \mathbf{UCT}\end{array} \Rightarrow \text{invariant on } \widetilde{X} = 0.$$

For example, it was proved by Colliot-Thélène and Pirutka that, for any universally Chow-zero trivial smooth k-variety X, the m-torsion subgroup of $\operatorname{Br}(X)$ is universally trivial if m is coprime to $\operatorname{char} k = p$: that is to say, for every field extension F/k, the natural map $\operatorname{Br}(F)[m] \to \operatorname{Br}(X_F)[m]$ is an isomorphism. We have extended this result to general torsion orders in [**ABBvB-19**], showing the following.

THEOREM. *Let X be a smooth proper variety over a field k. If X is* **UCT***, then the Brauer group of X is universally trivial.*

Assume now that X is the generic member of a family of varieties $\mathfrak{X}$, Voisin shows that the middle condition is preserved after degenerating to a special member X_0 with at worst nodal singularities:

$$\begin{array}{lll} X \text{ is stably rational} & \Rightarrow \begin{array}{c}\text{every smooth model}\\ \text{of } X \text{ admits a } \mathbf{CDD}\end{array} & \Rightarrow \text{invariant on } \widetilde{X}=0 \\ & \Downarrow \text{(degeneration)} & \\ & \begin{array}{c}\text{every smooth model}\\ \text{of } X_0 \text{ admits a } \mathbf{CDD}\end{array} & \Rightarrow \text{invariant on } \widetilde{X_0}=0 \end{array}$$

Further generalisations have widened the scope of the method by allowing worse classes of singularities on X_0. Ultimately, there is a new tentative extension of the degeneration method aimed to allow very singular special fibres to come into play.

But now the point is that, by considering the negative statements of the above chain of implications and using the characterisation of the Brauer group in terms of unramified elements, it is possible to obstruct stable rationality of *very general elements* in the family $\mathfrak{X}$ by inspecting invariants on a *single, special member* X_0. The choice of the invariant and how to check its non-triviality depends on the particular geometry of X_0; the Brauer group is a candidate, but in the literature there are also examples of applications of differential forms. For a conic bundle, we have already explained that the unramified Brauer group is particularly effective since non-trivial elements can be produced via geometric considerations.

As a matter of example, this would be a sketch of the typical application of such strategy for a family of varieties $\mathfrak{X} \to B$, where X_0 admits a conic bundle structure over some smooth, projective surface S.

$$\begin{array}{lll} \begin{array}{c}\text{for very general } t \in B\\ \mathfrak{X}_t \text{ does not admit a } \mathbf{CDD}\end{array} & \Rightarrow \begin{array}{c}\text{the very general member}\\ \text{of } \mathfrak{X} \text{ is not stably rational}\end{array} & \\ \Uparrow & & \\ \widetilde{X_0} \text{ does not admit a } \mathbf{CDD} & \Leftarrow \mathrm{Br}(\widetilde{X_0})[2] \neq 0 & \\ & \qquad\qquad \Uparrow & \\ \begin{array}{c} X_0 \to S \text{ conic bundle}\\ \text{satisfying certain conditions}\\ \text{on discriminant locus and}\\ \text{singularities of } X_0\end{array} & \Rightarrow \mathrm{Br}_{\mathrm{nr}}(k(X_0))[2] \neq 0 & \end{array}$$

The degeneration method has led to a plethora of new counter-examples, often of great impact and unexpected generality; we will illustrate the most important of them in the course of the thesis. However, the usage of the Brauer group is only a possibility and its applicability is limited to the situations in which unramified elements can be concretely calculated.

We point out that such strategy can be used in at least two versions: either in a "global" form, in which $\mathfrak{X}$ is a family of varieties over $\mathbb{C}$, parametrised by some base smooth variety B, or in a "local" form, in which $\mathfrak{X}$ is flat over the curve trait $B = \mathrm{Spec}(R)$ for some discrete valuation ring R.

The degeneration method applied to this latter situations aims therefore to determine stable irrationality of a variety defined over an algebraic closure of the fraction field K of R (the geometric generic fibre) by reduction to the residue field k. An interesting instance of this is the case in which R is of mixed characteristic: K has characteristic 0 and k has characteristic $p > 0$, a situation in which the special fibre can be rightfully thought as to be the "reduction modulo p" of the generic fibre.

Such context of degeneration of mixed characteristic has inspired our treatment of conic bundles over fields of characteristic 2. It is not the first attempt to exploit the special features of such setting, as already Totaro ([**Tot-15**]) and Ahmadinezhad-Okada ([**AO-18**]) produced some notable examples by degeneration to special fibres defined over fields of characteristic 2. However, ours is the first approach which uses conic bundles in characteristic 2 and a new theory of residue maps to establish stable irrationality of a new threefold over $\mathbb{C}$. More specifically, we prove the following.

THEOREM. *Let*

$$M := \begin{pmatrix} 2uv + 4v^2 + 2uw + 2w^2 & u^2 + uw + w^2 & uv \\ u^2 + uw + w^2 & 2u^2 + 2vw + 2w^2 & u^2 + vw + w^2 \\ uv & u^2 + vw + w^2 & 2v^2 + 2uw + 2w^2 \end{pmatrix}$$

be a 3×3 *symmetric matrix with coefficients in* $\mathbb{Z}[u, v, w]$ *and let* $X \subseteq \mathbf{P}^2_{\mathbb{Z}} \times \mathbf{P}^2_{\mathbb{Z}}$ *be the zero locus of the quadratic form*

$$q(x, y, z, u, v, w) = \frac{1}{2}(x, y, z)M(x, y, z)^t$$

Then the threefold conic bundle $\pi\colon X \times \mathrm{Spec}(\mathbb{C}) \to \mathbf{P}^2_{\mathbb{C}}$ *induced by projection onto* $\mathbf{P}^2_{\mathbb{Z}} = \mathrm{Proj}\,\mathbb{Z}[u, v, w]$ *is not stably rational (over* $\mathbb{C}$*).*

Such result was obtained by applying the above strategy to the conic bundle $X_{(2)} \to \mathbf{P}^2_{\mathbb{F}_2}$ obtained by reducing modulo 2 the equation $q = 0$. Moreover, the claim cannot be obtained by reducing to any prime $p \neq 2$, as we show through the thesis.

We shall now describe the roadmap of the various chapters.

In the first chapter we introduce the geometric context of the problem, explaining the differences between the various notions of "near rationality"; we also list the most recent developments of the subject.

The second chapter treats the degeneration principle. The chapter begins with a recollection of definitions and properties of the new Chow-theoretic invariants introduced by Voisin, followed by a brief summary of the idea of "mild desingularisation" introduced by Colliot-Thélène and Pirutka and used in a generalisation of Voisin's method. The following two sections contain a survey of Voisin's original method and the subsequent improvement made by Colliot-Thélène and Pirutka. Then, the last section presents an overview of a possible new development of the degeneration method to include "more degenerate" cases like toric degenerations, inspired by techniques from log geometry.

Chapter three contains all the technical background for characteristic not 2: Brauer groups of fields, Brauer groups of schemes and their representation as unramified cohomology groups via residue maps. In Section four, conic bundles are introduced and the techniques developed in the previous sections are specialised to this class of varieties. We prove Colliot-Thélène formula for the 2-torsion of the Brauer group of a conic bundle (in characteristic $\neq 2$) and the Chapter ends with a computation showing explicitly that the unramified cohomology of conic bundle with quintic discriminant is trivial − a result which is usually not proved explicitly.

Chapter four and Chapter five are respectively based on our original papers [**ABBvB-19**, **ABBvB-18**].

In particular, Chapter four is devoted to prove universal triviality of the p-torsion subgroup in the Brauer group of a smooth variety defined over a field of characteristic p by exploiting a Weil reciprocity-like argument. The Chapter also contains a more general treatment of the concept of unramified elements than Chapter three, with the specific purpose of working with p-primary torsion elements instead of p-prime torsion elements.

Finally, Chapter five contains the new most technical aspects of the thesis and the main concrete example. After a review of the geometry of conics and conic bundles in characteristic 2, we introduce conditionally defined residue maps (following Merkurjev) and we characterise unramified elements in terms of these maps. Then, after giving a geometric description of residues for the case of tamely ramified conic bundles, we prove a new formula for the 2-torsion in the Brauer group of a conic bundle defined over an algebraically closed field of characteristic 2. The Chapter ends with a new example of threefold conic bundle which is not stably rational: this example is worked out applying in synergy Voisin's degeneration method by reduction modulo 2, and the new formula established in this Chapter.

CHAPTER 1

Nearly rational varieties and the stable Lüroth problem.

1.1. Rationality and unirationality.

The study of rational varieties dates back to the origins of geometry itself; it is reasonable to think about rational varieties as some of the simplest objects in algebraic geometry. Yet, rationality problems still occupy a prominent position in modern research, with an overwhelming number of open problems. In order to study rationality more closely, a number of closely related notions has been introduced.

We begin recalling the precise definitions.

DEFINITION 1.1.1. Let X be a projective variety defined over any field k. One says that X is

- **R:** *rational over* k if there is a birational map $\mathbf{P}^n_k \dashrightarrow X$ for some $n > 0$;
- **UR:** *(separably) unirational over* k if there is a (separable) dominant rational map $\mathbf{P}^m_k \dashrightarrow X$;

REMARK 1.1.2. For unirational varieties, it is possible to choose $m = \dim X$ (see [**Oja-90**, Proposition 1.1]).

It is important to realise that these definitions may depend on the ground field. First of all, if X is (uni)rational over k, then it remains so after performing a change of base to any field extension L of k. However, the same property does not hold restricting the ground field: for example, every conic over $\mathbb{C}$ is rational (over $\mathbb{C}$) because it has plenty of $\mathbb{C}$-points (see later), but it might not have any $\mathbb{R}$-point and, consequently, cannot be rational over $\mathbb{R}$. This example explains why in the following we will assume that all rationality questions are discussed over a fixed ground field.

REMARK 1.1.3. In **UR** above, we distinguished between two definitions of unirationality. If $\operatorname{char} k = 0$ the distinction between separable unirationality and unirationality disappears, as every extension of fields in characteristic 0 is separable. The two notion, however, may differ significantly if $\operatorname{char} k = p > 0$. For example let $F_d \subseteq \mathbf{P}^n_k$ be the Fermat hypersurface of degree d, cut out by the equation

$$X_0^d + \ldots + X_n^d = 0.$$

Choose $k = \overline{\mathbb{F}_p}$ and let $d = p^r + 1$ for some integer $r > 0$ such that $p^r > n$. Then F_{p^r+1} is smooth and admits a rational p^r-covering $\widetilde{F}_{p^r+1} \longrightarrow F_{p^r+1}$ for every $n \geq 3$ (see [**Deb-01**, Exercise 2.5.1] and notice that this covering is purely inseparable) hence it is unirational. However, it is not *separably* unirational. Indeed, it is easy to show that, for a separably unirational variety X, the pluri-genera $\dim H^0(X, \omega_X^{\otimes m})$ vanish for $m > 0$ (see [**KSC-04**, Theorem 1.52] and the following discussion); however, the canonical sheaf of F_{p^r} is ample.

It is natural to ask whether rationality and unirationality are in fact equivalent; the answer depends both on the ground field and the dimension of the variety under consideration. For curves (over any field), equivalence holds and is a classical result.

THEOREM 1.1.4. (LÜROTH) *Let k be an arbitrary field (possibly algebraically non-closed). Then every subfield of $k(t)$ is a purely transcendental field extension of k. In other words, a k-curve is rational if and only if it is unirational.*

PROOF. A proof addressing the most general case can be found in [**Oja-90**]. □

For surfaces, equivalence holds if the ground field is algebraically closed of characteristic 0, but fails if the ground field has positive characteristic. To see how this is proved, let us first recall a well-known rationality criterion for fields of arbitrary characteristic.

THEOREM 1.1.5. (CASTELNUOVO, ZARISKI) *Let X be a projective surface over an algebraically closed field k (of arbitrary characteristic). Then X is rational if and only if $q = \dim H^1(X, \mathscr{O}_X) = 0 = P_2 = \dim H^0(X.\omega_X^{\otimes 2})$.*

PROOF. A proof for the classical case ($\operatorname{char} k = 0$) can be found in [**Kod-68**]; generalisations to positive characteristic, can be found in [**Zar-58**]. □

We have the following consequence.

COROLLARY 1.1.6. *Let X be a surface defined over an algebraically closed field k of characteristic 0. Then X is rational if and only if it is unirational.*

Sketch of proof. If X is unirational, there exists a dominant map $\varphi : \mathbf{P}^2_k \dashrightarrow X$. Then the pull-back induces an inclusion $\varphi^*\Omega^r_X \subseteq \Omega^r_{\mathbf{P}^2_k}$ for every r. Note that in order to get this, the assumption on the characteristic is essential. Then Theorem 1.1.5 implies that X is rational. □

REMARK 1.1.7. The hypothesis on the characteristic of k cannot be removed. Several families of counter-examples in characteristic $p > 0$ are given by Zariski in [**Zar-58**], one of which is sketched here. The example shows that the failure of this statement is due to the difference between unirational and separably unirational varieties in positive characteristic; indeed, the same statement above would work over any field after replacing unirationality with separable unirationality. Let k be a perfect field of characteristic $p > 0$ (with $p \neq 2$) and let Y be the affine surface cut out in $\mathbf{A}^3_k$ by the following equation:

$$f(x_0, x_1, x_2) = x_0^p + x_1^{p+1} - \frac{(x_1^2 + x_2^2)}{2} = 0$$

and let $X = \overline{Y} \subseteq \mathbf{P}^3_k$ be its projective closure. Since $\operatorname{char} k = p \neq 2$, on the whole of Y one has

$$x_0 = -x_1^{(p+1)/p} + \frac{x_1^{2/p} + x_2^{2/p}}{2^{1/p}}$$

and it follows

$$k(\mathbf{P}^2_k) = k(\mathbf{A}^2_k) \simeq k(x_1^{1/p}, x_2^{1/p}) = k(x_0, x_1^{1/p}, x_2^{1/p}) \supseteq k(X)$$

so X is unirational. However, X cannot be rational as the 2-form defined by

$$\omega = \frac{dx_0 \wedge dx_2}{x_1^p - x_1}$$

is regular on X and does not vanish on X; hence $P_2 \neq 0$ for X and by Theorem 1.1.5, the variety X cannot be rational. Notice that the field extension $k(X)/k(\mathbf{P}^2_k)$ is *not* separable, so the variety X is not separably unirational.

Remark 1.1.8. Neither the hypothesis on k being algebraically closed can be removed: consider a cubic surface C over the field of rational numbers $\mathbb{Q}$ such that $C(\mathbb{Q}) \neq \varnothing$. Then by a result of Segre, C is unirational but it need not be rational (for example, it might have Picard number 1).

Whether or not equivalence of rationality and unirationality held for higher dimensional varieties has been an open question for decades, until it was answered negatively in the Seventies of the XX century, through a series of different examples ([**AM-72**, **CG-72**, **IM-71**]) illustrated later in Section 1.3.

1.2. The stable Lüroth problem.

There are two further notion of being "nearly rational". In order to simplify the discussion, we assume our fields to be algebraically closed unless stated differently.

Definition 1.2.1. Let X be a projective variety defined over a field k. One says that X is

- **SR:** *stably rational* if $X \times \mathbf{P}^m_k$ is rational;
- **RC:** *rationally connected* if, for any two general points $p, q \in X$, there exists a rational curve $C_{p,q} \subseteq X$ such that $p, q \in X$.

Remark 1.2.2. Both notions above can be defined over algebraically non-closed ground fields but the definition of rational connectedness stated here is not appropriate in such context (for example, the curve $C_{p,q}$ might fail to be defined over the ground field). For a more general definition see [**Deb-01**, Section 4]; which is equivalent to ours for ground fields which are algebraically closed ([**Deb-01**, Remark 4.4.(3)]), a situation which is general enough for the purpose of this text.

If a variety satisfies one of the conditions **UR**, **SR**, **RC** we say, informally, that it is *nearly rational*, as these properties indeed express the fact that the variety in question is very close to be rational.

Proposition 1.2.3. *With notations as above, one has* **R** $\Rightarrow$ **SR** $\Rightarrow$ **UR** $\Rightarrow$ **RC**.

Proof. Some of these implications have already been proved earlier for some special cases; we will give here a proof for arbitrary varieties. Let X be a k-variety. Suppose that X is rational; then $X \times \mathbf{P}^n_k$ is rational for every n, so X is stably rational. Now suppose that X is stably rational; then there exists $m > 0$ such that $X \times \mathbf{P}^m_k$ is rational, that is to say, there is a birational map $\mathbf{P}^{m+\dim X}_k \dashrightarrow X \times \mathbf{P}^m_k$. Composing with the projection $p\colon X \times \mathbf{P}^m_k \longrightarrow X$ we obtain the required dominant map.

Now let us prove the last implication, assuming that k is algebraically closed. Let X be unirational and let $\varphi\colon \mathbf{P}^m_k \dashrightarrow X$ be a rational, dominant map, let x, y be two general point in X and let x', y' be two points in the fibres, respectively, $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$. Then, call L the line in $\mathbf{P}^m_k$ joining x' to y' and consider the restriction $\varphi|_L\colon L \simeq \mathbf{P}^1_k \dashrightarrow X$; the image of $\varphi|_L$ is a rational curve in X that joins x to y. $\square$

As we pointed out earlier, one does not expect these condition to be equivalent in high dimensions. We will mostly concentrate on the relationship between definitions **R**, **UR** and **SR**. This choice is heuristically justified by the relative difficulty to characterise unirationality and stable rationality, as opposed to rational connectedness, which has several strong characterisations see [**Deb-01**, Section 4.1]).

Stable rationality provides an intermediate notion between rationality and unirationality. The study of stably rational varieties has not received much attention for long time, since it was not even clear whether unirationality and rationality were two distinct notions. This was resolved with the seminal paper [**AM-72**] of Michael Artin and David Mumford, where the first example of unirational but stably irrational variety was exhibited.

Distinguishing unirational (or rationally connected) but stably irrational varieties from stably rational ones takes the name of *stable Lüroth problem* while distinguishing unirational and rational varieties is usually called *generalised Lüroth problem.*

1.3. Currently known results.

We have already pointed out in Section 1 that Lüroth problems in dimensions 1 and 2 have already been fully addressed. We will now present what is known for higher dimensional varieties over an algebraically closed field.

1.3.1. Quadric hypersurfaces. A hypersurface of degree 2 which is not a union of hyperplanes is rational over any algebraically closed field, regardless of the dimension. This is a consequence of the following rationality criterion for quadrics.

THEOREM 1.3.1. ([**KSC-04**, Theorem 1.11]) *Let k be an arbitrary field of characteristic not 2 and let X be a quadric hypersurface in a projective space such that X is not the union of two hyperplanes. Then the following conditions are equivalent:*

(1) *X is rational;*
(2) *X has a smooth k-rational point;*
(3) *X has a smooth L-rational point for some odd-degree field extension L/k.*

Despite this strong characterisation, the birational classification of quadrics over algebraically non-closed field is not complete. This is due to the difficulty to decide whether a quadric has a k-rational point or not; there are some shortcuts, for example equivalence of rationality for $k = \mathbb{Q}$ and $k = \mathbb{R}$ by the Hasse-Minkowski theorem ([**KSC-04**, Theorem 1.15]) and a concrete answer if k is a finite field ([**KSC-04**, Section 1.4]).

1.3.2. Cubic hypersurfaces. The following elementary result shows rationality for all singular cubics.

THEOREM 1.3.2. *Let $X \subseteq \mathbf{P}_k^{n+1}$ be a hypersurface of degree 3 which is not a cone over a lower-dimension hypersurface. Assume that X has a singular point; then X is rational.*

PROOF. See [**KSC-04**, Example 1.28]; it is enough to project away from the singular point. □

For smooth cubics, however, the situation is much more complicated. The following general result can be obtained with classical methods.

THEOREM 1.3.3. *Let $X \subseteq \mathbf{P}_k^{n+1}$ be a smooth cubic hypersurface that contains a line. Then X is unirational.*

PROOF. The general proof, using classical techniques, can be found in [**Hass-16**, Proposition 10]. The argument is essentially the same of the case $n = 3$, which is in [**CG-72**, Appendix B]. □

In particular, cubic 3-folds are unirational (see the discussion at [**CG-72**, Section 3.8] for the geometry of lines in a cubic 3-fold hypersurface), but they cannot be rational.

THEOREM 1.3.4. *Let $X \subseteq \mathbf{P}^4_k$ be a smooth cubic hypersurface. Then X is not rational.*

PROOF. The case $k = \mathbb{C}$ was solved by Clemens and Griffiths in [**CG-72**, Theorem 13.12] using the intermediate Jacobian criterion (*ibid.* Corollary 3.26). For algebraically closed fields k of characteristic $\neq 2$, the same result was obtained by Murre ([**Mur-73**]) using the Prym variety in place of the intermediate Jacobian. $\square$

Stable rationality of this class of 3-dimensional varieties is still an open problem and a key area of modern research in algebraic geometry. It is conjectured that the very general cubic 3-fold is not stably rational.

Cubic n-folds are likely to be even more varied in behaviour for $n \geq 4$. It is currently unknown whether they are rational or not and, although it is conjectured that the very general cubic n-fold hypersurface is not rational, there is no known individual example of such a variety. Using classical tools, it is possible to demonstrate that certain special cubic n-folds are rational. See the discussion at [**Hass-16**, Section 1] for a list of classical results on some special cubic 4-folds; some of these techniques can be generalised to show that certain cubic $(2m)$-folds are rational, but there are no known examples of odd-dimensional, smooth, cubic hypersurfaces.

1.3.3. Quartic hypersurfaces. Quartic hypersurfaces have been successfully studied since the late XIX century but there are still many unsolved questions in low dimension. Indeed, in 1936 Ugo Morin proved the following result ([**Mo-36**]) for high-dimensional varieties.

THEOREM 1.3.5. *The generic quartic hypersurfaces $X \subseteq \mathbf{P}^{n+1}_k$ is unirational if $n \geq 6$, over any field k.*

This result was extended to $n = 5$, still by Morin in [**Mo-52**] and Conte and Murre in [**CM-98**] simplified and improved it using a result by Segre from 1954 which was not available to Morin.

Unirationality for quartic 4-folds and 3-fold hypersurfaces is still an open question. Some of them, actually, are unirational and can be obtained via classic constructions ([**Seg-60**]).

It is known, however, that any smooth quartic 3-fold hypersurface is not rational.

THEOREM 1.3.6. *Let $X \subseteq \mathbf{P}^4_k$ be a smooth, quartic hypersurface. Then X is not rational.*

PROOF. It was shown by Iskovskikh and Manin ([**IM-71**]) introducing the method of *birational rigidity*. In essence, they proved that every birational automorphism of such a variety X extends to a regular automorphism of X; but the birational automorphism group of rational varieties is very big, while X has only few (regular) automorphisms. $\square$

Historically, the first relevant contribution to both Lüroth problems was given by Artin and Mumford in [**AM-72**]: as mentioned before, they constructed an unirational variety which is not stably rational (and *a fortiori ratione*, not even rational). The example can be viewed as a double covering X of $\mathbf{P}^3$, defined by an equation of the form

$$y^2 = f(x_0, x_1, x_2, x_3)$$

where $f = 0$ defines a quartic hypersurface in $\mathbf{P}^3$ with prescribed singular locus. This description permits one to deduce unirationality with elementary manipulations. However, X can also be viewed as conic bundle over $\mathbf{P}^2$ and this allowed the authors to calculate non-trivial elements in the 2-torsion subgroup of $H^3(X, \mathbb{Z})$, which is a stable birational invariant and vanishes for stably rational varieties.

With the introduction of Voisin's degeneration method, the above results have been improved in view of the stable Lüroth problem.

After the development of her own method, Voisin proved in [**Vois-15a**] the following result.

THEOREM 1.3.7. *The very general quartic double solid is not stably rational.*

The proof considered a flat family of double solids, whose special fibre is birational to the Artin-Mumford double solid. With a similar technique, Beauville proved that the very general sextic double solid is not stably rational ([**Beau-16**]); however, unirationality of this class of 3-folds is still unknown.

Colliot-Thélène and Pirutka proved the following result in [**CTP-16b**], using a generalisation of Voisin's degeneration method.

THEOREM 1.3.8. *The very general quartic* 3*-fold is not stably rational.*

The authors considered a flat family of 3-folds whose special fibre is a variety birational to the Artin-Mumford quartic, but with worse singularities, hence the need of an extension of Voisin's method. Those quartics which are proved to be unirational provide, therefore, a counterexample to the stable Lüroth problem.

1.3.4. Varieties of large degree or dimension. Stable irrationality for hypersurfaces of large degree compared to dimension was established by Totaro in [**Tot-15**].

THEOREM 1.3.9. *A very general hypersurface of degree $d \geq 2(n+2)/3$ in $\mathbf{P}^n_k$ is not stably rational for $n \geq 3$.*

This remarkable result has been obtained combining Voisin's degeneration method with Kollár's non-rationality criterion, which employs p-cyclic coverings and differential forms in positive characteristic (see [**KSC-04**, Chapter 4] for a survey).

More recently, Schreider ([**Schr-19b**]) improved the bound found by Totaro with the following result.

THEOREM 1.3.10. *Let k be an uncountable field of characteristic not 2. Then a very general hypersurface $X \subseteq \mathbf{P}_{k}^{n+1}$ of dimension $n \geq 3$ and degree $d \geq \log_2 n + 2$ is not stably rational over $\overline{k}$.*

For $n = 3$, one recovers the result of Colliot-Thélène-Pirutka; for $n = 4$, the bound coincides with Totaro's. But for $n \geq 5$ this new bound is much smaller than what was previously known. For example, Schreider's result implies that a very general quintic 5-fold is not stably rational (over an algebraically closed field). Also, since this bound grows only logarithmically (as opposed to the linear growth of Totaro's bound), one can find rather strong results as dimension increases.

Little is known about unirationality of these varieties, however. There are some results concerning unirationality of hypersurfaces whose degree is small compared with the dimension. More precisely, in [**HMP-98**], the following result is proved.

THEOREM 1.3.11. *For any $d \geq 3$ there exists $N(d) \in \mathbb{N}$ such that, for every $n > N(d)$, any smooth, degree d hypersurface of $\mathbf{P}^n$ is unirational.*

1.3.5. Quadric fibrations. A quadric bundle is a flat morphism of projective varieties $f : X \longrightarrow B$ whose generic fibre is a smooth quadric of dimension r. Assuming that B is a rational variety of dimension n, it has always been an interesting question going back at least to Artin and Mumford to determine whether X is rational or stably rational as well.

Rationality criteria for quadric bundles (that is, for the total space X) are known: by Spinger's Theorem ([**Spr-52**]), X is rational if f admits an odd-degree multisection, which exists if $r > 2^n - 2$ in virtue of a theorem of Lang ([**La-52**, Theorem 6 and Corollary]).

For $r = 1, \ldots, 6$, the existence of stably irrational quadric bundles over rational bases was proved by Artin Mumford (in [**AM-72**]) for $n = 2$ and, with similar examples, by Colliot-Thélène and Ojanguren (in [**CTO-89**]) for $n = 3$.

Stable irrationality has been proved for a number of different types of quadric bundles after the introduction of the degeneration method by Claire Voisin.

In [**BvB-18**], Böhning and von Bothmer have shown stable irrationality for a very general divisor of bi-degree $(2, n)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ for $n \geq 2$, exploiting its conic bundle structure induced by projecting onto the second factor.

The same authors, jointly with Auel and Pirutka, proved in [**ABvBP-19**] a closed formula for the unramified Brauer group of a conic bundle over a rational 3-fold; furthermore, this formula employed in synergy with Voisin-Colliot-Thélène-Pirutka's degeneration method to prove stable irrationality of a very general divisor of bi-degree $(2, 2)$ in $\mathbf{P}^2 \times \mathbf{P}^3$, considered along with its conic bundle structure induced by projecting onto the second factor.

Other important result in this paper include stable irrationality for some class of graded-free type conic bundles ([**ABvBP-19**, Theorem 6.6]), which are still obtained with similar methods.

This same example was studied in [**HPT-18**] as a quadric bundle (with 2-dimensional fibres) over $\mathbf{P}^2$, via the first projection. They also proved that the very general quadric bundle over $\mathbf{P}^2$, with octic discriminant locus, are not stably rational. Moreover, such family contains a dense set of smooth rational 4-folds, being thus the first examples of rational varieties with irrational deformations.

A recent result by Schreieder ([**Schr-19a**, Theorem 1]) gave other interesting counterexamples to the stable Lüroth problem.

Theorem 1.3.12. *Let n, m be positive integers with $r \leq 2^n - 2$ and let $m \leq n$ be the unique integer such that $2^{m-1} - 1 \leq r \leq 2^m - 2$. Then there exist smooth, unirational complex quadric bundles with r-dimensional fibres X over $B = \mathbf{P}^{n-m} \times \mathbf{P}^m$ such that X is not stably rational.*

The relevance of this result is that these examples are constructed using techniques borrowed from [**CTO-89**] and [**Vois-15a, CTP-16b**], but unlike in the latter reference, the author avoids the use of CH_0 -universally trivial resolution of singularities, simplifying the argument.

Finally, although we will not be interested in this problem, it is worth mentioning that there exist stably rational varieties that are not rational; several examples are in [**BCTSSD-85**]. These are also 3-folds with conic bundle structure over a rational surface; irrationality is shown again by the intermediate Jacobian criterion, while stable rationality was shown using universal torsors technique ([**CTS-80**]).

CHAPTER 2

Chow-theoretic invariants and the degeneration method.

2.1. Decomposition of the diagonal and zero-cycles.

Let X be an algebraic scheme of dimension n over a field k. At this point we do not assume k to be algebraically closed, neither do we put any restriction on the characteristic of k.

Recall that the *diagonal* of X is the unique morphism

$$\Delta_{X/k} \colon X \longrightarrow X \times_k X$$

which satisfies the universal property $\mathrm{pr}_i \circ \Delta_{X/k} = \mathrm{id}_X$ for $i = 1, 2$ where pr_i are the canonical projection morphism. This is the universal property of fibre products applied to the identity morphism $X \longrightarrow X$. Since affine and projective schemes are all separated ([**Hart-74**, II.4]), in any further discussion, we will omit to distinguish the morphism $\Delta_{X/k}$ from the closed sub scheme of $X \times_k X$ which is its scheme-theoretic image.

Let us give the following definitions (see also [**Vois-16**, Definitions 4.1, 4.5], [**Pir-16**, Definitions 2.1, 2.3])

Definition 2.1.1. Let X be a projective k-scheme.

(1) We say that X *admits a Chow-theoretic decomposition of the diagonal* **(CDD)** if there are a zero-cycle $z_0 \in \mathrm{CH}_0(X)$ of degree 1 and an n-cycle $Z \in \mathrm{CH}_n(X \times X)$ supported on $D \times X$ for some prime divisor $D \subset X$ such that

$$[\Delta_X] = [X \times x] + [Z] \in \mathrm{CH}_n(X \times X). \tag{2.1.1}$$

(2) We say that X *is universally* CH_0*-trivial* **(UCT)** if, for every field extension L/k, the degree map induces a isomorphism $\mathrm{CH}_0(X_L) \simeq \mathbb{Z}$.

Notice that this definition does not require any assumption on the singularities of the varieties under consideration.

Proposition 2.1.2. *Let X be a projective k-variety which is* **UCT**. *Then it admits a* **CDD**. *Furthermore, the converse holds if X is also smooth.*

Proof. The statement of equivalence for X smooth can be found at [**CTP-16b**, Proposition 1.4.(iii)], which cites [**ACTP-17**, Lemma 1.3].

The same proof, with minor amendments, can be rearranged to show that if X is a projective k-variety which is **UCT**, then it admits a **CDD**, with no assumptions on the singularities of X. □

The above Proposition can also be seen as a consequence of Bloch-Srinivas' criterion for the decomposition of the diagonal (see [**BS-83**])

Our interest in these properties of Chow-theoretic type is justified by the fact that they all detect stable rationality.

THEOREM 2.1.3. *Let X, Y be two smooth, projective k-varieties that are stably birational.*

(1) *X is* **UCT** *(equivalently, has a* **CDD***) if and only if Y is* **UCT** *(equivalently, has a* **CDD***)*
(2) *If X is stably rational, then X is* **UCT** *(equivalently, has a* **CDD***).*

PROOF. These facts are well known amongst experts but we replicate here the parts of proof which are not explicitly stated in the literature. Statement (1) for **CDD** is [**Vois-16**, Lemma 4.6, Proposition 2.4]. To deduce (2) from (1), it is enough to show that $\mathbf{P}^n_k$ admits a **CDD**: indeed, recall that there is a surjective map ([**Fult-98**, Example 1.10.2])

$$\bigoplus_{i+j=n} \mathrm{CH}_i(\mathbf{P}^n_k) \otimes \mathrm{CH}_j(\mathbf{P}^n_k) \longrightarrow \mathrm{CH}_n(\mathbf{P}^n_k \times \mathbf{P}^n_k)$$

so $\mathrm{CH}_n(\mathbf{P}^n_k \times \mathbf{P}^n_k)$ is generated by cycles $[L_i \times L_{n-i}]$ where L_i are linear spaces of dimension i. Therefore one has

$$[\Delta_{\mathbf{P}^n_k}] = \sum_{j=0}^{n} a_i [L_i \times L_{n-i}]$$

for unique integers $a_0, \ldots, a_n$. Let $\pi \colon \mathbf{P}^n_k \times \mathbf{P}^n_k \longrightarrow \mathbf{P}^n_k$ be the projection onto the first component. Therefore

$$[\mathbf{P}^n_k] = \pi_*[\Delta_{\mathbf{P}^n_k}] = a_n \pi_*[L_n \times L_0] = a_n[\mathbf{P}^n_k]$$

so $a_n = 1$. This implies that

$$[\Delta_{\mathbf{P}^n_k}] = [\mathbf{P}^n_k \times L_0] + Z$$

where $L_0 = \{x\}$ for some $x \in \mathbf{P}^n_k(k)$ and Z is a cycle supported on $H \times \mathbf{P}^n_k$ for some hyperplane $H \subseteq \mathbf{P}^n_k$.

Now we consider statements (1) and (2) for the **UCT** property. Let us prove (1): assuming that X and Y are stably birational, then given any field extension L/k we have that the base changes X_L and Y_L are stably birational and since the Chow group of zero-cycles is a stable birational invariant ([**Vois-16**, Lemma 2.11]) we have that $\mathrm{CH}_0(X_L) \simeq \mathrm{CH}_0(Y_L)$, which implies the assertion.

To prove (2), it is sufficient to recall that $\mathrm{CH}_0(\mathbf{P}^n_L) = \mathbb{Z}$ for every field extension L/k and for every n. This follows from excision exact sequence ([**Fult-98**, Proposition 1.8]):

$$\mathrm{CH}_0(\mathbf{P}^{n-1}_L) \to \mathrm{CH}_0(\mathbf{P}^n_L) \to \mathrm{CH}_0(\mathbf{P}^n_L \setminus \mathbf{P}^{n-1}_L) \to 0$$

and $\mathrm{CH}_0(\mathbf{P}^n_L \setminus \mathbf{P}^{n-1}_L) \simeq \mathrm{CH}_0(\mathbf{A}^n_L) = 0$. Here we have denoted by $\mathbf{P}^{n-1}_L$ a hyperplane of $\mathbf{P}^n_L$. So one has a surjective map $\mathrm{CH}_0(\mathbf{P}^{n-1}_L) \longrightarrow \mathrm{CH}_0(\mathbf{P}^n_L)$. The proof is concluded by proceeding by induction on n, recalling that $\mathrm{CH}_0(\mathbf{P}^1_L) = \mathrm{Div}(\mathbf{P}^1_L) \simeq \mathbb{Z}$. □

2.2. Mild desingularisation

Let us introduce the following relative version of universal CH_0-triviality.

Definition 2.2.1. ([**CTP-16b**, Définition 1.1]) Let $f\colon X \longrightarrow Y$ be a proper morphism of k-varieties. We say that f is *universally* CH_0*-trivial* (**UCT**) if, for every field extension L/k of the ground field, the induced push-forward $f_*\colon \mathrm{CH}_0(X_L) \longrightarrow \mathrm{CH}_0(Y_L)$ is an isomorphism.

Remark 2.2.2. It is worth mentioning that Definition 2.1.1,(2) is a particular instance of this new Definition 2.2.1 obtained by choosing f as the structure morphism $X \longrightarrow \mathrm{Spec}\,(k)$. Therefore, if X is **UCT**, there are isomorphisms $\mathrm{CH}_0(X_L) \simeq \mathbb{Z}x_L$, where $x_L \in X_L(L)$, for every field extension L/k.

Let us introduce the following terminology.

Definition 2.2.3. Let X be a projective k-variety. A **UCT** *desingularisation* of X is a morphism $\sigma\colon \widetilde{X} \longrightarrow X$ such that

(1) $\sigma\colon \widetilde{X} \longrightarrow X$ is birational and $\widetilde{X}$ is a smooth, projective k-variety;
(2) σ is an **UCT** morphism.

It is a natural question to ask whether a **UCT** desingularisation exists or not for a given variety X of arbitrary dimension. While in general this question is difficult to answer, there is the following sufficient criterion to determine universal CH_0-triviality for morphisms.

Proposition 2.2.4. *Let $f\colon V \longrightarrow W$ be a projective morphism of varieties defined over any field k.*

(1) *If the fibre V_ξ above each scheme-point ξ of W is a (possibly reducible)* **UCT** *variety over $k(\xi)$, then f is a* **UCT** *morphism.*
(2) *If V_ξ is a projective, reduced and geometrically connected $k(\xi)$-variety, whose irreducible components V_i are all* **UCT**, *geometrically irreducible and such that each intersection $X_i \cap X_j$ is either empty or has a zero-cycle of degree* 1, *then f is a* **UCT** *morphism.*

Proof. Part (1) of the statement was proved in [**CTP-16b**, Proposition 1.8], while part (2) is a combination of part (1) and [**CTP-16a**, Lemma 2.4]. □

In order to simplify the application of this criterion, we will also also use of the following property.

Proposition 2.2.5. ([**CTP-16a**, Lemma 2.2]) *Let k be an algebraically closed field and let Y be a projective rational k-variety. If the singular locus of Y consists of a finite number of isolated points, then Y is* **UCT**.

2.3. The degeneration method.

Let us start by recalling the following well-known terminology. Let $\mathscr{V}$ be an algebraic variety over an algebraically closed field k.

- We say that a property $\mathbb{P}$ holds for *general points* of $\mathscr{V}$ if there exists a Zariski closed proper subset $Z \subseteq \mathscr{V}$ such that $\mathbb{P}$ holds for every $p \in \mathscr{V} \setminus Z$.
- We say that a property $\mathbb{P}$ holds for *very general* points of $\mathscr{V}$ if there exists a countable family of Zariski closed proper subsets $Z_n \subseteq \mathscr{V}$ such that $\mathbb{P}$ holds for every $p \in \mathscr{V} \setminus \bigcup_n Z_n$.

We have already made use of this terminology in Chapter 1, where $\mathscr{V}$ was implicitly assumed to be the parameter space of some class of projective varieties.

Example 2.3.1. For example, a degree d hypersurface inside $\mathbf{P}^n_k$ is the locus of the points cut out by an homogeneous, degree d polynomial equation in $n+1$ variables. Such polynomial is determined by the choice of its coefficients $a_{i_0,\ldots,i_n} \in k$, where $i_0, \ldots, i_n \in \mathbb{N}$ are such that $i_0 + \ldots + i_n = d$. Therefore, each such hypersurface corresponds to the choice of $\binom{n+d}{d}$ coefficients not simultaneously equal to 0, hence a point $(a_{i_1,\ldots,i_n}) \in k^{\binom{n+d}{d}} \setminus \{0\}$. Moreover, the homogeneity property implies that rescaling each of these equations gives rise to the same hypersurface. Therefore, one identifies the parameter space of degree d hypersurface inside $\mathbf{P}^n_k$ with the projective space $\mathbf{P}_k^{\binom{n+d}{d}-1}$. A *(very) general hypersurface* of degree d in $\mathbf{P}^n_k$ is therefore assumed to be a (very) general point in this projective space.

The following result is the original version of the "degeneration method".

Theorem 2.3.2. ([**Vois-15a**, Theorem 2.1]) *Let $f\colon \mathfrak{X} \longrightarrow B$ be a flat projective morphism of relative dimension at least 2, where B is a smooth curve. Assume that*

(1) *the fibre $\mathfrak{X}_t$ is smooth for $t \neq 0$;*
(2) *$\mathfrak{X}_0$ has at worst ordinary quadratic singularities;*
(3) *for general $t \in B$, the fibre $\mathfrak{X}_t$ admits a* **CDD**.

Then every desingularisation $\widetilde{\mathfrak{X}}_0$ of $\mathfrak{X}_0$ admits a **CDD** .

Remark 2.3.3. Due to Proposition 2.1.2, the Theorem works even replacing existence of **CDD** with the **UCT** property everywhere.

Since we are interested in proving stable irrationality for families of varieties, we will rephrase Theorem 2.3.2 into the following negative statement.

Theorem 2.3.4. *Let $f\colon \mathfrak{X} \longrightarrow B$ be a flat projective morphism of relative dimension at least 2, where B is a smooth curve. Assume that*

(1) *the fibre $\mathfrak{X}_t$ is smooth for $t \neq 0$;*
(2) *$\mathfrak{X}_0$ has at worst ordinary quadratic singularities;*

(3) *a smooth model* $\widetilde{\mathfrak{X}}_0$ *of* $\mathfrak{X}_0$ *does not admit a* **CDD** (equivalently, is not **UCT**).

