A Thesis Submitted for the Degree of PhD at the University of Warwick

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Mathematics and Statistics Centre for Doctoral Training

# Random Walks on Decorated Galton-Watson Trees 

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Thesis

Submitted for the degree of
Doctor of Philosophy

## Mathematics Institute <br> The University of Warwick

June 2020


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## Acknowledgments

Firstly, I would like to thank my supervisor, David Croydon, who had the initial vision for this thesis, and who has given me the freedom to take ownership of this project whilst quietly keeping me on track from the sidelines. The past three years have been a challenging but hugely enjoyable learning curve, have opened up unexpected opportunities and have enabled me to exceed all expectations I had for myself three years ago. This would not have been possible without David's guidance and encouragement.

I am extremely fortunate to have benefitted from multiple visits to Kyoto University during my PhD and I would also like to thank everyone who welcomed me to Japan on these occasions: in particular Takashi Kumagai, Ryoki Fukushima, and Naotaka Kajino. Not only was this an incredibly mathematically-enriching experience, but I am also very grateful for their generous hospitality during this time, which really went above and beyond.

I would also like to thank those at Warwick who have offered advice and support throughout the past few years and who have contributed to maintaining a stimulating research environment in the department: in particular Daniel Ueltschi, Nikos Zygouras and Stefan Adams. This list would not be complete without the names of my office-mates (friends!), Shannon and Quirin, who have just about managed to keep me sane throughout.

I probably would not have attempted a PhD in the first place had it not been for an undergraduate summer project supervised by Perla Sousi, and the subsequent encouragement that she gave me. I would also like to thank Igor Spasojevic, with whom I worked on the project, for his willingness to share ideas and for making that summer a very enjoyable one.

I am also greatly indebted to Nicolas Curien and Larbi Alili for taking the
time to read and examine this thesis. I am very privileged to have you as examiners. Throughout the last few years, I have also been humbled and inspired by many members of the international research community who have generously taken the time to discuss my research with me. There are too many people to name individually.

Finally, I am always grateful to my friends and family, for keeping me grounded, and encouraging me to spread my wings.

## Declarations

The results of this thesis are entirely my own work although I have benefitted many helpful discussions with my supervisor David Croydon along the way. The content of Chapter 4 comprises the preprint [Arc19] and the content of Chapter 5 is the basis of the article [Arc20]. The presentation of both of these chapters has benefitted from helpful comments from the anonymous referees for these articles.

The results that are summarised in Section 4.1 were obtained in my 2017 MSc thesis [Arc17]. This is clearly stated in the text and these results are meant to be the springboard for the main results of this thesis, rather than a major part.

## Abstract

The purpose of this thesis is to study random walks on "decorated" Galton-Watson trees with critical offspring distribution in the domain of attraction of an $\alpha$-stable law for some $\alpha \in(1,2)$.

In Chapters 2 and 3 we give some background on the topologies and models used in the thesis. In Chapter 4 we consider a specific example: stable looptrees. We prove a scaling limit result for convergence of random walks on discrete looptrees to convergence of Brownian motion on continuum looptrees. We then construct a detailed investigation of the limiting Brownian motion, in particular obtaining detailed bounds on the transition density and on the spectrum of the associated Laplacian. Along the way, we also prove precise volume asymptotics.

In Chapter 5 we construct the local limit of compact stable looptrees which we call infinite stable looptrees. In particular, this allows us to show that the operations of taking scaling limits and local limits of discrete and continuum looptrees can be done in either order or in combination. As a result, we are also able to prove similar limit results for stochastic processes on these spaces. Moreover, we are able to apply the local limit result to obtain limiting results for the volume of a small ball and the small-time on-diagonal transition density for compact stable looptrees.

In Chapter 6 we consider the main general model of interest: that of a decorated Galton-Watson tree. In this chapter we formulate some assumptions regarding the graphs used for "decoration", and then prove some results establishing the volume growth exponent, random walk displacement exponent and spectral dimension for these decorated Galton-Watson trees.

In Chapter 7 we give some brief comments on future research directions.

## Chapter 1

## Introduction

The random walk has been a fundamental object in probability theory since Karl Pearson, now generally acknowledged to be the founder of modern-day statistics, first coined the term over one hundred years ago, in a 1905 edition of Nature [Pea05].

## The Problem of the Random Walk.

Can any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point $O$ and walks $l$ yards in a straight line; he then turns through any angle whatever and walks another $l$ yards in a second straight line. He repeats this process $n$ times. I require the probability that after these $n$ stretches he is at a distance between $r$ and $r+\delta r$ from his starting point, $O$.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for two stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1 / n$, when $n$ is large.

The Gables, East Ilsley, Berks.
Figure 1.1: PEARSON, K. The Problem of the Random Walk. Nature 72, 294(1905).

Pearson's call was answered by Lord Rayleigh [Ray05], who had in fact solved the problem 25 years previously whilst studying sound waves with random phases. In a two-dimensional setting and for large $n$, the required density is asymptotic to $\frac{2 r}{n} e^{\frac{-r^{2}}{n}} d r$. This is now known as the Rayleigh distribution and naturally describes a range of physical phenomena.

Although Pearson's problem has now been solved, one hundred years later the case is by no means closed. Rayleigh's solution holds for a random walk on the two-dimensional lattice, but the world is not flat, nor is it uniform. What if the
walker encounters a mountain range, or a valley?
Motivated by these considerations, there has since been a concerted mathematical effort to understand the behaviour of random walks on a variety of different graphs, including trees, fractals, and Cayley graphs, to name but a few. The properties of the random walk depend on the properties of the underlying graphs: for example, while a random walk on the lattice typically moves distance $\sqrt{n}$ after $n$ steps, on the binary tree we would expect it to move distance of order $n$. On the other hand, a diffusion on the Sierpinski gasket moves roughly distance $t^{\frac{\log 2}{\log 5}}$ in time $t$. We give some further examples in Figure 1.2, letting $\left(X_{n}\right)_{n \geq 0}$ denote a simple random walks on the given graphs, and $d\left(0, X_{n}\right)$ denote the distance of $X_{n}$ from its starting point with respect to the graph distance on these graphs.

(a) $d\left(0, X_{n}\right) \approx \sqrt{n}$.

(d) $d\left(0, X_{n}\right) \approx n^{\frac{5}{13}}$, image by Mike Bostock.

(b) $d\left(0, X_{n}\right) \approx n^{\frac{\log 2}{\log 5}}$.

(e) $d\left(0, X_{n}\right) \approx n^{\frac{1}{3}}$.

(c) $d\left(0, X_{n}\right) \approx n$.

(f) $d\left(0, X_{n}\right) \approx n^{\frac{1}{4}}$, image by Igor Kortchemski.

Figure 1.2: Examples of different spaces with different random walk exponents. References: [BP88], [BCK17], [Cro12], [GM17a] [GH18].

The speed of a random walk is in general quantified through the introduction of two random walk exponents (if they exist):
(i) The spectral dimension, $d_{S}=-2 \lim _{n \rightarrow \infty} \frac{\log p_{2 n}(x, x)}{\log n}$,
(ii) The displacement exponent, $d_{\text {dis }}=\lim _{n \rightarrow \infty} \frac{\log \sup _{k \leq n} d\left(0, X_{k}\right)}{\log n}$.

As we will see in this thesis, on sufficiently homogeneous spaces, these limits are not random, even if the underlying graph is. The extra factor of 2 is included in the definition of spectral dimension so that it is equal to $d$ on $\mathbb{Z}^{d}$. Here $p_{n}(x, y)=$ $\frac{\mathbb{P}_{x}\left(X_{n}=y\right)}{\operatorname{deg} y}$ is the transition density of the simple random walk, and for connected graphs the limit in (i) does not depend on the choice of $x$. The quantity $d_{w}=\frac{1}{d_{\mathrm{dis}}}$
is also known as the walk dimension since in natural cases it corresponds to the dimension of the range of the walk.

These notions of dimension can be contrasted with the natural definition of dimension in terms of the volume growth of the space, given by

$$
d_{f}=\frac{\lim _{r \rightarrow \infty} \log (\operatorname{Vol}(B(x, r)))}{\log r}
$$

Here $B(x, r)$ denotes the open ball of radius $r$ around $x$ (with respect to graph distance), and $\operatorname{Vol}(B(x, r))$ its volume (i.e. the number of vertices it contains). On $\mathbb{Z}^{d}$, it is the case that $d_{f}=d_{S}$, and this remains true for a wide class of "wellbehaved" graph models. However, this is not always the case, and it is commonly observed that $d_{S}<d_{f}$ on graphs of a fractal nature, for example as mentioned for the Sierpinski gasket above [BP88]. Heuristically, this is because the random walk gets "trapped" in the fractal parts of the graphs for non-negligible amounts of time, which has the overall effect of slowing it down. More concretely, whilst we expect Brownian motion in $\mathbb{R}^{d}$ to move distance $t^{\frac{1}{2}}$ in time $t$ (known as diffusive behaviour), we can see from Figure 1.2 that on fractal-type graphs it is often the case that the walk dimension is strictly greater than 2 (for example, in the Sierpinski gasket it is $\frac{\log 5}{\log 2}=2.32 \ldots$ ), in which case we say the random walk is subdiffusive, or undergoes anomalous diffusion.

On sufficiently homogeneous graphs it is usually the case that $d_{w}=\frac{2 d_{f}}{d_{S}}$, and we will see that this also holds for all the examples considered in this thesis. Heuristically, this is because at time $t$, we expect a random walk started at $x$ to be distributed roughly uniformly in the ball of radius $t^{\frac{1}{d_{w}}}$ around $x$, so that $p_{t}(x, x)$ is of order $t^{\frac{-d_{f}}{d_{w}}}$.

Random walks are now well-understood on several (but by no means all!) natural graph models, and in recent decades the scope has widened to include random walks on graphs that are random, as well as deterministic. In this context, subdiffusive behaviour was first rigorously established by Kesten in 1986, who considered a random walk on a critical percolation cluster both in two dimensions and on a regular tree [Kes86a, Kes86b]. On such a tree, the critical cluster is more tractable (it is simply a critical Galton-Watson tree with binomial offspring distribution), and Kesten showed that $d_{w}=3$. The study of the two-dimensional "ant in a labyrinth" (as De Gennes called it [dG76]) is more subtle, but as a first step Kesten managed to show that there is some $\varepsilon>0$ such that the Euclidean distance to the root grows slower than $n^{\frac{1}{2}-\varepsilon}$. However, this restriction to the Euclidean metric is still somewhat suboptimal, and establishing the same result with respect to the intrinsic metric on the cluster has been an open problem for some time. It is only very recently that Ganguly and Lee proved the first result in this direction
[GL20].
Critical percolation clusters are a classical example of random structures that are naturally "fractal-like", and Alexander and Orbach conjectured in 1982 [AO82] that a critical cluster on $\mathbb{Z}^{d}$ should have $d_{S}=\frac{4}{3}$ for all $d \geq 2$. This is the same as the spectral dimension of an (unpercolated) finite variance critical Galton-Watson tree. The Alexander-Orbach conjecture resisted attack for some time, with results mainly restricted to critical clusters on trees [Kes86b, BK06], and then extended to oriented percolation in high dimensions in 2008 [BJKS08]. The full result in high dimensions by was proved by Kozma and Nachmias in their celebrated paper [KN09] in 2009 using lace expansion techniques. The intuition for the result is that in high dimensions, the lattice is sufficiently spread out that the critical cluster looks a lot like a critical Galton-Watson tree, so that their spectral dimensions agree. For comparison, $d_{f}=2$ in this case. In low dimensions, there is now evidence to suggest that the conjecture is false when $d<6$ [JN14, JL20], and the question of the true exponent remains an open problem.

After well as establishing the exponents, the next aim would be to prove a full random walk scaling limit on the critical cluster. In high dimensions this is conjectured to be Brownian motion on the integrated super-Brownian excursion ( $\left.\mathrm{B}^{\mathrm{ISE}}\right)$, informally Brownian motion on an embedded spatial tree and formally constructed in [Cro09]. Substantial progress was made in this direction in [BACF19b], in which the authors establish four conditions for convergence to $\mathrm{B}^{\text {ISE }}$. Verifying these conditions still poses a challenge, however, and as of today has been achieved in the case of a random walk on the range of a branching random walk [BACF19a], but not yet for the full percolation cluster.

It is now well-known that in low-dimensional, sufficiently recurrent regimes, the exponents for a random walk on a given graph depend on two key properties of the graph: its volume and (electrical) resistance growth. The first of these is not surprising (intuitively, it makes sense that a random walk should take longer to escape a denser subgraph), but the second of these is perhaps less obvious. However, it turns out that viewing a graph as an electrical network with given edge conductances is both useful and natural: due to Kirchoff's Laws, voltages are harmonic functions, and it turns out that the notion of "effective resistance" is precisely what determines the escape probabilities for an electron (i.e. a random walker). We will introduce the necessary quantities more precisely in Section 2.4, but see [Kum14, Nac] for excellent surveys covering these concepts in more detail.

Resistance is easiest to study on trees since the effective resistance between two points agrees with the graph distance in this setting. It is therefore of no surprise that substantial progress has been made in understanding random walks on graphs that are trees or "tree-like", whereas often less is known about random
walks on other graphs (though there are notable exceptions to this). In particular, our understanding of random walks on critical percolation clusters, and many other structures arising naturally in statistical physics, is far from complete.

Physical models such as percolation have classically been studied on the integer lattice, but in recent years there has been increased mathematical interest in adding an extra layer of randomness to the underlying graphs, with a particular emphasis on random surfaces. In many ways, random graphs are a more natural model for the real world but another advantage is that this opens up more techniques for mathematical analysis since we are not constricted by the rigidity of the underlying space. In particular, random graphs often enjoy a spatial Markov property which means that they can be studied using Markovian exploration processes; see [Cur] for more details.


Figure 1.3: Planar maps. The first two are the same map; the third is different.

In recent decades the study of random surfaces has blossomed into a very fruitful area of probabilistic research, and currently we perhaps know more about random walks on random critical structures when they are defined on random planar maps, rather than on the two-dimensional lattice. Formally, a planar map is an equivalence class of the set of graphs embedded into the plane such that no two edges cross, where we say that two embedded graphs are equivalent to each other if one can be obtained from the other through an orientation-preserving homeomorphism, as illustrated in Figure 1.3. We view a planar map as a metric space by giving each edge length 1 , and endowing it with the graph metric. It is also possible to add a measure supported on the vertices of the map.

We do not attempt to give a full introduction to random planar maps, as the theory will not be necessary for this thesis; instead we show a simulation in Figure 1.4 , and refer to [Mie] for an introduction. We also note that there is a parallel continuum theory for studying random surfaces embedded into the plane known as Liouville Quantum Gravity (LQG): we refer to [Gwy, GHS19b] for recent surveys.

Just as Brownian motion can be thought of as a "canonical" random path and any discrete time random walk with finite variance jump distribution falls into its universality class, there is a notion of a Brownian surface and a corresponding Brownian universality class of discrete random planar maps, and there is consider-


Figure 1.4: A triangulation of the sphere. Image by Thomas Budzinski.
able interest in studying statistical physics models on random planar maps in this universality class.

For example, take the model of site percolation on a large uniform triangulation. It is known that this model exhibits a phase transition with critical probability $p_{c}=\frac{1}{2}$ [Ang03, BCM19]. Moreover, although the critical cluster is not a tree (recall that even on the lattice it is only believed to be asymptotically tree-like in high dimensions), it is known to have macroscopic faces which are individually glued along a tree structure, and the boundary is described by an object known as a "looptree" [CK15, Ric18a, BCM19]. Each macroscopic face has its own internal structure, but one would hope to exploit the underlying tree structure in order to understand resistance and random walks on the critical cluster.

Critical percolation clusters are not the only model that can be described by a "decorated tree" structure. In fact, there is a much wider class of maps, known as stable maps (each corresponding to a different universality class of random planar maps) that can be described in this way. In the so-called dense phase (corresponding to stability parameter $\alpha \in\left(1, \frac{3}{2}\right)$ ), these are also known to have a decorated tree structure [Ric18b]. The critical percolation cluster discussed above is a special case since it is believed to correspond to the case $\alpha=\frac{7}{6}$ [BCM19, Section 5.4]. Another special case is the $\mathrm{O}(\mathrm{n})$ loop model [Ric18b, Section 6], and we see similar structures appearing from other Fortuin-Kasteleyn models, and also [BLR17] from quadrangulations with skewness [BR18].

The purpose of this thesis is to study random walks on a general "decorated tree" model, along the lines of that pictured in Figure 1.5, in the hope that it will


Figure 1.5: An example of a decorated tree, and its underlying Galton-Watson tree.
apply to the models described above. Before commencing, we briefly comment on other results regarding random walk exponents on random planar maps.

Subdiffusivity of a random walk on random planar maps was first established by Benjamini and Curien [BC13], who showed that the displacement exponent on the Uniform Infinite Planar Quadrangulation (UIPQ) is at most $\frac{1}{3}$ by decomposing at so-called "pioneer points"; however, in line with the KPZ relations they conjectured that the correct exponent should in fact be $\frac{1}{4}$, and this was subsequently proved in [GM17a, GH18] (lower and upper bounds respectively). The proofs that this exponent is $\frac{1}{4}$ also encompass several other subclasses of the Brownian universality class. More recently, random walks on stable maps were considered by Curien and Marzouk [CM19b, CM19a], who extended the upper bound of $\frac{1}{3}$ to (bipartite) stable maps with parameter $\alpha \in(1,2)$, although this is not believed to be sharp except for in the limit as $\alpha \downarrow 1$. Similar results were also obtained by Lee [Lee17] for random walks on the wider class of unimodular planar graphs, who presents a method for establishing subdiffusivity based only on the volume growth properties of the space (although the corresponding displacement exponent is not necessarily sharp).

Finally, we mention that stochastic processes on random surfaces have also been constructed in the continuum and the natural analogue of Brownian motion on a Brownian surface is known as Liouville Brownian motion (LBM). This was independently constructed in [Ber15, GRV16] using techniques of LQG, and has recently been shown to arise as the scaling limit of simple random walks on certain classes of random planar map models [BG20]. It is also known to exhibit subGaussian behaviour (e.g. sub-Gaussian heat kernel estimates were established in [AK16]), but establishing the relevant displacement exponent is a difficult task since it was only very recently that the intrinsic LQG metric was constructed [GM19].

In terms of this thesis: the full model is postponed to Chapter 6, and we start in Chapter 4 by considering the simpler model of a looptree. This essentially corresponds to the case where each macroscopic face is empty, and can be represented


Figure 1.6: Relations between discrete/continuum and compact/infinite looptrees.
by a loop. Random walks on this model were already considered in [BS15], but we extend their result to give a scaling limit and perform a detailed analysis of the continuum limit, in particular obtaining finer fluctuation results. Our methods for the looptree model are also instructive for the general model in Chapter 6; in particular to prove the volume upper bounds we introduce an iterative decomposition procedure that we then generalise in Chapter 6.

Chapter 5 concerns a model of infinite stable looptrees. We originally considered these as a tool to understand the limiting behaviour of small balls in compact stable looptrees, but this eventually evolved into a separate project. The purpose of the chapter is essentially to complete the picture in Figure 1.6. We also obtain similar limit theorems for stochastic processes on these spaces.

The full decorated tree model is considered in Chapter 6. In view of applications, we only consider critical Galton-Watson trees, though one could define a similar model on supercritical trees and ask different questions about the random walk (e.g. regarding its speed: we elaborate on this in Chapter 7), and we assume that the offspring distribution $\xi$ satisfies $\xi([x, \infty)) \sim c x^{-\alpha}$ as $x \rightarrow \infty$ for some $\alpha \in(1,2)$. To understand volume and resistance growth on the decorated tree, it is clearly necessary to make some assumptions regarding volume and resistance growth on the inserted graphs. The main contribution of the chapter is to explain how the overall random walk exponents are obtained from the relevant exponents for the inserted graphs and that of the offspring distribution in the underlying tree.

We will see that the model undergoes several phase transitions, at the points where certain important quantities transition from finite to infinite expectation. For example, if the diameter of a graph inserted at a vertex of degree $n$ is approximately $n^{\frac{1}{d}}$, and the volume of this graph is approximately $n^{v}$ (we will make this precise in Chapter 6), then the volume growth exponent behaves as shown in Figure 1.7.

At the end of Chapter 6 we consider some examples of graphs to insert. We conclude the thesis with a brief discussion of the future outlook in Chapter 7.


Figure 1.7: Different phases of the decorated volume exponent. We do not necessarily see all three phases for one given model.

## Chapter 2

## Preliminaries

In this chapter we go over some technical preliminaries that will be useful for the rest of the thesis.

### 2.1 Hausdorff and packing measures on metric spaces

We start by introducing Hausdorff and packing measures, which are closely linked to Hausdorff and packing dimensions which provide a natural notion of dimension on fractal metric spaces. Our exposition here is similar to that of [Duq10, Section 1], but for further background Hausdorff and packing measures see [Fal14, Chapter 2] and [TT85, Sections 3,5] respectively.

Let $\left(M, d_{M}\right)$ be a metric space. For any $x \in M$, let $\bar{B}(x, r)$ denote the closed ball of radius $r$ around $x$. For any $\varepsilon>0$ and any $S \subset M$, recall that an $\varepsilon$-packing of $S$ is a collection of disjoint balls $\left(\bar{B}\left(x_{i}, r_{i}\right)\right)_{i \geq 0}$ with $\bar{B}\left(x_{i}, r_{i}\right) \subset S$ and $r_{i} \leq \varepsilon$ for all $i$.

Now take a function $g:\left(0, r_{0}\right) \rightarrow \mathbb{R}^{+}$for some $r_{0}>0$. We say that $g$ is a regular gauge function if it is continuous, non-decreasing, $\lim _{r \downarrow 0} g(r)=0$ and there exists a constant $C \in(1, \infty)$ such that $g(2 r) \leq C g(r)$ for all $r \in\left(0, \frac{r_{0}}{2}\right)$. We then define a measure $\mathcal{P}_{g}^{*}(S)$ by

$$
\mathcal{P}_{g}^{*}(S)=\lim _{\varepsilon \downarrow 0} \sup \left\{\sum_{i \geq 0} g\left(r_{i}\right):\left(\bar{B}\left(x_{i}, r_{i}\right)\right)_{i \geq 0} \text { is an } \varepsilon \text {-packing of } S\right\} .
$$

$\mathcal{P}_{g}^{*}(S)$ is know as the $g$-packing pre-measure of $S$. We define the $g$-packing outermeasure of $S$ by

$$
\mathcal{P}_{g}(S)=\inf \left\{\sum_{i \geq 0} \mathcal{P}_{g}^{*}\left(E_{i}\right): S \subset \cup_{i \geq 0} E_{i}\right\} .
$$

It can be shown (e.g. ref) that $\mathcal{P}_{g}$ is a Borel regular metric outer measure.
The $g$-Hausdorff measure is defined similarly, but by considering coverings
rather than packings. We set

$$
\mathcal{H}_{g}(S)=\liminf _{\varepsilon \downarrow 0}\left\{\sum_{i \geq 0} g\left(\operatorname{Diam}\left(E_{i}\right)\right): \operatorname{Diam}\left(E_{i}\right)<\varepsilon \text { and } S \subset \cup_{i \geq 0} E_{i}\right\} .
$$

Here $\operatorname{Diam}(S)=\sup _{x, y \in S} d_{M}(x, y)$ denotes the diameter of $S$.
As above, $\mathcal{H}_{g}$ is a Borel regular metric outer measure on $M$.
In the case where we take $g(r)=g_{n}(r)=r^{n}$ for some $n \in \mathbb{R}^{+}$, it is staightforward to show that there is at most one value of $n$ for which $\mathcal{P}_{g}(S) \notin\{0, \infty\}$, and similarly for $\mathcal{H}_{g}(S)$ (though the two values may not be the same). We respectively define the packing and Hausdorff dimensions of $S$ by

$$
\begin{aligned}
\operatorname{dim}_{\mathrm{P}}(S) & =\sup \left\{n \geq 0: \mathcal{P}_{g_{n}}(S)=\infty\right\} \\
\operatorname{dim}_{\mathrm{H}}(S) & \left.=\sup \left\{n \geq 0: \operatorname{Pinf}_{g_{n}}(S)=\infty\right\}=0: \mathcal{P}_{g_{n}}(S)=0\right\}, \\
& \inf \left\{n \geq 0: \mathcal{P}_{g_{n}}(S)=0\right\},
\end{aligned}
$$

where in this case we take $\sup \emptyset=0, \inf \emptyset=\infty$. In the case of $\mathbb{R}^{d}$ with the usual Euclidean metric, the Hausdorff and packing dimensions are both equal to $d$.

In the case of fractal metric spaces endowed with a particular measure, say $\mu_{M}$, it is sometimes the case that $\mu_{M}$ will be equal to the packing or Hausdorff measure (up to a constant) for a specific choice of $g$. In this case, it is unlikely that $g$ will be polynomial. For example, in the case of the continuum random tree endowed with its usual uniform volume measure (to be formally introduced in Section 3.1.1), it was shown in [Cro08], [DLG06] and [Duq12] that we need to take $g(r)=r^{2} \log \log r^{-1}$ and $g(r)=r^{2}\left(\log \log r^{-1}\right)^{-1}$ to get agreement for the Hausdorff and packing measures respectively. In the case of stable trees, Duquesne showed the form of the exact packing measure in [Duq12], and showed in [DLG06] that there is no such exact Hausdorff measure.

In the case of stable looptrees we have not been able obtain the exact functions needed to obtain non-trivial Hausdorff and packing measures. However, in Chapter 4 we prove very precise bounds for the gauge functions at which the associated Hausdorff and packing measures jump from infinity to zero, accurate up to log-logarithmic terms.

### 2.2 Gromov-Hausdorff-Prohorov topologies

A key part of this thesis will be to prove convergence results for sequences of metric spaces endowed with measures. In this section we introduce the Gromov-HausdorffProhorov topology, which is an appropriate topology for this convergence.

Firstly, let $(E, d)$ be a metric space. The Hausdorff distance $d_{H}$ between two
sets $A, A^{\prime} \subset E$ is defined as

$$
d_{H}^{E}\left(A, A^{\prime}\right)=\max \left\{\sup _{a \in A} \inf _{a^{\prime} \in A^{\prime}} d\left(a, a^{\prime}\right), \sup _{a^{\prime} \in A^{\prime}} \inf _{a \in A} d\left(a^{\prime}, a\right)\right\} .
$$

Now let ( $E^{\prime}, d^{\prime}$ ) be a second compact metric space. The Gromov-Hausdorff distance between $E$ and $E^{\prime}$ is defined as

$$
d_{G H}\left(E, E^{\prime}\right)=\inf \left\{d_{H}^{F}\left(\varphi(E), \varphi^{\prime}\left(E^{\prime}\right)\right)\right\},
$$

where the infimum is taken over all isometric embeddings $\varphi: E \rightarrow F, \varphi^{\prime}: E^{\prime} \rightarrow F$ into some common metric space $(F, \delta)$.

The Gromov-Hausdorff distance can also alternatively be defined using correspondences. A correspondence between the metric spaces $(E, d)$ and $\left(E^{\prime}, d^{\prime}\right)$ is a subset $\mathcal{R} \subset E \times E^{\prime}$ such that for every $x_{1} \in E$ there exists $x_{2} \in E^{\prime}$ with $\left(x_{1}, x_{2}\right) \in \mathcal{R}$, and similarly for every $y_{2} \in E^{\prime}$ there exists $y_{1} \in E$ with $\left(y_{1}, y_{2}\right) \in \mathcal{R}$. The distortion of the correspondence $R$ is defined as

$$
\operatorname{dis}(R)=\sup \left\{\left|d\left(x_{1}, y_{1}\right)-d^{\prime}\left(x_{2}, y_{2}\right)\right|:\left(x_{1}, x_{2}\right),\left(y_{1}, y_{2}\right) \in R\right\} .
$$

The following result is standard.

## Lemma 2.2.1.

$$
d_{G H}\left(E, E^{\prime}\right)=\frac{1}{2} \inf \{d i s(R)\} .
$$

where the infimum is taken over all correspondences $R$ between $E$ and $E^{\prime}$.
We will mainly be working with trees and related objects which normally have a distinguished root vertex and are equipped with a natural volume measure, so suppose additionally that $\mu$ and $\nu$ are two finite measures on $E$, and let $A^{\varepsilon}=$ $\{x \in E: d(x, A)<\varepsilon\}$ be the $\varepsilon$-fattening of $A$ in $E$. We define the Prohorov distance $d_{P}^{E}(\mu, \nu)$ between $\mu$ and $\nu$ by

$$
\inf \left\{\varepsilon>0: \mu(A) \leq \nu\left(A^{\varepsilon}\right)+\varepsilon \text { and } \nu(A) \leq \mu\left(A^{\varepsilon}\right)+\varepsilon \text { for any closed set } A \subset E\right\} .
$$

For two rooted measure metric spaces $(E, d, \mu, \rho)$ and $\left(E^{\prime}, d^{\prime}, \mu^{\prime}, \rho^{\prime}\right)$ we define the pointed Gromov-Hausdorff-Prohorov distance between them as

$$
d_{G H P}\left(E, E^{\prime}\right)=\inf \left\{d_{H}^{F}\left(\varphi(E), \varphi^{\prime}\left(E^{\prime}\right)\right)+d^{F}\left(\varphi(\rho), \varphi^{\prime}\left(\rho^{\prime}\right)\right)+d_{P}^{F}\left(\mu \circ \varphi^{-1}, \mu^{\prime} \circ \varphi^{\prime-1}\right)\right\},
$$

where $\rho$ and $\rho^{\prime}$ are the roots of $\mathcal{T}$ and $\mathcal{T}^{\prime}$ respectively and the infimum is taken over all isometric embeddings $\varphi: E \rightarrow F, \varphi^{\prime}: E^{\prime} \rightarrow F$ into some metric space $(F, \delta)$.

It is also worth noting that we can simply define the pointed Gromov-Hausdorff
distance as

$$
d_{G H}^{*}\left(E, E^{\prime}\right)=\inf \left\{d_{H}^{F}\left(\varphi(E), \varphi^{\prime}\left(E^{\prime}\right)\right)+d^{F}\left(\varphi(\rho), \varphi^{\prime}(\rho)\right\},\right.
$$

where again the infimum is taken over all isometric embeddings into a common metric space $(F, \delta)$. Similarly to before, it can then be shown that

$$
d_{G H}^{*}\left(E, E^{\prime}\right)=\frac{1}{2} \inf \{\operatorname{dis}(R)\}
$$

where the infimum is instead taken over all correspondences $R$ between $E$ and $E^{\prime}$ that include the pair ( $\rho, \rho^{\prime}$ ).

### 2.2.1 Gromov-Hausdorff-vague topology

In order to extend this to convergence of non-compact measured metric spaces, we will use the Gromov-Hausdorff-vague topology of [ALW16]. We will restrict our attention to boundedly finite, Heine-Borel metric spaces endowed with measures of full support. In this case, weak convergence is metrized by the Prohorov metric [Bil68, Theorem 6.8] so we can write the weak convergence of [ALW16, Definition 5.8] as Prohorov convergence in our Definition 2.2.2.

This topology will be used to prove a local limit theorem stating that an increasing sequence of compact metric spaces converge in distribution to an infinite metric space. To make sense of this kind of convergence, we will consider all of our metric spaces to be pointed, in that they will have a distinguished vertex (normally the root) which plays a special role. Gromov-Hausdorff-vague convergence then says that for almost every $r>0$, balls of radius $r$ around the root converge with respect to the usual Gromov-Hausdorff-Prohorov topology defined in previous section. We formalise this below.

Recall that a Heine-Borel space is a metric space in which every bounded, closed set is compact (so in particular, a Heine-Borel space is complete, separable and locally compact). As in [ALW16], we let $\mathbb{X}$ be the space of equivalence classes of boundedly finite measure metric space, and $\mathbb{X}_{\mathrm{HB}}$ be the set of equivalence classes of boundedly finite Heine-Borel measure metric spaces, where we say that two elements $\left(X_{1}, \tilde{d}_{1}, \rho_{1}, \mu_{1}\right),\left(X_{2}, \tilde{d}_{2}, \rho_{2}, \mu_{2}\right) \in \mathbb{X}$ are equivalent if and only if there exists an isometry $\varphi: X_{1} \rightarrow X_{2}$ such that $\varphi\left(\rho_{1}\right)=\rho_{2}$ and $\mu_{1} \circ \varphi^{-1}=\mu_{2}$.

For any $\mathfrak{X}=(X, \tilde{d}, \rho, \mu) \in \mathbb{X}$ we let $\mathcal{B}_{r}(\mathfrak{X})$ denote the closed subspace $\left(B(\rho, r),\left.\tilde{d}\right|_{B(\rho, r)}, \rho,\left.\mu\right|_{B(\rho, r)}\right)$.

Definition 2.2.2. (cf [ALW16, Definition 5.8]). Let $\mathfrak{X}=(X, \tilde{d}, \rho, \mu)$ and $\left(\mathfrak{X}_{n}=\right.$ $\left.\left(X_{n}, \tilde{d}_{n}, \rho_{n}, \mu_{n}\right)\right)_{n \geq 1} \in \mathbb{X}_{H B}$. We say that $\mathfrak{X}_{n} \rightarrow \mathfrak{X}$ in the Gromov-Hausdorff-vague topology if and only if $\mathcal{B}_{r}\left(\mathfrak{X}_{n}\right) \rightarrow \mathcal{B}_{r}(\mathfrak{X})$ with respect to the Gromov-Hausdorff-

Prohorov topology for Lebesgue almost every $r>0$.
Remark 2.2.3. For compact spaces, it should be clear that Gromov-Hausdorff-vague convergence is equivalent to the usual Gromov-Hausdorff convergence. In particular, if $\mathcal{B}_{R}\left(\mathfrak{X}_{n}\right) \rightarrow \mathcal{B}_{R}(\mathfrak{X})$ for a given $R>0$, then the convergence also holds for Lebesgue almost every $r \in(0, R)$. To prove full Gromov-Hausdorff-vague convergence, it is therefore sufficient to prove convergence along a countable sequence $r_{n}$ diverging to infinity. We will do this in Chapter 5 when we prove several limit theorems for sequences of looptrees that converge to an infinite stable looptree.

We also have the following proposition, which will allow us to apply the Skorokhod Representation theorem in Chapter 5.

Proposition 2.2.4. (cf [ALW16, Proposition 5.12], [ADH13, Lemma 2.9]). The space of Heine-Borel boundedly finite measure spaces equipped with the Gromov-Hausdorff-vague topology is a Polish space.

Remark 2.2.5. In keeping with [GM17b], we say that a $r$ is a"good radius" if $\mu(\partial B(\rho, r))=0$. Since the sets $\partial B(\rho, r)$ are disjoint for different values of $r$, it follows that the set $\{r \geq 0: r$ not good $\}$ must have zero Lebesgue measure. In particular, the Prohorov convergence (or lack of it) at these values of $r$ will not have any effect on Gromov-Hausdorff-vague convergence. It can in fact be shown [BBI01] that if $\mathfrak{X}_{n} \rightarrow \mathfrak{X}$ Gromov-Hausdorff vaguely, then the (at most countable) set of $r$ for which $\mathcal{B}_{r}\left(\mathfrak{X}_{n}\right) \nrightarrow \mathcal{B}_{r}(\mathfrak{X})$ is a subset of the set $S=\{r>0: \mu(\partial B(\rho, r))>0\}$.

### 2.3 Skorokhod- $J_{1}$ topology

In this section we briefly introduce the Skorokhod- $J_{1}$ topology, first defined in [Sko56], and used to give a notion of convergence for càdlàg functions that are not continuous. In this setting, the usual uniform convergence is too strong for our purposes since if $f_{n} \rightarrow f$ uniformly and $f$ is not continuous, then this requires that any jump point of $f$ will also be a jump point of $f_{n}$ for all sufficiently large $n$. Convergence in the Skorokhod- $J_{1}$ topology instead allows for the jump locations of the $f_{n}$ to converge to those of $f$.

The Skorokhod- $J_{1}$ distance function is defined as follows. First let

$$
\Lambda=\{\lambda:[0,1] \rightarrow[0,1]: \lambda(0)=0, \lambda(1)=1, \lambda \text { a homeomorphism }\}
$$

The Skorokhod- $J_{1}$ distance is then defined as

$$
\begin{equation*}
d_{J_{1}}(f, g)=\inf _{\lambda \in \Lambda}\left\{\|f \cdot \lambda-g\|_{\infty}+\|\lambda-I\|_{\infty}\right\} \tag{2.1}
\end{equation*}
$$



Figure 2.1: Tree.

It should be clear that this defines a metric on $D(I, \mathbb{R})$ for any compact interval $I \subset \mathbb{R}$. If $f_{n}$ is a sequence with $d_{J_{1}}\left(f_{n}, f\right) \rightarrow 0$, then the function $\lambda_{n}$ is used to align the jump locations of the $f_{n}$ to those of $f$. The requirement that $\left\|\lambda_{n}-I\right\|_{\infty} \rightarrow 0$ ensures that the locations of the jumps of $f_{n}$ uniformly approach those of $f$, and the requirement that $\left\|f_{n} \circ \lambda_{n}-f\right\|_{\infty} \rightarrow 0$ ensures that $f_{n} \rightarrow f$ uniformly away from the jumps.

### 2.4 Stochastic processes and electrical resistance

Throughout this thesis, the theory of resistance forms will be a key technique used to define and analyse stochastic processes. Here we give a brief overview of the salient points of the theory. For a full account, consult [Kig01] and [Kig12]. A good introduction can also be found in [Cro17].

For intuition, we start by considering random walks on trees. Let $X$ be a simple random walk on the tree in Figure 2.4, and let $d$ be the shortest distance metric on the tree. It can be verified by direct calculation that

$$
\begin{aligned}
& \mathbb{P}_{a}(X \text { hits } b \text { before returning to } a)=\frac{1}{2}, \\
& \mathbb{P}_{a}(X \text { hits } c \text { before returning to } a)=\frac{1}{4} .
\end{aligned}
$$

Notice also that

$$
\begin{aligned}
& \frac{1}{\operatorname{deg}(a) d(a, b)}=\frac{1}{2} \\
& \frac{1}{\operatorname{deg}(a) d(a, c)}=\frac{1}{4} .
\end{aligned}
$$

In fact, it is a general result (e.g. see [LPW09, Chapter 9]) that for any tree $T$, if $a$ and $b$ are two vertices in the tree and $X$ is a simple random walk on the tree, then

$$
\begin{equation*}
\mathbb{P}_{a}(X \text { hits } b \text { before returning to } a)=\frac{1}{\operatorname{deg}(a) d(a, b)} . \tag{2.2}
\end{equation*}
$$

Moreover, it is also true that for any vertices $a, b \in T$, we have the following occupation density formula:

$$
\begin{equation*}
\mathbb{E}_{a}\left[\int_{0}^{\tau_{b}} f\left(X_{s}\right) d s\right]=2 \int_{T} d(b, \operatorname{branch}(a, b, c)) f(c) \operatorname{deg}(c) d c, \tag{2.3}
\end{equation*}
$$

where $\operatorname{branch}(a, b, c)$ is the unique branch point of $a, b$ and $c$.
This suggests that the shortest-distance metric $d$ on any tree $T$ is somehow characterising the behaviour of a simple random walk on $T$. We would therefore expect that properties of the metric can be used to give information about the simple random walk. This useful because the behaviour of the metric is often easier to analyse than the random walk itself.

It is clear that a relationship in the form of (2.2) will not hold for a general non-tree-like graph $G$ (consider adding an edge between $b$ and $c$ in Figure 2.4, for example), however it is natural to ask whether there is a different metric that we can use for a more general graph that characterises a random walk in the same way. It turns out that the appropriate metric is given by the effective resistance metric, defined as follows.

Take a discrete graph $G=(V, E)$, with edge weights $\left(c_{e}\right)_{e \in E}$. We view $G$ as an electrical network where each edge $e$ has electrical conductance $c_{e}$. The resistance of each edge is given by $R_{e}=c_{e}^{-1}$. Take two vertices $a, b \in V$, and apply a unit voltage at $a$. This induces a flow of current from $a$ to $b$. Now suppose we replace the whole of $G$ by a single edge joining $a$ to $b$. The effective resistance between $a$ and $b$, denoted $R(a, b)$ is equal to the resistance that we must give this one edge so that on again applying a unit voltage to $a$, the total current flowing from $a$ to $b$ is unchanged.

Effective resistance corresponds to the usual physical notion of electrical resistance and can be calculated for a specific graph $G$ using the series and parallel laws. These are as follows (we have lifted the definitions from [LPW09, Section 9.4]).

Parallel Law. Conductances in parallel add. Suppose edges $e_{1}$ and $e_{2}$ have conductances $c_{1}$ and $c_{2}$ respectively, and share vertices $v_{1}$ and $v_{2}$ as endpoints. Then both edges can be collectively replaced by a single edge of conductance $c_{1}+c_{2}$ without affecting the effective resistance in the rest of the network.

Series Law. Resistances in parallel add. If $v$ is a node of degree 2 with neighbours $v_{1}$ and $v_{2}$, then the edges $\left(v, v_{1}\right)$ and $\left(v, v_{2}\right)$ can be collectively replaced by a single edge of resistance $r_{\left(v, v_{1}\right)}+r_{\left(v, v_{2}\right)}$ without changing the effective resistance in the rest of the network.

Letting $X$ be a simple random walk on $G$, we then have the following result,
which generalises (2.2).

$$
\begin{equation*}
\mathbb{P}_{a}(X \text { hits } b \text { before returning to } a)=\frac{1}{c(a) R(a, b)}, \tag{2.4}
\end{equation*}
$$

where for any $c \in V, c(a)=\sum_{x \sim a} c(a, x)$. For a proof, see [LPW09, Chapter 9].
We formalise these heuristics in the next section.

### 2.4.1 Stochastic processes associated with resistance metrics

To study Brownian motion and random walks on metric spaces we will be using the theory of resistance forms and resistance metrics, developed by Kigami [Kig01, Kig12].

Let $G=(V, E)$ be a discrete graph equipped with non-negative symmetric edge conductances $c(x, y)_{(x, y) \in E}$ and a measure $(\mu(x))_{x \in V}$. The associated random walk on $G$ is the continuous time Markov chain on $V$ with generator $\Delta$, i.e. such that

$$
\begin{equation*}
\Delta f(x)=\frac{1}{\mu(x)} \sum_{y \in V: y \sim x} c(x, y)(f(y)-f(x)) \tag{2.5}
\end{equation*}
$$

for any function $f: V \rightarrow \mathbb{R}$; in other words, this is the continuous time random walk that jumps from $x$ to $y$ with rate $\frac{c(x, y)}{\mu(x)}$. The conductances therefore determine the transition probabilities of the random walk, and we can use $\mu$ to vary the timescaling.

If we view $G$ as an electrical network where each vertex $x \in V$ has potential (i.e. voltage) $f(x)$, then the energy dissipated by the network is equal to $\mathcal{E}(f, f)$, where $\mathcal{E}(f, g)$ is an energy functional given by

$$
\begin{equation*}
\mathcal{E}(f, g)=\frac{1}{2} \sum_{x, y \in V} c(x, y)(f(y)-f(x))(g(y)-g(x)) . \tag{2.6}
\end{equation*}
$$

Clearly, this does not depend on $\mu$; however, we can also write it as a Dirichlet form on the space $L^{2}(V, \mu)$ as

$$
\begin{equation*}
\mathcal{E}(f, g)=-\sum_{x \in V}(\Delta f)(x) g(x) \mu(x) . \tag{2.7}
\end{equation*}
$$

Moreover, we can write effective resistance on $G$ as a function $R$ on $V \times V$ by setting

$$
\begin{equation*}
R(x, y)^{-1}=\inf \{\mathcal{E}(f, f) \mid f: V \rightarrow \mathbb{R}, f(x)=1, f(y)=0\} \tag{2.8}
\end{equation*}
$$

where we take the convention that $\inf \emptyset=\infty . R(x, y)$ corresponds to the usual physical notion of electrical resistance between $x$ and $y$ in $G$. It can be shown (e.g.
see $[\operatorname{Tet} 91])$ that $R$ is a metric on $G$, so we call it the resistance metric.
The notion of a resistance metric can be extended to the continuum as follows.
Definition 2.4.1. [Kig01, Definition 2.3.2]. Let $F$ be a set. A function $R: F \times F$ is a resistance metric on $F$ if and only if for every finite subset $V \subset F$, there exists a weighted graph with vertex set $V$ such that $\left.R\right|_{V \times V}$ is the effective resistance on $V$, i.e. is given by (2.8).

A resistance metric on a set $F$ can be naturally associated with an energy functional $\mathcal{E}$ constructed analogously to (2.6) above. We let $\mathcal{F}$ denote a subspace of real-valued functions on $F$ with finite energy, and call the pair $(\mathcal{E}, \mathcal{F})$ a resistance form. (Technically, $(\mathcal{E}, \mathcal{F})$ must also satisfy the so-called Markov property, see [Kig12, Definition 3.1] for details). Given a measure $\mu$ on $F$, the energy functional can be written as a Dirichlet form on the space $L^{2}(F, \mu)$ analogously to (2.7) above, and therefore can be naturally associated with a stochastic process on $F$, provided the resistance form is regular as per the definition below.

Definition 2.4.2. [Kig12, Definition 6.2]. A resistance form $(\mathcal{E}, \mathcal{F})$ is regular if $\mathcal{F} \cup C_{0}(F)$ is dense in $C_{0}(F)$ with respect to the supremum norm, where $C_{0}(F)$ represents the space of continuous functions on $F$ with compact support.

By [Kig01, Theorems 2.3.4 and 2.3.6], there is a one-to-one correspondence between resistance metrics and resistance forms on $F$, given analogously to (2.8). Moreover, if the corresponding resistance form is regular, then it induces a regular Dirichlet form on the space $L^{2}(F, \mu)$ (analogous to (2.6)), which in turn is naturally associated with a Hunt process on $F$ as a consequence of [FOT11, Theorem 7.2.1]. This is automatically the case when $(F, R)$ is a compact resistance metric space endowed with a finite Borel measure $\mu$ of full support, for example, but in the case of the infinite looptrees considered in Chapter 5 we will have to put some extra work into proving that the associated resistance form is regular. This is done in Proposition 5.5.2.