Then, for very general $t \in B$*, the fibre* $\mathfrak{X}_t$ *does not admit a* **CDD** *(equivalently, is not* **UCT***).*

Proof. This is a consequence of applying the degeneration method by contrapositive. The hypotheses (1), (2) and (3) above imply that condition (3) in Theorem 2.3.2 must be violated. But

$$\mathscr{C} = \{t \in B : \mathfrak{X}_t \text{ is smooth and admits a } \mathbf{CDD}\}$$

is countable union of Zariski closed proper subsets of B, in virtue of [**Vois-15a**, Proposition 2.4]. This means precisely that the very general $\mathfrak{X}_t$ has no **CDD**.

Finally, notice that one can replace **CDD** with **UCT** everywhere due to smoothness. $\square$

Remark 2.3.5. The theorem can also be viewed as a statement about the geometric generic fibre of f in virtue of [**Vi-13**, Lemma 2.1]. More precisely, let $\overline{\mathfrak{X}_\eta} := \mathfrak{X} \times_B \operatorname{Spec}(\overline{k(B)})$ be the geometric generic fibre; then there exists a subset $U \subseteq B$ consisting of the countable intersection of non-empty Zariski open subsets such that, for every $u \in U$, there exists an isomorphism of fields $\omega_u : k(u) = k \longrightarrow \overline{k(B)}$ such that the following diagram

$$\begin{tikzcd}
\overline{\mathfrak{X}_\eta} \arrow[r] \arrow[d] & \mathfrak{X}_u \arrow[d] \\
\operatorname{Spec}(\overline{k(B)}) \arrow[r, "\omega_u"] & \operatorname{Spec}(k)
\end{tikzcd}$$

is Cartesian. This also implies that $\overline{\mathfrak{X}_\eta}$ and $\mathfrak{X}_u$ are isomorphic as abstract schemes.

Condition (2) in Theorem 2.3.2 was improved in the influential work [**CTP-16b**], considerably relaxing the requirements for singularities of the special fibre by introducing “mild” desingularisations in the sense of Definition 2.2.3.

We present this enhanced version in a “local” form.

Theorem 2.3.6. *Let* A *be a discrete valuation ring,* $K = \operatorname{Quot}(A)$ *its field of fractions and* k *its residue field, which we assume to be algebraically closed. Let* $\mathfrak{X} \longrightarrow \operatorname{Spec}(A)$ *be a proper, flat morphism with integral geometric fibres. Let us call* X *the generic fibre over* K *and* Y *the special fibre over* k*. Assume:*

(1) Y *admits a* **UCT** *desingularisation* $\widetilde{Y} \longrightarrow Y$*;*
(2) *the geometric generic fibre* $\widetilde{X} := X \times \operatorname{Spec}(\overline{K})$ *admits a desingularisation* $\widetilde{X} \longrightarrow \overline{X}$*;*
(3) $\widetilde{X}$ *is* **UCT**.

Then $\widetilde{Y}$ *is* **UCT.**

Proof. This statement is an extract of [**CTP-16b**, Théorème 1.14]. □

Remark 2.3.7. Condition (1) is much weaker than condition (2) of Theorem 2.3.2. Notice that if Y has ordinary quadratic singularities, then a resolution of singularities will satisfy (1) thanks to Proposition 2.2.4.

(2) We will distinguish between two types of degeneration arguments, depending on the nature of the ring A. For the case in which the characteristics of residue and fraction field of A agree, we will speak of *equal characteristic degeneration*; for the case in which these differ (notably, for the case $\operatorname{char} K < \operatorname{char} k$) we will speak of *mixed* or *unequal characteristic degeneration*. If A is complete, the corresponding two classes of discrete valuation rings can be classified up to isomorphism, see [**Ser-79**, Chapter II].

We finally illustrate how these degeneration principles are combined in applications.

Strategy 2.3.8. Suppose one wants to prove that the very general member of a family of smooth varieties $\mathscr{V}$ (defined over $\mathbb{C}$) is not stably rational.

(1) Firstly, one identifies a proper, flat morphism $f\colon \mathfrak{X} \longrightarrow \operatorname{Spec}(A)$ with A some discrete valuation ring with geometric generic fibre $X/\overline{K}$, a base change of which is (stably) birational to a variety in $\mathscr{V}$.
(2) Then one requires that the special fibre Y/k admits a **UCT** desingularisation $\widetilde{Y} \longrightarrow Y$ as in Definition 2.2.3.
(3) Additionally, one requires that $\widetilde{Y}/k$ is not **UCT** (in practice it is sufficient that there is some non-zero obstruction to universal CH_0-triviality).
(4) Then, by Theorem 2.3.6, the variety $X/\overline{K}$ is not **UCT**, hence X is stably irrational over $\overline{K}$ and, *a fortiori*, X is not stably birational over $\mathbb{C}$.
(5) By the reasoning explained in Remark 2.3.5, this shows that the very general member of the family $\mathscr{V}$ is stably irrational.

In the applications, one seeks to check condition (3) by computing a suitable invariant that detects existence of **CDD** or **UCT**.

2.4. Further generalisation: semi-stable degeneration method.

This section introduces a tentative extension of the scope of the degeneration method; the ideas behind what follows originated from the desire to find a way to attack the (currently unsolved) stable Lüroth problem for cubic threefolds using a degeneration technique.

We start by having a closer look at the original proof of Voisin's theorem [**Vois-15a**, Theorem 2.1]. The setup is again a flat projective morphism $f\colon \mathfrak{X} \longrightarrow B$, where B is a smooth curve, with the additional hypotheses that $\mathfrak{X}_t$ is smooth for $t \neq 0$ and $\mathfrak{X}_t$ admits a **CDD** for general $t \in B$, that is to say

$$[\Delta_{\mathfrak{X}_t}] = [\mathfrak{X}_t \times \{x_t\}] + [Z] \tag{2.4.1}$$

as cycle classes in $\mathfrak{X}_t \times \mathfrak{X}_t$, where x_t is a closed point of $\mathfrak{X}_t$ and Z is a cycle on $\mathfrak{X}_t \times_B \mathfrak{X}_t$ supported on $D \times \mathfrak{X}_t$, where D is a divisor of $\mathfrak{X}_t$. Voisin shows that, possibly after shrinking B and passing to a branched covering $B \longrightarrow B$ (induced by replacing the deformation parameter t with a suitable power, $t \mapsto t^k$), one can find a **CDD** in the family: more precisely, there are a section $\sigma : B \longrightarrow \mathfrak{X}$, a divisor $\mathfrak{D} \subseteq \mathfrak{X}$ and a cycle $\mathfrak{Z} \subseteq \mathfrak{X} \times_B \mathfrak{X}$ supported on $\mathfrak{D} \times_B \mathfrak{X}$ such that, for general $t \in B$,

$$[\Delta_{\mathfrak{X}_t}] = [\mathfrak{X}_t \times \{\sigma(t)\}] + [\mathfrak{Z}_t] \tag{2.4.2}$$

as cycle classes in $\mathfrak{X}_t \times \mathfrak{X}_t$. Now, the set of points $t \in B$ such that (2.4.2) holds is countable union of closed Zariski proper subsets of B; since it contains an open Zariski subset of B (the decomposition holds for general $t \in B$), one concludes that (2.4.2) holds for all $t \in B$.

In particular, one has a **CDD** on $\mathfrak{X}_0$, without any assumption on its singularities. Hence, in the application of this result to the stable Lüroth problem, one finds useful a criterion of the following type.

Corollary 2.4.1. *Let $f\colon \mathfrak{X} \longrightarrow B$ be a flat projective morphism where B is a smooth curve. Assume that:*

(1) *the fibre $\mathfrak{X}_t$ is smooth for $t \neq 0$;*
(2) *the special fibre $\mathfrak{X}_0$ has no* **CDD.**

Then, for very general $t \in B$, the fibre $\mathfrak{X}_t$ has no **CDD**, *hence it is not stably rational.*

In theory such statement could potentially be applied to a wide spectrum of situations due to the absence of any restriction on the special fibre: for example, one might seek to employ this strategy to prove stable irrationality of cubic 3-folds by specialising to a very singular special fibre $\mathfrak{X}_0$ − for example, with toric components as considered in the Gross-Siebert program. However, this approach clashes with the fact that condition (2) is hard to check directly; moreover, all the known invariants obstruct existence of **CDD** for smooth varieties only.

Indeed, Voisin's proof continues by exploiting the hypothesis that $\mathfrak{X}_0$ has ordinary quadratic singularities (although a wider class of singularities has been allowed successively) to show the following implication:

$$\mathfrak{X}_0 \text{ has a } \mathbf{CDD} \Rightarrow \widetilde{\mathfrak{X}}_0 \text{ has a } \mathbf{CDD} \tag{2.4.3}$$

where $\widetilde{\mathfrak{X}}_0 \longrightarrow \mathfrak{X}_0$ is a desingularisation. Now, as anticipated, one is able to apply the result in concrete situations by computing a non-trivial obstruction on the smooth model $\widetilde{\mathfrak{X}}_0$, hence violating (2.4.2) and, *a fortiori*, (2.4.1) too.

In this section, instead, we illustrate how one could try to obstruct (2.4.2) directly without recurring to (2.4.3). In the above situation, one considers the product family $\mathfrak{Y} := \mathfrak{X} \times_B \mathfrak{X}$ which is again flat over B, and there is a morphism

$$\mathrm{CH}_*(\mathfrak{Y}_K) \longrightarrow \mathrm{CH}_*(\mathfrak{Y}_0) \tag{2.4.4}$$

which is the *Fulton-MacPherson specialisation morphism* (see [**Fult-75**, Section 4]). The idea is to obstruct the existence of a decomposition (2.4.2) for $t = 0$ *directly on* $\mathfrak{Y}_0$, by considering homomorphic images of equation (2.4.2) via the specialisation morphism and seeking for a contradiction within the group of cycles on $\mathfrak{Y}_0$.

This new strategy, however, clashes with a crucial problem: if the degeneration $\mathfrak{X} \longrightarrow B$ leads to a special fibre $\mathfrak{X}_0$ which is highly singular (for example, a simple normal crossing divisor with toric components), then several **CDD**s may exist in $\mathfrak{X}_0$ as cycles in $\mathfrak{Y}_0$ but possibly none of them arises as flat limit of a decomposition (2.4.2). In other words, $\mathrm{CH}_*(\mathfrak{Y}_0)$ contains "too many" cycles that we do not need.

Example 2.4.2. An example of this behaviour can be seen by considering a degeneration of a smooth elliptic curve E to the union E_0 of three lines meeting transversely (such degeneration arises as elliptic surface $\mathfrak{E}$ of Kodaira type $\mathbf{I_3}$, see [**SS-10**, Section 4]). It is clear that E_0 has a **CDD** inherited by the **CDD** on each irreducible component; however, this decomposition cannot come as flat limit of a **CDD** on the generic fibre E, since a smooth elliptic curve does not admit a **CDD**. Indeed, assume by contradiction that one has a relation of the following form, up to rational equivalence:

$$\Delta_E = E \times \{p\} + Z \tag{2.4.5}$$

where $p \in E$ is a closed point and Z is a cycle of the form

$$Z = \sum_i n_i \{q_i\} \times E$$

for finitely many closed points $q_i \in E$ and integers $n_i \in \mathbb{Z}$. Intersecting both sides of (2.4.5) with $E \times \{p'\}$ for $p' \neq p$ another point in E implies that one can choose $Z = \{q\} \times E$ for $q \in E$ a single closed point. Then, let $\Gamma = \{(x, x + p') : x \in E\}$ be a cycle on $E \times E$ with $x + p'$ indicating a non-trivial translation in E; intersecting both sides of (2.4.5) with Γ gives a contradiction, since the left hand side would have degree 0 and the right hand side would have degree 2.

In order to solve this discrepancy, one needs to introduce some notion that, solely in terms of the geometry of $\mathfrak{Y}_0$, recognises whether a cycle can be expressed as flat limit of a suitable cycle in the family. The necessity to single out these particular cycle classes naturally leads to introduce some techniques from log geometry: one endows $\mathfrak{Y}_0$ with its natural log structure, which is the restriction to $\mathfrak{Y}_0$ of the divisorial log structure for $(\mathfrak{Y}, \mathfrak{Y}_0)$ (see [**Gross-11**, Section 3.2]): this log structure is heuristically thought as remembering the way how $\mathfrak{Y}_0$ is embedded in $\mathfrak{Y}$ (see, for example, the discussion at [**Gross-11**, Example 3.12]), therefore it is a natural starting point to develop some kind of log Chow theory, on which cycles carry some extra information about the way they can arise as limits. Such kind of approach, although seemingly very promising and possibly being the effective, general way to approach these problems, has not been developed yet. Instead, we will illustrate a more practical approach, that takes inspiration from the observation that, for a degeneration $\mathfrak{Y} \longrightarrow B$ in which $\mathfrak{Y}_0$ is simply normal crossing, cycles classes in $\mathrm{CH}_*(\mathfrak{Y}_0)$ that arise as specialisations through the Fulton-MacPherson map come from cycle classes in the normalisation of $\mathfrak{Y}_0$ that satisfy an obvious compatibility condition, named *pre-log condition* following [**NS-06**, **Nish-15**].

In the following we will give a survey of these new ideas, which have been developed in the new paper [**BvBvG-19**].

2.4.1. Compatible classes and pre-log cycles. We will adopt the following terminology: a k-scheme X of dimension n is said to be *simple normal crossing* if for every $p \in X$ such that X is regular at p, there are local coordinates $x_1, \ldots, x_n$ at p such that X is Zariski locally around p given by $x_1 \cdots x_r = 0$ for some $r \leq n$.

Let X be a simple normal crossing k-scheme and let X_i for $i \in I$ be its irreducible components. For each non-empty subset $J \subseteq I$ we set

$$X_J := \bigcap_{j \in J} X_j.$$

Because of the hypotheses put on X, one automatically has that X_J is a smooth (but possibly not connected) k-variety.

For non-empty sets $J_1 \subseteq J_2 \subseteq I$ and $J \subseteq I$ we let

$$\iota_{J_2 > J_1} : X_{J_2} \longrightarrow X_{J_1}, \quad \iota_J : X_J \longrightarrow X$$

be the natural inclusions. Now let

$$\nu : \widehat{X} \simeq \bigsqcup_{i \in I} X_i \longrightarrow X$$

be the normalisation morphism.

Definition 2.4.3. Denote by

$$R(X) \subseteq \mathrm{CH}^*(\widehat{X}) = \bigoplus_{i \in I} \mathrm{CH}^*(X_i)$$

the subring defined by tuples $(\alpha_i)_{i \in I}$ of cycle classes $\alpha_i \in \mathrm{CH}^*(X_i)$ that satisfy the property

$$\iota^*_{\{j,k\} > \{k\}}(\alpha_k) = \iota^*_{\{j,k\} > \{j\}}(\alpha_j) \tag{2.4.6}$$

for all distinct $j, k \in I$. The ring $R(X)$ is called *ring of compatible classes* and equation (2.4.7) is called *pre-log condition*.

REMARK 2.4.4. Notice that the pull-back maps ι^* are well defined since all the components X_i and their mutual intersections are smooth by construction.

Now, the Chow group $\mathrm{CH}_*(X)$, although not having a well-defined inner product, inherits a $R(X)$-module structure via the normalisation morphism.

DEFINITION 2.4.5. Let $\alpha = (\alpha_i)_{i\in I} \in R(X)$ and let Z be an irreducible subvariety of X. Let $J \subseteq I$ be the largest subset of indices such that $Z \subseteq X_J$. Then we define

$$\langle\!\langle \alpha, Z \rangle\!\rangle := (\iota_J)_*(\iota^*_{J>\{j_0\}}(\alpha_{j_0}) \cdot [Z])$$

where:

(1) $(\iota_J)_*$ is the push-forward of the inclusion ι_J;
(2) $[Z]$ is the cycle class of Z in $\mathrm{CH}^*(X_J)$;
(3) $\cdot$ is the intersection product in $\mathrm{CH}^*(X_J)$;
(4) j_0 is an arbitrary index chosen in J.

Notice that this definition is independent of the choice of j_0: if $j_0' \in J$ is another index, then

$$\iota^*_{\{j_0,j_0'\}>\{j_0'\}}(\alpha_{j_0'}) = \iota^*_{\{j_0,j_0'\}>\{j_0\}}(\alpha_{j_0})$$

so *a fortiori*

$$\iota^*_{J>\{j_0'\}}(\alpha_{j_0'}) = \iota^*_{J>\{j_0\}}(\alpha_{j_0}).$$

If Z is an arbitrary cycle (*i.e.* formal sum of irreducible subvarieties with integer coefficients), then $\langle\!\langle \alpha, Z \rangle\!\rangle$ is defined extending by linearity the above formula.

PROPOSITION 2.4.6. *If Z_1 and Z_2 are two rationally equivalent cycles on X, then*

$$\langle\!\langle \alpha, Z_1 \rangle\!\rangle = \langle\!\langle \alpha, Z_2 \rangle\!\rangle$$

for every $\alpha \in R(X)$. In particular, the pairing descends to rational equivalence on X and turns $\mathrm{CH}_*(X)$ into a $R(X)$-module with the following product:

$$r \cdot m := \langle\!\langle r, m \rangle\!\rangle$$

for all $r \in R(X)$ and $m \in \mathrm{CH}_*(X)$. The restriction of the push-forward $\nu_*\mathrm{CH}_*(\widehat{X}) \longrightarrow \mathrm{CH}_*(X)$ is, moreover, a morphism of $R(X)$-modules.

We then give the following definitions.

DEFINITION 2.4.7. In the above setting, we call

$$P(X) := R(X)/\ker(\nu_*)$$

the *pre-log Chow ring* of X.

A description of $\ker(\nu_*)$ can be found using basic intersection theory (see [**BvBvG-19**, Proposition 2.5]): there is an exact sequence

$$\bigoplus_{i<j} \mathrm{CH}_*(X_i \cap X_j) \xrightarrow{\delta} \bigoplus_{i\in I} \mathrm{CH}_*(X_i) \xrightarrow{\nu_*} \mathrm{CH}_*(X) \longrightarrow 0$$

where δ is defined in terms of push-forward maps of the inclusions $\iota_{\{ij\}>\{i\}}$ and $\iota_{\{ij\}>\{j\}}$. However, this is not enough to calculate concretely $P(X)$ in most cases. It is expected that $P(X)$ will depend on the groups $\mathrm{CH}_*(X_J)$ as well, but the relations amongst their generator are not easy to determine. A description of $P(X)$ in the case of X having at worst triple normal crossing singularities can be found by assuming that it satisfies certain local compatibility properties that arise from Friedman's theory of infinitesimal normal bundles (see [**BvBvG-19**, Proposition 2.8]).

2.4.2. Specialisation morphism and degenerations. Let $\pi : \mathfrak{X} \longrightarrow B$ be a flat morphism from a (possibly singular) variety $\mathfrak{X}$ to a *curve trait* B; with this we mean that $B = \operatorname{Spec} R$ where R is a discrete valuation ring which is the (completion of the) local ring at a point of some smooth curve. For such map we have a well defined specialisation morphism (2.4.4), whose construction is explained in [**Fult-75**, Section 4].

Definition 2.4.8. The datum of a flat morphism $\pi : \mathfrak{X} \longrightarrow B$ where $\mathfrak{X}$ is a regular scheme, B is a curve trait with closed point t_0, such that the special fibre $\mathfrak{X}_{t_0}$ is a simple normal crossing reduced scheme is called *strictly semi-stable degeneration.*

The advantage for $\mathfrak{X}$ being regular is that the irreducible components of $\mathfrak{X}_{t_0}$ are Cartier divisors and we can take intersections with each of them. This is essential for the following property.

Proposition 2.4.9. ([**BvBvG-19**, Theorem 3.2]) *Let $\pi : \mathfrak{X} \longrightarrow B$ be a strictly semi-stable degeneration. Then the specialisation morphism takes values into $P(X)$.*

2.4.3. Ramified base change and saturated group of pre-log cycles. Let $\pi : \mathfrak{X} \longrightarrow B$ be a strictly semi-stable degeneration and let $B' \longrightarrow B$ be a covering of curve traits, ramified at the closed point $t_0 \in B$. Then one can consider the base change

$$\mathfrak{X}' := \mathfrak{X} \times_B B'$$

where the fibre product is taken through the morphism $B' \longrightarrow B$. Notice that $\mathfrak{X}'$ is still flat over B' and its special fibre is still simple normal crossing but $\mathfrak{X}'$ will be singular in general, hence one can not apply Proposition 2.4.9; however, the specialisation morphism associated to this new degeneration takes values in a group which is very similar to the pre-log ring. The main idea, here, is that the irreducible components of the special fibre $\mathfrak{X}_{t_0}$, viewed as special fibre of the family $\mathfrak{X}' \longrightarrow B'$ are now $\mathbb{Q}$-Cartier divisors: they are

Cartier divisors on $\mathfrak{X}$ and local equations pull back (under the induced base-change map $\mathfrak{X}' \longrightarrow \mathfrak{X}$) to local equations in $\mathfrak{X}'$ with some multiplicity.

Definition 2.4.10. Recalling the formalism of Paragraph 2.4.1, let X be a simple normal crossing scheme and let $\nu : \widehat{X} \longrightarrow X$ be the normalisation morphism. We define the *ring of rational compatible classes* as the subring

$$R(X)^{\mathbb{Q}} \subseteq \mathrm{CH}^*(\widehat{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which consists of tuples $(\alpha_i)_{i\in I}$ with $\alpha_i \in \mathrm{CH}^*(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that each α_i and α_j pull-back to the same class in $\mathrm{CH}^*(X_i \cap X_j) \otimes_{\mathbb{Z}} \mathbb{Q}$. In other words, $R(X)^{\mathbb{Q}}$ consists of those tuples which satisfy the pre-log condition (2.4.6) with rational coefficients.

$R(X)^{\mathbb{Q}}$ may differ from $R(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ but there is a natural map $R(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow R(X)^{\mathbb{Q}}$. Let also $\nu_* : R(X)^{\mathbb{Q}} \longrightarrow \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the restriction of the push-forward $\nu_* : \mathrm{CH}_*(\widehat{X}) \longrightarrow \mathrm{CH}_*(X)$.

Definition 2.4.11. We define

$$P_{\mathrm{sat}}(X) := \mathrm{im}(\nu_*) \cap (\mathrm{CH}_*(X)/(\text{torsion}))$$

and we call it *saturated pre-log Chow group* of X.

Finally one is able to prove the following.

Proposition 2.4.12. ([**BvBvG-19**, Proposition 4.2]) *With notations as above, the specialisation morphism associated to the flat morphism $\pi' : \mathfrak{X}' \longrightarrow C'$ takes values in $P_{\mathrm{sat}}(X)$.*

2.4.4. Specialisations of CDDs into (saturated) pre-log Chow groups. We finally explain how this new machinery fits together into a more general version of the degeneration method established by Voisin and improved by Colliot-Thélène, Pirutka *et alii*. The goal is to develop this method to study stable irrationality of very general fibres in families of varieties by looking at degenerations with very singular special fibres.

Let $\pi : \mathfrak{X} \longrightarrow B$ be a degeneration and suppose that a very general fibre of $\pi : \mathfrak{X} \to B$ is stably rational (to avoid uninteresting cases we might also assume that a general fibre of π is rationally connected); equivalently, the geometric generic fibre $\mathfrak{X}_{\overline{K}}$, where $K = k(B)$, is stably rational, hence it admits a **CDD**. By the reasoning explained at the beginning of the Section, there is a finite covering $B' \to B$ of smooth curves with function field $L = k(B')$ such that the base-change $\mathfrak{X}_{\overline{L}}$ admits a **CDD** too.

Assume further that $\mathfrak{X}$ is semi-stable (possibly after having performed a semi-stable reduction on a more general total space); we have a diagram

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \mathfrak{X} \times_B \mathfrak{X} \arrow[d] \\ & B \end{tikzcd}$$

where, with a slight abuse of notation, we have indicated with π the base-change of the natural morphism $\mathfrak{X} \times_B \mathfrak{X} \to B$ induced from π, and ρ is a morphism which is birational outside the special fibre and such that $\mathfrak{Y}$ is again semi-stable. This can be obtained, in practice, by blowing up repeatedly those components in the special fibre of $\mathfrak{X} \times_B \mathfrak{X} \to B$ which are not Cartier divisors in the product, according to [**Ha-01**, Proposition 2.1]. With such construction, each irreducible components $A_i \times A_j$ of the special fibre $X_0 \times X_0$ of $\mathfrak{X} \times_B \mathfrak{X} \to B$ is birational to an irreducible component Y_{ij} of the special fibre Y of $\mathfrak{Y} \to B$. Then we consider the base-change

$$\begin{tikzcd} \mathfrak{Y}' = \mathfrak{Y} \times_B \times B' \arrow[r] \arrow[d, "\pi"] & \mathfrak{Y} \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

By Proposition 2.4.12, the specialisation map $\sigma : \mathrm{CH}_*(\mathfrak{X}_L \times_L \mathfrak{X}_L) \to P_{\mathrm{sat}}(Y)$ induced by $\mathfrak{Y} \to B$ is well defined. In particular, we can apply σ to an equation of the form

$$[\Delta_{\mathfrak{X}_L}] = [\mathfrak{X}_L \times x] + [Z]$$

as in Definition 2.1.1.(1) and try to establish a contradiction from assuming the resulting equality in $P_{\mathrm{sat}}(Y)$.

In these notations, following [**BvBvG-19**, Definition 5.2.(a)], we say that Y has a *pre-log decomposition of the diagonal* in $P_{\mathrm{sat}}(Y)$ if there is a class $[Z] \in P_{\mathrm{sat}}(Y)$ represented by a cycle Z that does not dominate any component of X_0 mapped via

$$Y \xrightarrow{\rho|_Y} X_0 \times X_0 \xrightarrow{\mathrm{pr}_1} X_0$$

and satisfies

$$\sigma([\Delta_{\mathfrak{X}_L}]) - \sigma([\mathfrak{X}_L \times x]) = [Z]$$

in the group $P_{\mathrm{sat}}(Y)$. The following "semi-stable degeneration principle" can therefore be established.

PROPOSITION 2.4.13. ([**BvBvG-19**, Proposition 5.3.(1)]) *In the above notation, if the geometric generic fibre $\mathfrak{X}_{\overline{K}}$ of the original semi-stable degeneration $\mathfrak{X} \to B$ is stably rational, then the special fibre Y of the induced semi-stable degeneration $\mathfrak{Y} \to B$ admits a pre-log decomposition of the diagonal in $P_{\mathrm{sat}}(Y)$.*

However, what we really want to inspect is the new degeneration $\mathfrak{Y}' \to B'$, obtained by base-change along the finite covering $B' \to B$. The total space $\mathfrak{Y}'$ need not be semi-stable but it can be brought to such form by blowing up components of the special fibre ([**Ha-01**, Proposition 2.2]) and it is possible to recover an analogous notion of pre-log decomposition of the diagonal ([**BvBvG-19**, Definition 5.2.(b)] along with a degeneration argument which is similar to the one presented above ([**BvBvG-19**, Proposition 5.3.(2)]).

Hence, in light of the above results, the degeneration method can be extended to a wide class of degenerations, provided we are able to obstruct the existence of pre-log decompositions of the diagonal by looking at the relevant saturated pre-log groups. Computations aiming to a better understanding of this technique for degenerations of cubic threefolds were undertaken in the paper [**BvBvG-19**].

CHAPTER 3

Brauer groups, unramified invariants and applications to conic bundles.

3.1. Brauer group of fields.

In this section we will work with an arbitrary field k, unless stated otherwise.

3.1.1. Quaternion algebras. All of the following definitions are compatible with the usual ones given in literature; our main references are [**GS-06**] for the results over fields with $\operatorname{char} k \neq 2$ and [**Vign-80**, **Voig-18**] for fields with $\operatorname{char} k = 2$.

DEFINITION 3.1.1. Let k be a field and let $a, b \in k$.

(1) If $\operatorname{char} k \neq 2$ and $a, b \neq 0$, the *quaternion algebra* (a, b) is the 4-dimensional k-algebra generated by symbols ξ, η subject to relations $\xi^2 = a, \eta^2 = b$ and $\xi\eta = -\eta\xi$.
(2) If $\operatorname{char} k = 2$ and $b \neq 0$, the *quaternion algebra* $[a, b)$ is the 4-dimensional k-algebra generated by symbols ξ, η subjected to relations $\xi^2 + \xi = a$, $\eta^2 = b$ and $\xi\eta = \eta\xi + \eta$.

More concretely, both (a, b) and $[a, b)$ have a structure of vector space over k, with basis given by $\{1, \xi, \eta, \xi\eta\}$. The additional algebra structure is given by the multiplication rules in Definition 3.1.1.

REMARK 3.1.2. By definition, it is clear that the isomorphism class of a quaternion algebra (a, b) depends only on the class of a, b in $k^\times/(k^\times)^2$. Indeed, if $a = u^2\alpha$ and $b = v^2\beta$, the substitutions $i \mapsto ui$, $j \mapsto vj$ yield an isomorphism $(\alpha, \beta) \simeq (u^2\alpha, v^2\beta) = (a, b)$. In particular, this shows that $(a, b) \simeq (b, a)$.

Instead, the isomorphism class of a quaternion algebra $[a, b)$ depends on the class of a in $k/\wp(k)$ and on the class of b in $k^\times/(k^\times)^2$, where $\wp : k \longrightarrow k$ is the map $x \mapsto x^2 + x$. Indeed, if $a = \alpha + (u^2 + u)$ and $b = \beta v^2$, setting $\xi \mapsto \xi + u$, $\eta \mapsto v\eta$ we have

$$(\xi + u)^2 + (\xi + u) = \xi^2 + u^2 + \xi + u = \alpha + a + \alpha = a$$

and

$$(\eta v)^2 = \eta^2 v^2 = v^2\beta = b$$

so the above function defines an isomorphism $[\alpha, \beta) \simeq [\alpha + (u^2 + u), v^2\beta) \simeq [a, b)$. This shows also that, unlike in the char $\neq 2$ case, in general $[a, b) \not\simeq [b, a)$.

Let Q be a quaternion k-algebra. Then we can assign to Q a projective plane conic $C_Q \subseteq \mathbf{P}^2_k := \operatorname{Proj} k[X_0, X_1, X_2]$, called the *associated conic*, in the following way:

(1) if $\operatorname{char} k \neq 2$, let $Q = (a, b)$ and we define C_Q by means of the equation

$$aX_1^2 + bX_2^2 - X_0^2 = 0; \tag{3.1.1}$$

(2) if $\operatorname{char} k = 2$, let $Q = [a, b)$ and we define C_Q by means of the equation

$$aX_1^2 + bX_2^2 + X_0X_1 + X_0^2 = 0. \tag{3.1.2}$$

It is possible to define quaternion algebras (and therefore even C_Q) intrinsically without using bases and coordinates (see the first chapter of [**GS-06**, **Vign-80**] for more details about this approach). This is done by defining the *conjugation* map, an involution morphism $Q \longrightarrow Q$ for each quaternion algebra Q, and then introducing a reduced norm map. The reduced norm also defines a quadratic form f_Q on Q itself, viewed as a k-vector space. Then the vanishing of f_Q defines C_Q and proves that such conic is canonically attached to Q, independently of the choice of a basis for Q.

Definition 3.1.3. A quaternion k-algebra Q is said to *neutralise* if there exists an isomorphism of k-algebras $Q \simeq M(2, k)$.

Remark 3.1.4. This terminology was borrowed from [**Vign-80**]; it is perhaps not the standard one nowadays − in more recent literature like [**GS-06**] and [**Voig-18**] the same phenomenon is called "splitting" − but it has been preferred to avoid over-use of the latter term. The convention we have chosen to follow is the following: "neutralisation" occurs for algebraic objects (*e.g.* quaternion algebras), "splitting" occurs for geometric objects (*e.g.* coverings).

We have the following criterion for neutralisation.

Proposition 3.1.5. *Let Q be a quaternion k-algebra. The following properties are equivalent:*

(1) *Q neutralises;*
(2) *Q is not a division algebra;*
(3) *the associated conic $C_{(a,b)}$ has a k-rational point.*

Proof. See [**GS-06**, Propostion 1.1.7, Proposition 1.3.2] for $\operatorname{char} k \neq 2$ and [**Vign-80**, Corollary 1.2.4] to integrate the $\operatorname{char} k = 2$ case. □

Recall that a conic k is rational if and only if it has a k-rational point (Theorem 1.3.1). This characterisation establishes a close relationship between quaternion algebras and their associated conics: a result of Witt ([**GS-06**, Theorem 1.4.2] and, more generally, [**EKM-08**, Corollary 23.5]), shows that two quaternion algebras are isomorphic if and only if their associated conics are birational (which is also equivalent to say that they are isomorphic).

3.1.2. Central simple algebras. Recall that the *centre* of a ring A is the set

$$Z(A) := \{x \in A : xy = yx \text{ for all } y \in A\}.$$

Let k be a field; a k-algebra A is called *central* if $Z(A) = k$. Following Definition 3.1.3, we say that a k-algebra A *neutralises* if $A \simeq M(n, k)$ as k-algebras.

DEFINITION 3.1.6. An algebra A is called *central simple* over k if it is central as a k-algebra and it is simple as a ring, namely it has no non-trivial two-sided ideals.

For example, every division algebra D is simple and, since $Z(D)$ is a field, D is central simple over $Z(D)$; every non-neutralising quaternion k-algebra is of this kind. Moreover, the matrix ring $M(n, D)$ over any division algebra D is simple; the centre of $M(n, D)$ is an isomorphic copy of $Z(D)$ formed by scalar matrices, so $M(n, D)$ is a central simple $Z(D)$-algebra.

Wedderburn's Theorem ([**GS-06**, Theorem 2.1.3]) implies that for a finite-dimensional simple algebra A over a field k the number $n = \sqrt{\dim_k A}$ is well defined and; this is called *degree* of A. Also, by Wedderburn's Theorem, there is an unique D such that $A \simeq M(n, D)$, and the degree of D is called *index* of A. If k is algebraically closed, then any finite-dimensional division k-algebra is k itself, hence every finite-dimensional simple algebra over such k is isomorphic to $M(n, k)$ for some n. This fact suggests an alternative definition of central simple algebra.

THEOREM 3.1.7. ([**GS-06**, Theorem 2.2.1]) *Let k be a field and A a finite-dimensional k-algebra. Then A is central simple if and only if there exist an integer $n \geq 1$ and a finite field extension K/k such that $A \otimes_k K$ neutralises over K.*

The field K in the statement is called a *neutralising field* of A and is a separable extension of k by Nöther-Köthe's theorem ([**GS-06**, Proposition 2.2.5]). Since every finite, separable extension is contained in a finite Galois extension, the above results imply that a finite-dimensional k-algebra A is central simple if and only if it neutralises as a K-algebra, after extending scalars to a finite Galois extension K/k (we will say that A *neutralises over* K).

LEMMA 3.1.8. *If A, B are two central simple k-algebras that neutralise over K, then so is $A \otimes_k B$.*

PROOF. We have isomorphisms

$$A \otimes_k K \simeq M(n, K),\ B \otimes_k K \simeq M(m, K)$$

for appropriate integers $n, m > 0$, so

$$(A \otimes_k B) \otimes_k K \simeq (A \otimes_k K) \otimes_K (B \otimes_k K) \simeq \\ \simeq M(n, K) \otimes_K M(m, K) \simeq M(nm, K)$$

which is the assertion. □

3.1.3. Brauer equivalence. Let us give the following definition.

DEFINITION 3.1.9. Let A, B be two central simple k-algebras. Then we say A and B are *Brauer equivalent* if there exist integers $m, n > 0$ such that $A \otimes_k M(m,k) \simeq B \otimes_k M(n,k)$.

We can make Definition 3.1.9 work as an equivalence relation. Call $\mathscr{C}(K/k, n)$ the set of central simple k-algebras of degree n that neutralise over a finite Galois extension K/k. Then Brauer equivalence induces an equivalence relation on the union of all sets $\mathscr{C}(K/k, n)$ amongst positive integers n: if A, B and B, C are pairwise Brauer equivalent, we have isomorphisms

$$A \otimes_k M(n,k) \simeq B \otimes_K M(m,k), \quad B \otimes_k M(p,k) \simeq C \otimes_k M(q,k)$$

that is to say

$$A \otimes_k M(np,k) \simeq B \otimes_k M(p,k) \simeq C \otimes_k M(q,k)$$

since $M(np,k) \simeq M(n,k) \otimes_k M(p,k)$. Every equivalence class is called *Brauer class neutralising over* K.

We will denote $\mathrm{Br}(K/k)$ the set of all Brauer classes neutralising over K and $\mathrm{Br}(k)$ the union of all $\mathrm{Br}(K/k)$ amongst all the finite Galois extensions K of k. Since tensor product manifestly preserves Brauer equivalence of k-algebras (see also Lemma 3.1.8), the sets defined above admit a natural inner operation.

PROPOSITION 3.1.10. *The sets* $\mathrm{Br}(K/k)$ *and* $\mathrm{Br}(k)$ *admit an abelian group structure with operation induced by tensor product of k-algebras.*

PROOF. The basic properties of tensor product make clear that the induced operation is commutative and associative on Brauer classes. The identity element is given by the class of $M(n,k)$. Let us show that every element in $\mathrm{Br}(K/k)$ has an inverse. Recall that, given a k-algebra A, its *opposite algebra* is the k-algebra A° with inner product $(x,y) \mapsto y \cdot x$ where $\cdot$ is the product in A. It is easy to see that A° is central simple as $Z(A^\circ) = Z(A)$ and also neutralises over K since the algebra $A^\circ \otimes_k K$ is the opposite algebra of $A \otimes_k K$; we need only to prove that $A \otimes_k A^\circ$ is Brauer equivalent to some matrix algebra so it would act as identity in $\mathrm{Br}(K/k)$; but indeed, define a k-linear map

$$\begin{aligned} A \otimes_k A^\circ &\longrightarrow M(\dim(A), k) \\ x \otimes y &\mapsto \varphi_{x,y} \end{aligned}$$

where $\varphi_{x,y}(a) := xay$; it is immediate to see that this defines a morphism of k-algebras. This map is also non-zero, therefore injective as $A \otimes_k A^\circ$ is simple by Lemma 3.1.8. Hence by dimension reasons it is an isomorphism and this proves the assertion. $\square$

The set $\mathrm{Br}(K/k)$ equipped with the tensor product is called *relative Brauer group of* K/k, while $\mathrm{Br}(k)$ is the *(absolute) Brauer group of* k. We state here an important characterisation of the 2-torsion subgroup of $\mathrm{Br}(k)$; we will also give a cohomological characterisation later.

THEOREM 3.1.11. *Let k be an arbitrary field. Then $\mathrm{Br}(k)[2]$ is generated by quaternion k-algebras.*

PROOF. If $\mathrm{char}\, k \neq 2$, the assertion follows from the Merkurjev-Suslin theorem, but this particular result was already known earlier ([**Mer-81**]). If $\mathrm{char}\, k = 2$, the assertion follows from a result due to Albert ([**GS-06**, Theorem 9.1.8]). $\square$

3.1.4. Brauer group via Galois descent. The results illustrated in this paragraph follow from the application of Grothendieck's "theory of descent" in a particular situation. See [**GS-06**, Section 2.4] or [**Ser-79**, Chapter X, Section 2] for a complete treatment.

The aim is to identify the set $\mathscr{C}(K/k, n)$, of central simple algebras of degree n that neutralise over the Galois extension K/k, with some group cohomology with coefficients given by a certain automorphism module. Considering certain classes of *twisted forms* (tensors of type (p, q)), it is possible to obtain results [**Ser-79**, Chapter X, Section 2, Proposition 4] and [**GS-06**, Theorem 2.4.3] which construct a bijective correspondence

$$\mathscr{C}(K/k, n) \leftrightarrow H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K M(n, K)).$$

It is well known that every automorphism of $M(n, K)$ is inner ([**GS-06**, Lemma 2.4.1]), hence one can also identify

$$\mathrm{Aut}_K M(n, K) \simeq \mathrm{GL}(n, K)/K^\times := \mathrm{PGL}(n, K)$$

and thus one has proved that

$$\mathscr{C}(n, K/k) \leftrightarrow H^1(\mathrm{Gal}(K/k), \mathrm{PGL}(n, K)) \tag{3.1.3}$$

as a set-theoretic bijection.

From now on, we will denote $G := \mathrm{Gal}(K/k)$. Since tensor product of central simple algebras is a central simple algebra (Lemma 3.1.8), cohomology inherits a binary operation via identification (3.1.3):

$$\otimes : H^1(G, \mathrm{PGL}(n, K)) \times H^1(G, \mathrm{PGL}(m, K)) \longrightarrow H^1(G, \mathrm{PGL}(mn, K)).$$

Now note that, for every $n, m > 0$, there are injective maps $\mathrm{GL}(n, K) \longrightarrow \mathrm{GL}(mn, K)$ by sending each $M \in \mathrm{GL}(n, K)$ to the block matrix of size $mn \times mn$ obtained placing m copies of M along the diagonal. These maps are equivariant with respect to the multiplication action of $K^\times$, hence they induce

$$\lambda_{m,n} : H^1(G, \mathrm{PGL}(n, K)) \longrightarrow H^1(G, \mathrm{PGL}(mn, K)).$$

The class of each central simple algebra A in $H^1(G, \mathrm{PGL}(n, K))$ is mapped to the class of $A \otimes_k M(m, k)$ through $\lambda_{m,n}$, via the identification (3.1.3). It is a straightforward consequence of Wedderburn's Theorem to prove that the maps $\lambda_{m,n}$ are again injective for all m, n ([**GS-06**, Lemma 2.4.5]).

This allows one to consider the set

$$H^1(G, \mathrm{PGL}_\infty) := \bigcup_{n>0} H^1(G, \mathrm{PGL}(n, K))$$

or, more formally, the direct limit taken using the injective maps $\lambda_{m,n}$. We can equip $H^1(G, \mathrm{PGL}_\infty)$ with the product defined above via tensor product of algebras, which is also compatible with the injections $\lambda_{m,n}$. Moreover, functoriality of Galois groups and group cohomology imply that, for every Galois extensions tower $L/K/k$ there is an injection

$$\iota_{L,K} : H^1(\mathrm{Gal}(K/k), \mathrm{PGL}_\infty) \longrightarrow H^1(\mathrm{Gal}(L/k), \mathrm{PGL}_\infty). \tag{3.1.4}$$

Let us now fix a separable closure k_{sep} of k; using the maps (3.1.4) we can define $H^1(k, \mathrm{PGL}_\infty)$ as the union of all the $H^1(\mathrm{Gal}(L/k), \mathrm{PGL}_\infty)$ over all the finite Galois extensions L/k which are contained in k_{sep} (again, one formally takes the direct limit via injections $\iota_{L,M}$). This leads to the following central result.

THEOREM 3.1.12. *The sets $H^1(G, \mathrm{PGL}_\infty)$ and $H^1(k, \mathrm{PGL}_\infty)$ are abelian groups and we have the following identifications:*

$$\mathrm{Br}(K/k) \simeq H^1(G, \mathrm{PGL}_\infty), \quad \mathrm{Br}(k) = H^1(k, \mathrm{PGL}_\infty).$$

PROOF. The group structure is obtained by extending the $\otimes$ operation defined before to the whole union of the cohomology sets. To prove the two isomorphisms recall that

$$\mathrm{Br}(K/k) = \bigcup_n \mathscr{C}(K/k, n) \Big/ \sim$$

where $\sim$ is Brauer equivalence. We want to use correspondence (3.1.3) to show the desired isomorphism. In fact, let $A \in \mathscr{C}(K/k, n)$ and $B \in \mathscr{C}(K/k, m)$. By the stated bijection, A corresponds uniquely to the class of a 1-cocycle χ_A in $H^1(G, \mathrm{PGL}(n, K))$ and B to the class of a 1-cocycle χ_B in $H^1(G, \mathrm{PGL}(m, K))$; it remains to show that this identification is compatible with Brauer equivalence and with the direct limit. Suppose that A and B are Brauer equivalent: therefore there exist $n', m' \geq 0$ such that

$$A \otimes_k M(n', k) \simeq B \otimes_k M(m', k)$$

namely

$$\lambda_{n,n'}(\bar{\chi}_A) = \lambda_{m,m'}(\bar{\chi}_B)$$

In other words, the cohomology classes of the cocycles χ_A and χ_B agree in the direct limit $H^1(G, \mathrm{PGL}_\infty)$ which implies, by injectivity, $\bar{\chi}_A = \bar{\chi}_B$. Conversely, equality of classes in $H^1(G, \mathrm{PGL}_\infty)$ implies Brauer equivalence between the corresponding central simple algebras by the same argument. Finally, it is straightforward to prove that this correspondence respects tensor product operations and this proves $H^1(G, \mathrm{PGL}_\infty) \simeq \mathrm{Br}(K/k)$. The second isomorphism is straightforwardly obtained recalling that $\mathrm{Br}(k)$ is the union of all $\mathrm{Br}(K/k)$ where K/k are finite Galois extensions and

$H^1(k, \mathrm{PGL}_\infty)$ is similarly obtained by a union of copies of $\mathrm{Br}(K/k)$ through the maps $\iota_{L,K}$ for finite Galois extensions $L/k, K/k$. □

3.1.5. Brauer group via Galois cohomology.

NOTATION 3.1.13. Recall that, given a profinite group G, a *continuous G-module* is an abelian group M equipped with an action of G which is continuous with respect to the profinite topology of G. From now on, given a Galois extension K/k and a continuous $\mathrm{Gal}(K/k)$-module M, we will denote

$$H^i(K/k, M) := H^i(\mathrm{Gal}(K/k), M)$$

and

$$H^i(k, M) := H^i(\mathrm{Gal}(k), M)$$

to indicate Galois cohomology with coefficients in M.

Let K/k be a finite Galois extension with $G = \mathrm{Gal}(K/k)$ and consider the exact sequence of G-modules

$$0 \longrightarrow K^\times \longrightarrow \mathrm{GL}(m, K) \to \mathrm{PGL}(m, K) \to 1 \tag{3.1.5}$$

for every positive integer m. Since $K^\times$ is abelian and contained in the centre of $\mathrm{GL}(n, K)$, there is a coboundary operator (see [GS-06, Proposition 4.4.1])

$$\partial_m : H^1(K/k, \mathrm{PGL}(m, K)) \longrightarrow H^2(K/k, K^\times).$$

It can be proved that the maps ∂_m pass to direct limit of the system formed by $H^1(G, \mathrm{PGL}(m, K))$ and injections $\lambda_{m,n}$ (see [GS-06, Lemma 4.4.3]) and thus they lead to a map

$$\partial_\infty : H^1(K/k, \mathrm{PGL}_\infty) \longrightarrow H^2(K/k, K^\times)$$

which, recalling Theorem 3.1.12, can be rewritten as

$$\partial_\infty : \mathrm{Br}(K/k) \longrightarrow H^2(K/k, K^\times). \tag{3.1.6}$$

It can also be proved that ∂_∞ is an isomorphism of abelian groups (see [GS-06, Theorem 4.4.5]). This map has also an explicit description in terms of cocycles (see again [GS-06, Lemma 4.4.4]). Combining all the results, one gets the following characterisation.

THEOREM 3.1.14. *Let K/k be a finite Galois extension. Then*

$$\mathrm{Br}(K/k) \simeq H^2(K/k, K^\times),\ \mathrm{Br}(k) \simeq H^2(k, k_{\mathrm{sep}}^\times).$$

Using this Galois cohomology characterisation, it becomes trivial to establish the following result.

COROLLARY 3.1.15. *Let K/k be a Galois extension of degree n. Then the Brauer group $\mathrm{Br}(K/k)$ has order dividing n. Moreover, $\mathrm{Br}(k)$ is a torsion group.*

Each subgroup with prescribed torsion order of $\mathrm{Br}(k)$ can be recovered in terms of Galois cohomology as well. This is an easy consequence of Theorem 3.1.14 for the torsion part coprime to the characteristic.

PROPOSITION 3.1.16. ([GS-06, Corollary 4.4.9]) *Let k be any field of characteristic p and let m be an integer. If $p \nmid m$ then $\mathrm{Br}(k)[m] \simeq H^2(k, \mu_m)$.*

The isomorphism in Proposition 3.1.16 is determined by taking cohomology of the following exact sequence

$$1 \longrightarrow \mu_m \longrightarrow k_{\mathrm{sep}}^{\times} \xrightarrow{m} k_{\mathrm{sep}}^{\times} \longrightarrow 1$$

and applying Hilbert's Theorem 90 (see [GS-06, Example 2.3.4]). We also state the following result for the p-torsion of the first Galois cohomology group.

LEMMA 3.1.17. ([GS-06, Proposition 4.3.6, Proposition 4.3.10]) *Let k be any field of characteristic p and let ℓ be a prime number. Then:*

1. (KUMMER THEORY) *if $\ell \neq p$, then $H^1(k, \mu_\ell) \simeq k^\times/(k^\times)^\ell$;*
2. (ARTIN-SCHREIER THEORY) *if $\ell = p$, then $H^1(k, \mu_p) \simeq k/\wp(k)$.*

Here we have denoted with μ_ℓ the multiplicative group of ℓ-roots of units and $\wp(x) = x^p - x$ the Artin-Schreier morphism.

There are two different ways to obtain a concrete description of $\mathrm{Br}(k)[p^r]$. The classical method is due to Hochschild and employs directly central simple algebras, while the second method relies on giving a presentation of $\mathrm{Br}(k)[p]$ via logarithmic p-adic differentials. See [GS-06, Chapter 9] for a complete discussion.

The idea of the latter method is to exploit the natural exact sequence of $\mathrm{Gal}(k)$-modules

$$0 \longrightarrow k_{\mathrm{sep}}^{\times} \xrightarrow{p} k_{\mathrm{sep},}^{\times} \longrightarrow C \longrightarrow 0$$

where p denotes raising to the p-th power and C is the cokernel. Now, C can be described explicitly by means of the universal logarithmic differential $\mathrm{dlog} : k_{\mathrm{sep}}^{\times} \longrightarrow \Omega^1_{k_{\mathrm{sep}}}$ defined by $a \mapsto \mathrm{d}\,a/a$, where $\Omega^1_{k_{\mathrm{sep}}} := \Omega^1_{k_{\mathrm{sep}/\mathbb{Z}}}$ denotes the module of absolute differentials. More precisely, let $\nu(1)_{k_{\mathrm{sep}}}$ be the image of dlog. Then one can rewrite the above exact sequence as

$$0 \longrightarrow k_{\mathrm{sep}}^{\times} \longrightarrow k_{\mathrm{sep}}^{\times} \longrightarrow \nu(1)_{k_{\mathrm{sep}}} \longrightarrow 0.$$

Taking Galois cohomology, one obtains the desired description.

PROPOSITION 3.1.18. *Let k be any field of characteristic p. Then $\mathrm{Br}(k)[p] \simeq H^1(k, \nu(1)_{k_{\mathrm{sep}}})$.*

The description of $H^1(k, \nu(1)_{k_{\mathrm{sep}}})$ can be made even more explicit by the use of logarithmic differentials for a presentation of $\nu(1)$. Indeed, for any field extension K/k, there exists an exact sequence ([GS-06, Theorem 9.2.3])

$$0 \longrightarrow \nu(1)_K \longrightarrow \Omega^1_K \xrightarrow{\Gamma} \Omega^1_K/B^1_K \longrightarrow 0 \tag{3.1.7}$$

where B^1_K is the image of the usual de Rham differential $\mathrm{d} : K \longrightarrow \Omega^1_K$ and Γ is a generalisation of the Artin-Schreier map, defined as

$$\Gamma\left(a \cdot \frac{\mathrm{d}x}{x}\right) = (a^p - a)\frac{\mathrm{d}x}{x} \mod B^1_K$$

whee we have used the fact that logarithmic differential $\mathrm{d}x/x$ is a generator for Ω^1_K.

This description for $K = k_{\mathrm{sep}}$ leads to the following important characterisation, attributed to Kato.

THEOREM 3.1.19. *Let k be a field of characteristic $p > 0$. There is an unique isomorphism*

$$\mathrm{Br}(k)[p] \simeq \frac{\Omega^1_k}{B^1_k + \Gamma(\Omega^1_k)}.$$

PROOF. See [**GS-06**, Theorem 9.2.4]. □

REMARK 3.1.20. More generally, one can define $\mathrm{Gal}(k)$-modules $\nu(r)_K$ for any $r \geq 0$ and any field extension K/k that fit into an exact sequence analogous to (3.1.7):

$$0 \to \nu(r)_K \longrightarrow \Omega^r_K \xrightarrow{\Gamma} \Omega^r_K/B^r_K \to 0.$$

3.2. Brauer group of schemes.

3.2.1. Definition and main properties. Let k be any field and let X be a noetherian scheme over k. It is natural to give the following definitions.

DEFINITION 3.2.1. Let X be a scheme and let $\mathscr{A}$ be a sheaf of $\mathscr{O}_X$-algebras such that $\mathscr{A}$ is of finite presentation as a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{A}$ is an *Azumaya algebra* over X if there exists an étale open cover $\{\varphi_i : U_i \longrightarrow X\}_{i \in I}$ of X such that $\varphi_i^* \mathscr{A} \simeq \mathscr{E}nd_{\mathscr{O}_{U_i}}(\mathscr{O}_{U_i}^{\oplus r_i})$ for some integer $r_i > 0$, where $\mathscr{E}nd_{\mathscr{O}_X}\mathscr{F}$ is the sheaf of $\mathscr{O}_X$-algebras defined by sending each open set U to the algebra $M(\mathrm{rank}(\mathscr{F}(U)), \mathscr{O}_X(U))$ of square matrices of size $\mathrm{rank}(\mathscr{F}(U))$ with entries in $\mathscr{O}_X(U)$.

Equivalently ([**Grot-64**, Vol. I, Théorème 5.1]), one can say that a sheaf of $\mathscr{O}_X$-algebras $\mathscr{A}$ is an Azumaya algebra on X if, for every $x \in X$, the stalk $\mathscr{A}_x$ is an Azumaya algebra in the sense of [**Grot-64**, Proposition 3.1].

DEFINITION 3.2.2. Let X be a scheme and let $\mathscr{A}, \mathscr{B}$ be two Azumaya algebras on X. We say that $\mathscr{A}$ and $\mathscr{B}$ are *(Morita) equivalent* if there exist two locally free sheaves of $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$ such that

(1) $\mathscr{F}$ and $\mathscr{G}$ have positive rank at every $x \in X$;
(2) there is an $\mathscr{O}_X$-algebras isomorphism

$$\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}) \simeq \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{G}).$$

Denote $\mathrm{Br}(X)$ the set of equivalence classes of Azumaya algebras on X modulo Morita equivalence; this is an abelian group with inner operation induced by tensor product of sheaves over $\mathscr{O}_X$ and it is called *Brauer group of* X.

This construction can be easily related to étale cohomology in the following way.

LEMMA 3.2.3. *Let X be a scheme and let $\mathscr{A}$ be an Azumaya algebra which is locally free of rank d^2. Then its Brauer class is d-torsion.*

PROOF. Since $\mathscr{A}$ is locally free of constant rank d^2, the assertion can be proved locally: for every $x \in X$ the stalk $\mathscr{A}_x$ is an Azumaya algebra, hence it is sufficient to prove that $\mathscr{A}_x^{\otimes d} \simeq \mathrm{End}_{\mathscr{O}_{X,x}}(V)$ for some V. This is done in the short note [**Salt-81**]. $\square$

Mimicking the construction of isomorphism (3.1.6), which in turn comes from the connecting morphism in non-abelian cohomology (3.1.4), one can define a map

$$\delta : \mathrm{Br}(X) \longrightarrow H^2_{\text{ét}}(X, \mathbf{G}_m)$$

which happens to be an injective group homomorphism. However, it is not true that δ is an isomorphism in general. One only knows that $\mathrm{Br}(X) \subseteq \mathrm{tors}\, H^2_{\text{ét}}(X, \mathbf{G}_m) =: \mathrm{Br}'(X)$; the latter is called *cohomological Brauer group.* In general, one has $\mathrm{Br}(X) \neq \mathrm{Br}'(X)$ but in many situation they agree.

PROPOSITION 3.2.4. *Let X be a scheme.*

(1) ([**Grot-64**, II, Proposition 1.4]) *If X is regular, then $H^2_{\text{ét}}(X, \mathbf{G}_m)$ is a torsion group.*
(2) ([**deJo-03**, Theorem 1.1]) *If X is quasi-projective over an affine base, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Examples showing failure of the above statements removing regularity or quasi-projectivity hypotheses are known even in low dimension; see the discussion at [**Grot-64**, II, Remarque 1.11] and [**Grot-64**, II, Section 2]. Since we will deal with quasi-projective varieties, we can work with either $\mathrm{Br}(X)$ or $\mathrm{Br}(X')$ with no distinction.

REMARK 3.2.5. If $X = \mathrm{Spec}(A)$ is an affine scheme, then we will write $\mathrm{Br}(A)$ for $\mathrm{Br}(\mathrm{Spec}(A))$. In particular, if $X = \mathrm{Spec}(k)$ is the spectrum of a field, then X is a regular scheme and one has that

$$\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbf{G}_m) = H^2_{\mathrm{Gal}}(k, k^\times_{\mathrm{sep}}) \simeq \mathrm{Br}(k).$$

NOTATION 3.2.6. Let X be a scheme and let η_X be the generic points of X. We will denote by $i_X : \{\eta_X\} \longrightarrow X$ the natural inclusion and with $i_X^* : \mathrm{Br}(X) \longrightarrow \mathrm{Br}\, k(X)$ the pull-back on Brauer classes (in other words, i_X^* restricts every Azumaya algebra over X to the generic point of X).

The following two results will be essential to develop our techniques in the next sections.

THEOREM 3.2.7. *Let X be a regular scheme. Then the natural map $i_X^* : \mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$ is injective.*

PROOF. This is a consequence of [**Grot-64**, Vol. II, Corollaire 1.10], which holds under the more general hypotheses stated in [**Grot-64**, Vol. II, Proposition 1.4]. □

THEOREM 3.2.8. *Let k be an arbitrary field and let C be a smooth conic over k such that $C(k) = \varnothing$. Then there is a short exact sequence*

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C) \longrightarrow 0$$

where the central arrow is pull-back along the structure morphism of C and its kernel is generated by the 2-torsion Brauer class represented by the quaternion algebra associated to C. Moreover, if $\mathrm{char}\, k \neq 2$ and -1 is a square in k, taking 2-torsion yields another exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Br}(k)[2] \longrightarrow \mathrm{Br}(C)[2] \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

such that any class not in the image of $\mathrm{Br}(k)[2] \longrightarrow \mathrm{Br}(C)[2]$ is 4-torsion; that is to say, it belongs to the image of $\mathrm{Br}(k)[4] \longrightarrow \mathrm{Br}(C)[4]$.

PROOF. See [**CTO-89**, Proposition 1.5] for a proof which attributes the identification of the kernel to Ernst Witt ([**Witt-35**]). □

REMARK 3.2.9. The assumption $C(k) = \varnothing$ is equivalent to require that the Brauer class associated to C is non-trivial, as the quaternion algebra associated to C does not neutralise (Proposition 3.1.5).

We also recall the following result concerning the Brauer group of the projective space.

THEOREM 3.2.10. *Let k be an arbitrary field and let $n \geq 1$ be an integer. Then $\mathrm{Br}(\mathbf{P}^n_k) \simeq \mathrm{Br}(k)$.*

PROOF. This is implied by the main theorem in [**Gabb-81**], which states that given a Brauer-Severi scheme $\pi : X \to S$, the kernel of the natural pull-back $\pi^* : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is generated by $\delta(\alpha_X)$ where $\alpha_X \in H^1(S, \mathrm{PGL}_{n+1})$ is the class of the scheme X and $\delta : H^1_{\text{ét}}(S, \mathrm{PGL}_{n+1}) \to H^2_{\text{ét}}(S, \mathbf{G}_m)$ is the boundary map of the long étale cohomology sequence asssociated to $1 \to \mathbf{G}_m \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1$. In particular, for the projective space, one can show that α_X lies in the image of $H^1(S, \mathrm{GL}_{n+1}) \to H^1(S, \mathrm{PGL}_{n+1})$, see *ibidem*. □

3.2.2. Invariance properties. One of the first appearances of the Brauer group as a stable birational invariant occurs in the influential paper [**AM-72**], where the following more geometric realisation is used.

PROPOSITION 3.2.11. *Let X be a smooth, projective k-variety.*

(1) *If $k = \mathbb{C}$, then the subgroup $\mathrm{tors}\, H^3(X, \mathbb{Z})$ of the third singular cohomology group is a stable birational invariant;*

(2) *if* $\operatorname{char} k > 0$*, then the subgroup* $\operatorname{tors} H^3_{\text{ét}}(X, \mathbb{Z}_\ell)$ *of the third étale* ℓ*-adic cohomology group is a stable birational invariant for every prime* $\ell \neq \operatorname{char} k$*.*

PROOF. For (1), observe first that $T_X := \operatorname{tors} H^3(X, \mathbb{Z})$ does not change if we replace X with $X \times \mathbf{P}^m$; indeed, recalling the Künneth exact sequence

$$0 \longrightarrow \bigoplus_{p+q=3} H^p(X, \mathbb{Z}) \otimes H^q(\mathbf{P}^r, \mathbb{Z}) \longrightarrow H^3(X \times \mathbf{P}^r, \mathbb{Z}) \longrightarrow$$

$$\longrightarrow \bigoplus_{p+q=3} \operatorname{Tor}_1^{\mathbb{Z}}(H^p(X, \mathbb{Z}), H^q(\mathbf{P}^r, \mathbb{Z})) \longrightarrow 0.$$

However one has

$$H^q(\mathbf{P}^r, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & 0 \leq s \leq 2r, q = 2s \\ 0 & \text{otherwise} \end{cases}$$

and therefore

$$\begin{aligned}
\operatorname{Tor}_1^{\mathbb{Z}}(H^0(X, \mathbb{Z}), H^3(\mathbf{P}^r, \mathbb{Z})) &= \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, 0) = 0 \\
\operatorname{Tor}_1^{\mathbb{Z}}(H^1(X, \mathbb{Z}), H^2(\mathbf{P}^r, \mathbb{Z})) &= \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{m_0} \oplus (\mathbb{Z}/p_1)^{m_1} \oplus \ldots \oplus (\mathbb{Z}/p_s)^{m_s}, \mathbb{Z}) = \\
&= \bigoplus_{n=1}^{s} \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_n, \mathbb{Z})^{m_n} = 0 \\
\operatorname{Tor}_1^{\mathbb{Z}}(H^2(X, \mathbb{Z}), H^1(\mathbf{P}^r, \mathbb{Z})) &= \operatorname{Tor}_1^{\mathbb{Z}}(H^2(X, \mathbb{Z}), 0) = 0 \\
\operatorname{Tor}_1^{\mathbb{Z}}(H^3(X, \mathbb{Z}), H^0(\mathbf{P}^r, \mathbb{Z})) &= 0
\end{aligned}$$

where we have used that $H^1(X, \mathbb{Z})$ is a finitely generated abelian group. In summary, we have proved that the usual Künneth formula holds. However

$$\begin{aligned}
H^0(X, \mathbb{Z}) \otimes H^3(\mathbf{P}^r, \mathbb{Z}) &= 0 \\
H^1(X, \mathbb{Z}) \otimes H^2(\mathbf{P}^r, \mathbb{Z}) &= H^1(X, \mathbb{Z}) \otimes \mathbb{Z} \\
H^2(X, \mathbb{Z}) \otimes H^1(\mathbf{P}^r, \mathbb{Z}) &= 0 \\
H^3(X, \mathbb{Z}) \otimes H^0(\mathbf{P}^r, \mathbb{Z}) &= H^3(X, \mathbb{Z}) \otimes \mathbb{Z}
\end{aligned}$$

therefore

$$\operatorname{tors} H^3(X \times \mathbf{P}^r, \mathbb{Z}) = \operatorname{tors}(H^1(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z})).$$

However, the group $H^1(X, \mathbb{Z})$ has no torsion: considering the sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

and taking cohomology one has

$$\cdots \longrightarrow H^0(X, \mathbb{Z}/n) \longrightarrow H^1(X, \mathbb{Z}) \xrightarrow{n} H^1(X, \mathbb{Z}) \longrightarrow \cdots$$

namely

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow H^1(X, \mathbb{Z}) \xrightarrow{n} H^1(X, \mathbb{Z}) \longrightarrow \cdots.$$

Since n is arbitrary, $H^1(X,\mathbb{Z})$ has no torsion, hence the assertion. It remains to show that T_X is a birational invariant and this is done in [**AM-72**, Proposition 1].

For (2), an analogous argument as the above can be carried over in virtue of the étale Künneth formula ([**Mil-80**, Section 8]) and then the conclusion follows from birational invariance of tors $H^3_{\text{ét}}(X,\mathbb{Z}_\ell)$, which is proved in [**AM-72**, Proposition 1*]. Notice that the proof is given for varieties of dimension ≤ 3, as existence of resolutions of singularities for higher dimensional varieties over fields of positive characteristic has not been fully established yet. The preprint [**Hir-17**] claims that such a desingularisation exists for varieties of arbitrary dimension, but the community still debates whether this proof is valid or not. □

In a more general setting, one has the following result.

PROPOSITION 3.2.12. *Let k be a field of characteristic 0 and let X, Y be smooth, projective k-varieties that are stably birational. Then* $\mathrm{Br}(X) \simeq \mathrm{Br}(Y)$*, namely the Brauer group is a stable birational invariant for smooth varieties.*

PROOF. Birational invariance is proved in [**Grot-68**, Corollaire 7.3] for the cohomological Brauer group (which equals the Azumaya Brauer group for smooth projective varieties) and relies on existence of resolution of singularities. Showing that the Brauer group does not change replacing X with $X \times \mathbf{P}^n$ can be done using the Künneth formula ([**Mil-80**, Section 8]) for étale cohomology, with an argument which is essentially identical to the one done in Proposition 3.2.11. □

REMARK 3.2.13. For smooth, projective varieties X defined over a field k of characteristic $p > 0$, the torsion sub-groups $\mathrm{Br}(X)[m]$ for m coprime to p are stable birational invariants ([**Grot-68**, Corollaire 7.5]). In Chapter 5, we obtain birational invariance of the whole Brauer group for schemes defined over field of arbitrary characteristic thanks to a “purity” property proved by Gabber.

Note also that, for rationally connected, smooth varieties defined over $\mathbb{C}$, the Brauer group agrees with the torsion subgroup of the third singular cohomology group with $\mathbb{Z}$-coefficients (see later, Theorem 3.3.3).

It is known that neither the Brauer group not the torsion in third cohomology are invariant within flat families of varieties (see discussion in [**Vois-15a**, Theorem 1.6]), hence they appear to be of little use for an approach involving degeneration methods (for example, condition (3) in Strategy 2.3.8 might be inconclusive, since desingularising the special fibre might “destroy” the invariant). However, what we really need for the Brauer group is that it controls the Chow-theoretic invariants defined in Section 2.1 for smooth varieties.

Theorem 3.2.14. *Let X be a smooth, proper, k-variety which is* **UCT** *and let m be an integer coprime to* $\operatorname{char} k$. *Then the m-torsion subgroup of* $\mathrm{Br}(X)$ *is universally trivial; more precisely, the natural map* $\mathrm{Br}(F)[m] \longrightarrow \mathrm{Br}(X_F)[m]$ *is an isomorphism for every field extension F/k.*

Proof. The statement is an extract of [**CTP-16b**, Théorème 1.12]. $\square$

3.3. Residue maps and unramified cohomology.

Theorem 3.2.14 provides an effective way to obstruct universal CH_0-triviality for an arbitrary variety X: it is sufficient to prove that $\mathrm{Br}(\widetilde{X}) \neq 0$ for some desingularisation $\widetilde{X}$ of X.

However, in various applications, one is frequently given some highly singular model of X for which explicit desingularisation is not practical. Therefore, it is desirable to be able to determine $\mathrm{Br}(X)$ purely in terms of data associated with the function field $k(X)$. We have already seen in Theorem 3.2.7 that Brauer classes over X correspond to certain Brauer classes over the function field $k(X)$; now what remains to do is to determine which classes of $\mathrm{Br}(k(X))$ come from $\mathrm{Br}(X)$, by means of a valuation-theoretic criterion. In particular, we will show how to produce m-torsion classes in $\mathrm{Br}(X)$ for m coprime to the characteristic of the ground field k.

In the context of the degeneration method, following Strategy 2.3.8, this technique provides a concrete way to check condition (3). This technique does not apply to calculate p-torsion classes in $\mathrm{Br}(X)$ for $\operatorname{char} k = p$; this case, which is not covered in the literature, is particularly significant in the application of mixed-characteristic degenerations and will be developed separately in Chapters 4 and 5.

3.3.1. Gersten conjecture and residue maps. Let X be a scheme over a field k and let m be an integer invertible in k. Let μ_m denote the group scheme of m-th roots of unity on X; this is the base change to X of the group scheme

$$\mu_m = \mathrm{Spec}(\mathbb{Z}[x]/(x^m - 1))$$

with multiplication induced by the ring morphism $x \mapsto x \otimes x$. This group scheme represents a sheaf for the étale topology on X ([**Mil-80**, Corollary 1.7]), the locally constant sheaf that sends each étale open set U to the multiplicative groups of m-th roots of unity of $\mathscr{O}_X(U)$. Equivalently, we can define μ_m as the étale sub-sheaf of $\mathbf{G}_m$ forming the kernel of the m-th power map. For every integer j, we also define the étale sheaves

$$\mu_m^{\otimes j} := \begin{cases} \overbrace{\mu_m \otimes \cdots \otimes \mu_m}^{j \text{ times}} & \text{if } j > 0 \\ \underline{\mathbb{Z}/m} & \text{if } j = 0 \\ \mathrm{hom}(\mu_m^{\otimes(-j)}, \underline{\mathbb{Z}/m}) & \text{if } j < 0 \end{cases}$$

where $\underline{\mathbb{Z}/m}$ denotes the constant sheaf with stalk $\mathbb{Z}/m$ and $\mathrm{hom}(-,-)$ is in the category of étale sheaves.

Now let $Y \subseteq X$ be a closed sub-scheme of X; to any morphism $f\colon V \longrightarrow X$ and any integer $i \geq 0$ we associate the cohomology group with support on $f^{-1}(Y)$:

$$H^i_{f^{-1}(Y)}(V, \mu_m^{\otimes j})$$

(see [**Mil-80**, page 91-92] for the definition of cohomology of an étale sheaf with support). As f varies through étale morphisms, we obtain an étale presheaf on Y, whose sheafification we denote $\mathscr{H}^i_Y(\mu_m^{\otimes j})$. There is a spectral sequence (a variant of the local-global exact sequence, [**Grot-68**, Section 6]):

$$H^p_{\text{ét}}(Y, \mathscr{H}^q_Y(\mu_m^{\otimes j})) \Rightarrow H^{p+q}_Y(X, \mu_m^{\otimes j}).$$

This spectral sequence occupies a central role in the study of Gersten conjecture (see [**CT-95**] for a complete exposition), the area from which the techniques explainned in this section originate.

Assume that X and Y are regular and Y has pure codimension $c \geq 1$; if the cohomological purity conjecture holds for Y (see [**CT-95**, Conjecture 3.2.1]), the above spectral sequence degenerates and, recalling the isomorphism $\mathscr{H}^i_Y(\mu_m^{\otimes j}) \simeq \mu_m^{\otimes(j-c)}$ ([**CT-95**, beginning of Section 3.2], one gets a Gysin exact sequence

$$\begin{aligned}(3.3.1)\quad \cdots \to H^i_{\text{ét}}(X, \mu_m^{\otimes j}) \to H^i_{\text{ét}}(U, \mu_m^{\otimes j}) \to H^{i+1-2c}_{\text{ét}}(Y, \mu_m^{\otimes(j-c)}) \to \\ \to H^{i+1}_{\text{ét}}(X, \mu_m^{\otimes j}) \to \cdots\end{aligned}$$

where U is the complement of Y in X (see again *ibidem*). Cohomological purity is known in a variety of situations (see [**CT-95**, Chapter 3] for a complete treatment), in particular it holds for regular schemes of dimension 1 ([**Grot-68**, Théorème 6.1]).

Let X be the spectrum of a discrete valuation ring A with residue field $\kappa = A/\mathfrak{m}_A$ and fraction field K. Choosing $Y = \{\mathfrak{m}_A\}$ leads to $U \simeq \operatorname{Spec}(K)$ and sequence (3.3.1) reads as follows:

(3.3.2)

$$\cdots \to H^i_{\text{ét}}(A, \mu_m^{\otimes j}) \to H^i(K, \mu_m^{\otimes j}) \xrightarrow{\partial^i_A} H^{i-1}(\kappa, \mu_m^{\otimes(j-1)}) \to H^{i+1}(A, \mu_m^{\otimes j}) \to \cdots$$

for all integers $i, j > 0$. Here we have identified étale cohomology of a spectrum of a field with the Galois cohomology of the field and, with a slight abuse of notation, for any field F we have still denoted μ_m the continuous module $\mu_m(F)$, the group of m-th roots of unit in a separable closure F_{sep}. The map

$$\partial^i_A : H^i(K, \mu_m^{\otimes j}) \longrightarrow H^{i-1}(\kappa, \mu_m^{\otimes j})$$

is called *i-th residue map along A.*

We do not proceed further into the general constructions of these maps; however, they can be expressed in terms of Galois symbol ([**GS-06**, Section 4.6]): by Bloch-Kato conjecture ([**GS-06**, Conjecture 4.6.5]), elements in $H^i(K, \mu_m^{\otimes j})$ come from cup products $a_1 \smile \cdots \smile a_i$ of symbols $a_h \in$

$H^1(K, \mu_m)$ modulo the Tate relations ([GS-06, Proposition 4.6.1]). One has the following formula for residues of certain 2-fold cup products.

PROPOSITION 3.3.1. ([CTO-89, Proposition 1.3]) *Let A be a discrete valuation ring, with fraction field K and residue field κ; assume that m is an integer invertible in A. Let $\alpha \in H^i_{\text{ét}}(A, \mu_m^{\otimes j})$ and let $\alpha_0 \in H^i(\kappa, \mu_m^{\otimes j})$ be the image of α under the corestriction map induced by $A \longrightarrow \kappa$. Let $b \in K^\times$ and let $\beta \in H^1(K, \mu_m) \simeq K^\times/(K^\times)^m$ be its associated class. Then, for the cup product $\alpha \smile \beta \in H^{i+1}(K, \mu_m^{(j+1)})$ we have*

$$\partial_A^{i+1}(\alpha \smile \beta) = v_A(b) \cdot \alpha_0 \in H^i(\kappa, \mu_m^{\otimes j})$$

where v_A is the valuation map of A.

Several different definitions of residue maps are scattered throughout the literature, each one adapted to a specific context; depending on the source, residue maps can actually differ by a sign.

Nevertheless, the Gersten conjecture guarantees that all of these maps have the same kernel: by [CT-95, Section 3.5] and purity [CT-95, Section 3.2], the morphisms $H^{i-1}(\kappa, \mu_m^{\otimes(j-1)}) \longrightarrow H^{i+1}(A, \mu_m^{\otimes j})$ in sequence (3.3.2) are the zero maps, so for all $i > 0$ we have short exact sequences

$$0 \longrightarrow H^i_{\text{ét}}(A, \mu_m^{\otimes j}) \longrightarrow H^i(K, \mu_m^{\otimes j}) \xrightarrow{\partial_A^i} H^{i-1}(\kappa, \mu_m^{\otimes(j-1)}) \longrightarrow 0. \tag{3.3.3}$$

Let us introduce the following terminology: we say that $\alpha \in H^i(K, \mu_m^{\otimes j})$ is *unramified with respect to* a discrete, rank 1 valuation v of K with valuation ring A, if α belongs to the image of the natural map $H^i(A, \mu_m^{\otimes j}) \to H^i(K, \mu_m^{\otimes j})$. In virtue of (3.3.3), one can characterise unramified elements in $H^i(X, \mu_m^{\otimes j})$ as those classes α such that $\partial_A^i(\alpha) = 0$.

It is then natural to give the following definition.

DEFINITION 3.3.2. Let k be a field and let X be a smooth, complete, integral k-variety. Let m be an integer coprime to $\operatorname{char} k$ and let DISCR be the set of all discrete, rank 1 valuations on $k(X)$ which are trivial on k; for each $v \in \mathsf{DISCR}$ we denote A_v the valuation ring of v and $k(v)$ its residue field. Then, the *i-th unramified cohomology group of X/k with coefficients in $\mu_m^{\otimes j}$* is defined as

$$H^i_{\text{nr}}(k(X)/k, \mu_m^{\otimes j}) := \bigcap_{v \in \mathsf{DISCR}} \ker(\partial^2_{A_v})$$

where $\partial^2_{A_v} : H^2(k(X), \mu^{\otimes j}) \longrightarrow H^1(k(v), \mu^{\otimes(j-1)})$ is the residue map introduced in (3.3.3) with $i = 2$, $K = k(X)$ and $\kappa = k(v)$.

Note that the definition of unramified cohomology depends on the ground field k, as we require all our valuations to fix k; the choice of the notation aims to emphasise this dependence and is not related to the similar notation for the Brauer group of field extensions used in Section 3.1. It is also immediate to verify that unramified cohomology groups are birational invariants in X, as they depend only on its function field. We also have the following important result. One can also describe elements in $H^i_{\mathrm{nr}}(k(X)/k, \mu_m^{\otimes j})$ as those classes in $H^i(k(X), \mu_m^{\otimes j})$ which are unramified with respect to all discrete valuations in DISCR. See also [**CT-95**, Theorem 4,1,1] for other equivalent definitions of unramified cohomology that can be deduced from the sequence (3.3.3).

We will be interested in unramified cohomology groups of degree 2, as the following result explains.

PROPOSITION 3.3.3. ([**CT-95**, Proposition 4.2.3]) Let k be a field and let X be a smooth, projective variety.

(1) *If m is an integer coprime to* $\operatorname{char} k$*, then the isomorphism inducing* $H^2(k(X), \mu_m) \simeq \mathrm{Br}(k(X))[m]$ *as in Proposition 3.1.16 induces an isomorphism*
$$H^2_{\mathrm{nr}}(k(X)/k, \mu_m) \simeq \mathrm{Br}(X)[m].$$
(2) *If k is algebraically closed and m is power of a prime number $\ell \neq$* $\operatorname{char} k$*, then*
$$H^2_{\mathrm{nr}}(k(X)/k, \mu_m) \simeq (\mathbb{Z}/m)^{B_2 - \rho_X} \oplus H^3_{\text{ét}}(X, \mathbb{Z}_\ell)[m].$$
where $B_2 = \operatorname{rk} H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ *and* $\rho_X = \operatorname{rk}(\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$ *is the ℓ-adic Picard number.*

REMARK 3.3.4. Statement (2) was already known to Grothendieck (see [**Grot-68**, Section 6]). Now let us compare it with Proposition 3.2.11: in particular, if k is algebraically closed of characteristic 0, and X is rationally connected, then it can be shown ([**Nyg-78**]) that $B_2 = \rho_X$. Thus, the only possible non-trivial elements in the Brauer group come from the torsion subgroup of $H^3_{\text{ét}}(X, \mathbb{Z}_\ell)$. This shows how unramified cohomology is, morally, a generalised version of the invariant employed in [**AM-72**].

REMARK 3.3.5. Sometimes, one calls $H^2_{\mathrm{nr}}(k(X)/k, \mu_m)$ the *m-torsion part of the unramified Brauer group* of X and denotes it $\mathrm{Br}_{\mathrm{nr}}(k(X))[m]$.

3.3.2. Residue maps in low degrees. Let us focus our attention on on the construction of residue maps in the case $m = 2, i = 1, 2$, putting them in the geometric context of Definition 3.3.2. We repeat that our definitions agree with those given in [**Rost-96**], but we do not need such degree of generality.

Let X be an integral k-variety, with $\operatorname{char} k \neq 2$ and let $K = k(X)$ be its function field. Since k contains both 2-roots of unity, we can identify $\mu_2^{\otimes j} \simeq \mathbb{Z}/2$ as Galois modules for all $j > 0$ (in general, $\mu_m \simeq \mathbb{Z}/m$ holds if the ground field contains a primitive root of unity). Kummer theory (Lemma 3.1.17) induces an isomorphism
$$H^1(K, \mathbb{Z}/2) \simeq K^\times/(K^\times)^2.$$
Similarly, by Proposition 3.1.16,
$$H^2(K, \mathbb{Z}/2) \simeq \mathrm{Br}(K)[2]. \tag{3.3.4}$$

Suppose D is a prime divisor in X, such that X is regular at the generic point of D. Note that D corresponds to an unique divisorial valuation v_D over the field $K = k(X)$, with residue field $k(D)$, the function field of D, and discrete valuation ring exactly $A = \mathscr{O}_{X,\eta_D}$ the local ring of X at the generic point of D. We want to define the group morphisms

$$\partial_D^1 : H^1(K, \mathbb{Z}/2) \longrightarrow H^0(k(D), \mathbb{Z}/2) = \mathbb{Z}/2$$
$$\partial_D^2 : H^2(K, \mathbb{Z}/2) \longrightarrow H^1(k(D), \mathbb{Z}/2)$$

that we have called ∂_A^i in the previous paragraph. Let us proceed in the following way: for each $a \in K$, denote $\bar{a} \in H^1(K, \mathbb{Z}/2) = K^\times/(K^\times)^2$ and define

$$\partial_D^1(\bar{a}) := v_D(a) \mod 2.$$

Then, let $\alpha \in H^2(K, \mathbb{Z}/2)$; according to isomorphism (3.3.4) and Theorem 3.1.11, α can be represented by tensor product of quaternion algebras; therefore, let (a, b) be such an algebra, for some $a, b \in K$ with $b \neq 0$. Following Proposition 3.3.1 for $i = 1$ and $A = \mathscr{O}_{X,\eta_D}$, one gets the following expression

$$\partial_D^2(a, b) := (-1)^{v_D(a)v_D(b)} \left(a^{v_D(b)} b^{-v_D(a)}\right)_D \tag{3.3.5}$$

where $(-)_D$ indicates the class modulo squares in $k(D)^\times/(k(D)^\times)^2$. One could check that these two maps fit into exact sequence (3.3.2) above.

If X is not regular at the generic point of D, we also have an alternative description of ∂_D^1. In this case, the local ring $\mathscr{O}_{X,\eta_D}$ is not necessarily a discrete valuation ring. Suppose $\nu : \widehat{X} \longrightarrow X$ is the normalisation and suppose $D_1, \ldots, D_s$ are the irreducible components lying over D. Then each D_i defines a discrete divisorial valuation over $k(\widehat{X}) = K$ with residue field $k(D_i)$ and we define, for each $a \in L^\times$

$$\partial_D^1(\bar{a}) := \sum_{i=1}^{s} |k(D_i) : k(D)| v_{D_i}(a) \mod 2 \tag{3.3.6}$$

where $|k(D_i) : k(D)|$ is the degree of the extension induced by the dominant morphism $D_i \to D$ obtained by restricting v.

3.3.3. Purity and reciprocity sequence. Keeping notation and setting from the previous paragraph, we will now explain how one can calculate $H^2_{\mathrm{nr}}(k(X)/k, \mathbb{Z}/2)$ without having to check the triviality of residue maps along *all* discrete valuations. This kind of results is known as a “purity property”.