We have tried to keep background on resistance forms and Dirichlet forms to a minimum, but see [Kig12] for more on this. The key point is that, under appropriate regularity conditions on the underlying space (which will always be fulfilled in this thesis), there is a one-to-one correspondence between resistance metrics and stochastic processes. The reader should feel free to skip the proof of Proposition 5.5.2, which proves the required regularity in the setting of Chapter 5, and merely use this correspondence as a black box throughout the thesis.

This correspondence allows us to use results about scaling limits of measured resistance metric spaces to prove results about scaling limits of stochastic processes as detailed in the following result of [Cro18]. Before stating it, we note that the
notion of effective resistance between points given in (2.8) can be extended to that of effective resistance between two sets $A, B \subset F$ by setting

$$
R(A, B)^{-1}=\inf \{\mathcal{E}(f, f) \mid f: F \rightarrow \mathbb{R}, f(x)=1 \forall x \in A, f(y)=0 \forall y \in B\}
$$

Theorem 2.4.3. [Cro18, Theorem 1.2]. Suppose that $\left(F_{n}, R_{n}, \mu_{n}, \rho_{n}\right)_{n \geq 0}$ is a sequence in $\mathbb{F}$ such that

$$
\left(F_{n}, R_{n}, \mu_{n}, \rho_{n}\right) \rightarrow(F, R, \mu, \rho)
$$

Gromov-Hausdorff-vaguely for some $(F, R, \mu, \rho) \in \mathbb{F}$, and $R,\left(R_{n}\right)_{n \geq 1}$ are resistance metrics on the respective spaces. Assume further that

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \liminf _{n \rightarrow \infty} R_{n}\left(\rho_{n}, B_{n}\left(\rho_{n}, r\right)^{c}\right)=\infty \tag{2.9}
\end{equation*}
$$

Let $\left(Y_{t}\right)_{t \geq 0}$ and $\left(Y_{t}^{n}\right)_{t \geq 0}$ be the stochastic processes associated with $(F, R, \mu, \rho)$ and $\left(F_{n}, R_{n}, \mu_{n}, \rho_{n}\right)$ as described above. Then it is possible to isometrically embed $(F, R)$ and $\left(F_{n}, R_{n}\right)_{n \geq 1}$ into a common metric space $\left(M, d_{M}\right)$ so that

$$
\mathbb{P}_{\rho_{n}}^{n}\left(\left(Y_{t}^{n}\right)_{t \geq 0} \in \cdot\right) \rightarrow \mathbb{P}_{\rho}\left(\left(Y_{t}\right)_{t \geq 0} \in \cdot\right)
$$

weakly as probability measures as $n \rightarrow \infty$ on the space $D\left(\mathbb{R}_{+}, M\right)$ equipped with the Skorokhod $J_{1}$-topology.

The intuition behind the result above is that the convergence of metrics and measures respectively give the appropriate spatial and temporal convergences of the stochastic processes. We will apply it several times in this thesis to take limits of stochastic processes on looptrees.

We also give an annealed version, which applies in the case where the metric spaces are random, say defined under the probability measure $\mathbf{P}$, and we can define a stochastic process by averaging over the state space. In this case we let

$$
\mathbb{P}_{\rho}\left((\tilde{X})_{t \geq 0} \in \cdot\right):=\int P_{\rho}\left(\left(X_{t}\right)_{t \geq 0} \in \cdot\right) d \mathbf{P}
$$

denote the annealed law obtained by averaging over the random state space. In this case we have a similar result.

Theorem 2.4.4. [Cro18, Theorem 7.2]. Suppose that $\left(F_{n}, R_{n}, \mu_{n}, \rho_{n}\right)_{n \geq 0}$ is a sequence in $\mathbb{F}$ such that

$$
\left(F_{n}, R_{n}, \mu_{n}, \rho_{n}\right) \xrightarrow{(d)}(F, R, \mu, \rho)
$$

Gromov-Hausdorff-vaguely for some $(F, R, \mu, \rho) \in \mathbb{F}$. Assume further that

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \liminf _{n \rightarrow \infty} \mathbb{P}\left(R_{n}\left(\rho_{n}, B_{n}\left(\rho_{n}, r\right)^{c}\right) \geq \lambda\right)=1 \tag{2.10}
\end{equation*}
$$

for all $\lambda>0$. Then it is possible to isometrically embed $\left(F_{n}, R_{n}\right)_{n \geq 1}$ and $(F, R)$ into a common metric space $\left(M, d_{M}\right)$ so that

$$
\mathbb{P}_{\rho_{n}}\left(\left(\tilde{X}^{n}\right)_{t \geq 0} \in \cdot\right) \rightarrow \mathbb{P}_{\rho}\left(\left(\tilde{X}_{t}\right)_{t \geq 0} \in \cdot\right)
$$

weakly as probability measures as $n \rightarrow \infty$ on $D\left(\mathbb{R}_{+}, M\right)$ (i.e. on the space of càdlàg functions on $M$ equipped with the Skorokhod $J_{1}$-topology).

### 2.4.2 Simple random walk estimates

In Chapter 6 we will be analysing simple random walks on a discrete graph $G=$ $(V, E)$. To fit this into the framework of electrical networks introduced above, we therefore give every edge in the graph conductance 1 , and set $\mu(x)=\operatorname{deg} x$ for all $x \in V$. The Markov process with generator as given by (2.5) above is therefore a constant speed random walk, in that it waits at each vertex for an $\exp (1)$ random time before jumping to the next vertex. This is not quite the same as the simple random walk; however, it seems quite reasonable to expect that the two processes should have the same asymptotics, and this is indeed the case: in Chapter 6 we will study the measure $\mu$ along with resistance, and apply results of [KM08] to use these to understand a simple random walk.

Note that, if $U \subset V$, then $\mu(U)$ essentially counts the number of edges in $\mu$. This matches the intuition that the overall time it takes for a simple random walk to cross a set depends on how many edges it is required to cross. If one instead takes $\mu(x)=1$ for all $x \in V$, then the volume estimates correspond to counting vertices, and the associated random walk is instead a (continuous time) variable speed random walk.

### 2.5 Stable Lévy processes

In Chapter 3 we will introduce stable trees and looptrees. These are constructed from stable Lévy processes, which we now introduce, along with some of their key properties. The material presented here is classical and may be found in greater detail in [Ber96].

A Lévy process is a càdlàg process starting from 0 with stationary independent increments. We say $(a, b, K)$ is a Lévy triple if $a \in[0, \infty), b \in \mathbb{R}$ and $K$ is a Borel measure on $\mathbb{R}$ with $K(\{0\})=0$ and

$$
\int_{\mathbb{R}}\left(1 \wedge|y|^{2}\right) K(d y)<\infty
$$

We call $a$ the diffusivity, $b$ the drift and $K$ the Lévy measure. We can define a process corresponding to a given Lévy triple as follows. First let $B$ be a Brownian
motion in $\mathbb{R}$ and let $M$ be a Poisson random measure, independent of $B$ and with intensity $\mu$ on $(0, \infty) \times \mathbb{R}$, where $\mu(d t, d y)=d t K(d y)$. Then let

$$
\begin{equation*}
X_{t}=\sqrt{a} B_{t}+b t+\int_{(0, t] \times\{|y| \leq 1\}} y \tilde{M}(d s, d y)+\int_{(0, t] \times\{|y|>1\}} y M(d s, d y) \tag{2.11}
\end{equation*}
$$

where $\tilde{M}:=M-\mu$. We set the final integral to be zero on the null set

$$
\{M((0, t] \times\{|y|>1\})=\infty \text { for some } t \geq 0\}
$$

Then $X$ is a Lévy process with characteristic function

$$
\mathbb{E}\left[e^{i u X_{t}}\right]=e^{t \psi(u)}
$$

for all $t \geq 0$, where

$$
\psi(u)=\psi_{a, b, K}(u)=i b u-\frac{1}{2} a u^{2}+\int_{\mathbb{R}}\left(e^{i u y}-1-i u y \mathbb{1}\{|y| \leq 1\}\right) K(d y)
$$

The next theorem tells us that all Lévy processes are of this form. A proof may be found in [Ber96].

Theorem 2.5.1 (Lévy-Khinchin Theorem). Let $X$ be a Lévy process. Then there exists a unique Lévy triple $(a, b, K)$ such that

$$
\mathbb{E}\left[e^{i u X_{t}}\right]=e^{t \psi_{a, b, K}(u)}
$$

for all $t \geq 0$ and all $u \in \mathbb{R}$. Furthermore,

$$
\mathbb{E}\left[e^{-\lambda X_{t}}\right]=\exp \left\{t\left(-b \lambda+\frac{1}{2} a \lambda^{2}+\int_{\mathbb{R}}\left(e^{-\lambda y}-1+\lambda y \mathbb{1}\{|y| \leq 1\}\right) K(d y)\right)\right\}
$$

provided that the right hand side is finite.
We say that a Lévy process $X$ is spectrally positive if it has no negative jumps. Additionally, we say it is $\alpha$-stable if we can normalise $X$ so that $\mathbb{E}\left[e^{-\lambda X_{t}}\right]=e^{\lambda^{\alpha} t}$. In this thesis we will be focusing on the case when $X$ is both spectrally positive and $\alpha$-stable. As we show below, it then follows that $\mathbb{E}\left[e^{-\lambda X_{t}}\right]$ is finite for all $\lambda \geq 0$, and that $X$ satisfies the scaling property $\left(c^{-\frac{1}{\alpha}} X_{c t}\right)_{t \geq 0} \stackrel{d}{=}\left(X_{t}\right)_{t \geq 0}$ for any constant $c>0$. Its transition density consequently satisfies the relation $p_{t}(x)=t^{\frac{-1}{\alpha}} p_{1}\left(x t^{\frac{-1}{\alpha}}\right)$.

Corollary 2.5.2. Let $X$ be a spectrally positive Lévy process. Then $\mathbb{E}\left[e^{-\lambda X_{t}}\right]<\infty$ for all $\lambda \geq 0$ and all $t \geq 0$.

Proof. This follows directly from the representation in 2.5.1.

### 2.5.1 Lévy excursions and bridges

To code stable trees and looptrees, we will be using Lévy excursions rather than a general Lévy process. We now explain how these can be constructed from Lévy processes.

Let $X$ be an $\alpha$-stable spectrally positive Lévy process, normalised so that $\mathbb{E}\left[e^{-\lambda X_{t}}\right]=e^{\lambda^{\alpha} t}$, and let $\underline{X}_{t}=\inf _{s \in[0, t]} X_{s}$ denote its running infimum process. Define $g_{1}$ and $d_{1}$ by

$$
\begin{aligned}
g_{1} & =\sup \left\{s \leq 1: X_{s}=\underline{X}_{s}\right\} \\
d_{1} & =\inf \left\{s>1: X_{s}=\underline{X}_{s}\right\} .
\end{aligned}
$$

Note that $X_{g_{1}}=X_{d_{1}}$ almost surely since $X$ almost surely has no jump at time $g_{1}$ and $X$ has no negative jumps. We define the normalised excursion $X^{\text {exc }}$ of $X$ above its infimum at time 1 by

$$
X_{s}^{\mathrm{exc}}=\left(d_{1}-g_{1}\right)^{\frac{-1}{\alpha}}\left(X_{g_{1}+s\left(d_{1}-g_{1}\right)}-X_{g_{1}}\right)
$$

for every $s \in[0,1]$. Then $X^{\text {exc }}$ is almost surely an $\alpha$-stable càdlàg function on $[0,1]$ with $X^{\text {exc }}(s)>0$ for all $s \in(0,1)$, and $X_{0}^{\text {exc }}=X_{1}^{\text {exc }}=0$.

Most of the results regarding volume will be obtained by careful analysis of the Lévy excursions coding the looptrees. In most cases, it is easier to analyse an $\alpha$-stable Lévy process rather than an excursion. Results regarding the Lévy process can then be transferred first to a Lévy bridge, and then to an excursion via a sequence of transformations which we now describe.

An $\alpha$-stable Lévy bridge is informally an $\alpha$-stable Lévy process conditioned to return to 0 at time 1 (see [Ber96, Chapter VIII]). It has a density with respect to the law of a standard Lévy process, which is equal to

$$
\frac{p_{1-t}\left(-X_{t}\right)}{p_{1}(0)},
$$

with $p_{t}$ the density of $X_{t}$ and $t \in(0,1)$. Note that it follows from [Zol86, Section I.4] that $p_{1}$ is bounded on $\mathbb{R}$. This means that if $F$ is a bounded continuous function $D[0, t] \rightarrow \mathbb{R}$, then

$$
\begin{equation*}
\mathbb{E}\left[F\left(X_{s}^{\mathrm{br}}: 0 \leq s \leq t\right)\right]=\mathbb{E}\left[F\left(X_{s}: 0 \leq s \leq t\right) \frac{p_{1-t}\left(-X_{t}\right)}{p_{1}(0)}\right] . \tag{2.12}
\end{equation*}
$$

The excursion can then be obtained from the bridge via the Vervaat transform, given below. For a full treatment see [Cha97].

Theorem 2.5.3. Let $X^{\text {br }}$ be a Lévy bridge, and let $m$ be the almost surely unique
time when $X^{b r}$ attains its minimum. Then the process defined by

$$
X_{t}^{e x c}= \begin{cases}X_{m+t}^{b r}-X_{m}^{b r} & \text { if } m+t \leq 1 \\ X_{m+t-1}^{b r}-X_{m}^{b r} & \text { if } m+t>1\end{cases}
$$

has the law of a normalised Lévy excursion.
Conversely, if $X^{e x c}$ is a normalised Lévy excursion and $U$ is independent and uniformly distributed on $[0,1]$, then the process

$$
X_{t}^{b r}= \begin{cases}X_{U+t}^{e x c}-X_{U}^{e x c} & \text { if } U+t \leq 1 \\ X_{U+t-1}^{e x c}-X_{U}^{e x c} & \text { if } U+t>1\end{cases}
$$

is distributed as a Lévy bridge.
We use these results throughout to transform a statement regarding $X^{\text {exc }}$ to a similar statement regarding $X$, which is usually easier to prove.

### 2.5.2 Descents

When Lévy excursions are used to code stable trees and looptrees, the sizes of the jumps of the processes correspond to the density of the hubs in the stable tree, and the lengths of loops in the stable looptree. The following proposition is useful in characterising the behaviour of the jumps, and consequently the behaviour of loops in the looptree. It follows from Proposition 3.1 of [CK14], which is proved using results from [Ber92a].

Proposition 2.5.4. Let $\left(X_{s}: s \in \mathbb{R}\right)$ be a two-sided spectrally positive $\alpha$-stable Lévy process. For each point $s \in \mathbb{R}$ with $X_{s} \neq X_{s^{-}}$let $\Delta_{s}=X_{s}-X_{s^{-}}$, and for $t>s$ let $x_{s}^{t}=\inf _{s \leq r \leq t} X_{r}-X_{s^{-}} \vee 0$. Also let $L_{s}$ denote the local time of the process $\left(X_{(t-s)^{-}}\right)_{s \geq 0}$ at its infimum, normalised so that $\mathbb{E}\left[\exp \left(-\lambda X_{\left(-L^{-1}(s)\right)^{-}}\right]=\right.$ $\exp \left(-s \lambda^{\alpha-1}\right)$. Then the point measure

$$
\sum_{s \preceq 0} \delta\left(L_{s}, \Delta_{s}, \frac{x_{s}^{0}}{\Delta_{s}}\right)
$$

is a Poisson point measure of intensity $d l \cdot x \Pi(d x) \cdot \mathbb{1}_{[0,1]}(r) d r$, where here we write $s \preceq t$ if $s \leq t, \Delta_{s}>0$, and $x_{s}^{t}>0$.

We also give a technical lemma which will later be used at various points in Chapter 4. This appeared previously in [CK14, Section 3.3.1] and uses an argument from [Ber96].

For a function $f:[0, \infty) \rightarrow \mathbb{R}$ and $[a, b] \subset[0, \infty)$, we first define

$$
\operatorname{Osc}_{[a, b]} f:=\sup _{s, t \in[a, b]}|f(t)-f(s)| .
$$

We also let $S_{t}=\sup _{0 \leq s \leq t} X_{s}$ denote the running supremum process of $X$, and $I_{t}=\inf _{0 \leq s \leq t} X_{s}$ its running infimum process.

Lemma 2.5.5. Let $\mathcal{E}$ be an exponential random variable with parameter 1 , and let $X$ be a spectrally positive $\alpha$-stable Lévy process conditioned to have no jumps of size greater than 1 on $[0, \mathcal{E}]$, independent of $\mathcal{E}$. Let $\tilde{O s c}=\operatorname{Osc}_{[0, \mathcal{E}]} X$. Then there exists $\theta>0$ such that $\mathbb{E}\left[e^{\theta \tilde{\sigma s c}}\right]<\infty$. Moreover, $\mathbb{E}\left[e^{\theta \tilde{O s c}}\right] \downarrow 1$ as $\theta \downarrow 0$.
Proof. Since $\operatorname{Osc}_{[0, \mathcal{E}]}=S_{\mathcal{E}}-I_{\mathcal{E}}$, and $S_{\mathcal{E}}-I_{\mathcal{E}}$ is stochastically dominated by $S_{\mathcal{E}}^{(1)}-I_{\mathcal{E}}^{(2)}$ where $S_{\mathcal{E}}^{(1)}$ and $I_{\mathcal{E}}^{(2)}$ are independent copies of the corresponding random variables, it is sufficient to show the existence of a $\theta>0$ such that both $\mathbb{E}\left[e^{\theta S_{\varepsilon}}\right]$ and $\mathbb{E}\left[e^{-\theta I_{\varepsilon}}\right]<$ $\infty$.

As noted in [Ber96, Section VII.1], the second of these is quite straightforward. Let $T_{-a}=\inf \left\{t \geq 0: X_{t} \in[-a, \infty)\right\}$. Since $X$ has no negative jumps, conditionally on $\left\{T_{-a}<\infty\right\}$, we have that $X_{T_{-a}}=-a$ almost surely. Moreover, we have by the memoryless property that

$$
\mathbb{P}\left(T_{a+b}<\mathcal{E}\right)=\mathbb{P}\left(T_{a}<\mathcal{E}\right) \mathbb{P}\left(T_{b}<\mathcal{E}\right)
$$

Equivalently,

$$
\mathbb{P}\left(-I_{\mathcal{E}}>a+b\right)=\mathbb{P}\left(-I_{\mathcal{E}}>a\right) \mathbb{P}\left(-I_{\mathcal{E}}>b\right)
$$

and hence $-I_{\mathcal{E}}$ has an exponential distribution, say with parameter $\Phi(\mathcal{E})$, which is clearly non-zero, and so $\mathbb{E}\left[e^{-\theta I_{\mathcal{E}}}\right]<\infty$ for all $\theta<\Phi(\mathcal{E})$.

To bound $\mathbb{E}\left[e^{\theta S_{\varepsilon}}\right]$, first note that since $X$ has no jumps greater than 1 on $[0, \mathcal{E}]$ it follows that for any $m \geq 1, \mathbb{P}\left(S_{\mathcal{E}}>m+2\right) \leq \mathbb{P}\left(S_{\mathcal{E}}>m\right) \mathbb{P}\left(S_{\mathcal{E}}>1\right)$ and hence $\mathbb{P}\left(S_{\mathcal{E}}>2 n\right) \leq \mathbb{P}\left(S_{\mathcal{E}}>1\right)^{n}$ for all $n \geq 1$. By direct computation it then follows that, provided $0<\theta<\frac{-1}{2} \log \left(\mathbb{P}\left(S_{\mathcal{E}}>1\right)\right)$, we have

$$
\mathbb{E}\left[e^{\theta S_{\mathcal{E}}}\right] \leq \frac{2}{\left|2 \theta+\log \left(\mathbb{P}\left(S_{\mathcal{E}}>1\right)\right)\right|}
$$

The final claim follows by bounded convergence.
Remark 2.5.6. Note that the same results holds if $\mathcal{E}$ is set to be deterministically equal to 1 rather than an exponential random variable. The proof is almost identical to the one above, but we instead have that $I_{\mathcal{E}}$ is stochastically dominated by an exponential random variable. The result also holds if the exponential random variable $\mathcal{E}$ has any other constant parameter, by exactly the same proof as above.

### 2.5.3 Itô excursion measure

We can alternatively define $X^{\text {exc }}$ using the Itô excursion measure. For full details, see [Ber96, Chapter IV], but the measure is defined by applying excursion theory to the process $X-\underline{X}$, which is strongly Markov and for which the point 0 is regular for itself. We normalise local time so that $-\underline{X}$ denotes the local time of $X-\underline{X}$ at its infimum, and let $\left(g_{j}, d_{j}\right)_{j \in \mathcal{I}}$ denote the excursion intervals of $X-\underline{X}$ away from zero. For each $i \in \mathcal{I}$, the process $\left(e^{i}\right)_{0 \leq s \leq d_{i}-g_{i}}$ defined by $e^{i}(s)=X_{g_{i}+s}-X_{g_{i}}$ is an element of the excursion space

$$
E=\bigcup_{\ell>0} D^{\mathrm{exc}}\left([0, \ell], \mathbb{R}^{\geq 0}\right)
$$

We let $\zeta(e)=\sup \{s>0: e(s)>0\}$ denote the lifetime of the excursion $e$. It was shown in [Itô72] that the measure

$$
N(d t, d e)=\sum_{i \in \mathcal{I}} \delta\left(-\underline{X}_{g_{i}}, e^{i}\right)
$$

is a Poisson point measure of intensity $d t N(d e)$, where $N$ is a $\sigma$-finite measure on the set $E$ known as the Itô excursion measure.

Moreover, the measure $N(\cdot)$ inherits a scaling property from the $\alpha$-stability of $X$. Indeed, for any $\lambda>0$ we define a mapping $\Phi_{\lambda}: E \rightarrow E$ by $\Phi_{\lambda}(e)(t)=\lambda^{\frac{1}{\alpha}} e\left(\frac{t}{\lambda}\right)$, so that $N \circ \Phi_{\lambda}^{-1}=\lambda^{\frac{1}{\alpha}} N$ (e.g. see [Wat10]). It then follows from the results in [Ber96, Section IV.4] that we can uniquely define a set of conditional measures $\left(N_{(s)}, s>0\right)$ on $E$ such that:
(i) For every $s>0, N_{(s)}(\zeta=s)=1$.
(ii) For every $\lambda>0$ and every $s>0, \Phi_{\lambda}\left(N_{(s)}\right)=N_{(\lambda s)}$.
(iii) For every measurable $A \subset E$

$$
N(A)=\int_{0}^{\infty} \frac{N_{(s)}(A)}{\alpha \Gamma\left(1-\frac{1}{\alpha}\right) s^{\frac{1}{\alpha}+1}} d s
$$

$N_{(s)}$ is therefore used to denote the law $N(\cdot \mid \zeta=s)$. The probability distribution $N_{(1)}$ coincides with the law of $X^{\text {exc }}$ as constructed above.

## Chapter 3

## Random trees and looptrees

In this chapter, we introduce random trees, random looptrees, and their coding functions.

### 3.1 Discrete random trees and their coding functions

We start with random trees. Random trees have been the subject of considerable mathematical research over the past few decades. They are fundamental objects in probability theory in their own right and have applications in the study of population genetics and percolation. More recently, they have been crucial in the study of random maps, which can be constructed via various bijections with random trees.

We start with discrete trees. Trees are connected graphs with no cycles, and are most commonly defined using the Ulam-Harris formalism of [Nev86]. First let

$$
\mathcal{U}=\cup_{n=0}^{\infty} \mathbb{N}^{n}
$$

By convention, $\mathbb{N}^{0}=\{\emptyset\}$. If $u=\left(u_{1}, \ldots, u_{n}\right)$ and $v=\left(v_{1}, \ldots, v_{m}\right) \in \mathcal{U}$, we let $u v=\left(u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{m}\right)$ be the concatenation of $u$ and $v$.

Definition 3.1.1. [Nev86, Section 2]. A plane tree $\theta$ is a finite subset of $\mathcal{U}$ such that
(i) $\emptyset \in \theta$,
(ii) If $v \in \theta$ and $v=u j$ for some $j \in \mathbb{N}$, then $u \in \theta$,
(iii) For every $u \in \theta$, there exists a number $k_{u}(\theta) \geq 0$ such that $u j \in \theta$ if and only if $1 \leq j \leq k_{u}(\theta)$.

Informally, a plane tree $\theta$ is a branching process started from some initial root vertex labelled $\emptyset$ (or $\rho$ ), and the offpring of vertex $u$ are of the form $u j$ where $j \in\left\{1,2, \ldots, k_{u}(\theta)\right\}$.

We denote by $\mathbb{T}^{\text {disc }}$ the set of all plane trees. Additionally, if $\theta \in \mathbb{T}^{\text {disc }}$, we let $\tau_{u}(\theta)=\{v \in \mathcal{U}: u v \in \theta\}$ be the shift of $\theta$ at $u . \tau_{u}(\theta)$ can be understood as the subtree grown from the root $u$.

The canonical examples of random discrete trees are Galton-Watson trees. To define these, first let $\mu$ be a probability measure on $\mathbb{Z}^{\geq 0}$. We will refer to $\mu$ as the offspring distribution.

Definition 3.1.2. A Galton-Watson tree with offspring distribution $\mu$ is a plane tree $\mathcal{T}^{\theta}$ satisfying the following properties.
(i) $\mathbb{P}_{\mu}\left(k_{\emptyset}=j\right)=\mu(j)$ for all $j \in \mathbb{Z}^{\geq 0}$,
(ii) For every $j \geq 1$ with $\mu(j)>0$, the shifted trees $\theta_{1}\left(\mathcal{T}^{\theta}\right), \ldots, \theta_{j}\left(\mathcal{T}^{\theta}\right)$ are independent under the conditional probability $\mathbb{P}_{\mu}\left(\cdot \mid k_{\emptyset}=j\right)$, with law $\mathbb{P}_{\mu}$.

In other words, a Galton-Watson tree with offspring distribution $\mu$ is a branching process with a single root $\emptyset$ where the trees emanating from each vertex are independently distributed according to $\mathbb{P}_{\mu}$. It is shown in $[\mathrm{Nev} 86$, Section 3] that for any probability measure $\mu$ on $\mathbb{Z}^{\geq 0}$, there is a unique probability measure $\mathbb{P}_{\mu}$ on $\mathbb{T}$ satisfying the above two properties. In this thesis, we will mainly be considering the critical case where $\sum_{k=0}^{\infty} k \mu(k)=1$.

Suppose $\mathcal{T}$ is a tree with $|\mathcal{T}|=n+1$. We can define two functions which encode its structure: the height function and the contour function. Both are illustrated in Figure 3.1, and are defined as follows. The height function $H^{\mathcal{T}}$ is defined by considering the vertices $u_{0}, u_{1}, \ldots, u_{n}$ in lexicographical (i.e. depth-first) order, and then setting $H_{i}^{\mathcal{T}}$ to be equal to the generation of vertex $u_{i}$. The contour function $C^{\mathcal{T}}$ is defined by considering the motion of a particle that starts at the root $\emptyset$ at time zero, and then continuously traverses the boundary of $\mathcal{T}$ at speed one, respecting the lexicographical order where possible, until returning to the root. $C^{\mathcal{T}}(t)$ is equal to the height of the particle at time $t$. Since each edge is traversed twice in this process, the contour function is defined in this way up until time equal to $2 n$. It will be convenient to set it equal to zero after this point. By contrast, the height function is defined precisely up until time $n$.

Note that this definition ensures that the contour function will be nonnegative after time zero, and the height function will be strictly positive.

In order to gain some intuition about the height and contour functions we have marked two vertices on the tree in Figure 3.1 along with the points corresponding to the excursion around $\tau_{1}(\mathcal{T})$ of the height and contour processes. For the contour function, note that this excursion lasts precisely until the time that $C^{\mathcal{T}}$ drops strictly below $C^{\mathcal{T}}(1)$. Any subsequent visits to the same level as $C^{\mathcal{T}}(1)$ correspond to visits of siblings of vertex 1 . By similar logic, for two points $s, t \leq 2 n$ it follows that the most recent common ancestor of $u(s)$ and $u(t)$ corresponds to the


Figure 3.1: Example of contour function and height function for the given tree.
point between $s$ and $t$ where $C^{\mathcal{T}}$ takes its minimum value, i.e. if $u(r)=u(s) \wedge u(t)$, then $C^{\mathcal{T}}(r)=\min _{s \wedge t \leq k \leq s \vee t} C^{\mathcal{T}}(k)$. We can therefore define a function

$$
\begin{equation*}
d(s, t)=C^{\mathcal{T}}(s)+C^{\mathcal{T}}(t)-2 \min _{s \wedge t \leq k \leq s \vee t} C^{\mathcal{T}}(k) \tag{3.1}
\end{equation*}
$$

which is equal to the graph distance between $u(s)$ and $u(t)$ and hence is a pseudometric on $\{0,1, \ldots, 2 n\}$. Also, $d(s, t)=0$ if and only if $u(s)=u(t)$, so we can define an equivalence relation on $\{0,1, \ldots, n\}$ by setting $s \sim t$ if and only if $d(s, t)=0$. $\mathcal{T}$ is then isomorphic to the quotient space $(\{0,1, \ldots, 2(n-1)\} / \sim, d)$. This construction of $\mathcal{T}$ will be useful when we extend the construction to the continuum.

One major difference between the height and contour functions is that whilst the contour function must move by $\pm 1$ at each time interval, the height function can drop by an arbitrary amount (provided it remains positive). For example, in Figure 3.1 we see that there is a place where the height function drops by 4.

Now let $\xi$ be an offspring distribution with $\mathbb{E}[\xi]=1$ and $0<\operatorname{Var}(\xi)=$ $\sigma^{2}<\infty$, and such that there exists $\lambda>0$ such that $\mathbb{E}\left[e^{\lambda \xi}\right]<\infty$. Consider a sequence of Galton-Watson trees $\left(\tau_{n}\right)_{n=1}^{\infty}$ started from an initial root $\rho$ with offspring distribution $\xi$ conditioned on $\left|\tau_{n}\right|=n$. Let $H^{n}$ and $C^{n}$ be the corresponding height and contour processes. The following results of Marckert and Mokkadem in [MM03] (and previously by Aldous via alternative methods in [Ald93]), show that the height function and contour function converge to the same process when appropriately rescaled.

Theorem 3.1.3. Let $e(t)_{t \in[0,1]}$ be a standard normalised Brownian excursion, con-
structed from a Brownian motion as described in Section 2.5.1. Then

$$
\begin{aligned}
& \left(\frac{C^{n}(\lfloor 2 n t\rfloor)}{\sqrt{n}}\right)_{t \in[0,1]} \Rightarrow\left(\frac{2}{\sigma} e(t)\right)_{t \in[0,1]} \text { and } \\
& \left(\frac{H^{n}(\lfloor n t\rfloor)}{\sqrt{n}}\right)_{t \in[0,1]} \Rightarrow\left(\frac{2}{\sigma} e(t)\right)_{t \in[0,1]}
\end{aligned}
$$

as $n \rightarrow \infty$, weakly as processes on the space $D([0,1], \mathbb{R})$ equipped with the Skorokhod$J_{1}$ topology.

The height process and the contour process are both very useful characterisations of trees but neither are Markovian. However, it turns out that both can be written as functionals of a Markovian process, known as the Lukasiewicz path. For a random tree $\mathcal{T}^{\theta}$ with height and contour processes $H^{\theta}$ and $C^{\theta}$, this is a random walk $W^{\theta}=\left\{W_{n}\left(\mathcal{T}^{\theta}\right): 1 \leq n \leq\left|\mathcal{T}^{\theta}\right|\right\}$, defined by first setting $W_{0}=0$, and then for $0 \leq n \leq\left|\mathcal{T}^{\theta}\right|$, setting $W_{n+1}=W_{n}+k_{u(n)}-1$, where $u(n)$ refers to the $n^{\text {th }}$ vertex in the lexicographical ordering of $\mathcal{T}^{\theta}$, and $k_{u(n)}$ is the number of offspring of $u(n)$, as in Defition 3.1.2.

An example for the tree pictured in Figure 3.1 is shown in Figure 3.2.


Figure 3.2: Lukasiewicz Path for the tree above
We remark on some properties of the Lukasiewicz path. Firstly, note that when we are dealing with a Galton-Watson tree at criticality, it follows that

$$
\mathbb{E}\left[W_{n+1}-W_{n}\right]=\mathbb{E}\left[k_{u(n)}\right]-1=0,
$$

so $W$ is a centred random walk.
Secondly, we have that $W_{i} \geq 0$ for all $0 \leq i \leq n-2, W_{n-1}=0$ and
$W_{n}=-1$. This can be understood by the following argument. We say that a vertex $v$ is "visible" from another vertex $u$ if $v$ is a child of $u$, and in this case we say that $v$ can be "seen" from vertex $u$. Then if we let $\left(X_{i}\right)_{0 \leq i \leq n}$ be the process which visits $u(i)$ at time $i$, then $W_{i}$ gives the total number of vertices seen up until time $i$, minus the total number of vertices visited. Since each vertex other than the root is seen before it is visited, it follows that this quantity will be strictly positive until $X$ hits the final vertex, at which it will be 0 , and then fall to -1 since at this point we will have visited all vertices and seen all except the root vertex.

The next lemma (taken from [LGLJ98] but independently shown in [BK00] and [BV96]) shows how to recover the height function from the Lukasiewicz path.

Lemma 3.1.4. [LGLJ98, Corollary 2.2]. Let $\mathcal{T}$ be a $\mu$-Galton Watson tree, with associated height process $\left(H_{n}\right)_{0 \leq n \leq|\mathcal{T}|-1}$. Then

$$
\begin{equation*}
H_{n}=\left|\left\{j \in\{0,1, \ldots, n-1\}: W_{j}=\inf _{j \leq k \leq n} W_{k}\right\}\right| . \tag{3.2}
\end{equation*}
$$

The intuition behind the result is that the set of points $j \in\{0,1, \ldots, n-1\}$ satisfying $W_{j}=\inf _{j \leq k \leq n} W_{k}$ correspond to the ancestors of the vertex $u(n)$.

In the next section we will construct $\alpha$-stable Lévy trees and show that they arise as scaling limits of discrete trees with appropriately defined offspring distributions. The main idea behind the proof of this result is to show that the Lukasiewicz paths of the discrete trees converge to an $\alpha$-stable Lévy process, and that this in turn is enough to imply convergence of the height and contour processes. This approach depends crucially on the Markovian structure of the Lukasiewicz path and its definition in terms of the offspring distribution.

## Multi-type Galton-Watson trees

We will consider scaling limits of looptrees defined from both one and two-type Galton-Watson trees in Chapter 5. Accordingly, let $\xi, \xi_{\circ}$ and $\xi$ • be probability distributions on $\mathbb{Z}^{\geq 0}$.

Definition 3.1.5. A Galton-Watson tree with offspring distribution $\xi$ is a random plane tree $\mathcal{T}$ with law $\mathbb{P}_{\xi}$ satisfying the following properties.
(i) $\mathbb{P}_{\xi}\left(k_{\emptyset}=j\right)=\xi(j)$ for all $j \in \mathbb{Z}^{\geq 0}$,
(ii) For every $j \geq 1$ with $\xi(j)>0$, the shifted trees $\theta_{1}(\mathcal{T}), \ldots, \theta_{j}(\mathcal{T})$ are independent under the conditional probability $\mathbb{P}_{\xi}\left(\cdot \mid k_{\emptyset}=j\right)$, with law $\mathbb{P}_{\xi}$, where $\theta_{i}(\mathcal{T})=\{v \in \mathcal{U}: i v \in \mathcal{T}\}$.

We say that $\mathcal{T}$ is critical if $\mathbb{E}[\xi]=1$. Additionally, we say a random plane tree is an alternating two-type Galton-Watson tree with offspring distribution $\left(\xi_{0}, \xi_{\bullet}\right)$ if
all vertices at even (respectively odd) height have offspring distribution $\xi_{\circ}$ (respectively $\left.\xi_{\bullet}\right)$. We say that the tree is critical if $\mathbb{E}\left[\xi_{0}\right] \mathbb{E}\left[\xi_{\bullet}\right]=1$.

### 3.1.1 Lévy trees

To define continuum random trees, we mimic the construction of discrete trees as quotient spaces given in the previous section. We begin with a definition of a random real tree, originally given in [DMT96], and then explain how they can be explicitly constructed by analogy with the discrete case.

Definition 3.1.6. A metric space $(\mathcal{T}, d)$ is a real tree if the following two properties hold for every $\sigma_{1}, \sigma_{2} \in \mathcal{T}$.
(i) There is a unique isometric map $f_{\sigma_{1}, \sigma_{2}}$ from $\left[0, d\left(\sigma_{1}, \sigma_{2}\right)\right]$ into $\mathcal{T}$ such that $f_{\sigma_{1}, \sigma_{2}}(0)=\sigma_{1}$ and $f_{\sigma_{1}, \sigma_{2}}\left(d\left(\sigma_{1}, \sigma_{2}\right)\right)=\sigma_{2}$.
(ii) If $q$ is a continuous injective map from $[0,1]$ into $\mathcal{T}$ such that $q(0)=\sigma_{1}$ and $q(1)=\sigma_{2}$, we have

$$
q([0,1])=f_{\sigma_{1}, \sigma_{2}}\left(\left[0, d\left(\sigma_{1}, \sigma_{2}\right)\right]\right)
$$

A rooted real tree additionally has a distinguished vertex $\rho$ called the root.
We define an equivalence relation on the set of real trees by saying that $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are equivalent if there exists a root-preserving isometry between them, and denote by $\mathbb{T}$ the space of isometry classes of compact rooted real trees. We endow $\mathbb{T}$ with the pointed Gromov-Hausdorff topology. Additionally, we let $\mathbb{T}_{\mathcal{M}}$ be the set of quadruples $(\mathcal{T}, d, \rho, \mu)$ with $(\mathcal{T}, d) \in \mathbb{T}, \rho$ its root and $\mu$ a probability measure on $\mathcal{T}$. It is shown in [EPW06] that $\mathbb{T}$ endowed with the metric $d_{G H}$ is complete and separable and their proof can be extended slightly to include measures as well. This is given in the theorem below, and a proof including measures was given in [Arc17].

Theorem 3.1.7. [EPW06, Theorem 1]. $\mathbb{T}_{\mathcal{M}}$ endowed with the metric $d_{G H P}$ is complete and separable.

The canonical example of a random real tree is the Brownian Continuum Random Tree (CRT), denoted by $\mathcal{T}_{e}$ and introduced by Aldous in [Ald91b]. It arises naturally as a scaling limit of discrete trees in that, if $\mathcal{T}_{n}$ is a discrete tree with critical offspring distribution $\xi$ with variance $\sigma^{2}$, then

$$
\frac{1}{\sigma \sqrt{n}} \mathcal{T}_{n} \xrightarrow{(d)} \mathcal{T}_{e}
$$

as $n \rightarrow \infty$. A simulation is shown in Figure 3.1.1. The CRT possesses many fractal and self-similarity properties but we do not go into detail here. An extensive account


Figure 3.3: Stable trees.
is given in the three Aldous papers [Ald91b], [Ald91c] and [Ald93], including several alternative constructions.

In this thesis, we will be most interested in stable Lévy trees which arise as scaling limits of discrete trees when the offspring distribution has infinite variance. This has the effect that in the limit, we get vertices (or "hubs") of infinite degree. This contrasts strongly with the CRT where all branch points are binary. Lévy trees were studied extensively by Duquesne and Le Gall in [DLG05], building on the work [LGLJ98] of Le Gall and Le Jan where the authors introduced Lévy trees to code the genealogy of continuous state branching processes. More specifically, again let $\mathcal{T}_{n}$ be a discrete tree conditioned to have $n$ vertices, but this time with critical offspring distribution $\xi$ in the domain of attraction of an $\alpha$-stable law, i.e. a law $\xi$ such that $\xi([k, \infty)) \sim c k^{-\alpha}$ as $k \rightarrow \infty$ for some $\alpha \in(1,2)$. We then have that

$$
\begin{equation*}
n^{-\left(1-\frac{1}{\alpha}\right)} \mathcal{T}_{n} \xrightarrow{(d)} c \cdot \mathcal{T}_{\alpha} \tag{3.3}
\end{equation*}
$$

in the Gromov-Hausdorff topology as $n \rightarrow \infty$, where $\mathcal{T}_{\alpha}$ is known as the stable tree of index $\alpha$, and $c$ is just a positive constant. A simulation of $\mathcal{T}_{\alpha}$ for $\alpha=1.5$ is also shown in Figure 3.1.1.

Just as in the previous section, continuum trees can be coded by functions that play the same role as the Lukasiewicz path for discrete trees. In the case of $\alpha$-stable trees, this path is a spectrally positive $\alpha$-stable Lévy excursion $X^{\text {exc }}$. The height function $H_{\alpha}$ can then be defined analogously to (3.2) by setting it to be the continuous modification of the process defined for $t \in[0,1]$ by

$$
\begin{equation*}
H_{\alpha}(t)=\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{0}^{t} \mathbb{1}\left\{X_{s}^{\mathrm{exc}}<I_{s}^{t}+\varepsilon\right\} d s, \tag{3.4}
\end{equation*}
$$

where $I_{s}^{t}=\inf _{s \leq r \leq t} X_{r}^{\text {exc }}$. This limit exists in probability - for more details see
[DLG02]. To define the stable tree, we first define a pseudodistance on $[0,1]$ by

$$
d_{\alpha}(s, t)=H_{\alpha}(s)+H_{\alpha}(t)-2 \inf _{s \leq r \leq t} H_{\alpha}(r)
$$

whenever $s \leq t$. Note the resemblance to (3.1). We then define an equivalence relation on $[0,1]$ by saying $s \sim_{\alpha} t$ if and only if $d_{\alpha}(s, t)=0$, and set $\mathcal{T}_{\alpha}$ to be the quotient space $\left([0,1] / \sim_{\alpha}, d_{\alpha}\right)$.

This construction also provides a natural way to define a measure on $\mathcal{T}_{\alpha}$ as the image of Lebesgue measure on $[0,1]$ under the quotient operation. The scaling relation (3.3) can then be extended to Gromov-Hausdorff-Prokhorov convergence by endowing the discrete trees with the renormalised degree measure.

We remark that this construction can also be mimicked more generally by taking a function $g$ (not identically 0 ) from $[0, \infty)$ to $[0, \infty)$ with $g(0)=0$ and of compact support. For every $s, t \geq 0$, we can define a distance function $d_{g}$ by

$$
d_{g}(s, t)=g(s)+g(t)-2 \inf _{s \wedge t \leq r \leq s \backslash t} g(r),
$$

and an equivalence relation $\sim$ on $[0, \infty)$ by setting $s \sim t$ if and only if $d_{g}(s, t)=0$. Then, letting $\zeta_{g}=\sup \{x \in[0, \infty): g(x)>0\}$, we set

$$
\mathcal{T}_{g}=\left[0, \zeta_{g}\right) / \sim
$$

Theorem 3.1.8. [DLG05, Theorem 2.1]. The metric space $\left(T_{g}, d_{g}\right)$ is a real tree.
As in the discrete case, properties of $\mathcal{T}_{\alpha}$ are encoded by the process $X^{\text {exc }}$. Letting $\pi:[0,1] \rightarrow \mathcal{T}$ be the canonical projection, we have a distinguished vertex $\rho=\pi(0)$ which is the root of $\mathcal{T}$. Also, for $u, v \in \mathcal{T}$, we denote by $[[u, v]]$ the unique geodesic between $u$ and $v$, and we say $u \preceq v$ if and only if $u \in[[\rho, v]]$. For given $u, v \in \mathcal{T}$, we say $z$ is the unique common ancestor of $u$ and $v$, written $z=u \wedge v$, if $z$ is the unique element in $\mathcal{T}$ with $[[\rho, u]] \cap[[\rho, v]]=[[\rho, z]]$.

The relation $\preceq$ on $\mathcal{T}$ can be recovered from $X^{\text {exc }}$ by defining

$$
s \preceq t \text { if and only if } s \leq t \text { and } X_{s^{-}}^{\mathrm{exc}} \leq I_{s}^{t} .
$$

This is the same as in Proposition 2.5.4. It can be shown that the two definitions of $\preceq$ given above define partial orders on $\mathcal{T}$ and $[0,1]$ respectively, and are compatible with $p$ in the sense that, if $u, v \in \mathcal{T}$, then $u \preceq v$ if and only if there exist $s, t \in[0,1]$ with $p(s)=u, p(t)=v$ and such that $s \preceq t$. Additionally, if we let $s \wedge t$ be the most recent common ancestor of $s$ and $t$ (with respect to $\preceq$ ), then it can be verified that $p(s \wedge t)=p(s) \wedge p(t)$.

As in the discrete case, the multiplicity of a vertex $u \in \mathcal{T}$ is defined as the
number of connected components of $\mathcal{T} \backslash\{u\}$. Vertex $u$ is called a leaf if it has multiplicity one, and a branch point if it has multiplicity at least three. It can then be shown that $u \in \mathcal{T}$ is a branch point if and only if there exists $s \in[0,1]$ such that $\pi(s)=u$ and $\Delta_{s}:=X_{s}^{\mathrm{exc}}-X_{s^{-}}^{\mathrm{exc}}>0 . \Delta_{s}$ gives some measure of the number of children of $u$. Finally, if $s, t, \in[0,1]$ with $s \preceq t$, the quantity

$$
x_{s}^{t}:=I_{s}^{t}-X_{s^{-}}^{\mathrm{exc}}
$$

can be regarded as the position of the ancestor of $p(t)$ amongst the " $\Delta_{s}$ " children of $p(s)$.