Corollary 3.3.6. *Suppose X is a smooth variety over a field k such that* $\mathrm{char}(k) \neq 2$, *let $\mathscr{O}$ be the local ring at some point of X. Then:*

(1) *every class $\alpha \in H^2(k(X), \mathbb{Z}/2)$ which is unramified with respect to all divisorial valuations corresponding to height 1 prime ideals in $\mathscr{O}$ is unramified with respect to all valuations with centre on* $\mathrm{Spec}(\mathscr{O})$*;*

(2) *every class $\alpha \in H^2(k(X), \mathbb{Z}/2)$ which is unramified with respect to divisorial valuations corresponding to prime divisors on X is unramified with respect to all divisorial valuations with centres on X.*

PROOF. The local statement in (1) is proved directly in [**CT-95**, Theorem 3.8.2], while the geometric statement (2) can be found at [**CT-95**, Proposition 2.1.8(d)]. $\square$

In practice, one applies statement (2) of Corollary 3.3.6 in the following way. Assume we are given a singular, projective variety Y such that the singular locus Y_{sing} has small codimension; then $X := Y \setminus Y_{\text{sing}}$ is a smooth variety birational to Y and to check that $\alpha \in H^2(k(Y), \mathbb{Z}/2)$ is unramified with respect to all valuations over $k(Y)$ it is enough to check α is unramified with respect to divisorial valuations corresponding to prime divisors on X.

We now give, without proof, two results that are useful to manipulate residue maps in practice: the first one is a local comparison tool for residue maps, while the second one a sort of "local-global" principle.

LEMMA 3.3.7. ([**CT-95**, Proposition 3.3.1]) *Let $A \subseteq B$ be discrete valuation rings, with fields of fractions $K \subseteq L$ respectively and let $\pi_A \in A$, $\pi_B \in B$ two uniformisers. Call k_A and k_B the residue fields of A and B respectively and let $e_{B/A} := v_B(\pi_A)$, where v_B is the valuation map of B. Then we have the following commutative diagram:*

$$\begin{tikzcd}
H^i(L, \mathbb{Z}/2) \arrow[r, "\partial^i_{v_B}"] & H^{i-1}(k_B, \mathbb{Z}/2) \\
H^i(K, \mathbb{Z}/2) \arrow[r, "\partial^i_{v_A}"] \arrow[u, "i^*_{L/K}"] & H^{i-1}(k_A, \mathbb{Z}/2) \arrow[u, "e_{B/A} i^*_{k_B/k_A}"']
\end{tikzcd}$$

where i^* on both columns denotes the pull-back in Galois cohomology.

PROPOSITION 3.3.8. *Let k be an algebraically closed field with* $\operatorname{char} k \neq 2$ *and let B be a smooth algebraic surface over k. Suppose that $H^1_{\text{ét}}(B, \mathbb{Z}/2) = 0$; then the following sequence is exact:*

$$0 \longrightarrow \operatorname{Br}(B)[2] \longrightarrow \operatorname{Br}(k(B))[2] \xrightarrow{\partial} \bigoplus_{D \in B^{(1)}} H^1(k(D), \mathbb{Z}/2) \xrightarrow{s}$$
$$\longrightarrow \bigoplus_{p \in B^{(2)}} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

where the first arrow is induced by the restriction to the generic point of B, the sets $B^{(i)}$ consist of i-codimensional points of B, the last arrow is the sum, while

$$\partial(\alpha) := (\partial^2_D(\alpha))_{D \in B^{(1)}}$$

and

$$s((f_D)_{D \in B^{(1)}}) := \left(\sum_{D \in B^{(1)}} \partial^1_p(f_D) \right)_{p \in B^{(2)}}.$$

Remark 3.3.9. See [**AM-72**, Theorem 1] for a self-contained proof. The sequence is actually a 2-dimensional version of the local-global exact sequence for the Brauer group ([**GS-06**, Theorem 6.5.1]), which is the function field analogue of the Hasse's principle for class field theory. Both sequences can be seen as special case of a more general construction: they are precisely some proved parts of a conjecture by Kato, concerning the exactness of a Bloch-Ogus type complex for étale cohomology. This complex was already considered by Grothendieck ([**Grot-64**, vol. III, p. 165]) and was discussed by Bloch and Ogus in [**BO-74**] with greater details. A more explicit construction, which avoids the use of Grothedieck's duality, and a proof of the conjecture was then presented in [**Kat-86**]. Compare also with [**CT-93**] for higher-dimensional analogues of this sequence, in the case of number fields.

3.4. Conic bundles.

Throughout this section, we will work with algebraically closed fields of characteristic different than 2.

3.4.1. Definitions and properties. Here we introduce the notion of *conic fibration*, which we will distinguish from the concept of *conic bundle* on a first instance.

Definition 3.4.1. Let X be a projective k-variety and let B be a smooth, projective k-variety. A *conic fibration* is a flat, projective morphism $\pi : X \longrightarrow B$ with every geometric fibre isomorphic to a plane conic and with smooth geometric generic fibre.

The following definitions explain how most conic fibrations will be presented in practice.

Definition 3.4.2. Let $\mathscr{E}$ be a vector bundle of rank 3 over B and let $\mathscr{L}$ be a line bundle over B. We define a *$\mathscr{L}$-valued quadratic form* as a map $q : \mathscr{E} \longrightarrow \mathscr{L}$ such that, for each $x \in B$, the induced maps on the fibres

$$q_x : \mathscr{E}_x/\mathfrak{m}_x \longrightarrow \mathscr{L}_x/\mathfrak{m}_x$$

are non-zero quadratic forms.

Remark 3.4.3. Note that by hypothesis $\mathscr{E}_x$ is a 3-dimensional k-vector space and $\mathscr{L}_x$ is a 1-dimensional k-vector space, so the datum of a $\mathscr{L}$-valued quadratic form amounts to assign an ordinary quadratic form to each point of B. It is meaningful to consider the set of zeros of q in the total space of $\mathscr{E}$:

$$\mathscr{C}_q := \{p \in \mathbf{P}(\mathscr{E}) \mid q(p) = 0\}$$

where we set $\mathbf{P}(\mathscr{E}) := \mathbf{Proj}\,\mathscr{S}ym^\bullet(\mathscr{E}^\vee)$, as the global Proj of $\mathscr{S}ym^\bullet(\mathscr{E}^\vee)$, the sheaf associated to the presheaf of graded $\mathscr{O}_B$-modules $U \mapsto \mathrm{Sym}^\bullet(\mathscr{E}^\vee(U))$.

DEFINITION 3.4.4. A *conic bundle* over B is the data of the variety $\mathscr{C}_q$ together with the projective morphism $\pi : \mathscr{C}_q \longrightarrow B$ obtained by restricting the canonical projection $\pi : \mathbf{P}(\mathscr{E}) \longrightarrow B$.

Note that for any k-point $x \in B$ the fibre $\pi^{-1}(x)$ corresponds to a quadric curve cut out by the non-zero quadratic form q_x on the k-vector space $\mathscr{E}_x$. More precisely, $\pi^{-1}(x)$ is the set of the zeros of q_x in

$$\mathbf{P}(\mathscr{E})|_{\pi^{-1}(x)} \simeq \operatorname{Proj}\operatorname{Sym}^\bullet(\mathscr{E}_x^\vee).$$

Since $\dim \mathscr{E}_x = 3$, it follows that that $\pi^{-1}(x)$ is a conic in $\mathbf{P}^2_k$. In other words, a conic bundle defines a conic fibration.

REMARK 3.4.5. The non-vanishing hypothesis for q guarantees that *all* fibres have dimension 1. Dropping this hypothesis would allow the presence of fibres isomorphic to copies of $\mathbf{P}^2$, hence the morphism π would not be flat.

It can be shown, however, that the notion of conic fibration and conic bundle are equivalent.

PROPOSITION 3.4.6. *Let $f \colon X \longrightarrow B$ a conic fibration, where X and B are smooth, projective k-varieties. Then there exist a locally free sheaf $\mathscr{E}$ on B such that Y can be identified with the zero locus in $\mathbf{P}(\mathscr{E})$ of the global section*

$$\sigma \in H^0(\mathbf{P}(\mathscr{E}), \mathscr{M}^{\otimes 2} \otimes p^*(\mathscr{L}))$$

where $\mathscr{M} := \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$, $\mathscr{L} = \det(\mathscr{E}^\vee) \otimes \omega_B^{-1}$ and $p : \mathbf{P}(\mathscr{E}) \longrightarrow B$ is the natural projection.

PROOF. See [**Sar-83**, Paragraph 1.5] and [**Beau-77**, Proposition 1.2]. More precisely, one can choose $\mathscr{E} := f_*\omega_X^{-1}$ where ω_X is the canonical sheaf. Furthermore, there is an isomorphism

$$H^0(\mathbf{P}(\mathscr{E}), \mathscr{M}^{\otimes 2} \otimes p^*(\mathscr{L})) \simeq H^0(B, \mathscr{S}\!ym^2(\mathscr{E}) \otimes \mathscr{L})$$

which means that one can see X as the zero locus of a $\mathscr{L}$-valued quadratic form. $\square$

REMARK 3.4.7. The same holds without regularity assumptions on X; in this case it is enough to replace the canonical sheaf ω_X with the relative dualising sheaf $\omega_{X/B}$; it is an invertible $\mathscr{O}_X$-module in this case, since f is a flat family with 1-dimensional, Gorenstein fibres.

Given a conic bundle $\pi : X \longrightarrow B$ we define the *discriminant locus* Δ to be the sub-variety of B consisting of points $p \in B$ such that the scheme-theoretic fibre X_p is singular. It is also possible to give a description of Δ as vanishing locus of some sheaf section ([**Sar-83**, Definition 1.6]).

Here we state some properties of the discriminant locus in the special case of conic bundles with smooth total space. This will not be our ideal working situation, however, so we state the following result for reference purposes only.

Proposition 3.4.8. *Let $\pi : X \longrightarrow B$ be a conic bundle, where X is a smooth k-variety and B is a smooth, projective k-variety, and let Δ be the discriminant locus. Then:*

(1) *Δ has at worst ordinary quadratic singularities;*
(2) *for each $p \in B \setminus \Delta$, the fibre X_p is smooth;*
(3) *if $p \in \Delta$ is non-singular, the fibre X_p has exactly one singular point;*
(4) *if $p \in \Delta$ is singular, the fibre X_p has a whole line of singular points.*

Proof. See [**Beau-77**, Proposition 1.2]. □

3.4.2. Geometric description of residue maps. In this paragraph we will concentrate on the geometric meaning of residue maps in the case of conic bundles.

Let k be an algebraically closed field and let $\pi : X \longrightarrow B$ be a conic bundle over a smooth, projective variety B. Notice that the generic fibre of π is a smooth conic C over $k(B)$, up to isomorphism cut out in $\mathbf{P}^2_{k(B)}$ by an equation of the form

$$aX^2 + bY^2 - Z^2 = 0$$

with $a, b \in k(B)^\times$. Following the reasoning in Paragraph 3.1.1, we can canonically attach to C the quaternion $k(B)$-algebra (a, b) so that C becomes the associated conic of (a, b). Furthermore, one can associate to C a Brauer class $\alpha_\pi \in \mathrm{Br}(k(B))$ of order 2, which is the Brauer class of (a, b). More generally, it is possible to prove that conic bundles are in one-to-one correspondence with *maximal orders* on the base B; this correspondence is described in [**AM-72**, Section 4] in the case B is a surface and has been generalised in [**Sar-83**]. However, the main idea is due to Grothendieck ([**Grot-68**]): letting Δ be the discriminant locus and $U = B \setminus \Delta$, then $\pi|_U : \pi^{-1}(U) \longrightarrow U$ corresponds to a quaternion Azumaya algebra $\mathscr{A}$ over U, hence to a Brauer class in $\mathrm{Br}(U)$, which by Theorem 3.2.7 injects into $\mathrm{Br}(k(U)) = \mathrm{Br}(k(B))$.

Recall that residue maps can be defined on B following Paragraph 3.3.2; we give the following definition.

Definition 3.4.9. Denote by $B^{(1)}$ the set of all valuations of $k(B)$ corresponding to prime divisors on B. Let $\pi \colon X \longrightarrow B$ be a conic bundle and let $\alpha_\pi \in \mathrm{Br}(k(B))[2]$ be its associated Brauer class. We call *residue profile* of π the family $(\alpha_x)_{x \in B^{(1)}}$ such that:

$$(\alpha_v = \partial^2_x(\alpha_\pi))_{v \in B^{(1)}} \in \bigoplus_{x \in B^{(1)}} k(v)^\times / (k(v)^\times)^2$$

where $k(v)$ is the residue field of v.

Remark 3.4.10. Every residue profile fits into the reciprocity sequence (Proposition 3.3.8) and the sequence can be used to construct conic bundles with prescribed residue profile; more precisely, given a smooth, projective variety B and a residue profile $(\alpha_v)_{v \in B^{(1)}}$ that fits into the reciprocity sequence, there exists a Brauer class $\alpha \in \mathrm{Br}\, k(B)[2]$. By Theorem 3.1.11, the class α must be tensor product of some quaternion algebras; in general, α might be represented by algebras of degree 2^d for some $d > 1$ but in some notable cases α is actually the class of a quaternion algebra (this happens, for example, if B is an algebraic surface, see [**deJo-04**]). Finally, this quaternion algebra defines a conic bundle $\pi : X_U \longrightarrow U$ where U is an open subset of X, following Grothendieck's correspondence ([**AM-72**, Section 4, initial discussion]).

We will now characterise the residue profile entirely in terms of the geometry of the discriminant locus. We first introduce some notation. Given a conic bundle $\pi : X \longrightarrow B$ and a reduced prime divisor D, assume that above the generic point of D the fibre of π is geometrically a union of two distinct lines. Then $\pi|_{\pi^{-1}(D)} : X \times_B D \longrightarrow D$ defines a double covering of an open subset of D, which can be identified with a (Galois) extension of degree 2 $L/k(D)$. We call *associated double covering* of D such double covering defined by $\pi|_{\pi^{-1}(D)}$.

Now, for any prime divisor D, one has a residue map $\partial^2_D : H^2(k(B), \mathbb{Z}/2) \longrightarrow k(D)^\times/(k(D)^\times)^2$ as defined in (3.3.5).

Proposition 3.4.11. *Let k be an algebraically closed field and let $\pi : X \longrightarrow B$ be a conic bundle with $\alpha_\pi \in \mathrm{Br}(k(B))[2]$ being the associated Brauer class. Let D be a prime divisor over B such that the fibre over a general point of D consists of two distinct lines. Let $\overline{X_D}$ be the geometric generic fibre of the associated double covering of D. Then:*

(1) *if $\overline{X_D}$ is a smooth conic, then $\partial^2_D(\alpha_\pi) = 0$;*
(2) *if $\overline{X_D}$ is a singular conic, then D is an irreducible component of the discriminant locus and the associated double covering of D is a quadratic extension $k(D)(a_D^{1/2})/k(D)$ for some $a_D \in K(D)$. Then*
   (a) *if D has even multiplicity, $\partial^2_D(\alpha_\pi) = 0$;*
   (b) *if D has odd multiplicity, $\partial^2_D(\alpha_\pi) = [a_D] \in k(D)^\times/(k(D)^\times)^2$.*

Proof. Let D be a divisor in B and let $P \in D$ be a point such that the fibre X_P is a cross of line; then, Zariski locally around P, the conic bundle is defined by the vanishing in $\mathbf{P}^2$ (with fibre coordinates x, y, z) of a quadratic form associated to a symmetric matrix

$$M_P = \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{xy} & a_{yy} & a_{yz} \\ a_{xz} & a_{yz} & a_{zz} \end{pmatrix}$$

with entries being regular function in $\mathscr{O}_{B,P}$. Since X_P is reduced, after a suitable coordinate change in x, y, z (with coefficients in $k(P)$) one can assume that all the mixed terms are 0 and at least two amongst a_{xx}, a_{yy}, a_{zz} are equal to 1 at P; without loss of generality, assume $a_{xx}(P) = 1 = a_{zz}(P)$. Then one has that a_{xx}, a_{zz} are units in $\mathscr{O}_{B,P}$ and the same coordinate change with coefficients in $\mathscr{O}_{B,P}$ yields a normal form of M_P as $\mathrm{diag}(a, b, -c)$ where

a, c are units in $\mathcal{O}_{B,P}$ and $b = u \cdot \xi^m$ where $u \in \mathcal{O}_{B,P}^\times$ and ξ is a local equation for D in $\mathcal{O}_{B,P}$.

In particular, above the generic point of D, the conic bundle X is defined by an equation of the form

$$ax^2 + by^2 - cz^2 = 0$$

where a, b are units in $\mathcal{O}_{B,\eta_D}$ and $b = u \cdot \xi^m$ for some unit u and ξ a local equation for D. Notice first that after dividing by c, we can assume $c = 1$. Hence, we can identify X above the generic point of D with the symbol $(a, b) \in \mathrm{Br}(k(B))[2]$, representing the quaternion algebra associated to the conic cut out by $ax^2 + by^2 - z^2 = 0$.

(1) If $\overline{X_D}$ is smooth, then D is not a discriminant component, and in the above matrix M_P the entry a_{yy} is a unit locally around P as well, hence $b \in \mathcal{O}_{B,\eta_D}^\times$. By the residue formula ((3.3.5), it follows

$$\partial_D^2(a, b) = a^{v_D(b)} b^{-v_D(a)} = 1$$

which is the trivial class in $k(D)^\times/(k(D)^\times)^2$.

(2) If, on the contrary, $\overline{X_D}$ is singular, then D is a discriminant component and a_{yy} is not a unit locally around P. Hence $b = u \cdot \xi^m$ for some unit u and some integer m. We have the following two cases:
   (a) if D has even multiplicity, then $m = 2m'$ is even and performing the coordinate change $y \mapsto \xi^{m'} y$ we have that $b \in \mathcal{O}_{B,\eta_D}^\times$, hence again $\partial_B^2(a, b)$ is trivial;
   (b) if D has odd multiplicity, then m is odd and, after absorbing even powers of ξ into the coordinate y as before, we can assume ξ to be a local parameter for $\mathcal{O}_{B,\eta_D}$ and we have $b = u \cdot \xi$, hence

$$\partial_D^2(a, b) = \partial_D^2(a, \xi) = [a] \in k(D)^\times/(k(D)^\times)^2$$

   which is the claim; indeed, in our description, the associated double covering of $D = \{b = 0\}$ is defined by

$$ax^2 - z^2 = 0$$

   In the affine chart $x = 1$, one retrieves the equation of the covering $z^2 = a$, which corresponds to the Kummer extension $k(D)(a^{1/2})$ as wished.

□

The main consequence of Proposition 3.4.11 is that residue profiles of conic bundles are determined by the geometry of the discriminant locus; if, indeed, D is a divisor not contained in the discriminant locus of a conic bundle $\pi : X \longrightarrow B$, then the geometric generic fibre of π above D is a smooth conic and, thus, $\partial_D^2(\alpha_\pi) = 0$. This suggests a more geometric reformulation of Remark 3.4.10: the reciprocity sequence allows to deduce existence of conic bundles with prescribed discriminant locus, Indeed, for a finite family of reduced divisors $\{\Delta_i\}_{i \in I}$ on B and non-square rational functions $\alpha_i \in k(\Delta_i)^\times$ it is enough that $\partial^1((\bar{\alpha}_i)_{i\in I}) = 0$ (where $\bar{\alpha}_i$ is the class of α_i modulo squares) to determine a conic bundle $\pi : X \longrightarrow B$ with discriminant locus being the union of the Δ_i and residue profile being exactly $(\bar{\alpha}_i)_{i\in I}$.

3.4.3. Formulae for unramified cohomology. We are finally ready to present some formulae that compute the unramified cohomology of a projective variety over a field of characteristic $\neq 2$ which admits a conic bundle structure over a smooth, projective base variety.

Let us put ourselves in the viewpoint of checking condition (3) in Strategy 2.3.8, for the case in which the special fibre has a conic bundle structure $\pi : Y \longrightarrow B$: we have seen that only a non-trivial Brauer class in $\mathrm{Br}(Y)$ is needed (see indeed Theorem 3.2.14). Now, Proposition 3.3.3 implies that determining non-trivial 2-torsion classes in the Brauer group of Y can be accomplished by computing the unramified cohomology group $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$ whose elements, in turn, can be singled out from those of $\mathrm{Br}(k(Y))[2] = H^2(k(Y), \mathbb{Z}/2)$ by looking at residues that vanish at divisorial valuations on $k(Y)$ with centre a prime divisor on Y (Proposition 3.3.6). Moreover, the sequence in Proposition 3.3.8 and the local-global comparison Lemma 3.3.7, move the core of the residue computations from Y to B; and this is convenient, since we have a geometric characterisation for the vanishing of residue maps along divisors in the base space B, provided by Proposition 3.4.11, which essentially limits the valuations we need to check to those which have centre on a component of the discriminant.

It has to be noted that several techniques for the computation of non-trivial elements in the Brauer group of certain conic bundles appear to have been known in various forms by experts since decades. However, it is difficult to find precise statements of these formulae with the desired geometric setting. We have then endeavoured to re-arrange, in line with our formalism, the existing material scattered in the literature.

We start by recalling a result obtained by Colliot-Thélène and Ojanguren which takes into account the Artin-Mumford example with the unramified cohomology formalism and extends the same technique to “similar” conic bundles.

Proposition 3.4.12. *Let $\pi : Y \longrightarrow \mathbf{P}^2 := \mathbf{P}^2_{\mathbb{C}}$ be a conic bundle and let $L := \mathbb{C}(x, y)$ be the function field of $\mathbf{P}^2$. Suppose that the generic fibre of π is a conic cut out in $\mathbf{P}^2_L$ by an equation of the form*

$$fX_1^2 + g_1g_2X_2^2 - X_0^2 = 0$$

where $f, g_1, g_2 \in L$ are non-zero rational functions. Assume, moreover, that

(1) *there exist prime divisors D_1, D_2 on $\mathbf{P}^2$ such that $\partial^2_{D_i}(f, g_i) \neq 0$ for $i = 1, 2$;*
(2) *for every prime divisor D in $\mathbf{P}^2$ either $\partial^2_D(f, g_1) = 0$ or $\partial^2_D(f, g_2) = 0$;*

(3) *for every point* $p \in \mathbf{P}^2$*, at least one of the functions* f, g_1 *or* g_2 *is a square in* $\mathscr{O}_{\mathbf{P}^2,p}$*.*

Then the image of $(f, g_1) \in \mathrm{Br}(L)[2]$ *in* $H^2(\mathbb{C}(Y), \mathbb{Z}/2)$ *is non-trivial and unramified, namely* $H^2_{\mathrm{nr}}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2)[2] \neq 0$*.*

Proof. This is a reformulation of [**CTO-89**, Proposition 2.1] together with [**CTO-89**, Assertion 2.1.1] and [**CTO-89**, Complément 2.2]. The statement is originally given in terms of discrete valuation rings and the original conclusion is given in terms of function fields of the generic fibre. □

Remark 3.4.13. Condition (1) above can be rephrased in a more geometric fashion. The discriminant locus of the conic bundle in hypothesis is $\Delta = \{fg_1g_2 = 0\} \subseteq \mathbf{P}^2$ and let $C_i = \{g_i = 0\}$ for $i = 1, 2$ be the two subvarieties defined by g_i (note that these might be reducible but they must be distinct by hypothesis (2)). Now let D_1, D_2 be two irreducible components of C_1 and C_2 respectively. Then condition (1) is satisfied if each associated double covering to D_i is irreducible. Indeed, if such is the case, then

$$0 \neq \partial^2_{D_i}(f, g_1g_2) = f^{v_{D_i}(g_i)} = \partial^2_{D_i}(f, g_i)$$

as $v_{D_i}(f) = 0$ and $v_{D_i}(g_1g_2) = v_{D_i}(g_i)$ since g_j does not vanish on D_i if $i \neq j$.

Notice also that condition (3) is nothing but requiring that the residue profiles (f, g_i) fit into the reciprocity sequence shown in Proposition 3.3.8 by satisfying condition $s(f, g_i) = 0$.

Remark 3.4.14. Proposition 3.4.12 aims to address stable irrationality for conic bundles modelled on the Artin-Mumford example ([**AM-72**]) and can be used to produce some further examples ([**CTO-89**, Exemple 2.3, 2.4]). It is worth mentioning that in the same work ([**CTO-89**, Section 3]) such formula was generalised to conic bundles over rational threefolds by employing residues of higher order.

A closed formula for unramified cohomology of the total space can be found by rewriting the conditions of Proposition 3.4.12 in a group-theoretic fashion. This result is attributed to Jean-Louis Colliot-Thélène in [**Pir-16**], but we were unable to find an explicit proof in the literature.

Theorem 3.4.15. *Let* B *be a smooth, projective rational surface over an algebraically closed field* k *of characteristic* 0 *and let* $L := k(B)$*; let moreover* Y *be a 3-fold equipped with a conic bundle structure* $\pi : Y \longrightarrow B$*. Suppose* $\alpha \in \mathrm{Br}(K)[2], \alpha \neq 0$ *is the Brauer class corresponding to the conic bundle. Assume:*

(1) *the discriminant locus* $\Delta \subseteq B$ *has at worst ordinary quadratic singularities;*
(2) Δ *decomposes into reduced components* $\Delta_1, \ldots, \Delta_n$ *for* $n > 1$*;*
(3) *the fibre of* π *above a general point of* Δ_i *consists of two distinct lines;*

(4) *each associated double covering* $\pi^{-1}(\Delta_i) \longrightarrow \Delta_i$ *is non-split.*

Consider the subgroup

$$H := \left\{ (\sigma_1, \ldots, \sigma_n) \in (\mathbb{Z}/2)^n : \begin{array}{c} \sigma_i = \sigma_j \text{ if for } i \neq j \text{ there is } p \in \Delta_i \cap \Delta_j \\ \text{such that } \partial^1_p(\alpha_i) = \partial^1_p(\alpha_j) \neq 0 \end{array} \right\}.$$

Then

$$H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2) \simeq H/\langle 1, \ldots, 1 \rangle.$$

Proof. We show the assertion in two steps; first we prove that all unramified elements in $k(Y)$ come from $H/\langle 1, \ldots, 1 \rangle$ and then we prove that every element in this quotient gives rise to an unramified Brauer class.

Part I: inducing unramified classes through elements in *H*. Combining the reciprocity exact sequence in Proposition 3.3.8, applied to the base $\mathbf{P}^2$, with Theorem 3.2.8, we obtain the following commutative diagram: (3.4.1)

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & 0 \arrow[d] & H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)[2] \arrow[d] \\
0 \arrow[r] & \langle \alpha \rangle \arrow[r] & \mathrm{Br}(L)[2] \arrow[r, "\tau"] \arrow[d, "\partial"] & \mathrm{Br}(Y_\eta)[2] \arrow[d, "\partial'"] \\
0 \arrow[r] & \ker(\pi^*) \arrow[r] & \bigoplus_{D \in B^{(1)}} H^1(k(D), \mathbb{Z}/2) \arrow[r, "\pi^*"] \arrow[d, "s"] & \bigoplus_{C \in Y^{(1)}_{\mathrm{d}}} H^1(k(C), \mathbb{Z}/2) \\
 & & \bigoplus_{p \in B^{(2)}} \mathbb{Z}/2 &
\end{tikzcd}$$

Let us explain all the notation in the diagram:

- the first row is exact, as it follows from Theorem 3.2.8 directly; here Y_η is the generic fibre of π, which is smooth and has no L-points since $\alpha \neq 0$. Note that, although τ is not surjective, each class in $\mathrm{Br}(Y_\eta)[2]$ which is not 4-torsion comes from a class in $\mathrm{Br}(L)[2]$;
- clearly, the second row is exact by construction. The set $Y^{(1)}_{\mathrm{d}}$ contains those prime divisors in Y which do not dominate the base B and π^* is the pull-back map induced by restricting π to each such divisor (we have indicated the direct sum of these maps again by π^*, with a slight abuse of notation). At each summand π^* is the restriction in Galois cohomology and is induced by the usual pull-back map: if $C \in Y^{(1)}$ is a divisor above $D \in B^{(1)}$ and we denote

$L_D := k(D)$ and $L_C := k(C)$, then there is a quadratic extension $L_D \subset L_C$ and by Kummer Theory (Lemma 3.1.17), the induced π^* is just the inclusion modulo squares.

- The first column is the reciprocity exact sequence (Proposition 3.3.8), having noted that
$$\mathrm{Br}(B) \simeq \mathrm{Br}(\mathbf{P}^2_k) \simeq \mathrm{Br}(k) = 0$$
by birational invariance and by Theorem (3.2.10);
- We now illustrate the second column, where $H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$ denotes the group of classes in $\mathrm{Br}(k(Y))[2]$ which are unramified with respect to all valuations corresponding to prime divisors on Y; note that this is *larger* than $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$ as we point out later. We need to show that this is $\ker(\partial')$; first of all, note that, by purity (Proposition 3.3.6, see also [**CT-95**, par. 2.2.2])
$$\mathrm{Br}(Y_\eta)[2] = H^2_{\mathrm{nr}}(k(Y)/L, \mathbb{Z}/2) \tag{3.4.2}$$
So $\xi \in \mathrm{Br}(Y_\eta)[2]$ if and only if $\partial^2_v(\xi) = 0$ for all divisorial valuations v on $k(Y)$ that are trivial on L; equivalently, since π is of relative dimension 1, we have $\xi \in \mathrm{Br}(Y_\eta)[2]$ if and only if $\partial^2_C(\xi) = 0$ for all prime divisors on Y which dominate the base B via π. By construction, if $\xi \in \mathrm{Br}(Y_\eta)[2]$ belongs to $\ker(\partial')$, then it annihilates all the residues maps along those prime divisors on Y which do *not* dominate the base via π, therefore by the description of $\mathrm{Br}(Y_\eta)[2]$ given in (3.4.2), ξ must also annihilate all the residue maps along *all* prime divisors on Y, that is to say $\xi \in H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$.
- Note that, since Y is not smooth, in general $H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$ differs from $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$, since a class ξ belongs to the latter group only if it is unramified with respect to all valuations on $k(Y)$ which are trivial on k or, more geometrically, only if $\partial^2_v(\xi) = 0$ for all divisorial valuations v corresponding to prime divisors C on some *smooth model* $\widetilde{Y}$ of Y.

We first establish that $H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2) \simeq H/\langle 1, \ldots, 1\rangle$and this will be done in the following technical result.

LEMMA 3.4.16. *In the above setting:*

1. $\ker(\pi^*) = \langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_n \rangle$, *where* $\alpha_i := \partial^2_{\Delta_i}(\alpha)$ *for* $i = 1, \ldots, n$*;*
2. $\ker(\partial')$ *is contained in the image of* τ*;*
3. *there exists a well-defined morphism* $\varphi : H \to H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$*;*
4. φ *is surjective and its kernel equals* $\langle 1, \ldots, 1\rangle$*.*

PROOF. (1) First of all note that $\alpha_i \neq 0$ because of assumption (4) in the statement. Now, if the generic fibre of $\pi|_C : C \longrightarrow D$ is geometrically integral (this is the case if D is not contained in the discriminant locus of π, since the generic fibre of $\pi|_C : C \longrightarrow D$ is a smooth conic) then L_D is integrally closed in L_C and this means that π^*_D is injective: if $[a] \in L_D^\times/(L_D^\times)^2$ is such that $\pi^*[a] = 0$ then it means that the element a is a square in L_C, namely there is $b \in L_C$ such that $a = b^2$; however, $b \in L_D$ by integral closure, as root of the monic polynomial $T^2 - a \in L_D[☺T]$. Thus $[a] = 0$ and π^* is injective.

If, instead, D is one of the irreducible components Δ_i of Δ, then the geometric generic fibre of each $C \longrightarrow D$ is a union of two lines. The map $C \longrightarrow D$ is precisely one of the associated double covering and $L_C \simeq L_D(\sqrt{\alpha_D})$ where $\alpha_D \in L_D^\times$ is a representative of $\partial^2_D(\alpha)$, The restriction-corestriction exact sequence in Galois cohomology implies that

$$0 \to H^1(L_D/L_C, \mathbb{Z}/2)) \to H^1(L_D, \mathbb{Z}/2) \stackrel{\pi_D^*}{\to} H^1(L_C, \mathbb{Z}/2)$$

and since $\mathrm{Gal}(L_D/L_C)$ is generated by the order-2 automorphism of L_C induced by $\sqrt{\alpha_D}$, one has that $\ker(\pi^*_D) \simeq \langle \alpha_D \rangle$. Putting everything together, we have shown that $\ker(\pi^*) \simeq \langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_n \rangle$.

(2) Assume that there is a non-zero $\xi \in \ker(\partial') = H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$ such that $\xi \notin \mathrm{im}(\tau)$. Then, by Theorem 3.2.8, there is $\xi' \in H^2(L, \mathbb{Z}/4)$ such that $\tau(\xi') = \xi$. Since the map ∂ in diagram (3.4.1) is injective, there exists $D \in B^{(1)}$ such that $\xi'' = \partial^2_D(\xi') \in H^1(k(D), \mathbb{Z}/4)$, namely ξ'' is 4-torsion. By part (1), we know that $\pi^*(\xi'') \neq 0$ as $\ker(\pi^*)$ is a 2-group; more precisely, one can prove that the same diagram above commutes replacing 2-torsion coefficients with 4-torsion coefficients, hence formula (1) still describes the kernel of π^* with 4-torsion coefficients. Then, by commutativity, we have

$$0 \neq \pi^*(\partial^2_D(\xi')) = \partial^2_{\pi^{-1}(D)}(\xi)$$

which means $\xi \notin H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$. This proves the claim.

(3) Every element ξ of H belongs to $\ker(s)$, because Δ has at worst quadratic singularities, hence for each point $p \in \Delta$ there are, locally around p, at most two branches meeting transversely at p and the further ramification adds up to 0 modulo 2 at all these points. Hence, since $\ker(s) = \mathrm{im}(\partial)$, there exists $x \in H^2(L, \mathbb{Z}/2)$ such that $\partial(x) = \xi$; notice that by injectivity x is uniquely determined, so it only remains to check that $\varphi(\xi) := u_\xi = \tau(x)$ belongs to $H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2) = \ker(\partial')$. But this is immediate by commutativity:

$$\partial'(u_\xi) = \pi^*(\partial(x)) = \pi^*(\xi) = 0$$

since $\xi \in \ker(\pi^*)$.

(4) To show surjectivity, the diagram chasing required is similar: let $\xi \in \ker(\partial')$; then we know by part (2) that there is $\xi' \in H^2(L, \mathbb{Z}/2)$ such that $\tau(\xi') = \xi$. By commutativity, it must be that

$$\pi^*(\partial(\xi')) = \partial'(\tau(\xi')) = \partial'(\xi) = 0$$

hence $\partial(\xi') \in \ker(\pi^*)$. However, the element $\xi'' = \partial(\xi')$ also participates to the reciprocity exact sequence, so $s(\xi'') = 0$, which means that, for every $p \in B^{(2)}$, it must be

$$\sum_{D \in B^{(1)}} \partial^1_p(\xi''_D) \equiv 0 \mod 2.$$

Since $\xi''_D = 0$ if D is not a component of Δ (Proposition 3.4.11) the condition is equivalent to

$$\sum_{i=1}^{n} \partial_p^1(\xi''_{\Delta_i}) \equiv 0 \mod 2$$

Hence, this implies that, for a fixed, arbitrary point p,

- either $\partial_p^1(\xi''_{\Delta_i}) = \partial_p^1(\xi'_{\Delta_j}) = 1$ for each $i \neq j$;
- either $\partial_p^1(\xi''_{\Delta_i}) = \partial'_p(\xi''_{\Delta_j}) = 0$ for each $i \neq j$.

It is clear that the second condition is trivially satisfied if p is not an intersection point of two components Δ_i, Δ_j.This reasoning shows that $\xi'' \in H$ and $\varphi(\xi'') = \xi$.
Finally, let $\zeta \in H$ be an element of $\ker(\varphi)$, namely $\varphi(\zeta) = u_\zeta = 0$. Recall that $u_\zeta = \tau(x)$, where x is the unique lift of ζ to $H^2(L, \mathbb{Z}/2)$, hence $u_\zeta = 0$ if and only if $x \in \ker(\tau) = \langle \alpha \rangle$. It follows that $\ker(\varphi) = \langle \partial(\alpha) \rangle \simeq \langle 1, \dots, 1 \rangle$. □

Putting everything together, we have constructed an isomorphism

$$H/\langle 1, \dots, 1 \rangle \simeq H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$$

as wished.

Part II: checking unramifiedness with respect to all valuations. It now remains to check that elements in $H/\langle 1, \dots, 1 \rangle$ give rise to elements in the unramified cohomology $H^2(k(Y)/k, \mathbb{Z}/2)$; more precisely, we need to show that all classes of the form $\varphi(\xi)$ for $\xi \in H$ are unramified with respect to *all* discrete valuations on $k(Y)$ and not only with respect to those having divisorial centre on Y.

Recall, from diagram chasing, that there exists $\beta \in H^2(L, \mathbb{Z}/2)$ such that $\partial(\beta) = \xi$ and one sets $\varphi(\xi) := \tau(\beta) \in \mathrm{Br}(k(Y))$. Now let v be a discrete, rank 1 valuation on $k(Y)$; this corresponds to a prime divisor C_v on a smooth projective model $\widetilde{Y}$ for Y, which yields a desingularisation $\sigma : \widetilde{Y} \longrightarrow Y$. We will need to check that $\tau(\beta)$ is unramified along C_v and we will divide the proof in several cases, depending on the image $D_v := \pi_*\sigma_*(C_v)$ of C_v on the base B. Now, D_v can have codimension 1 or 2.

Case 1. If D_v has codimension 1, then $\sigma_*(C_v)$ is a is a prime divisor on Y hence the class $\tau(\beta)$ is unramified by the previous argument.

Case 2. Assume now that $D_v = \{p_v\}$ has codimension 2; we will show that β is unramified locally around p_v, or, equivalently, that there is a Zariski open neighbourhood U_v of p_v such that β belongs to the image of the natural map $\mathrm{Br}(U_v) \longrightarrow \mathrm{Br}(L)$. Then we will argue, with a sought-for argument for each v, that the class $\tau(\beta)$ is unramified as well. Firstly, since $\partial(\beta) = \xi \in H \subseteq \ker(\pi^*)$ notice that the residue profile of β is of the form

$$\partial(\beta) = (\varepsilon_1\alpha_1, \dots, \varepsilon_n\alpha_n)$$

for some $\varepsilon_i \in \{0,1\}$. Here we are abusing the definition of ∂ by omitting all the residues that vanish on a first instance (*i.e.* those along divisors not meeting the discriminant locus). Hence, without loss of generality we can assume p_v lies in some irreducible component Δ_i of the discriminant locus Δ. We need to distinguish three sub-cases.

Case 2a. Assume that $p_v \in \Delta_i \setminus \bigcup_{j\neq i} \Delta_j$ and $\partial^2_{\Delta_i}(\beta) = \alpha_i$; then α and β have the same residue profile Zariski locally around p_v. More precisely, there is a Zariski open neighbourhood U_v of p_v on which $\partial(\beta) = \partial(\alpha)$; for instance, one can choose

$$U_v := B \setminus \bigcup_{j\neq i} \Delta_j.$$

Set $\beta' = \beta - \alpha \in \mathrm{Br}(L)[2]$; then β' is unramified around U_v, namely for each curve C in U_v passing through p one has

$$\partial^2_C(\beta') = \partial^2_C(\beta) - \partial^2_C(\alpha) = \begin{cases} 0 & \text{if } C \neq \Delta_i \\ \alpha_i - \alpha_i = 0 & \text{if } C = \Delta_i \end{cases}$$

so β comes from a class in $\mathrm{Br}(U_v)$. Since $\tau(\beta') = \tau(\beta)$, it is immediate to conclude that $\tau(\beta) = \varphi(\xi)$ is unramified with respect to v too.

Case 2b. Assume now that $p_v \in \Delta_i \cap \Delta_j$ for some $j \neq i$ and $\partial^2_{\Delta_h}(\beta) = \alpha_h$ for $h = i, j$. Even in this case, α and β have the same residue profile Zariski locally around p_v: choose

$$U_v := B \setminus \bigcup_{h\neq i,j} \Delta_h$$

and $\partial(\beta) = \partial(\alpha)$ holds for curve on U_v. Then setting $\beta' := \beta - \alpha$ as before implies that $\partial^2_C(\beta') = 0$ for every curve through p_v in U_v, so β' comes from a class in $\mathrm{Br}(U_v)$. Moreover, since $\tau(\beta') = \tau(\beta)$, this is true even for $\tau(\beta) = \varphi(\xi)$.

Case 2c. Finally, assume $p_v \in \Delta_i \cap \Delta_j$ for some $j \neq i$ and $\partial^2_{\Delta_i}(\beta) = \alpha_i$ but $\partial^2_{\Delta_j}(\beta) = 0$. In this case, the residue profile of β and α do not agree in any Zariski neighbourhood of p_v because of the assumption that $\alpha_j = \partial^2_{\Delta_j}(\alpha) \neq 0$.

However, since β must fit into the reciprocity sequence and the only curves through p_v along which α has non-trivial residue are Δ_i and Δ_j, it must be that $\partial^1_{p_v}(\alpha_i) = 0$; this means that α_i – as rational function in $k(\Delta_i)$ representing the residue $\partial^2_{\Delta_i}(\alpha)$ – has even order at p_v. It follows that for a suitable $c \in k(\Delta_i)^\times$ we can make $u = c^2\alpha_i$ to be non-zero at p_v and this means u defines a regular function around p_v which is a unit in the local ring $\mathscr{O}_{\Delta_i,p_v}$, such that $[u] = [\alpha_i]$ as classes modulo squares in $k(\Delta_i)^\times/(k(\Delta_i)^\times)^2$.