This intuition can be seen to be consistent with the discrete case by the following argument. Suppose $u$ is a vertex in a discrete tree with $k_{u}$ children, listed as $u_{1}, u_{2}, \ldots, u_{k_{u}}$, and suppose that $u$ is the $s^{t h}$ vertex of the tree when the vertices are written in lexicographical order. The Lukasiewicz path thus has a jump of size $k_{u}-1$ between times $s-1$ and $s$. Now suppose $u \preceq u_{j} \preceq v$ for some $j \in\left\{1,2, \ldots, k_{u}\right\}$ and that $v$ is the $t^{t h}$ vertex of the tree when the vertices are written in lexicographical order. Then the Lukasiewicz path $W$ must travel around the vertices of the subtrees $\tau_{u_{1}}(\mathcal{T}), \tau_{u_{2}}(\mathcal{T}), \ldots, \tau_{u_{j-1}}(\mathcal{T})$ before coming to the subtree $\tau_{u_{j}}(\mathcal{T})$ containing $v$. Each time it traverses a complete subtree, it finishes at a height one lower than when it starts, so the total number of such subtrees traversed before reaching $\tau_{u_{j}}(\mathcal{T})$ is equal to $W_{s}-\inf _{s \leq r \leq t} W_{r}$, and the total number of such subtrees yet to be traversed is equal to

$$
w_{s}^{t}:=\inf _{s \leq r \leq t}\left\{W_{r}-W_{r_{-}}\right\}
$$

Hence $w_{s}^{t}$ gives the number of siblings of $u_{j}$ "to the right" of $u_{j}$, and is precisely the discrete analogue of $x_{s}^{t}$.

This construction also provides a natural way to define a measure on $\mathcal{T}_{\alpha}$ as the image of Lebesgue measure on $[0,1]$ under the projection $\pi$.

Finally, we note that the law of the stable tree is characterised by the normalised Itô excursion measure $N_{(1)}$ for the $\alpha$-stable excursion that we introduced in Section 2.5.3.

### 3.2 Random looptrees

A large part of this thesis is concerned with stable looptrees, which can be informally thought of as the dual graphs of stable trees. They were first formally introduced by Curien and Kortchemski in [CK14], motivated by their appearance as scaling limits in various planar map percolation models and building in particular on the 2011 work [LGM11] of Le Gall and Miermont. We will shortly give their formal definition in terms of an $\alpha$-stable Lévy excursion, but we start by defining discrete
looptrees for better intuition.
Accordingly, let $\mathcal{T}$ be a discrete tree. The discrete looptree $\operatorname{Loop}(\mathcal{T})$ is constructed by replacing each vertex $u \in \mathcal{T}$ with a cycle of length equal to the degree of $u$ in $\mathcal{T}$, and then gluing these cycles at various points according to the tree structure of $\mathcal{T}$. This is best illustrated by an example, such as that in Figure 3.4.



Figure 3.4: Discrete tree and its associated looptree.
We will be focusing on the case of $\alpha$-stable looptrees. In [CK14, Theorem 4.1], Curien and Korthemski showed that if $\mathcal{T}_{n}$ is a critical Galton Watson tree conditioned to have $n$ vertices and with offspring distribution $\xi$ satisfying $\xi([k, \infty)) \sim$ $c k^{-\alpha}$ for some $\alpha \in(1,2)$, then we can define the stable looptree $\mathcal{L}_{\alpha}$ as the random compact metric space satisfying

$$
n^{\frac{-1}{\alpha}} \operatorname{Loop}\left(\mathcal{T}_{n}\right) \xrightarrow{(d)} C_{\alpha} \mathcal{L}_{\alpha}
$$

as $n \rightarrow \infty$, where $C_{\alpha}=(c|\Gamma(-\alpha)|)^{\frac{-1}{\alpha}}$. (Their result also allows for the inclusion of slowly-varying functions in the offspring distribution, but for sake of clarity we will mostly not include these in this thesis).

Recall from Section 3.1 that $\mathbb{T}^{\text {disc }}$ denotes the set of all plane trees. We set $\mathbb{L}^{\text {disc }}=\{\operatorname{Loop}(\mathcal{T}): \mathcal{T} \in \mathbb{T}\}$ to be the corresponding set of discrete looptrees.

### 3.2.1 Continuum looptrees

By comparison with the convergence for stable trees in (3.3), we would like to construct $\mathcal{L}_{\alpha}$ as the looptree version of the Lévy tree $\mathcal{T}_{\alpha}$. This gives the intuitive picture of stable looptrees: heuristically, we replace each branch point of $\mathcal{T}_{\alpha}$ by a loop with length proportional to the size of the branch point, and glue these loops along the tree structure of $\mathcal{T}_{\alpha}$. We explain below how $\mathcal{T}_{\alpha}$ and $\mathcal{L}_{\alpha}$ can be coded from the same Lévy excursion to reflect this intuition.

It was shown in [Mie05, Proposition 2] that if we define the width of a branch point at $t \in[0,1]$ by

$$
\lim _{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mu\left(\left\{v \in \mathcal{T}_{\alpha}, d(\pi(t), v) \leq \varepsilon\right\}\right),
$$

then the limit almost surely exists and is equal to $\Delta_{t}$. This is therefore the natural candidate for the length of the loop at the point $t$ in $\mathcal{L}_{\alpha}$.

In the construction of stable trees, recall that for a point $t \in[0,1]$, the ancestors of $t$ correspond to points $s<t$ such that $X_{s}^{\mathrm{exc}}=\inf _{s \leq r \leq t} X_{r}^{\mathrm{exc}}$. In [CK14, Section 2.3], the authors define a distance function between two points $s, t \in[0,1]$ with $s \preceq t$ by taking all points $s \preceq r \preceq t$ (which corresponds to points that are ancestors of $t$ but not of $s$ ), and summing the shortest distance across each of the corresponding loops. Since the loop corresponding to the point $r$ has length $\Delta_{r}$, this corresponds to setting:

$$
d(s, t)=\sum_{s \prec u \preceq t} \min \left(x_{u}^{t}, \Delta_{u}-x_{u}^{t}\right) .
$$

We now outline this formal construction given in [CK14]. We make the construction slightly more general by giving the definition for any excursion $f$ with only positive jumps, but the construction is identical to that of [CK14, Section 2.3].

Firstly, we define the set of excursions of lifetime $\zeta \in(0, \infty)$ with only positive jumps to be
$E_{\zeta}:=\left\{e \in D\left((0, \zeta), \mathbb{R}_{+}\right): e(0)=e(\zeta)=0, e(x)>0\right.$ and $\left.e(x)-e\left(x^{-}\right)>0 \forall x \in(0, \zeta)\right\}$.

We then define $E^{+}=\bigcup_{\zeta \in(0, \infty)} E_{\zeta}$ to be the set of finite excursions with only positive jumps.

Then take any $f \in E^{+}$, say with lifetime $\zeta_{f}$. The function $f$ plays the role of the Lukasiewicz path for the underlying tree structure. Recall that jumps represented branch points in the usual tree coding system. For each $t \in\left[0, \zeta_{f}\right]$, if $f$ has a jump at time $t$ let $\Delta_{t}$ denote the size of that jump, and otherwise let $\Delta_{t}=0$. For every $t \in\left[0, \zeta_{f}\right]$ with $\Delta_{t}>0$, we equip the segment $\left[0, \Delta_{t}\right]$ with the pseudodistance

$$
\begin{equation*}
\delta_{t}(a, b)=\min \left\{|a-b|\left(\Delta_{t}-|a-b|\right), \Delta_{t}\right\}, \quad \text { for } a, b \in\left[0, \Delta_{t}\right] \tag{3.5}
\end{equation*}
$$

The quantity $\delta_{t}$ corresponds to the distance associated with traversing the loop associated to the branch point at $t$.

In keeping with the notation for trees, for $s \leq t$ we set $I_{s}^{t}(f)=\inf _{r \in[s, t]} f_{r}$, and $x_{s}^{t}(f)=I_{s}^{t}(f)-f_{s^{-}}$. We use these quantities to define a pseudodistance $d$ on $[0,1]$ which will ultimately be used to define $\mathcal{L}_{f}$ as a quotient space. For $s, t \in\left[0, \zeta_{f}\right]$ we again write $s \prec t$ if $s \preceq t$ and $s \neq t$. Then, suppressing the notational dependence on $f$, if $s \preceq t$ set

$$
\begin{equation*}
d_{0}(s, t)=\sum_{s \prec u \preceq t} \delta_{u}\left(0, x_{u}^{t}\right) \tag{3.6}
\end{equation*}
$$

For general $s, t \in[0,1]$, set

$$
\begin{equation*}
d_{f}(s, t)=d_{s \wedge t}\left(x_{s \wedge t}^{s}, x_{s \wedge t}^{t}\right)+d_{0}(s \wedge t, s)+d_{0}(s \wedge t, t) . \tag{3.7}
\end{equation*}
$$

Heuristically, the second term represents the total distance along the interior of the path between the points corresponding to $s$ and $t$ in our looptree (and similarly for the third term), whilst the term $d_{s \wedge t}\left(x_{s \wedge t}^{s}, x_{s \wedge t}^{t}\right)$ represents the distance between the ancestors of $s$ and $t$ across the loop corresponding to $s \wedge t$. Note that the sum in (3.6) is countable since a càdlàg function can only have countably many jumps.

The proof of [CK14, Proposition 2.2] also applies in this general framework to show that $d_{f}$ is a continuous pseudodistance on $\left[0, \zeta_{f}\right]$. We can therefore define an equivalence relation $\sim_{f}$ on $[0, \zeta]$ by setting $s \sim_{f} t$ if $d_{f}(s, t)=0$. We define the continuum looptree associated with $f$ by

$$
\mathcal{L}_{f}=\left(\left[0, \zeta_{f}\right] / \sim_{f}, d_{f}\right) .
$$

We will use $p_{f}:\left[0, \zeta_{f}\right] \rightarrow \mathcal{L}_{f}$ to denote the canonical projection for our looptree $\mathcal{L}_{f}$, and we let $\nu_{f}$ be the projection of Lebesgue measure on $\left[0, \zeta_{f}\right]$ onto $\mathcal{L}_{f}$ via $p_{f}$.

Definition 3.2.1. (Stable looptree, of [CK14, Definition 2.3]). For $\alpha \in(1,2)$, the $\alpha$-stable looptree is the random looptree $\mathcal{L}_{X^{\text {exc }}}$, where $X^{\text {exc }}$ is an $\alpha$-stable, spectrally positive Lévy excursion conditioned to have length 1 . We denote it by $\mathcal{L}_{\alpha}$.

A simulation by Igor Kortchemski is shown in Figure 3.2.1.
We let $\mathbb{L}^{c}$ denote the space of continuum looptrees, i.e.

$$
\mathbb{L}^{c}=\left\{\mathcal{L}_{f}: f \in E^{+}\right\} .
$$

We also set $\mathbb{L}=\mathbb{L}^{\text {disc }} \cup \mathbb{L}^{c}$.
The proof of [CK14, Theorem 4.1] can be extended to this general framework to give the following (deterministic) result.

Proposition 3.2.2. (cf [CK14, Theorem 4.1]). Let $\left(\tau_{n}\right)_{n=1}^{\infty}$ be a sequence of trees with $\left|\tau_{n}\right|=n$ and corresponding Lukasiewicz paths $\left(W^{n}\right)_{n=1}^{\infty}$, and let $f$ be a function in $D^{\text {exc }}([0, \zeta], \mathbb{R})$ for some $\zeta \in(0, \infty)$. Additionally let $\nu_{n}$ be the uniform measure that gives mass 1 to each vertex of $\operatorname{Loop}\left(\tau_{n}\right)$. Suppose that $\left(C_{n}\right)_{n=1}^{\infty}$ is a sequence of positive real numbers such that
(i) $\left(\frac{1}{C_{n}} W_{\lfloor n\rfloor\rfloor}^{n}\left(\tau_{n}\right)\right)_{0 \leq t \leq 1} \rightarrow f$ in the Skorokhod- $J_{1}$ topology as $n \rightarrow \infty$,
(ii) $\frac{1}{C_{n}} \operatorname{Height}\left(\tau_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$.


Figure 3.5: Stable looptree with $\alpha=1.07$, by Igor Kortchemski.

Then

$$
d_{G H P}\left(\left(\operatorname{Loop}\left(\tau_{n}\right), \frac{1}{C_{n}} d_{n}, \frac{\zeta}{n} \nu_{n}, \rho_{n}\right),\left(\mathcal{L}_{f}, d_{f}, \nu_{f}, \rho_{f}\right)\right) \rightarrow 0
$$

as $n \rightarrow \infty$.
Proof. The result follows exactly as in the proof of [CK14, Theorem 4.1] by defining a correspondence $\mathcal{R}_{n}$ between $\mathcal{L}_{f}$ and $\mathcal{L}_{f_{n}}$ to consist of all pairs $\left(t, \lambda_{n}(t)\right)$, where $\lambda_{n}$ is the Skorokhod homeomorphism that minimises the Skorokhod distance between $f_{n}$ and $f$.

The following proof that the measures also converge appeared previously in [ $\operatorname{Arc} 17$, Proposition 5.3.5]. Let $F_{n}=\operatorname{Loop}\left(\tau_{n}\right) \sqcup \mathcal{L}_{f}$ endowed with the metric

$$
D_{F}(x, y)= \begin{cases}\frac{1}{C_{n}} d_{n}(x, y) & \text { if } x, y \in \operatorname{Loop}\left(\tau_{n}\right) \\ d_{f}(x, y) & \text { if } x, y \in \mathcal{L}_{f} \\ \inf _{u, v \in \mathcal{R}_{n}}\left(\frac{1}{C_{n}} d_{n}(x, u)+d(y, v)+\frac{1}{2} r_{n}\right) & \text { if } x \in \operatorname{Loop}\left(\tau_{n}\right), y \in \mathcal{L}_{f}\end{cases}
$$

where $r_{n}=\operatorname{dis}\left(\mathcal{R}_{n}\right)$.
We claim that $d_{P}^{F_{n}}\left(\nu_{n}, \nu_{f}\right) \rightarrow 0$ as $n \rightarrow \infty$. Recall that $\left|\tau_{n}\right|=n$, and let $I_{n, i}=\left[\frac{i}{n}-\frac{1}{2 n}, \frac{i}{n}+\frac{1}{2 n}\right]$. Take a set $A_{n}$ of vertices in $L_{n}$, and let

$$
A_{n}^{\prime}=\cup_{u_{i} \in A_{n}} I_{n, i} .
$$

Let $A^{\prime}=p\left(A_{n}^{\prime}\right)$. We will show that $A^{\prime} \subset A_{n}^{r_{n}}$. For any $v \in A^{\prime}, \exists s \in A_{n}^{\prime}$ with $v=\pi(s)$ and $s \in I_{n, i}$ for some $u_{i} \in A_{n}$. It follows that $i=\lfloor n s\rfloor$ or $\lceil n s\rceil$, and hence $\left(u_{i}, v\right) \in \mathcal{R}_{n}$, so $D_{F}\left(u_{i}, v\right)=\frac{1}{2} r_{n}$ and $A^{\prime} \subset A_{n}^{r_{n}}$.

Also note that $\nu_{n}\left(A_{n}\right)=\nu\left(A^{\prime}\right)$ by construction, and so $\nu_{n}\left(A_{n}\right) \leq \nu\left(A_{n}^{r_{n}}\right)$.
Similarly, take any set $B \subset \mathcal{L}_{f}$. We use the same argument to show that $\nu(B) \leq \nu_{n}\left(B^{r_{n}}\right)$. Let $B^{\prime}=p^{-1}(B)$, and

$$
B_{n}=\left\{u_{i} \in L_{n}: \exists s \in B^{\prime} \text { with } s \in I_{n, i}\right\} .
$$

Clearly $B^{\prime} \subset \cup_{u_{i} \in B_{n}} I_{n, i}$ and so

$$
\nu(B)=\operatorname{Leb}\left(B^{\prime}\right) \leq \frac{\left|B_{n}\right|}{n}=\nu_{n}\left(B_{n}\right) .
$$

If $u_{i} \in B_{n}$, then there exists $s \in B^{\prime}$ with $s \in I_{n, i}$ and so $\left(u_{i}, \pi(s)\right) \in \mathcal{R}_{n}$. Hence $B_{n} \subset B^{r_{n}}$, so $\nu_{n}\left(B_{n}\right) \leq \nu_{n}\left(B^{r_{n}}\right)$ and $\nu(B) \leq \nu_{n}\left(B^{r_{n}}\right)$.

It follows that $d_{P}^{F_{n}}\left(\nu_{n}, \nu\right) \leq r_{n} \rightarrow 0$ as $n \rightarrow \infty$.
The second condition that $\frac{1}{C_{n}} \operatorname{Height}\left(\tau_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$ is important because it ensures that in the limit, distances in the rescaled discrete looptrees come from the loop structure and not from the height of the corresponding tree. More formally, in the proof of the theorem it is used to make a comparison between the expressions $\frac{1}{C_{n}} \sum_{u_{n} \preceq v_{n}} x_{u_{n}}^{v_{n}}$ and $\sum_{u \preceq v} x_{u}^{v}$ for the discrete and continuum trees respectively. For any $v_{n} \in \operatorname{Loop}\left(\tau_{n}\right)$ and $v \in \mathcal{L}_{f}$ we have

$$
\begin{align*}
\sum_{u_{n} \preceq v_{n}} x_{u_{n}}^{v_{n}} & =\operatorname{Height}\left(v_{n}\right)+W^{n}\left(v_{n}\right), \\
\sum_{u \preceq v} x_{u}^{v} & =f(v) . \tag{3.8}
\end{align*}
$$

Then if $v$ and $v_{n}$ are in correspondence with each other, after being careful with left and right limits we can essentially apply the result that $\frac{1}{C_{n}} W^{n}\left(v_{n}\right) \rightarrow f(v)$ to deduce that the $\frac{1}{C_{n}} \sum_{u_{n} \preceq v_{n}} x_{u_{n}}^{v_{n}}$ also converges to $\sum_{u \preceq v} x_{u}^{v}$ in the limit. To obtain this result, it is therefore crucial that the contribution from the height function goes to zero.

If, however, we replace the sequence of rescaled discrete looptrees with a sequence of continuum looptrees, say coded by the functions $\left(f_{n}\right)_{n=1}^{\infty}$ each with support $[0,1]$ and such that $f_{n} \rightarrow f$ in the Skorokhod topology as $n \rightarrow \infty$, then the height function won't appear in any of the new terms in (3.8) and so the continuum analogue of condition (ii) of Theorem 3.2.2 is not required for convergence of the corresponding looptrees.

In this sense, condition (ii) reflects the fact the looptree $\operatorname{Loop}\left(\tau_{n}\right)$ isn't quite
the same as the looptree $\mathcal{L}_{W^{n}}$. Condition (ii) is precisely what is required to say that the difference between $\operatorname{Loop}\left(\tau_{n}\right)$ and $\mathcal{L}_{W^{n}}$ becomes negligible in the limit.

Hence, in the continuum, the same proof gives the following result.
Proposition 3.2.3. Let $\left(f_{n}\right)_{n \geq 1}$ be a sequence in $E^{+}$, and $f \in E^{+}$be such that $f_{n} \rightarrow f$ as $n \rightarrow \infty$ with respect to the Skorokhod- $J_{1}$ topology. Additionally let $\nu$ and $\nu_{n}$ be the appropriate projections of Lebesgue measure onto the spaces $\mathcal{L}_{f}$ and $\mathcal{L}_{f_{n}}$ respectively. Then

$$
d_{G H P}\left(\left(\operatorname{Loop}\left(\tau_{n}\right), d_{n}, \nu_{n}, \rho_{n}\right),\left(\mathcal{L}_{f}, d_{f}, \nu_{f}, \rho_{f}\right)\right) \rightarrow 0
$$

as $n \rightarrow \infty$.
Again, the result follows exactly as in the proof of [CK14, Theorem 4.1] by defining a correspondence between $\mathcal{L}_{f}$ and $\mathcal{L}_{f_{n}}$ to consist of all pairs $\left(t, \lambda_{n}(t)\right)$, where $\lambda_{n}$ is the Skorokhod homeomorphism that minimises the Skorokhod distance between $f_{n}$ and $f$.

### 3.2.2 Re-rooting invariance for stable trees (and looptrees)

In [DLG05], Duquesne and Le Gall prove that stable Lévy trees are invariant under uniform rerooting. More formally, if $U$ is a uniform point in $[0,1]$, and we define a new height function $H^{[U]}:[0,1] \rightarrow \mathbb{R}$ from the original height function $H$ by

$$
H^{[U]}(x)= \begin{cases}H(U)+H(U+x)-2 \min _{U \leq s \leq U+x} H(s) & \text { if } U+x \leq 1 \\ H(U)+H(U+x-1)-2 \min _{U+x-1 \leq s \leq U} H(s) & \text { if } U+x>1\end{cases}
$$

then $H^{[U]} \stackrel{(d)}{=} H$. This property is just saying that if we pick a uniform point $U \in[0,1]$, and reroot the tree $\mathcal{T}_{\alpha}$ at $\pi(U)$, then the resulting tree has the same distribution as the original one.

A substantial part of Chapter 4 will be devoted to proving precise volume bounds for stable looptrees. We will prove most of these for the volume of a ball at a uniform point in $\mathcal{L}_{\alpha}$, and then extend to almost all of $\mathcal{L}_{\alpha}$ by Fubini's theorem. In the proof, we will use a couple of decompositions for stable trees based on defining a "spine" from this uniform point to another point in the tree; the rerooting invariance result means that we can equivalently consider our uniform point to be the root, when convenient.

Note that the problem of uniform rerooting invariance of continuum fragmentation trees was also considered in the paper [HPW09], where the authors additionally show that stable trees are the only fragmentation trees for which this property holds. Duquesne and Le Gall also prove a similar result for rerooting at
an (independent) deterministic point $u \in[0,1]$ in the paper [DLG09]. Moreover, Curien and Kortchemski consider a similar property for stable looptrees in [CK14, Remark 4.6].

### 3.2.3 Notions of height

At some points in this thesis, we will refer to the "corresponding" or "underlying" stable tree of $\mathcal{L}_{\alpha}$, by which we mean the stable tree $\mathcal{T}_{\alpha}$ coded by the same excursion that codes $\mathcal{L}_{\alpha}$. We let $\mathcal{L}_{\alpha}$ denote a compact stable looptree conditioned on $\nu\left(\mathcal{L}_{\alpha}\right)=$ 1 , but at various points we will let $\tilde{\mathcal{L}_{\alpha}}$ denote a generic stable looptree coded by an excursion under the Itô measure but without any conditioning on its total mass. We will also let $\mathcal{L}_{\alpha}^{1}$ denote a stable looptree but conditioned so that its underlying tree has height 1. However, we will make this notation explicit at the time of writing.

The height of a stable tree $\tilde{\mathcal{T}}_{\alpha}$ is defined as $H_{\max }=\sup _{u \in \tilde{\mathcal{T}}_{\alpha}} d_{\tilde{\mathcal{T}}_{\alpha}}(\rho, u)$. As the height process is almost surely continuous, this maximum is almost surely realised by at least one $u \in \tilde{\mathcal{T}}_{\alpha}$. Moreover, we see from [DW17, Equation (23)] (and references therein) that there is almost surely a unique $u \in \tilde{\mathcal{T}}_{\alpha}$ that attains this maximum, which we denote by $u_{H}$. If $\tilde{\mathcal{L}_{\alpha}}$ is the corresponding stable looptree, we define three notions of its height:
(i) We define its $L^{W}$-Height to be the looptree distance from $\rho$ to $u_{H}$,
(ii) We define its $L$-Height to be $\sup _{u \in \tilde{\mathcal{L}_{\alpha}}} d_{\tilde{\mathcal{L}_{\alpha}}}(\rho, u)$.
(iii) We define its $L^{m}$-Height to be $\max \tilde{X}_{s}^{\text {exc }}$, where $\tilde{X}^{\text {exc }}$ is the Lévy excursion coding $\tilde{\mathcal{L}_{\alpha}}$.

In general, these are not the same. Note however that the $L^{m}$-Height is at least as big as the $L$-Height, since $\tilde{X}_{s}^{\text {exc }}$ gives the distance to the point in $\tilde{\mathcal{L}}_{\alpha}$ represented by $s$ but going "clockwise" around all loops. At times, we will also use the notation $T^{W}$-Height and $T^{m}$-Height to denote the length of the corresponding spine in the underlying tree, which we respectively denote by W-spine or m-spine.

### 3.3 Infinite critical trees and looptrees

We now introduce a construction of infinite critical trees and looptrees, which arise naturally in the study of infinite critical percolation clusters. In the case where $\xi$ is a supercritical offspring distribution, it is easy to define an infinite tree $T_{\infty}$ with offspring distribution $\xi$ simply by defining $T$ to be a standard Galton-Watson tree with offspring distribution $\xi$ (as in Definition 3.1.2) and conditioning on $|T|=\infty$.

In the case where $\xi$ is critical (or subcritical), this conditioning does not really make sense since $\{|T|=\infty\}$ is a null event. However, motivated by the
study of critical percolation clusters on the planar lattice, Kesten in [Kes86b] made a sensible definition of an infinite critical Galton-Watson tree $T_{\infty}^{c r i t}$ and showed that it arises as a local limit as $n \rightarrow \infty$ of critical Galton-Watson trees conditioned to have height at least $n$. Informally, $T_{\infty}^{c r i t}$ consists of an infinite spine (or backbone) of vertices which all have a size-biased version of $\xi$ as their offspring distribution. All other vertices have children according to $\xi$.

Definition 3.3.1. [AD15, Definition 1.3.1]. Let $\xi$ be a critical offspring distribution, and define its size biased version $\xi^{*}$ by

$$
\xi^{*}(n)=n \xi(n) .
$$

The Kesten's tree $T_{\infty}^{c r i t}$ associated to the probability distribution $\xi$ is a two-type Galton-Watson tree distributed as follows:

- Individuals are either normal or special.
- The root of $T_{\infty}^{c r i t}$ is special.
- A normal individual produces only normal individuals according to $\xi$.
- A special individual produces individuals according to the size-biased distribution $\xi^{*}$. Of these, one of them is chosen uniformly at random to be special, and the rest are normal.

Almost surely, the special vertices form a unique infinite backbone of $T_{\infty}$. Note that this is one-ended. Aldous in [Ald91a] coined the term sin-trees for such trees, since they have a single infinite spine.

The reason for taking a size-biased distribution is because this arises naturally on conditioning the height of a finite Galton-Watson tree to be large, e.g. see [GK99, Remark, p.5]. In fact, we have the following local limit theorem. This was originally proved by Kesten in [Kes86b] under a second moment condition, but was proved with the given condition in [Jan12, Theorem 7.1]. In fact the result there is stated for a different topology, but the Gromov-Hausdorff convergence follows as a consequence.

Theorem 3.3.2. ([Kes86b], [AD15, Theorem 2.1.1], [Jan12, Theorem 7.1]). Let $\xi$ be a critical offspring distribution with $\xi(0)+\xi(1)<1$ and define $T_{\infty}$ as in Definition 3.3.1. Let $T_{n}$ be a Galton-Watson tree with offspring distribution $\xi$ conditioned on having height at least $n$. Then

$$
T_{n} \xrightarrow{(d)} T_{\infty}^{c r i t}
$$

with respect to the Gromov-Hausdorff-vague topology as $n \rightarrow \infty$.
Remark 3.3.3. We can take a similar local limit in the subcritical case but in this case the limiting tree will almost surely have a finite spine, ending with a vertex of infinite degree. See [Jan12] for more on this case.

Kesten originally made this definition in [Kes86b] to make a comparison to the incipient infinite cluster (IIC) of critical percolation on the planar lattice. We will see that similarly constructed infinite discrete looptrees have similar connections to the IIC on random planar triangulations.

Kesten's construction has been imitated in the continuum by Duquesne in [Duq09], who constructs continuum sin-trees and shows that these arise as the appropriate local limit of compact continuum trees conditioned on being large. By analogy with the compact continuum case, Duquesne's construction involves defining two height functions from two independent Lévy processes in the same way as done with the excursion in (3.4). These respectively code the tree structure on the left and right sides of the spine in the usual way.

The construction was further extended to infinite discrete looptrees in [BS15], where the authors define the infinite looptree associated with a critical offspring distribution $\xi$ to simply be $\operatorname{Loop}\left(T_{\infty}^{\text {crit }}\right)$, where $T_{\infty}^{\text {crit }}$ is constructed as in Definition 3.3.1. This infinite looptree thus inherits the structure of having a loopspine with loop sizes determined by a size-biased version of $\xi$, to which usual compact discrete looptrees are grafted. The local limit theorem of Theorem 3.3.2 thus passes directly to the looptree case by continuity of the Loop operation (see [BS15, Corollary 2.3]).

Finally, Kesten's construction of Definition 3.3.1 was extended to critical multi-type Galton Watson trees in [Ste18, Theorem 3.1] satisfying an analogous local limit theorem. Richier in [Ric18a] then used this to define an infinite twotype looptree and showed in [Ric18b] this also arises as a similar local limit under appropriate conditions.

The concept of an infinite stable looptree has thus left a gap in the literature which is now filled by the construction given in Chapter 5 . This extends the construction of infinite discrete looptrees in the same way that Duquesne's continuum sin-trees extend the construction of their discrete counterpart. The resulting local limit theorem allows us to prove various volume convergence results for compact stable looptrees in Chapter 4.

## Chapter 4

## Brownian Motion on Compact Stable Looptrees

The main purpose of this chapter is to study Brownian motion on stable looptrees, and in particular to get precise bounds on its heat kernel. This is useful to gain initial insights into the more general decorated tree model considered in Chapter 6 , but looptrees are also an interesting fractal model in their own right: they have close connections to many random planar map models, such as causal maps and stable shredded spheres [BCS19], as well as describing the boundary structure of many statistical mechanics models.

Our analysis of the Brownian motion is achieved by defining a resistance metric on stable looptrees as outlined in Section 4.1.1. The key ingredient in the proofs of the heat kernel bounds will be to prove precise volume bounds for looptrees with respect to this resistance metric and indeed the bulk of the chapter is concerned with proving Theorems 4.0.5 and 4.0.4. These are also interesting results in their own right as they give insight into the fractal properties of stable looptrees, such as behaviour of Hausdorff and packing measures. There are two main approaches used to prove the volume results: one uses self-similarity properties of looptrees obtained from spinal decomposition results, and the other uses fluctuation results for Lévy excursions and map these over to stable looptrees using their construction in terms of a stable Lévy process excursion.

The main results of this chapter are as follows. We assume that $\alpha \in(1,2)$ throughout. Firstly, for $x \in \mathcal{L}_{\alpha}$, let $B(x, r)$ denote the open ball of radius $r$ around the point $x$. The volume results in this chapter will be true regardless of whether this is defined using the shortest distance metric introduced in Section 3.2, or using the effective resistance metric which we define shortly in Section 4.1.1. Recall that $\nu$ can be thought of as a uniform probability measure on $\mathcal{L}_{\alpha}$.

We will use the bold font $\mathbf{P}$ to denote the law of $\mathcal{L}_{\alpha}$ on $\Omega$, and $\mathbf{E}$ the
corresponding expectation.
We first recall the following random walk scaling limit result that was proved in [Arc17]. Throughout, we fix some critical offspring distribution $\xi$ in the domain of attraction of an $\alpha$-stable law, and let $a_{n}$ be such that

$$
\frac{\sum_{i=1}^{n} \xi^{(i)}-n}{a_{n}} \stackrel{(d)}{\rightarrow} Z_{\alpha}
$$

as $n \rightarrow \infty$, where $\left(\xi^{(i)}\right)_{i=1}^{\infty}$ are independent copies of $\xi$, and $Z_{\alpha}$ is an $\alpha$-stable random variable. Since $\alpha \in(1,2)$, this entails that $\xi((n, \infty])=n^{-\alpha} L(n)$ and that $a_{n}=n^{\frac{1}{\alpha}} \tilde{L}(n)$, for some (related) slowly-varying functions $L$ and $\tilde{L}$. We let $T_{n}$ denote a discrete Galton-Watson tree with offspring distribution $\xi$, conditioned to have $n$ vertices.

Theorem 4.0.1. [Arc17, Theorem 5.4.1], [Arc19, Theorem 1.1]. Let $T_{n}$ be as above, let $Z^{(n)}$ denote a discrete-time simple random walk on $\operatorname{Loop}\left(T_{n}\right)$, and let $\left(B_{t}\right)_{t \geq 0}$ denote Brownian motion on $\mathcal{L}_{\alpha}$. There exists a probability space $\left(\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbf{P}^{\prime}\right)$ on which we can (pointwise) define isometric embeddings of $\left(a_{n}^{-1} \operatorname{Loop}\left(T_{n}\right)\right)_{n \geq 1}$ and $\mathcal{L}_{\alpha}$ into a common metric space $\left(M, d_{M}\right)$ so that

$$
a_{n}^{-1} \operatorname{Loop}\left(T_{n}\right) \rightarrow \mathcal{L}_{\alpha}
$$

almost surely with respect to the Hausdorff metric. In this metric space, we also have that

$$
\left(a_{n}^{-1} Z_{\left\lfloor 4 n a_{n} t\right\rfloor}^{(n)}\right)_{t \geq 0} \xrightarrow{(d)}\left(B_{t}\right)_{t \geq 0}
$$

as $n \rightarrow \infty$, by which we mean that, almost surely on $\left(\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbf{P}^{\prime}\right)$, the laws of these processes converge weakly on the space $D([0, \infty), M)$ endowed with the uniform topology.

Theorem 4.0.2. There exists $C \in(0, \infty)$ such that $r^{-\alpha} \mathbf{E}[\nu(B(\rho, r))] \rightarrow C$ as $r \downarrow 0$.
This is actually just a corollary of the following convergence result:
Theorem 4.0.3. There exists a random variable $V: \Omega \rightarrow(0, \infty)$ such that

$$
r^{-\alpha} \nu(B(\rho, r)) \xrightarrow{(d)} V
$$

as $r \downarrow 0$. Moreover, for any $p \in[1, \infty), r^{-\alpha p} \mathbf{E}\left[\nu(B(\rho, r))^{p}\right] \rightarrow \mathbf{E}\left[V^{p}\right]$ as $r \downarrow 0$.
Here $V$ denotes the volume of a unit ball in an infinite stable looptree, as will be introduced later in Chapter 5.

We start with the following global (uniform) volume bounds for small balls in $\mathcal{L}_{\alpha}$, which demonstrate both upper and lower fluctuations of logarithmic order.

Theorem 4.0.4. $\mathbf{P}$-almost surely, there exist constants $C_{1}, C_{2} \in(0, \infty)$ such that for all $r \in\left(0, \operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)\right)$ :

$$
\begin{align*}
\inf _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) & \geq C_{1} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha}  \tag{4.1}\\
\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) & \leq C_{2} r^{\alpha}\left(\log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}  \tag{4.2}\\
\limsup _{r \downarrow 0}\left(\frac{\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r))}{r^{\alpha} \log r^{-1}}\right) & >0 .  \tag{4.3}\\
\liminf _{r \downarrow 0}\left(\frac{\inf _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r))}{r^{\alpha}\left(\log r^{-1}\right)^{-(\alpha-1)}}\right) & <\infty . \tag{4.4}
\end{align*}
$$

We also have the following local (pointwise) results.
Theorem 4.0.5. $\mathbf{P}$-almost surely, for $\nu$-almost every $u \in \mathcal{L}_{\alpha}$ we have:

$$
\begin{align*}
\liminf _{r \downarrow 0}\left(\frac{\nu(B(u, r))}{r^{\alpha}\left(\log \log r^{-1}\right)^{-\alpha}}\right) & >0,  \tag{4.5}\\
\underset{r \downarrow 0}{\limsup }\left(\frac{\nu(B(u, r))}{r^{\alpha}\left(\log \log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}}\right) & <\infty,  \tag{4.6}\\
\limsup _{r \downarrow 0}\left(\frac{\nu(B(u, r))}{r^{\alpha} \log \log r^{-1}}\right) & >0  \tag{4.7}\\
\liminf _{r \downarrow 0}\left(\frac{\nu(B(u, r))}{r^{\alpha}\left(\log \log r^{-1}\right)^{-(\alpha-1)}}\right) & <\infty . \tag{4.8}
\end{align*}
$$

By applying results of [Cro07], we are also able to deduce the following heat kernel bounds for Brownian motion on $\mathcal{L}_{\alpha}$. We start by giving the quenched results.

Theorem 4.0.6. Almost surely, there exists $t_{0} \in(0, \infty)$ such that

$$
c t^{\frac{-\alpha}{\alpha+1}}\left(\log t^{-1}\right)^{-(3+2 \alpha)(2+\alpha)\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)} \leq p_{t}(x, x) \leq C t^{\frac{-\alpha}{\alpha+1}}\left(\log t^{-1}\right)^{\alpha}
$$

for all $x \in \mathcal{L}_{\alpha}$ and all $t \in\left(0, t_{0}\right)$. Moreover, it holds almost surely that

$$
\begin{aligned}
& \liminf _{t \downarrow 0} \frac{\inf _{x \in \mathcal{L}_{\alpha}} p_{t}(x, x)}{t^{\frac{-\alpha}{\alpha+1}}\left(\log t^{-1}\right)^{-1}}<\infty, \\
\limsup _{t \downarrow 0} & \frac{\sup _{x \in \mathcal{L}_{\alpha}} p_{t}(x, x)}{t^{\frac{-\alpha}{\alpha+1}}\left(\log t^{-1}\right)^{\alpha-1}}>0 .
\end{aligned}
$$

We can also use the local volume bounds of Theorem 4.0.5 to deduce pointwise heat kernel estimates. Note however that one of the lower bounds in Theorem 4.0.7 is missing. Heat kernel lower bounds are generally more subtle to obtain than upper bounds, and in particular in this case we need some global volume control to
apply the chaining arguments of [Cro07] that are used to prove the corresponding global bound in Theorem 4.0.6.

Theorem 4.0.7. Almost surely, for any $\varepsilon>0$ we have for $\nu$-almost every $x \in \mathcal{L}_{\alpha}$ that

$$
\begin{array}{r}
\liminf _{t \downarrow 0} \frac{p_{t}(x, x)}{t^{\frac{-\alpha}{\alpha+1}}\left(\log \log t^{-1}\right)^{\frac{-1}{\alpha+1}}}<\infty \\
\quad \limsup _{t \downarrow 0} \frac{p_{t}(x, x)}{t^{\frac{-\alpha}{\alpha+1}}\left(\log \log t^{-1}\right)^{\frac{\alpha}{\alpha+1}}}<\infty \\
\limsup \\
\sup _{t \downarrow 0} \frac{p_{t}(x, x)}{t^{\frac{-\alpha}{\alpha+1}}\left(\log \log t^{-1}\right)^{\frac{\alpha-1-\varepsilon}{\alpha+1}}}>0
\end{array}
$$

We can similarly apply the results to get off diagonal heat kernel bounds. The constants $\theta_{1}, \theta_{2}$ and $\theta_{3}$ are deterministic and we will give their values in Section 4.4.

Theorem 4.0.8. Almost surely, there exists $t_{0}^{\prime} \in(0, \infty)$ such that for all $x, y \in \mathcal{L}_{\alpha}$ and all $t \in\left(0, t_{0}^{\prime}\right)$, we have

$$
\begin{aligned}
& p_{t}(x, y) \leq C t^{\frac{-\alpha}{\alpha+1}}\left(\log t^{-1}\right)^{\alpha} \exp \left\{-\tilde{c} \tilde{d}^{1+\frac{1}{\alpha}} t^{\frac{-1}{\alpha}}\left(\log t^{-1} \tilde{d}\right)^{-\theta_{3}\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)}\right\} \\
& p_{t}(x, y) \geq c t^{\frac{-\alpha}{\alpha+1}}\left(\log t^{-1}\right)^{-\theta_{1}\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)} \exp \left\{\tilde{c}^{\prime} \tilde{d}^{1+\frac{1}{\alpha}} t^{\frac{-1}{\alpha}}\left(\log t^{-1} \tilde{d}\right)^{\theta_{2}\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)}\right\}
\end{aligned}
$$

for all $x, y \in \mathcal{L}_{\alpha}$ and all $t \in\left(0, t_{0}^{\prime}\right)$. Here $\tilde{d}=\tilde{d}(x, y)$ can denote the distance between $x$ and $y$ with respect to either the shortest distance metric on $\mathcal{L}_{\alpha}$, or the effective resistance metric, since the two are equivalent.

A key step in these heat kernel estimates are bounds on the expected exit times from balls, for which we can show the following. Here $\tau_{A}=\inf \left\{t \geq 0: B_{t} \notin A\right\}$ for any $A \subset \mathcal{L}_{\alpha}$.

## Proposition 4.0.9.

$$
\begin{aligned}
& \mathbf{E}_{x}\left[\tau_{B(x, r)}\right] \geq c r^{\alpha+1}\left(\log r^{-1}\right)^{-2\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)(\alpha+1)}\left(\log \left(r^{-1}\left(\log r^{-1}\right)^{2\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)}\right)\right)^{-\alpha} \\
& \mathbf{E}_{x}\left[\tau_{B(x, r)}\right] \leq C r^{\alpha+1}\left(\log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}
\end{aligned}
$$

We also give an annealed result for the transition density at the root, averaged over the law of $\mathcal{L}_{\alpha}$.

Theorem 4.0.10. There exists $C^{\prime} \in(0, \infty)$ such that

$$
t^{\frac{\alpha}{\alpha+1}} \mathbf{E}\left[p_{t}(\rho, \rho)\right] \rightarrow C^{\prime}
$$

as $t \downarrow 0$.

In light of these results, it also natural to investigate the associated eigenvalue counting function of the Laplacian $\Delta$ associated with $\left(B_{t}\right)_{t \geq 0}$. More precisely, let $R$ be the effective resistance metric on $\mathcal{L}_{\alpha}$ (we will construct this properly in Section 4.1.1), and let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with the space $L^{2}\left(\mathcal{L}_{\alpha}, \nu\right)$ through the relation

$$
R(x, y)^{-1}=\inf \{\mathcal{E}(f, f): f \in \mathcal{F}, f(x)=0, f(y)=1\}
$$

(this is $\mathbf{P}$-almost surely well-defined: see [CH10, Section 1] for more details on the construction for stable trees; the same principles apply for stable looptrees). We say that $\lambda>0$ is an eigenvalue of $(\mathcal{E}, \mathcal{F}, \nu)$ with eigenfunction $f$ (assumed to be non-trivial) if

$$
\mathcal{E}(f, g)=\lambda \int_{\mathcal{L}_{\alpha}} f g d \nu
$$

for all $g \in \mathcal{F}$. The eigenvalue counting function $N(\lambda)$ is then defined as the number of eigenvalues of $(\mathcal{E}, \mathcal{F}, \nu)$ that are less than or equal to $\lambda$. Due to the representation $\mathcal{E}(f, g)=-\int_{\mathcal{L}_{\alpha}}(\Delta f) g d \nu$, any eigenvalue of the operator $\Delta$ is also an eigenvalue of $(\mathcal{E}, \mathcal{F}, \nu)$. Since $\mathcal{L}_{\alpha}$ is compact and $(\mathcal{E}, \mathcal{F})$ is consequently regular, the converse also holds. Similar arguments to [CH10] then lead to the following result.

Theorem 4.0.11. (i) For any $\varepsilon>0$, as $\lambda \rightarrow \infty$,

$$
\mathbf{E}[N(\lambda)] \sim C \lambda^{\frac{\alpha}{\alpha+1}}+O\left(\lambda^{\frac{1}{\alpha+1}+\varepsilon}\right)
$$

(ii) $\mathbf{P}$-almost surely, $N(\lambda) \sim C \lambda^{\frac{\alpha}{\alpha+1}}$ as $\lambda \rightarrow \infty$. More over, in $\mathbf{P}$-probability, the second order estimate of part (i) holds.

We start by explaining how resistance can be used to define Brownian motion on stable looptrees, and use the theory to prove that it arises as an appropriate scaling limit of random walks on discrete looptrees. We also discuss convergence of some associated quantities, including transition densities, mixing times and blanket times of these random walks. We then move on and prove the volume and heat kernel bounds outlined above, and conclude by discussing the spectral asymptotics in Section 4.5.

### 4.1 Scaling limits for random walks and associated quantities

Looptrees fall nicely in to the framework of Section 2.4. The main purpose of the present section is to apply the results outlined there to firstly define Brownian motion on stable looptrees, and then prove various scaling limit results about it.

### 4.1.1 Construction of a resistance metric on stable looptrees

To define Brownian motion on stable looptrees, we first define a resistance metric on them. This is similar in spirit to the metric constructed by Curien and Kortchemski that we introduced in Section 3.2, but we will sum the effective resistance across loops rather than the shortest-path distance. It turns out that these resistance looptrees are in fact homeomorphic to those of Curien and Kortchemski, which means that the shortest distance metric can equivalently be used to prove the precise volume bounds of Theorems 4.0.4 and 4.0.5, making the problem far more tractable. Additionally this means that the invariance principle of Theorem 4.1.6 arises as a direct consequence of [CK14, Theorem 4.1] and Theorem 3.2.2. However, the advantage of writing the convergence with respect to the resistance metric means that we get similar random walk convergence results as a direct consequence.

To define a resistance looptree for a discrete looptree $\operatorname{Loop}(\mathcal{T})$, we view $\operatorname{Loop}(\mathcal{T})$ as an electrical network and equip it with the resulting effective resistance metric. Each edge of each loop has unit length and the distance between two points $x$ and $y$ in $\operatorname{Loop}(\mathcal{T})$ is defined to be the effective resistance between them (calculated using the series and parallel laws).

In the continuum, again let $f \in E^{+}$, with lifetime $\zeta_{f}$. This time, if $f$ has a jump of size $\Delta_{t}>0$ at point $t$, equip the segment $\left[0, \Delta_{t}\right]$ with the pseudodistance

$$
\begin{equation*}
r_{t}(a, b)=\left(\frac{1}{|a-b|}+\frac{1}{\Delta_{t}-|a-b|}\right)^{-1}=\frac{|a-b|\left(\Delta_{t}-|a-b|\right)}{\Delta_{t}} \tag{4.9}
\end{equation*}
$$

for $a, b \in\left[0, \Delta_{t}\right]$. The quantity $r_{t}$ corresponds to the resistance across the loop associated to the branch point at $t$. Note that $r_{t}(a, b)$ corresponds to the effective resistance of two parallel edges of resistance $|a-b|$ and $\Delta_{t}-|a-b|$, and by Rayleigh's Monotonicity Principle it follows that $r_{t}(a, b) \leq \min \left\{|a-b|, \Delta_{t}-|a-b|\right\}$ (this is also shown algebraically in Lemma 4.1.1).

Now recall that for $s \leq t$ we set $I_{s}^{t}(f)=\inf _{r \in[s, t]} f_{r}$, and $x_{s}^{t}(f)=I_{s}^{t}(f)-f_{s^{-}}$. We use these quantities to define a pseudodistance $R$ on $[0,1]$ which will ultimately be used to define our resistance looptree as a quotient space. Note in particular the similarity to expression (3.5) in Section 3.2. For $s, t \in[0,1]$ we again write $s \prec t$ if $s \preceq t$ and $s \neq t$. Then, if $s \preceq t$ set

$$
\begin{equation*}
R_{0}(s, t)=\sum_{s \prec u \unlhd t} r_{u}\left(0, x_{u}^{t}\right) . \tag{4.10}
\end{equation*}
$$

For general $s, t \in[0,1]$, set

$$
\begin{equation*}
R(s, t)=r_{s \wedge t}\left(x_{s \wedge t}^{s}, x_{s \wedge t}^{t}\right)+R_{0}(s \wedge t, s)+R_{0}(s \wedge t, t) . \tag{4.11}
\end{equation*}
$$

It will also follow from Lemma 4.1.1 that can use the same projection $p$ : $[0,1] \rightarrow \mathcal{L}_{\alpha}^{R}$ to denote the canonical projection for our resistance looptree. Heuristically, the second term represents the total effective resistance along the interior of the path $[[p(s \wedge t), p(s)]]$ in our looptree (and similarly for the third term), whilst the term $r_{s \wedge t}\left(x_{s \wedge t}^{s}, x_{s \wedge t}^{t}\right)$ represents the resistance between the ancestors of $s$ and $t$ across the loop corresponding to $\pi(s \wedge t)$.

Note that our metrics $R_{0}$ and $R$ are then defined analogously to the metrics $d_{0}$ and $d$ of Section 3.2.1. The metric $d$ is essentially a shortest path metric on the looptrees and we give a comparison with $R$ in the following lemma.

Lemma 4.1.1. ([Arc17, Lemma 5.1.4]). For any $s, t \in[0,1]$, we have

$$
\frac{1}{2} d(s, t) \leq R(s, t) \leq d(s, t)
$$

Proof. Note that, for any $x, y \in[0,1]$ :

$$
\left(\frac{2}{\min \{x, y\}}\right)^{-1} \leq\left(\frac{1}{x}+\frac{1}{y}\right)^{-1} \leq\left(\frac{1}{\min \{x, y\}}\right)^{-1}
$$

Thus taking $x=|a-b|, y=\Delta_{t}-|a-b|$ we obtain $\frac{1}{2} \delta_{t}(a, b) \leq r_{t}(a, b) \leq \delta_{t}(a, b)$ for all $t \in[0,1]$ and for all $a, b \in\left[0, \Delta_{t}\right]$.

It follows that for two points $s, t \in[0,1], d(s, t)=0$ if and only if $R(s t)=$ 0 . Moreover, the following proposition is therefore a direct consequence of the corresponding result for $d$ given in [CK14, Proposition 2.2].

Proposition 4.1.2. Almost surely, the function $R(\cdot, \cdot):[0,1]^{2} \rightarrow \mathbb{R}_{+}$is a continuous pseudodistance.

We can therefore make the following definition.
Definition 4.1.3. Let $X$ be an $\alpha$-stable Lévy excursion. The corresponding $\alpha$-stable resistance looptree is defined to be the quotient metric space

$$
\mathcal{L}_{\alpha}^{R}=([0,1] / \sim, R) .
$$

Recall from Section 3.2.1 that the authors define the $\alpha$-stable looptree as

$$
\mathcal{L}_{\alpha}=([0,1] / \sim, d)
$$

in [CK14]. In many places we will abuse notation slightly to write $\mathcal{L}_{\alpha}^{R}=\left(\mathcal{L}_{\alpha}, R\right)$, to minimise new notation.

The next result is a direct consequence of Lemma 4.1.1.

Corollary 4.1.4. [Arc17, Corollary 5.1.5], [Arc19, Corollary 4.3]. The looptrees $\mathcal{L}_{\alpha}$ and $\mathcal{L}_{\alpha}^{R}$ are homeomorphic.

The proof of the next proposition is not enlightening so we just give the statement.

Proposition 4.1.5. [Arc17, Proposition 5.1.6], [Arc20, Proposition 4.4]. $R$ is a resistance metric in the sense of Definition 2.4.1.

Given a realisation of $\mathcal{L}_{\alpha}$, recall we defined its root to be the equivalence class of 0 in $\mathcal{L}_{\alpha}$, and define the measure $\nu$ to be the projection of Lebesgue measure on $[0,1]$ onto $\mathcal{L}_{\alpha}$ via $p$.

This proposition allows us to define a Brownian motion on $\mathcal{L}_{\alpha}$ to be the diffusion naturally associated with $\left(\mathcal{L}_{\alpha}, R, \nu, \rho\right)$ in the sense of Section 2.4. The results of the next section give further justification to this definition.

As a result of Lemma 4.1.1 we also have the following result by exactly the same proof as Proposition 3.2.2

Proposition 4.1.6. Let $\left(\tau_{n}\right)_{n=1}^{\infty}$ be a sequence of trees with $\left|\tau_{n}\right| \rightarrow \infty$ and corresponding Lukasiewicz paths $\left(W^{n}\right)_{n=1}^{\infty}$, and let $f$ be a function in $D^{\text {exc }}([0, \zeta], \mathbb{R})$ for some $\zeta \in(0, \infty)$. Additionally let $\nu_{n}$ be the uniform measure that gives mass 1 to each vertex of $\operatorname{Loop}\left(\tau_{n}\right)$. Suppose that $\left(C_{n}\right)_{n=1}^{\infty}$ is a sequence of positive real numbers such that
(i) $\left(\frac{1}{C_{n}} W_{\left\lfloor\left|\tau_{n}\right| t\right\rfloor}^{n}\left(\tau_{n}\right)\right)_{0 \leq t \leq \zeta} \rightarrow f$ as $n \rightarrow \infty$,
(ii) $\frac{1}{C_{n}} \operatorname{Height}\left(\tau_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$.

Then

$$
d_{G H P}\left(\left(\operatorname{Loop}\left(\tau_{n}\right), \frac{1}{C_{n}} R_{n}, \frac{1}{\left|\tau_{n}\right|} \nu_{n}, \rho_{n}\right),\left(\mathcal{L}_{f}, R_{f}, \nu_{f}, \rho_{f}\right)\right) \rightarrow 0
$$

as $n \rightarrow \infty$.

### 4.1.2 Random walk scaling limits

In light of Proposition 5.5.1, we define Brownian motion on $\mathcal{L}_{\alpha}$ to be the diffusion associated with $\left(\mathcal{L}_{\alpha}, R, \nu, \rho\right)$ as in Section 2.4.1. We now show that this is the scaling limit of random walks on discrete looptrees.

Proof of Theorem 4.0.1. It follows from Proposition 4.1.6, separability and the Skorokhod Representation Theorem that there exists a probability space on which the looptree convergence Proposition 4.1.6 holds almost surely. Without loss of generality, we assume this space is $(\Omega, \mathcal{F}, \mathbf{P})$ : we will show that $\mathbf{P}$-almost surely on this space, the laws of the given stochastic processes converge weakly.

The stochastic process $Y^{(n)}$ naturally associated with $\left(\operatorname{Loop}\left(T_{n}\right), R_{n}, \nu_{n}, \rho_{n}\right)$ in the sense of Section 2.4 .1 is a continuous time random walk that jumps from its present state to each of its neighbouring vertices at rate 1 . Since every vertex of these discrete looptrees has degree 4 (we consider self-loops as undirected), this amounts to an $\exp (4)$ waiting time at every vertex.

Now define processes $\left(\tilde{Z}_{t}^{(n)}\right)_{t \geq 0}$ and $\left(\tilde{Y}_{t}^{(n)}\right)_{t \geq 0}$ by $\tilde{Z}_{t}^{(n)}=a_{n}^{-1} Z_{\left\lfloor 4 n a_{n} t\right\rfloor}^{(n)}$, and $\tilde{Y}_{t}^{(n)}=a_{n}^{-1} Y_{n a_{n} t}^{(n)}$. It follows from Theorem 2.4.3 that almost surely as $n \rightarrow \infty$, we have the weak convergence

$$
\begin{equation*}
\left(\tilde{Y}_{t}^{(n)}\right)_{t \geq 0} \rightarrow\left(B_{t}\right)_{t \geq 0} \tag{4.12}
\end{equation*}
$$

To deduce the result for $\tilde{Z}^{(n)}$ in place of $\tilde{Y}^{(n)}$, we will show that we can couple the processes $Y^{(n)}$ and $Z^{(n)}$ so that they almost surely have the same limit. To do this, note that we can obtain $Y^{(n)}$ from $Z^{(n)}$ by sampling a sequence of independent exponential(4) random variables $\left(w_{i}^{(n)}\right)_{i=1}^{\infty}$, letting $S_{m}^{(n)}=\sum_{i=1}^{m} w_{i}^{(n)}$ for all $m \in \mathbb{N}$, and setting $Y_{t}^{(n)}=Z_{m}^{(n)}$ for all $t \in\left[S_{m}^{(n)}, S_{m+1}^{(n)}\right)$. In particular, $Y_{S_{m}^{(n)}}^{(n)}=Z_{m}^{(n)}$ for all $m$.

Fix some $T<\infty$. Since the limit process $\left(B_{t}\right)_{t \geq 0}$ is almost surely continuous, the convergence of (4.12) actually holds with respect to the uniform topology. By again appealing to the Skorokhod representation theorem along with a functional law of large numbers, we can therefore restrict to a probability space where $\left(\left(\tilde{Y}_{t}\right)_{t \in[0, T]},\left(\left(n a_{n}\right)^{-1} S_{\left\lfloor 4 C_{\alpha} n^{\left.1+\frac{1}{\alpha} t\right\rfloor}\right.}^{(n)}\right)_{t \in[0 . T]}\right) \rightarrow\left(\left(B_{t}\right)_{t \in[0, T]}, t\right)$ jointly almost surely.

By composing these continuous limits, we therefore deduce that

$$
\left(\tilde{Z}_{t}\right)_{t \in[0, T]}=\left(\tilde{Y}_{n a_{n} S_{\left\lfloor 4 n a_{n}\right\rfloor}^{(n)}}\right)_{t \in[0, T]} \rightarrow\left(B_{t}\right)_{t \in[0, T]}
$$

uniformly almost surely. This proves that the distributional result holds for arbitrary $T<\infty$, and we extend to all time by applying [Bil68, Lemma 16.3].

Remark 4.1.7. 1. It also follows from [CH08, Theorem 1 and Proposition 14] that the transition densities of the discrete time random walks on any compact time interval will converge to those of $\left(B_{t}\right)_{t \geq 0}$ under the same rescaling when we isometrically embed in the space $\left(M, d_{M}\right)$ as described above. This can be metrized using the spectral Gromov-Hausdorff distance, introduced in [CHK12, Section 2]. It also follows by an application of [CHK12, Theorem 1.4] that for any $p \in[1, \infty)$, the rescaled $L^{p}$-mixing times for $\operatorname{Loop}\left(\tau_{n}\right)$ will converge to those of $\mathcal{L}_{\alpha}$. We expect that we can prove similar convergence results for blanket times using ideas of [And20], and that the sequence of cover times will be Type 2 in the sense of [Abe14, Definition 1.1].
2. Now that we have constructed a resistance metric on $\mathcal{L}_{\alpha}$, it is possible to
adapt the arguments of [SS19, Theorem 3.2] and combine with Proposition 4.1.6 to show that, in certain regimes, random outerplanar maps endowed with the effective resistance metric and uniform measure converge to $\left(\mathcal{L}_{\alpha}, R, \nu\right)$, which implies a similar scaling limit for variable speed random walks on these outerplanar maps.

### 4.2 Extremal volume bounds

In this section we prove the extremal bounds of Theorems 4.0 .4 and 4.0.5, namely (4.6), (4.5), (4.1), (4.2). Recall that $\mathbf{P}$ denotes the law of $\mathcal{L}_{\alpha}$, and we let $U$ be Uniform $([0,1])$. For ease of intuition, we define the open ball $B(u, r)$ using the metric $d$ rather than $R$.

### 4.2.1 Infimal lower bounds

We prove the lower volume bounds of Theorems 4.0 .4 and 4.0 .5 via the following proposition.

Proposition 4.2.1. There exist constants $c, C, r_{0} \in(0, \infty)$ such that for all $r \in$ $\left(0, r_{0}\right)$ and all $\lambda \in\left(0, \frac{1}{2} r^{-\alpha}\right)$,

$$
\mathbf{P}\left(\nu(B(p(U), r))<r^{\alpha} \lambda^{-1}\right) \leq C \exp \left\{-c \lambda^{\frac{1}{\alpha}}\right\} .
$$

The proof of Proposition 4.2.1 uses ideas from the proof of the upper bound on the Hausdorff dimension of $\mathcal{L}_{\alpha}$ that was given in [CK14, Section 3.3.1]. It relies on the fact that for any $s, t \in[0,1]$ with $s \leq t$,

$$
\begin{equation*}
d(p(s), p(t)) \leq X_{s}^{\mathrm{exc}}+X_{t}^{\mathrm{exc}}-2 \inf _{s \leq r \leq t} X_{r}^{\mathrm{exc}} \tag{4.13}
\end{equation*}
$$

This result appears as [CK14, Lemma 2.1]. Consequently, we can lower bound the volume of small balls in $\mathcal{L}_{\alpha}$ by upper bounding the oscillations of $X^{\text {exc }}$. We use the notation $\operatorname{Diam}_{f}(p([a, b])$ to denote the diameter of the set $p(a, b)$ defined from $f$ using the distance function of (3.7), but with $f$ in place of $X^{\text {exc }}$.

We first give a technical lemma which appeared previously in [CK14, Section 3.3.1] and uses an argument from [Ber96]. The final claim follows by bounded convergence.

First recall that for a function $f:[0, \infty) \rightarrow \mathbb{R}$ and $[a, b] \subset[0, \infty)$, we define

$$
\operatorname{Osc}_{[a, b]} f:=\sup _{s, t \in[a, b]}|f(t)-f(s)| .
$$

Lemma 4.2.2. Let $\mathcal{E}$ be an exponential random variable with parameter 1 , and let $X$ be a spectrally positive $\alpha$-stable Lévy process conditioned to have no jumps of size greater than 1 on $[0, \mathcal{E}]$. Let $\tilde{O s c}=\operatorname{Osc}_{[0, \mathcal{E}]} X$. Then there exists $\theta>0$ such that $\mathbb{E}\left[e^{\theta \tilde{O} s c}\right]<\infty$. Moreover, $\mathbb{E}\left[e^{\theta \tilde{O s c}}\right] \downarrow 1$ as $\theta \downarrow 0$.

Remark 4.2.3. The same results holds if $\mathcal{E}$ is set to be deterministically equal to 1 rather than an exponential random variable. The proof is almost identical to the one above, with one minor modification.

Proof of Proposition 4.2.1. First, note the inclusion

$$
\begin{aligned}
\left\{\nu(B(p(U), r))<r^{\alpha} \lambda^{-1}\right\} & \subset\left\{p\left(\left[U, U+r^{\alpha} \lambda^{-1}\right]\right) \cap B^{c}(p(U), r) \neq \emptyset\right\} \\
& \subset\left\{\operatorname{Diam}_{\left.X^{\operatorname{exc}}\left(p\left[U, U+r^{\alpha} \lambda^{-1}\right]\right)>r\right\}}\right.
\end{aligned}
$$

By applying the Vervaat transform, the absolute continuity relation (2.12) and scaling invariance, we get that
$\mathbb{P}\left(\operatorname{Diam}_{X^{\operatorname{exc}}}\left(p\left[U, U+r^{\alpha} \lambda^{-1}\right]\right)>r\right) \leq \frac{\left(1-r^{\alpha} \lambda^{-1}\right)^{\frac{-1}{\alpha}}\left\|p_{1}\right\|_{\infty}}{p_{1}(0)} \mathbb{P}\left(\operatorname{Diam}_{X}(p[0,1])>\lambda^{\frac{1}{\alpha}}\right)$.
To bound the latter quantity, let $N$ be the cardinality of the set $\left\{t \in[0,1]: \Delta_{t}>1\right\}$, where $\Delta_{t}=X_{t}-X_{t^{-}}$now denotes the jump size of $X$ rather than $X^{\text {exc }}$, and let $t_{1}, \ldots, t_{N}$ be its members in increasing order of size. Additionally let $t_{0}=0$ and $t_{N+1}=1$, and $\tilde{C}_{\alpha}=\frac{\alpha-1}{\Gamma(2-\alpha)}$, so that $N \sim \operatorname{Poi}\left(\tilde{C}_{\alpha}\right)$. We then have:

$$
\begin{aligned}
\mathbb{P}\left(\operatorname{Diam}_{X}(p[0,1])>\lambda^{\frac{1}{\alpha}}\right) & \leq \sum_{n=1}^{\infty} \frac{e^{-\tilde{C_{\alpha}}}\left(\tilde{C_{\alpha}}\right)^{n}}{n!} \mathbb{P}\left(\left.\sum_{i=1}^{N} \operatorname{Osc}_{\left[t_{i}, t_{i+1}\right]} X>\lambda^{\frac{1}{\alpha}} \right\rvert\, N=n\right) \\
& \leq \sum_{n=1}^{\infty} \frac{e^{-\tilde{C_{\alpha}}}\left(\tilde{C_{\alpha}}\right)^{n}}{n!} \mathbb{E}\left[e^{\theta \tilde{\mathrm{Osc}}]^{n} \exp \left\{-\theta \lambda^{\frac{1}{\alpha}}\right\}}\right.
\end{aligned}
$$

where $\tilde{O s c}$ is as in Remark 4.2.3. Note that $N$ and $\left(\tilde{\mathrm{Osc}_{\left[t_{i}, t_{i+1}\right]}}\right)_{i \leq N}$ are not independent, but we certainly have $t_{i+1}-t_{i} \leq 1$ for all $i$, and hence by Lemma 4.2.2 and Remark 4.2 .3 we can choose $\theta$ small enough that $C_{\theta}:=\mathbb{E}\left[e^{\theta \text { Osc }}\right]<\infty$. The result follows from noting that

$$
\mathbb{P}\left(\operatorname{Diam}_{X}(p[0,1])>\lambda^{\frac{1}{\alpha}}\right) \leq e^{\left(C_{\theta}-1\right) \tilde{C_{\alpha}}} e^{-\theta \lambda^{\frac{1}{\alpha}}}
$$

By taking a union bound, the same argument can be used to give a bound on the global infimum.

Proposition 4.2.4. There exist constants $c, C, r_{0} \in(0, \infty)$ such that for all $r \in$ $\left(0, r_{0}\right)$ and all $\lambda \in\left(0, \frac{1}{2} r^{-\alpha}\right)$,

$$
\mathbf{P}\left(\inf _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r))<r^{\alpha} \lambda^{-1}\right) \leq C r^{-\alpha} \lambda \exp \left\{-c \lambda^{\frac{1}{\alpha}}\right\}
$$

Proof. By the same reasoning as in the proof of Proposition 4.2.1, we have:

$$
\begin{aligned}
& \left\{\inf _{u \in \mathcal{L}} \nu(B(u, r))<r^{\alpha} \lambda^{-1}\right\} \\
& \quad \subset\left\{\operatorname{Diam}_{X^{\operatorname{br}}}\left(p\left[k r^{\alpha} \lambda^{-1},(k+1) r^{\alpha} \lambda^{-1} \wedge 1\right]\right)>\frac{1}{2} r \text { for some } k=0, \ldots,\left\lfloor r^{-\alpha} \lambda\right\rfloor\right\},
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \mathbf{P}\left(\inf _{u \in \mathcal{L}} \nu(B(u, r))<r^{\alpha} \lambda^{-1}\right) \\
& \leq \mathbf{P}\left(\operatorname{Diam}_{X^{\operatorname{br}}}\left(p\left[k r^{\alpha} \lambda^{-1},(k+1) r^{\alpha} \lambda^{-1} \wedge \frac{1}{2}\right]\right)>\frac{1}{2} r \text { for some } k=0, \ldots,\left\lfloor\frac{1}{2} r^{-\alpha} \lambda\right\rfloor\right) \\
& +\mathbf{P}\left(\operatorname{Diam}_{X^{\operatorname{br}}}\left(p\left[\frac{1}{2} \vee k r^{\alpha} \lambda^{-1},(k+1) r^{\alpha} \lambda^{-1} \wedge 1\right]\right)>\frac{1}{2} r \text { for some } k=\left\lfloor\frac{1}{2} r^{-\alpha} \lambda\right\rfloor, \ldots,\left\lfloor r^{-\alpha} \lambda\right\rfloor\right) \\
& \leq C_{\theta} r^{-\alpha} \lambda \frac{\left\|p_{\frac{1}{2}}\right\|_{\infty}}{p_{1}(0)} e^{-\theta \lambda^{\frac{1}{\alpha}}},
\end{aligned}
$$

where the final line follows by Proposition 4.2.1.
Proof of infimal lower bounds in Theorems 4.0.4 and 4.0.5. Take $c$ as in Proposition 4.2.1, and $M>c^{-1}$. Set

$$
g(r)=M r^{\alpha}\left(\log \log r^{-1}\right)^{-\alpha}, \quad \text { and } \quad J_{r}=\{\nu(B(p(U), r))<g(r)\}
$$

Taking $\lambda=M\left(\log \log r^{-1}\right)^{\alpha}$ in Proposition 4.2 .1 we see that $\mathbf{P}\left(J_{r}\right) \leq C\left(\log r^{-1}\right)^{-c M}$, and since $M>c^{-1}$ we have by Borel-Cantelli that $\mathbf{P}\left(J_{2^{-k}}\right.$ i.o. $)=0$. Hence there almost surely exists $K \in \mathbb{N}$ such that $J_{2^{-k}}^{c}$ occurs for all $k \geq K$. On this event, $\nu(B(p(U), r)) \geq 2^{-\alpha} g(r)$ for all sufficiently small $r$, or equivalently,

$$
\begin{equation*}
\underset{r \downarrow 0}{\liminf }\left(\frac{\nu(B(p(U), r))}{r^{\alpha}\left(\log \log r^{-1}\right)^{-\alpha}}\right) \geq 2^{-\alpha} M \tag{4.14}
\end{equation*}
$$

To deduce the result for $\nu$-almost every $u \in \mathcal{L}_{\alpha}$ we apply Fubini's theorem.
Letting

$$
F\left(\mathcal{L}_{\alpha}, u\right)=\mathbb{1}\left\{\liminf _{r \downarrow 0}\left(\frac{\nu(B(u, r))}{r^{\alpha}\left(\log \log r^{-1}\right)^{-\alpha}}\right) \geq 2^{-\alpha} M\right\}
$$

we have from above that

$$
\int_{0}^{1} \mathbf{E}\left[F\left(\mathcal{L}_{\alpha}, u\right)\right] d u=\mathbf{E}\left[F\left(\mathcal{L}_{\alpha}, p(U)\right)\right]=1
$$

By Fubini's theorem, this implies that almost surely, $F\left(\mathcal{L}_{\alpha}, u\right)=1$ for Lebesgue almost every $u \in[0,1]$, and consequently for $\nu$-almost every $u \in \mathcal{L}_{\alpha}$. This proves (4.5).

The proof of the global bound (4.1) is similar. Take $c$ as in Proposition 4.2.1, choose some $A>\alpha c^{-1}$, and set $\varepsilon=A-\alpha c^{-1}$. Then, setting $\lambda=\left(A \log r^{-1}\right)^{\alpha}$ we have by Proposition 4.2 .4 that:

$$
\mathbf{P}\left(\inf _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r))<r^{\alpha}\left(A \log r^{-1}\right)^{-\alpha}\right) \leq C r^{\varepsilon}\left(\log r^{-1}\right)^{\alpha}
$$

Consequently, letting

$$
K_{r}=\left\{\inf _{u \in \mathcal{L}_{\alpha}} \nu\left(B_{r}(u)\right)<r^{\alpha}\left(A \log r^{-1}\right)^{-\alpha}\right\}
$$

we have by Borel-Cantelli that $\mathbf{P}\left(K_{2^{-k}}\right.$ i.o. $)=0$. Hence, there almost surely exists a $K_{0}<\infty$ such that for any $r<2^{-K_{0}}$ we have (4.1), or more precisely that:

$$
\inf _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) \geq 2^{-\alpha} r^{\alpha}\left(A \log r^{-1}\right)^{-\alpha}
$$

### 4.2.2 Supremal upper bounds

In this section we prove (4.2) and (4.6) using the following Williams' Decomposition. By appealing to uniform re-rooting invariance, we will treat $p(U)$ as the root of the looptree throughout.

## Williams' decomposition

The Williams' Decomposition of [AD09] gives a decomposition of a stable tree $\tilde{\mathcal{T}}_{\alpha}$ along its spine of maximal height. In the Brownian case $\alpha=2$, this corresponds to Williams' decomposition of Brownian motion. Letting $H_{\max }=\sup _{u \in \tilde{\mathcal{T}}_{\alpha}} d_{\tilde{\mathcal{T}}_{\alpha}}(\rho, u)$, we see from [DW17, Equation (23)] (and references therein) that there is almost surely a unique $u_{h} \in \tilde{\mathcal{T}}_{\alpha}$ such that $d_{\tilde{\mathcal{T}}_{\alpha}}\left(\rho, u_{h}\right)=H_{\text {max }}$. We define the Williams' spine (or W-spine) of $\tilde{\mathcal{T}}_{\alpha}$ to be the segment $\left[\left[\rho, u_{h}\right]\right]$, and take the Williams' loopspine (or W-loopspine) in the corresponding looptree $\mathcal{L}_{\alpha}$ to be the closure of the set of loops coded by points in $\left[\left[\rho, u_{h}\right]\right]$. A main result of [AD09] is a theorem which firstly gives the distribution of the loop lengths along the W-loopspine, and additionally the
distribution of the fragments obtained by decomposing along it.
Given the spine from $\rho$ to $u_{h}$, and conditional on $H_{\max }=H$, the loops along the W-loopspine can be represented by a Poisson point measure $\sum_{j \in J} \delta\left(l_{j}, t_{j}, u_{j}\right)$ on $\mathbb{R}^{+} \times[0, H] \times[0,1]$ with a certain intensity. A point $(l, t, u)$ corresponds to a loop of length $l$ in the W-loopspine, occurring on the W-spine at distance $t$ from the root in the corresponding tree $\tilde{\mathcal{T}}_{\alpha}$, and such that a proportion $u$ of the loop is on the "left" of the W-loopspine, and a proportion $1-u$ is on the "right". In [AD09], this is written in terms of the exploration process on $\tilde{\mathcal{T}}_{\alpha}$, but we interpret their result below in the context of looptrees.

We note that when stating this result, we are not conditioning on the total mass of $\tilde{\mathcal{T}}_{\alpha}$ : only the maximal height. The mass will depend on its height via the joint laws for these under the Itô excursion measure.

Theorem 4.2.5. (Follows directly from [AD09, Lemma 3.1 and Theorem 3.3]).
(i) Conditionally on $H_{\max }=H$, the set of loops in the $W$-loopspine forms a Poisson point process $\mu_{W \text {-loopspine }}=\sum_{j \in \mathcal{J}} \delta\left(l_{j}, t_{j}, u_{j}\right)$ on the $W$-spine in the corresponding tree with intensity

$$
\mathbb{1}_{\{[0,1]\}}(u) \mathbb{1}_{\{[0, H]\}}(t) l \exp \left\{-l(H-t)^{\frac{-1}{\alpha-1}}\right\} d u d t \Pi(d l),
$$

where $\Pi$ is the underlying Lévy measure, with $\Pi(d l)=\frac{1}{|\Gamma(-\alpha)|} l^{-\alpha-1} \mathbb{1}_{(0, \infty)}(l) d l$ in the stable case. We will denote the atom $\delta\left(l_{j}, t_{j}, u_{j}\right)$ by Loop $_{j}$.
(ii) Let $\delta(l, t, u)$ be an atom of the Poisson process described above. The set of sublooptrees grafted to the $W$-loopspine at a point in the corresponding loop can be described by a random measure $M^{(l)}=\sum_{i \in I} \delta^{(l)}\left(\mathcal{E}_{i}, D_{i}\right)$, where $\mathcal{E}_{i}$ is a Lévy excursion that codes a looptree in the usual way, and $D_{i}$ represents the distance going clockwise around the loop from the point at which this sublooptree is grafted to the loop, to the point in the loop that is closest to $\rho$. This measure has intensity

$$
N\left(\cdot, H_{\max } \leq H-t\right) \times \mathbb{1}_{\{[0, l]\}}(D) d D .
$$

In particular, the sublooptrees are just rescaled copies of our usual normalised compact stable looptrees, and each of these is grafted to the loop on the $W$ loopspine at a uniform point around the loop lengths.

Remark 4.2.6. Point (ii) is a slight extension of the results of [AD09], where the authors write that the intensity of subtrees incident to the $W$-spine at a node of "degree" $l$ has intensity $l N\left(\cdot, H_{\max } \leq m-t\right)$. However, it follows from [DLG05, Equation (11)] and the remarks below it that the corresponding sublooptrees are actually distributed uniformly around the corresponding loop.

## Encoding the looptree structure in a branching process

The Williams' decomposition suggests a natural way to encode the fractal structure of $\mathcal{L}_{\alpha}$ in a branching process or cascade. Specifically, we let $\emptyset$ denote the root vertex of our cascade. This represents the whole looptree $\mathcal{L}_{\alpha}$ (in particular, $\emptyset$ should not be confused with $\rho$, which is the root of $\mathcal{L}_{\alpha}$ ). By performing the Williams' decomposition on $\mathcal{L}_{\alpha}$ and removing the W -loopspine, the fragments obtained are countably many smaller copies of $\mathcal{L}_{\alpha}$, which we view as the children of $\emptyset$ in our branching process, and index by $\mathbb{N}$. Moreover, to each edge joining $\emptyset$ to one of its offspring $i$, we associate a random variable $m(\emptyset, i)$ which gives the mass of the sublooptree corresponding to index $i$. The root of a sublooptree is the point at which it is grafted to the W-loopspine of its parent.

We can then perform further Williams' decompositions of these sublooptrees. More precisely, if $i$ is a child of $\emptyset$, we can decompose along its W-loopspine from its root to its point of maximal tree height to obtain a countable collection of offspring of $i$ that correspond to the fragments obtained on removing this W-loopspine, and label the offspring as $(i j)_{j \geq 1}$. By repeating this procedure again and again on the resulting subsublooptrees, we can keep iterating to obtain an infinite branching process.

Remark 4.2.7. It may seem more straightforward to use a spinal decomposition to a uniform point (as in [HPW09]) as the basis of this iteration, rather than the Williams' decomposition. However, this leads to technical difficulties in the case when $V$ is chosen so that $p(V)$ is a point too close to $p(U)$, and it is convenient to avoid this by instead decomposing along the maximal spine in the underlying tree.

We index this process using the Ulam-Harris labelling convention defined in Section 3.1. Using the notation of [Nev86], an element of our branching process will be denoted by $u=u_{1} u_{2} u_{3} \ldots u_{j}$, and corresponds to a smaller sublooptree $L \subset \mathcal{L}_{\alpha}$. Its offspring will all be of the form $(u i)_{i \in \mathbb{N}}$, with corresponding roots $\left(\rho_{u i}\right)_{i \in \mathbb{N}}$, where $u i$ here abbreviates the concatenation $u_{1} u_{2} u_{3} \ldots u_{j} i$, and each will correspond to one of the further sublooptrees obtained on performing a Williams' decomposition of $L$.

Moreover, to each edge joining $u$ to its child $u i$ we associate a random variable $m(u, u i)$. These give the ratios of the masses of each of the sublooptrees that correspond to the offspring of $u$, so that $\sum_{i=1}^{\infty} m(u, u i)=1$ for all $u \in \mathcal{U}$. Given a particular element $u=u_{1} u_{2} \ldots u_{j}$ of the branching process, the overall mass of the corresponding sublooptree is then given by $M_{u}=\prod_{i=0}^{j-1} m\left(u_{i}, u_{i+1}\right)$, where here we let $u_{0}$ denote the root $\emptyset$.

## Main argument for supremal upper bound

The simplest way to upper bound the volume is to sum the masses of all the sublooptrees that are incident to the W -loopspine at a point within distance $r$ of $p(U)$, giving

$$
\begin{equation*}
\nu(B(p(U), r)) \leq \sum_{i=1}^{\infty} M_{i} \mathbb{\mathbb { 1 }}\left\{\rho_{i} \in B(p(U), r)\right\} . \tag{4.15}
\end{equation*}
$$

We would like to use this to bound $\mathbf{P}\left(\nu(B(p(U), r)) \geq r^{\alpha} \lambda\right)$. However, this approach is not very sharp since the probability that there is such an incident sublooptree of mass greater than $r^{\alpha} \lambda$ is of order $\lambda^{\frac{-1}{\alpha}}$, and when this happens the bound on the right hand side of (4.15) is immediately too large. However, if this event occurs, it is actually likely that this sublooptree is not completely contained in $B(p(U), r)$, and so we are not really capturing the right asymptotics for the behaviour of $\nu(B(p(U), r))$ by applying (4.15).

To refine the argument we instead repeat the same procedure around the W-loopspine of the larger sublooptree. If there are no larger (sub)sublooptrees incident to the (sub)W-loopspine close to the (sub)root, then we conclude by summing the smaller terms; otherwise, we can keep repeating the same procedure and iterating further until eventually we reach a stage where there are no more "large" sublooptrees to consider.

This iterative process corresponds to selecting a finite subtree $T$ of $\mathcal{U}$ in such a way that the elements of $T$ correspond to the large sublooptrees around which we perform further iterations. The offspring distribution of $T$ will be sufficiently subcritical that the process will die out fairly quickly. Conditioning on the extinction time and then on the total progeny of $T$, we bound the volume of the ball $B(p(U), r)$ by the sum of the masses of all the small sublooptrees that are grafted to the Wloopspine of each of the large sublooptrees.

Below, we describe how we select $T$ generation by generation as a subtree of $\mathcal{U}$. Throughout, we take:

$$
\beta_{1}=\frac{\alpha-1}{4 \alpha-3}, \quad \beta_{2}=\frac{\alpha-1}{4 \alpha-3}, \quad \beta_{3}=\frac{2 \alpha-1}{2 \alpha(4 \alpha-3)}, \quad \beta_{4}=\frac{1}{4 \alpha-3} .
$$

Note that $2 \beta_{3}-\frac{1}{\alpha}\left(1-\beta_{1}-\beta_{2}\right)=0$. Also fix some $\kappa \in\left(0,\left(\frac{1}{3 e^{2}} \Gamma\left(1-\frac{1}{\alpha}\right)\right)^{\alpha}\right)$. We need $\kappa$ to be sufficiently small to ensure that $T$ is sufficiently subcritical, but we will not be taking any kind of limit as $\kappa \downarrow 0$.

## Iterative Algorithm 1

Start by taking $\emptyset$ to be the root of $T$. Recall this represents the whole looptree $\mathcal{L}_{\alpha}$.

1. Perform a Williams' decomposition of $\mathcal{L}_{\alpha}$ along its W-loopspine.
2. Consider the resulting fragments. To choose the offspring of $\emptyset$, select the fragments that have mass at least $\kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$, and such that the subroots of the corresponding looptrees are within distance $r$ of the root of $\emptyset$.
3. Repeat this process to construct $T$ one generation at a time. Given an element $u=u_{1} u_{2} \ldots u_{j} \in T$, there is a corresponding sublooptree $L_{u}$ in $\mathcal{L}_{\alpha}$ with root $\rho_{u}$ and $M_{u}:=\nu\left(L_{u}\right) \geq \kappa r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$. Consider the fragments obtained in a Williams' decomposition of $L_{u}$, and select those that correspond to further sublooptrees that are within distance $r$ of $\rho_{u}$, and also such that they have mass at least $\kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$ (i.e. with $\left.M_{u_{1} u_{2} \ldots u_{j} u_{j+1}}=\prod_{k=0}^{j} m\left(u_{k}, u_{k+1}\right) \geq \kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right)$, to be the offspring of $u$.
4. For each $u=u_{1} u_{2} \ldots u_{j} \in T$, set

$$
S_{u}=\sum_{i=1}^{\infty} M_{u i} \mathbb{1}\left\{\rho_{u i} \in B\left(\rho_{u}, r\right)\right\} \mathbb{1}\left\{M_{u i}<\kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right\} .
$$

As explained above, in the event that $T$ is finite we then have that:

$$
\nu(B(p(U), r)) \leq \sum_{u \in T} S_{u}
$$

Since the Williams' decomposition involves conditioning on the height of the corresponding stable tree rather than its mass, we will prove this theorem by rescaling each sublooptree corresponding to an element of $T$ to have underlying tree height 1 , and then using Theorem 4.2.5 to analyse the fragments. Most of the effort in proving the supremal upper bounds is devoted to proving the following proposition.

Proposition 4.2.8. There exist constants $\tilde{c}, \tilde{C} \in(0, \infty)$ such that for all $r<1$ and all $\lambda>1$,

$$
\mathbf{P}\left(\nu(B(p(U), r)) \geq r^{\alpha} \lambda\right) \leq \tilde{C} \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-\tilde{c} \lambda^{\frac{\alpha-1}{4 \alpha-3}}}
$$

The volume results (4.2) and (4.6) follow from Proposition 4.2.8 by BorelCantelli, similarly to those in the previous section. We sketch this below, and prove Proposition 4.2.8 afterwards.

Proof of supremal upper bounds, assuming Proposition 4.2.8. Take $\tilde{c}$ as in Proposition 4.2.8, and choose $A>\tilde{c}^{-1}$. Taking $\lambda_{r}=A\left(\log \log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}$ in Proposition 4.2.8 and applying Borel-Cantelli we deduce that $\mathbf{P}\left(I_{2^{-k}}\right.$ i.o. $)=0$, where

$$
I_{r}=\left\{\nu(B(p(U), r)) \geq r^{\alpha} \lambda_{r}\right\}
$$

Similarly to the proof of the infimal bounds, it follows that

$$
\limsup _{r \downarrow 0}\left(\frac{\nu(B(p(U), r))}{r^{\alpha}\left(\log \log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}}\right) \leq 2^{\alpha} A
$$

almost surely, and we extend to $\nu$-almost every $u \in \mathcal{L}_{\alpha}$ using Fubini's theorem as before. This proves (4.6).

To prove the global bound (4.2), we have to do a bit more work. First take some $\varepsilon>0$, and define $\mathcal{W}$ to be the set of sets

$$
\begin{array}{r}
\left\{p\left(\left[n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)},(n+1) c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}\right)\right):\right. \\
\left.n \in\left\{0,1, \ldots\left\lfloor c^{-\alpha}(\alpha+\varepsilon)^{\alpha} r^{-\alpha}\left(\log r^{-1}\right)^{\alpha(1+\varepsilon)}\right\rfloor\right\}\right\}
\end{array}
$$

where $c$ takes the same value as it did in Proposition 4.2.4. It then follows from Proposition 4.2.4 that

$$
\begin{equation*}
\mathbf{P}\left(\mathcal{W} \text { is an } r \text {-covering of } \mathcal{L}_{\alpha}\right) \geq 1-C c^{-\alpha}(\alpha+\varepsilon)^{\alpha} r^{\varepsilon}\left(\log r^{-1}\right)^{\alpha(1+\varepsilon)} \tag{4.16}
\end{equation*}
$$

for all sufficiently small $r$. Moreover, assuming that $\mathcal{W}$ is indeed an $r$-covering of $\mathcal{L}_{\alpha}$, and letting

$$
W^{r}=\left\{x \in \mathcal{L}_{\alpha}: d(x, y) \leq r \text { for some } y \in W\right\}
$$

be the $r$-fattening of $W$ for any set $W \in \mathcal{W}$, say with

$$
W=p\left(\left[n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)},(n+1) c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}\right)\right)
$$

we have that $W^{r} \subset B\left(p\left(n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}, 2 r\right)\right.$. It hence follows that

$$
\left\{\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) \leq r^{\alpha} \lambda_{r}\right\} \subset\left\{\left\{\mathcal{W} \text { is an } r \text {-covering of } \mathcal{L}_{\alpha}\right\} \cap\left\{\nu\left(W^{r}\right) \leq r^{\alpha} \lambda_{r} \forall W \in \mathcal{W}\right\}\right\}
$$

and consequently,

$$
\begin{align*}
& \mathbf{P}\left(\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) \geq r^{\alpha} \lambda_{r}\right) \\
& \leq \mathbf{P}\left(\mathcal{W} \text { is not an } r \text {-covering of } \mathcal{L}_{\alpha}\right)  \tag{4.17}\\
& \quad+\mathbf{P}\left(\exists n: \nu\left(B\left(p\left(n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}\right), 2 r\right)\right) \geq r^{\alpha} \lambda_{r}\right) .
\end{align*}
$$

It follows from re-rooting invariance at deterministic points that for any $n$,

$$
\begin{aligned}
\mathbf{P}\left(\nu\left(B\left(p\left(n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}\right), 2 r\right)\right) \geq r^{\alpha} \lambda_{r}\right) & =\mathbf{P}\left(\nu(B(\rho, 2 r)) \geq r^{\alpha} \lambda_{r}\right) \\
& =\mathbf{P}\left(\nu(B(p(U), 2 r)) \geq r^{\alpha} \lambda_{r}\right),
\end{aligned}
$$

and hence by applying a union bound and Proposition 4.2.8, we see that

$$
\begin{aligned}
& \mathbf{P}\left(\exists n: \nu\left(B\left(p\left(n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}\right), 2 r\right)\right) \geq r^{\alpha} \lambda_{r}\right) \\
& \quad \leq C^{\prime} r^{-\alpha}\left(\log r^{-1}\right)^{\alpha(1+\varepsilon)} \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-\tilde{c} \lambda^{\frac{\alpha-1}{4 \alpha-3}}} .
\end{aligned}
$$

In particular, taking $\lambda=\lambda_{r}=\left((\alpha+\varepsilon) \tilde{c}^{-1} \log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}$, where $\tilde{c}$ is as it was in Proposition 4.2.8, we obtain

$$
\begin{equation*}
\mathbf{P}\left(\exists n: \nu\left(B\left(p\left(n c^{\alpha}(\alpha+\varepsilon)^{-\alpha} r^{\alpha}\left(\log r^{-1}\right)^{-\alpha(1+\varepsilon)}\right), 2 r\right)\right) \leq r^{\alpha} \lambda_{r}\right) \leq C^{\prime} r^{\varepsilon}\left(\log r^{-1}\right)^{1+\alpha(1+\varepsilon)} . \tag{4.18}
\end{equation*}
$$

By combining equations (4.16), (4.17) and (4.18), we therefore see that

$$
\mathbf{P}\left(\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) \geq r^{\alpha} \lambda_{r}\right) \leq C^{\prime} r^{\varepsilon}\left(\log r^{-1}\right)^{1+\alpha(1+\varepsilon)} .
$$

Hence, letting $J_{r}=\left\{\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) \geq r^{\alpha} \lambda_{r}\right\}$, we have that $\mathbf{P}\left(J_{2^{-k}}\right.$ i.o. $)=0$ as before. This implies (4.2), since similarly to before, we deduce that there exists $r_{0}>0$ such that for all $r \in\left(0, r_{0}\right)$,

$$
\sup _{u \in \mathcal{L}_{\alpha}} \nu(B(u, r)) \leq 2^{\alpha} r^{\alpha}\left(\log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}} .
$$

For a given looptree $\tilde{L}_{\alpha}$ and a given $R>0$, we let $I_{R}$ denote the set of points in the W-loopspine that also fall within distance $R$ of the root. Formally,

$$
I_{R}=\bigcup_{s \preceq u_{H}}\left\{t \geq s: \tilde{X}_{t}^{\mathrm{exc}}=\inf _{s<r \leq t} \tilde{X}_{r}^{\text {exc }}, d(\rho, p(t))<R\right\},
$$

where $\tilde{X}^{\text {exc }}$ is the Lévy excursion coding $\tilde{L}_{\alpha} . I_{R}$ can be endowed with a natural notion of length, denoted $\left|I_{R}\right|$, which can be thought of as the sum of the lengths of loop fragments contained in $I_{R}$. Formally, this can be defined as the Lebesgue measure of the closure of the set $\left\{\tilde{X}_{t}^{\text {exc }}: t \in I_{R}\right\}$.

To bound the progeny of $T$, we can then use the Williams' decomposition to view the sublooptrees grafted to the W -loopspine as a Poisson process on $D([0, \infty),[0, \infty)) \times I_{R}$. In particular, the number of sublooptrees with mass greater than $m$ will be stochastically dominated by a Poisson with parameter $\left|I_{R}\right| N(\zeta>m)$, where $N$ denotes the Itô excursion measure and $\zeta$ denotes the length of an excursion under this measure. $\left|I_{R}\right|$ will be roughly of order $R$, but the purpose of the next lemma is to control this more precisely.

Lemma 4.2.9. Let $\left(\mathcal{L}_{\alpha}^{1}, \rho^{1}, d^{1}, \nu^{1}\right)$ be a compact stable looptree conditioned so that its underlying tree has height 1, but with no conditioning on its mass. Take $R \leq$ $\lambda^{-\beta_{4}}$, and let $I_{R}$ and $\left|I_{R}\right|$ be as above. Then

$$
\mathbf{P}\left(\left|I_{R}\right| \geq 3 R \lambda^{2 \beta_{3}}\right) \leq C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2 \beta_{3}}}\right) \leq C e^{-c \lambda^{\frac{\alpha-1}{4 \alpha-3}}}
$$

Proof. It is possible that $\left|I_{R}\right|$ may be of order greater than $R$ if, for example, many of the loops close to the root have spinal branch points distributed such that they split the loop into two very unequal segments. We show that this occurs only with very low probability.