Now let ω be a local equation for Δ_i around p_v and let us consider the class $\beta' = (u, \omega) \in \mathrm{Br}(L)[2]$. Then β' has the same residue profile of β

Zariski locally around p_v: first of all, for any curve C that passes through p one has

$$\partial^2_C(u,\xi) = u^{v_C(\xi)}/\xi^{v_C(u)} \mod (k(C)^\times)^2$$

Since u is a unit at p, we have $v_C(u)=0$ for any C; moreover, $v_C(\xi)\neq 0$ if and only if $C=\Delta_i$ so β' has non-trivial residue along Δ_i only, implying that

$$\partial^2_{\Delta_i}(\beta') = [u] = [\alpha_i] = \partial^2_{\Delta_i}(\beta).$$

Hence, the class $\beta'' = \beta - \beta'$ is unramified in some Zariski open neighbourhood U_v of p_v; here we can choose U_v as in Case 2a.

Now it remains to show that $\tau(\beta')$ is unramified with respect to v, so that $\gamma = \tau(\beta) = \tau(\beta''+\beta')$ will be unramified as well. Observe that the class $\beta' = (u,\omega)$ can be represented as a conic bundle $Y' \to U_v$, whose generic fibre is defined by

$$uX^2 + \omega Y^2 - Z^2 = 0$$

and the fibre above p_v is precisely a cross of lines. Hence the pull-back class $\tau(\beta')$ corresponds to the pull-back of this conic bundle to the desingularisation $\widetilde{Y}$; the discriminant locus of this conic bundle contains the centre C_v (possibly with some multiplicity m_i). But this conic bundle induces a split double covering along C_v by construction, hence the class $\tau(\beta')$ is unramified. □

Remark 3.4.17. In the formula above, if Δ is irreducible, then $H = \langle \partial^2_\Delta(\alpha)\rangle \simeq \mathbb{Z}/2$ and thus $H^2_{\mathrm{nr}}(k(Y)/k,\mathbb{Z}/2)=0$. Of course, there are situations in which unramified cohomology vanishes even if Δ is reducible.

3.5. The case of cubic threefold hypersurfaces.

3.5.1. Overview of the problem. In this section we will formally establish the fact that it is not possible to address the stable Lüroth problem for 3-dimensional hypersurfaces of degree 3 by applying the techniques explained so far. More precisely, we will show that cubic hypersurfaces in $\mathbf{P}^4 := \mathbf{P}^4_{\mathbb{C}}$ are not amenable to the technique described in Strategy 2.3.8 if one seeks to check condition (3) by computing unramified cohomology with 2-torsion coefficients: indeed, in this situation, there is no 2-torsion in the unramified cohomology group at all.

It is a widely known fact that a smooth cubic hypersurface $Y \subseteq \mathbf{P}^4$ admits a conic bundle structure over $\mathbf{P}^2$ with quintic discriminant; also, Y is unirational but not rational (see Section 1.3.2), while stable rationality is still unknown. Since in the previous sections we have developed a great deal machinery for varieties with a conic bundle structure, it is natural to ask whether one could employ Strategy 2.3.8 jointly with the formulae from Paragraph 3.4.3 to establish stable irrationality of very general cubic hypersurfaces.

It has to be noted, on a first instance, that any application of the degeneration method to cubic *hypersurfaces* is doomed to be inconclusive, by a simple deformation-theoretic argument: embedded deformations of cubic 3-fold hypersurfaces are still cubic hypersurfaces, so the special fibre of such a family will be rational and will carry no relevant obstruction that could possibly be employed to obstruct stable rationality. In summary, Strategy 1.3.2 is inconclusive if one considers degenerations of cubic threefolds as *embedded deformations* of smooth cubic hypersurfaces.

In particular, the 2-torsion subgroup in the unramified cohomology group of the induced conic bundle is trivial *a priori*. Nevertheless, in this section we will obtain this result by computing this subgroup *directly* for a general conic bundle with quintic discriminant (possibly with worse singularities in the discriminant locus than those prescribed by Theorem 3.4.15) and proving that it is trivial.

It is worth noting that we do not rule out amenability of the degeneration method to establish stable irrationality of cubic threefolds *at all*: indeed, we only show that the particular choice of cubic *hypersurfaces* as a birational model of cubic threefolds *and* the use of unramified cohomology to check condition (3) in Strategy 2.3.8 yield no satisfactory conclusion. It could be entirely possible, for example, that degenerations of another birational model for cubic threefolds yield a central fibre on which condition (3) in Strategy 1.3.2 can be checked through the non-vanishing of some known invariant.

3.5.2. Geometric constructions. We begin with some classical results about cubic hypersurfaces of dimension 3 and we derive some easy consequences.

PROPOSITION 3.5.1. *Let $X \subseteq \mathbf{P}^4$ be a smooth cubic 3-fold hypersurface. There is a conic bundle structure $\pi : Y \longrightarrow \mathbf{P}^2$ such that Y is smooth and birational to X.*

PROOF. Let ℓ be a line in X and consider the projection away from ℓ onto $\mathbf{P}^2$; the restriction of this map to X gives a morphism $\pi : X \setminus \ell \longrightarrow \mathbf{P}^2$. One can then embed $\mathbf{P}^2$ in $\mathrm{Grass}(3, 5)$, namely the set of 2-planes in $\mathbf{P}^4$; in this way every $p \in \mathbf{P}^2$ corresponds to a 2-plane $\Pi \subseteq \mathbf{P}^4$ that contains ℓ and the fibre $\pi^{-1}(p)$ is given by the intersection $\Pi \cap (X \setminus \ell)$. Now each 2-plane containing ℓ cuts X into a reducible cubic curve and, more precisely, the intersection breaks down into the line ℓ and a residual quadric curve C_p. It follows that $\pi^{-1}(p) = C_p$ is isomorphic to a plane conic. This gives $X \setminus \ell$ a structure of *rational* conic bundle. To obtain a conic fibration morphism, one lets $Y = \mathrm{Blow}_\ell(X)$; in this way the rational map $X \dashrightarrow \mathbf{P}^2$ extends uniquely to a morphism $\pi : Y \longrightarrow \mathbf{P}^2$ with generic fibre isomorphic to a conic, which is the desired conic bundle. $\square$

LEMMA 3.5.2. *Let X be a smooth cubic threefold and let $\pi : Y \longrightarrow \mathbf{P}^2$ be the associated conic bundle. Then there is an embedding $Y \hookrightarrow \mathbf{P}(\mathscr{O}_{\mathbf{P}^2}^{\oplus 2} \oplus \mathscr{O}_{\mathbf{P}^2}(1))$.*

Proof. Consider homogeneous coordinates:

$$\mathbf{P}^4 = \operatorname{Proj} k[U, V, X_0, X_1, X_2]$$

in a way that $\ell = \{X_0 = X_1 = X_2 = 0\}$ and X is cut out by a homogeneous equation of the form

$$s = a_{20}U^2 + 2a_{11}UV + a_{02}V^2 + 2a_{10}U + 2a_{01}V + a_{00} = 0$$

where $a_{ij} \in k[X_0, X_1, X_2]$ are homogeneous polynomial of degree $\deg(a_{ij}) = 3 - (i + j)$. This equation can be associated with a matrix

$$A = \begin{pmatrix} a_{20} & a_{11} & a_{10} \\ a_{11} & a_{02} & a_{01} \\ a_{10} & a_{01} & a_{00} \end{pmatrix}$$

and noting that $\deg(a_{ij}) = (\deg(a_{ii}) + \deg(a_{jj}))/2$ the conic fibration is *of free-graded type* (following [**ABvBP-19**]). Define

$$\mathscr{E} = \mathscr{O}_{\mathbf{P}^2}(-2) \oplus \mathscr{O}_{\mathbf{P}^2}(-2) \oplus \mathscr{O}_{\mathbf{P}^2}(-1)$$

so that multiplication by A gives rise to a bilinear morphism of sheaves

$$\psi : \mathscr{E} \otimes \mathscr{E} \longrightarrow \mathscr{O}(5)$$

and consequently a quadratic form $q : \mathscr{E} \longrightarrow \mathscr{O}(5)$, with the property that, for every $p \in \mathbf{P}^2$, the induced quadratic form q_p cuts out the fibre X_p in $\mathbf{P}(\mathscr{E}_p)$. Let now $\mathbf{P}(\mathscr{E})$ be the projective $\mathbf{P}^2$-bundle and let $\pi : \mathbf{P}(\mathscr{E}) \longrightarrow \mathbf{P}^2$ be the canonical morphism. By the above argument we see that Y is identified with the set of zeroes of q, which is naturally embedded in $\mathbf{P}(\mathscr{E})$, as wished.

An analogous proof can be found by recalling that Y embeds in $\mathrm{Blow}_l(\mathbf{P}^4)$. Note also that $\mathscr{E}$ is determined only up to the tensor product of some $\mathscr{O}_{\mathbf{P}^2}(k)$ due to invariance of projective bundles under twists. □

Proposition 3.5.3. *Let X be a smooth cubic threefold and let $\pi : Y \longrightarrow \mathbf{P}^2$ be the corresponding conic fibration. Therefore the discriminant locus $\Delta \subseteq \mathbf{P}^2$ is a quintic curve.*

Proof. Following the reasoning in Lemma 3.5.2, we see that the (affine cone of the) fibre above a generic $p = (x_0, x_1, x_2)$ is cut out by the affine equation

$$a_{20}U^2 + 2a_{11}UV + a_{02}V^2 + 2a_{10}U + 2a_{01}V + a_{00} = 0$$

where we called, with a slight abuse of notation, $a_{ij} = a_{ij}(x_0, x_1, x_2)$. For this equation to describe a singular conic, we must impose that the symmetric matrix

$$\begin{pmatrix} a_{20} & a_{11} & a_{10} \\ a_{11} & a_{02} & a_{01} \\ a_{10} & a_{01} & a_{00} \end{pmatrix}$$

is not full rank. Hence, the determinant of this matrix is the equation for the discriminant locus Δ; we have

$$a_{20}(a_{02}a_{00} - a_{01}^2) - a_{11}(a_{11}a_{00} - a_{01}a_{10}) + a_{10}(a_{11}a_{01} - a_{02}a_{10}) = 0$$

and recalling that $\deg(a_{ij}) = 3 - (i + j)$ it follows that this equation is homogeneous of degree 5. Hence, Δ is a quintic plane curve.

Equivalently, one could proceed by using the embedding $Y \hookrightarrow \mathbf{P}(\mathscr{E})$, realise Δ as the degeneracy locus of a symmetric morphism of sheaves $\mathscr{E}^\vee \longrightarrow \mathscr{E}(m)$ and thus apply the Giambelli formula to compute its fundamental class, from which the degree follows. $\square$

Lemma 3.5.4. *Let $\Delta \subseteq \mathbf{P}^2$ be a reducible quintic curve. Then one of its irreducible components is isomorphic to $\mathbf{P}^1$.*

Proof. Clearly, if $C_1, \ldots, C_n$ are the irreducible components of Δ, one has that $\deg C_1 + \ldots + \deg C_n = 5$. We have that $\deg C_i \geq 1$ for all i, so in every case there is either a i_0 such that $\deg C_{i_0} = 2$ or i_1 such that $\deg C_{i_1} = 1$. In the first case, Δ contains a smooth conic, which is isomorphic to $\mathbf{P}^1$, while in the second case Δ contains a line, which is again isomorphic to $\mathbf{P}^1$. This gives the assertion. $\square$

Lemma 3.5.5. *Let $f \in k(\mathbf{P}^1)^\times$ be a non-square rational function. Then there are two distinct points $p_1, p_2 \in \mathbf{P}^1$ such that f has further ramification at p_1 and p_2, in the sense that $\partial^1_{p_j}(f) \neq 0$ for $j = 1, 2$.*

Proof. Let us consider the divisor of zeroes and poles for f:

$$\operatorname{div} f = \sum_i n_i P_i$$

where the sum is finite and $n_i \in \mathbb{Z}$. Since f is a rational function on $\mathbf{P}^1$, it must be that

$$\sum_i n_i = 0. \tag{3.5.1}$$

Since f is non-zero and non-square, there must exists an i_1 such that n_{i_1} is odd, hence a point P_1 such that $\operatorname{ord}_{P_1}(f) = n_{i_1}$. But by equation (3.5.1) there must be points $P_2, \ldots, P_s$ such that $\operatorname{ord}_{P_2}(f) + \ldots + \operatorname{ord}_{P_s}(f) = -n_{i_1}$, with $s \geq 2$; since n_{i_1} is odd, there must be some $i_2 \in \{2, \ldots, s\}$ such that $\operatorname{ord}_{P_{i_2}}(f)$ is odd as well $-$ assume $i_2 = 2$ without loss of generality.

But this implies that

$$\partial^1_{P_1}(f) = \operatorname{ord}_{P_1}(f) \mod 2 = 1 = \operatorname{ord}_{P_2}(f) \mod 2 = \partial^1_{P_2}(f).$$

$\square$

3.5.3. Vanishing of unramified cohomology. We are now ready to prove the main result of this section.

Proposition 3.5.6. *Let $\pi : Y \longrightarrow \mathbf{P}^2_k$ be a conic bundle with discriminant a reducible quintic curve. Then $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2) = 0$*

Proof. Let $\Delta = \Delta_0 \cup \dots \Delta_l$ be the decomposition of the discriminant into its irreducible components, with $1 \leq l \leq 4$. Let $L = k(\mathbf{P}^2_k)$ and let $\alpha \in \mathrm{Br}(L)[2]$ be the associated Brauer class of π; we have a residue profile

$$(\alpha_0, \dots, \alpha_l) \in \bigoplus_{i=0}^{l} H^1(k(\Delta_i), \mathbb{Z}/2)$$

where $\alpha_i = \partial^2_{\Delta_i}(\alpha)$ is a rational function modulo squares. Recall that this discriminant profile must fit into the reciprocity sequence (Proposition 3.3.8), hence for every $p \in \mathbf{P}^2$, the condition for the further ramification

$$s_p(\alpha_1, \dots, \alpha_l) = \sum_{i=0}^{l} \partial^1_p(\alpha_i) = 0 \tag{3.5.2}$$

must hold.

Recalling the strategy employed in the proof of Theorem 3.4.15, our aim is to show that for any $k \in \{2, \dots, l\}$ and for any choice of $\varepsilon_1, \dots, \varepsilon_k \in \{0, \dots, l\}$ there is no sub-profile

$$(\alpha_{\varepsilon_1}, \dots, \alpha_{\varepsilon_k}) \in \bigoplus_{j=1}^{k} H^1(k(\Delta_{i_j}), \mathbb{Z}/2)$$

such that $s(\alpha_{\varepsilon_1}, \dots, \alpha_{\varepsilon_k}) = 0$ and the pull-back of the class $\beta_{\varepsilon_1,\dots,\varepsilon_k} := r^{-1}(\alpha_{\varepsilon_1}, \dots, \alpha_{\varepsilon_k})$ to $\mathrm{Br}\, k(Y)[2]$ is unramified. As it will be clear in the course of the proof, we will not need to check this latter condition, since all candidate sub-profiles will fail to give rise to a class in $\mathrm{Br}(L)[2]$.

Suppose preliminary that all components of Δ are reduced. Since we have assumed that Δ_0 is isomorphic to $\mathbf{P}^1$, by Lemma 3.5.5 we know that α_0 has further ramification at two distinct points $p_1, p_2 \in \Delta_0$, namely $\partial^1_{p_1}(\alpha_0) = 1 = \partial^1_{p_2}(\alpha_0)$. Hence condition (3.5.2) implies that, for all $h \in \{1, 2\}$

$$0 = s_{p_h}(\alpha_0, \dots, \alpha_l) = \underbrace{\partial^1_{p_h}(\alpha_0)}_{=1} + \sum_{i=1}^{l} \partial^1_{p_h}(\alpha_i).$$

Hence there exist $i_1, i_2 \in \{1, \dots, l\}$ such that α_{i_1} has further ramification at p_1 and α_{i_2} has further ramification at p_2 (this also implies that $p_1 \in \Delta_{i_1} \cap \Delta_0$, $p_2 \in \Delta_{i_2} \cap \Delta_0$).

Case 1. Suppose that $i_1 = i_2 = 1$, up to relabelling the components; we have that $p_1, p_2 \in \Delta_1 \cap \Delta_0$ and $\partial^1_{p_1}(\alpha_1) = 1 = \partial^1_{p_2}(\alpha_1)$. Notice that, in this case, we cannot have $\deg \Delta_0 = \deg \Delta_1 = 1$. Let us distinguish several cases depending on the number of components.

Case 1a. If $l = 1$, then $\Delta = \Delta_0 \cup \Delta_1$ and we have two possibilities (up to relabelling the components): either Δ_0 is a line and Δ_1 is an irreducible quartic curve or Δ_0 is a smooth conic and Δ_1 is an irreducible cubic curve. In any case, none of the two candidate sub-profiles (α_0) or (α_1) satisfies condition (3.5.2) at both p_1 and p_2.

Case 1b. If $l = 2$ then $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$ and we have $\deg \Delta_0 = 1$ and $\deg \Delta_1 = 2 = \deg \Delta_2$ up to relabelling the components. Note that α_2 cannot have further ramification at p_1 or p_2, otherwise condition (3.5.2) would fail at one of these two points. But since $\Delta_2 \simeq \mathbf{P}^1$, there must exist $q_1, q_2 \in \Delta_2 \setminus \{p_1, p_2\}$, such that $\partial^1_{q_1}(\alpha_2) = 1 = \partial^1_{q_2}(\alpha_2)$. Therefore, condition (3.5.2) again implies, for all $h \in \{1, 2\}$,

$$0 = s_{q_h}(\alpha_0, \alpha_1, \alpha_2) = \underbrace{\partial^1_{q_h}(\alpha_2)}_{=1} + \partial^1_{q_h}(\alpha_0) + \partial^1_{q_h}(\alpha_1)$$

hence α_0 or α_1 must have further ramification at q_h as well. But then for every choice of $i_1, i_2 \in \{0, 1, 2\}$ the pair $(\alpha_{i_1}, \alpha_{i_2})$ will not be an admissible sub-profile since it will fail to satisfy (3.5.2) at at least one point. The choices are:

- (α_0, α_1) – fails at q_h for all $h \in \{1, 2\}$;
- (α_0, α_2) or (α_1, α_2) – fail at p_1, p_2 and q_h for precisely one $h \in \{1, 2\}$.

Case 1c. If $l = 3$ then $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$ and the only possible configuration is $\deg \Delta_0 = 1$, $\deg \Delta_1 = 2$ and $\deg \Delta_2 = \deg \Delta_3 = 1$ up to relabelling the components. Hence the residues α_2, α_3 will have further ramification at at least two distinct points each. Further ramification might occur at the following places:

***i)*:** at the single point $q_1 \in \Delta_2 \cap \Delta_3$ for both α_2 and α_3,
***ii)*:** at multiple points in $\Delta_2 \setminus \Delta_3$ for α_2,
***iii)*:** at multiple points $\Delta_3 \setminus \Delta_2$ for α_3.

We do not rule out the case $q_1 = p_i$ for some $i \in \{1, 2\}$, in which case q_1 would be a quadruple intersection point; but notice that while q_1 need not be further ramification for α_2 or α_3, in any circumstance there must exist at least two points $q_2 \in \Delta_2 \setminus \Delta_3$ and $q_3 \in \Delta_3 \setminus \Delta_2$ which are further ramification for α_2 and α_3 respectively (if q_1 is not further ramification, there will have to be more further ramification points in these intersections). Thus, by condition (3.5.2) for all $h \in \{2, 3\}$ it must be

$$0 = s_{q_h}(\alpha_0, \ldots, \alpha_3) = \underbrace{\partial^1_{q_h}(\alpha_2) + \partial^1_{q_h}(\alpha_3)}_{=1} + \partial^1_{q_h}(\alpha_0) + \partial^1_{q_{h,1}}(\alpha_1)$$

so each q_h must be a further ramification point for either α_0 or α_1. But this implies that for any choice of $i_1, \ldots, i_k \in \{0, 1, 2, 3\}$ for $k \in \{2, 3\}$ the candidate sub-profile $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ will fail to satisfy condition (3.5.2) at at least one point: if $k = 2$ the choices are

- (α_0, α_1) – fails at q_h for some $h \in \{2, 3\}$;
- (α_0, α_2), (α_0, α_3), (α_1, α_2) or (α_1, α_3) – fail at p_1, p_2 and q_h for precisely one $h \in \{2, 3\}$;
- (α_2, α_3) – fails at $q_{h,1}$ for all $h \in \{2, 3\}$.

For $k = 3$ the choices are

- $(\alpha_0, \alpha_1, \alpha_2)$ – fails at q_3;

- $(\alpha_0, \alpha_1, \alpha_3)$ – fails at q_2;
- $(\alpha_0, \alpha_2, \alpha_3)$ or $(\alpha_2, \alpha_3, \alpha_4)$ – fail at p_1, p_2 and q_h for precisely one $h \in \{2, 3\}$;

Case 2. Assume now that $i_1 \neq i_2$. This forces the discriminant locus to split into at least three irreducible components $\Delta_0, \Delta_1, \Delta_2$ with the requirements that $p_1 \in \Delta_1 \cap \Delta_0$, $p_2 \in \Delta_2 \cap \Delta_0$ are also further ramification points for the residues α_1, α_2 respectively. The only two amenable cases that have not been addressed in Case 1 yet (up to relabelling the components) are the following:

Case 2a. Assume $\deg \Delta_0 = \deg \Delta_1 = 1$ and $\deg \Delta_2 = 3$. Then there is at least another point $q \in \Delta_1 \setminus \{p_1\}$ such that $\partial_q^1(\alpha_1) = 1$. Thus, condition (3.5.2) implies that

$$0 == \underbrace{\partial_q^1(\alpha_1)}_{=1} + \partial_q^1(\alpha_0) + \partial_q^1(\alpha_2)$$

and, since $\Delta_1 \cap \Delta_0 = \{p_1\}$, it must be $q \in \Delta_2 \cap \Delta_1 \setminus \Delta_0$ and $\partial_q^1(\alpha_2) = 1$. Then again there are no amenable sub-profiles since the only choices are:

- (α_0, α_1) – fails at q;
- (α_0, α_2) – fails at p_1, p_2 and q;
- (α_1, α_2) – fails at p_1, p_2.

Note also that we can allow Δ_2 to be singular at some point $r \in \Delta_2$; however, this means that $\Delta_2 \simeq \mathbf{P}^1$ so the residue α_2 has at least another further ramification point q' other than p_2. If $q' \neq r$, then q' will be further ramification for either α_0, α_1 or α_2 and any candidate sub-profile not containing α_3 will not satisfy condition (3.5.2) at q'. If $q' = r$, then by formula 3.3.6 further ramification has to occur at both points r_1, r_2 in the normalisation $\widehat{\Delta}_3$ which lie over r; but even in this case no sub-profile can be found without violating condition (3.5.2) somewhere.

Case 2b. Assume that $\Delta = \Delta_0 \cup \ldots \cup \Delta_4$ is union of 5 distinct lines. We have mandatory further ramification in the following way:

***i)*:** for α_0 at p_1, p_2
***ii)*:** for α_1 at p_1, q_1
***iii)*:** for α_2 at p_2, q_2
***iv)*:** for α_3 at $r_{3,1}, r_{3,2}$
***v)*:** for α_4 at $r_{4,1}, r_{4,1}$

where the points $q_j, r_{h,j}$ above need not be all distinct nor all distinct from p_1, p_2. Since all components are lines and condition (3.5.2) must be respected, each of the above points is either a double or a quadruple intersection point.

(1) Assume that there is a further ramification point which is a quadruple intersection; without loss of generality we can take $\Delta_0 \cap \Delta_1 \cap \Delta_2 \cap \Delta_3 = \{p_1\} \nsubseteq \Delta_4$ so $r_{3,1} = p_1 = q_2$. Then the only possibility

for $(\alpha_0, \dots, \alpha_4)$ to satisfy condition (3.5.2) is that $\Delta_4 \cap \Delta_0 = \{p_2\}$, $\Delta_4 \cap \Delta_1 = \{q_1\}, \Delta_4 \cap \Delta_2 = \{q_1\}$, $\Delta_4 \cap \Delta_3 = \{r_{3,2}\}$ but this also implies that any candidate sub-profile will fail to satisfy (3.5.2) at some point.

(2) Assume then that each further ramification point is a double intersection $\Delta_i \cap \Delta_j$, hence $\partial^1_{\Delta_i \cap \Delta_j}(\alpha_i) = 1 = \partial^1_{\Delta_i \cap \Delta_j}(\alpha_j)$. There are 10 possible choices for these intersection points, of which only 5 are mandatory further ramification points; thus one has $\binom{10}{5}$ different residue profiles that fit into the reciprocity sequence. However, for each of these configurations, there is no sub-profile that satisfies condition (3.5.2): suppose we consider the sub-profile $(\alpha_0, \dots, \widehat{\alpha}_{i_0}, \dots, \alpha_5)$ obtained by remove the single residue α_{i_0} with further ramification at points r_1, r_2: then by construction there are two indices $i_1, i_2 \in \{1, \dots, 5\} \setminus \{i_0\}$ such that α_{i_1} has further ramification at r_1 and α_{i_2} has further ramification at r_2, but condition (3.5.2) gives

$$\underbrace{\partial^1_{r_j}(\alpha_{i_1}) + \partial^1_{r_j}(\alpha_{i_2})}_{=1} + \underbrace{\sum_{k \neq i_0, i_1, i_2} \partial^1_{r_j}(\alpha_k)}_{=0} \neq 0.$$

Considering sub-profile with even less components produces the same obstruction.

This concludes the case-by-case analysis under the hypothesis that all components of Δ are reduced. It remains to rule out the possibility that some components might be non-reduced. Suppose $C \subseteq \Delta$ is a non-reduced component of multiplicity m; by Proposition 3.4.11, if m is odd then the residue along C is equal to the residue along C_{red} (in other words, one can assume $m = 1$) while if m is even we have $\partial^2_C(\alpha) = 0$. In this latter case, one is therefore forced to seek for amenable sub-profiles amongst the remaining components; but the above case break-down shows that such configuration is even less likely to give rise to any admissible sub-profile. $\square$

CHAPTER 4

Purity and universal triviality for the p-torsion of the Brauer group in characteristic p.

4.1. Introduction.

Let X be a smooth proper variety over a field k; following Definition 2.1.1 we recall that X is **UCT** if the degree map $\mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism for every field extension L/k; similarly we say that the Brauer group of X is *universally trivial* if the natural map $\mathrm{Br}(L) \to \mathrm{Br}(X_L)$ is an isomorphism for every field extension L/k.

In this chapter, we illustrate our paper [**ABBvB-19**] which gives a proof of the following result.

THEOREM 4.1.1. *Let X be a smooth proper variety over a field k. If X is* **UCT***, then the Brauer group of X is universally trivial.*

For torsion in the Brauer group coprime to $\mathrm{char}\, k$, Theorem 4.1.1 takes the form of the previously stated Theorem 3.2.14, which is claimed in [**CTP-16b**, Théorème 1.12] without restrictions on the characteristic. However, since it relies on the results in [**Mer-08**], it is only proved for torsion coprime to the characteristic. Our proof, which covers the case of p-primary torsion in the Brauer group when k has characteristic $p > 0$, follows a simplified version of an argument in [**Mer-08**] utilising a pairing between the Chow group of 0-cycles and the Brauer group, but with several non-trivial new ingredients.

The result of Merkurjev [**Mer-08**] is that for a smooth proper variety X over a field k, the universal triviality of CH_0 is equivalent to the condition that for all Rost cycle modules M (see [**Rost-96**]) and all field extensions L/k, the subgroup $M_{\mathrm{nr}}(L(X)/L) \subseteq M(L(X))$ of unramified elements of the function field $L(X)$, is trivial, meaning that the natural map $M(L) \to M_{\mathrm{nr}}(L(X)/L)$ is an isomorphism. In particular, taking M as Galois cohomology with finite torsion coefficients μ_ℓ where ℓ is coprime to the $\mathrm{char}\, p$, the group of unramified elements is precisely the usual unramified cohomology $H^i_{\mathrm{nr}}(L(X)/L, \mu_\ell^{\otimes(i-1)})$, which we have described in terms of residue maps in Paragraph 3.3.1 For $i = 2$, one gets the ℓ-torsion in the Brauer group $\mathrm{Br}(X)[\ell]$ (Proposition 3.3.3).

Now suppose that k has characteristic $p > 0$, and we consider the p-torsion $\mathrm{Br}(X)[p]$ in the Brauer group of X. In fact, $\mathrm{Br}(X)[p]$ is no longer a group of unramified elements in any graded piece of any Rost cycle module, since residue maps are not generally defined, see [**GMS-03**, Appendix

A]. Hence one can no longer appeal directly to [**Mer-08**] to deduce universal triviality of $\mathrm{Br}(X)[p]$ under the assumption that X is **UCT**. However, $\mathrm{Br}(X)[p]$ is isomorphic to the subgroup of "unramified classes" in $H^2(k(X), \mathbb{Z}/p(1))$, where $\mathbb{Z}/p(j)$ is defined as in [**Kat-86**] via the logarithmic part of the de Rham–Witt complex. Here, we say that a class in $H^2(k(X), \mathbb{Z}/p(1))$ is *unramified* if it comes from $H^i_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/p(j))$ for every discrete valuation ring A with fraction field $k(X)$. More generally, we have the following.

Problem 4.1.2. Let X be a smooth proper variety over an algebraically closed field k of characteristic $p > 0$. Assume that X is **UCT**. Is the subgroup of unramified classes in the Galois cohomology group $H^i(k(X), \mathbb{Z}/p(j))$ trivial?

The main result in this chapter gives a positive solution to Problem 4.1.2 for $i = 2$ and $j = 1$, namely, for the p-torsion in the Brauer group.

4.2. Unramified elements and purity.

4.2.1. Functorial properties of the Brauer group. We start with recalling some functorial properties for the cohomological Brauer group of a scheme (see Section 3.2).

Proposition 4.2.1. *Assume that all the schemes below are quasi-projective over some ring.*

(1) *For a morphism $g : X \longrightarrow Y$ there is an induced pull-back morphism*
$$g^* : \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X).$$
(2) *For a finite, flat morphism $f \colon X \longrightarrow Y$ there is a push-forward (or corestriction, or norm) morphism*
$$\mathrm{cor}_{X/Y} : \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y)$$
also denoted f_*, which satisfies the following properties:
   (a) *for every pair of composable finite, flat morphisms $X \longrightarrow Y \longrightarrow Z$, we have*
$$\mathrm{cor}_{X/Z} = \mathrm{cor}_{Y/Z} \circ \mathrm{cor}_{X/Y}$$
   (b) *for any Cartesian square*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
where f is finite flat and g is any morphism, for any $\alpha \in \mathrm{Br}(X)$ we have
$$g^* \circ f_*(\alpha) = (f')_* \circ (g')^*(\alpha).$$

PROOF. Part (1) is immediate by functorial properties of étale cohomology. To prove part (2), let $f\colon X \longrightarrow Y$ be a finite, flat morphism of quasi-projective schemes. Then, by [**Mum-66**, Lecture 4] there is a norm morphism $f_*\mathscr{O}_X \longrightarrow \mathscr{O}_Y$ and, consequently, there is an induced morphism of étale sheaves $f_*((\mathbf{G}_m)_X) \longrightarrow (\mathbf{G}_m)_Y$. Then taking cohomology one has a morphism

$$\Gamma\colon H^2_{\text{ét}}(Y, f_*((\mathbf{G}_m)_X)) \longrightarrow H^2_{\text{ét}}(Y, (\mathbf{G}_m)_Y).$$

Now, the Leray spectral sequence [**Mil-80**, Theorem 1.18] for the morphism $f\colon X \longrightarrow Y$ yields a map

$$\Delta\colon H^2_{\text{ét}}(Y, f_*((\mathbf{G}_m)_X) \longrightarrow H^2_{\text{ét}}(X, (\mathbf{G}_m)_X)$$

which is an isomorphism since, by the finiteness of f, we have $R^i f_*((\mathbf{G}_m)_X) = 0$ for $i > 0$. Therefore, we can set

$$\mathrm{cor}_{X/Y} := \Gamma \circ \Delta^{-1}.$$

□

NOTATION 4.2.2. For $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ and a morphism $\varphi\colon X \longrightarrow Y$, we will abbreviate $\mathrm{cor}_{X/Y}$ with $\mathrm{cor}_{A/B}$. Moreover, if $i : \{x\} \hookrightarrow X$ is the inclusion of a point with residue field $k(x)$, then for a class $\alpha \in \mathrm{Br}(X)$ we denote by $\alpha(x)$ the class $i^*(\alpha) \in \mathrm{Br}\, k(x)$.

NOTATION 4.2.3. Given a finite, surjective morphism $\varphi : X \longrightarrow Y$ of k-varieties and given rational functions $f \in k(X), g \in k(Y)$ we denote $\varphi^*(g) := g \circ f$ the usual pull-back of g via φ and $\varphi_*(f)$ the *norm* of f, which is defined as the determinant of the $k(Y)$-linear endomorphism $m_f\colon k(X) \longrightarrow k(X)$ given by $m_f(a) = fa$.

4.2.2. Unramified elements and purity. Let k be an arbitrary field, let X be a smooth proper k-variety and let Λ be a family of Krull valuations of the function field $k(X)$ which are trivial on k. For $v \in \Lambda$, let $A_v \subset k(X)$ be the valuation ring of v.

We denote by $\mathrm{Br}_\Lambda(k(X)) \subset \mathrm{Br}(k(X))$ the set of all Brauer classes $\alpha \in \mathrm{Br}(k(X))$ that, for all $v \in \Lambda$, belong to the image of the natural restriction map $i^*_{A_v}\colon \mathrm{Br}(A_v) \longrightarrow \mathrm{Br}(k(X))$ induced as a pull-back of the natural inclusion $\mathrm{Spec}(k(X)) \longrightarrow \mathrm{Spec}(A_v)$; explicitly, 0

$$\mathrm{Br}_\Lambda(k(X)) := \bigcap_{v\in\Lambda} \mathrm{im}\,(i^*_{A_v}).$$

We say that an element $\alpha \in \mathrm{Br}(k(X))$ is *unramified with respect to v* if $\alpha \in \mathrm{im}(i^*_{A_v})$. In particular, we will consider the following sets Λ.

(1) The set DISCR of all discrete, rank 1 valuations over $k(X)$.
(2) The set DIV of all *divisorial* valuations over $k(X)$, corresponding to a prime divisor D on a variety $X' \simeq_{\mathrm{bir}} X$ which is generically smooth above D.

(3) The set DIV_X of all divisorial valuations of $k(X)$ corresponding to a prime divisor on X. In this case, to $v_D \in \mathsf{DIV}_X$ corresponds the valuation ring $\mathscr{O}_{X,\eta_D}$, being η_D the generic point of D.

Remark 4.2.4. Notice that we have inclusions $\mathsf{DISCR} \supset \mathsf{DIV} \supset \mathsf{DIV}_X$, recalling that divisorial valuations on $k(X)$ are discrete rank 1 valuations with the additional property that the transcendence degree (over k) of their residue field is precisely $\dim X - 1$ ([**ZS-76**, Chapter VI, Section 14]). However, these inclusions are strict in general. By construction, one also has natural inclusions

$$\mathrm{Br}_{\mathsf{DISCR}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}_X}(k(X)).$$

In Paragraph 3.3.1, we have characterised elements in $\mathrm{Br}(k(X))[m]$, for m coprime to $\mathrm{char}\, k$, which are unramified with respect to valuations in DISCR; these are precisely those Brauer classes that annihilate all residue maps. Thus we have

$$\mathrm{Br}_{\mathsf{DISCR}}(k(X))[m] = H^2_{\mathrm{nr}}(k(X)/k, \mathbb{Z}/m)$$

for such a choice of m. We have subsequently shown that elements in the unramified cohomology can also be characterised by checking triviality of residues along divisorial valuations only; more precisely, Proposition 3.3.6 proves that

$$\mathrm{Br}_{\mathsf{DIV}_X}(k(X))[m] = H^2_{\mathrm{nr}}(k(X)/k, \mathbb{Z}/m) = \mathrm{Br}_{\mathsf{DIV}}(k(X))[m].$$

We seek, now, to reproduce a version of this statement without any constraint on m in term of $\mathrm{char}\, k$.

Theorem 4.2.5. *Let X be a smooth projective k-variety. Then all of the natural inclusions*

$$\mathrm{Br}(X) \subset \mathrm{Br}_{\mathsf{DISCR}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}_X}(k(X))$$

are equalities. More generally, if X is smooth but not necessarily proper, then we still have the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}_{\mathsf{DIV}_X}(k(X))$ and this is an equality.

Remark 4.2.6. This result was proved for the torsion part prime to $\mathrm{char}\, k$ in [**CT-95**, Proposition 2.1.8]. We will provide a proof that covers all cases with no distinctions.

The key result is the following "purity" statement for the Brauer group.

Lemma 4.2.7. *Let V be a smooth k-variety and let $U \subseteq V$ be an open sub-variety such that $V \setminus U$ has at least codimension 2 in V. Then the natural restriction $\mathrm{Br}(V) \longrightarrow \mathrm{Br}(U)$ is an isomorphism.*

Proof. For the m-torsion part, if m is coprime to $p = \mathrm{char}\, k$, the assertion follows from the absolute cohomological purity conjecture, whose proof appears in [**Fuji-02**] and is attributed to Ofer Gabber. For the p-primary torsion part, the assertion is a consequence of the complimentary result of

Gabber which appears [**Gabb-04**, Theorem 5]. Even more generally, a similar statement was recently proved for arbitrary schemes in [**Česk-19**]. □

We are now ready to prove Theorem 4.2.5.

PROOF (OF THEOREM 4.2.5). First of all, notice that the properness hypothesis is necessary for inclusion $\mathrm{Br}(X) \subseteq \mathrm{Br}_{\mathsf{DISCR}}(k(X))$ to hold: indeed, let $\alpha \in \mathrm{Br}(X)$ and let $\alpha' \in \mathrm{Br}(k(X))$ be the pull-back to the generic point, namely $\alpha' = i_X^*\alpha$ following Notation 3.2.6. Since X is proper, for every $v \in \mathsf{DIV}$ the morphism i_X extends to a morphism $j_X : \mathrm{Spec}(A_v) \longrightarrow X$ (see [**Hart-74**, II, Theorem 4.7]) and denoting with $i_{A_v}^* : \mathrm{Br}(A_v) \longrightarrow \mathrm{Br}(k(X))$ the natural pull-back, we have that

$$\alpha' = i_X^*\alpha = u^* j_X^* \alpha$$

hence $\alpha' \in \mathrm{Br}(A_v)$. Inclusions $\mathrm{Br}_{\mathsf{DISCR}}(k(X)) \subset \mathrm{Br}_{\mathsf{DIV}}(X) \subset \mathrm{Br}_{\mathsf{DIV}_X}(X)$ hold by construction.

We now prove that every class in $\mathrm{Br}_{\mathsf{DIV}_X}(k(X))$ belongs to $\mathrm{Br}(X)$. Let $\alpha \in \mathrm{Br}(k(X))$ be any class; by Theorem 3.2.7 there exists an open subset $V \subseteq X$ and a Brauer class $\alpha_V \in \mathrm{Br}(V)$ such that $\alpha = i_V^*\alpha_V$. Moreover, one can choose V such that $X \setminus V$ is union of prime divisors D_i of X, $i = 1, \dots, s$.

Now suppose that $\alpha \in \mathrm{Br}_{\mathsf{DIV}_X}(k(X))$. By construction, we have that $\alpha \in \mathrm{Br}(\mathscr{O}_{X,\eta_i})$ for η_i the generic point of D_i; hence there are open sets U_i in X such that $U_i \cap D \neq \varnothing$ and there exist Brauer classes $\alpha_i \in \mathrm{Br}(U_i)$ such that $i_{U_i}^*\alpha_i = \alpha$.

We then use the étale Mayer-Vietoris exact sequence [**Mil-80**, Exercise 2.24] repeatedly. Let $i = 1$; we have

$$\mathrm{Br}(V \cup U_1) \xrightarrow{a} \mathrm{Br}(V) \oplus \mathrm{Br}(U_1) \xrightarrow{b} \mathrm{Br}(U \cap V_1).$$

Since

$$\iota_V^*\alpha_V = \alpha = i_1^*\alpha_1$$

we also have that

$$i_{V\cap U_1}^*\alpha_V = i_{V\cap U_1}^*\alpha_1$$

and it follows by Theorem 3.2.7 that $\alpha_V = \alpha_1$ in $\mathrm{Br}(U \cap V_1)$, namely $b(\alpha_v, \alpha_1) = 0$. Hence there exists a Brauer class $\beta_1 \in \mathrm{Br}(V \cup U_1)$ such that $a(\beta_1) = (\alpha_V, \alpha_1)$ and in particular $i_{V\cup U_1}^*\beta_1 = \alpha$.

We repeat this process for every $i = 1, \dots, s$, choosing at each step the open subsets

$$\Omega_1 = V \cup U_1 \cup \ldots \cup U_{i-1},\ \Omega_2 = U_i.$$

This yields an open set $W \subseteq X$ and a Brauer class $\beta \in \mathrm{Br}(W)$ such that $i_W^*\beta = \alpha$. Since by construction $X \setminus W$ has codimension ≥ 2, by Lemma 4.2.7 it follows that $\mathrm{Br}(W) \simeq \mathrm{Br}(X)$ hence $\alpha = i_X^*\beta'$ for some $\beta' \in \mathrm{Br}(X)$. □

NOTATION 4.2.8. In the setting of Theorem 4.2.5, we will agree to denote the group $\mathrm{Br}_{\mathsf{DIV}}(k(X))$ by $\mathrm{Br}_{\mathrm{nr}}(k(X))$ and call this the *unramified Brauer group* of the function field $k(X)$. We will also keep this notation if X is not smooth. According to [**Hir-17**], a desingularisation should always exist, but one does not need such generality: indeed, in every example it is enough to produce an explicit desingularisation $\widetilde{X}$ and then one knows that $\mathrm{Br}_{\mathrm{nr}}(k(X)) = \mathrm{Br}(\widetilde{X})$.