First note that, by Theorem 4.2.5(i), the loops that fall on the first half of the W -spine stochastically dominate a Poisson point measure $\sum_{j \in \mathcal{J}} \delta\left(l_{j}, t_{j}, u_{j}\right)$ with intensity

$$
\begin{equation*}
\mathbb{1}_{\{[0,1]\}}(u) \mathbb{1}_{\left\{\left[0, \frac{1}{2}\right]\right\}}(t) l \exp \left\{-l 2^{\frac{1}{\alpha-1}}\right\} d u d t \Pi(d l) \tag{4.19}
\end{equation*}
$$

Elements of the set $\left(t_{j}\right)_{j \in \mathcal{J}}$ correspond to distances along the spine in the underlying tree, but we will consider them as time indices throughout the remainder of this proof. We will model the loop lengths using a subordinator, where a jump of the subordinator of size $\Delta$ at time $t$ corresponds to a loop of length $\Delta$ which in turn corresponds to a node at a distance $t$ along the W-spine in the associated stable tree.

To prove the bound, we first condition on existence of a loop in the W loopspine with length $l$ greater than $4 R$ and with $u \in\left[\frac{1}{4}, \frac{3}{4}\right]$. We say that such a loop is "good". We also say that a loop is "goodish" if it just has length at least $4 R$, with no restriction on $u$. We then select the closest good loop to $\rho$. Given such a loop, the number of goodish loops between $\rho$ and the first good loop is stochastically dominated by a Geometric $\left(\frac{1}{2}\right)$ random variable. Letting this number be $N,\left|I_{R}\right|$ can
then be upper bounded by the random variable

$$
2 R(N+1)+\sum_{i=1}^{N+1} Q^{(i)}
$$

where $Q^{(i)}$ denotes the sum of the lengths of all the smaller loops on the W-loopspine that are between the $(i-1)^{\text {th }}$ and $i^{\text {th }}$ goodish loops, and the term $2 R(N+1)$ comes from selecting a segment of length at most $R$ in each direction from the "base point" around each of the goodish loops. Each $Q^{(i)}$ can be independently approximated by an ( $\alpha-1$ )-stable subordinator run up until an exponential time and conditioned not to have any jumps greater than $4 R$.

First let the number of good loops on the first half of the W -spine be equal to $M$. From (4.19), it follows that $M$ stochastically dominates a Poisson random variable with parameter

$$
\kappa_{R}=\frac{1}{4} \int_{4 R}^{8 R} l \exp \left\{-l 2^{\frac{1}{\alpha-1}}\right\} \Pi(d l) \geq \frac{1}{4} \int_{4 R}^{8 R} l^{-\alpha} \exp \left\{-8 R 2^{\frac{1}{\alpha-1}}\right\} d l \geq \tilde{C} R^{1-\alpha}
$$

where $\tilde{C}=\frac{1}{4(\alpha-1)}\left(4^{1-\alpha}-8^{1-\alpha}\right) \exp \left\{-8 \cdot 2^{\frac{1}{\alpha-1}}\right\}$ is just a constant. Hence,

$$
\begin{equation*}
\mathbf{P}(M=0) \leq e^{-c R^{1-\alpha}} \leq e^{-c \lambda^{\beta_{4}(\alpha-1)}} \tag{4.20}
\end{equation*}
$$

We henceforth condition on $M>0$. Next, note that for any loop of length at least $4 R$, the probability that it is good is at least $\frac{1}{2}$ (independently of the other loops), and so if we examine all such loops of the W-loopspine in order from $\rho$, as described in the previous paragraph, we have that $N+1$ is stochastically dominated by a Geo( $\frac{1}{2}$ ) random variable. Hence, for any $\theta>0$, we have by a Chernoff bound that

$$
\begin{equation*}
\mathbf{P}\left(N+1 \geq \lambda^{2 \beta_{3}}\right) \leq \mathbf{P}\left(\operatorname{Geo}\left(\frac{1}{2}\right) \geq \lambda^{2 \beta_{3}}\right) \leq C e^{-\lambda^{2 \beta_{3}}} \tag{4.21}
\end{equation*}
$$

To bound $\sum_{i=1}^{N+1} Q^{(i)}$, we again use (4.19). Conditionally on $M>0$, (4.19) implies that the times between each successive pair of goodish loops in the Wloopspine will each be independently stochastically dominated by an $\exp \left(2 \kappa_{R}\right)$ random variable, which we denote by $\mathcal{E}_{R}$. Hence, the sum of the smaller jumps between each pair can be stochastically dominated by $\operatorname{Sub}_{\mathcal{E}_{R}}$, where $\operatorname{Sub}$ is a subordinator with Lévy measure

$$
c l^{-\alpha} \mathbb{1}_{\{l \leq 4 R\}} d l
$$

Also let $\mathcal{E}$ be an $\exp (2 \tilde{C})$ random variable (recall that $\left.\kappa_{R}=\tilde{C} R^{1-\alpha}\right)$. It further
follows by rescaling that

$$
\mathbf{P}\left(\sum_{i=1}^{N+1} Q^{(i)} \geq R \lambda^{2 \beta_{3}}\right) \leq \mathbf{P}\left(\sum_{i=1}^{N+1} \operatorname{Sub}_{\mathcal{E}_{R}}^{(i)} \geq R \lambda^{2 \beta_{3}}\right) \leq \mathbf{P}\left(\sum_{i=1}^{N+1} \operatorname{Sub}_{\mathcal{E}}^{(i)^{\prime}} \geq \lambda^{2 \beta_{3}}\right)
$$

where $\operatorname{Sub}^{(i)}$ are independent copies of Sub, and Sub ${ }^{(i)^{\prime}}$ are independent copies of a subordinator similar to Sub but with Lévy measure

$$
c l^{-\alpha} \mathbb{1}_{\{l \leq 4\}} d l
$$

It then follows by Lemma 4.2 .2 that there exists $\theta>0$ with $C_{\theta}:=\mathbf{E}\left[e^{\theta \operatorname{Sub}_{\mathcal{E}}}\right]<\frac{3}{2}$. For such $\theta$, we hence have

$$
\begin{align*}
\mathbf{P}\left(\sum_{i=1}^{N+1} Q^{(i)} \geq R \lambda^{2 \beta_{3}}\right) & =\sum_{n=1}^{\infty} \mathbf{P}\left(\sum_{i=1}^{N+1} \operatorname{Sub}_{\mathcal{E}}^{(i)^{\prime}} \geq R \lambda^{2 \beta_{3}} \mid N+1=n\right) \mathbf{P}(N+1=n) \\
& \leq \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n} e^{-\theta \lambda^{2 \beta_{3}}}\left(\frac{1}{2}\right)^{n} \\
& =C_{\theta}^{\prime \prime} e^{-\theta \lambda^{2 \beta_{3}}} . \tag{4.22}
\end{align*}
$$

To conclude, we combine the results of (4.20), (4.21) and (5.12) by writing

$$
\begin{aligned}
& \mathbf{P}\left(\left|I_{R}\right| \geq 3 R \lambda^{2 \beta_{3}}\right) \leq \mathbf{P}(M=0)+\mathbf{P}\left(N+1 \geq \lambda^{2 \beta_{3}} \mid M>0\right) \\
&+\mathbf{P}\left(\sum_{i=1}^{N+1} Q^{(i)} \geq R \lambda^{2 \beta_{3}} \mid M>0\right) \\
& \leq C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+C_{\theta}^{\prime} e^{-c \lambda^{2 \beta_{3}}}\right)
\end{aligned}
$$

The second technical lemma will allow us to bound the total progeny of $T$ by comparing it to a subcritical Galton-Watson tree with Poisson offspring distribution.

Lemma 4.2.10. Let $\tilde{\mathcal{T}}_{\alpha}$ be a compact stable tree, and $\tilde{\mathcal{L}_{\alpha}}$ be its corresponding compact stable looptree, both coded by the same excursion $\mathcal{E}$ under the Ito measure $N(\cdot)$ but conditioned to have lifetime $\zeta$ at least $\kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$. Let $\rho$ be the root of $\tilde{\mathcal{L}_{\alpha}}$, and perform a Williams' decomposition of $\tilde{\mathcal{L}_{\alpha}}$ along its $W$-loopspine. Let $N$ denote the number of resulting sublooptrees obtained that are of mass at least $\kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$ and are also grafted to the $W$-loopspine within distance $r$ of the root of $\tilde{\mathcal{L}_{\alpha}}$. Then
where $K_{\alpha}=3\left(\Gamma\left(1-\frac{1}{\alpha}\right)\right)^{-1} \kappa^{\frac{1}{\alpha}}$. The constants $c$ and $C$ also depend on $\kappa$, but $\kappa$ is fixed and the precise dependence will not be important, so we suppress this notationally.

Proof. Let $H$ be the height of $\tilde{\mathcal{T}}_{\alpha}$, and let $\mathcal{E}^{(H)}$ be the rescaled excursion given by

$$
\mathcal{E}^{(H)}=\left(H^{\frac{-1}{\alpha-1}} \mathcal{E}_{H^{\frac{\alpha}{\alpha-1}} t}\right)_{0 \leq t \leq H^{\frac{-\alpha}{\alpha-1}} \zeta}
$$

The excursion $\mathcal{E}^{(H)}$ codes a tree conditioned to have height 1 (this can be seen from combining [GH10, Lemma 5.8, Part 3] with [DW17, Equation (26)], for example). Moreover, in the corresponding looptree, $N$ now denotes the number of sublooptrees of mass at least $H^{\frac{-\alpha}{\alpha-1}} \kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$ that are grafted to the W-loopspine within distance $R:=H^{\frac{-1}{\alpha-1}} r$ of $\rho$.

We wish to bound $R$ so that we can apply Lemma 4.2.9. To do this, note by monotonicity and scaling invariance that

$$
\begin{aligned}
\mathbf{P}\left(R \geq \lambda^{-\beta_{4}} \mid \zeta \geq \kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right) & \leq \mathbf{P}\left(\left.H \leq \kappa^{\frac{\alpha-1}{\alpha}} \lambda^{\frac{-(\alpha-1)\left(1-\beta_{1}-\beta_{2}\right)}{\alpha}} \lambda^{\beta_{4}(\alpha-1)} \right\rvert\, \zeta=1\right) \\
& \leq C e^{-c \lambda^{1-\beta_{1}-\beta_{2}-\alpha \beta_{4}}},
\end{aligned}
$$

where the final line holds by [DW17, Theorem 1.8]. Then, conditioning on $R \leq \lambda^{-\beta_{4}}$ (i.e. $H \geq r^{\alpha-1} \lambda^{\beta_{4}(\alpha-1)}$ ), we have by Lemma 4.2 .9 that

$$
\begin{aligned}
\mathbf{P}\left(\left|I_{R}\right| \geq 3 R \lambda^{2 \beta_{3}} \mid R \leq \lambda^{-\beta_{4}}\right) & =\mathbf{P}\left(\left.\left|I_{R}\right| \geq 3 H^{\frac{-1}{\alpha-1}} r \lambda^{2 \beta_{3}} \right\rvert\, H \geq r^{\alpha-1} \lambda^{-\beta_{4}(\alpha-1)}\right) \\
& \leq C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2 \beta_{3}}}\right)
\end{aligned}
$$

By Theorem 4.2.5(ii), the sublooptrees grafted to the W-loopspine at points in $I_{R}$ form a Poisson process of sublooptrees coded by the Itô excursion measure, but thinned so that none have height large enough to violate the condition that the end of the W-spine corresponds to the point of maximal height in the tree. We can therefore stochastically dominate this by the classical, unthinned version of the Itô excursion measure of Section 2.5.3. Since $N(\zeta \geq t)=\hat{C}_{\alpha} t^{\frac{-1}{\alpha}}$, where $\hat{C}_{\alpha}=\left(\Gamma\left(1-\frac{1}{\alpha}\right)\right)^{-1}$ (e.g. see [GH10, Proposition 5.6$\left.]\right)$, it follows that conditionally on $\left|I_{R}\right| \leq 3 R \lambda^{2 \beta_{3}}=3 H^{\frac{-1}{\alpha-1}} r \lambda^{2 \beta_{3}}, N$ is stochastically dominated by a Poisson random variable with parameter:

$$
3 \hat{C}_{\alpha}\left(\kappa^{-1} H^{\frac{-\alpha}{\alpha-1}} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right)^{\frac{-1}{\alpha}} H^{\frac{-1}{\alpha-1}} r \lambda^{2 \beta_{3}}=3 \hat{C}_{\alpha} \kappa^{\frac{1}{\alpha}}
$$

To conclude, we write:

$$
\begin{aligned}
& \mathbf{P}(N \geq n) \leq \mathbf{P}\left(H \leq r^{\alpha-1} \lambda^{\beta_{4}(\alpha-1)} \mid \zeta \geq \kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right) \\
&+\mathbf{P}\left(\left.\left|I_{R}\right| \geq 3 H^{\frac{-1}{\alpha-1}} r \lambda^{2 \beta_{3}} \right\rvert\, H \geq r^{\alpha-1} \lambda^{-\beta_{4}(\alpha-1)}\right) \\
& \quad+\mathbb{P}\left(\operatorname{Poisson}\left(3 \hat{C}_{\alpha} \kappa^{\frac{1}{\alpha}} \lambda^{2 \beta_{3}-\frac{1}{\alpha}\left(1-\beta_{1}-\beta_{2}\right)}\right) \geq n\right) \\
& \leq C\left(e^{-c \lambda^{1-\beta_{1}-\beta_{2}-\alpha \beta_{4}}}+e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2 \beta_{3}}}\right)+\mathbb{P}\left(\operatorname{Poisson}\left(3 \hat{C}_{\alpha} \kappa^{\frac{1}{\alpha}}\right) \geq n\right) .
\end{aligned}
$$

Armed with these lemmas, there are now two key steps to the main argument. One of these is to bound the number of times we need to reiterate around larger sublooptrees as described by the algorithm, and the other is to bound the contributions of smaller terms from each of these iterations.

As is usual, we will let $|T|$ denote the total progeny of the tree $T$. The first main result is the following.

Proposition 4.2.11. There exist constants $c, C \in(0, \infty)$ such that

$$
\mathbf{P}\left(|T| \geq \lambda^{\beta_{1}}\right) \leq C \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-c \lambda^{\frac{\alpha-1}{4 \alpha-3}}}
$$

Proof. The main ingredient in this proof is the main theorem of Dwass from [Dwa69], that for a Galton-Watson tree with total progeny Prog and offspring distribution $\xi$, it holds that

$$
\mathbb{P}(\operatorname{Prog}=k)=\frac{1}{k} \mathbb{P}\left(\sum_{i=1}^{k} \xi^{(i)}=k-1\right),
$$

where the $\xi^{(i)}$ are i.i.d. copies of $\xi$. In particular, if $\xi \sim$ Poisson $(\theta)$ for some $\theta<\frac{1}{e^{2}}$ we see by writing the sum explicitly and applying Stirling's formula that

$$
\begin{equation*}
\mathbb{P}(\text { Prog } \geq k)=\sum_{j \geq k} \frac{1}{j} \mathbb{P}(\text { Poisson }(j \theta)=j-1) \leq \frac{c}{\theta} \sum_{j \geq k} j^{\frac{-3}{2}}(e \theta)^{j} \leq \frac{c}{\theta} k^{\frac{-3}{2}}(e \theta)^{k} . \tag{4.23}
\end{equation*}
$$

This isn't a priori applicable since in our case $T$ is not quite a Galton-Watson tree. However, it follows from Lemma 4.2.9 that for any $k>0$, we have

$$
\mathbf{P}(|T| \geq k) \leq k\left[C e^{-c \lambda^{1-\beta_{1}-\beta_{2}-\alpha \beta_{4}}}+C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2 \beta_{3}}}\right)\right]+\mathbb{P}\left(\left|T^{\prime}\right| \geq k\right),
$$

where $T^{\prime}$ is a Galton-Watson tree with Poisson $\left(K_{\alpha}\right)$ offspring distribution. Accordingly, setting $\theta=K_{\alpha}$ (which is less than $\frac{1}{e^{2}}$ by our choice of $\kappa$ ) and $k=\lambda^{\beta_{1}}$ we see that

$$
\mathbb{P}\left(\left|T^{\prime}\right| \geq k\right) \leq C e^{-\lambda^{\beta_{1}}}
$$

so combining with the above we deduce that

$$
\mathbb{P}\left(|T| \geq \lambda^{\beta_{1}}\right) \leq \lambda^{\beta_{1}}\left[C e^{\left.-c \lambda^{1-\beta_{1}-\beta_{2}-\alpha \beta_{4}}+C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2 \beta_{3}}}\right)\right]+C e^{-\lambda^{\beta_{1}}},, ~}\right.
$$

which gives the result on substituting for the $\beta_{i}$.
Proposition 4.2.12. Conditional on $|T| \leq \lambda^{\beta_{1}}$, we have that

$$
\mathbf{P}\left(\exists u \in T: S_{u} \geq r^{\alpha} \lambda^{1-\beta_{1}}\right) \leq C \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-c \lambda^{\frac{\alpha-1}{4 \alpha-3}}}
$$

Proof. Take $u \in T$, and let $L_{u}$ be the corresponding (sub)looptree that forms part of $\mathcal{L}_{\alpha}$. By the same arguments used in Proposition 4.2.11, we can use Lemma 4.2.9 to show that, letting $R=H^{\frac{-\alpha}{\alpha-1}} r$, we have

$$
\begin{aligned}
\mathbf{P}\left(\left|I_{R}\right| \geq 3 R \lambda^{2 \beta_{3}}\right) & \leq \mathbf{P}\left(R \geq \lambda^{-\beta_{4}}\right)+\mathbf{P}\left(\left|I_{R}\right| \geq 3 R \lambda^{2 \beta_{3}} \mid R \geq \lambda^{-\beta_{4}}\right) \\
& \leq C e^{-c \lambda^{1-\beta_{1}-\beta_{2}-\alpha \beta_{4}}+C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2 \beta_{3}}}\right)}
\end{aligned}
$$

We now condition on $\left\{\left|I_{R}\right| \geq 3 R \lambda^{2 \beta_{3}}\right\}$. Again dominating the thinned Itô excursion measure by the classical Itô excursion measure as we did in the proof of Proposition 4.2.12, we have that $H^{\frac{-\alpha}{\alpha-1}} S_{u}$ is stochastically dominated by a subordinator with Lévy measure $C_{\alpha} x^{\frac{-1}{\alpha}-1} \mathbb{1}\left\{x \leq H^{\frac{-\alpha}{\alpha-1}} \kappa^{-1} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right\} d x$, run up until the time $3 H^{\frac{-1}{\alpha-1}} r \lambda^{2 \beta_{3}}$. Note that the Lévy measure coincides with that of an $\alpha^{-1}$-stable subordinator, conditioned to have no jumps greater than $\kappa^{-1} H^{\frac{-\alpha}{\alpha-1}} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}$.

Hence, letting Subord be an $\alpha^{-1}$-stable subordinator, and conditioning on $\left|I_{R}\right| \leq 3 R \lambda^{2 \beta_{3}}$, we have by scaling invariance that:
$\mathbf{P}\left(S_{u} \geq r^{\alpha} \lambda^{1-\beta_{1}}| | I_{R} \mid \leq R \lambda^{2 \beta_{3}}\right)$
$=\mathbf{P}\left(\left.H^{\frac{-\alpha}{\alpha-1}} S_{u} \geq H^{\frac{-\alpha}{\alpha-1}} r^{\alpha} \lambda^{1-\beta_{1}}| | I_{R} \right\rvert\, \leq R \lambda^{2 \beta_{3}}\right)$
$\leq \mathbf{P}\left(\right.$ Subord $\left.{ }_{H^{\frac{-1}{\alpha-1}} r \lambda^{2 \beta_{3}}} \geq H^{\frac{-\alpha}{\alpha-1}} r^{\alpha} \lambda^{1-\beta_{1}} \right\rvert\,$ no jumps more than $\left.\kappa^{-1} H^{\frac{-\alpha}{\alpha-1}} r^{\alpha} \lambda^{1-\beta_{1}-\beta_{2}}\right)$ $\leq \mathbf{P}\left(\right.$ Subord $_{1} \geq \kappa \lambda^{1-\beta_{1}-2 \beta_{3} \alpha} \mid$ no jumps more than 1$)$.

By the arguments of Lemma 4.2.2, it follows that there exists $\theta>0$ such that $\mathbf{E}\left[e^{\theta \text { Subord }_{1}}\right]<\infty$ when conditioned to have no jumps greater than 1 , so, as before, the latter probability can be bounded by $C e^{-c \lambda^{1-\beta_{1}-2 \beta_{3} \alpha}}$.

Combining these, we see that
$\mathbf{P}\left(S_{u} \geq r^{\alpha} \lambda^{1-\beta_{1}}\right) \leq C e^{-c \lambda^{1-\beta_{1}-\beta_{2}-\alpha \beta_{4}}}+C\left(e^{-c \lambda^{\beta_{4}(\alpha-1)}}+e^{-c \lambda^{2} \beta_{3}}\right)+C e^{-c \lambda^{1-\beta_{1}-2 \beta_{3} \alpha}}$.
The result follows on taking a union bound and substituting the value of each $\beta_{i}$.

We are now able to prove Proposition 4.2.8.
Proof of Proposition 4.2.8. Note that, on the event $\left\{|T| \leq \lambda^{\beta_{1}}, S_{u} \leq r^{\alpha} \lambda^{1-\beta_{1}} \forall u \in\right.$ $T\}$, we have that

$$
\nu(B(\rho, r)) \leq \sum_{u \in T} S_{u} \leq|T| \sup _{u \in T} S_{u} \leq \lambda^{\beta_{1}} r^{\alpha} \lambda^{1-\beta_{1}}=r^{\alpha} \lambda .
$$

Hence, by combining the results of Propositions 4.2.11 and 4.2.12, we see that

$$
\begin{align*}
\mathbf{P}\left(\nu(B(\rho, r)) \geq r^{\alpha} \lambda\right) & \leq \mathbf{P}\left(|T| \geq \lambda^{\beta_{1}} \text { or } S_{u} \geq r^{\alpha} \lambda^{1-\beta_{1}} \text { for some } u \in T\right) \\
& \leq C \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-c \lambda^{\frac{\alpha-1}{4 \alpha-3}}} . \tag{4.24}
\end{align*}
$$

### 4.3 Attaining extremal volumes

In this section, we show that up to logarithmic and log-logarithmic factors, the extremal volume bounds of Theorems 4.0.4 and 4.0.5 are attained, namely proving (4.3), (4.4), (4.7) and (4.4).

In order to apply the second Borel-Cantelli lemma to prove these results, we will need a level of independence across different parts of the looptree. We achieve this by using a spinal decomposition result which enables us to split $\mathcal{L}_{\alpha}$ into a series of smaller sublooptrees, which are independent of each other after rescaling. We detail this decomposition below.

### 4.3.1 Spinal decomposition from the root to a uniform point

In [HPW09], it was shown that if we define the spine of a stable Lévy tree $\mathcal{T}_{\alpha}$ to be the unique path from the root to a uniform point, then $\mathcal{T}_{\alpha}$ can be broken along this spine and that the resulting fragments form a collection of smaller Lévy trees. This gives a similar decomposition result for looptrees.

We define the decomposition formally as follows. Let $U \sim$ Uniform $([0,1])$, so that $p(U)$ is a uniformly chosen vertex in $\mathcal{L}_{\alpha}$, and let $\rho$ be its root. We say that the loopspine from $\rho$ to $p(U)$, denoted $S_{U}$, is the closure of the set of loops corresponding to ancestors of $U$. To form the fine spinal decomposition, first let $\left(L_{i}^{o}\right)_{i=1}^{\infty}$ be the connected components of $\mathcal{L}_{\alpha} \backslash S_{U}$, and then for each $i \in \mathbb{N}$ let $L_{i}$ be the closure of $L_{i}^{o}$ in $\mathcal{L}_{\alpha}$. Then almost surely, each $L_{i}$ can be written in the form $L_{i}^{o} \dot{\cup} \rho_{i}$ for some $\rho_{i} \in \mathcal{L}_{\alpha} \backslash L_{i}^{o}$. Note that by uniform rerooting invariance, we can also replace the root with an independent uniform point in $\mathcal{L}_{\alpha}$.

If the fragment $L_{i}$ has mass $\alpha_{i}$, define a metric $d_{i}$ and a measure $\nu_{i}$ on $L_{i}$ by

$$
d_{i}=\left.\alpha_{i}^{\frac{-1}{\alpha}} d\right|_{L_{i}}, \quad \nu_{i}=\frac{\nu\left(\cdot \cap L_{i}\right)}{\alpha_{i}} .
$$

Additionally let $p\left(U_{i}\right)$ be a vertex in $L_{i}$ chosen uniformly according to $\nu_{i}$. We then have the following result, which is a consequence of [HPW09, Corollary 10], which gives the corresponding result for Lévy trees.

Theorem 4.3.1. $\left\{\left(L_{i}, d_{i}, \nu_{i}, \rho_{i}, p\left(U_{i}\right)\right)\right\}_{i \in \mathbb{N}}$ is a collection of independent copies of $\left(\mathcal{L}_{\alpha}, d, \rho, \nu, p(U)\right)$. Moreover, the entire family is independent of $\left(\alpha_{i}\right)_{i \in \mathbb{N}}$, which has a Poisson-Dirichlet $\left(\alpha^{-1}, 1-\alpha^{-1}\right)$ distribution.

In the looptree case, we can add some further information about the arrangement of the sublooptrees around the loopspine. It follows as a direct consequence of equation (11) and the paragraph following it in [DLG05] that they are distributed uniformly around the loopspine (in the natural way with respect to the "length" of the loopspine).

## Representations of the Poisson-Dirichlet distribution

In order to apply Theorem 4.3.1, we will use the following constructive characterisation of the Poisson-Dirichlet distribution. This construction is the GEM construction of [Ewe90], so named after Griffiths, Engen and McCloskey.

Firstly, let $\left(Z_{k}\right)_{k=1}^{\infty}$ be a sequence of independent beta random variables with respective parameters $\left(1-\alpha^{-1}, 1+(k-1) \alpha^{-1}\right)$. Set

$$
M_{k}=\left(1-Z_{1}\right)\left(1-Z_{2}\right) \ldots\left(1-Z_{k-1}\right) Z_{k} .
$$

Then the random vector $\left(M_{1}, M_{2}, \ldots\right)$ is distributed as a size-biased ordering of a Poisson-Dirichlet $\left(\alpha^{-1}, 1-\alpha^{-1}\right)$ random variable. See [PY97, Proposition 4] or [Pit96] for a proof.

We will use the following lemma in Section 4.3.2 to prove lower bounds on extremal supremal volume values.

Lemma 4.3.2. Let $\left(M_{1}, M_{2}, \ldots\right)$ be a sequence of random variables constructed via a stick-breaking construction from an independent sequence of random variables $\left(Z_{1}, Z_{2}, \ldots\right)$, each taking values in $[0,1]$; that is

$$
\begin{aligned}
& M_{1}=Z_{1}, \\
& M_{2}=\left(1-Z_{1}\right) Z_{2}, \\
& M_{n}=\left(1-Z_{1}\right)\left(1-Z_{2}\right) \ldots\left(1-Z_{n-1}\right) Z_{n}
\end{aligned}
$$

for all $n \geq 1$. Let $(g(n))_{n \geq 1}$ be any sequence of numbers taking values in $[0,1]^{\mathbb{N}}$. Then

$$
\mathbb{P}\left(M_{n} \geq g(n) \mid M(l)<g(l) \forall l<n\right) \geq \mathbb{P}\left(M_{n} \geq g(n)\right) .
$$

Proof. This is immediate on noting that $M_{n}=\left(1-\sum_{i=1}^{n-1} M_{i}\right) Z_{n}$.
Lemma 4.3.3. Letting $M_{k}$ be the $k^{\text {th }}$ GEM random variable, we have for any $c^{\prime}>0$ that:

$$
\mathbb{P}\left(M_{k} \geq c^{\prime} k^{-\alpha}\right) \geq c^{-1}\left(1-c^{\prime}\right)^{2} .
$$

Proof. The proof is an application of the Paley-Zigmund inequality, which says that for any non-negative random variable $X$ with finite variance, and any $\theta \in[0,1]$,

$$
\begin{equation*}
\mathbb{P}(X \geq \theta \mathbb{E}[X]) \geq(1-\theta) \frac{\mathbb{E}[X]^{2}}{\mathbb{E}\left[X^{2}\right]} \tag{4.25}
\end{equation*}
$$

By taking $X=M_{k}$, we have (recalling that $M_{k}=Z_{k} \prod_{i=1}^{k-1}\left(1-Z_{i}\right)$, where $Z_{i}$ are independent $\operatorname{Beta}\left(1-\alpha^{-1}, 1+(i-1) \alpha^{-1}\right)$ random variables) that:

$$
\begin{aligned}
\mathbb{E}\left[M_{k}\right]=\mathbb{E}\left[Z_{k}\right] \prod_{i=1}^{k-1} \mathbb{E}\left[1-Z_{i}\right] & \geq \frac{1-\alpha^{-1}}{2+(k-2) \alpha^{-1}} \prod_{i=1}^{k-1} \frac{1+(i-1) \alpha^{-1}}{2+(i-2) \alpha^{-1}} \\
& =\frac{\alpha-1}{k+2(\alpha-1)} \prod_{i=1}^{k-1} \frac{i+\alpha-1}{i+2(\alpha-1)} \\
& \geq\left(\frac{3}{2}\right)^{\alpha} k^{-1} k^{-(\alpha-1)} \\
& =\left(\frac{3}{2}\right)^{\alpha} k^{-\alpha}
\end{aligned}
$$

whenever $k \geq K$, say, and similarly

$$
\begin{aligned}
\mathbb{E}\left[M_{k}^{2}\right] & =\mathbb{E}\left[Z_{k}^{2}\right] \prod_{i=1}^{k-1} \mathbb{E}\left[\left(1-Z_{i}\right)^{2}\right] \\
& \leq \frac{\alpha-1}{(3 \alpha+k-2)(2 \alpha+k-2)} \prod_{i=1}^{k-1} \frac{2 \alpha+i-1}{3 \alpha+i-2} \frac{\alpha+i-1}{2 \alpha+i-2} \\
& =\frac{\alpha-1}{(3 \alpha+k-2)(2 \alpha+k-2)} \frac{2 \alpha+k-2}{2 \alpha-2} \prod_{i=1}^{k-1} \frac{\alpha+i-1}{3 \alpha+i-2} \\
& \leq c k^{-2} k k^{-(2 \alpha-1)} \\
& =c k^{-2 \alpha} .
\end{aligned}
$$

It therefore follows from (4.25) that

$$
\mathbb{P}\left(M_{k} \geq c^{\prime} k^{-\alpha}\right) \geq \mathbb{P}\left(M_{k} \geq c^{\prime}\left(\frac{3}{2}\right)^{\alpha} k^{-\alpha}\right) \geq c^{-1}\left(1-c^{\prime}\right)^{2}
$$

### 4.3.2 Supremal lower bounds

We now prove (4.3) and (4.7), starting with a probabilistic bound. The proof relies on using the relation (4.13) to compare volume fluctuations with stable Lévy oscillations.

Proposition 4.3.4. There exist constants $c, C \in(0, \infty)$ such that for all $r<1$ and all $\lambda>1$,

$$
\mathbf{P}\left(\nu\left(B\left(p(U), \frac{1}{2} r\right)\right) \geq r^{\alpha} \lambda\right) \geq C e^{-c \lambda}
$$

Proof. As explained in Section 4.2.1, we know that

$$
\left\{\operatorname{Osc}_{\left[p(U), p(U)+r^{\alpha} \lambda\right]} X^{\mathrm{exc}} \leq r\right\} \subset\left\{\nu(B(p(U), r)) \geq r^{\alpha} \lambda\right\}
$$

It follows from the relation $p_{t}(x)=t^{\frac{-1}{\alpha}} p_{1}\left(x t^{\frac{-1}{\alpha}}\right)$ that $\frac{p_{1-r^{\alpha} \lambda_{r}}(r)}{p_{1}(0)} \wedge \frac{p_{1-r^{\alpha} \lambda_{r}}(-r)}{p_{1}(0)} \rightarrow 1$ as $r \downarrow 0$ whenever $\lambda_{r}=o\left(r^{-\alpha}\right)$. Consequently, by applying the Vervaat transform and the absolute continuity relation, we have for all sufficiently small values of $r$ that

$$
\begin{aligned}
\mathbf{P}\left(\nu(B(p(U), r)) \geq r^{\alpha} \lambda\right) & \geq \mathbf{P}\left(\operatorname{Osc}_{\left[p(U), p(U)+r^{\alpha} \lambda\right]} X^{\operatorname{exc}} \leq r\right) \\
& \geq\left\{\frac{p_{1-r^{\alpha} \lambda}(r)}{p_{1}(0)} \wedge \frac{p_{1-r^{\alpha} \lambda}(-r)}{p_{1}(0)}\right\} \mathbf{P}\left(\operatorname{Osc}_{\left[0, r^{\alpha} \lambda\right]} X \leq r\right) \\
& \geq \frac{1}{2} \mathbf{P}\left(T_{[-1,1]}^{0}>2^{\alpha} \lambda\right),
\end{aligned}
$$

where $T_{I}^{x}$ denotes the exit time of $X$ from the interval $I$, conditioned on $X_{0}=x$.
It follows from the discussion below [Ber97, Theorem 2] that there exist deterministic constants $c_{1}, c_{2}$ such that $\mathbf{P}\left(T_{[-1,1]}^{0}>2^{\alpha} \lambda\right) \sim c_{1} e^{-c_{2} \lambda}$. The proposition follows.

We cannot directly use Proposition 4.3 .4 to prove the lower supremal bounds since we do not have the necessary independence to immediately apply the second Borel-Cantelli lemma. However, we can achieve this by performing a spinal decomposition, and considering volumes in different fragments, which are independent of each other. To do this, we will use a spinal decomposition to a uniform point, detailed below. The advantage of this over the Williams' decomposition in this case is that it allows us to control the masses of individual fragments more explicitly.

## Main argument for supremal lower bound

Using Theorem 4.3.1, we can construct an argument as follows. First take some $\varepsilon>0$ with $0<\varepsilon \ll 1$. Given $r \in(0,1)$, and $\lambda_{r}$ a decreasing function of $r$ such that $\frac{\lambda_{r}}{\lambda_{2 r}} \rightarrow 1$ as $r \downarrow 0$, define the interval $J_{r}=\left[r^{-1} \lambda_{r}^{\frac{-(1+\varepsilon)}{\alpha}}, \frac{3}{2} r^{-1} \lambda_{r}^{\frac{-(1+\varepsilon)}{\alpha}}\right]$. It is easy to verify that for all sufficiently small $r$, the intervals $J_{r}$ and $J_{2 r}$ are disjoint.

Our strategy is as follows. We use the spinal decomposition of Section 4.3.1, between $p(U)$ and an independent uniform point $p(V)$. Recall the GEM distribution introduced there, that gives a size biased representation ( $M_{1}, M_{2}, \ldots$ ) of the PoissonDirichlet distribution. Letting $I_{r}^{\prime}$ denote the segment of loopspine that intersects $B(p(U), r)$ (analagously to $I_{R}$ defined in Section 4.2.2), there is probability of order at least $\lambda^{-\left(\frac{1}{\alpha}+\varepsilon\right)}$ that there is a $n \in J_{r}$ such that the sublooptree with PoissonDirichlet mass given by the GEM random variable $M_{n}$ is grafted to the loopspine at a point in $I_{\frac{r}{2}}^{\prime}$. Say this sublooptree is $L_{i, r}$, with root $\rho_{i}$ being the point at which it is grafted to the loopspine. The mass of the ball $B(p(U), r)$ is then lower bounded by the mass of $B\left(\rho_{i}, \frac{1}{2} r\right) \cap L_{i, r}$. We can then rescale the looptree $L_{i, r}$, and the corresponding unit ball, to compute that this mass is at least $r^{\alpha} \lambda$ with at least polynomial probability. We repeat this argument along the sequence $r_{n}=2^{-n}$. Since the corresponding intervals $J_{r_{n}}$ are disjoint (provided we start at a sufficiently large value of $n$ ), and the rescaled looptrees from the spinal decomposition of Section 4.3.1 are independent, we obtain the necessary independence to apply the second Borel-Cantelli Lemma.

Proof of supremal lower bound in Theorem 4.0.5. Let

$$
L=\sum_{U \wedge V \prec t \preceq U} \Delta_{t}+\sum_{U \wedge V \prec t \preceq V} \Delta_{t}+\delta_{U \wedge V}\left(x_{U \wedge V}^{U}, x_{U \wedge V}^{V}\right)
$$

be the length of the loopspine, and let $N_{r}$ be the total number of sublooptrees in the spinal decomposition that are incident to the loopspine at a point in $I_{\frac{r}{2}}^{\prime}$ and have mass corresponding to a GEM index in $J_{r}$. Then, conditional on $L=l, N_{r}$ stochastically dominates a random variable that is $\operatorname{Binomial}\left(\left\lfloor\frac{1}{2} r^{-1} \lambda_{r}^{\frac{-(1+\varepsilon)}{\alpha}}\right\rfloor, r l^{-1}\right)$. Hence, the probability that this number is non-zero is at least of order $l^{-1} \lambda_{r}^{\frac{-(1+\varepsilon)}{\alpha}}$.

Conditional on $\left\{N_{r} \geq 1\right\}$, let $n_{r}$ be an index in $J_{r}$ with corresponding sublooptree $L_{r}$ that is incident to the loopspine at a point in $I_{\left(\frac{r}{2}\right)}^{\prime}$. Note that $\nu\left(L_{r}\right)$ stochastically dominates the Poisson-Dirichlet GEM weight $M_{k_{r}}$, where $k_{r}=$ $\frac{3}{2} r^{-1} \lambda_{r}^{\frac{-(1+\varepsilon)}{\alpha}}$, and hence we have by Lemma 4.3.3 that there exists $c_{p}>0$ such that

$$
\mathbf{P}\left(\nu\left(L_{r}\right) \geq \frac{1}{2} r^{\alpha} \lambda_{r}^{1+\varepsilon}\right) \geq \mathbf{P}\left(M_{k_{r}} \geq \frac{1}{2} r^{\alpha} \lambda_{r}^{1+\varepsilon}\right) \geq c_{p} .
$$

Conditional on there being such a sublooptree $L_{r}$, say of mass $m \geq \frac{1}{2} r^{\alpha} \lambda_{r}^{1+\varepsilon}$, we know that

$$
\mathbf{P}\left(\nu\left(B\left(\rho_{i}, \frac{1}{2} r\right) \cap L_{i, r}\right) \geq r^{\alpha} \lambda_{r}\right)=\mathbf{P}\left(\nu\left(B\left(\rho, \frac{1}{2} r m^{\frac{-1}{\alpha}}\right)\right) \geq m^{-1} r^{\alpha} \lambda_{r}\right) \geq C e^{-c \lambda_{r n}}
$$

by Proposition 4.3 .4 (note in particular that $m^{-1} r^{\alpha} \lambda_{r} \leq 2 \lambda_{r}^{-\varepsilon} \rightarrow 0$ as $r \downarrow 0$ so it is fine to apply the result here).

Hence, letting $A_{r}$ be the event that there exists a sublooptree incident to the loopspine at a point in $I_{\frac{r}{2}}^{\prime}$ with GEM index $n_{r} \in J_{r}$, and such that the ball of radius $\frac{1}{2} r$ in this sublooptree has mass at least $r^{\alpha} \lambda_{r}$, we deduce that $\mathbf{P}\left(A_{r}\right) \geq$ $C l^{-1} \lambda_{r}^{\frac{-(1+\varepsilon)}{\alpha}} e^{-c \lambda_{r}}$.

Now, letting $r_{n}=2^{-n}$, we have that there exists a finite $N$ such that the intervals $J_{r_{n}}$ and $J_{r_{m}}$ are disjoint whenever $m, n \geq N$, and hence since each sublooptree is distributed uniformly around the perimeter of the loopspine independently of the others, then the events that there exist sublooptrees with GEM index in $J_{r_{n}}$ (respectively $J_{r_{m}}$ ) within distance $\frac{1}{2} r_{n}$ (respectively $\frac{1}{2} r_{m}$ ) from the root are independent events. Moreover, if the sublooptrees described in the events $A_{r_{n}}$ and $A_{r_{m}}$ exist, then they are independent once rescaled by Proposition 4.3.1. Thus the only dependence between the events $A_{r_{n}}$ and $A_{r_{m}}$ is in whether these sublooptrees have masses greater than $c^{\prime} r_{n}^{\alpha} \lambda_{r_{n}}^{1+\varepsilon}$ (respectively $c^{\prime} r_{m}^{\alpha} \lambda_{r_{m}}^{1+\varepsilon}$ ), but here the only dependence is that all the Poisson-Dirichlet masses must sum to 1 , and hence in actual fact $\mathbf{P}\left(A_{r_{n}} \mid A_{r_{m}}^{c}\right.$ for all $\left.N \leq m<n\right) \geq \mathbf{P}\left(A_{r_{n}}\right)$ by an application of Lemma 4.3.2.

Hence,

$$
\sum_{n=N}^{\infty} \mathbf{P}\left(A_{r_{n}} \mid A_{r_{m}}^{c} \text { for all } N \leq m<n\right) \geq \sum_{n=N}^{\infty} \mathbf{P}\left(A_{r_{n}}\right) \geq \sum_{n=N}^{\infty} C l^{-1} \lambda_{r_{n}}^{\frac{-(1+\varepsilon)}{\alpha}} e^{-c \lambda}
$$

so setting $\lambda_{r}=\frac{C_{2}^{-1}}{2}\left(\log \log r^{-1}\right)$ we see that $\mathbf{P}\left(A_{r_{n}}\right.$ i.o. $\left.\mid \operatorname{Length}\left(S_{\sigma}\right)=l\right)=1$. Since $L$ is almost surely finite, we can integrate over possible values of $l$ to deduce that $\mathbf{P}\left(A_{r_{n}}\right.$ i.o. $)=1$.

The result at (4.7) follows by applying Fubini's theorem similarly to the previous extremal volume bounds.

The proof of the global bound given in (4.3) uses a similar decomposition approach, but this time we perform two subsequent spinal decompositions. This is illustrated in Figure 4.1. Firstly, let $\left(M_{1}, M_{2}, \ldots\right)$ denote the GEM masses obtained on performing a first spinal decomposition of $\mathcal{L}_{\alpha}$. Then, for each of the resulting fragments $\left(L_{1}, L_{2}, \ldots\right)$, rescale to obtain a sequence of independent stable looptrees $\left(\mathcal{L}_{\alpha}^{1}, \mathcal{L}_{\alpha}^{2}, \ldots\right)$, each with mass 1 . For each $n \in \mathbb{N}$, we perform a further spinal decomposition of $\mathcal{L}_{\alpha}^{n}$ and denote the resulting GEM masses by $\left\{M_{n, 1}, M_{n, 2}, \ldots\right\}$, and


Figure 4.1: Illustration of the Double Decomposition. NB Between any two loops, there are countably many more loops, so loops do not really touch each other like this.
corresponding looptrees by $\left\{L_{n, 1}, L_{n, 2}, \ldots\right\}$, and by $\left\{\mathcal{L}_{\alpha}^{n, 1}, \mathcal{L}_{\alpha}^{n, 2}, \ldots\right\}$ after rescaling again to have mass 1. We take $r_{n}=2^{-n}, R_{n}=M_{n}^{\frac{-1}{\alpha}} r_{n}, \lambda_{n}=C^{*} \log r_{n}^{-1}$ and $\lambda_{n}^{\prime}=C^{*} \log R_{n}^{-1}$, where $C^{*}$ is a specific constant to be specified later. We also define the events:

$$
\begin{aligned}
& B_{n}=\left\{r_{n}^{\alpha} \leq M_{n}^{2}\right\}, \\
& C_{n, m}=\left\{M_{n, m} \geq R_{n}^{\alpha} \lambda_{n}\right\}, \\
& D_{n, m}=\left\{\nu\left(B\left(\rho_{n, m}, R_{n}\right) \cap L_{n, m}\right) \geq R_{n}^{\alpha} \lambda_{n}\right\}, \quad A_{n, m}=C_{n, m} \cap D_{n, m} .
\end{aligned}
$$

Also set $N_{n}=2^{\frac{-1}{\alpha}} r_{n}^{-1}\left(\log r_{n}^{-1}\right)^{\frac{-1}{\alpha}}$, and define the event

$$
A_{n}=B_{n} \cap\left(\bigcup_{m=1}^{N_{n}} A_{n, m}\right)
$$

The key point is to observe that $A_{n} \subset\left\{\sup _{u \in \mathcal{L}_{\alpha}} \nu\left(B\left(u, r_{n}\right)\right) \geq r_{n}^{\alpha} \lambda_{n}\right\}$, so it is sufficient to show that $\mathbf{P}\left(A_{n}\right.$ i.o. $)=1$. The next lemma gives a means to overcome the dependencies between the GEM masses and apply the second Borel-Cantelli Lemma. It should be intuitively clear, but we give a proof for completeness.

Lemma 4.3.5. Let $A_{n}, B_{n}, A_{n, m}, C_{n, m}, D_{n, m}, N_{n}$ be as above. Then
(i) $\mathbf{P}\left(A_{n} \mid A_{m}^{c} \forall m<n\right) \geq \mathbf{P}\left(A_{n}\right)$,
(ii) $\mathbf{P}\left(A_{n, m} \mid A_{n, l}^{c} \forall l<m\right) \geq \mathbf{P}\left(A_{n, m}\right)$.

Proof. First, note that since the individual looptrees in the spinal decomposition are independent of each other and of their original masses once rescaled, we can make the following observations:

- $A_{n, m}$ is independent of $B_{l}$ for all $m$ and all $l \leq n$,
- $B_{n}$ is independent of $A_{l, m}$ for all $m$ and all $l \leq n$,
- Conditional on $C_{n, l}, D_{n, l}$ is independent of $D_{n, k}$ for all $k<l$.

Figure 4.1 may be helpful to keep track of the dependencies. In fact, the only dependence between these events is of the form described by Lemma 4.3.2.

We start by proving (i). First note that by the first independence stated above, we have that

$$
\begin{align*}
& \mathbf{P}\left(A_{n} \mid A_{m}^{c} \forall m<n\right) \\
& =\mathbf{P}\left(B_{n} \cap\left(\bigcup_{m=1}^{N_{n}} A_{n, m}\right) \mid A_{m}^{c} \forall m<n\right) \\
& =\mathbf{P}\left(B_{n} \mid A_{m}^{c} \forall m<n\right) \mathbf{P}\left(\bigcup_{m=1}^{N_{n}} A_{n, m} \mid A_{m}^{c} \forall m<n\right)  \tag{4.26}\\
& =\mathbf{P}\left(B_{n} \mid A_{m}^{c} \forall m<n\right) \mathbf{P}\left(B_{n} \mid A_{m}^{c} \forall m<n\right) \mathbf{P}\left(\bigcup_{m=1}^{N_{n}} A_{n, m} \mid A_{m}^{c} \forall m<n\right) .
\end{align*}
$$

We focus on the first term in the final line above. By the second independence stated above, we have that

$$
\begin{aligned}
& \mathbf{P}\left(B_{n} \mid A_{l}^{c} \forall l<n\right) \\
& =\mathbf{P}\left(B_{n} \mid\left(\cup_{m} A_{l, m}\right)^{c} \sqcup\left(\left(\cup_{m} A_{l, m}\right) \cap B_{l}^{c}\right) \forall l<n\right) \\
& =\sum_{\omega \in\{0,1\}^{n-1}} \mathbf{P}\left(B_{n} \mid E(\omega)\right) \mathbf{P}\left(E(\omega) \mid\left(\cup_{m} A_{l, m}\right)^{c} \sqcup\left(\left(\cup_{m} A_{l, m}\right) \cap B_{l}^{c}\right) \forall l<n\right) \\
& =\sum_{\omega \in\{0,1\}^{n-1}} \mathbf{P}\left(B_{n} \mid E^{\prime}(\omega)\right) \mathbf{P}\left(E(\omega) \mid\left(\cup_{m} A_{l, m}\right)^{c} \sqcup\left(\left(\cup_{m} A_{l, m}\right) \cap B_{l}^{c}\right) \forall l<n\right),
\end{aligned}
$$

where for $\omega \in\{0,1\}^{n-1}$ :

$$
\begin{aligned}
E(\omega) & =\left(\bigcup_{l: \omega_{l}=1}\left(\cup_{m} A_{l, m}\right)^{c}\right) \cap\left(\bigcup_{l: \omega_{l}=0}\left(\left(\cup_{m} A_{l, m}\right) \cap B_{l}^{c}\right)\right) \\
E^{\prime}(\omega) & =\left(\bigcup_{l: \omega_{l}=0}\left(\left(\cup_{m} A_{l, m}\right) \cap B_{l}^{c}\right)\right) .
\end{aligned}
$$

Since $B_{n}$ is independent of $\cup_{m} A_{l, m}$, we can apply Lemma 4.3.2 to deduce that $\mathbf{P}\left(B_{n}\right) \geq \mathbf{P}\left(B_{n} \mid E^{\prime}(\omega)\right)$ for all $\omega$. Substituting this into the final line, we obtain

$$
\begin{aligned}
\mathbf{P}\left(B_{n} \mid A_{l}^{c} \forall l<n\right) & \geq \sum_{\omega \in\{0,1\}^{n-1}} \mathbf{P}\left(B_{n}\right) \mathbf{P}\left(E(\omega) \mid\left(\cup_{m} A_{l, m}\right)^{c} \sqcup\left(\left(\cup_{m} A_{l, m}\right) \cap B_{l}^{c}\right) \forall l<n\right) \\
& =\mathbf{P}\left(B_{n}\right) .
\end{aligned}
$$

We can use the same kind of expansion and apply Lemma 4.3.2 to show that

$$
\mathbf{P}\left(\bigcup_{m=1}^{N_{n}} A_{n, m} \mid A_{m}^{c} \forall m<n\right) \geq \mathbf{P}\left(\bigcup_{m=1}^{N_{n}} A_{n, m}\right) .
$$

Point ( $i$ ) then follows from the final line of (4.26). The proof of the point $(i i)$ is almost identical, so we omit it.

Armed with the lemma, we prove the global infimum upper bound as follows.
Proof of supremal lower bound in Theorem 4.0.4. Recall from Theorem 4.3.1 that the rescaled looptree $\mathcal{L}_{\alpha}^{n}$ is independent of $M_{n}$. It follows that the event $B_{n}$ is independent of $\cup_{m=1}^{N_{n}} A_{n, m}$, and hence

$$
\begin{equation*}
\mathbf{P}\left(A_{n}\right)=\mathbf{P}\left(B_{n}\right) \mathbf{P}\left(\cup_{m=1}^{N_{n}} A_{n, m}\right) \tag{4.27}
\end{equation*}
$$

We bound each of these terms separately. Firstly, by Lemma 4.3.3, we have that there exists $\tilde{c}_{p}>0$ such that

$$
\mathbf{P}\left(M_{k} \geq \frac{1}{2} k^{-\alpha}\right) \geq \tilde{c}_{p}
$$

for all $k \geq 1$. Recalling that $r_{n}=2^{-n}$, we see that

$$
\begin{equation*}
\mathbf{P}\left(B_{n}\right)=\mathbf{P}\left(M_{n} \geq r_{n}^{\frac{\alpha}{2}}\right) \geq \mathbf{P}\left(M_{n} \geq \frac{1}{2} n^{-\alpha}\right) \geq \tilde{c}_{p} \tag{4.28}
\end{equation*}
$$

To bound the second term in (4.27), we apply point (ii) of Lemma 4.3.2, which implies that

$$
\mathbf{P}\left(\bigcup_{m=1}^{N_{n}} A_{n, m}\right) \geq 1-\prod_{m=1}^{N_{n}}\left(1-\mathbf{P}\left(A_{n, m}\right)\right) .
$$

Recalling that $N_{n}=\left\lfloor 2^{\frac{-(2 \alpha+1)}{\alpha}} r_{n}^{-1}\left(\log r_{n}^{-1}\right)^{\frac{-1}{\alpha}}\right\rfloor \leq 2^{\frac{-(\alpha+1)}{\alpha}} r_{n}^{-1}\left(\log r_{n}^{-1}\right)^{\frac{-1}{\alpha}}$, we again apply (4.28) to deduce that

$$
\mathbf{P}\left(C_{n, m} \mid B_{n}\right)=\mathbf{P}\left(C_{n, m}\right)=\mathbf{P}\left(M_{n, m} \geq R_{n}^{\alpha}\left(\log r_{n}^{-1}\right)\right) \geq \mathbf{P}\left(M_{n, m} \geq \frac{1}{2} m^{-\alpha}\right)>c_{p}
$$

whenever $m<N_{n}$. To conclude, note that conditional on $C_{n, m}$, we have that $M_{n}^{-1} R_{n}^{\alpha}\left(\log r_{n}^{-1}\right) \leq 1$ and hence we can apply Proposition 4.3.4 to deduce that

$$
\mathbf{P}\left(D_{n, m} \mid C_{n, m}, B_{n}\right) \geq \mathbf{P}\left(\nu\left(B\left(\rho, M_{n}^{\frac{-1}{\alpha}} R_{n}\right)\right) \geq M_{n}^{-1} R_{n}\left(\log r_{n}^{-1}\right)\right) \geq C e^{-\hat{c} \lambda_{n}}
$$

Here we are specifically taking $\hat{c}$ to be the constant in the exponent of Proposition 4.3.4. Combining, we see that

$$
\begin{aligned}
\mathbf{P}\left(\bigcup_{m=1}^{N_{n}} A_{n, m}\right) \geq 1-\prod_{m=1}^{N_{n}}\left(1-\mathbf{P}\left(A_{n, m}\right)\right) & \geq 1-\left(1-C e^{-2 \hat{c} \lambda_{n}^{\prime}}\right)^{N_{n}} \\
& \geq 1-\exp \left\{2^{\frac{-(\alpha+1)}{\alpha}} r_{n}^{-1}\left(\log r_{n}^{-1}\right)^{\frac{-1}{\alpha}} C r_{n}^{2 \hat{c} C^{*}}\right\}
\end{aligned}
$$

Hence, by choosing $C^{*}>(2 \hat{c})^{-1}$, we see that $\mathbf{P}\left(\cup_{m=1}^{N_{n}} A_{n, m}\right) \rightarrow 1$ as $n \rightarrow \infty$, and in particular that we can lower bound it by a non-negative constant uniformly in $n$. Combining this with (4.27) and (4.28), we see that there exists a constant $c>0$ such that $\mathbf{P}\left(A_{n}\right) \geq c$ for all $n \geq 1$. It then follows from Lemma 4.3.5 and Borel-Cantelli that $\mathbf{P}\left(A_{n}\right.$ i.o. $)=1$.

The conclusion follows since on the event $D_{n, m}$, we can rescale the ball $B\left(\rho_{n, m}, R_{n}\right) \cap L_{n, m}$ back to its original size in the original looptree to obtain a ball of radius $r_{n}$ with volume at least $r_{n}^{\alpha} 2 \lambda_{n}^{\prime}$. Moreover, on the event $B_{n}$ we also have that $\lambda_{n} \leq 2 \lambda_{n}^{\prime}$, so this volume is actually lower bounded by $r_{n}^{\alpha} \lambda_{n}=r_{n}^{\alpha} \log r_{n}^{-1}$.

### 4.3.3 Infimal upper bounds

We now prove (4.4) and (4.8). The method we use to prove upper bounds on infimal extrema is a simpler version of that used in Section 4.2.2 based on the Williams' decomposition. We can again control the masses of fragments in the decomposition by comparison with an $\alpha^{-1}$-stable subordinator. In this case however, we do not need to worry about reiterating around larger fragments since the presence of such fragments is a rare event and thus should not affect the infimal behaviour of the subordinator.

Let $H$ be the height of the spine in the corresponding tree $\mathcal{T}_{\alpha}$. As in Section 4.2.2, we start by rescaling $\mathcal{L}_{\alpha}$ by $H$ to form the looptree $\left(\mathcal{L}_{\alpha}^{1}, d^{1}, \rho^{1}, \nu^{1}\right)$, which now has mass $H^{\frac{-\alpha}{\alpha-1}}$ and has a corresponding underlying stable tree that has height 1. Note that

$$
\left\{\nu(B(\rho, r)) \leq r^{\alpha} \lambda^{-1}\right\}=\left\{\nu^{1}\left(B^{1}\left(\rho^{1}, r H^{\frac{-1}{\alpha-1}}\right)\right) \leq R^{\alpha} \lambda^{-1}\right\}
$$

where again $R=r H^{\frac{-1}{\alpha-1}}$. As explained in the Lemma 4.2.10, and using the notation we introduced there, it follows from properties of the Itô excursion measure that $\nu^{1}\left(B^{1}\left(\rho^{1}, R\right)\right)$ is stochastically dominated by $Y\left(\left|I_{R}\right|\right)$, where $Y$ is an $\alpha^{-1}$-stable subordinator, and $I_{R}$ denotes the length of W -loopspine that intersects $B^{1}\left(\rho^{1}, R\right)$. A jump of $Y$ of size $\Delta$ at a time $t$ corresponds to a sublooptree coded by an Itô excursion of lifetime equal to $\Delta$, and grafted to the W -loopspine at a point that informally is at a clockwise distance $t$ "through" $I_{R}$. Moreover, since we have
rescaled the looptree to have tree height 1 , there is no constraint on its total mass, and therefore no dependence between different jumps of $Y$.

For technical reasons we will in fact model this by two independent $\alpha^{-1}$ stable subordinators, $Y^{(l)}$ and $Y^{(r)}$, corresponding to the left and right sides of the W -loopspine respectively. We set $Y=Y^{(l)}+Y^{(r)}$.

The comparison relies on the following result, which gives the limiting behaviour of the infimum of an $\alpha^{-1}$-stable Lévy subordinator.

Theorem 4.3.6. [Ber96, Section III.4, Theorem 11]. Let $\left(W_{t}\right)_{t \geq 0}$ be an $\alpha^{-1}$-stable Lévy subordinator. Then, almost surely,

$$
\liminf _{t \downarrow 0^{+}} \frac{W_{t}}{t^{\alpha}\left(\log \log t^{-1}\right)^{-(\alpha-1)}}=\alpha^{-1}\left(1-\alpha^{-1}\right)^{\alpha-1}
$$

To deduce a similar result for $\left(Y_{t}\right)_{t \geq 0}$ in place of $\left(W_{t}\right)_{t \geq 0}$, note that the only difference between the two subordinators is the constant in the Lévy measure. Hence we have the same result for $\left(Y_{t}\right)_{t \geq 0}$, but just with a different constant on the right hand side. We will denote this constant by $c_{\alpha}$.

Proof of local infimal upper bound in Theorem 4.0.5. Set $f(t)=t^{\alpha}\left(\log \log t^{-1}\right)^{-(\alpha-1)}$ for $t>0$. By Theorem 4.3.6, there almost surely exists a sequence $\left(r_{n}\right)_{n \geq 1}$ with $r_{n} \downarrow 0$ such that

$$
Y\left(3 r_{n} H^{\frac{-1}{\alpha-1}}\right) \leq\left(c_{\alpha}+1\right) f\left(3 r_{n} H^{\frac{-1}{\alpha-1}}\right)
$$

for all $n$. Since $f\left(3 r_{n} H^{\frac{-1}{\alpha-1}}\right) \leq 2 \cdot 3^{\alpha} r_{n}^{\alpha} H^{\frac{-\alpha}{\alpha-1}}\left(\log \log r_{n}^{-1}\right)^{-(\alpha-1)}$ whenever $r_{n} \leq H^{\frac{-1}{\alpha-1}}$, we can extract a subsequence if necessary so that

$$
Y\left(3 r_{n} H^{\frac{-1}{\alpha-1}}\right) \leq 2 \cdot 3^{\alpha}\left(c_{\alpha}+1\right) r_{n}^{\alpha} H^{\frac{-\alpha}{\alpha-1}}\left(\log \log r_{n}^{-1}\right)^{-(\alpha-1)}
$$

and also $r_{n+1}<\frac{1}{2} r_{n}$ for all $n \geq 1$. Set $R_{n}=r_{n} H^{\frac{-1}{\alpha-1}}$.
Note that since the process $Y$ depends only on the total length of the Wloopspine, and not on its microscopic structure, it follows from Lemma 4.2.9 that there exists a constant $C_{p}>0$ such that

$$
\mathbf{P}\left(\left|I_{R_{n}}\right| \leq 3 R_{n}\right) \geq C_{p}
$$

for all $n$. More specifically, we let $A_{n}$ be the event described by taking $\lambda=1$ in the proof of Lemma 4.2 .9 that ensures that $\left|I_{R_{n}}\right| \leq 3 R_{n}$, consisting of the three subevents:
$(i)_{n}$ There exists a good loop in the W-loopspine with total length in $\left[4 R_{n}, 8 R_{n}\right]$.
$(i i)_{n}$ There are no goodish loops in the W-loopspine occurring between the root and the first good one.
$(i i i)_{n}$ The sum of the lengths of the smaller loops up until the first good loop is upper bounded by $R_{n}$.

The proof of Lemma 4.2.9 ensures that $\mathbf{P}\left(A_{n}\right) \geq C_{p}$ for all $n$, but to apply the second Borel-Cantelli Lemma we need to lower bound $\mathbf{P}\left(A_{n} \mid A_{m}^{c} \forall m<n\right)$ instead. To do this, note that conditional on $A_{m}^{c} \forall m<n$ :

- The probability of the event described in $(i)_{n}$ is unaffected by the events of $A_{m}$ for $m<n$, since the sets $\left[4 R_{n}, 8 R_{n}\right]$ are disjoint for different $n$ and therefore can be viewed as independent thinned Poisson processes along the W-spine of the tree.
- Conditional on $(i)_{m}^{c}$ occurring for all $m<n$, the probability that there is only one goodish loop before the first good one at level $n-1$ is lower bounded by $\mathbb{P}\left(\operatorname{Geo}\left(\frac{1}{2}\right)=1 \left\lvert\, \operatorname{Geo}\left(\frac{1}{2}\right) \neq 0\right.\right)=\frac{1}{2}$.
- Conditional on there only being one such goodish loop at level $n-1$, the probability that the good loop at level $n$ occurs before the goodish loop at level $n-1$ is at least $\frac{1}{2}$. If this occurs, then the probability of the events in $(i i)_{n}$ and $(i i i)_{n}$ is unaffected.

It follows that

$$
\mathbf{P}\left(A_{n} \mid A_{m}^{c} \forall m<n\right) \geq \frac{1}{4} C_{p}
$$

for all $n$, and therefore $\mathbf{P}\left(A_{n}\right.$ i.o. $)=1$ by the second Borel-Cantelli Lemma.
To conclude, note that on the event $A_{n}$ we have

$$
\nu^{1}\left(B^{1}\left(\rho^{1}, R_{n}\right)\right) \leq Y\left(3 R_{n}\right) \leq 2 \cdot 3^{\alpha}\left(c_{\alpha}+1\right) R_{n}^{\alpha}\left(\log \log r_{n}^{-1}\right)^{-(\alpha-1)},
$$

and hence scaling back to the original looptree we see that

$$
\nu\left(B\left(\rho, r_{n}\right)\right) \leq 3^{\alpha}\left(c_{\alpha}+1\right) r_{n}^{\alpha}\left(\log \log r_{n}^{-1}\right)^{-(\alpha-1)} .
$$

for all sufficiently large $n$. This proves the local result (4.8).
To prove the global bound, we perform two subsequent spinal decompositions of $\mathcal{L}_{\alpha}$, exactly as illustrated in Figure 4.1 in the previous section. Recall from there that we let ( $M_{1}, M_{2}, \ldots$ ) denote the GEM masses obtained on performing a first spinal decomposition of $\mathcal{L}_{\alpha}$, as described in Section 4.3.1. Then, for each of the resulting fragments ( $L_{1}, L_{2}, \ldots$ ), rescale to obtain a sequence of independent stable looptrees $\left(\mathcal{L}_{\alpha}^{1}, \mathcal{L}_{\alpha}^{2}, \ldots\right)$, each with mass 1 . For each $n \in \mathbb{N}$, we perform a further spinal decomposition of $\mathcal{L}_{\alpha}^{n}$ and denote the resulting GEM masses by $\left\{M_{n, 1}, M_{n, 2}, \ldots\right\}$, and corresponding looptrees by $\left\{L_{n, 1}, L_{n, 2}, \ldots\right\}$, and by $\left\{\mathcal{L}_{\alpha}^{n, 1}, \mathcal{L}_{\alpha}^{n, 2}, \ldots\right\}$ after rescaling. We also let $U_{n, m}$ denote a point chosen uniformly in $L_{n, m}$ according to the natural
volume measure. We take $r_{n}=2^{-n}, R_{n}=M_{n}^{\frac{-1}{\alpha}} r_{n}, \lambda_{n}=\left(C^{*} \log r_{n}^{-1}\right)^{\alpha-1}$ and $\lambda_{n}^{\prime}=\left(C^{*} \log R_{n}^{-1}\right)^{\alpha-1}$, where $C^{*}$ is a constant to be specified later. We also define the events:

$$
\begin{aligned}
B_{n} & =\left\{r_{n}^{\alpha} \leq M_{n}^{2}\right\}, & & C_{n, m}=\left\{d_{\mathcal{L}_{\alpha}^{n}}\left(\rho_{m, n}, U_{m, n}\right) \geq R_{n}\right\}, \\
D_{n, m} & =\left\{\nu_{\mathcal{L}_{\alpha}^{n}}\left(B\left(U_{n, m}, R_{n}\right) \cap L_{n, m}\right) \leq R_{n}^{\alpha} \lambda_{n}^{-1}\right\}, & A_{n, m} & =C_{n, m} \cap D_{n, m} .
\end{aligned}
$$

We also set $N_{n}=r_{n}^{\frac{-1}{2}}$. We then define the event

$$
A_{n}=B_{n} \cap\left(\bigcup_{m=1}^{N_{n}} A_{n, m}\right) .
$$

The key point is to observe that $A_{n} \subset\left\{\sup _{u \in \mathcal{L}_{\alpha}} \nu\left(B\left(u, r_{n}\right)\right) \leq r_{n}^{\alpha} \lambda_{n}{ }^{-1}\right\}$, and hence it is sufficient to only show that $\mathbf{P}\left(A_{n}\right.$ i.o. $)=1$. Similarly to the previous section, the next lemma gives us a means to overcome the dependencies between the GEM masses and apply the second Borel-Cantelli Lemma. Its proof is almost identical to that of Lemma 4.3.5, so is omitted.

Lemma 4.3.7. Let $A_{n}, B_{n}, A_{n, m}, C_{n, m}, D_{n, m}, N_{n}$ be as above. Then
(i) $\mathbf{P}\left(A_{n} \mid A_{m}^{c} \forall m<n\right) \geq \mathbf{P}\left(A_{n}\right)$,
(ii) $\mathbf{P}\left(A_{n, m} \mid A_{n, l}^{c} \forall l<m\right) \geq \mathbf{P}\left(A_{n, m}\right)$.

Proof of global infimal upper bound in Theorem 4.0.4. Now, note that it follows from [Ber96, Section III.4, Theorem 12] and the local argument given above that

$$
\begin{equation*}
\mathbf{P}\left(\nu\left(B(p(U), r) \leq r^{\alpha} \lambda^{-1}\right) \geq C e^{-c \lambda^{\frac{1}{\alpha-1}}} .\right. \tag{4.29}
\end{equation*}
$$

We will apply this to prove that $\mathbf{P}\left(A_{n}\right) \geq C e^{-c \lambda^{\frac{1}{\alpha-1}}}$ as well. Firstly, note that by Lemma 4.3.3 there exists a constant $c>0$ such that $\mathbf{P}\left(B_{n}\right)>c$ for all $n$. Then, since the looptrees in the spinal decomposition are independent of their original masses after rescaling (see Theorem 4.3.1), it follows that $\bigcup_{m=1}^{N_{n}} A_{n, m}$ is independent of $B_{n}$.

Next, we note that:

$$
\begin{aligned}
& \mathbf{P}\left(C_{n, m} \mid B_{n}, m \leq r_{n}^{\frac{-1}{2}}\right) \\
& =\mathbf{P}\left(d_{\mathcal{L}_{\alpha}^{n}}\left(\rho_{m, n}, U_{m, n}\right) \geq R_{n} \mid B_{n}, m \leq r_{n}^{\frac{-1}{2}}\right) \\
& \geq \mathbf{P}\left(\left.d_{\mathcal{L}_{\alpha}^{n}}\left(\rho_{m, n}, U_{m, n}\right) \geq r_{n}^{\frac{1}{2}} \right\rvert\, m \leq r_{n}^{\frac{-1}{2}}\right) \\
& \geq \mathbf{P}\left(\left.\nu_{\mathcal{L}_{\alpha}}\left(L_{m}\right) \geq \frac{1}{2} r_{n}^{\frac{\alpha}{2}} \right\rvert\, m=r_{n}^{\frac{-1}{2}}\right) \mathbf{P}\left(d_{\mathcal{L}_{\alpha}}\left(\rho_{m}, U_{m}\right) \geq r_{n}^{\frac{1}{2}} \left\lvert\, \nu_{\mathcal{L}_{\alpha}}\left(L_{m}\right)=\frac{1}{2} r_{n}^{\frac{\alpha}{2}}\right.\right) \\
& \geq C,
\end{aligned}
$$

where $C>0$. The final line follows since by Lemma 4.3.3 the first term in the penultimate line above can be uniformly lower bounded by a constant, and the second term can also be uniformly lower bounded by a constant by scaling invariance.

To conclude, we note from (4.29) that $\mathbf{P}\left(D_{n, m} \mid C_{n, m}, B_{n}\right) \geq C e^{-c \lambda_{n} \frac{1}{\alpha-1}}$ for all $n$, and all $m \leq N_{n}$. Combining these, we see that $\mathbf{P}\left(A_{m, n}\right) \geq C e^{-c \lambda_{n} \frac{1}{\alpha-1}}$. We therefore deduce from Lemma 4.3.7(ii) that

$$
\begin{aligned}
\mathbf{P}\left(A_{n}\right) \geq \mathbf{P}\left(B_{n}\right)\left(1-\left(1-\mathbf{P}\left(A_{n, m} \mid B_{n}\right)\right)^{N_{n}}\right) & \geq C^{\prime}\left(1-\left(1-C e^{-c \lambda_{n} \alpha^{\frac{1}{\alpha-1}}}\right)^{N_{n}}\right) \\
& \geq C^{\prime}\left(1-\exp \left\{-N_{n} C e^{\left.\left.-c \lambda_{n}{ }^{\frac{1}{\alpha-1}}\right\}\right)}\right.\right. \\
& \geq C^{\prime}\left(1-\exp \left\{-r_{n}^{\frac{-1}{2}} C e^{-c C^{*} \log r_{n}^{-1}}\right\}\right) .
\end{aligned}
$$

Choosing $C^{*}$ so that $C^{*}<\frac{1}{4} c^{-1}$, we obtain that

$$
\mathbf{P}\left(A_{n}\right) \geq C^{\prime}\left(1-\exp \left\{-r_{n}^{\frac{-1}{4}} C\right\}\right) \geq \frac{1}{2} C^{\prime}
$$

for all sufficiently large $n$. Applying Lemma 4.3.7(i) and the second Borel-Cantelli Lemma, we deduce that $\mathbf{P}\left(A_{n}\right.$ i.o. $)=1$, which implies (4.4).

### 4.3.4 Volume convergence results

Here we briefly note a convergence result for $\nu(B(\rho, r))$. In Chapter 5 , we introduce the infinite stable looptree $\mathcal{L}_{\alpha}^{\infty}$, which is defined from two stable Lévy processes rather than a Lévy excursion, arises as the local distributional limit of compact stable looptrees as their mass goes to infinity (Theorem 5.0.1), and provides the machinery to prove the following result. We prove this in Section 5.4.1.

Theorem 4.3.8. There exists a random process $\left(V_{t}\right)_{t \geq 0}: \Omega \rightarrow D([0, \infty),[0, \infty))$
such that the finite dimensional distributions of the process

$$
\left(r^{-\alpha} \nu(\bar{B}(\rho, r t))\right)_{t \geq 0}
$$

converge to those of $\left(V_{t}\right)_{t \geq 0}$ as $r \downarrow 0$, and $V_{t}$ denotes the volume of a closed ball of radius $t$ around the root in $\mathcal{L}_{\alpha}^{\infty}$. Moreover, for any $p \in[1, \infty)$, we have that

$$
r^{-\alpha p} \mathbf{E}\left[\nu(\bar{B}(\rho, r))^{p}\right] \rightarrow \mathbf{E}\left[V_{1}^{p}\right]
$$

as $r \downarrow 0$, and $V_{1}$ is a $(0, \infty)$-valued random variable with all moments finite.
This is proved as Theorem 5.4.5.

### 4.4 Heat kernel estimates

Although we used the shortest distance metric to prove the volume results of Theorems 4.0.5 and 4.0.4, the result of Lemma 4.1.1 ensures that they also hold true with respect to the resistance metric $R$. This allows us to apply results of [Cro07] to deduce the heat kernel bounds of Theorems 4.0.6 and 4.0.7. Most of our results follow from a direct application of those of [Cro07], so we refer the reader there for further background.

To get some off-diagonal results, we need to verify the Chaining Condition (CC) of [Cro07, Section 4.2].

Definition 4.4.1. (Chaining Condition (CC), [Cro07, Section 4.2]). A metric space $(X, R)$ is said to satisfy the chaining condition if there exists a constant $c$ such that for all $x, y \in X$ and all $n \in \mathbb{N}$, there exists $\left\{x_{0}, x_{1}, \ldots, x_{n}\right\} \subset X$ with $x_{0}=x$ and $x_{n}=y$ such that

$$
R\left(x_{i}, x_{i+1}\right) \leq c \frac{R(x, y)}{n} .
$$

It is easy to verify that CC holds for ( $\mathcal{L}_{\alpha}, R, \rho, \nu$ ). Recall from [CK14, Corollary 4.4] that $\mathcal{L}_{\alpha}$ is almost surely a length space when endowed with the shortest distance metric $d$. The chaining condition for $\left(\mathcal{L}_{\alpha}, d, \rho, \nu\right)$ therefore holds as a straightforward extension of the midpoint condition for length spaces, with $c=1+\varepsilon$ for any $\varepsilon>0$ (though it actually holds with $c=1$ ). It hence follows from Corollary 4.1.1 that $\mathcal{L}_{\alpha}$ endowed with the resistance metric $R$ also satisfies the condition, with $c=2(1+\varepsilon)$ (in fact $c=2$ works) instead.

In the notation of [Cro07], we can take any $\varepsilon>0$ to satisfy point $(i)$ of the conditions given in Section 2 of that paper, and take $b=\varepsilon$ to satisfy point (iii). We also let $f_{l}(r)=C\left(\log r^{-1}\right)^{-\alpha}, f_{u}(r)=C\left(\log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}$, and $\beta_{l}=\beta_{u}=\alpha$, and $\theta_{1}=(3+2 \alpha)(2+\alpha)$. The first part of Theorem 4.0.6 then follows by a direct application of [Cro07, Theorem 1], with $\gamma_{1}=\theta_{1}\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)$.

We can similarly apply the results to get off diagonal heat kernel bounds. Again in the notation of [Cro07], take $\theta_{2}$ and $\theta_{3}$ satisfying

$$
\theta_{2}>\theta_{1}(1+\alpha), \quad \theta_{3}>(3+2 \alpha)\left(1+2 \alpha^{-1}\right)
$$

and let $\gamma_{i}=\theta_{i}\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)$ for $i=2,3$. Theorem 4.0.8 then follows by a direct application of [Cro07, Theorem 3].

The results of [Cro07, Proposition 11] can also be applied to give bounds on expected exit times from a ball of radius $r$. Indeed, letting $\tau_{A}=\inf \left\{t \geq 0: B_{t} \notin A\right\}$ for any $A \subset \mathcal{L}_{\alpha}$, we deduce the following.

## Proposition 4.4.2.

$$
\begin{aligned}
& \mathbf{E}_{x}\left[\tau_{B(x, r)}\right] \geq c r^{\alpha+1}\left(\log r^{-1}\right)^{-2\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)(\alpha+1)}\left(\log \left(r^{-1}\left(\log r^{-1}\right)^{2\left(\alpha+\frac{4 \alpha-3}{\alpha-1}\right)}\right)\right)^{-\alpha} \\
& \mathbf{E}_{x}\left[\tau_{B(x, r)}\right] \leq C r^{\alpha+1}\left(\log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}
\end{aligned}
$$

The results of the propositions above all follow from the fact that the global volume fluctuations are at most logarithmic. We can also use the fact that these logarithmic fluctuations are indeed attained infinitely often as $r \downarrow 0$ to deduce that the heat kernel will indeed experience similar fluctuations.

The volume results as stated in Theorem 4.0.5 do not quite fall into the framework of [Cro07, Theorem 2], since we have only shown that the infimal and supremal volumes achieve extremal logarithmic fluctuations values infinitely often as $r \downarrow 0$, rather than eventually, which is what is required to apply the theorem. However, by repeating the proof given there with our weaker volume assumptions instead we are able to deduce the (weaker) results that make up the second part of Theorem 4.0.6.

Again using [Cro07], the local volume fluctuation results of Theorem 4.0.5 can also be used to bound pointwise fluctuations for the transition density $p_{t}(x, x)$. However, the conclusions of [Cro07, Theorem 20] also require the condition

$$
\liminf _{r \downarrow 0} \frac{R\left(x, B(x, r)^{c}\right)}{r}>0
$$

to hold for $\nu$-almost every $x \in \mathcal{L}_{\alpha}$ in order to get lower bounds on the heat kernel. This does not quite hold in our case but from the proof of [Cro07], we see that the following proposition is sufficient. For clarity in the next proof, we let $B_{R}(x, r)$ (respectively $\left.B_{d}(x, r)\right)$ denote the open ball of radius $r$ at $x$ defined with respect to the resistance (respectively geodesic) metric.

Proposition 4.4.3. Almost surely, taking $c_{\alpha}$ as in Section 4.3.3, we have that for $\nu$-almost every $x \in \mathcal{L}_{\alpha}$, there exists a sequence $r_{n} \downarrow 0$ such that both of the following
conditions hold:
(i) $\nu\left(B_{R}\left(x, r_{n}\right)\right) \leq 2\left(c_{\alpha}+1\right) r_{n}^{\alpha}\left(\log \log r_{n}^{-1}\right)^{-(\alpha-1)}$ for all $n$,
(ii) $R_{e f f}\left(x, B_{R}\left(x, r_{n}\right)^{c}\right) \geq \frac{1}{64} r_{n}$.

Proof. The proof uses a standard technique for lower bounding the effective resistance as given in [BK06, Lemma 4.5], by defining $M(\rho, r)$ to be the smallest number $m$ such that there exists a set $A_{r}=\left\{z_{1}, z_{2}, \ldots, z_{m}\right\}$ such that $d\left(\rho, z_{i}\right) \in\left[\frac{r}{4}, \frac{3 r}{4}\right]$ for each $i$, and every path $\gamma$ from $\rho$ to $B_{d}(\rho, r)^{c}$ must pass through at least one of the points in $A$. The proof of [BK06, Lemma 4.5] combined with Lemma 4.1.1 then entails that

$$
\begin{equation*}
R_{\mathrm{eff}}\left(\rho, B_{R}(\rho, r)^{c}\right) \geq \frac{r}{16 M(\rho, r)} . \tag{4.30}
\end{equation*}
$$

The result exactly as stated in [BK06] is written for discrete trees. However, by combining with Lemma 4.1.1, the same proof shows that (4.30) holds for $\mathcal{L}_{\alpha}$, just with an extra factor of 2 .

In what follows, we will therefore assume that all distances are defined with respect to the shortest-distance metric $d$. As in earlier sections, we will prove the result at a uniform point $p(U)$, which we can suppose to be the root, and extend to $\nu$-almost every $x \in \mathcal{L}_{\alpha}$ by Fubini's theorem. As in Sections 4.2.2 and 4.3.3, let $\lambda_{r}=2\left(c_{\alpha}+1\right)\left(\log \log r^{-1}\right)^{\alpha-1}$, choose $H$ to be the height of the stable tree associated with $\mathcal{L}_{\alpha}$, and rescale time by $H^{\frac{-\alpha}{\alpha-1}}$ and space by $H^{\frac{-1}{\alpha-1}}$ in the Lévy excursion coding $\mathcal{L}_{\alpha}$ to give a new looptree $\mathcal{L}_{\alpha}^{1}$ such that the new underlying tree associated to $\mathcal{L}_{\alpha}^{1}$ has height 1. From the arguments of Section 4.3.3, it follows that almost surely, there exists a sequence $\left(r_{n}\right)_{n \in \mathbb{N}}$ with $r_{n} \downarrow 0$ such that $\left|I_{\frac{1}{4} r_{n} H^{\frac{-1}{\alpha-1}}}\right| \leq \frac{3}{4} r_{n} H^{\frac{-1}{\alpha-1}}$, and all sublooptrees grafted to the W-loopspine at a point in $I_{\frac{3}{4} R}$ have mass at most $r_{n}^{\alpha} \lambda_{r_{n}}^{-1}$ for all $n$. We will show that, with high probability, we also have $R\left(x, B_{d}\left(x, r_{n}\right)^{c}\right) \geq$ $c r_{n}$ for each $n \in \mathbb{N}$.

Now let $r=r_{n}$ for some $n \in \mathbb{N}$, and $R=r H^{\frac{-1}{\alpha-1}}$. By construction, we then have:

- $\left|I_{\frac{1}{4} R}\right| \leq \frac{3}{4} R$,
- Any sublooptrees grafted to the W-loopspine at a point in $I_{\frac{3}{4} R}$ have mass at most $r^{\alpha} \lambda_{r}^{-1}$.

To bound $M(\rho, r)$, first let $N_{r}$ denote the number of sublooptrees grafted to the Wloopspine of $\mathcal{L}_{\alpha}^{1}$ at a point in $I_{\frac{3}{4} R}$ and with diameter at least $\frac{3}{4} R$. It follows by construction that any such sublooptrees also have mass at most $R^{\alpha} \lambda_{r}^{-1}$. Consequently, $N_{r}$ is stochastically dominated by a Poisson random variable with parameter:

$$
\left|I_{\frac{1}{4} R}\right| N\left(\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{3}{4} R, \zeta \leq R^{\alpha} \lambda_{r}^{-1}\right),
$$

where $N(\cdot)$ here denotes the Itô excursion measure, and $\tilde{\mathcal{L}_{\alpha}}$ is a looptree coded by an (unconditioned) excursion under $N$. The point is that the two events $\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq$ $\frac{3}{4} R$ and $\zeta \leq R^{\alpha} \lambda_{r}^{-1}$ are in conflict with each other and hence the Itô measure of the given set is small. Indeed, since $N(\cdot)$ codes a Poisson point process, we have the necessary independence from the Poisson thinning property so that:

$$
\begin{aligned}
& N\left(\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{3}{4} R, \zeta \leq R^{\alpha} \lambda_{r}^{-1}\right) \\
& \leq N\left(\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{3}{4} R\right) \mathbf{P}\left(\nu\left(B\left(\rho^{\prime}, R\right)\right) \leq R^{\alpha} \lambda_{r}^{-1} \left\lvert\, \operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{3}{4} R\right.\right) .
\end{aligned}
$$

To bound each of these terms, note first by the scaling property of looptrees and the Itô measure that

$$
N\left(\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq t\right)=\hat{C}_{\alpha} t^{-1}
$$

for some constant $\hat{C}_{\alpha} \in(0, \infty)$, and hence $N\left(\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{3}{4} R\right)=\hat{C}_{\alpha} R^{-1}$. Then, by the same arguments used to prove Proposition 4.2.1, we can bound the second term by $C e^{-c \lambda_{r}^{\frac{1}{\alpha}}}$, and therefore obtain that

$$
N\left(\operatorname{Diam}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{3}{4} R, \zeta \leq R^{\alpha} \lambda_{r}^{-1}\right) \leq \hat{C}_{\alpha} R^{-1} C e^{-c \lambda_{r}^{\frac{1}{\alpha}}}
$$

It hence follows that $N_{r}$ is stochastically dominated by a Poisson $\left(C^{\prime} e^{-c \lambda_{r}^{\frac{1}{\alpha}}}\right)$ random variable, so

$$
\mathbf{P}\left(N_{r}>0\right) \leq C^{\prime} e^{-c \lambda_{r}^{\frac{1}{x}}}
$$

By restricting to a subsequence $\left(r_{n_{l}}\right)_{l \geq 1}$ such that $r_{n_{l}} \leq e^{-e^{l}}$ for all $l$, we see by Borel-Cantelli that $\mathbf{P}\left(N_{r_{l}}>0\right.$ i.o. $)=0$.

On the event $N_{r}=0$, it follows that any path $\gamma$ from $\rho$ to $B_{d}(\rho, R)$ must leave the ball $B_{d}\left(\rho, \frac{1}{4} R\right)$ at a point on the W -loopspine. We conclude the argument by showing that we can then take a set $A_{r}$ (which we denote by $A_{R}$ in the rescaled looptree) with cardinality 2 .

Recall that, by assumption, we also have that $\left|I_{\frac{1}{4} R}\right| \leq \frac{3}{4} R$. In particular, we can assume that the particular event defined in Lemma 4.2.9 and then in Section 4.3.3 which leads to this length bound occurs. Moreover, taking $\lambda=1$ in that proof and $\frac{1}{4} r$ in place of $r$, and defining "good" and "goodish" loops as we did there, the proof ensures that the number of goodish loops encountered before we reach a good one is at most 1 . We claim that this implies that $\left|A_{R}\right| \leq 4$.

To see why, we refer to Figure 5.5, which shows a representation of (a discrete approximation of) the W-loopspine. Defining good and goodish loops for the radius $\frac{1}{4} R$ as in Lemma 4.2.9, we will assume a "worst-case scenario": that there does indeed exist a goodish loop, and that the smaller of the two segments that it is
broken into along the W-loopspine is less than $\frac{1}{4} R$ in length. Since $\left|I_{\frac{1}{4} R}\right| \leq \frac{3}{4} R$, it follows that all of the loops that fall between the root and this goodish loop, and also between this goodish loop and the good loop, are completely contained within $B_{d}\left(\rho, \frac{3}{4} R\right)$, and hence we cannot exit $B_{d}\left(\rho, \frac{1}{4} R\right)$ at a point within these sequences of smaller loops. We can therefore only exit at points on either the goodish loop or the good loop pictured, so we can add two points in $A_{R}$ in each of these loops to cover all possible exit routes, as shown. We rescale back to the original looptree to get $A_{r}$. Note that for any $\varepsilon>0$, it also follows that we can choose these points to be within distance $\frac{1}{4}+\varepsilon$ of $\rho$.

In the case that the smaller of the two segments of the goodish loop actually has length larger than $\frac{1}{4} R$, we can repeat the argument by treating the goodish loop as the good loop, and the same result holds.


Figure 4.2: How to select $A_{R}$. The red segment is a strict subset of $B\left(\rho, \frac{3}{4} R\right)$ and contains $B\left(\rho, \frac{1}{4} R\right)$.

This proves (4.30), and we deduce the result as claimed.
Remark 4.4.4. In [Arc20, Theorem 6.2], we prove for infinite stable looptrees that there almost surely exists a constant $c>0$ such that, for all $r>0$ :

$$
c r\left(\log \log r^{-1}\right)^{\frac{-(3 \alpha-2)}{\alpha-1}} \leq R\left(\rho, B(\rho, r)^{c}\right)
$$

The argument given there also applies in the compact case, so we deduce the same result for $\mathcal{L}_{\alpha}$.

Repeating the proof of [Cro07, Theorem 20] along the subsequence of Proposition 4.4.3 gives Theorem 4.0.7. Finally, we refer to Section 5.4.1 for the details of the proof of Theorem 4.0.10.

### 4.5 Spectral properties of stable looptrees

This section concerns results on the eigenvalue spectrum of stable looptrees. The proofs are just modifications of similar proofs for stable trees given in [CH10], and are not majorly new. We therefore just outline the heuristics for the proofs, and explain which parts are different for stable looptrees in place of stable trees.

### 4.5.1 Brief background

In light of the results of the preceding sections, the purpose of the present section is to investigate the eigenvalue counting function of the Laplacian $\Delta$ of the Dirichlet form associated with Brownian motion on $\mathcal{L}_{\alpha}$

More precisely, let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with the space $L^{2}\left(\mathcal{L}_{\alpha}, \nu\right)$ through the relation

$$
R(x, y)^{-1}=\inf \{\mathcal{E}(f, f): f \in \mathcal{F}, f(x)=0, f(y)=1\}
$$

We say that $\lambda>0$ is an eigenvalue of $(\mathcal{E}, \mathcal{F}, \nu)$ with eigenfunction $f$ (assumed to be non-trivial) if

$$
\mathcal{E}(f, g)=\lambda \int_{\mathcal{L}_{\alpha}} f g d \nu
$$

for all $g \in \mathcal{F}$. The eigenvalue counting function $N(\lambda)$ is then defined as the number of (in our case they are $\mathbf{P}$-almost surely all distinct) eigenvalues of $(\mathcal{E}, \mathcal{F}, \nu)$ that are less than or equal to $\lambda$.

The transition density $p_{t}(\cdot, \cdot)$ analysed in Section 4.4 is the heat kernel of the associated Laplacian, and due to the representation

$$
\mathcal{E}(f, g)=-\int_{\mathcal{L}_{\alpha}}(\Delta f) g d \nu
$$

we see that any eigenvalue of the operator $\Delta$ is also an eigenvalue of $(\mathcal{E}, \mathcal{F}, \nu)$. Since $\mathcal{L}_{\alpha}$ is compact and $(\mathcal{E}, \mathcal{F})$ is consequently regular, the converse also holds.

The main result is as follows.

Theorem 4.5.1. (i) For any $\varepsilon>0$,

$$
\mathbf{E}[N(\lambda)] \sim C \lambda^{\frac{\alpha}{\alpha+1}}+O\left(\lambda^{\frac{1}{\alpha+1}+\varepsilon}\right)
$$

as $\lambda \rightarrow \infty$.
(ii) $\mathbf{P}$-almost surely, $N(\lambda) \sim C \lambda^{\frac{\alpha}{\alpha+1}}$ as $\lambda \rightarrow \infty$. Moreover, in $\mathbf{P}$-probability, the second order estimate of part (i) holds.

The proofs of the results given in this section closely follow the ideas used
in [CH10] to deduce analogous results for stable trees. More specifically, we use the spinal decomposition of Theorem 4.3 . 1 to write the eigenvalues of $\mathcal{L}_{\alpha}$ in terms of the eigenvalues of all the sublooptrees $\left(\mathcal{L}_{\alpha}^{(i)}\right)_{i=1}^{\infty}$, and iterate this relation to deduce the asymptotics. In particular, the techniques used in [CH10] involve analysing both the mass and the diameter of the fragments obtained on performing subsequent decompositions. In the analogous argument for looptrees, the masses are exactly the same as the tree case and follow a Poisson-Dirichlet distribution, so the main aspect of the proofs carry over directly. However, there is one crucial difference in that the diameter of a (normalised) stable tree has finite moments of all orders, whereas for looptrees the moments are only finite up to a power of $\alpha$ (as we will show below). As a result, in some places some fine-tuning of the arguments of [CH10] is required.

To avoid double-counting the eigenvalue 0 when performing subsequent iterations, it will be convenient to define the shifted eigenvalue counting function $\tilde{N}$, given by $\tilde{N}(\lambda)=N(\lambda)-1$. It will also be useful to consider the Dirichlet eigenvalue counting function $N^{D}$, obtained when we consider $\mathcal{L}_{\alpha}$ to have a boundary consisting of two distinguished vertices, $\rho$ and $\sigma$, which can be thought of as two ends of the loopspine. $N^{D}$ is defined as the eigenvalue counting function for the restriction of $(\mathcal{E}, \mathcal{F})$ to the set $\mathcal{F}^{D}=\{f \in \mathcal{F}: f(\rho)=f(\sigma)=0\}$. Applying [KL93, Corollary 4.7] entails that

$$
N^{D}(\lambda) \leq N(\lambda) \leq N^{D}(\lambda)+2 .
$$

In what follows we will only give proper proofs in the places where the arguments deviate from those of [CH10]. For the most part, we will just give the intuition behind each step of the general proof strategy employed there.