REMARK 4.2.9. For all integers m coprime to $\operatorname{char} k$, we have that

$$\mathrm{Br}_{\mathrm{nr}}(k(X))[m] \simeq H^2_{\mathrm{nr}}(k(X)/k, \mathbb{Z}/m)$$

as in Definition 3.3.2; this describes the p-prime part of $\mathrm{Br}(X)$ in terms of residue maps. We will describe the p-primary torsion part in terms of certain residue maps in Chapter 5.

4.3. A variant of Weil reciprocity and a pairing.

Let V be a proper (but possibly singular) k-variety, with k arbitrary (possibly, algebraically non-closed), and let $Z_0(V)$ be the group of 0-cycles on V; we first define a pairing

$$Z_0(V) \times \mathrm{Br}(V) \longrightarrow \mathrm{Br}(k) \tag{4.3.1}$$

as follows: let

$$z = \sum_i a_i z_i$$

be a 0-cycle for finitely many integers a_i and closed points $z_i \in V$, and let $\alpha \in \mathrm{Br}(V)$ be a Brauer class. Then

$$(z, \alpha) \mapsto \alpha(z) := \sum_i a_i \mathrm{cor}_{k(z_i)/k}(\alpha(z_i))$$

where, accordingly with Notation 4.2.2, we have denoted with $\alpha(z_i)$ the pull-back of α to $\mathrm{Br}\, k(z_i)$ and with $\mathrm{cor}_{k(z_i)/k}$ the corestriction induced by the morphism $\operatorname{Spec} k(z_i) \longrightarrow \operatorname{Spec}(k)$; notice that this latter map is finite as z_i is a closed point. We will use without distinction both the notations $\alpha(z)$ and $\langle z, \alpha \rangle$ for this pairing.

We would like the above pairing to descend to the quotient modulo rational equivalence; namely, we would like that the pairing were trivial on cycles rationally equivalent to zero. Recall ([**Fult-98**, Chapter 2]) that zero-cycles rationally equivalent to zero are cycles of the form $\nu_*(\operatorname{div} f)$, where $f \in k(C)^*$ and $C \subseteq V$ is a curve with normalisation $\nu : \widehat{C} \longrightarrow C$.

LEMMA 4.3.1. *If $\varphi\colon C \longrightarrow D$ is a finite morphism of proper curves and if $\alpha \in \mathrm{Br}(D)$, $z \in Z_0(C)$ then*

$$\langle z, \varphi^*(\alpha) \rangle = \langle \varphi_*(z), \alpha \rangle.$$

PROOF. Notice that the pairing is bilinear and that the push-forward map $\varphi_*\colon Z_0(C) \longrightarrow Z_0(D)$ is linear as well; hence, is is sufficient to prove the statement for a cycle consisting of a single closed point $z \in C$. Let $w = \varphi(z)$ and let us consider the following commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Spec} k(z) \arrow[r, "i_z"] \arrow[dl, "\varepsilon_z"'] \arrow[d, "\varphi_z"] & C \arrow[d, "\varphi"] \\
\operatorname{Spec}(k) & \operatorname{Spec} k(w) \arrow[l, "\varepsilon_w"'] \arrow[r, "i_w"] & D
\end{tikzcd}$$

which in turn induces a commutative diagram

$$\begin{tikzcd}
 & \operatorname{Br}(k(z)) \arrow[dl, "\operatorname{cor}_{k(z)/k}"'] & \operatorname{Br}(C) \arrow[l, "i_z^*"'] \\
\operatorname{Br}(k) & \operatorname{Br}(k(w)) \arrow[l, "\operatorname{cor}_{k(w)/k}"'] \arrow[u, "\varphi_z^*"'] & \operatorname{Br}(D) \arrow[l, "i_w^*"'] \arrow[u, "\varphi^*"']
\end{tikzcd}$$

Therefore we have

$$\begin{aligned}
\langle z, \varphi^*(\alpha)\rangle &= \operatorname{cor}_{k(z)/k}(i_z^*(\varphi^*(\alpha))) = \operatorname{cor}_{k(z)/k}(\varphi_z^*(\iota_w^*\alpha)) = \\
&= \operatorname{cor}_{k(w)/k} \circ \operatorname{cor}_{k(z)/k(w)}(\varphi_z^*(\iota_w^*\alpha))
\end{aligned}$$

where the last equality makes use of Proposition 4.2.1, (2).(a) for the composition of morphisms

$$\operatorname{Spec} k(z) \xrightarrow{\varphi_z} \operatorname{Spec} k(w) \xrightarrow{\varepsilon_w} \operatorname{Spec} k.$$

Now note that by 4.2.1, (2).(b) we have that

$$\operatorname{cor}_{k(z)/k(w)} \circ \varphi_z^*(a) = [k(z) : k(w)]a.$$

It follows that

$$\langle z, \varphi^*(\alpha)\rangle = [k(z) : k(w)] \cdot \operatorname{cor}_{k(z)/k(w)}(\alpha(w)) = \langle \varphi_*(z), \alpha\rangle$$

since

$$\varphi_*(z) = [k(z) : k(w)]w.$$

$\square$

We point out that the above Lemma holds for finite morphisms between arbitrary proper varieties; in virtue of this it is enough to prove our claim for curves only. The next technical result proves a new version of the projection formula.

Lemma 4.3.2. *Let $\varphi\colon C \longrightarrow D$ be a finite, flat covering of proper k-curves, let $z \in Z_0(D)$ and let $\alpha \in \operatorname{Br}(C)$. Then*

$$\langle z, \varphi_*(\alpha)\rangle = \langle \varphi^*(z), \alpha\rangle.$$

Proof. Since the pairing is bilinear and φ_* is linear, we can assume that z is a closed point in D with residue field $k(z)$. Consider the diagram

$$\begin{tikzcd} C_z \arrow[r] \arrow[d, "\varphi_z"'] & C \arrow[d, "\varphi"] \\ \operatorname{Spec} k(z) \arrow[r] & D \end{tikzcd}$$

where C_z is the fibre above z. By 4.2.1.2.(b) and by noting that $\varphi^*(z) = [C_z] \in Z_0(C)$, we see that the assertion holds if

$$\langle z, (\varphi_z)_*(\alpha_z)\rangle = \langle \varphi_z^*(z), \alpha_z\rangle \tag{4.3.2}$$

where α_z is the restriction of α to C_z and we have identified z with the image of φ_z. Hence, it is enough to prove the assertion replacing C with C_z, the map φ with φ_z and α with α_z.

By assumption on φ, the fibre C_z is disjoint union of

$$Y_x := \operatorname{Spec} \mathscr{O}_{C,x}/(\pi_x^{e_x})$$

where x is a closed point lying over z, with local uniformiser π_x and multiplicity e_x. By definition, the residue field $k(x)$ of x is a finite extension of $k(z)$. Consider, for every x as above, the following diagram

$$\begin{tikzcd} & Y_x \arrow[dl] \arrow[d, "\varphi_z|_{Y_x}"] & \operatorname{Spec} k(x) \arrow[l, "\rho"'] \arrow[dl, "\psi"] \\ \operatorname{Spec} k & \operatorname{Spec} k(z) \arrow[l] & \end{tikzcd} \tag{4.3.3}$$

where ρ is the inclusion of the reduced scheme $\operatorname{Spec} k(x) = \operatorname{Spec} \mathscr{O}_{C,x}/(\pi_x)$ of Y_x and ψ is the morphism induced by inclusion $k(z) \subset k(x)$. Since, again, the pairing is bilinear and $\varphi_z|_{Y_x}$ is linear, it is enough to prove that, for every Brauer class $\alpha \in \operatorname{Br}(Y_x)$, the following holds:

$$(\varphi_z|_{Y_x})_*(\alpha) = e_x \cdot \psi_*(\rho^*(\alpha)).$$

Proving this will automatically demonstrate 4.3.2.

Recall Remark 4.2.3: corestriction maps can be described as determinants of matrices. Call $A = \mathscr{O}_{C,x}$, $\pi = \pi_x, e = e_x$, $A' = A/(\pi^e)$ and $L = k(z)$, $F = k(x) = A/(\pi)$. Recall that F is a finite extension of L, and we assume $[F : L] = d \in \mathbb{N}$. Let $\{\beta_1, \dots, \beta_d\}$ be a L-basis for F and let $b_1, \dots, b_d \in A$ be lifts to the valuation ring (namely, $\beta_i = b_i \mod (\pi)$). Therefore, we can extend this set of elements to form a L-basis of A' by considering the elements

$$b_1, \dots, b_d, \pi b_1, \dots, \pi b_d, \dots, \pi^{e-1} b_1, \dots, \pi^{e-1} b_d \tag{4.3.4}$$

modulo (π^e). Every element $a \in A'$ can be written as

$$\begin{aligned} a = {} & a_{0,1}b_1 + \ldots + a_{0,d}b_d + \\ & + (a_{1,1}b_1 + \ldots + a_{1,d}b_d)\pi + \\ & \vdots \\ & + (a_{e-1,1}b_1 + \ldots + a_{e-1,d}b_d)\pi^{e-1} \end{aligned}$$

for certain coefficients $a_{i,j} \in L$. For such element a, its image in $F = A'/\mathrm{nil}(A')$ is of course written as

$$\bar{a} = a_{0,1}\beta_1 + \ldots + a_{0,d}\beta_d.$$

Denote by m_a and by $m_{\bar{a}}$ multiplication by a and $\bar{a}$, respectively in A' and F, and let M be the $d \times d$ matrix with entries in L that represents $m_{\bar{a}}$ as a L-linear endomorphism of F with respect to the basis $\{\beta_1, \ldots, \beta_d\}$; then m_a can be represented, with respect to the basis of A' formed by elements (4.3.4), by the following lower triangular block matrix:

$$\begin{pmatrix} M & 0 & \cdots & \cdots & 0 \\ N_{21} & M & 0 & \cdots & 0 \\ N_{31} & N_{32} & M & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ N_{d1} & N_{d2} & \cdots & N_{d,d-1} & M \end{pmatrix}$$

where N_{ij} are $d \times d$ matrix with entries in L. Then we see that

$$\mathrm{cor}_{A'/L}(a) = (\mathrm{cor}_{F/L}(\bar{a}))^e.$$

We can now globalise the above calculation, following the proof of Proposition 4.2.1: the norm map

$$N = (\varphi_z|_{Y_x})_* : \varphi_*((\mathbf{G}_m)_{Y_x}) \longrightarrow (\mathbf{G}_m)_{\mathrm{Spec}(F)}$$

can be factored into three maps, according to diagram (4.3.3),

$$\begin{array}{c} \varphi_*((\mathbf{G}_m)_{Y_x}) \longrightarrow \varphi_*\rho^*((\mathbf{G}_m)_{\mathrm{Spec}(F)}) = \psi_*((\mathbf{G}_m)_{\mathrm{Spec}(F)}) \longrightarrow \psi_*((\mathbf{G}_m)_{\mathrm{Spec}(F)}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow \bar{N} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathbf{G}_m)_{\mathrm{Spec}(L)} \end{array}$$

where the first map is induced by reduction $Y_x \to (Y_x)_{\mathrm{red}} = \mathrm{Spec}(F)$, the second arrow is induced by $\bar{a} \to \bar{a}^e$ and $\bar{N}$ is the usual norm map associated to ψ. This is a sequence of morphisms of sheaves of abelian groups over the étale topology; we can thus take $H^2_{\text{ét}}(\mathrm{Spec}(L), -)$. Now, note that

$$H^2_{\text{ét}}(\mathrm{Spec}(L), \varphi_*((\mathbf{G}_m)_{Y_x})) \simeq \mathrm{Br}(Y_x)$$

and

$$H^2_{\text{ét}}(\mathrm{Spec}(L), \psi_*((\mathbf{G}_m)_{\mathrm{Spec} F})) \simeq \mathrm{Br}(\mathrm{Spec}(F)) = \mathrm{Br}(F)$$

by finiteness of φ and ψ like in the proof of Proposition 4.2.1. So we have

$$\begin{tikzcd} \mathrm{Br}(Y_x) \arrow[r, "\rho"] \arrow[drr, "N"'] & \mathrm{Br}(F) \arrow[r, "e"] & \mathrm{Br}(F) \arrow[d, "\psi_*"] \\ & & \mathrm{Br}(L) \end{tikzcd}$$

which means $N = e \cdot \psi_* \circ \rho^*$. $\square$

We are now ready to prove the main ingredient of this section.

Proposition 4.3.3. *Let C be a normal, proper k-curve and let $f \in k(C)^\times$, $\alpha \in \mathrm{Br}(C)$. Then $\alpha(\operatorname{div} f) = 0$.*

Proof. The pairing (4.3.1) induces a bilinear pairing $k(C)^\times \times \mathrm{Br}(C) \longrightarrow \mathrm{Br}(k)$ by setting

$$(f, \alpha) \mapsto \langle \operatorname{div} f, \alpha \rangle$$

where we identify

$$\operatorname{div} f = \sum_i a_i z_i$$

We want to show that this pairing is trivial and we will do this in two steps. Notice first that the assertion is obvious is f is a constant function, so we will exclude this case in the following.

(1) We first prove the assertion for $C = \mathbf{P}^1_k$. In this case, by Theorem 3.2.10, we know that $\mathrm{Br}(\mathbf{P}^1_k) \simeq \mathrm{Br}(k)$ so $\alpha = \sigma^*(\alpha')$ for some $\alpha' \in \mathrm{Br}(k)$, where σ is the structure morphism of $\mathbf{P}^1_k$. Hence, identifying z_i with the image of the morphism $\varphi_{z_i} \colon \operatorname{Spec}(k(z_i)) \longrightarrow C$, one has

$$\begin{aligned} \langle f, \alpha \rangle = \sum_i a_i \mathrm{cor}_{k(z_i)/k}(\alpha(z_i)) = \sum_i a_i \mathrm{cor}_{k(z_i)/k} \circ \varphi^*_{z_i} \circ \sigma^*(\alpha') = \\ = \sum_i a_i [k(z_i) : k] \alpha' = \deg(\operatorname{div} f)\alpha' = 0 \end{aligned}$$

where we have used 4.2.1, (2).(b) to get the third to last equality along with the fact that principal divisors have degree 0.

(2) We now prove the general case by reduction to part (1). Let $f \in k(C)^\times$ be a non-constant rational function and denote by $\varphi_f \colon C \longrightarrow \mathbf{P}^1_k$ the finite morphism induced by f. Then one has

$$\langle f, \alpha \rangle = \langle \varphi_f^*(\mathrm{id}_{\mathbf{P}^1_k}), \alpha \rangle = \langle \mathrm{id}_{\mathbf{P}^1_k}, (\varphi_f)_*(\alpha) \rangle = 0$$

because of Lemma 4.3.1 and part (1).

$\square$

Remark 4.3.4. Note that, if α has order ℓ coprime to $\operatorname{char} k$, then one can write

$$\langle f, \alpha \rangle = \sum_{z \in C^{(1)}} \mathrm{cor}_{k(z)/k}(\mathrm{ord}_z(f)\alpha(z)) = \sum_{z \in C^{(1)}} \mathrm{cor}_{k(z)/k} \partial^1_z (f \smile \alpha)$$

because of Proposition 4.2.3; here $C^{(1)}$ denotes the set of codimension 1 points of C and the residue formula is given in Proposition 3.3.1. Hence Proposition 4.3.3 holds because of the reciprocity property in [**Rost-96**, Theorem 2.2], since $\mathrm{Br}(C)[\ell]$ is a Rost cycle module for this choice of ℓ.

Thus, we have an induced pairing $\mathrm{CH}_0(V_L) \times \mathrm{Br}(V_L) \longrightarrow \mathrm{Br}(L)$ for any field extension L/k.

4.4. Proof of universal triviality of the Brauer group.

We are now ready to prove Theorem 4.1.1: if X is smooth, proper and **UCT** then $\mathrm{Br}(X)$ is universally trivial.

Suppose that X is **UCT**; then there is a zero-cycle z_0 of degree 1 in X, whose support we can assume to consist of closed points whose residue fields are separable extensions of k: it is a result in [**GLL-13**, Theorem 9.2] that a regular, generically smooth, non-empty scheme of finite type over k admits a 0-cycle of minimal positive degree supported on closed points with separable residue fields. In our case, the minimal positive degree of a 0-cycle on the smooth, proper k-variety X is 1.

Let L be a field extension of the ground field k and let $\omega_L\colon X_L \longrightarrow \mathrm{Spec}(L)$ be the structure morphism. We want to prove that any class of $\mathrm{Br}(X_L)$ comes from $\mathrm{Br}(L)$ through the natural map $\omega_L^*\colon \mathrm{Br}(L) \longrightarrow \mathrm{Br}(X_L)$. With a slight abuse of notation, we still denote with z_0 the 0-cycle of degree 1 on X_L obtained by scalar extension.

Let $\alpha \in \mathrm{Br}(X_L)$ and denote $\alpha_0 := \alpha(z_0) \in \mathrm{Br}(L)$. Then it follows that

$$(\alpha - \omega_L^*(\alpha_0))(z_0) = 0$$

because $(\omega_L^*(\alpha_0))(z_0) = \deg(z_0) \cdot \alpha_0 = \alpha_0$.

Let z_0' be the 0-cycle determined by z_0 via scalar extension on $X_{L(X)}$. Now, since X is **UCT**, it must be that $\eta_L = z_0'$ as zero-cycles in $\mathrm{CH}_0(X_{L(X)}) \simeq \mathbb{Z}$; denote by α' and α_0' the pull-backs of α and $\omega^*(\alpha_0)$ to $\mathrm{Br}(X_{L(X)})$ respectively. Then by bi-linearity

$$0 = (\alpha' - \alpha_0')(\eta_L - z_0') = (\alpha' - \alpha_0')(\eta_L) - (\alpha' - \alpha_0')(z_0') = (\alpha' - \alpha_0')(\eta_L)$$

since we know that $(\alpha' - \alpha_0')(z_0')$ is the pull-back from $\mathrm{Br}(L)$ to $\mathrm{Br}(L(X))$ of $(\alpha - \omega_L^*(\alpha_0))(z_0)$, which we know to be 0. Here we used the fact that z_0 is supported on closed points whose residue fields are separable extensions of k; in this way z_0 pulled back to $X_{L(X)}$ is supported on reduced closed points, and hence restricting a Brauer class and pushing forward to $\mathrm{Spec}(L(X))$ is the same as pairing with the underlying cycle.

But $(\alpha' - \alpha_0')(\eta_L)$ is just the image of the class $\alpha - \omega_L^*(\alpha_0)$ through the natural pull-back morphism $i_{X_L}^* : \mathrm{Br}(X_L) \longrightarrow \mathrm{Br}(L(X))$, which is injective by Theorem 3.2.7. Thus $i_{X_L}^*(\alpha - \omega_L^*(\alpha_0))$ is trivial in $\mathrm{Br}(L(X))$, and by injectivity this means that $\alpha - \omega_L^*(\alpha_0)$ is trivial in $\mathrm{Br}(X_L)$; that is to say, the class $\alpha \in \mathrm{Br}(X_L)$ is the image of $\alpha_0 \in \mathrm{Br}(L)$ via the natural map $\omega_L^*\colon \mathrm{Br}(L) \longrightarrow \mathrm{Br}(X_L)$.

CHAPTER 5

The unramified Brauer group of conic bundle threefolds in characteristic 2.

5.1. Preliminaries on conic bundles in characteristic 2.

5.1.1. Quadratic forms. Let us recall some basic facts about quadratic forms, rephrased in a characteristic-free approach. We refer to [**EKM-08**] as our main reference.

DEFINITION 5.1.1. Let V be a finitely generated vector space over a field k. A *quadratic form* is a map $q\colon V \longrightarrow k$ with the following properties:

(1) for each $\alpha \in k$ and $v \in V$, we have $q(\alpha v) = \alpha^2 q(v)$;
(2) the *associated polar form* $\varphi_q\colon V \times V \longrightarrow k$ defined by
$$\varphi_q(v,w) = q(v+w) - q(v) - q(w)$$
is a bilinear form.

Note that, if $\varphi : V \times V \longrightarrow k$ is a bilinear form (not necessarily symmetric), the map $q_\varphi\colon V \longrightarrow k$ defined by $q_\varphi(v) = \varphi(v,v)$ is a quadratic form.

REMARK 5.1.2. The compositions
$$q \mapsto \varphi_q \mapsto q_{\varphi_q}, \quad \varphi \mapsto q_\varphi \mapsto \varphi_{q_\varphi}$$
are both multiplication by 2. Thus, if 2 is invertible in k, any property formulated in terms of quadratic forms can be equivalently rephrased in terms of bilinear forms. This is not the case if $\operatorname{char} k = 2$, since the aforementioned compositions are the zero function; in this situation, a quadratic form does not determine its polar form (and neither the converse occurs) and it must be specified explicitly.

Let $q_1\colon V_1 \longrightarrow k, q_2\colon V_2 \longrightarrow k$ be quadratic forms. An *isometry* is a linear isomorphism $f\colon V_1 \longrightarrow V_2$ such that $q_1(v) = q_2(f(v))$ for each $v \in V$; we use the symbol $\simeq$ to denote the relation of isometry between quadratic forms. Similarly, we will write $\varphi_1 \simeq \varphi_2$ to indicate that two bilinear forms φ_1, φ_2 are isometric, meaning that $\varphi_1(v,w) = \varphi_2(f(v), f(w))$ for some linear isomorphism f.

Let V be a finitely generated vector space over k. We define the *hyperbolic quadratic form* $q_{\mathbf{H}}\colon V \oplus V^\vee \longrightarrow k$ by setting
$$q_{\mathbf{H}}(v \oplus \xi) := \xi(v).$$

Note that the polar form of $q_{\mathbf{H}}$ is exactly

$$\varphi_{\mathbf{H}}(v_1 \oplus \xi_1, v_2 \oplus \xi_2) := \xi_1(v_2) + \xi_2(v_1).$$

Any other form which is isometric to $q_{\mathbf{H}}$ is called *hyperbolic*. Letting $\{e_1, \dots, e_n\}$ be a basis for V and letting $\{\varepsilon_1, \dots \varepsilon_n\}$ be its dual basis we find that

$$q_{\mathbf{H}}\left(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j \varepsilon_j\right) = \sum_{j=1}^n y_j \varepsilon_j \left(\sum_{i=1}^n x_i e_i\right) = \sum_{i,j=1}^n x_i y_j \varepsilon_j(e_i) = \sum_{i=1}^n x_i y_i.$$

Identifying $V^\vee \simeq V$ we can also express the hyperbolic form on any even-dimensional vector space with the above expression.

Definition 5.1.3. Let $a, b \in k$ be non-zero scalars. We denote by $\langle a \rangle$ the *diagonal quadratic form* on k (as k-vector space over itself) defined by $v \mapsto a \cdot v^2$. Also, we denote by $[a, b]$ the quadratic form on k^2 defined as by $(x, y) \mapsto ax^2 + xy + by^2$.

Note that if $\operatorname{char} k = 2$, then the polar form of $[a, b]$ is isometric to the hyperbolic form $\varphi_{\mathbf{H}}$ while $[a, b]$ may not be isometric to $q_{\mathbf{H}} = [0, 0]$.

Definition 5.1.4. Let q be a quadratic form on V. A vector $v \in V$ is called *anisotropic* if $q(v) \neq 0$ and *isotropic* if $q(v) = 0$. The form q is thus called *anisotropic* if every non-zero $v \in V$ is so and *isotropic* otherwise.

Given a subspace $W \subseteq V$, we define the orthogonal complement $W^\perp$ as the orthogonal complement taken with respect to the polar form φ_q, namely

$$W^\perp := \{v \in V \mid \varphi_q(v, w) = 0 \text{ for each } w \in W\}.$$

We say that W is a *totally isotropic subspace* if $q|_W \equiv 0$; in this case we also have $\varphi_q|_W \equiv 0$.

If k is algebraically closed, then each non-constant polynomial over k has a root; hence, the only anisotropic forms over k are the zero form and $\langle 1 \rangle$. Note also that if $\operatorname{char} k \neq 2$, then a quadratic form q is isotropic if and only if its polar form b_q is isotropic; however, if $\operatorname{char} k = 2$, this need not be true: a diagonal anisotropic form has identically zero polar form.

Let q be a quadratic form on V. If $V = W \oplus U$ and $W \subseteq U^\perp$ (or $U \subseteq W^\perp$ equivalently), then we write $q = q|_W \perp q|_U$ for their *orthogonal sum*.

A quadratic form q on V is called *totally singular* if its associated polar form φ_q is identically zero. Totally singular forms are relevant only if $\operatorname{char} k = 2$; indeed, if $\operatorname{char} k \neq 2$, then a form q is totally singular if and only if q is the zero quadratic form.

For any bilinear form φ, define the *radical* as

$$\operatorname{rad}(\varphi) := \{v \in V \mid \varphi(v, w) = 0 \text{ for all } w \in V\}.$$

Clearly φ is non-degenerate if and only if $\operatorname{rad}(\varphi) = 0$. Now, define the *quadratic radical* as

$$\operatorname{rad}(q) := \{v \in \operatorname{rad}(\varphi_q) \mid q(v) = 0\}$$

which is a vector subspace of $\mathrm{rad}(\varphi_q)$. A quadratic form q is called *regular* if $\mathrm{rad}(q) = 0$. In general, this is not the same as requiring $\mathrm{rad}(\varphi_q) = 0$: if $\operatorname{char} k \neq 2$, then $\mathrm{rad}(q) = \mathrm{rad}(\varphi_q)$ and thus q is regular if and only if φ_q is non-degenerate but if $\operatorname{char} k = 2$, this need not be true. For instance, any anisotropic quadratic form is regular, but the associated polar form may have extra zeroes.

Let K/k be any field extension. We define $V_K := V \otimes_k K$ as a K-vector space obtained by scalar extension and, consequently, the induced quadratic form $q_K\colon V_K \longrightarrow K$ defined by $q_K(v \otimes \alpha) := \alpha^2 q(v)$. The polar form of q_K is $\varphi_{q_K} = (\varphi_q)_K$, namely it is obtained by extending scalars in the original polar form. We have

$$\mathrm{rad}(q_K) \supseteq \mathrm{rad}(q)_K := \mathrm{rad}(q) \otimes_k K$$

and strict inclusion is possible. Note that if q_K is regular by the above inclusion q is regular as well.

The first step to classify and decompose a quadratic form q is to determine an orthogonal decomposition of the underlying vector space that sets aside all the regular sub-forms of q.

LEMMA 5.1.5. *Let q be a quadratic form on V and let $W \subseteq V$ be any subspace such that $V = \mathrm{rad}(q) \oplus W$. Then we have the decomposition*

$$q = q|_{\mathrm{rad}(q)} \perp q|_W$$

and $q|_W$ is anisotropic. Moreover, $q|_W$ is unique up to isometry.

The sub-form $q|_W$ is called *regular part* of q. However, this is unsatisfactory as the polar form $\varphi_{q|_W}$ may be degenerate if $\operatorname{char} k = 2$. The key is to understand how to classify anisotropic forms; this notion is equivalent to non-degeneracy if $\operatorname{char} k \neq 2$, but in our case we need to use a more general definition for non-degeneracy. We need the following result.

PROPOSITION 5.1.6. *Let q be a quadratic form on a finitely generated k-vector space V. The following are equivalent:*

(1) *q_K is regular for each field extension K/k;*
(2) *q_K is regular for some algebraically closed field K containing k;*
(3) *q is regular and* $\dim \mathrm{rad}(\varphi_q) \leq 1$.

Then one gives the following definition.

DEFINITION 5.1.7. A quadratic form q is called *non-degenerate* if any of the above conditions is satisfied.

We have the following immediate result.

PROPOSITION 5.1.8. *Let k be any field.*

(1) *The form $\langle a \rangle$ is non-degenerate if and only if $a \neq 0$.*
(2) *The form $[a, b]$ is non-degenerate if and only if $1 - 4ab \neq 0$. In particular, $[a, b]$ is non-degenerate for all $a, b \in k$ if* $\operatorname{char} k = 2$.
(3) *Hyperbolic forms are non-degenerate.*

(4) *Every binary isotropic and non-degenerate quadratic form is isometric to* $q_{\mathbf{H}}$.

Let us then state the following result.

PROPOSITION. *Let q be a quadratic form on V and let $W \subseteq V$ be a vector subspace such that $\varphi_{q|_W}$ is non-degenerate. Then $q|_W$ is non-degenerate and $q = q|_W \perp q|_{W^\perp}$. Furthermore, if q is also non-degenerate, then $q|_{W^\perp}$ is non-degenerate.*

We say that a quadratic form q is *diagonalisable* if there exists a direct sum decomposition $V = V_1 \oplus \ldots \oplus V_n$ such that $V_i \subseteq V_j^\perp$ for every $i \neq j$ and $q|_{V_i} \simeq \langle a_i \rangle$ so that

$$q \simeq \langle a_1, \ldots, a_n \rangle := \langle a_1 \rangle \perp \ldots \perp \langle a_n \rangle.$$

We will also write

$$n \cdot q := \underbrace{q \perp \ldots \perp q}_{n \text{ times}}.$$

If $\operatorname{char} k = 2$, then q is diagonalisable if and only if q is totally singular. In particular, there are no non-degenerate diagonalisable quadratic forms in dimension greater than 1. This is in stark contrast with the well known case of $\operatorname{char} k \neq 2$, since in this latter hypothesis every quadratic form is diagonalisable.

We are finally ready to state the structure theorem for quadratic forms in characteristic 2.

THEOREM 5.1.9. *Let k be a field with $\operatorname{char} k = 2$ and let q be a quadratic form over a vector k-space V. Then there exists a m-dimensional vector subspace $W \subseteq \operatorname{rad}(b_q)$ and 2-dimensional vector subspaces $V_1, \ldots, V_s \subseteq V$ such that the following orthogonal decomposition is realised:*

$$q = q|_{\operatorname{rad}(q)} \perp q|_W \perp q|_{V_1} \perp \ldots \perp q|_{V_s}$$

with $q|_{V_i} \simeq [a_i, b_i]$ for some $a_i, b_i \in k$ and they are all non-degenerate. Moreover, $q|_W$ is anisotropic, diagonalisable and unique up to isometry. In particular,

$$q \simeq r \cdot \langle 0 \rangle \perp \langle c_1, \ldots, c_m \rangle \perp [a_1, b_1] \perp \ldots \perp [a_s, b_s]$$

where $r = \dim \operatorname{rad}(q)$.

SKETCH OF PROOF. Let $U \subseteq V$ such that $V = \operatorname{rad}(q) \perp U$; then by Lemma 5.1.5 one has that $q \simeq q|_{\operatorname{rad}(q)} \perp q|_U$ and $q|_U$ is unique and anisotropic. In particular, the polar form of $q|_W$ is $\varphi|_W$ where φ is the polar form of q. Let $W = \operatorname{rad}(\varphi|_U)$, hence $W \oplus W^\perp = U$; then the sub-form $q|_{W^\perp}$ is non-degenerate: by Proposition 5.1.1 we have $q|_U \simeq q|_W \oplus q|_{W^\perp}$ and $q|_W$ is anisotropic and totally singular, hence diagonalisable. Finally one has

$$q \simeq q|_{\operatorname{rad}(q)} \perp q|_W \perp q|_{W^\perp}$$

which gives the desired decomposition after splitting $W^\perp$ into 2-dimensional subspaces. $\square$

$\dim \operatorname{rad}(q)$	$\dim \operatorname{rad}(b_q)$	*normal form of q*	*geometry of Q*
0	1	$ax^2 + by^2 + xz + z^2$	*smooth conic*
0	3	$ax^2 + by^2 + z^2,\ a, b \notin (K^\times)^2$	*regular conic, geom. double line*
1	1	$ax^2 + xz + z^2$	*cross of lines over K_a*
1	1	xz	*cross of lines*
1	3	$ax^2 + z^2,\ a \notin (K^\times)^2$	*singular conic, geom. double line*
2	3	z^2	*double line*

5.1.2. Geometry of quadratic forms. Quadratic forms have an important geometric realisation as projective quadrics. We sketch here their construction as k-schemes. For most geometric purposes, it will be necessary to choose k algebraically closed.

Let q be a quadratic form over a k-vector space V of dimension $n+1$. Then q is an element of the symmetric square $\operatorname{Sym}^2(V^\vee) \subseteq \operatorname{Sym}^\bullet(V^\vee)$. We define the associated projective quadric of q as the scheme

$$X_q := \operatorname{Proj}\left(\operatorname{Sym}^\bullet(V^\vee)/(q)\right).$$

More informally, as $\operatorname{Proj}\operatorname{Sym}^\bullet(V^\vee) \simeq \mathbf{P}^n_k$, the quadric X_q is the subset of $\mathbf{P}^n_k$ cut out by the equation $q(X_0, \ldots, X_n) = 0$, where $q \in \operatorname{Sym}^2(V^\vee) \simeq k[X_0, \ldots, X_n]_{\mathrm{hom},2}$ is identified with a homogeneous polynomial of degree 2.

If $q \neq 0$ and $n \geq 1$, the scheme X_q has dimension $n-1$ and its geometric properties are related to q. In particular, we recall the following.

Proposition 5.1.10. *Let q be a non-zero quadratic form over a vector space of dimension at least 2.*

(1) *X_q is regular if and only if q is regular;*
(2) *X_q is smooth (over k) if and only if q is non-degenerate.*

Remark 5.1.11. Notice that a non-degenerate form is also regular, while the converse may fail to hold if $\operatorname{char} k = 2$; this agrees with the fact that smooth k-schemes are regular, but a regular scheme may not be smooth over k depending on the choice of k.

Corollary 5.1.12. *Let K be a field of characteristic 2, let q be a non-zero quadratic form in three variables over K, and let $Q \subset \mathbf{P}^2_K$ be the associated conic. In the following table, we give the classification of normal forms of q, up to similarity, and the corresponding geometry of Q.*

Here, $(x : y : z)$ are homogeneous coordinates on $\mathbf{P}^2_K$; by cross of lines, we mean a union of two distinct lines in $\mathbf{P}^2$; and by K_a, we mean the Artin–Schreier extension of K obtained adjoining a root of $x^2 - x - a$.

Proof. According to the classification in Theorem 5.1.9, we have the following normal forms for q up to isometry over K.

$\dim \operatorname{rad}(q)$	$\dim \operatorname{rad}(b_q)$	normal form of q up to isometry	
0	1	$ax^2 + by^2 + xz + cz^2$	$a, c \in K,\ b \in K^\times$
0	3	$ax^2 + by^2 + cz^2$	$a, b, c \in K^\times$
1	1	$ax^2 + xz + cz^2$	$a, c \in K$
1	3	$ax^2 + cz^2$	$a, c \in K^\times$
2	3	cz^2	$c \in K^\times$

Here, in the cases $\dim \operatorname{rad}(q) \leq 1$ and $\dim \operatorname{rad}(b_q) = 3$, we are assuming that the associated diagonal quadratic forms $\langle a, b, c\rangle$ in 3 variables or $\langle a, c\rangle$ in 2 variables, respectively, are anisotropic. Otherwise, these cases are not necessarily distinct.

We remark that up to the change of variables $z \mapsto c^{-1}z$ and multiplication by c, the quadratic forms $[a, c]$ and $[ac, 1]$ are similar. Hence up to similarity, we can assume that $c = 1$ in the above table of normal forms up to isometry.

The fact that if q is totally singular then Q is geometrically a double line follows since any diagonal quadratic form over an algebraically closed field of characteristic 2 is the square of a linear form.

Thus, the only case requiring attention is the case $\dim \operatorname{rad}(q) = \dim \operatorname{rad}(b_q) = 1$, where we claim that if $a = \alpha^2 - \alpha$ for some $\alpha \in K$, then q is similar to xz and thus Q is a cross of lines. Indeed, after assuming that $c = 1$, as above, we change variables $z \mapsto z - \alpha x$ and $x \mapsto x - z$. In particular, when $a \in K/\wp(K)$ is non-zero, where $\wp\colon K \to K$ is given by $\wp(x) = x^2 - x$, then Q becomes a cross of lines over the Artin–Schreier extension K_a/K defined by adjoining a root of $x^2 - x - a$ to K. □

5.1.3. Conic bundles. Let k be a field of characteristic 2; in this setting, the definitions give in Section 3.4 are still legitimate for the notion of conic bundle; what changes is the theory of quadratic forms in the underlying vector spaces, that now obey to the results illustrated earlier.

Therefore, for us a conic bundle will be a flat, proper morphism of projective k-varieties $\pi\colon X \to B$, where B is smooth, such that every geometric fibre is isomorphic to a plane conic and with smooth geometric generic fibre; according to Definition 3.4.4, these can be also given by the zero-locus of a sheaf-theoretic quadratic form $q\colon \mathscr{E} \to \mathscr{L}$, where $\mathscr{E}$ is a rank 3 vector bundle over B and $\mathscr{L}$ is a line bundle over B with the properties that q does not vanish on any stalk and q is non-degenerate on the generic point of B.

We point out that the smoothness assumption for the geometric generic fibre is not superfluous. Indeed, let $\pi\colon X \longrightarrow B$ be a flat, proper morphism such that every geometric fibre is isomorphic to a plane conic, and assume the ground field k has characteristic 2. Let η be the generic point of B and let $K = k(B)$; note first that the geometric generic fibre $\overline{X}_K$ is isomorphic to a conic in $\mathbf{P}^2_{\overline{K}}$, defined by the vanishing of some quadratic form $q_\eta = 0$ over the field $\overline{K}$. By Proposition 5.1.12, we conclude that $\overline{X}_K$ is cut out by one of the following equations:

$$aX^2 + bY^2 + cZ^2 = 0 \tag{5.1.1}$$

or

$$aX^2 + bY^2 + YZ + cZ^2 = 0 \tag{5.1.2}$$

where $a, b, c \in \overline{K}$ and X, Y, Z are homogeneous coordinates on the projective plane. Examples of type (5.1.1) are known in literature as *wild conic bundles*: since we are working over the algebraic closure of K, the equation can be rewritten as

$$(\sqrt{a}X + \sqrt{b}Y + \sqrt{c}Z)^2 = 0$$

which is the equation of an inseparable double line. The assumption of smoothness allows us to rule out this possibility in every case.

In the sequel, we will assume, unless differently specified, that B is a smooth projective surface (most frequently, one can set $B = \mathbf{P}^2_k$). Let us not rephrase the notion of discriminant in the setting of conic bundles defined over field of characteristic 2. We must define discriminants of conic bundles together with their scheme-structure with a notion known as "semi-discriminant" (see [**Kn-91**, IV.3.1]). First we discuss the discriminant of the generic conic.

Remark 5.1.13. Let $\mathcal{Q} := \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ be the 5-dimensional projective space of all conics in $\mathbf{P}^2_k$ and let

$$X_{\mathrm{univ}} \subseteq \mathcal{Q} \times \mathbf{P}^2_k \longrightarrow \mathcal{Q}$$

be the universal conic bundle defined as incidence correspondence; the universal conic (the generic fibre) can be written as hypersurface of bi-degree $(1, 2)$ in $\mathcal{Q} \times \mathbf{P}^2_k$ cut out by the equation

$$a_{00}X_0^2 + a_{11}X_1^2 + a_{22}X_2^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{12}X_1X_2 \tag{5.1.3}$$

where we consider $(a_{00} : a_{11} : a_{22} : a_{01} : a_{02} : a_{12})$ as a system of homogeneous coordinates on $\mathcal{Q}$ and $(X_0 : X_1 : X_2)$ as a system of homogeneous coordinates on $\mathbf{P}^2_k$. In these coordinates, the equation of the universal discriminant $\Delta_{\mathrm{univ}} \subseteq \mathcal{Q}$ is

$$4a_{00}a_{11}a_{22} + a_{01}a_{02}a_{12} - a_{02}^2a_{11} - a_{12}^2a_{00} - a_{01}^2a_{22} \tag{5.1.4}$$

which, over a field of characteristic 2, reduces to

$$a_{01}a_{02}a_{12} + a_{02}^2a_{11} + a_{12}^2a_{00} - a_{01}^2a_{22}. \tag{5.1.5}$$

In any characteristic, $\Delta_{\mathrm{univ}} \subseteq \mathcal{Q}$ is a geometrically integral hypersurface which parametrises the locus of singular conics in $\mathbf{P}^2_k$.

We also define the *universal sub-scheme of double lines*. For $\mathrm{char}(k) \neq 2$ this is the closed, reduced sub-scheme of Δ_{univ} defined by the vanishing of the order 2 minors of the associated symmetric matrix obtained from equation (5.1.3) yielding the generic conic (this coincides with the image of the Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^5 \simeq \mathcal{Q}$). For $\mathrm{char}\, k = 2$, it is defined as the closed, reduced sub scheme of Δ_{univ} defined by $a_{01} = a_{02} = a_{12} = 0$.

Definition 5.1.14. Let $\pi\colon X \longrightarrow B$ be a conic bundle given as zero-set of a $\mathscr{L}$-valued non-degenerate and non-zero quadratic form $\mathscr{E} \longrightarrow \mathscr{L}$ for a rank 3 vector bundle $\mathscr{E}$ and a line bundle $\mathscr{L}$, as in Definition 3.4.4. We define the *discriminant* of π in the following way. Locally around each point of B, let U be an open neighbourhood such that $\mathscr{E}|_U \simeq \mathscr{O}_U^{\oplus 3}$ and $\mathscr{L}|_U \simeq \mathscr{O}_U$; then there is an unique morphism $f\colon U \longrightarrow \mathcal{Q}$ such that $\pi|_{\pi^{-1}(U)}\colon X \times_B U \to U$ is isomorphic to the pull-back via f of the universal conic bundle. We define $\Delta \cap U$ as the scheme-theoretic pull-back of Δ_{univ} via f. By the uniqueness of f, these local descriptions glue to give a sub scheme Δ of B.

In a similar way, we define the sub-scheme of Δ of double lines of the conic bundle locally as the scheme-theoretic pull-back of the universal sub-scheme of double lines.