### 4.5.2 Setup and main ideas

In what follows we set $\gamma=\frac{\alpha}{\alpha+1}$ (though note that in [CH10] it has the different value of $\frac{\alpha}{2 \alpha-1}$ ). This is half the spectral dimension of the space. In keeping with the notation of [CH10], we also let $\Sigma_{k}$ denote the word-space of $k$-tuples in $\mathbb{N}^{k}$, and $\Sigma_{*}=$ $\cup_{k \geq 0} \Sigma_{k}$. For $i=\left(i_{1}, \ldots, i_{k}\right) \in \Sigma_{k}$ and $j \leq k$, let $\left.i\right|_{j}$ denote the truncation $\left(i_{1}, \ldots, i_{j}\right)$. We will use $\Sigma_{*}$ to index the sublooptrees obtained by performing subsequent spinal decompositions of $\mathcal{L}_{\alpha}$ in the standard way; more formally, let $L_{\emptyset}=\mathcal{L}_{\alpha}$, and for $i \in$ $\Sigma_{k}$ with $k \geq 1$, let $L_{i}$ be the $i_{k}^{\text {th }}$ sublooptree obtained by performing another spinal decomposition on $L_{\left.i\right|_{k-1}}$. Also let $\Delta_{i}$ be the Poisson-Dirichlet weight associated to $L_{i}$ in this decomposition, and set $D_{i}=\prod_{j=1}^{k} \Delta_{\left.j\right|_{k}}$ to be the mass of $L_{i}$ in the original looptree before rescaling.

We let $\mathcal{L}_{i}$ denote the normalised version of $L_{i}$, i.e. where the measure of $L_{i}$ is rescaled by a factor of $D_{i}^{-1}$, and the distance is rescaled by $D_{i}^{\frac{-1}{\alpha}}$. We also let
$N_{i}, N_{i}^{d}, \tilde{N}_{i}$ denote the analogous quantities for the looptree $\mathcal{L}_{i}$.
The main argument in the proof of Theorem 4.5.1 rests principally on the following two propositions.

Proposition 4.5.2. cf [CH10, Proposition 3.1]. P-almost surely, for all $\lambda \geq 0$ :

$$
\sum_{i=1}^{\infty} N_{i}^{D}\left(\Delta_{i}^{\frac{1}{\gamma}} \lambda\right) \leq N^{D}(\lambda) \leq N(\lambda) \leq 1+\sum_{i=1}^{\infty} \tilde{N}_{i}\left(\Delta_{i}^{\frac{1}{\gamma}} \lambda\right)
$$

The intuition behind the first result is that the loopspine, along with the sublooptrees $\left(L_{i}\right)_{i=1}^{\infty}$, form a partition of $\mathcal{L}_{\alpha}$. Any eigenfunction on $\mathcal{L}_{\alpha}$ can be mapped to an eigenfunction on one of these subspaces by restriction, and conversely any eigenfunction on one of these subspaces can be extended to an eigenfunction on $\mathcal{L}_{\alpha}$ by setting it to be zero elsewhere. Since the loopspine has $\nu$-measure zero, it has no non-trivial eigenvalues so we can discount its contribution. Moreover, $\mathbf{P}$-almost surely, the rescaled eigenvalues from different sublooptrees will all be distinct, so this is a genuine correspondence. The extra +1 term on the right hand side arises from counting the eigenvalue 0 .

The scaling factor of $\Delta^{\frac{1}{\gamma}}=\Delta^{1+\frac{1}{\alpha}}$ arises since on rescaling a looptree of mass $\Delta$ to have total mass one, $\mathcal{E}(f, g)=\int f g d \nu$ will pick up a factor of $\Delta$ from rescaling the measure, and a factor $\Delta^{\frac{1}{\alpha}}$ from rescaling the distance.

This is proved formally in [CH10, Propositions 3.1-3.4] and the same proof carries over to the looptree case.

The second proposition is the following, which will be our main tool to bound the number of eigenvalues at each level of the iteration. We give the proof since it is short.

Proposition 4.5.3. (cf [CH10, Lemma 2.1]). $N^{D}(\lambda)=\tilde{N}(\lambda)=0$ whenever

$$
\lambda<\frac{1}{\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right) \nu\left(\mathcal{L}_{\alpha}\right)}
$$

Proof. If $f$ is an eigenfunction of $(\mathcal{E}, \mathcal{F}, \nu)$ with eigenvalue $\lambda>0$ then

$$
(f(x)-f(y))^{2} \leq \mathcal{E}(f, f) R(x, y) \leq \lambda \operatorname{Diam}_{R}\left(\mathcal{L}_{\alpha}\right) \int_{\mathcal{L}_{\alpha}} f^{2} d \nu
$$

Integrating over both $x$ and $y$ yields $\lambda \geq \frac{2}{\operatorname{Diam}_{R}\left(\mathcal{L}_{\alpha}\right) \nu\left(\mathcal{L}_{\alpha}\right)}$ (a slightly stronger result). The Dirichlet case is similar.

We will also require the following result on the moments of the diameter of a stable looptree.

Proposition 4.5.4. $\mathbf{E}\left[\left(\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)\right)^{p}\right]<\infty$ if and only if $p<\alpha$.

Proof. Firstly, note that it follows from (2.12) that

$$
\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right) \leq 2 \sup _{[0,1]} X^{\mathrm{exc}} \stackrel{(d)}{=} 2\left(\sup _{[0,1]} X^{\mathrm{br}}-\inf _{[0,1]} X^{\mathrm{br}}\right) .
$$

Hence, by considering the time reversal of $X^{\text {br }}$ along with the absolute continuity relation (2.12), we deduce that

$$
\mathbf{P}\left(\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right) \geq 4 x\right) \leq 2 \frac{\left\|p_{1}\right\|_{\infty}}{p_{1}(0)}\left[\mathbf{P}\left(\sup _{\left[0, \frac{1}{2}\right]} X \geq x\right)+\mathbf{P}\left(-\inf _{\left[0, \frac{1}{2}\right]} X \geq x\right)\right] \leq C x^{-\alpha} .
$$

Here, the last line follows from [Ber96, Section VIII, Proposition 4] for the supremal term, and the infimal term has exponential tails since $X$ is spectrally positive. It follows that $\mathbf{E}\left[\left(\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)\right)^{p}\right]$ is finite whenever $p<\alpha$.

To prove the converse, note that $\mathbf{E}\left[\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)\right] \geq \frac{1}{2} \mathbf{E}\left[\sup _{t \in\left[0, \frac{1}{2}\right]} \Delta_{t}^{\text {exc }}\right]$, where $\Delta_{t}^{\text {exc }}=X_{t}^{\text {exc }}-X_{t^{-}}^{\text {exc }}$. Moreover, letting $\Delta_{t}$ denote the corresponding quantity for the unconditioned process $X$, we have:

$$
\mathbf{P}\left(\sup _{t \in\left[0, \frac{1}{2}\right]} \Delta_{t} \geq 2 x\right) \geq \mathbb{P}\left(\operatorname{Poi}\left(c x^{-\alpha}\right) \geq 1\right)=O\left(x^{-\alpha}\right)
$$

as $x \rightarrow \infty$, since the number of jumps on $[0,1]$ of size exceeding $x$ is a Poisson random variable with the given parameter. We deduce that $\mathbf{E}\left[\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)\right]=\infty$ whenever $p \geq \alpha$.

### 4.5.3 Annealed results

We with the proof of the annealed result (i.e. Theorem 4.5.1(i)). We first note that the first order term can be obtained directly from Theorem 4.0.10 by applying the Tauberian theorem of [Kor04, Theorem 8.1, Chapter IV]. However, to obtain the second order term we must give a longer argument.

To apply the results of Propositions 4.5.2 and 4.5.3, it is convenient to think of the argument $\lambda$ as a time index and bound the number of eigenvalues as the population size of a branching process with a certain offspring distribution. In this setting, it will also be natural to reparametrise time to enable a comparison with a supercritical Crump-Mode-Jagers process. Accordingly, for all $i \in \Sigma_{*}$, we set:

$$
X_{i}(t)=N_{i}^{D}\left(e^{t}\right),
$$

and

$$
\eta_{i}(t)=X_{i}(t)-\sum_{j=1}^{\infty} X_{i j}\left(t+\gamma^{-1}\left(\log \Delta_{i j}\right)\right)=N_{i}^{D}\left(e^{t}\right)-\sum_{j=1}^{\infty} N_{i j}^{D}\left(\Delta_{i j}^{\frac{1}{7}} e^{t}\right)
$$

$\eta_{i}$ is the "error" term acquired at level $i$ in the approximation suggested by Proposition 4.5.2. The point is that on iterating the relation of Proposition 4.5.2, we obtain

$$
\begin{equation*}
X(t)=\sum_{i:|i|<k} \eta_{i}\left(t+\gamma^{-1}\left(\log D_{i}\right)\right)+\sum_{i \in \Sigma_{k}} X_{i}\left(t+\gamma^{-1}\left(\log D_{i}\right)\right) \tag{4.31}
\end{equation*}
$$

We can then apply Propositions 4.5.3 and 4.5.4 to show that $\mathbf{P}$-almost surely, the final term on the right hand side converges to zero as $k \rightarrow \infty$. Analysing $X(t)$ therefore reduces to a study of the sum $\sum_{i \in \Sigma_{*}} \eta_{i}\left(t+\gamma^{-1}\left(\log D_{i}\right)\right)$.

This can again be bounded using Proposition 4.5.3: indeed, Propositions 4.5.2 and 4.5.3 can be used to show that

$$
\sum_{i \in \Sigma_{*}} \eta_{i}\left(t+\gamma^{-1}\left(\log D_{i}\right)\right) \leq 2 \sum_{i \in \Sigma_{*}} \mathbb{1}\left\{D_{i}^{\frac{1}{\gamma}} \operatorname{Diam}\left(\mathcal{L}_{i}\right) \geq e^{-t}\right\}
$$

and is therefore stochastically dominated by (twice) the expected population at time $\gamma\left(t+\log \operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)\right)$ of a Crump-Mode-Jagers branching process in which an individual gives birth at times $\left(-\log \Delta_{i}\right)_{i=1}^{\infty}$. This has Malthusian parameter 1. Standard results on the expectation of this population then entail that $e^{-\gamma t} \sum_{i \in \Sigma_{*}} \eta_{i}(t+$ $\left.\gamma^{-1}\left(\log D_{i}\right)\right)$ is bounded by $C \mathbf{E}\left[\left(\operatorname{Diam} \mathcal{L}_{\alpha}\right)^{\gamma}\right]$, which motivates the introduction of the following quantities.

We set $m(t)=e^{-\gamma t} \mathbf{E}\left[X_{t}\right], u(t)=e^{-\gamma t} \mathbf{E}[\eta(t)]$, and define measures $\nu$ and $\nu_{\gamma}$ by $\nu[0, t]=\sum_{i=1}^{\infty} \mathbf{P}\left(\Delta_{i}>e^{-\gamma t}\right)$, and $\nu_{\gamma}(d t)=e^{-\gamma t} \nu(d t)$. Exactly as in [CH10], one can then prove the following result.

Lemma 4.5.5. (cf [CH10, Lemma 3.5]).
(i) $m$ is bounded.
(ii) $u$ is in $L^{1}(\mathbb{R})$, and for any $\varepsilon>0, u(t)=O\left(e^{-\left(\frac{\alpha-1}{\alpha+1}-\varepsilon\right)}\right)$ as $t \rightarrow \infty$.
(iii) $\nu_{\lambda}$ is a Borel probability measure on $[0, \infty)$, with finite expectation.

We further define the limit

$$
m(\infty)=\frac{\int_{\infty}^{\infty} u(t) d t}{\int_{0}^{\infty} t \nu_{\gamma}(d t)}
$$

By imitating the proofs of [CH10], we can show that $m(t) \rightarrow m(\infty)$ as $t \rightarrow \infty$. We do not go into details here, but we explain why $m(\infty)$ is the natural candidate for the limit. Indeed, it can be shown by direct computation that if we set $I=\int_{0}^{\infty} t \nu_{\gamma}(d t)$, then

$$
I=-\gamma^{-1} \mathbf{E}\left[\sum_{i=1}^{\infty} \Delta_{i} \log \Delta_{i}\right]
$$

Moreover, we can write $m(t)$ as

$$
\begin{equation*}
m(t)=\int_{0}^{\infty} u(t-s) \sum_{i=1}^{\infty} e^{-\gamma s} \mathbf{P}\left(-\gamma^{-1} \log \Delta_{i} \in d s\right) . \tag{4.32}
\end{equation*}
$$

It is then straightforward to show (by inverting a Laplace transform) that

$$
\sum_{i=1}^{\infty} e^{-\gamma s} \mathbf{P}\left(-\gamma^{-1} \log \Delta_{i} \in d s\right)-I^{-1} d s=\delta_{0}(s) d s
$$

i.e. the Dirac mass at 0 . Substituting into (4.32) we therefore deduce that $m(t)-$ $m(\infty)=u(t)-I^{-1} \int_{t}^{\infty} u(s) d s$. The result of Theorem 4.5.1(i), including the second order term in the expansion, then follows from Lemma 4.5.5(ii).

### 4.5.4 Quenched results

To prove the almost sure convergence of $e^{-\gamma t} X(t)$ to $m(t)$, we follow [CH10, Section 4] and introduce the characteristics below (so-named in the CMJ literature), which are effectively a truncated version of the quantities of interest. More specifically, we set $\eta_{i}^{c}(t)=\eta_{i}(t) \mathbb{1}\{t \leq c n\}$, and

$$
X_{i}^{c}(t)=\sum_{j \in \Sigma_{*}} \eta_{i j}^{c}\left(t+\gamma^{-1} \Delta_{i j}\right) .
$$

We will eventually take a limit to deduce convergence of the untruncated version.
We also define the cutsets

$$
\begin{aligned}
\Lambda_{t} & =\left\{i \in \Sigma_{*}: D_{i}^{\frac{1}{\gamma}} \leq e^{-t} \leq D_{\left.i\right|_{|i|-1}}^{\frac{1}{\gamma}}\right\} \\
\Lambda_{t, c} & =\left\{i \in \Sigma_{*}: D_{i}^{\frac{1}{\gamma}} \leq e^{-t}, e^{-(t+c)} \leq D_{i| |_{|i|-1}}^{\frac{1}{\gamma}}\right\} .
\end{aligned}
$$

In what follows, rather than repeatedly applying the iteration (4.31) up to a fixed level $\Sigma_{k}$, we will apply it up to the level of a certain cutset $\Lambda_{t}$.

We also write $m^{c}(\infty)=e^{-\gamma t} \mathbf{E}\left[X^{c}(t)\right]$, which can be checked to converge to a limit $m^{c}(\infty)$ as $t \rightarrow \infty$. As in [CH10], we then write:

$$
\left|e^{-\gamma c\left(n+n_{1}\right)} X^{c}\left(c\left(n+n_{1}\right)\right)-m^{c}(\infty)\right| \leq S_{1}+S_{2}+S_{3}
$$

where

$$
\begin{aligned}
& S_{1}=\left|\sum_{i \in \Lambda_{c n} \backslash \Lambda_{c n, c n_{1}}} e^{-\gamma c\left(n+n_{1}\right)} X_{i}^{c}\left(c\left(n+n_{1}\right)+\gamma^{-1} \log D_{i}\right)-D_{i} m^{c}\left(c\left(n+n_{1}\right)+\gamma^{-1} \log D_{i}\right)\right| \\
& S_{2}=\left|\sum_{i \in \Lambda_{c n} \backslash \Lambda_{c n, c n_{1}}} D_{i} m^{c}\left(c\left(n+n_{1}\right)+\gamma^{-1} \log D_{i}\right)-m^{c}(\infty)\right| \\
& S_{3}=\mid \sum_{i \in \Lambda_{c n}} e^{-\gamma\left(n+n_{1}\right)} X_{i}^{c}\left(c\left(n+n_{1}\right)+\gamma^{-1} \log D_{i}\right) .
\end{aligned}
$$

As in [CH10, Proposition 4.5], the first two sums both converge to zero as $n$ and then $n_{1}$ go to infinity: the first using the strong law of large numbers, and the second using branching process techniques.

To show that $\lim _{n_{1} \rightarrow \infty} \lim \sup _{n \rightarrow \infty} S_{3}\left(n, n_{1}\right)=0$, we cannot directly repeat the proof of [CH10, Proposition 4.5] since this a priori requires existence of higher moments of $\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)$. However, this can be fixed by the following modifications of [CH10, Lemmas 4.2-4.4] which show that $X(t)$ has finite second moment.

Lemma 4.5.6. of [CH10, Lemma 4.2]. Let $i \in \Sigma_{k}, j \in \Sigma_{l}$ with $k \leq l$, and let $\theta \in\left(0, \frac{\alpha}{2}\right.$. Let $m=m(i, j)$ denote $i \wedge j, \delta_{i}=\operatorname{Diam}\left(\mathcal{L}_{i}\right)$ and $\delta=\operatorname{Diam}\left(\mathcal{L}_{\alpha}\right)$. Then:
(i) If $m(i, j)<k$, then:

$$
\begin{aligned}
& \mathbf{P}\left(D_{i}^{\frac{1}{\gamma}} \delta_{i} \geq e^{-t}, D_{j}^{\frac{1}{\gamma}} \delta_{j} \geq e^{-t}\right) \\
& \leq e^{2 \theta t} \mathbf{E}\left[\delta^{\theta}\right]^{2} \mathbf{E}\left[\Delta_{\left.i\right|_{m+1}}^{\frac{2 \theta}{\gamma}}\right]^{\frac{1}{2}} \mathbf{E}\left[\Delta_{\left.j\right|_{m+1}}^{\frac{2 \theta}{\gamma}}\right]^{\frac{1}{2}} \mathbf{E}\left[D_{\left.i\right|_{m}}^{\frac{2 \theta}{\gamma}}\right] \mathbf{E}\left[\left(D_{i}^{\left.i\right|_{m+1}}\right)^{\frac{\theta}{\gamma}}\right] \mathbf{E}\left[\left(D_{j}^{\left.j\right|_{m+1}}\right)^{\frac{\theta}{\gamma}}\right] .
\end{aligned}
$$

(ii) If $m(i, j)<k$, then:

$$
\mathbf{P}\left(D_{i}^{\frac{1}{\gamma}} \delta_{i} \geq e^{-t}, D_{j}^{\frac{1}{\gamma}} \delta_{j} \geq e^{-t}\right) \leq e^{2 \theta t} \mathbf{E}\left[\delta^{2 \theta}\right] \mathbf{E}\left[D_{i}^{\frac{2 \theta}{\gamma}}\right] \mathbf{E}\left[\left(D_{j}^{i}\right)^{\frac{\theta}{\gamma}}\right]
$$

Proof. (i) In the case $m<k$, which means that $j$ is not a descendant of $i$, we proceed as in [CH10, Lemma 4.4] to show that:

$$
\begin{aligned}
& \mathbf{P}\left(D_{i}^{\frac{1}{\gamma}} \delta_{i} \geq e^{-t}, D_{j}^{\frac{1}{\gamma}} \delta_{j} \geq e^{-t}\right) \\
& \leq e^{2 \theta t} \mathbf{E}\left[\delta_{i}^{\theta} \delta_{j}^{\theta} \Delta_{\left.i\right|_{m+1}}^{\frac{\theta}{\gamma}} \Delta_{\left.j\right|_{m+1}}^{\frac{\theta}{\gamma}}\right] \mathbf{E}\left[D_{\left.i\right|_{m}}^{\frac{2 \theta}{\gamma}}\right] \mathbf{E}\left[\left(D_{i}^{\left.i\right|_{m+1}}\right)^{\frac{\theta}{\gamma}}\right] \mathbf{E}\left[\left(D_{j}^{\left.j\right|_{m+1}}\right)^{\frac{\theta}{\gamma}}\right] .
\end{aligned}
$$

Since $j$ is not a descendant of $i$, and by Theorem 4.3.1 the looptrees $\mathcal{L}_{\left.i\right|_{m+1}}$ and $\mathcal{L}_{\left.j\right|_{m+1}}$ are independent, it follows that $\delta_{i}$ is independent of $\delta_{j}$, and moreover
that they are both independent of $\Delta_{\left.i\right|_{m+1}}^{\frac{\theta}{\gamma}}$ and $\Delta_{\left.j\right|_{m+1}}^{\frac{\theta}{\gamma}}$. We deduce that

$$
\begin{aligned}
\mathbf{E}\left[\delta_{i}^{\theta} \delta_{j}^{\theta} \Delta_{\left.i\right|_{m+1}}^{\frac{\theta}{\gamma}} \Delta_{\left.j\right|_{m+1}}^{\frac{\theta}{\gamma}}\right] & \leq \mathbf{E}\left[\delta_{i}^{\theta}\right] \mathbf{E}\left[\delta_{j}^{\theta}\right] \mathbf{E}\left[\Delta_{\left.i\right|_{m+1}}^{\frac{\theta}{\gamma}} \Delta_{\left.j\right|_{m+1}}^{\frac{\theta}{\gamma}}\right] \\
& \leq \mathbf{E}\left[\delta_{i}^{\theta}\right] \mathbf{E}\left[\delta_{j}^{\theta}\right] \mathbf{E}\left[\Delta_{\left.i\right|_{m+1}}^{\frac{2 \theta}{\gamma}}\right]^{\frac{1}{2}} \mathbf{E}\left[\Delta_{\left.j\right|_{m+1}}^{\frac{2 \theta}{\gamma}}\right]^{\frac{1}{2}},
\end{aligned}
$$

and the result follows.
(ii) If $m=k$ then again following [CH10, Lemma 4.4] we have that

$$
\mathbf{P}\left(D_{i}^{\frac{1}{\gamma}} \delta_{i} \geq e^{-t}, D_{j}^{\frac{1}{\gamma}} \delta_{j} \geq e^{-t}\right) \leq e^{2 \theta t} \mathbf{E}\left[\delta_{i}^{\theta} \delta_{j}^{\theta}\left(D_{j}^{i}\right)^{\frac{\theta}{\gamma}}\right] \mathbf{E}\left[D_{i}^{\frac{2 \theta}{\gamma}}\right]
$$

In this case $\delta_{i}$ and $\delta_{j}$ are not independent of each other, however by Theorem 4.3.1 they are independent of the Poisson-Dirichlet weight $D_{i}^{j}$. By factorising and then applying Cauchy-Schwarz, we therefore deduce that

$$
\mathbf{E}\left[\delta_{i}^{\theta} \delta_{j}^{\theta}\left(D_{j}^{i}\right)^{\frac{\theta}{\gamma}}\right] \leq \mathbf{E}\left[\delta^{2 \theta}\right] \mathbf{E}\left[\left(D_{j}^{i}\right)^{\frac{\theta}{\gamma}}\right]
$$

as claimed.

The next two lemmas then follow from Lemma 4.5.6 exactly as per the argument in [CH10], so we omit the proofs. In fact it is now even easier since we have removed the dependence on $\varepsilon$ considered there.

Lemma 4.5.7. cf [CH10, Lemma 4.3]. For $\theta \in\left(\frac{\gamma}{\alpha}, \frac{\alpha}{2}\right)$, there exists a finite constant $C$ such that

$$
\sum_{i \in \Sigma_{k}} \sum_{j \in \Sigma_{l}} \eta_{i}\left(t+\gamma^{-1} \log D_{i}\right) \eta_{j}\left(t+\gamma^{-1} \log D_{j}\right) \leq C e^{2 \theta t}(k+1) \psi_{1}^{k+1}\left(\frac{\psi_{2}}{\psi_{1}^{2}} \vee 1\right)^{k}
$$

Lemma 4.5.8. cf [CH10, Lemma 4.4]. For $\theta \in\left(\gamma, \frac{\alpha}{2}\right)$, there exists a finite constant $C$ such that

$$
\mathbf{E}\left[X(t)^{2}\right] \leq C e^{2 \theta t}
$$

In light of Lemma 4.5.8, we can modify the exponents given in the proof of [CH10, Proposition 4.5] to show that $S_{3}$ is finite. In particular, we define:

$$
\begin{aligned}
\varphi_{i}^{c, n_{1}}(t) & =\sum_{j=1}^{\infty} X_{i j}(t) \mathbb{1}\left\{t+c n_{1}+\log \delta_{i j} \geq-\gamma^{-1} \log \Delta_{i j} \geq t+c n_{1}\right\} \\
Y^{c, n_{1}}(t) & =\sum_{i \in \Sigma_{*}} \varphi_{i}^{c, n_{1}}\left(t+\gamma^{-1} \log D_{i}\right)
\end{aligned}
$$

and we can then check the conditions of [Ham00, Lemma 3.2] to confirm that there exists $\varepsilon>0$ such that

$$
e^{-\gamma t} Y^{c, n_{1}}(t) \leq C e^{-c n_{1}\left(\gamma \alpha^{-1}+\varepsilon\right)}
$$

for all sufficiently large $t$. This is done exactly as in [CH10], except that we take $h(t)=e^{-\left(\gamma\left(\alpha^{-1}+1\right)+\varepsilon\right) t}$ (for some sufficiently small $\varepsilon>0$ ), and in equation (25) there we replace the exponent $\frac{1+\alpha}{2(2 \alpha-1)}$ with $\gamma \alpha^{-1}+\varepsilon$, and we deduce that, $\mathbf{P}$-almost surely

$$
\lim _{n_{1} \rightarrow \infty} \limsup _{n \rightarrow \infty} S_{3}\left(n, n_{1}\right)=O\left(e^{-\left(\gamma\left(1-\alpha^{-1}\right)-\varepsilon\right) c n}\right)
$$

Combining with the results for $S_{1}$ and $S_{2}$ implies in particular that $\mathbf{P}$-almost surely,

$$
\lim _{n \rightarrow \infty}\left|e^{-\gamma n c} X^{c}(c n)-m^{c}(\infty)\right|=0
$$

Again by taking $h(t)=e^{-\left(\gamma\left(\alpha^{-1}+1\right)+\varepsilon\right) t}$, we can then proceed exactly as in [CH10] to take a limit along the subsequence $(c n)_{n \geq 1}$ and deduce that, in actual fact, $\mathbf{P}$-almost surely

$$
\limsup _{n \rightarrow \infty} e^{-\gamma n c}\left|X^{c}(c n)-X(c n)\right|=C e^{-n_{0}\left(\gamma\left(1-\alpha^{-1}\right)-\varepsilon\right) t}
$$

from which we can deduce Theorem 4.5 .1(ii) by taking $n_{0}$ to infinity and using monotonicity of $X$.

To prove the second part of Theorem 4.5.1(ii) about the second order term one can also just repeat the proofs of [CH10, Section 5]. The arguments are the same and just use the same principles outlined above, so we do not write the details.

## Chapter 5

## Infinite Stable Looptrees

The purpose of this chapter is to make a formal definition of an infinite stable looptree, which we denote by $\mathcal{L}_{\alpha}^{\infty}$ for $\alpha \in(1,2)$. Our construction is the natural one in light of the previous constructions of infinite trees and looptrees outlined in Section 3.3, and is further justified by the following local limit theorem, showing that $\mathcal{L}_{\alpha}^{\infty}$ can be characterised as a local limit of compact stable looptrees as their mass goes to infinity.

Theorem 5.0.1. Let $\mathcal{L}_{\alpha}^{\ell}$ be a compact stable looptree conditioned to have mass $\ell$, and let $\mathcal{L}_{\alpha}^{\infty}$ be as above. Then,

$$
\left(\mathcal{L}_{\alpha}^{\ell}, \tilde{d}^{\ell}, \nu^{\ell}, \rho^{\ell}\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

as $\ell \rightarrow \infty$, with respect to the Gromov-Hausdorff-vague topology. Here $\tilde{d}^{\ell}$ and $\tilde{d}^{\infty}$ can denote either the geodesic metrics, or the effective resistance metrics on the respective spaces.

We also prove a similar scaling limit result.
Theorem 5.0.2. Let $T_{\alpha}^{\infty}$ denote Kesten's tree with critical offspring distribution in the domain of attraction of an $\alpha$-stable law. Also let $\nu^{\text {disc }}$ denote the measure that gives mass 1 to every vertex of $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$. Then

$$
\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\text {disc }}, \rho\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

with respect to the Gromov-Hausdorff vague topology as $n \rightarrow \infty$. Here $\tilde{d}$ and $\tilde{d}^{\infty}$ can denote either the graph distances, or the effective resistance metrics on the respective spaces.

We will see in Section 5.3 that similar results hold for the infinite discrete looptrees defined in [BS15] and [Ric18a].


Figure 5.1: Relations between discrete/continuum and compact/infinite looptrees.

In fact, we prove all the remaining equivalences in the following diagram.
Given these two theorems, we are also in the right setting to apply results of [Cro18] regarding limits for stochastic processes on these spaces. In particular, we obtain the following results. We let $B^{\infty}$ denote Brownian motion on $\mathcal{L}_{\alpha}^{\infty}$; this is formally defined in Section 5.5 analogously to the compact case considered in Chapter 4.

Theorem 5.0.3. Let $\left(B_{t}^{\ell}\right)_{t \geq 0}$ be Brownian motion on $\mathcal{L}_{\alpha}^{\ell}$, and let $\left(B_{t}^{\infty}\right)_{t \geq 0}$ be Brownian motion on $\mathcal{L}_{\alpha}^{\infty}$. Then there exists a probability space $\left(\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbf{P}^{\prime}\right)$ on which we can almost surely define a metric space $\left(M, R_{M}\right)$ in which the spaces $\left(\mathcal{L}_{\alpha}^{\ell}, R^{\ell}, \nu^{\ell}, \rho^{\ell}\right)$ and $\left(\mathcal{L}_{\alpha}^{\infty}, R^{\infty}, \nu^{\infty}, \rho^{\infty}\right)$ can all be embedded and such that

$$
\left(\mathcal{L}_{\alpha}^{\ell}, R^{\ell}, \nu^{\ell}, \rho^{\ell}\right) \rightarrow\left(\mathcal{L}_{\alpha}^{\infty}, R^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

with respect to the Gromov-Hausdorff-vague topology as $\ell \rightarrow \infty$, and the required Hausdorff convergence specifically holds in the metric space $\left(M, R_{M}\right)$. Letting $\left(B^{\ell}\right)_{\ell \geq 1}$ and $B^{\infty}$ be as above, we have that

$$
\left(B_{t}^{\ell}\right)_{t \geq 0} \xrightarrow{(d)}\left(B_{t}^{\infty}\right)_{t \geq 0}
$$

as $\ell \rightarrow \infty$, considered on the space $C\left(\mathbb{R}^{+}, M\right)$ endowed with the topology of uniform convergence on compact time intervals.

Theorem 5.0.4. Let $\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\text {disc }}, \rho\right)$ be as in Theorem 5.0.2. Then there exists a probability space $\left(\Omega^{\prime \prime}, \mathcal{F}^{\prime \prime}, \mathbf{P}^{\prime \prime}\right)$ on which we can almost surely define a metric space $\left(M, R_{M}\right)$ in which the spaces $\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right), C a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\text {disc }}, \rho\right)$ and $\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)$ can all be embedded and such that

$$
\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\text {disc }}, \rho\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

with respect to the Gromov-Hausdorff-vague topology as $n \rightarrow \infty$, and the required Hausdorff convergence specifically holds on the metric space $\left(M, R_{M}\right)$. Letting $Y$ be
a simple random walk on $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$, and $B^{\infty}$ be as above, we have that

$$
\left(a_{n}^{-1} Y_{\left\lfloor 4 n a_{n} t\right\rfloor}\right)_{t \geq 0} \xrightarrow{(d)}\left(B_{t}^{\infty}\right)_{t \geq 0}
$$

on the space $D\left(\mathbb{R}^{+}, M\right)$ endowed with the Skorokhod- $J_{1}$ topology as $n \rightarrow \infty$.
Again, we will prove a similar result for random walks on the other infinite discrete looptrees in Section 5.5, along with annealed versions, but the one above is easiest to state as all vertices have degree 4 in $\operatorname{Loop}\left(T_{\infty}\right)$.

The process $B^{\infty}$ is considered further in Section 5.5 where we prove the following results about the spectral dimension of $\mathcal{L}_{\alpha}^{\infty}$. Recall that the spectral dimension of $\mathcal{L}_{\alpha}^{\infty}$ is defined as

$$
\begin{equation*}
d_{S}\left(\mathcal{L}_{\alpha}^{\infty}\right)=\lim _{t \rightarrow \infty} \frac{-2 \log \left(p_{t}^{\infty}\left(\rho^{\infty}, \rho^{\infty}\right)\right)}{\log t} \tag{5.1}
\end{equation*}
$$

where $p_{t}^{\infty}(\cdot, \cdot)$ is the transition density of the Brownian motion $B^{\infty}$ defined above, i.e. a symmetric $\nu^{\infty} \times \nu^{\infty}$-measurable function on $\mathcal{L}_{\alpha}^{\infty} \times \mathcal{L}_{\alpha}^{\infty}$ such that

$$
\mathbf{E}_{x}\left[f\left(B_{t}\right)\right]=\int_{\mathcal{L}_{\alpha}^{\infty}} f(y) p_{t}(x, y) \nu^{\infty}(d y)
$$

for all bounded, $\nu^{\infty}$-measurable functions $f$ on $\mathcal{L}_{\alpha}^{\infty}$ and $\nu^{\infty}$-almost every $x \in \mathcal{L}_{\alpha}^{\infty}$.
We assume that $\mathcal{L}_{\alpha}^{\infty}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and let $\mathbf{E}$ denote expectation on this space.

Theorem 5.0.5. $\mathbf{P}$-almost surely, $d_{S}\left(\mathcal{L}_{\alpha}^{\infty}\right)=\frac{2 \alpha}{\alpha+1}$.
In light of Theorem 5.0.5, we call $d_{S}\left(\mathcal{L}_{\alpha}^{\infty}\right)$ the quenched spectral dimension. We also define the annealed spectral dimension as

$$
d_{S}^{a}\left(\mathcal{L}_{\alpha}^{\infty}\right)=\lim _{t \rightarrow \infty} \frac{-2 \log \left(\mathbf{E}\left[p_{t}^{\infty}\left(\rho^{\infty}, \rho^{\infty}\right)\right]\right)}{\log t} .
$$

For a general space, the annealed heat kernel is trickier to bound than the quenched one defined above, since the expected transition density may not be finite. This is the case, for example, for the trees with heavy-tailed offspring distributions considered in [CK08]. In the case of stable looptrees however we are able to bound this using the volume and resistance estimates of Section 5.4, and then utilise scaling invariance of $\mathcal{L}_{\alpha}^{\infty}$ to prove the following (more precise) result.

Theorem 5.0.6. There exists $c_{1} \in(0, \infty)$ such that $\mathbf{E}\left[p_{t}^{\infty}\left(\rho^{\infty}, \rho^{\infty}\right)\right]=c_{1} t^{\frac{-\alpha}{\alpha+1}}$. In particular,

$$
d_{S}^{a}\left(\mathcal{L}_{\alpha}^{\infty}\right)=\frac{2 \alpha}{\alpha+1}
$$

Both the quenched and annealed spectral dimensions match those obtained for the infinite discrete looptrees defined from offspring distributions in the domain of attraction of an $\alpha$-stable law in [BS15].

We start by giving the construction of infinite stable looptrees in Section 5.1. We then prove Theorem 5.0.1 in Section 5.2. In Section 5.3 we prove several scaling limit results, and in Section 5.4 we consider volume and resistance bounds on $\mathcal{L}_{\alpha}^{\infty}$, which enables to prove various results for random walk limits in Section 5.5.

We prove the limiting results of Theorems 4.0.2, 4.3.8 and 4.0.10 of Chapter 4 in Sections 5.4.1 and 5.5.3 respectively.

### 5.1 Construction of infinite stable looptrees

It seems clear that the natural construction of infinite stable looptrees should use two stable Lévy processes to code each side of the loopspine, in place of the excursion. This is also the approach suggested in [Ric18a, Section 6] and our construction is merely the continuum version of the discrete construction of [Ric18a, Section 3], except that we have essentially turned this construction "upside down" to match the original coding mechanism for compact looptrees.

We will prove Theorem 5.0.1 for stable looptrees rooted at a uniform point. By taking a stable looptree coded by an excursion $X^{\text {exc, } \ell}$ of length $\ell$, and taking a uniform point in $U \in[0, \ell]$, it follows from the Vervaat transformation that the processes $\left(X_{t}^{\text {exc, }, \ell}\right)_{0 \leq t \leq U}$ and $\left(X_{t}^{\text {exc, } \ell}\right)_{U \leq t \leq \ell}$ are distributed respectively as the the post- and pre-minimum parts of a stable Lévy bridge. Standard convergence results then imply that on any compact interval, these converge in distribution to stable Lévy processes as $\ell \rightarrow \infty$, and results of [Mil77] imply that these two processes are independent of each other. Moreover, if we think of the loopspine as the sequence of loops coded by jump points at times $0 \preceq t \preceq U$, then $\left(X_{t}^{\text {exc }, \ell}\right)_{0 \leq t \leq U}$ codes for the loopspine along with everything grafted to the left hand side of it, and $\left(X_{t}^{\mathrm{exc}, \ell}\right)_{U \leq t \leq \ell}$ codes for everything grafted to the right hand side of it. It therefore seems natural to replace each of these by unconditioned Lévy process in the infinite volume limit.

We start by writing this below as an equivalent construction of compact stable looptrees. We give the construction for a looptree of mass $\ell$.

## Two-sided Construction of Compact Stable Looptrees

1. Let $X^{\text {br }, \ell}$ be a spectrally positive, $\alpha$-stable Lévy bridge of lifetime $\ell$. Let $m=m_{\ell}$ be the (almost surely unique) time at which $X^{\text {br }, \ell}$ attains its infimum.
2. Let $\left(X_{t}^{(2, \ell)}\right)_{t \geq 0}$ be the pre-infimum process, and $\left(X_{t}^{(1, \ell)}\right)_{t \geq 0}$ be the timereversed post-infimum process, extended to stay constant after times $m$ and $1-m$ respectively. That is,

$$
X_{t}^{(2, \ell)}=\left\{\begin{array}{l}
X_{t}^{\mathrm{br}} \text { for } t \in[0, m], \\
X_{m}^{\mathrm{br}} \text { for } t>m ;
\end{array} \quad X_{t}^{(1, \ell)}=\left\{\begin{array}{l}
X_{\ell-t}^{\mathrm{br}} \text { for } t \in[0,1-m], \\
X_{m}^{\mathrm{br}} \text { for } t>\ell-m .
\end{array}\right.\right.
$$

3. Define a function $X^{\ell}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
X_{t}^{\ell}= \begin{cases}X_{t}^{(2, \ell)} & \text { if } t \geq 0 \\ X_{-t}^{(1, \ell)} & \text { if } t<0\end{cases}
$$

It should be clear from the Vervaat transform that $X^{\ell}$ is just a shifted Lévy excursion.
4. For $s, t \in \mathbb{R}$, we define resistances $r^{\ell}, R_{0}^{\ell}$ and $R^{\ell}$ from $X^{\ell}$ exactly as in (4.9), (4.10) and (4.11), but with the superscript $\ell$ on all the quantities involved. We can similarly define distances $\delta^{\ell}, d_{0}^{\ell}$ and $d^{\ell}$ exactly as in (3.7). Analogously to the normalised case, we then set $\mathcal{L}_{\alpha}^{\ell}=\left(\mathbb{R} / \sim, d^{\ell}\right)$, and $\mathcal{L}_{\alpha}^{\ell}{ }^{R}=\left(\mathbb{R} / \sim, R^{\ell}\right)$, and let $p^{\ell}: \mathbb{R} \rightarrow \mathcal{L}_{\alpha}^{\ell}$ denote the canonical projection.

Due to the Vervaat transformation, this construction is entirely equivalent to the original construction of looptrees using the Lévy excursion, but we have now split the coding into two functions which define each side of the loopspine. To code the infinite looptree, we will take limits of each of these functions and use these to code each side of the infinite loopspine.

## Construction of Infinite Stable Looptrees

1. Let $X$ be an $\alpha$-stable, spectrally positive Lévy process, and let $X^{\prime}$ be an $\alpha$-stable, spectrally negative Lévy process.
2. Define a function $X^{\infty}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
X_{t}^{\infty}= \begin{cases}X_{t} & \text { if } t \geq 0 \\ X_{-t^{-}}^{\prime} & \text { if } t<0\end{cases}
$$

3. Analogously to the compact construction above, if $t$ is a jump point of $X^{\infty}$ with jump size $\Delta_{t}$ and $a, b \in\left[0, \Delta_{t}\right]$, set

$$
\begin{aligned}
& \delta_{t}^{\infty}(a, b)=\min \left\{|a-b|, \Delta_{t}-|a-b|\right\}, \\
& r_{t}^{\infty}(a, b)=\left(\frac{1}{|a-b|}+\frac{1}{\Delta_{t}-|a-b|}\right)^{-1}=\frac{|a-b|\left(\Delta_{t}-|a-b|\right)}{\Delta_{t}} .
\end{aligned}
$$

Additionally, as before, for $s, t \in \mathbb{R}$ with $s \leq t$ set $I_{s, t}^{\infty}=\inf _{r \in[s, t]} X_{r}^{\infty}$, and $x_{s, t}^{\infty}=I_{s, t}^{\infty}-X_{s^{-}}^{\infty}$. For $s, t \in \mathbb{R}$ we again write $s \prec t$ if $s \preceq t$ (meaning that $x_{s, t}^{\infty} \geq 0$ ) and $s \neq t$. Then, if $s \preceq t$ set

$$
\begin{aligned}
d_{0}^{\infty}(s, t) & =\sum_{s \prec u \_t} \delta_{u}^{\infty}\left(0, x_{u}^{t}\right), \\
R_{0}^{\infty}(s, t) & =\sum_{s \prec u \_t} r_{u}^{\infty}\left(0, x_{u}^{t}\right) .
\end{aligned}
$$

Then, for general $s, t \in \mathbb{R}$, set

$$
\begin{align*}
d^{\infty}(s, t) & =\delta_{s \wedge t}^{\infty}\left(x_{s \wedge t, s}^{\infty}, x_{s \wedge t, t}^{\infty}\right)+d_{0}^{\infty}(s \wedge t, s)+d_{0}^{\infty}(s \wedge t, t)  \tag{5.2}\\
R^{\infty}(s, t) & =r_{s \wedge t}^{\infty}\left(x_{s \wedge t, s}^{\infty}, x_{s \wedge t, t}^{\infty}\right)+R_{0}^{\infty}(s \wedge t, s)+R_{0}^{\infty}(s \wedge t, t)
\end{align*}
$$

Finally, define an equivalence relation $\sim$ on $\mathbb{R}$ by setting $s \sim t$ if and only if $d^{\infty}(s, t)=0$. We define the infinite looptrees $\mathcal{L}_{\alpha}^{\infty}$ and $\mathcal{L}_{\alpha}^{\infty, R}$ by

$$
\begin{aligned}
\mathcal{L}_{\alpha}^{\infty} & =\left(\mathbb{R} / \sim, d^{\infty}\right), \\
\mathcal{L}_{\alpha}^{\infty, R} & =\left(\mathbb{R} / \sim, R^{\infty}\right) .
\end{aligned}
$$

For ease of notation and intuition, we will focus on $\mathcal{L}_{\alpha}^{\infty}$ rather than $\mathcal{L}_{\alpha}^{\infty, R}$ in the following sections, but the results will hold in the resistance setting by exactly the same arguments.

As in the compact case, we can define the projection $p^{\infty}: \mathbb{R} \rightarrow \mathcal{L}_{\alpha}^{\infty}$, which is almost surely continuous, and endow $\mathcal{L}_{\alpha}^{\infty}$ with the measure $\nu^{\infty}$ which is defined
to be the pushforward of Lebesgue measure on the real line to $\mathcal{L}_{\alpha}^{\infty}$ via $p^{\infty}$.
We also have the following proposition, as a direct consequence of the scale invariance of the stable Lévy process.

Proposition 5.1.1 (Scale invariance of $\mathcal{L}_{\alpha}^{\infty}$ ). For any $c>0$,

$$
\left(\mathcal{L}_{\alpha}^{\infty}, c \tilde{d}, \rho^{\infty}, c^{\alpha} \nu^{\infty}\right) \stackrel{(d)}{=}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}, \rho^{\infty}, \nu^{\infty}\right)
$$

where $\tilde{d}$ here can be equal to either $d^{\infty}$ or $R^{\infty}$.

### 5.2 Proof of Theorem 5.0.1

The proof of Theorem 5.0.1 essentially stems from the fact that the two sides of the Lévy bridge used to code a compact stable looptree converge in distribution to a Lévy process on any compact time interval as $\ell \rightarrow \infty$. We first recap the following result, that also appeared as Proposition 3.2.3.

Proposition 5.2.1. Let $\left(f_{n}\right)_{n \geq 1}$ be a sequence in $D([0,1], \mathbb{R})$, and $f \in D^{\operatorname{exc}}([0,1], \mathbb{R})$ be such that $f_{n} \rightarrow f$ as $n \rightarrow \infty$ with respect to the Skorokhod topology. Additionally let $\nu$ and $\nu_{n}$ be the projections of Lebesgue measure onto the spaces $\mathcal{L}_{f}$ and $\mathcal{L}_{f_{n}}$ respectively. Then

$$
d_{G H P}\left(\left(\operatorname{Loop}\left(\tau_{n}\right), \tilde{d}_{n}, \nu_{n}, \rho_{n}\right),\left(\mathcal{L}_{f}, \tilde{d}_{f}, \nu_{f}, \rho_{f}\right)\right) \rightarrow 0
$$

as $n \rightarrow \infty$.
Here $\tilde{d}$ can denote either the shortest-distance metric of [CK14], or the resistance metric of (4.11), but defined using the function $f$ in place of $X^{\text {exc }}$.