5.2. Residue maps in characteristic 2.

Let E/F be a finite extension of local fields, with rings of integers $\mathscr{O}_E \supset \mathscr{O}_F$ respectively, and residue class fields $\mathfrak{e}/\mathfrak{f}$ respectively. Following [**Art-68**], we say that E/F is *unramified* if $\mathfrak{e}/\mathfrak{f}$ is a separable field extension; we also point out that there are several equivalent definitions (see *ibidem*). Let K be a field, let v be a discrete valuation over K and let K_v be the completion of K with respect to the absolute value induced by v. Denote by $k(v)$ the residue field of v, namely the residue field of the ring $\mathscr{O}_{K_v}$, in the previous notation. If $\overline{K}_v$ is an algebraic closure of K_v, one can extend v to an unique valuation on $\overline{K}_v$, which we will still denote v by a slight abuse of notation; we will, moreover, denote $\overline{k(v)}$ the residue field of this extended valuation.

Lemma 5.2.1. *There is a one-to-one correspondence between unramified subfields of $\overline{K}_v$ and separable sub-fields of $\overline{k(v)}$.*

Proof. This is precisely [**Art-68**, Chapter 4, Theorem 2A], noticing that K_v is a local field with residue class field $k(v)$. □

In particular, the discussion at [**Art-68**, p. 70] implies that there exists a maximal unramified extension of K_v, with residue field $k(v)_{\mathrm{sep}}$, the separable closure of $k(v)$, called the *inertia field*, and denoted by K_v^{nr}.

Lemma 5.2.2. $\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v) \simeq \mathrm{Gal}(k(v))$.

Proof. Compare with [**Art-68**, Chapter 4, Theorem 8]. □

5.2.1. Tame subgroup and residues. Let K be a field of characteristic p. We denote

$$\mathrm{Br}(K)[p^\infty] := \{\alpha \in \mathrm{Br}(K) : \mathrm{ord}(\alpha) = p^n \text{ for some } n > 0\}$$

the subgroup of p-primary torsion classes. Now, recall that by Theorem 3.1.14, the Brauer group $\mathrm{Br}(K_v)[p^\infty]$ is isomorphic to $H^2(K_v, (K_v)^\times_{\mathrm{sep}})[p^\infty]$

and there is a natural map

$$H^2(K_v^{\mathrm{nr}}/K_v, (K_v^{\mathrm{nr}})^\times)[p^\infty] \longrightarrow H^2(K_v, (K_v)_{\mathrm{sep}}^\times)[p^\infty] \tag{5.2.1}$$

([**GMS-03**, Appendix A]). Let us state the following technical result.

Lemma 5.2.3. ([**GMS-03**, Appendix A, Lemma A.6]) *Let E/F be a Galois field extension and let $G = \mathrm{Gal}(E/F)$. Suppose that G has cohomological p-dimension at most 1*[1]*. Then the natural map*

$$H^2(G, \mathscr{K}_d(E))[p^\infty] \to H^2(F, \mathscr{K}_d(F_{\mathrm{sep}}))[p^\infty]$$

is injective. Here $\mathscr{K}_d(-)$ denotes Milnor K-theory.

This leads to conclude that the map (5.2.1) is an injective morphism as it is a special case of the above, by choosing $E = K_v^{\mathrm{nr}}$, $F = K_v$ and $d = 2$.

Definition 5.2.4. With the above setting, we call the image of (5.2.1) the *tame subgroup* or *tamely ramified subgroup* of $\mathrm{Br}(K_v)[p^\infty]$ associated to v, and we denote it by $\mathrm{Br}_{\mathrm{tame},v}(K_v)[p^\infty]$. We denote its inverse image in $\mathrm{Br}(K)[p^\infty]$ via the natural map induced by inclusion $K \subseteq K_v$ as $\mathrm{Br}_{\mathrm{tame},v}(K)[p^\infty]$ and we similarly call it the *tame subgroup* of $\mathrm{Br}(K)[p^\infty]$ associated to v.

Let us write again v for the unique extension of v to K_v^{nr}; then we have a group morphism

$$v\colon (K_v^{\mathrm{nr}})^\times \to \mathbb{Z}$$

which is $\mathrm{Gal}(K_v^{\mathrm{nr}}/K_v)$-equivariant; by Lemma 5.2.2, the morphism is also $\mathrm{Gal}(k(v))$-equivariant.

Definition 5.2.5. Following [**GMS-03**, Appendix A] one can define a map as the composition

$$\begin{aligned} r_v \colon \mathrm{Br}_{\mathrm{tame}}(K_v)[p^\infty] \longrightarrow H^2(K_v^{\mathrm{nr}}/K_v, (K_v^{\mathrm{nr}})^\times)[p^\infty] \longrightarrow \\ \longrightarrow H^2(k(v), \mathbb{Z})[p^\infty] \simeq H^1(k(v), \mathbb{Q}/\mathbb{Z})[p^\infty] \end{aligned}$$

which we call the *residue map* with respect to the valuation v. We will say that the residue of an element $\alpha \in \mathrm{Br}(K)[p^\infty]$ with respect to a valuation v is *defined*, or equivalently that α is *tamely ramified* at v, if $\alpha \in \mathrm{Br}_{\mathrm{tame},v}(K)[p^\infty]$.

Remark 5.2.6. If $\alpha \in \mathrm{Br}(K)[p]$ and its residue with respect to v is defined as in Definition 5.2.5, then $r_v(\alpha) \in H^1(k(v), \mathbb{Z}/p)$. By Artin–Schreier theory (Lemma 3.1.17), one has $H^1(k(v), \mathbb{Z}/p) \simeq k(v)/\wp(k(v))$. This group classifies pairs, consisting of a finite $\mathbb{Z}/p$-Galois extension of $k(v)$ together with a chosen generator of the Galois group. Indeed, $\mathbb{Z}/p$-Galois extensions of $k(v)$ are Artin–Schreier extensions, namely they are generated by the

[1]For a group G and a prime p, the *cohomological p-dimension* of G is the smallest natural number n such that group cohomology $H^i(G, M)[p^\infty]$ vanishes for all $i > n$, for every G-module M, see [**GS-06**, Section 6.1].

roots of a polynomial $x^p - x - a$ for some $a \in k(v)$. The $k(v)$-isomorphism class of this extension is determined up to the transformations

$$a \mapsto \eta a + (c^p - c)$$

having set $\eta \in \mathbb{F}_p^\times \subseteq (k(v))^\times$ and $c \in k(v)$. In particular, for $p = 2$, one may also identify $H^1(k(v), \mathbb{Z}/2)$ with he set of isomorphism classes of étale algebras of degree 2 over $k(v)$, as seen in [**EKM-08**, Example 101.1].

More geometrically, if D is a prime divisor on a smooth algebraic variety over a field k and v_D the corresponding valuation, the residue r_{v_D} can be thought of as being given by a double covering which is étale over an open part of D; hence the group $H^1(k(v_D), \mathbb{Z}/2) = H^1(k(D), \mathbb{Z}/2)$ where $k(D)$ is the function field of D, classifies these double coverings up to birational isomorphism over D. With this we mean that, two such coverings $\psi_1 \colon D_1 \longrightarrow D$ and $\psi_2 \colon D_2 \longrightarrow D$ induce the same class in $H^1(k(D), \mathbb{Z}/2)$ if and only if there exists an open set $U \subseteq D$ such that the two étale double coverings $\psi_1|_{\psi_1^{-1}(U)}$ and $\psi_2|_{\psi_2^{-1}(U)}$ are isomorphic.

Remark 5.2.7. Keep the notation of Definition 5.2.5. The tame subgroup

$$\mathrm{Br}_{\mathrm{tame},v}(K_v)[p^\infty] = H^2(K_v^{\mathrm{nr}}/K_v, (K_v^{\mathrm{nr}})^\times)[p^\infty]$$

of $\mathrm{Br}(K_v)[p^\infty]$ has a simpler description in terms of Brauer classes thanks to Theorem 3.1.14: it is nothing but the p-primary torsion subgroup of the relative Brauer group $\mathrm{Br}(K_v^{\mathrm{nr}}/K_v)$. The latter, in turn, contains those Brauer classes represented by central simple K_v-algebras that neutralise over the inertia field K_v^{nr} (see Section 3.1.2).

The name "tame subgroup" was suggested by Burt Totaro and has the following explanation: this subgroup consists of those classes that become trivial in $\mathrm{Br}(V)$, where V is the maximal tamely ramified extension of K_v, named *ramification field* (*Verzweigungskörper*, see [**Art-68**, Chapter 4, Section 2]). Indeed, note that

$$V = K_v\big(\sqrt[m]{\pi} : p \nmid m\big) \subseteq K_v^{\mathrm{nr}}$$

where $\pi = \pi_{\mathcal{O}_{K_v}}$ is an uniformiser for the ring of integers $\mathcal{O}_{K_v}$ of K_v (see *ibidem*). Now let $\alpha \in \mathrm{Br}_{\mathrm{tame},v}(K_v)[p^\infty]$; then by definition the class α neutralises over some finite unramified extension L_v/K_v and $L_v = K_v(\sqrt[m]{\pi}) \subset V$ for some m which is not divisible by p. Then

$$\mathrm{cor}_{V/K_v}(\alpha) = \mathrm{cor}_{V/L_v} \circ \mathrm{cor}_{L_v/K_v}(\alpha) = 0$$

which means that α neutralises over V.

5.2.2. Residues for higher cohomology. The construction of residue maps r_v can be carried over in greater generality, mimicking the strategy used to define classical residues and unramified cohomology (Section 3.3.1). Let F be a field of characteristic $p > 0$; in this setting one can define a version of Galois cohomology with "mock p-adic coefficients", following Kato

[**Kat-86**] or Merkurjev [**GMS-03**, Appendix A], in the following way: we define

$$H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) := H^2(F, \mathscr{K}_n(F_{\mathrm{sep}}))[p^\infty] \tag{5.2.2}$$

where $\mathscr{K}_n(F_{\mathrm{sep}})$ is the n-th Milnor K-theory of a separable closure of F, and the cohomology on the right hand side is usual Galois cohomology with coefficients in this Galois module. Notice that the coefficients module $\mathbb{Q}_p/\mathbb{Z}_p(n)$ on the left hand side is just a symbol aimed to point out the similarity with the case of characteristic coprime to p. However, one could also define it via the logarithmic part of the de Rham–Witt complex ([**Kat-86, Suw-95**]), where this symbol is meaningful as an actual coefficients module.

Given a discrete rank 1 valuation v of F with residue field E, one can define a *tame subgroup* (or *tamely ramified subgroup*)

$$H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \subset H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$$

in this more general setting in such a way that one recovers the definition given for the Brauer group in Definition 5.2.5 above: let F_v be the completion, F_v^{nr} its inertia field as above, and put

$$H^{n+1}_{\mathrm{tame},v}(F_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) := H^2(F_v^{\mathrm{nr}}/F_v, \mathscr{K}_n(F_v^{\mathrm{nr}}))[p^\infty] \subset H^2(F_v, \mathscr{K}_n((F_v)_{\mathrm{sep}}))[p^\infty].$$

Note that, again, this is actually a subgroup by [**GMS-03**, Lemma A.6]. Then define the subgroup $H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ as the preimage of the group $H^{n+1}_{\mathrm{tame},v}(F_v, \mathbb{Q}_p/\mathbb{Z}_p(n))$ under the natural map

$$H^{n+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^{n+1}(F_v, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

There is a $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v)$-equivariant residue map in Milnor K-theory

$$\mathscr{K}_n(F_v^{\mathrm{nr}}) \to \mathscr{K}_{n-1}(E_{\mathrm{sep}})$$

and this induces a residue map, defined only on $H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$,

$$r_v \colon H^{n+1}_{\mathrm{tame},v}(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^n(E, \mathbb{Q}_p/\mathbb{Z}_p(n-1)).$$

Here we have used Lemma 5.2.2 in the form $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v) \simeq \mathrm{Gal}(E)$.

5.2.3. Residues and unramified Brauer group. Let us begin this paragraph with a couple of preliminary results.

LEMMA 5.2.8. *Let R be a complete discrete valuation ring with field of fractions K and let K^{nr} be the inertia field of K, as before. Let R^{nr} be the integral closure of R in K^{nr}. Then*

$$\mathrm{Br}(R) = H^2(K^{\mathrm{nr}}/K, (R^{\mathrm{nr}})^\times).$$

PROOF. This is contained in [**AN-68**], at page 289. □

LEMMA 5.2.9. *Let K be the function field of an algebraic variety and v a discrete rank 1 valuation of K. Let $A \subset K$ be the valuation ring, let K_v be the completion of K with respect to the absolute value induced by v and let $A_v \subset K_v$ be the valuation ring of the unique extension of v to K_v. Then* *a Brauer class* $\alpha \in \mathrm{Br}(K)$ *such that* $i^*_{K_v}(\alpha) \in \mathrm{Br}(K_v)$ *comes from a class* $\alpha' \in \mathrm{Br}(A_v)$ *already comes from a class in* $\mathrm{Br}(A)$*.*

Proof. This is a special case of [**CTPS-12**, Lemma 4.1] but we include a self-contained proof for reference.

Suppose the class α is represented by an Azumaya algebra $\mathscr{A}$ over K, and that α' is represented by an Azumaya algebra $\mathscr{B}$ over A_v. By assumption, $\mathscr{A}$ and $\mathscr{B}$ become Brauer-equivalent over K_v, and we can assume that they even become isomorphic over K_v by replacing $\mathscr{A}$ and $\mathscr{B}$ by matrix algebras over them so that they have the same degree. Let $\mathscr{A}_A$ be a maximal A-order of the algebra $\mathscr{A}$ in the sense of Auslander–Goldman [**AG-60**], which means that $\mathscr{A}_A$ is a subring of $\mathscr{A}$ that is finitely generated as an A-module, spans $\mathscr{A}$ over K and is maximal with these properties. We seek to prove that $\mathscr{A}_A$ is Azumaya. Now we know that the base change $(\mathscr{A}_A)_{A_v}$ is a maximal order, but also any Azumaya A_v-algebra is a maximal order, and by [**AG-60**, Proposition 3.5], any two maximal orders over a rank 1 discrete valuation ring are conjugate, so in fact the base change algebra $(\mathscr{A}_A)_{A_v}$ is Azumaya because $\mathscr{B}$ is. But then this implies that $\mathscr{A}_A$ is Azumaya since A_v is faithfully flat over A, so if the canonical algebra morphism $\mathscr{A}_A \otimes \mathscr{A}_A \to \mathrm{End}(\mathscr{A}_A)$ becomes an isomorphism over A_v, it is already an isomorphism over A. $\square$

We are finally ready to prove the p-primary analogue of Theorem 3.3.3.

Theorem 5.2.10. *Let X be a smooth and projective variety over an algebraically closed field k of characteristic p. Assume $\alpha \in \mathrm{Br}(k(X))[p^\infty]$ is such that the residue $r_{v_D}(\alpha)$ is defined in the sense of Definition 5.2.5 and is trivial for all divisorial valuations v_D corresponding to prime divisors D on X. Then $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(X))[p^\infty] = \mathrm{Br}(X)[p^\infty]$.*

Moreover, if $Z \subset X$ is an irreducible subvariety with local ring $\mathscr{O}_{X,Z}$ and the residue $r_{v_D}(\alpha)$ is defined and trivial for all divisors D passing through Z, the class α comes from $\mathrm{Br}(\mathscr{O}_{X,Z})$.

Proof. We show that, under the hypotheses above, we have that α belongs to $\mathrm{Br}_{\mathsf{DIV}_X}(k(X))$, which is enough in virtue of Theorem 4.2.5. Let $K = k(X)$, let $v = v_D$ and let $A = \mathscr{O}_{X,\eta_D}$; there is an exact sequence

$$H^2(K_v^{\mathrm{nr}}/K_v, (A_v^{\mathrm{nr}})^\times)[p^\infty] \to H^2(K_v^{\mathrm{nr}}/K_v, (K_v)^{\mathrm{nr}})^\times)[p^\infty] \xrightarrow{r_v} \\ \xrightarrow{r_v} H^1(k(v), \mathbb{Q}/\mathbb{Z})[p^\infty]$$

which is induced by taking cohomology of the coefficients exact sequence $1 \to (A_v^{\mathrm{nr}})^\times \to (K_v^{\mathrm{nr}})^\times \to \mathbb{Z} \to 0$ where A_v^{nr} is the valuation ring of the unique extension of v from K_v to K_v^{nr}.

Thus it suffices to show that those classes in $\mathrm{Br}_{\mathrm{tame},v}(K)[p^\infty] \subset \mathrm{Br}(K)[p^\infty]$ that are mapped onto the image of $H^2(K_v^{\mathrm{nr}}/K_v, (A_v^{\mathrm{nr}})^\times)[p^\infty]$ via the map

$$\mathrm{Br}_{\mathrm{tame},v}(K)[p^\infty] \to H^2(K_v^{\mathrm{ur}}/K_v, (K_v^{\mathrm{nr}})^\times)[p^\infty] \subset \mathrm{Br}(K_v)[p^\infty]$$

actually come from $\mathrm{Br}(A)[p^\infty]$. Now, by Lemma 5.2.8, we have

$$H^2(K_v^{\mathrm{nr}}/K_v, (A_v^{\mathrm{nr}})^\times) \simeq \mathrm{Br}(A_v).$$

But a class γ in $\mathrm{Br}(K)$ whose image γ_v in $\mathrm{Br}(K_v)$ is contained in $\mathrm{Br}(A_v)$ comes from the valuation ring A of v in K by Lemma 5.2.9, hence is unramified. $\square$

5.3. Residue maps and conic bundles in characteristic 2.

5.3.1. Geometric description of residue maps. Now we would like to develop a geometric description of residue maps for a conic bundle defined over a field of characteristic 2 on the lines of Proposition 3.4.11. This is necessitated by the desire, in the applications, to determine whether two resides are the same or not.

DEFINITION 5.3.1. Let $\pi\colon X \longrightarrow B$ be a conic bundle and let $D \subseteq B$ be a prime divisor. We say that π is *tamely ramified along* D if the geometric generic fibre of the restriction $\pi|_{\pi^{-1}(D)}\colon X \times_B D \longrightarrow D$ is a cross of lines (as in Corollary 5.1.12).

For a conic bundle $\pi\colon X \longrightarrow B$ which is tamely ramified along D, the restriction $\pi|_{\pi^{-1}(D)}\colon X \times_B D \longrightarrow D$ is a (non-split) double covering of D, which is étale on an open subset of D. Equivalently it corresponds to a non-trivial Artin-Schreier extension of $k(D)$ generated by a root ξ of the polynomial $x^2 + x + a_D$ for some $a_D \in k(D)$. Following the terminology introduced in Section 3.4 we will refer to this covering as to the *associated double covering of* D.

The next result is a computation that establishes a local normal form for tamely ramified conic bundles.

PROPOSITION 5.3.2. *Let $\pi\colon X \longrightarrow B$ be a conic bundle and let D be a prime divisor in B. Assume that π is either tamely ramified along D or the geometric generic fibre of $\pi|_{\pi^{-1}(D)}$ is a smooth conic; let also $P \in D$ be a point above which the fibre X_P is reduced. Then we can assume that Zariski locally around P the variety X is cut out by*

$$ax^2 + by^2 + xz + z^2 = 0$$

where x, y, z are fibre coordinates and a, b are functions on B which are regular locally around P, with b not identically zero.

PROOF. Locally around P, the fibre X_P is cut out by an equation of the following form

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{xz}xz + a_{yz}yz = 0$$

where $a_{\bullet,\bullet}$ are regular functions locally around P. Since X_P is reduced, it must be that one of the coefficients of the mixed terms does not vanish at P; without loss of generality we can assume $a_{xz}(P) \neq 0$. Let us now perform the change of coordinates $x \mapsto x/a_{xz}$; with a slight abuse of notation, we still call x, y, z the new coordinates and we get the equation

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + xz + a_{yz}yz = 0.$$

We now perform the change of coordinates $x \mapsto x + a_{yz}y, \quad y \mapsto y, \quad z \mapsto z + a_{xy}y$ and we get (again, committing the same abuse of notation)

$$a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + xz = 0.$$

Suppose first that one of a_{xx} and a_{zz} is non-zero at P; without loss of generality assume a_{zz} and apply the substitution $x \mapsto a_{zz}x$. This leads to the desired local equation:

$$a_{xx}x^2 + a_{yy}y^2 + xz + z^2 = 0. \tag{5.3.1}$$

Instead, if both a_{xx} and a_{zz} vanish at P, we change coordinates by means of $x \mapsto x + z$ and we get $a_{zz}(P) \neq 0$ in the new coordinates, so one can resort to the previous case. □

Let K be a field of characteristic $p > 0$; we recall that the product in Milnor K-theory $\mathscr{K}_i(K_{\text{sep}})$ and cup product in Galois cohomology induce a cup product

$$\mathscr{K}_i(K) \otimes H^{n+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^{n+i+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n+i)),$$

which restricts to the tamely ramified subgroups, as in [**GMS-03**, p.152, p. 154]. See Paragraph 5.2.2 for the definitions of these cohomology groups.

Proposition 5.3.3. *Let K be a field of characteristic 2 and let $Q \subseteq \mathbf{P}^2_K$ be the conic defined by $ax^2 + by^2 + xz + z^2 = 0$ for $a \in K$ and $b \in K^\times$. Then the Brauer class associated to Q is the cup product $b \smile a$ via the cup product homomorphism*

$$\mathscr{K}_1(K) \otimes H^1(K, \mathbb{Q}_2/\mathbb{Z}_2(0)) \to H^2(K, \mathbb{Q}_2/\mathbb{Z}_2(1)) = \operatorname{Br}(K)\{2\},$$

where we consider $a \in H^1(K, \mathbb{Q}_2/\mathbb{Z}_2)[2] = K/\wp(K)$ and $b \in \mathscr{K}_1(K) = K^\times$.

Proof. The Brauer class associated to Q is the quaternion algebra $[a, b)$ as in Definition 3.1.1. Let L/K be the neutralising field extension, which is an Artin-Schreier extension of K generated by the roots of $x^2 - x - a$; let now $\chi_{L/K} \colon \operatorname{Gal}(K) \to \mathbb{Z}/2$ be the canonically associated character of the absolute Galois group of K. By [**GS-06**, Corollary 2.5.5b], the quaternion algebra $[a, b)$ is K-isomorphic to the cyclic algebra $(\chi_{L/K}, b)$, generated as a K-algebra by L and an element y subject to the relations

$$y^2 = b, \quad \lambda y = y\sigma(\lambda)$$

where $\lambda \in L$ and σ is the generator of $\operatorname{Gal}(L/K)$.

Letting $\delta \colon H^1(K, \mathbb{Z}/2) \to H^2(K, \mathbb{Z})$ be the *Bockstein homomorphism*, namely the connecting morphism induced from taking Galois cohomology of the exact sequence of trivial Galois modules

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0,$$

then $\delta \colon H^1(K, \mathbb{Z}/2) \to H^2(K, \mathbb{Z})[2]$ is an isomorphism that gives meaning to Merkurjev's definition $H^1(K, \mathbb{Z}/2(0)) := H^2(K, K_0(F_{\text{sep}}))[2] = H^2(K, \mathbb{Z})[2]$.

In virtue of [**GS-06**, Proposition 4.7.3], the cup product pairing in Galois cohomology

$$H^2(K,\mathbb{Z}) \times H^0(K,K_{\mathrm{sep}}^\times) \to H^2(K,K_{\mathrm{sep}}^\times) = \mathrm{Br}(K)$$

has the property that the cup product of the class $\delta(a) \in H^2(K,\mathbb{Z})[2]$, where we consider $a \in K/\wp(K) = H^1(K,\mathbb{Z}/2)$, with the class $b \in H^0(K,K_{\mathrm{sep}}^\times) = K^\times$, results in the Brauer class of the cyclic algebra $(\chi_{L/K}, b) \in \mathrm{Br}(K)[2]$.

Finally, under the canonical identification $H^0(K,K_{\mathrm{sep}}^\times) = K^\times = \mathscr{K}_1(K)$, the isomorphism $\delta\colon H^1(K,\mathbb{Z}/2) \to H^2(K,\mathbb{Z})[2]$, and the definition of the action of $\mathscr{K}_1(K)$ on $H^1(K,\mathbb{Q}_2/\mathbb{Z}(0))$, the cup product

$$H^1(K,\mathbb{Z}/2(0)) \times \mathscr{K}_1(K) \to H^2(K,\mathbb{Z}/2(1)) = \mathrm{Br}(K)[2]$$

is identified with the 2-torsion part of the above cup product in Galois cohomology. Since the cup product commutes on 2-torsion classes, we get the desired formula. $\square$

We are finally ready to prove the envisaged geometric characterisation of residues for conic bundles.

THEOREM 5.3.4. *Let k be an algebraically closed field of characteristic 2 and let $\pi\colon X \to B$ be a conic bundle with $\alpha_\pi \in \mathrm{Br}(k(B))[2]$ being its associated Brauer class . Let D be a prime divisor on B and let v_D be the unique divisorial valuation over $k(B)$ corresponding to D. Then the residue $r_{v_D}(\alpha_\pi)$ is defined in the following two cases:*

(1) *if the geometric generic fibre of $\pi|_{\pi^{-1}(D)}$ is a smooth conic;*
(2) *if π is tamely ramified along D.*

In particular, one has:

(1) *in case (1), $r_{v_D}(\alpha_\pi) = 0$;*
(2) *in case (2) the divisor D is an irreducible component of the discriminant Δ of π and the associated double covering of D is an Artin-Schreier extension $k(D)(\xi)/k(D)$ where ξ is a root of $x^2 + x + a_D$ for some $a_D \in k(D)$; one has*
   (a) *if D has even multiplicity on Δ, then $r_{v_D}(\alpha_\pi) = 0$*
   (b) *if D has odd multiplicity on Δ, then $r_{v_D}(\alpha_\pi) = [a_D] \in k(D)/\wp(k(D))$*

PROOF. We have to check that α_π belongs to $\mathrm{Br}_{\mathrm{tame},v}(k(B))$ for $v = v_D$ in both cases described above; recalling Remark 5.2.7, this amounts to check that, in both cases, $\alpha_\pi \in \mathrm{Br}(K_v^{\mathrm{nr}}/K_v)$, namely, that α_π neutralises over K_v^{nr}. Since locally one has the description found in Proposition 5.3.2, the conic bundle splits over the Galois covering of the base obtained by adjoining to $k(B)$ the roots of $T^2 + T + a$; indeed in such case the quadratic form in Proposition 5.3.2 acquires a zero. Moreover, that Galois covering does not ramify at the generic point of D because a has no pole along D. Hence it defines an extension of K_v contained in K_v^{nr}.

By formula (5.1.5) we find that the discriminant of $ax^2 + by^2 + xz + z^2 = 0$ is cut out by $b = 0$, hence up to absorbing even powers of a local parameter for D into the fibre coordinate y, we can assume b itself is either a local parameter for D (in case (2)) or a unit generically along D (in case (1)). In particular $b \neq 0$ so by Proposition 5.3.3, the Brauer class $\alpha_\pi \in \mathrm{Br}(k(B))[2^\infty] = H^2(k(B), \mathbb{Q}_2/\mathbb{Z}_2(1))$ associated to the conic bundle defined by the preceding formula is the cup product $\alpha = b \smile a$ of the class $b \in \mathscr{K}_1(k(B)) = k(B)^\times$ and the class $a \in H^1(k(B), \mathbb{Q}_2/\mathbb{Z}_2) = k(B)/\wp(k(B))$.

Now let $\xi \in \mathscr{O}_{B,\eta_D}$ be a local equation for D. One has that a vanishes along D if and only if a is congruent to a non-zero scalar modulo ξ; since k is algebraically closed, after performing a change of coordinates, one can say that a polynomial in a with coefficients in k vanishes along D if and only $a = u \cdot \xi^m$ in $\mathscr{O}_{B,\eta_D}$ for some $u \in \mathscr{O}_{B,\eta_D}^\times$ and $m > 0$. Therefore, we have two cases:

Case 1. If a is not of the above form, then $v_D(a) \neq 0$ and thus $k(a) \subset k(B)$ is a subfield of the valuation ring $\mathscr{O}_{B,\eta_D}$ of v_D. By the reasoning at [**GMS-03**, (A.8)], the element that a induces in $H^1(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$ is in $H^1_{\mathrm{tame},v_D}(k(B), \mathbb{Q}_2/\mathbb{Z}_2)$, and then formula (A.8) *ibidem* implies

$$r_v(b \smile a) = \begin{cases} a|_D & \text{if } \mathrm{ord}_D(b) = 1 \\ 0 & \text{if } b \in \mathscr{O}_{B,\eta_D}^\times \end{cases}$$

Since $a|_D$ is precisely the element defining the associated double covering of $D = \{b = 0\}$, the residue is given by this geometrically defined double covering.

Case 2. If $a = u \cdot \xi^m$, since $\dim B \geq 2$, we can find a unit $a' \in \mathscr{O}_{B,\eta_D}^\times$ that is not congruent to an element of k modulo ξ, and write $a = (a - a') + a'$. Now we can apply Step 1 to $a - a'$ and a' and this finishes the proof since

$$b \smile a = b \smile (a - a') + b \smile a'$$

and

$$r_{v_D}(b \smile a) = r_{v_D}(b \smile (a - a')) + r_{v_D}(b \smile a') = (a - a')|_D + a'|_D = a|_D$$

as wished. □

5.3.2. Discriminant profile of conic bundles in characteristic 2. In this section, we work over an algebraically closed ground field k of arbitrary characteristic. Let $\pi\colon X \longrightarrow B$ be a conic bundle over k and let us first give the following definition.

Definition 5.3.5. Assume we are working over a field of characteristic 2. Denote by $B^{(1)}$ the set of all valuations of $k(B)$ corresponding to prime divisors on B. Let $\pi\colon X \longrightarrow B$ be a conic bundle and let $\alpha_\pi \in \mathrm{Br}(k(B))[2]$ be its associated Brauer class. Assume that that π satisfies one of the conditions (1) and (2) of Theorem 5.3.4 along the centre of each valuation v. We call *residue profile* of π the family $(\alpha_x)_{x \in B^{(1)}}$ such that:

$$(\alpha_v = r_v(\alpha_\pi))_{v \in B^{(1)}} \in \bigoplus_{v \in B^{(1)}} k(v)/\wp(k(v))$$

where $k(x)$ is the residue field of v.

Remark 5.3.6. Note that the valuations v for which the component in $H^1(k(v), \mathbb{Z}/2)$ of the residue profile of a conic bundle is non-trivial are a (possibly proper) subset of the divisorial valuations corresponding to the discriminant components of the conic bundle, accordingly to Theorem 5.3.4.

Let us compare the above definition with the analogue in characteristic not 2, Definition 3.4.9, in the case B is a smooth, projective rational surface. One main difference (besides the fact that the residue profiles are governed by Artin–Schreier theory in characteristic 2 as opposed as Kummer theory in characteristic not 2) is the following: for $\mathrm{char}(k) \neq 2$ the residue profiles of conic bundles that can occur can be characterised as elements in the kernel of another explicit morphism, induced by further residues; more precisely, there is a map

$$s\colon \bigoplus_{v\in B^{(1)}} H^1(k(v), \mathbb{Z}/2) \longrightarrow \bigoplus_{p\in B^{(2)}} \mathbb{Z}/2$$

where $B^{(2)}$ is the set of codimension 2 points of B, which fits into the sequence illustrated in Proposition The map s is induced by residue maps ∂^1_p as defined in Paragraph 3.3.2 and, on each summand, amounts to taking the order of zeros and poles of a function in $k(v)^\times/(k(v)^\times)$ at a point $p \in B$, and then taking the remainder modulo 2 (if the centre of the valuation v is not smooth at p one has to make a slightly more refined definition involving the normalisation).

As we have already pointed out in Remark 3.4.10, this sequence is a powerful existence result for conic bundles with prescribed discriminant profile over field of characteristic $\neq 2$. However, in characteristic 2, we cannot expect a sequence that naïvely has similar exactness properties. The following example, which will be essential in the concrete application of our machinery later on, exhibits this phenomenon.

Example 5.3.7. Let $Y \subset \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ be the conic bundle defined, over $k = \overline{\mathbb{F}_2}$, by an equation

$$Q = ax^2 + axz + by^2 + byz + cz^2 = 0,$$

where x, y, z are fibre coordinates in the "fibre copy" $\mathbf{P}^2$ in $\mathbf{P}^2 \times \mathbf{P}^2$, and a, b, c are general linear forms in the homogeneous coordinates u, v, w on the base $\mathbf{P}^2$.

We now determine the locus Δ of points $[u : v : w] \in \mathbf{P}^2$ such that

$$Q_x = az = 0,\ Q_y = bz = 0,\ Q_z = by + ax = 0$$
$$ax^2 + by^2 + cz^2 = 0$$

simultaneously; notice that the last relation is necessary since it is not implied by the previous ones due to $\mathrm{char}\, k > 0$. Since taking squares is an automorphism of k, the last equation can be rewritten as $(x\sqrt{a}+y\sqrt{b}+z\sqrt{c})^2 = 0$, hence we are looking for the values u, v, w that allow a non-trivial solution of the following linear system:

$$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \\ \sqrt{a} & \sqrt{b} & \sqrt{c} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This happens if and only if the associated matrix has rank at most 2, which in turn happens if $a = 0$, $b = 0$ or $a = b$. Hence

$$\Delta = \{ab(a+b) = 0\}$$

is the desired discriminant locus. This is the union of three distinct lines (with multiplicity 1 each) meeting at the single point $P = [0 : 0 : 1]$.

More precisely, the conic bundle is tamely ramified along each irreducible component of Δ: it induces Artin–Schreier double coverings ramified only in P on each of those lines: For $\Delta_1 = \{a = 0\}$ the total space is cut out by

$$b(y^2 + yz) + cz^2 = 0$$

which describes a non-trivial Artin–Schreier covering ramified only at $b = 0$. The associated residue is $[c/b] \in H^1(k(\Delta_1), \mathbb{Z}/2)$. Similarly, for $\Delta_2 = \{b = 0\}$ the total space is cut out by

$$a(x^2 + xz) + cz^2 = 0$$

and the residue is $[c/a] \in H^1(k(\Delta_2), \mathbb{Z}/2)$. Finally the total space above $\Delta_2 = \{a = b\}$ is cut out by:

$$\begin{aligned} &a(x^2 + xz + y^2 + yz) + cz^2 \\ &\qquad = a\big((x^2 + y^2) + (x+y)z\big) + cz^2 \\ &\qquad\qquad = a\big((x+y)^2 + (x+y)z\big) + cz^2. \end{aligned}$$

which is again a non-trivial Artin-Schreier covering ramified only at $a = 0$. The residue is again $[c/a] \in H^1(k(\Delta_3), \mathbb{Z}/2)$.

The preceding example shows that we can not expect a naïve analogue of Proposition 3.3.8 in characteristic 2: to define a reasonable further residue map to codimension 2 points, the only thing that springs to mind here would be to assign some measure of ramification at P for each of the three Artin–Schreier coverings. But the resulting ramification measures would have to add to zero (modulo 2), and would have to be the same for each of the coverings (as they are each birational to the other over $\mathbf{P}^1$), so that only the slightly non-geometric option to assign ramification zero would remain.

Note that the conic bundle in Example 5.3.7, when lifted to characteristic 0 by interpreting the coefficients in the defining equation as linear forms over $\mathbb{Z}$, has discriminant consisting of the triangle of lines $a = 0, b = 0, 4c - a - b = 0$, with double (Kummer) coverings over each of the lines ramified in the vertices of the triangle. That might suggest that we should define a further residue map also in characteristic 2 by using local lifts to characteristic 0 and then summing the ramification indices in those points that become identical when reducing modulo 2, an idea that is reminiscent of certain constructions in crystalline cohomology and log geometry. But we have not succeeded in carrying this out yet.

Moreover, the theory in [**Kat-86**], although developed also in cases where the characteristic equals the torsion order of the Brauer classes under consideration, gives no satisfactory solution either because the arithmetical Bloch–Ogus complex in [**Kat-86**, Section 1] we would need to study would be the one for parameters $i = -1, q = 0$ and then condition (1.1) *ibidem* is not satisfied, whence the further residue map we are looking for is undefined.

This seems to indicate that we have to do without an analogue of the reciprocity sequence for the case of conic bundles of characteristic 2, and this is exactly what we will do in Section 5.4: we will simply assume existence of certain Brauer classes with predefined residue profiles, and we will prove this existence in practice by writing down conic bundles over the bases under consideration that have the sought-for residue profiles.

In fact, the next result partly explains Example 5.3.7 and also shows that the situation in characteristic 2 can be even more interesting.

PROPOSITION 5.3.8. *Let $\pi\colon X \to B$ be a conic bundle in characteristic 2, where B is a smooth projective surface and let Δ be its discriminant. Then there is no point p of Δ locally analytically around which Δ consists of two smooth branches Δ_1, Δ_2 intersecting transversely at p and in a way such that, at p, the fibre of π is a double line, and near p, the fibres of π over points in $\Delta_1\backslash\{p\}$ and $\Delta_2\backslash\{p\}$ are crosses of lines.*

PROOF. Let $p \in \Delta$ be a point in the discriminant. Then, as in Remark 5.1.13 and Definition 5.1.14, let $\mathbf{P}^2$ have homogeneous coordinates $(X_0 : X_1 : X_2)$ and $X_{\mathrm{univ}} \longrightarrow \mathcal{Q}$ be the universal conic bundle, and let $U \subset B$ be a Zariski open neighbourhood of p such that $\Delta_i \cap U \neq \emptyset$ for every irreducible component of Δ passing through p, and such that there is a morphism $f\colon U \to \mathcal{Q}$ realizing $\pi|_{\pi^{-1}(U)}\colon X \times_B U \to U$ as the pull-back via f of the universal conic bundle.

Let $\mathfrak{R} \subseteq \Delta$ be the locus of double lines and recall that $\mathcal{Q} \simeq \mathbf{P}^5$ has homogeneous coordinates $(a_{00}, a_{11}, a_{22}, a_{01}, a_{02}, a_{12})$. Let $f(p) = q$ and assume $q \in \mathfrak{R}$; after a coordinate change we can assume (independently of $\mathrm{char}\, k$) that q has coordinates $a_{00} = 1$ and all other coordinates equal to zero. Expanding the equation (5.1.5) locally around the point q, we get the following local equation of Δ_{univ} around q (we denote the de-homogenisation of each coordinate with the same letter; also q becomes the origin in these affine coordinates)

$$a_{12}^2 + a_{01}^2 a_{22} + a_{01}a_{02}a_{12} + a_{02}^2 a_{11}. \tag{5.3.2}$$

The leading term is a_{12}^2, whereas in characteristic not equal to 2, the same procedure applied to (5.1.4) yields

$$\left(4a_{11}a_{22} - a_{12}^2\right) + a_{01}a_{02}a_{12} - a_{02}^2a_{11} - a_{01}^2a_{22} \tag{5.3.3}$$

with leading term $4a_{11}a_{22} - a_{12}^2$. Now the discriminant $\Delta \cap U$ is given, in the characteristic 2 case, by

$$f^*(a_{12})^2 + f^*(a_{01})^2 f^*(a_{22}) + f^*(a_{02})^2 f^*(a_{11}) + f^*(a_{01})f^*(a_{02})f^*(a_{12})$$

showing that the projectivised tangent cone to Δ at p is either non-reduced of degree 2 or has degree at least three (in Example 5.3.7 the latter possibility occurs). This proves the first assertion of the Theorem. $\square$

REMARK 5.3.9. Note that, in characteristic not 2, the configuration ruled out by Proposition 5.3.8 is precisely the generic local normal form of the discriminant of a conic bundle around a point above which the fibre is a double line: indeed, by (5.3.3) and the subsequent expansion, the tangent cone to Δ in p is generically equal to two distinct lines.

5.4. A formula for the unramified Brauer group of a conic bundle threefold in characteristic 2.

In this section we give a criterion for the non-triviality of the unramified Brauer group of conic bundle threefolds defined over fields of characteristic 2. The criterion seeks to emulate the statements of Proposition 3.4.12 and Theorem 3.4.15 but has substantial differences.

THEOREM 5.4.1. *Let $\pi\colon X \longrightarrow B$ a conic bundle defined over an algebraically closed field k of characteristic 2, where B is a smooth, projective surface. Let Δ be the discriminant component and let Δ_i for $i \in I$ its irreducible components. Suppose that π is tamely ramified along each Δ_i as in case (2) in Theorem 5.3.4. Let $\alpha \in \mathrm{Br}(k(B))[2]$ be the Brauer class defined by the conic bundle and let*

$$(\alpha_i)_{i\in I} \in \bigoplus_{i\in I} H^1(k(\Delta_i), \mathbb{Z}/2) \simeq \bigoplus_{i\in I} k(\Delta_i)/\wp(k(\Delta_i))$$

be the discriminant profile. Suppose that $I = I_1 \sqcup I_2$ and:

1. *there exist a tamely ramified conic bundle $\pi'\colon Y \longrightarrow B$ with residue profile $(\alpha_i)_{i\in I_1}$ and such that for each point $p \in \Delta_i \cap \Delta_j$ for $i \in I_1, j \in I_2$, the fibre Y_p is a cross of lines;*
2. *there exist $i_0 \in I_1$ and $j_0 \in I_2$ such that $\alpha_{i_0} \neq 0$ and $\alpha_{j_0} \neq 0$.*

Then, $\mathrm{Br}_{\mathrm{nr}}(k(X))[2]$ *is non-trivial.*

REMARK 5.4.2. Notice that the total space X need not be non-singular. By the work of Cossart and Piltant [**CP-08**], [**CP-09**], resolution of singularities is known for quasi-projective threefolds in arbitrary characteristic. Then a smooth projective model $\widetilde{X}$ of X exists and we have $\mathrm{Br}_{\mathrm{nr}}(k(X))[2] = \mathrm{Br}(\widetilde{X})[2]$. Still, in all applications we will exhibit such a resolution explicitly.

Proof. By Theorem 3.2.8 combined with Theorem 3.2.7, the kernel of the natural pull-back morphism

$$\pi^*\colon \operatorname{Br} k(B) \to \operatorname{Br}(C) \xrightarrow{i_C^*} \operatorname{Br} k(C) \simeq \operatorname{Br} k(Y)$$

is generated by the class $\alpha \in \operatorname{Br} k(B)$ corresponding to the generic fibre of the conic bundle itself. Denote by β the class of $\pi'\colon Y \to B$ in $\operatorname{Br}(k(B))$. We claim that $\pi^*(\beta) \in \operatorname{Br}(k(X))$ is non-trivial and unramified. It is non-trivial because $\beta \neq \alpha$ by assumption (2): α and β have different residues along some irreducible component Δ_{j_0} of Δ, more precisely $r_{\Delta_{j_0}}(\alpha) \neq 0$ while $r_{\Delta_{j_0}}(\beta) = 0$.

In order to check that $\pi^*(\beta)$ is unramified, it suffices to check that for any valuation $v = v_D$ corresponding to a prime divisor D on a model $X' \simeq_{\mathrm{bir}} X$ which is smooth at the generic point of D, we have that $\pi^*(\beta)$ is unramified with respect to that valuation, in the sense that it lies in the image of $\operatorname{Br}(\mathscr{O}_{X',D})$. Let

$$\Delta^{(1)} := \bigcup_{i \in I_1} \Delta_i, \quad \Delta^{(2)} := \bigcup_{j \in I_2} \Delta_j.$$

First of all, notice that the residue is defined for β and α with respect to any divisor on B, by the assumption that the geometric generic fibre of π is smooth and by Theorem 5.3.4. We need to distinguish two cases:

(1) The centre Z_v of v on B (in other words, the image of $D \subseteq X'$ in B) , is not contained in $\Delta^{(1)} \cap \Delta^{(2)}$. It follows that β or $\beta - \alpha$ have residue zero along every divisor $D' \subseteq B$ passing through Z_v. By Theorem 5.2.10, the class $\beta - \alpha$ comes from $\operatorname{Br}(\mathscr{O}_{B,Z})$. But $\pi^*(\beta - \alpha) = \pi^*(\beta)$, and hence $\pi^*(\beta)$ comes from $\operatorname{Br}(\mathscr{O}_{X',D})$ as desired.

(2) The centre Z of v on B is contained in $\Delta^{(1)} \cap \Delta^{(2)}$, hence it is a point $Z = P$ over which the fibre Y_P is a cross of lines by the assumption in 1) of the Theorem. Then the class $\pi^*(\beta)$ is represented by a conic bundle on X' whose residue along D is defined and trivial by Theorem 5.3.4. So $\pi^*(\beta)$ comes from $\operatorname{Br}(\mathscr{O}_{X',D})$ as desired by Theorem 5.2.10 again.

Thus $\pi^*(\beta) \in \operatorname{Br}_{\mathrm{nr}}(k(X))[2]$ is a non-trivial, unramified class. $\square$

Let us compare this new formula with the classical results for conic bundles in characteristic $\neq 2$ (Proposition 3.4.12).

Condition (1) in the statement is a necessary substitute for conditions (2) and (3) of Proposition 3.4.12, which guarantee that the sub-profiles selected to yield unramified classes give rise to conic bundles with the sought-for discriminant profile. Notice that, by the discussion following Example 5.3.7, it is hard to replace this condition with a cohomological or syzygy-theoretic criterion.

Condition (2) is the analogous of condition (3) in Proposition 3.4.12 and makes sure that the sub-profile chosen is not the entire original residue profile, which would give rise to $\beta = 0$. Such condition does not hold, for example, if the discriminant locus is irreducible.

5.5. A stably irrational conic bundle threefold.

In this section we present an application of Theorem 5.4.1 which also leads to a new example of conic bundle which is not stably rational over $\mathbb{C}$. The example is illustrated in the original paper [**ABBvB-18**] and was found using the computer algebra system `Macaulay2` with Jakob Kröker's packages `FiniteFieldExperiments` and `BlackBoxIdeals` [**Kr-15**].

5.5.1. Setup and overview. We begin with considering the following 3×3 symmetric matrix with entries in $\mathbb{Z}[u, v, w]$.

$$M(u,v,w) := \begin{pmatrix} 2uv + 4v^2 + 2uw + 2w^2 & u^2 + uw + w^2 & uv \\ u^2 + uw + w^2 & 2u^2 + 2vw + 2w^2 & u^2 + vw + w^2 \\ uv & u^2 + vw + w^2 & 2v^2 + 2uw + 2w^2 \end{pmatrix}$$

which defines a quadratic form via the vanishing of the polynomial

$$(x\ y\ z) M(u,v,w) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Notice that, since the diagonal elements of $M(u, v, w)$ are divisible by 2, the matrix is symmetric and all its entries are homogeneous of degree 2, the polynomial

$$f(x,y,z,u,v,w) := \frac{1}{2}(x\ y\ z) M(u,v,w) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a bi-homogeneous element of $\mathbb{Z}[x, y, z] \otimes_{\mathbb{Z}} \mathbb{Z}[u, v, w]$, hence its zero locus $X := \{f = 0\}$ is a divisor in $\mathbf{P}^2_{\mathbb{Z}} \times \mathbf{P}^2_{\mathbb{Z}}$ of bi-degree $(2, 2)$, where we conventionally set the coordinates to be u, v, w on the first factor and x, y, z on the second one.

The projection onto the first factor induces a morphism $\pi\colon X \longrightarrow \mathbf{P}^2_{\mathbb{Z}}$ which is a conic bundle; for each $[u_0, v_0, w_0] \in \mathbf{P}^2_{\mathbb{Z}}$ in the base, the corresponding fibre is the conic cut out by the quadratic equation

$$f(x,y,z,u_0,v_0,w_0) = \frac{1}{2}(x\ y\ z) M(u_0,v_0,w_0) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

in $\mathbf{P}^2_{\mathbb{Z}}$. Notice that the discriminant locus of π is given by $\Delta = \{[u : v : w] \in \mathbf{P}^2_{\mathbb{Z}} : \det M(u, v, w)/2 = 0\}$; we also point out that $\det M$ is a polynomial divisible by 2, hence $D(u, v, w) := \det M/2$ is still a polynomial with coefficients in $\mathbb{Z}$.

Let p be a prime number and let $\overline{\mathbb{F}_p}$ be an algebraic closure of the finite field $\mathbb{F}_p$. We define

$$X_{(p)} := X \times_{\mathbb{Z}} \operatorname{Spec}(\overline{\mathbb{F}_p})$$

the reduction modulo p of X.

Note that $X_{(p)}$ is still a projective subvariety of $\mathbf{P}^2_{\overline{\mathbb{F}_p}} \times \mathbf{P}^2_{\overline{\mathbb{F}_p}}$ which admits a conic bundle structure $\pi = \pi_{(p)} \colon X_{(p)} \longrightarrow \mathbf{P}^2_{\overline{\mathbb{F}_p}}$ inherited by X; the defining equation of $X_{(p)}$ is just $f \equiv 0 \mod p$ and similarly the discriminant locus is the divisor $\Delta_{(p)}$ cut out by the equation $D \equiv 0 \mod p$.

In the rest of the section we aim to prove the following result.

THEOREM 5.5.1. *The threefold conic bundle* $\pi \colon X' \colon = X \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{C}) \longrightarrow \mathbf{P}^2_{\mathbb{C}}$ *has total space* X' *which is not stably rational.*

We will divide the proof in the following intermediary results.

PROPOSITION 5.5.2. *The conic bundle* $\pi : X_{(2)} \longrightarrow \mathbf{P}^2_{\overline{\mathbb{F}_2}}$ *satisfied the hypotheses of Theorem 5.4.1 and in particular* $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}_2}(X_{(2)}))[2] \neq 0$.

PROPOSITION 5.5.3. *There exists a* CH_0*-desingularisation* $\sigma : \widetilde{X}_{(2)} \longrightarrow X_{(2)}$.

Once the above results have been established, Theorem 5.5.1 will be a consequence of Theorem 2.3.6 as following.

PROOF. (*Theorem 5.5.1*) Let $\mathrm{W}(\overline{\mathbb{F}_2})$ be the ring of Witt vectors over an algebraic closure of $\mathbb{F}_2$; it is known that this is a complete discrete valuation ring with residue field $\overline{\mathbb{F}_2}$ and fraction field K of characteristic 0 (this is actually the only complete DVR with these properties up to isomorphism, see [**Ser-79**, Chapter II, Section 5]).

Let $\mathfrak{X} := X \times_{\mathbb{Z}} \mathrm{Spec}\,\mathrm{W}(\overline{\mathbb{F}_2})$. The natural projection $\mathfrak{X} \longrightarrow \mathrm{Spec}\,\mathrm{W}(\overline{\mathbb{F}_2})$ is flat and is such that $\mathfrak{X}_0 = X_{(2)}$. Proposition 5.5.2 and Theorem 4.1.1 imply that $\widetilde{X}_{(2)}$ cannot be **UCT**. By Proposition 5.5.3 we can apply Theorem 2.3.6 in the contrapositive form: this implies that the geometric generic fibre $\mathfrak{X}_{\overline{K}} := X \times_{\mathbb{Z}} \mathrm{Spec}(\overline{K})$ is not **UCT**, and *a fortiori* it has to be stably irrational over $\overline{K}$. Now, since $\mathbb{Z} \subseteq \mathbb{Z}_2 \simeq \mathrm{W}(\mathbb{F}_2) \subseteq \mathrm{W}(\overline{\mathbb{F}_2})$ by functoriality of the ring of Witt vectors ([**Ser-79**, Chapter II, Section 6]), passing to the fraction field yields that $\mathbb{Q} \subseteq K$. Thus

$$\mathfrak{X}_{\overline{K}} = X \times_{\mathbb{Z}} \mathrm{Spec}(\overline{K}) \simeq X \times_{\mathbb{Z}} \mathrm{Spec}(\overline{\mathbb{Q}}) \times_{\overline{\mathbb{Q}}} \mathrm{Spec}(\overline{K})$$

is a base change of X; in more elementary terms, we have just said that $\mathfrak{X}_{\overline{K}}$ has equation given by the same equation of X viewed with coefficient in $\overline{K} \supset \overline{\mathbb{Q}}$. Hence, X cannot be stably rational over $\overline{\mathbb{Q}}$ as well: if it were, any base change would have been so (in particular, $\mathfrak{X}_{\overline{K}}$ would have been so). Finally, this proves that X is not stably rational over any algebraically closed field of characteristic 0 (see [**KSC-04**, Proposition 3.33]). $\square$

5.5.2. The case of $X_{(p)}$ for $p \neq 2$. One might wonder whether it would be possible to obtain the results claimed in Theorem 5.5.1 by applying the degeneration method to some other reduction $X_{(p)}$ for $p \neq 2$ and by using unramified cohomology to determine a non-zero Brauer class. Here we show that this is not the case: Proposition 5.5.7 shows that the degeneration method, using reduction modulo $p \neq 2$ and the unramified Brauer group, cannot yield this result. This follows from work of Colliot-Thélène, see [**Pir-16**, Theorem 3.13, Remark 3.14]; note that one only has to assume X is a threefold which is non-singular in codimension 1 in Theorem 3.13 *ibidem*. Likewise, usage of differential forms as in [**AO-18**], see in particular their Theorem 1.1 and Corollary 1.2, does not imply the result either.

Our first aim is to prove that $\Delta_{(p)}$ is irreducible for $p \neq 2$. This is easy for generic p since X is smooth over $\mathbb{Q}$ (by a straight-forward Gröbner basis computation [**ABBvB-18**, M2 files]). Since being singular is a codimension 1 condition, we expect that $\Delta_{(p)}$ is singular for a finite number of primes. So we need a more refined argument to prove irreducibility. Our idea is to prove that there is at most one singular point for each $p \neq 2$ (counted with multiplicity).

Lemma 5.5.4. *Let C be a reduced and reducible plane curve of degree at least 3 over an algebraically closed field. Then the length of the singular sub-scheme, defined by the Jacobi ideal on the curve, is at least 2.*

Proof. The only singularities of length 1 are those where two smooth branches of the curve cross transversely (étale locally at the point): if $f(x, y) = 0$ is a local equation for C with isolated singular point at the origin, then the length can only be 1 if

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

have leading terms consisting of linearly independent linear forms. This means two smooth branches cross transversely the origin. The only reducible curve that has only one transverse intersection is the union of two lines. $\square$

We also need the following technical lemma.

Lemma 5.5.5. *Let $I \subset \mathbb{Z}[u, v, w]$ be a homogeneous ideal, $\mathscr{B} = \{\lambda_1, \ldots, \lambda_n\}$ a Z-Basis of the space of linear forms $I_1 \subset I$, and M the $n \times 3$ matrix of coefficients of the λ_i. Let g be the minimal generator of the ideal of order 2 minors of M in $\mathbb{Z}$. If a prime p does not divide g, then I defines a finite scheme of degree at most 1 in characteristic p.*

Proof. If p does not divide g, there is at least one minor m of order 2 with $p \nmid m$. Therefore in characteristic p this minor is invertible and the matrix M has rank at least 2. It follows that I contains at least 2 independent linear forms in characteristic p and therefore the vanishing set is either empty or contains 1 reduced point. $\square$

Remark 5.5.6. Notice that the condition $p \nmid g$ is sufficient, but not necessary. For example the ideal (u^2, v^2, w^2) vanishes nowhere, but still has $g = 0$ and therefore $p \mid g$. The condition becomes necessary if I is saturated.

PROPOSITION 5.5.7. *For $p \neq 2$, $\Delta_{(p)}$ is an irreducible sextic curve. In particular,* $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}_p}(X_{(p)}))[2] = 0$ *for $p \neq 2$.*

PROOF. We apply Lemma 5.5.5 to the saturation of the ideal $J := (D, \frac{\mathrm{d}D}{\mathrm{d}u}, \frac{\mathrm{d}D}{\mathrm{d}v}, \frac{\mathrm{d}D}{\mathrm{d}w}) \subset \mathbb{Z}[u, v, w]$, where D is the equation of Δ with integer coefficients. A Macaulay2 computation gives $g = 2^{10}$ ([**ABBvB-18**, M2 files]). So we have at most one singular point over $p \neq 2$ and therefore $\Delta_{(p)}$ is irreducible by Lemma 5.5.4. $\square$

5.5.3. Non-triviality of the unramified Brauer group. Let us study more closely the geometry of $\Delta_{(2)}$ and its components.

PROPOSITION 5.5.8. *Let D be the equation of Δ. Then*

$$D \equiv uw(u + w)(ug(u, v, w) + v^3) \mod 2$$

where $g(u, v, w) = v^2 + uv + vw + w^2$. Moreover

(1) *the equation $ug + v^3 = 0$ defines a smooth elliptic curve $E \subseteq \mathbf{P}^2 = \mathbf{P}^2_{\overline{\mathbb{F}_2}}$;*
(2) *E intersect transversely each of the lines cut out by $w = 0, u + w = 0$:*
(3) *the line $u = 0$ is tangent to E at the triple point* $[0 : 0 : 1]$.

PROOF. This is an explicit calculation done via Macauly2 in [**ABBvB-18**, M2 script]. $\square$

Thus we have four distinct Artin-Schreier coverings induced by the morphism π over each Δ_i: let us denote them $c_i \colon \widetilde{\Delta}_i \longrightarrow \Delta_i$ where $\widetilde{\Delta}_i = X_{(2)} \times_{\Delta_i} \Delta_i$ for $i = 1, \ldots, 4$. Recall that these correspond to residues $[c_i] = k(\Delta_i)/\wp(k(\Delta_i))$. We need to show that these coverings induce non-trivial residues, namely that their total spaces are irreducible. To show that we employ the following criterion.

LEMMA 5.5.9. *Let $\pi \colon Y \to \mathbf{P}^2_{\mathbb{F}_2}$ be a conic bundle defined over $\mathbb{F}_2$ and let $C \subset \mathbb{P}^2$ be an irreducible curve, such that the fibre of π above the generic point of C consists of two distinct lines. Let $\widetilde{C} \to C$ be the natural double covering of C induced by π. Then $\widetilde{C}$ is irreducible if each of the following holds:*

(1) *There exists a $\mathbb{F}_2$-rational point $P_1 \in C$ such that Y_{P_1} splits into two $\mathbb{F}_2$-rational lines:*
(2) *There exists a $\mathbb{F}_2$-rational point $P_2 \in C$ such that Y_{P_2} is irreducible over $\mathbb{F}_2$ and $Y_{P_2} \times \mathrm{Spec}(\overline{\mathbb{F}_2})$ splits into two $\overline{\mathbb{F}_2}$-rational lines. .*

PROOF. Suppose, by contradiction, that $\widetilde{C}$ is geometrically reducible; then the action induced by the Frobenius morphism on $\widetilde{C}$ either swaps the two irreducible components or fixes each of them. But now, in the first case for each $P \in C(\mathbb{F}_2)$ the Frobenius action would swap the two lines of Y_P, while in the second case it would fix each of them. This contradicts, respectively, the existence of points P_1 and P_2. $\square$

Thanks to Lemma 5.5.9, in order to prove that $\widetilde{\Delta}_i$ are irreducible it is enough to inspect the fibres above each $\mathbb{F}_2$-rational point of $\mathbf{P}^2$.

PROPOSITION 5.5.10. *In the above notation, the associated double coverings $c_i : \widetilde{\Delta}_i \longrightarrow \Delta_i$ all have irreducible total space. In particular, the residue is defined along each Δ_i and it coincides with the class of the covering c_i in $k(\Delta_i)/\wp(k(\Delta_i))$.*

PROOF. This is proved by applying Lemma 5.5.9 directly. The list of $\mathbb{F}_2$-rational points of $\mathbf{P}^2$ comprises $2^3 - 1 = 7$ distinct points. It is an easy computation ([**ABBvB-18**, M2 files]) to retrieve the geometry of the fibres above each point; we have collected the results in this chart:

P	geometry of Y_P	Δ_1	Δ_2	Δ_3	Δ_4
$(0:1:0)$	double line	$\times$	$\times$	$\times$	
$(0:1:1)$	2 rational lines	$\times$			
$(1:0:0)$	2 rational lines		$\times$		$\times$
$(1:0:1)$	2 rational lines			$\times$	
$(0:0:1)$	2 conjugate lines	$\times$			$\times$
$(1:1:0)$	2 conjugate lines		$\times$		
$(1:1:1)$	2 conjugate lines			$\times$	

For the sake of brevity, we have written "2 rational lines" for Y_P as in case (1) and "2 conjugate lines" for Y_P as in case (2). Now we check Lemma 5.5.9 for each of the discriminant components.

- Case of $\Delta_1 = \{u = 0\}$: we choose $P_1 = [0 : 1 : 1]$ and $P_2 = [0 : 0 : 1]$.
- Case of $\Delta_2 = \{w = 0\}$: we choose $P_1 = [1 : 0 : 0]$ and $P_2 = [1 : 1 : 0]$.
- Case of $\Delta_3 = \{u + w = 0\}$: we choose $P_1 = [1 : 0 : 1]$ and $P_2 = [1 : 1 : 1]$.
- Case of $\Delta_4 = E = \{ug(u, v, w) + v^3 = 0\}$: we choose $P_1 = [1 : 0 : 0]$ and $P_2 = [0 : 0 : 1]$.

$\square$

It remains to verify that condition (1) in Theorem 5.4.1 is satisfied; to do this we need an auxiliary conic bundle with prescribed residue profile. We have already described this new conic bundle as an example of the peculiar behaviour of conic bundles in characteristic 2 and the following result only serves to recollect all the relevant information.

LEMMA 5.5.11. *Let $Y \subseteq \mathbf{P}^2 \times \mathbf{P}^2$ be the zero locus of the following equation:*

$$Q = ux^2 + uxz + wy^2 + wyz + vz^2 = 0$$

where as usual u, v, w and x, y, z denote coordinates on the two respective copies of $\mathbf{P}^2$. Moreover, let $\pi' : Y \longrightarrow \mathbf{P}^2$ be the morphism obtained by restricting the projection onto the first factor (coordinates u, v, w). Then:

(1) *$\pi'\colon Y \longrightarrow \mathbf{P}^2$ is a conic bundle;*
(2) *the discriminant locus of π' is cut out by the equation $uw(u+w) = 0$ in $\mathbf{P}^2$;*
(3) *the associated coverings of π' induce non-trivial Artin-Schreier classes.*

Proof. This is the same calculation done in Example 5.3.7. □

Now, in accordance with Theorem 5.4.1, we need to show that the residue profile of $\pi' : Y \longrightarrow \mathbf{P}^2$ agrees with part of the residue profile of $\pi : X_{(2)} \longrightarrow \mathbf{P}^2$; more precisely, we need to show that the coverings $c_i'\colon (\widetilde{\Delta}_i)' \longrightarrow \Delta_i$ and the coverings $c_i : \widetilde{\Delta}_i \longrightarrow \Delta_i$ determine the same class in $H^1(k(\Delta_i), \mathbb{Z}/2) \simeq k(\Delta_i)/\wp(k(\Delta_i))$. To do this, it is sufficient to show that each two coverings are birational over $\Delta_i \simeq \mathbf{P}^1$.

Lemma 5.5.12. *Let $Q \subset \mathbf{P}^2 \times \mathbf{P}^2$ be a divisor of bi-degree $(d, 2)$ defined over $\overline{\mathbb{F}_2}$, considered as a conic bundle over $\mathbf{P}^2$ via the first projection. Assume that the discriminant locus of Q contains a line L as irreducible component and assume also that the sub-scheme of double lines over L is a reduced single points r. Furthermore, assume that Q, L and r are all defined over $\mathbb{F}_2$. Then either the associated double covering of L is reducible or it is birational over L to the Artin-Schreier covering*

$$x^2 + x + \frac{\eta}{\xi} = 0$$

where ξ, η are homogeneous coordinates on $L \simeq \mathbf{P}^1$ chosen such that $r = [0 : 1]$. In particular all non-trivial double coverings of $\mathbf{P}^1$ satisfying the above conditions yield the same element $[\eta/\xi]$ in $H^1(\overline{\mathbb{F}_2}(L), \mathbb{Z}/2)$.

Proof. Since L and Q have $\mathbb{F}_2$-rational structure, it follows that $c : \widetilde{L} \longrightarrow L$ has $\mathbb{F}_2$-rational structure too. Now let Y be the relative Grassmannian of lines in the fibres of c (this is defined formally as Grassmann bundle of the sheaf of section associated to the double covering c, see [**Fult-98**, Section 14.6]); it defines a double covering $\pi : Y \to L$, because Q is generically a cross of lines above L.

Then $Y \to L$ has also $\mathbb{F}_2$-rational structure and is flat over L. Hence $\mathscr{E} := \pi_*(\mathscr{O}_Y)$ is a rank 2 vector bundle on L, and Y can be naturally embedded into $\mathbf{P}(\mathscr{E})$. By Grothendieck's theorem we have that $\mathscr{E}$ splits into a direct sum of line bundles, and Y is defined inside $\mathbf{P}(\mathscr{E})$ by an equation

$$ax^2 + bxy + cy^2 = 0$$

with a, b, c homogeneous polynomials with $\deg(a)+\deg(c) = 2\deg(b)$. Notice that $b = 0$ defines the locus of points of the base L over which the fibre is a double point. We have assumed in our hypotheses that $b = 0$ must be a single reduced point, so $\deg(b) = 1$. Notice that if the double covering Y is non-trivial, a and c are non-zero, hence $\deg(a) \geq 0, \deg(c) \geq 0$ and $\deg(a) + \deg(c) = 2$.

Let $[\xi : \eta]$ be homogeneous coordinates on L. Let us set $a' = a/\xi^{\deg(a)}, b' = b/\xi^{\deg(b)}, c' = c/\xi^{\deg c}$ and calculate over the function field of L. Apply $(x, y) \mapsto (b'x, a'y)$ to obtain

$$a'(b')^2x^2 + a'(b')^2xy + (a')^2c'y^2 = 0.$$

Divide by $a'(b')^2$ and de-homogenise by setting $y = 1$ to obtain the Artin–Schreier normal form

$$x^2 + x + \frac{ac}{b^2} = 0.$$

We now use the fact that we can choose coordinates $[\xi : \eta]$ such that $b(\xi, \eta) = \xi$. We can write $ac = \alpha\xi^2 + \beta\xi\eta + \gamma\eta^2$ with $\alpha, \beta, \gamma \in \mathbb{F}_2$:

$$x^2 + x + \alpha + \beta\frac{\eta}{\xi} + \gamma\left(\frac{\eta}{\xi}\right)^2 = 0.$$

At this point it becomes essential to work over $\mathbb{F}_2$: on a first instance, either $\alpha = 0$ or $\alpha = 1$. In the second case let $\rho \in \overline{\mathbb{F}_2}$ be a root of $x^2 + x + 1$ and apply the transformation $x \mapsto x + \rho$. This gives

$$x^2 + x + \beta\frac{\eta}{\xi} + \gamma\left(\frac{\eta}{\xi}\right)^2 = 0$$

in both cases. Notice that although we have performed a transformation defined over $\overline{\mathbb{F}_2}$ this does not change the fact that β and γ are elements of $\mathbb{F}_2$.

Secondly, either $\gamma = 0$ – and in this case we have

$$x^2 + x + \beta\frac{\eta}{\xi} = 0$$

– or $\gamma = 1$ – and in this case we apply $x \mapsto x + \eta/\xi$ to obtain

$$x^2 + x + (\beta + 1)\frac{\eta}{\xi} = 0.$$

In all cases the coefficient in front of η/ξ is either 0 or 1, thus the covering is either trivial or has the normal form

$$x^2 + x + \frac{\eta}{\xi} = 0.$$

□

REMARK 5.5.13. The hypothesis that Q, L, and r must be defined over $\mathbb{F}_2$ cannot be dropped. More precisely, in the above formalism one can choose an isomorphism $\overline{\mathbb{F}_2}(L) \simeq \overline{\mathbb{F}_2}(t)$ with $t = \xi/\eta$ and the proof shows that either the associated covering of L would be trivial or would induce an element in $H^1(\overline{\mathbb{F}_2}(t), \mathbb{Z}/2)$ of the form γ/t with $\gamma \in (\overline{\mathbb{F}_2})^\times$. However,

$$H^1(\overline{\mathbb{F}_2}(t), \mathbb{Z}/2) \simeq \overline{\mathbb{F}_2}(t)/\wp(\overline{\mathbb{F}_2}(t))$$

and γ/t and γ'/t, for $\gamma, \gamma' \in (\overline{\mathbb{F}_2})^\times$, will define distinct elements of $H^1(\overline{\mathbb{F}_2}(t), \mathbb{Z}/2)$ if $\gamma \neq \gamma'$. Recall Remark 5.2.6: the group $H^1(\overline{\mathbb{F}_2}(t), \mathbb{Z}/2)$ classifies degree

2 extensions of $\overline{\mathbb{F}_2}(t)$ up to $\overline{\mathbb{F}}_2(t)$-isomorphism and, geometrically, it parametrises generically étale double coverings $C \longrightarrow \mathbf{P}^1$, up to birational isomorphism over $\mathbf{P}^1$ in the sense that $C \longrightarrow \mathbf{P}^1$ and $C' \longrightarrow \mathbf{P}^1$ are considered equivalent if there is a diagram

$$\begin{tikzcd} C \arrow[rr, dashed, "\varphi"] \arrow[dr] & & C' \arrow[dl] \\ & \mathbf{P}^1 & \end{tikzcd}$$

where φ is a birational map. However, if we drop the assumption that Q, L and R are defined over $\mathbb{F}_2$, we could only conclude in the above situation that two non-trivial coverings $C \longrightarrow \mathbf{P}^1$ and $C_2 \to \mathbf{P}^1$ arising as in the Proposition would be related by a diagram

$$\begin{tikzcd} C \arrow[r, dashed, "\varphi"] \arrow[d] & C' \arrow[d] \\ \mathbf{P}^1 \arrow[r, dashed, "\varphi'"] & \mathbf{P}^1 \end{tikzcd}$$

where φ, φ' are both birational, but φ' is not necessarily the identity. That would not be sufficient for our purposes, since to apply Theorem 5.4.1 to get a non-trivial Brauer class in $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}_2}(X_{(2)}))[2]$, we need to check that certain residues, which belong in $H^1(\overline{\mathbb{F}_2}(t), \mathbb{Z}/2)$, are *the same*. This is achieved by requiring that Q, L and r are defined over $\mathbb{F}_2$, whence γ above will actually be in $\mathbb{F}_2$, hence $\gamma = 1$ if non-trivial.

We have finally put together all the material needed to apply Theorem 5.4.1 and prove non-triviality of the unramified Brauer group.

Proof (of Proposition 5.5.2). We know that the residue profile of $\pi : X_{(2)} \longrightarrow \mathbf{P}^2$ is

$$([c_1], [c_2], [c_3], [c_4]) \in \bigoplus_{i=1}^{4} k(\Delta_i)/\wp(k(\Delta_1))$$

where c_4 is the residue along the elliptic curve $E = \Delta_4$. By Lemma 5.5.9 we know that $[c_i] \neq 0$ for all i.

Define $I_1 = \{1, 2, 3\}$ and $I_2 = \{4\}$ in a way that $I_1 \sqcup I_2$ indexes the residue profile; this choice satisfies condition (2) in Theorem 5.4.1.

One then defines $\pi' \colon Y \longrightarrow \mathbf{P}^2$ as in Lemma 5.5.11; its residue profile is, accordingly,

$$([c_1'], [c_2'], [c_3']) \in \bigoplus_{i=1}^{3} k(\Delta_i')/\wp(k(\Delta_i'))$$

and by Lemma 5.5.12 one has $[c_i'] = [c_i]$ for all $i = 1, 2, 3$. Furthermore, for each $i = 1, 2, 3$ and for all $P \in \Delta_i \cap \Delta_4$ one has that Y_P is union of two distinct lines: indeed Y_P is a double line only for $P \in \Delta_1 \cap \Delta_2 \cap \Delta_3 = \{[0 :$

$1:0]\}$ which does not belong to Δ_4, as shown in Proposition 5.5.8. Hence $\mathrm{Br}_{\mathrm{nr}}(\overline{\mathbb{F}_2}(X_{(2)}))[2] \neq 0$. $\square$

5.5.4. A UCT desingularisation of $X_{(2)}$. In this section we explicitly determine a resolution of singularities for $X_{(2)}$ that is **UCT** in accordance with Definition 2.2.3. This will be done by applying the criterion described in Proposition 2.2.4. In this view, it is enough to take into account singularities of $X_{(2)}$ only locally; with this in mind, we now proceed to examine the behaviour of $X_{(2)}$ locally above each point $P \in \mathbf{P}^2$ in the base. For the cases in which $X_{(2)}$ is singular locally above P, we will exhibit an appropriate scheme that desingularises it and satisfies condition (1) or (2) in Proposition 2.2.4.

We begin with establishing some useful results about the local geometry of $X_{(2)}$.

LEMMA 5.5.14. *Let K be a field of characteristic 2 and let us define $\hat{\mathbf{A}}^2 := \mathrm{Spec}(K[\![u,v]\!])$. Moreover, let X be a conic bundle over $\hat{\mathbf{A}}^2$. Thus X is cut out by the following equation*

$$c_{xx}x^2 + c_{xy}xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

where $c_{\bullet,\bullet}$ denote formal power series in u and v with coefficients in K.

Assume that

1. *locally around $(0,0)$ the discriminant of X has local equation of the form $u(u+v^n) = 0$ for some $n \geq 1$;*
2. *the fibre over $(0,0)$ has equation of the form $x^2 + xy + y^2$*

Then, after a change in the fibre coordinates x, y and z, we can assume X is cut out by

$$x^2 + xy + c_{yy}y^2 + c_{zz}z^2 = 0$$

with c_{yy} a unit and $c_{zz} = \beta u(u+v^n)$ where β is a unit.

PROOF. Because of hypothesis (2) we can assume that c_{xx} is a unit, so after multiplying by c_{xx}^{-1} we can assume that we have the form

$$x^2 + c_{xy}xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

where c_{xy} and c_{yy} are now units. After the substitution of $x \mapsto c_{xy}x$ we can divide the whole equation by c_{xy}^2 and we can assume that we have the form

$$x^2 + xy + c_{yy}y^2 + c_{xz}xz + c_{yz}yz + c_{zz}z^2 = 0$$

with c_{yy} a unit because of the same hypothesis. Now substituting $x \mapsto x + c_{yz}z$ and $y \mapsto y + c_{xz}z$ we obtain the normal form

$$x^2 + xy + c_{yy}y^2 + c_{zz}z^2 = 0$$

with c_{yy} still a unit. The discriminant locus of this conic bundle is $c_{zz} = 0$. Since the discriminant defining equation was changed at most by a unit during the above changes of coordinates, it must be $c_{zz} = \beta u(u+v^n)$ for some unit β, in agreement with hypothesis 1. $\square$

LEMMA 5.5.15. *Let K be a field of characteristic 2, let Y be a hypersurface in $\widehat{\mathbf{A}}^4 := \operatorname{Spec}(K[\![x,y,u,v]\!])$ with equation*

$$x^2 + xy + \alpha y^2 + \beta u(u + v^n) = 0, \quad n \geq 1,$$

where α and β are units in $K[\![u,v]\!]$. Then Y is singular at the origin only.

Let $\widetilde{\mathbf{A}^4}$ be the blow up of $\widehat{\mathbf{A}}^4$ at the origin and let $\widetilde{Y} \subset \widetilde{\mathbf{A}^4}$ be the strict transform of Y. If $n = 1$, then $\widetilde{Y}$ is smooth. If $n > 1$, then $\widetilde{Y}$ is singular at only one point, which we can assume to be the origin again. Moreover around this singular point $\widetilde{Y}$ has local equation

$$x^2 + xy + \alpha' y^2 + \beta' u(u + v^{n-1}) = 0$$

with α' and β' units in $K[\![u,v]\!]$.

PROOF. The blow-up scheme $\widetilde{\mathbf{A}^4}$ has an affine open cover made of four subvarieties; more precisely, let

$$R = \bigoplus_{n=0}^{\infty} (x,y,u,v)^n$$

so that $\widetilde{\mathbf{A}^4} = \operatorname{Proj}(R)$ and the open sets

$$U_1 = \operatorname{Spec}(R_{(x)}), U_2 = \operatorname{Spec}(R_{(y)}), U_3 = \operatorname{Spec}(R_{(u)}), U_4 = \operatorname{Spec}(R_{(v)})$$

constitute an affine open covering, where x, y, u, v have been identified, up to a slight abuse of notation, with the corresponding degree 1 elements in the Rees algebra R. Now one has

$$R_x \simeq K[\![x,y,y,v]\!]_x, R_y = K[\![x,y,u,v]\!]_y, R_u = K[\![x,y,u,v]\!]_u, R_v = K[\![x,y,u,v]\!]_v$$

so we can work on each U_j by using appropriate charts. We will show that the strict transform $\widetilde{Y}$ is smooth on U_j for $j = 1, 2, 3$ and takes the above local form on U_4.

(1) To work on U_1, we apply the chart $(x,y,u,v) \mapsto (x, xy, xu, xv)$ to get the total transform

$$x^2 + x^2 y + \alpha' x^2 y^2 + \beta' xu(xu + x^n v^n) = 0$$

where α', β' are the transformed formal series under this chart. The strict transform is, instead,

$$1 + y + \alpha' y^2 + \beta' u(u + x^{n-1} v^n) = 0$$

Notice that α' and β' are power series that only involve u, v and x. Therefore the derivative with respect to y is 1 in both cases and the strict transform is smooth in this chart.

(2) To work on U_1, we apply the chart $(x,y,u,v) \mapsto (xy, y, yu, yv)$ to get the total transform

$$x^2 y^2 + xy^2 + \alpha' y^2 + \beta' yu(yu + y^n v^n) = 0$$

where α', β' are defined similarly as before; the strict transform is

$$x^2 + x + \alpha' + \beta' u(u + y^{n-1} v^n) = 0.$$

Notice that α' and β' are power series that only involve u, v and y. Therefore the derivative with respect to x is 1 in both cases and the strict transform is smooth in this chart.

(3) To work on U_3, we apply the chart $(x, y, u, v) \mapsto (xu, yu, u, uv)$ to get the total transform

$$x^2u^2 + xyu^2 + \alpha' y^2u^2 + \beta' u(u + u^n v^n) = 0$$

where α', β' are defined similarly as before; the strict transform is

$$x^2 + xy + \alpha' y^2 + \beta'(1 + u^{n-1}v^n) = 0$$

as the strict transform. This time α' and β' are power series that only involve u, v like the original ones. Therefore computing the derivatives with respect to x and y yields

$$\partial_x = y, \quad \partial_y = x$$

so the singular locus must satisfy equations $x = y = 0$. Substituting this into the equation of the strict transform yields

$$\beta'(1 + u^{n-1}v^n) = 0$$

which can not be satisfied anywhere since β' and $(1 + u^{n-1}v^n)$ are units in $K[\![u, v]\!]$. Therefore the strict transform is smooth in this chart.

(4) Finally, we work on U_4 applying the chart $(x, y, u, v) \mapsto (xv, yv, uv, v)$. The total transform is

$$x^2v^2 + xyv^2 + \alpha' y^2v^2 + \beta' uv(uv + v^n) = 0$$

and the strict transform is

$$x^2 + xy + \alpha' y^2 + \beta' u(u + v^{n-1}) = 0.$$

Note that this is the desired local equation for $\widetilde{Y}$, as α' and β' are power series that only involve u, v. and still units in $K[\![u, v]\!]$. An analogous calculation of the derivatives with respect to x and y shows that the singular locus lies on $x = y = 0$. Substituting this into the equation of the strict transform yields the additional constraint $\beta' u(u + v^{n-1}) = 0$. So the singular locus must satisfy equations

$$x = y = u(u + v^{n-1}) = 0. \tag{5.5.1}$$

Let us now look at the derivative with respect to u:

$$\frac{\mathrm{d}\alpha'}{\mathrm{d}u} y^2 + \frac{\mathrm{d}\beta'}{\mathrm{d}u} u(u + v^{n-1}) + \beta' v^{n-1} = 0.$$

And using the relation 5.5.1, the above equation reduces to $v^{n-1} = 0$. If $n = 1$, this shows that $\widetilde{Y}$ is smooth everywhere. If $n \geq 2$, we obtain that the strict transform is singular at most at $x = y = u =$

$v = 0$ in this chart. To check that this is indeed a singular point we also calculate the derivative with respect to v:

$$\frac{\mathrm{d}\alpha'}{\mathrm{d}v}y^2 + \frac{\mathrm{d}\beta'}{\mathrm{d}v}u(u+v^{n-1}) + \beta'(n-1)uv^{n-2} = 0$$

and this is automatically satisfied at $x = y = u = v = 0$.

This finishes the proof. □

LEMMA 5.5.16. *In the notation of Proposition 5.5.15, the exceptional divisor of $\widetilde{Y} \to Y$ is a quadric with at most one singular point.*

PROOF. It is enough to determine the nature of the singularities of Y locally around the origin. Recall that the equation of Y is

$$x^2 + xy + \alpha y^2 + \beta u(u+v^n) = 0.$$

We see immediately that around the origin, the term with smallest degree is

$$\begin{cases} x^2 + xy + \alpha_0 y^2 + \beta_0 u^2 + \beta_0 uv & \text{for } n = 1 \\ x^2 + xy + \alpha_0 y^2 + \beta_0 u^2 & \text{for } n > 1 \end{cases}$$

where α_0, β_0 are the (non-zero) constant terms of α, β respectively. The first equation defines a smooth conic, the second one defines a quadric cone with an isolated singularity at the origin. □

We now proceed to examine the behaviour of $X_{(2)}$ locally above each $p \in \mathbf{P}^2$.

PROPOSITION 5.5.17. *Let $P \in \mathbf{P}^2$ be a closed point and let $\pi\colon X = X_{(2)} \longrightarrow \mathbf{P}^2$ be the conic bundle which is object of study.*

1. *The singularities of X lie above the singular points of $\Delta_{(2)}$.*
2. *If $P = [0:1:0]$ is the intersection $\Delta_1 \cap \Delta_2 \cap \Delta_3$ then X is smooth locally above P.*
3. *If $P \in (\Delta_2 \cup \Delta_3) \cap E$ then X has an unique singular point above P and the blow up at this point has a smooth quadric as exceptional divisor.*
4. *If $P = [0:0:1]$ is the intersection $\Delta_1 \cap E$ then X has an unique singular point above P and the blow up at point has a quadric cone with an isolated singular point as exceptional divisor.*

PROOF. Parts (1) and (2) can be shown by direct calculation (see Macaulay2 scripts for [**ABBvB-18**]).

(3) In this case the intersection at P is transverse and the fibre X_P is a cross of lines; thus we can apply Lemma 5.5.14 with $n = 1$ and by Lemma 5.5.15 we know that there is only one singular point locally above P. Blowing up P and applying Lemma 5.5.16 we see that the exceptional divisor is a smooth quadric, hence it is **UCT** by Proposition 2.2.5.

(4) The point $P = [0:0:1]$ is a point of intersection multiplicity 3 and the fibre X_P is the union of two conjugate lines over $\overline{\mathbb{F}_2}$. Thus we have

the local equation as in Lemma 5.5.14 with $n = 2$ and by Lemma 5.5.15 we know that there is only one singular point locally above P. Blowing up P and applying Lemma 5.5.16 we see that the exceptional divisor is a quadric cone with one isolated singular point, hence it is **UCT** by Proposition 2.2.5.

In conclusion, blowing up the singular points of X yields a morphism $\widetilde{X} \longrightarrow X$ whose fibres are all **UCT** varieties (over the residue field of the base points), thus it realises a **UCT** desingularisation by Proposition 2.2.4, part (1). This proves Proposition 5.5.3 and concludes the proof of all claims needed to prove Theorem 5.5.1. $\square$

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68