Clearly the result of the proposition will still hold on any compact time interval, not just $[0,1]$.

To prove Theorem 5.0.1 we will make use of the following result and apply Theorem 3.2.3.

Theorem 5.0.1 is proved by applying Proposition 3.2.3 to the following convergence result. The Lévy processes are all normalised as in Section 2.5.1.

Proposition 5.2.2. Let $X^{b r, \ell}$ be a spectrally positive, $\alpha$-stable Lévy bridge of lifetime $\ell$, starting and ending at 0 , let $X$ be an $\alpha$-stable, spectrally positive Lévy process, and let $X^{\prime}$ be an independent $\alpha$-stable, spectrally negative Lévy process. Also let $m_{\ell}$ be the (almost surely unique) time at which $X^{b r, \ell}$ attains its minimum. Then, for any $T_{1}, T_{2}>0$, letting $f$ and $g$ be any bounded continuous functions $D\left(\left[0, T_{i}\right], \mathbb{R}\right) \rightarrow \mathbb{R}$, we have that

$$
\mathbb{E}\left[f\left(\left(X_{t \wedge m_{\ell}}^{b r, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{\left((\ell-t) \vee m_{\ell}\right)^{-}}^{b r, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right] \rightarrow \mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right)\right] \mathbb{E}\left[g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\right]
$$

as $\ell \rightarrow \infty$.
Before we prove the proposition, we show how we can apply Proposition 3.2.3 to the functions $X$ and $X^{\prime}$ on compact time intervals to prove Theorem 5.0.1.

Proof of Theorem 5.0.1, assuming Proposition 5.2.2. We need to show that for almost every $r>0$,

$$
\begin{equation*}
\mathcal{B}_{r}\left(\mathcal{L}_{\alpha}^{\ell}\right) \xrightarrow{(d)} \mathcal{B}_{r}\left(\mathcal{L}_{\alpha}^{\infty}\right) \tag{5.3}
\end{equation*}
$$

To this end, take some $r>0$. We define two times $t_{g}(r)$ and $t_{d}(r)$ by
$t_{g}(r)=\inf \left\{s \geq 0: \Delta_{-s} \geq 4 r, \delta_{-s}^{\infty}\left(x_{-s, 0}^{\infty}\right) \geq r\right\}, t_{d}(r)=\inf \left\{s \geq 0: X_{s}^{\infty} \leq X_{-t_{g}(r)^{-}}^{\infty}\right\}$.
The purpose of defining $t_{g}(r)$ and $t_{d}(r)$ like this is that $X^{\infty}$ codes a compact looptree on the interval $\left[-t_{g}(r), t_{d}(r)\right]$, and that $\mathcal{B}_{r}\left(\mathcal{L}_{\alpha}^{\infty}\right)$ is contained in this.

Note that $t_{g}(r)$ is $\mathbf{P}$-almost surely finite, since if $L_{s}$ is the local time spent by $\left(X_{-t^{+}}^{\infty}\right)_{t \geq 0}$ at its infimum by time $s$, normalised so that $\mathbb{E}\left[e^{\lambda X_{L^{-1}(t)}^{\infty}}\right]=e^{-\lambda^{\alpha-1} t}$, we have from Proposition 2.5.4 that the measure

$$
\sum_{s \in J} \delta_{\left(L_{s}, \Delta_{s}\right)}
$$

is a Poisson point measure of intensity $d l \cdot x \mathbb{1}\left\{x^{-\alpha} \geq 4 r\right\} d x$, where $J$ is the set $\left\{s \geq 0: \Delta_{-s} \geq 4 r, \delta_{-s}^{\infty}\left(x_{-s, 0}^{\infty}\right) \geq r\right\}$. Moreover, by [Ber96, Chapter VIII, Lemma 1] we know that $L^{-1}$ is a stable subordinator of parameter $1-\frac{1}{\alpha}$, and hence $L_{t} \rightarrow \infty$ $\mathbf{P}$-almost surely as $t \rightarrow \infty$. It follows that $t_{g}(r)$ is $\mathbf{P}$-almost surely finite for all $r>0$. Similarly, since $\liminf _{t \rightarrow \infty} X_{t}^{\infty}=-\infty \mathbf{P}$-almost surely, $t_{d}(r)$ is also $\mathbf{P}$-almost surely finite for all $r>0$.

For notational convenience, we write $t_{g}=t_{g}(r)$ and $t_{d}=t_{d}(r)$ from now on.
The compact looptree $\mathcal{L}_{\alpha}^{\ell}$ is coded by an excursion $X^{\text {exc, } \ell}$ of length $\ell$. To write this as a two-sided construction as described in the previous section, choose $U_{\ell}$ uniform on $[0, \ell]$, and define a function $X^{\mathrm{br}, \ell}:\left[-U_{\ell}, \ell-U_{\ell}\right]$ by

$$
X_{t}^{\mathrm{br}, \ell}=X_{t+U_{\ell}}^{\mathrm{exc}, \ell}-X_{U_{\ell}}^{\mathrm{exc}, \ell}
$$

for all $t \in\left[-U_{\ell}, \ell-U_{\ell}\right]$. Then $X^{\mathrm{br}, \ell} \operatorname{codes} \mathcal{L}_{\alpha}^{\ell}$. Moreover, we can extend $X^{\mathrm{br}, \ell}$ to $\mathbb{R}$ by taking it to be constant outside of $\left[-U_{\ell}, \ell-U_{\ell}\right]$, and by Proposition 5.2.2, it is then the case that $\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[-t_{g}-1, t_{d}+1\right]} \xrightarrow{(d)}\left(X_{t}^{\infty}\right)_{t \in\left[-t_{g}-1, t_{d}+1\right]}$.

Since the interval $\left[-t_{d}-1, t_{g}+1\right]$ is $\mathbf{P}$-almost surely compact, and the space of càdlàg functions with compact support endowed with the Skorokhod- $J_{1}$ topology is separable, it follows by the Skorokhod Representation Theorem and Proposition 5.2 .2 that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[-t_{g}-1, t_{d}+1\right]} \rightarrow$ $\left(X_{t}^{\infty}\right)_{t \in\left[-t_{g}-1, t_{d}+1\right]}$ almost surely. We henceforth work in this space.

For each $\ell>0$, let $\lambda_{\ell}$ be the Skorokhod homeomorphism (defined pointwise on $\Omega$ ) from $\left[-t_{g}-1, t_{d}+1\right] \rightarrow\left[-t_{g}-1, t_{d}+1\right]$ that minimises the Skorokhod distance between these $X^{\text {br }, \ell}$ and $X^{\infty}$ on this interval. Then set $t_{d}^{\ell}=\lambda_{\ell}\left(t_{d}\right)$, and similarly $t_{g}^{\ell}=\lambda_{\ell}\left(t_{g}\right)$.

The correspondence consisting of all pairs $\left[t, \lambda_{\ell}(t)\right]$ for $t \in\left[-t_{g}, t_{d}\right]$ is a subset of the correspondence used to minimise the Gromov-Hausdorff distance in the proof of Proposition 3.2.3, so letting $\mathcal{L}_{\alpha}^{\ell, r}=p^{\ell}\left(\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[-t_{g}^{\ell}, t_{d}^{\ell}\right]}\right)$ for each $\ell>0$ and $\mathcal{L}_{\alpha}^{\infty, r}=$ $p^{\infty}\left(\left(X_{t}\right)_{t \in\left[-t_{g}, t_{d}\right]}\right)$, it follows from Proposition 3.2.3 that $d_{G H P}\left(\mathcal{L}_{\alpha}^{\ell, r}, \mathcal{L}_{\alpha}^{\infty, r}\right) \rightarrow 0$ as $\ell \rightarrow \infty$. Since $\mathcal{B}_{r}\left(\mathcal{L}_{\alpha}^{\ell}\right) \subset \mathcal{L}_{\alpha}^{\ell, r}$ and $\mathcal{B}_{r}\left(\mathcal{L}_{\alpha}^{\infty}\right) \subset \mathcal{L}_{\alpha}^{\infty, r}$, it thus follows that $\mathcal{B}_{r}\left(\mathcal{L}_{\alpha}^{\ell}\right) \xrightarrow{(d)}$ $\mathcal{B}_{r^{\prime}}\left(\mathcal{L}_{\alpha}^{\infty}\right)$ for Lebesgue almost every $r^{\prime}<r$. By taking a countable sequence $r_{n} \rightarrow \infty$ we therefore deduce the result for Lebesgue almost-every $r>0$, and the theorem follows.

We now conclude the proof of Theorem 5.0.1 by proving Proposition 5.2.2.
Proof of Proposition 5.2.2. The key point is that the two sides of the bridge have a density with respect to the laws of $X$ and $X^{\prime}$, in that for any $f, g$ as in the statement of the proposition, and any $\ell>T_{1}+T_{2}$, it follows from a minor modification of (2.12) that

$$
\begin{align*}
& \mathbb{E}\left[f\left(\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{(\ell-t)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right] \\
& \quad=\mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right) \frac{p_{\ell-T_{1}-T_{2}}\left(X_{T_{2}^{-}}^{\prime}-X_{T_{1}}\right)}{p_{\ell}(0)}\right], \tag{5.4}
\end{align*}
$$

where $p_{t}(\cdot)$ here denotes the transition density of $X$. The proof then essentially just uses the fact that $m_{\ell}$ and $\ell-m_{\ell}$ tend to infinity in probability as $\ell \rightarrow \infty$, and then the fact that with high probability, $X_{T_{1}}$ and $X_{T_{2}}^{\prime}$ will also not be too large. There are two main steps. We first note that the quantity
$\mathbb{E}\left[f\left(\left(X_{t \wedge m_{\ell}}^{\mathrm{br},}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{\left((\ell-\ell) \vee m_{\ell}\right)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right]-\mathbb{E}\left[f\left(\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{(\ell-t)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right]$
is upper bounded by

$$
2\|f\|_{\infty}\|g\|_{\infty}\left(\mathbb{P}\left(m_{1}<\frac{T_{1}}{\ell}\right)+\mathbb{P}\left(m_{1}>1-\frac{T_{2}}{\ell}\right)\right)
$$

which converges to 0 as $\ell \rightarrow \infty$. This allows us to apply (5.4) as follows. First, note that it follows from the scaling relation $p_{t}(x)=t^{\frac{-1}{\alpha}} p_{1}\left(x t^{\frac{-1}{\alpha}}\right)$ that

$$
\frac{p_{\ell-T_{1}-T_{2}}\left(X_{T_{2}}^{\prime}-X_{T_{1}}\right)}{p_{\ell}(0)}=\left(\frac{\ell}{\ell-T_{1}-T_{2}}\right)^{\frac{1}{\alpha}} \frac{p_{1}\left(\left(\ell-T_{1}-T_{2}\right)^{\frac{-1}{\alpha}}\left(X_{T_{2}}^{\prime}-X_{T_{1}}\right)\right)}{p_{1}(0)}
$$

We denote this latter quantity by $p\left(\ell, X, X^{\prime}, T_{1}, T_{2}\right)$, so that

$$
\begin{array}{r}
\mathbb{E}\left[f\left(\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{(\ell-t)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right]-\mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\right] \\
=\mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\left(p\left(\ell, X, X^{\prime}, T_{1}, T_{2}\right)-1\right)\right] .
\end{array}
$$

Taking some $0<\varepsilon \ll \frac{1}{\alpha}$, we decompose on the event $\left\{\left|X_{T_{1}}\right| \vee\left|X_{T_{2}}^{\prime}\right| \leq\left(\ell-T_{1}-\right.\right.$ $\left.\left.T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}\right\}$ and its complement by writing the latter quantity as the sum

$$
\begin{align*}
& \mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\left(p\left(\ell, X, X^{\prime}, T_{1}, T_{2}\right)-1\right) \mathbb{1}\left\{\left|X_{T_{1}}\right| \vee\left|X_{T_{2}}^{\prime}\right| \leq\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}\right\}\right] \\
& +\mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\left(p\left(\ell, X, X^{\prime}, T_{1}, T_{2}\right)-1\right) \mathbb{1}\left\{\left|X_{T_{1}}\right| \vee\left|X_{T_{2}}^{\prime}\right|>\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}\right\}\right] . \tag{5.5}
\end{align*}
$$

We deal with each of these two terms separately. For the first term, note that by continuity of the transition density [Ber96, Section VIII.1],

$$
\sup _{|x| \leq 2\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}}\left\{p_{1}\left(x\left(\ell-T_{1}-T_{2}\right)^{\frac{-1}{\alpha}}\right)\right\} \rightarrow p_{1}(0)
$$

as $\ell \rightarrow \infty$. We apply this by writing:

$$
\begin{aligned}
& \|\left(p\left(\ell, X, X^{\prime}, T_{1}, T_{2}\right)-1\right) \mathbb{1}\left\{\left|X_{T_{1}}\right| \vee\left|X_{T_{2}}^{\prime}\right| \leq\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}\right\}| |_{\infty} \\
& \leq \frac{1}{p_{1}(0)}\left(\left|\left(\left(\frac{\ell}{\ell-T_{1}-T_{2}}\right)^{\frac{1}{\alpha}}-1\right) \sup _{|x| \leq 2\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}}\left\{p_{1}\left(x\left(\ell-T_{1}-T_{2}\right)^{\frac{-1}{\alpha}}\right)\right\}\right|\right. \\
& \left.\quad+\left|\sup _{|x| \leq 2\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}}\left\{p_{1}\left(x\left(\ell-T_{1}-T_{2}\right)^{\frac{-1}{\alpha}}\right)\right\}-p_{1}(0)\right|\right),
\end{aligned}
$$

from which we deduce that the first term in (5.5) converges to zero as $\ell \rightarrow \infty$, since $f$ and $g$ are also bounded. To deal with the second term, we upper bound it by

$$
\|f\|_{\infty}\|g\|_{\infty} \frac{1}{p_{1}(0)}\left\|p_{1}\right\|_{\infty} \mathbb{P}\left(\left|X_{T_{1}}\right| \vee\left|X_{T_{2}}^{\prime}\right|>\left(\ell-T_{1}-T_{2}\right)^{\frac{1}{\alpha}-\varepsilon}\right),
$$

which also vanishes as $\ell \rightarrow \infty$. (Note that $\left\|p_{1}\right\|_{\infty}$ by results of [Zol86, Section I.4]).
It therefore follows by an application of the triangle inequality and the bounds above that

$$
\begin{aligned}
\mathbb{E}[f( & \left.\left.\left(X_{t \wedge m_{\ell}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{\left((\ell-t) \vee m_{\ell}\right)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right]-\mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\right] \\
& \leq \mathbb{E}\left[f\left(\left(X_{\wedge \wedge m_{\ell}}^{\mathrm{br} \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{\left((\ell-\ell) \vee m_{\ell}\right)^{-}}^{\mathrm{br},}\right)_{t \in\left[0, T_{2}\right]}\right)\right]-\mathbb{E}\left[f\left(\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{(\ell-\ell)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right] \\
& +\mathbb{E}\left[f\left(\left(X_{t}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{(\ell-t)^{-}}^{\mathrm{br}, \ell}\right)_{t \in\left[0, T_{2}\right]}\right)\right]-\mathbb{E}\left[f\left(\left(X_{t}\right)_{t \in\left[0, T_{1}\right]}\right) g\left(\left(X_{t}^{\prime}\right)_{t \in\left[0, T_{2}\right]}\right)\right] \\
& \rightarrow 0
\end{aligned}
$$

as $\ell \rightarrow \infty$, as claimed. We can then factorise the final term by independence of $X$ and $X^{\prime}$.

### 5.3 Scaling limits of infinite discrete looptrees

In this section, we prove that infinite stable looptrees are scaling limits of infinite discrete looptrees. We start by proving the following proposition, from which Theorem 5.0.2 will follow. Note the analogy with Proposition 4.1.6, and [CK14, Theorem 4.1].

Given an infinite critical discrete tree $T_{\infty}$, we note that it can be coded by a two-sided Lukasiewicz path indexed by $\mathbb{Z}$ in the same way that an infinite critical continuum tree can be coded by a two-sided Lévy process.

As introduced in Section 3.3, the infinite discrete looptrees defined by Björnberg and Stefánsson in [BS15] are formed by first taking a critical offspring distribution $\xi$ in the domain of attraction of an $\alpha$-stable law, and then forming Kesten's tree $T_{\alpha}^{\infty}$ as outlined in Section 3.3. This tree has a unique infinite spine of vertices with a size-biased version of the offspring distribution. The authors define their looptree as Loop ${ }^{\prime}\left(T_{\alpha}^{\infty}\right)$. Here Loop' is an operation very similar to Loop, obtained as in Figure 5.2, and $d_{G H}\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right), \operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)\right) \leq 2$ (see [CK14, Proof of Theorem 4.1]). We let $L_{\alpha}^{\infty, 1}=\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$.


Figure 5.2: A tree $T$ and $\operatorname{Loop}^{\prime}(T)$, for the same underlying tree as in Figure 3.4.

Remark 5.3.1. In various places in other literature, the notation for Loop and Loop' is interchanged. We have used the notation of [CK14] since our paper follows on more naturally from the results there.

We also make one further definition. Given an infinite critical tree $T_{\infty}$ and $R>0$, we define $\operatorname{Loop}\left(T_{\infty}\right)^{R}$ to be the sublooptree of $\operatorname{Loop}\left(T_{\infty}\right)$ obtained by letting $L$ be the first loop on the infinite loopspine that is of length greater than $4 R$, and such that if we let $l_{1}$ and $l_{2}$ be the lengths of the two segments of this loop obtained by splitting the loop at the two points where it intersects its neighbouring loops in the infinite loopspine, we have that $\frac{l_{1}}{l_{1}+l_{2}} \in\left[\frac{1}{4}, \frac{3}{4}\right]$. We then let $\operatorname{Loop}\left(T_{\infty}\right)^{R}$ be the subset of Loop $\left(T_{\infty}\right)$ obtained by removing all descendants of all points in $L$ (but not removing $L$ itself). This definition is the discrete analogue to that of $\mathcal{L}_{\alpha}^{\infty, R}$ given in the proof of Theorem 5.0.1, and is useful since $\mathcal{B}_{R}\left(\operatorname{Loop}\left(T_{\infty}\right)\right) \subset \operatorname{Loop}\left(T_{\infty}\right)^{R}$, but $\operatorname{Loop}\left(T_{\infty}\right)^{R}$ has the advantage of being a full looptree, whereas $\mathcal{B}_{R}\left(\operatorname{Loop}\left(T_{\infty}\right)\right)$ may contain incomplete loops.

Proposition 5.3.2. Let $\left(\tau_{n}\right)_{n=1}^{\infty}$ be a sequence of infinite critical trees (in the sense of Kesten) with corresponding two-sided Lukasiewicz paths $\left(W^{n}\right)_{n=1}^{\infty}$, and let $\tilde{d}_{n}$ denote either the shortest-distance or effective resistance metric on $\operatorname{Loop}\left(\tau_{n}\right)$. Additionally let $\nu_{n}$ be the measure that gives mass 1 to each vertex in $\operatorname{Loop}\left(\tau_{n}\right)$, and let $\rho_{n}$ be the root of $\operatorname{Loop}\left(\tau_{n}\right)$, defined to be the vertex representing the edge joining the root of $\tau_{n}$ to its first child. Suppose that $\left(C_{n}\right)_{n=1}^{\infty}$ is a sequence of positive real numbers such that
(i) For any compact interval $K \subset \mathbb{R},\left(\frac{1}{C_{n}} W_{\lfloor n t\rfloor}^{n}\right)_{t \in K} \xrightarrow{(d)}\left(X_{t}^{\infty}\right)_{t \in K}$ as $n \rightarrow \infty$,
(ii) $\frac{1}{C_{n}} \operatorname{Height}\left(\operatorname{Tree}\left(\operatorname{Loop}\left(\tau_{n}\right)^{r C_{n}}\right)\right) \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$, for all $r>0$, where Tree is the inverse operation of Loop, and $\operatorname{Loop}\left(\tau_{n}\right)^{R}$ is defined above.

Then

$$
\left(\operatorname{Loop}\left(\tau_{n}\right), \frac{1}{C_{n}} \tilde{d}_{n}, \frac{1}{n} \nu_{n}, \rho_{n}\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

as $n \rightarrow \infty$ with respect to the Gromov-Hausdorff vague topology, where $\tilde{d}^{\infty}$ can denote either the shortest-distance or effective resistance metric on $\mathcal{L}_{\alpha}^{\infty}$, as appropriate. Moreover, the result also holds on replacing Loop by Loop' in all the statements above.

Proof. We start by proving the result for Loop. We will prove the result with $\tilde{d}=d$ and note that the corresponding result for $\tilde{d}=R$ follows by the same arguments. The proof is again a consequence of Proposition 4.1.6, given which, the proof is almost identical to the proof of Theorem 5.0.1 (i.e. by defining an increasing sequence of sublooptrees that exhaust the whole space, to each of which we then apply Proposition 4.1.6), so we omit the details. As we did there, take $r>0$, and define
two times $t_{g}(r)$ and $t_{d}(r)$ by

$$
\begin{aligned}
& t_{g}(r)=\inf \left\{s \geq 0: \Delta_{-s} \geq 4 r, \delta_{-s}\left(x_{-s}^{0}\right) \geq r\right\} \\
& t_{d}(r)=\inf \left\{s \geq 0: X_{s}^{\infty} \leq X_{-t_{g}(r)^{-}}^{\infty}\right\}
\end{aligned}
$$

It then follows by the Skorokhod Representation Theorem that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ upon which $\left(\frac{1}{C_{n}} W_{n t}^{n}\right)_{-\left(t_{g}+1\right) \leq t \leq t_{d}+1} \rightarrow\left(X^{\infty}\right)_{-\left(t_{g}+1\right) \leq t \leq t_{d}+1}$ almost surely with respect to the Skorokhod- $J_{1}$ topology. As in the proof of Theorem 5.0.1, for each $n \in \mathbb{N}$ let $\lambda_{n}$ be the Skorokhod homeomorphism $\left[-t_{g}-1, t_{d}+1\right] \rightarrow$ $\left[-t_{g}-1, t_{d}+1\right]$ that minimises the Skorokhod- $J_{1}$ distance between these two functions, and set $t_{d}^{n}=\lambda_{n}\left(t_{d}\right)$, and similarly $t_{g}^{n}=\lambda_{n}\left(t_{g}\right)$.

By repeating the arguments of the proof of Theorem 5.0.1, and noting that condition (ii) above ensures that condition (ii) of Proposition 4.1.6 is satisfied, we deduce that the looptrees coded by $\left(\frac{1}{C_{n}} W_{n t}^{n}\right)_{-t_{g}^{n} \leq t \leq t_{d}^{n}}$ converge to the looptree coded by $\left(X^{\infty}\right)_{t \geq 0}$. The result then follows as in the proof of Theorem 5.0.1.

To prove the same result for Loop' in place of Loop, first note that

$$
d_{G H}\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right), \operatorname{Loop}^{\prime}\left(T_{\infty}^{\alpha}\right)\right) \leq 2
$$

Therefore, the Gromov-Hausdorff convergence of Proposition 4.1.6 holds with Loop $\left(\tau_{n}\right)$ replaced by Loop ${ }^{\prime}\left(\tau_{n}\right)$, and the Prohorov convergence of measures of that proposition holds by the exactly the same arguments. As a consequence, we can just repeat exactly the same proof for Loop'.

In particular, the result applies taking $\tau_{n}=T_{\alpha}^{\infty}$ for all $n$, and $C_{n}=a_{n}$. In this case, $\frac{1}{C_{n}} \operatorname{Height}\left(\operatorname{Tree}\left(\operatorname{Loop}\left(\tau_{n}\right)\left(r C_{n}\right)\right)\right.$ ) will be of order $r^{\alpha-1} n^{-\frac{2-\alpha}{\alpha}} L(n)$ for some slowly-varying function $L$, so point (ii) of Proposition 5.3 .2 holds by an application of Markov's inequality. We therefore deduce both Theorem 5.0.2, and Theorem 5.3.3 below, as a corollary.

Theorem 5.3.3. Take $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$ as above, with $\nu^{\prime}$ the measure on $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$ such that $\nu^{\prime}(x)=1$ for all $x \in \operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$. Then

$$
\left(\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\prime}, \rho\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

with respect to the Gromov-Hausdorff vague topology as $n \rightarrow \infty$. Here $\tilde{d}$ (respectively $\tilde{d}^{\infty}$ ) can denote either the geodesic metric $d$ (respectively $d^{\infty}$ ), or the effective resistance metric $R$ (respectively $R^{\infty}$ ).

## Looptrees defined from two-type Galton Watson trees

In practice in the context of random planar maps, it is often convenient to define discrete looptrees from alternating two-type Galton-Watson trees. In particular, Richier in [Ric18a, Section 3] gives the following definition, illustrated in Figure 5.3. Given an infinite alternating two-type Galton-Watson tree $T$ (as defined in Section 3.1), say with white vertices at even height and black vertices at odd height, draw a loop around each black vertex by connecting its $i^{\text {th }}$ white child to its $(i+1)^{\text {th }}$ white child for all $i$, and join its parent to both its first and last white child. Then delete the black vertices and their incident edges; we denote the resulting structure by $\operatorname{Loop}^{2}(T)$.


Figure 5.3: A two-type tree and its looptree.
We now take a two-type tree $T_{\alpha}^{\infty, 2}$ with offspring distribution $\left(\xi_{\circ}, \xi_{\bullet}\right)$ such that:

- $\left(\xi_{0}, \xi_{\bullet}\right)$ is critical, i.e. $\mathbb{E}\left[\xi_{0}\right] \mathbb{E}\left[\xi_{\bullet}\right]=1$.
- $\xi_{\circ}$ is shifted geometric with parameter $1-p \in(0,1)$, i.e. $\xi_{\circ}(k)=(1-p) p^{k}$ for all $k \geq 0$.
- $\xi_{\bullet}$ is in the domain of attraction of an $\alpha$-stable law.

Before stating the scaling result, we briefly introduce two related concepts. One of these is the Janson-Stefánsson bijection of [JS15], which gives a bijection between alternating two-type Galton-Watson trees and one-type Galton-Watson trees. Given an alternating two-type Galton-Watson tree $T$, we denote its image under this bijection by $\Phi_{\mathrm{JS}}(T) . \Phi_{\mathrm{JS}}(T)$ has the same vertex set as $T$, but different edges, and is constructed as follows: for every white vertex that is not equal to the root, label its offspring as $u_{1}, \ldots, u_{k}$ in lexicographical order, and label its parent $u_{0}$. Then draw an edge joining $u_{i}$ to $u_{i+1}$ for each $i \in\{0, \ldots, k-1\}$, and draw an edge joining $u_{k}$ to $u$. See Figure 5.4.

The bijection is such that each white vertex in $T$ is therefore mapped to a leaf in $\Phi_{\mathrm{JS}}(T)$, and each black vertex in $T$ with $k$ offspring is mapped to a vertex in $\Phi_{\mathrm{JS}}(T)$ with $k+1$ offspring.

The second concept is a (final) related loop operation $\overline{\text { Loop. }}$. Given a (onetype) tree $T, \overline{\operatorname{Loop}}(T)$ is obtained by first forming $\operatorname{Loop}^{\prime}(T)$, and then for each vertex $u \in \operatorname{Loop}^{\prime}(T)$, contracting each edge joining $u$ to its rightmost child. $\overline{\operatorname{Loop}}(T)$ therefore has the property that multiple loops can be grafted at the same vertex, which is not the case with $\operatorname{Loop}(T)$ and $\operatorname{Loop}^{\prime}(T)$ (but is the case with the two-type operation Loop ${ }^{2}$ ).

(b) $\operatorname{Loop}^{\prime}\left(\Phi_{\mathrm{JS}}(T)\right)$ and $\overline{\operatorname{Loop}}\left(\Phi_{\mathrm{JS}}(T)\right)$.

Figure 5.4: Illustrations for the two-type tree $T$ in Figure 5.3.
The proof of the two-type scaling result then proceeds by applying the Janson-Stefánsson bijection to the two-type tree, and using the following facts, which we state without proof, but which should be plausible from looking at Figure 5.4.
(i) For any plane tree $T$ endowed with a measure giving mass 1 to every vertex, $d_{G H P}\left(\operatorname{Loop}^{\prime}(T), \overline{\operatorname{Loop}}(T)\right) \leq 4 \operatorname{Height}(T)$ (see [Ric18b, Equation (48)] for Gromov-Hausdorff version, then the Prohorov bound on measures follows by same reasoning).
(ii) If $T$ is an alternating two-type tree, then $\operatorname{Loop}^{2}(T)=\overline{\operatorname{Loop}}\left(\Phi_{\mathrm{JS}}(T)\right)$ ) (see [CK15, Lemma 4.3]).
(iii) Let $T$ be an alternating two-type Galton-Watson tree with offspring distributions $\xi_{\circ}$ and $\xi_{\bullet}$ such that $\xi_{\circ}$ is shifted geometric with parameter $1-p \in(0,1)$, i.e. $\xi_{0}(k)=(1-p) p^{k}$ for all $k \geq 0$, and $\mathbb{E}\left[\xi_{0}\right] \mathbb{E}\left[\xi_{\bullet}\right] \leq 1$. Then $\Phi_{\mathrm{JS}}(T)$ is a one-type Galton-Watson tree with offspring distribution $\xi$, where $\xi$ is such that $\xi(0)=1-p$ and $\xi(k)=p \xi \bullet(k-1)$ for all $k \geq 1$ (see [JS15, Appendix A]).

Moreover, under the criticality assumption, this implies that

$$
\begin{equation*}
\frac{\sum_{i=1}^{n} \xi^{(i)}-n}{a_{n}} \xrightarrow{(d)} Z_{\alpha} \quad \text { if and only if } \quad \frac{\sum_{i=1}^{n} \xi_{\bullet}^{(i)}-\frac{1-p}{p} n}{p^{\frac{-1}{\alpha}} a_{n}} \xrightarrow{(d)} Z_{\alpha} . \tag{5.6}
\end{equation*}
$$

We are now ready to state and prove the convergence result.
Theorem 5.3.4. Let $\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right)$ be above, with $\left(a_{n}\right)_{n \geq 1}$ as in (5.6), and let $\nu^{2}$ be the measure on $\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right)$ such that $\nu^{2}(x)=1$ for all $x \in \operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right)$. Then

$$
\left(\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{2}, \rho\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

with respect to the Gromov-Hausdorff vague topology as $n \rightarrow \infty$. Again, here $\tilde{d}$ (respectively $\tilde{d}^{\infty}$ ) can denote either the geodesic metric d (respectively $d^{\infty}$ ), or the effective resistance metric $R$ (respectively $R^{\infty}$ ).

Proof of Theorem 5.3.4. Using the points above, we will show that there exists a probability space on which we can define both $T_{\alpha}^{\infty, 2}$ and a one-type Galton Watson tree $\tilde{T}_{\alpha}$ satisfying the assumptions of Proposition 5.3.2 such that, for all $r>0$,

$$
\begin{equation*}
d_{G H P}\left(\mathcal{B}_{r}\left(\left(\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\prime}, \rho\right)\right), \mathcal{B}_{r}\left(\left(\operatorname{Loop}^{\prime}\left(\tilde{T}_{\alpha}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\prime}, \rho\right)\right) \rightarrow 0\right. \tag{5.7}
\end{equation*}
$$

almost surely as $n \rightarrow \infty$. As a result, we deduce that these two looptrees have the same Gromov-Hausdorff-Prohorov vague limit.

To do this, we first make a definition. As in the one-type case, it follows that $T_{\alpha}^{\infty, 2}$ almost surely has a unique infinite spine on which vertices instead have a size-biased offspring distribution (see [Ste18, Section 3.1]). Analogously to previous definitions, for any $R>0$ we say that a loop on the corresponding loopspine is $R$-good if it has length at least $4 R$ and if the two points at which it is connected to adjacent loops on the loopspine are separated by distance at least $R$. We then let $L_{\alpha}^{2}(R)$ denote the subspace obtained by taking the union of all the loops up to and including the first $R$-good loop on the loopspine, along with any sublooptrees grafted to them. The reason for this definition is that $\mathcal{B}_{R}\left(\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right)\right) \subset L_{\alpha}^{2}(R)$, and $L_{\alpha}^{2}(R)$ is a full looptree (i.e. does not contain partial loops). We also let $T_{\alpha}^{2}(R)$ denote the (two-type) tree such that $\operatorname{Loop}^{2}\left(T_{\alpha}^{2}(R)\right)=L_{\alpha}^{2}(R)$ (this is well-defined since Loop ${ }^{2}$ is a bijection).

Set $\tilde{T}_{\alpha}^{r, n}=\Phi_{\mathrm{JS}}\left(T_{\alpha}^{2}\left(r a_{n}\right)\right)$. We make the following observations, based on the facts above.

1. By Fact (ii) above, $\overline{\operatorname{Loop}}\left(\tilde{T}_{\alpha}^{r, n}\right)=L_{\alpha}^{2}\left(r a_{n}\right)$.
2. By Fact (i) above, $d_{G H P}\left(\overline{\operatorname{Loop}}\left(\tilde{T}_{\alpha}^{r, n}\right), \operatorname{Loop}^{\prime}\left(\tilde{T}_{\alpha}^{r, n}\right)\right) \leq 4 \operatorname{Height}\left(\tilde{T}_{\alpha}^{r, n}\right)$.

Moreover, $n^{\frac{-1}{\alpha}} \operatorname{Height}\left(\tilde{T}_{\alpha}^{r, n}\right) \rightarrow 0$ in probability as $n \rightarrow \infty$ since:

$$
\begin{aligned}
\mathbf{P}\left(\operatorname{Height}\left(\tilde{T}_{\alpha}^{r, n}\right) \geq \varepsilon n^{\frac{1}{\alpha}}+1\right) & \leq \mathbf{P}\left(\operatorname{Height}\left(T_{\alpha}^{2}\left(r n^{\frac{1}{\alpha}}\right)\right) \geq \varepsilon n^{\frac{1}{\alpha}}+1\right) \\
& =\left(1-p_{r, n}\right)^{\varepsilon n^{\frac{1}{\alpha}}} \\
& \leq \exp \left\{-C r^{-\alpha} n^{-\frac{\alpha-1}{\alpha}} \varepsilon n^{\frac{1}{\alpha}}\right\},
\end{aligned}
$$

where $p_{r, n}=\frac{1}{2} \mathbb{P}\left(\hat{\xi}_{\bullet} \geq r n^{\frac{1}{\alpha}}\right) \sim C r^{\alpha} n^{\frac{\alpha-1}{\alpha}}$ as $n \rightarrow \infty$ by assumption, since $\hat{\xi}_{\bullet}$ is a size-biased version of $\xi_{\bullet}$.
3. By construction and Fact (iii) above, $\mathcal{B}_{r}\left(\operatorname{Loop}^{\prime}\left(\tilde{T}_{\alpha}^{r, n}\right)\right)=\mathcal{B}_{r}\left(\operatorname{Loop}^{\prime}\left(\tilde{T}_{\alpha}\right)\right)$, where $\tilde{T}_{\alpha}=\lim _{n \rightarrow \infty} \tilde{T}_{\alpha}^{r, n}$ (the Janson-Stefánsson bijection is such that this is welldefined). Moreover, $\tilde{T}_{\alpha}$ is distributed as Kesten's critical tree with offspring distribution $\xi$.

These three points imply that (5.7) holds with $\tilde{T}_{\alpha}$ as in Point 3 above. Then, $\tilde{T}_{\alpha}$ satisfies the conditions of Proposition 5.3 .2 (in particular, condition (ii) of the Proposition holds by similar arguments to those in Point 2 above), so we deduce that

$$
\left(\operatorname{Loop}^{\prime}\left(\tilde{T}_{\alpha}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\prime}, \rho\right) \xrightarrow{(d)} \mathcal{L}_{\alpha}^{\infty}
$$

as $n \rightarrow \infty$. Since these $T_{\alpha}^{\infty, 2}$ and $\tilde{T}_{\alpha}$ are defined on a common probability space, (5.7) therefore implies the same distributional result for $\left(\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right), a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\prime}, \rho\right)$.

Remark 5.3.5. In [Ric18a], these two-type looptrees are coded by upward skip-free random walks in a similar way to the one-type case. It is also possible to write an analogous result to Proposition 5.3.2 in this case, under more general assumptions on the coding functions.

### 5.4 Volume bounds and resistance estimates for infinite stable looptrees

In this section, we prove precise estimates on the volume and resistance growth properties of infinite stable looptrees. These are of interest in their own right but in Section 5.5 we also use these to obtain bounds on the heat kernel, and use the resistance estimate to verify that the non-explosion conditions of Theorem 2.4.3 in order to deduce similar limiting results for stochastic processes.

Similarly to Chapter 4, we can prove the following volume results. The results holds regardless of whether we define the balls in terms of $R^{\infty}$ or $d^{\infty}$, since the two metrics are equivalent.

Theorem 5.4.1. (cf [Arc19, Theorem 1.4]). P-almost surely, we have:

$$
\begin{array}{ll}
\limsup _{r \uparrow \infty}\left(\frac{\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, r\right)\right)}{r^{\alpha}(\log \log r)^{\frac{4 \alpha-3}{\alpha-1}}}\right)<\infty, & \underset{r \uparrow \infty}{\limsup }\left(\frac{\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, r\right)\right)}{r^{\alpha} \log \log r}\right)>0, \\
\liminf _{r \uparrow \infty}\left(\frac{\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, r\right)\right)}{r^{\alpha}(\log \log r)^{-\alpha}}\right)>0, & \liminf _{r \uparrow \infty}\left(\frac{\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, r\right)\right)}{r^{\alpha}(\log \log r)^{-(\alpha-1)}}\right)<\infty .
\end{array}
$$

Moreover, $\mathbf{P}$-almost surely, for $\nu^{\infty}$-almost every $u \in \mathcal{L}_{\alpha}^{\infty}$ we have

$$
\begin{aligned}
& \limsup _{r \downarrow 0}\left(\frac{\nu^{\infty}\left(B^{\infty}(u, r)\right)}{r^{\alpha}\left(\log \log r^{-1}\right)^{\frac{4 \alpha-3}{\alpha-1}}}\right)<\infty, \quad \limsup _{r \downarrow 0}\left(\frac{\nu^{\infty}\left(B^{\infty}(u, r)\right)}{r^{\alpha} \log \log r^{-1}}\right)>0, \\
& \underset{r \downarrow 0}{\liminf }\left(\frac{\nu^{\infty}\left(B^{\infty}(u, r)\right)}{r^{\alpha}\left(\log \log r^{-1}\right)^{-\alpha}}\right)>0, \quad \quad \liminf _{r \downarrow 0}\left(\frac{\nu^{\infty}\left(B^{\infty}(u, r)\right)}{r^{\alpha}\left(\log \log r^{-1}\right)^{-(\alpha-1)}}\right)<\infty .
\end{aligned}
$$

Theorem 5.4.2. $\mathbf{P}$-almost surely, there exists a constant $c>0$ such that for all $r>0$,

$$
\operatorname{cr}\left(\log \log \left(r \vee r^{-1}\right)\right)^{\frac{-(3 \alpha-2)}{\alpha-1}} \leq R^{\infty}\left(\rho^{\infty}, B^{\infty}\left(\rho^{\infty}, r\right)^{c}\right) \leq r
$$

These results are obtained as a consequence of the following propositions.
Proposition 5.4.3. There exist constants $c, c^{\prime}, C, C^{\prime} \in(0, \infty)$ such that for all $r>0, \lambda>1$ :

$$
\begin{aligned}
C \exp \left\{-c \lambda^{\frac{1}{\alpha-1}}\right\} \leq \mathbf{P}\left(\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, r\right)\right)<r^{\alpha} \lambda^{-1}\right) \leq C^{\prime} \exp \left\{-c^{\prime} \lambda^{\frac{1}{\alpha}}\right\} \\
C e^{-c \lambda} \leq \mathbf{P}\left(\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, r\right)\right) \geq r^{\alpha} \lambda\right) \leq C^{\prime} \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-c^{\prime} \lambda^{\frac{\alpha-1}{4 \alpha-3}}}
\end{aligned}
$$

Proposition 5.4.4. There exist constants $C, c \in(0, \infty)$ such that for all $r>0, \lambda>$ 1:

$$
\mathbf{P}\left(R_{e f f}^{\infty}\left(\rho^{\infty}, B^{\infty}\left(\rho^{\infty}, r\right)^{c}\right) \leq r \lambda^{-1}\right) \leq C e^{-c \lambda^{\frac{1}{4}}}
$$

By applying Borel-Cantelli arguments along the sequence $r_{n}=2^{n}$ (respectively $r_{n}=2^{-n}$ ) in Propositions 5.4.3 and 5.4.4, we obtain the results of Theorems 5.4.1 and 5.4.2 for the regime $r \uparrow \infty$ (respectively $r \downarrow 0)$. For any $R \in(0, \infty)$, the local results can then be extended to $\nu^{\infty}$-almost every $u \in \mathcal{L}_{\alpha}^{\infty, R}$ by uniform rerooting invariance (recall that $\left(\mathcal{L}_{\alpha}^{\infty, R}\right)_{R \geq 0}$ is a sequence of nested compact looptrees that exhaust $\mathcal{L}_{\alpha}^{\infty}$ ). Taking $R \rightarrow \infty$ then gives the result.

We do not prove the volume results since the proofs are essentially the same as those of the analogous results in Chapter 4, except that at some stages we decompose along the infinite loopspine rather than the W-loopspine (which is technically more straightforward anyway), and we are already dealing with Lévy processes so we do not need to use absolute continuity to compare an excursion with an uncondtioned process.

### 5.4.1 Applications to volume limits in compact stable looptrees

As a result of Theorem 5.0.1, we are able to prove various volume convergence results that are exploited in [Arc19] to study Brownian motion on compact stable looptrees. The main applicable result is the following theorem. Here we let $\nu$ denote the intrinsic measure on a compact stable looptree $\mathcal{L}_{\alpha}$ as defined in Section 3.2.1, conditioned so that $\nu\left(\mathcal{L}_{\alpha}\right)=1$. We also let $B(\rho, r)$ denote the open ball of radius $r$ around the root in $\mathcal{L}_{\alpha}$, and $\bar{B}(\rho, r)$ its closure.

Theorem 5.4.5. There exists a random variable $\left(V_{t}\right)_{t \geq 0}: \Omega \rightarrow D([0, \infty),[0, \infty))$ such that the finite dimensional distributions of the process

$$
\left(r^{-\alpha} \nu(\bar{B}(\rho, r t))\right)_{t \geq 0}
$$

converge to those of $\left(V_{t}\right)_{t \geq 0}$ as $r \downarrow 0$, and $V_{t}$ denotes the volume of a closed ball of radius $t$ around the root in $\mathcal{L}_{\alpha}^{\infty}$. Moreover, for any $p \in[1, \infty)$, setting $V:=V_{1}$ we have that $\mathbf{E}\left[V^{p}\right]<\infty$, and that

$$
r^{-\alpha p} \mathbf{E}\left[\nu(\bar{B}(\rho, r))^{p}\right] \rightarrow \mathbf{E}\left[V^{p}\right]
$$

as $r \downarrow 0$.
Remark 5.4.6. We have taken closed balls rather than open ones simply so that $V$ is càdlàg. We conjecture that the volume processes are in fact continuous, and that the convergence of the theorem can be extended to hold uniformly on compacts. However, due to the complex nature of looptrees, this is not straightforward to prove. In particular it is difficult to replicate the argument used to prove a similar result for stable trees, since looptrees do not have such a straightforward regeneration structure around the boundary of a ball of radius $r$.

Proof. By the separability of Proposition 2.2.4, we can work on a probability space on which $\mathcal{L}_{\alpha}^{\ell} \rightarrow \mathcal{L}_{\alpha}^{\infty}$ almost surely as $\ell \rightarrow \infty$. By standard results on metric space convergence, it follows that almost surely on this space, $\nu^{\ell}\left(B^{\ell}\left(\rho^{\ell}, t\right)\right) \rightarrow$ $\nu^{\infty}\left(B^{\infty}\left(\rho^{\infty}, t\right)\right)$ for all $t$ such that $\nu^{\infty}\left(\partial B^{\infty}\left(\rho^{\infty}, t\right)\right)=0$ (e.g. see [GM17b, Lemma 2.11]), and therefore for Lebesgue almost every $t$. Moreover, by scaling invariance of $\mathcal{L}_{\alpha}^{\infty}$, there are no "special" values of $t$, so we deduce that for any fixed sequence $0<t_{0}<t_{1}<\ldots<t_{n}<\infty$, the convergence almost surely holds simultaneously for all of the points $t_{i}, 0 \leq i \leq n$.

Since $\left(\nu^{\ell}\left(B^{\ell}(\rho, t)\right)\right)_{t \geq 0} \stackrel{(d)}{=}\left(\ell \nu B\left(\rho, \ell^{\frac{-1}{\alpha}} t\right)\right)_{t \geq 0}$, by writing $\ell=r^{-\alpha}$ we therefore deduce the result as stated. In particular, it follows that $\nu^{\ell}\left(B^{\ell}\left(\rho^{\ell}, 1\right)\right) \xrightarrow{(d)} V$ as $\ell \rightarrow \infty$.

We claim that $V \in(0, \infty)$ almost surely, with all moments finite. This follows
immediately from the exponential upper tails of Proposition 5.4.3, namely that

$$
\begin{equation*}
\mathbf{P}(V \geq \lambda) \leq C \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-c \lambda^{\frac{\alpha-1}{4 \alpha-3}}} \tag{5.8}
\end{equation*}
$$

We now prove that the moments of $r^{-\alpha} \nu_{1}\left(B\left(\rho_{1}, r\right)\right)$ converge to those of $V$. To see this, we observe that the arguments used to prove (5.8) and the compact analogue in [Arc19, Proposition 5.4] can be applied uniformly along the sequence $\mathcal{L}_{\alpha}^{\ell}$ to give constants $c, C \in(0, \infty)$ such that

$$
\mathbf{P}^{\ell}\left(\nu^{\ell}\left(B^{\ell}(\rho, r)\right) \geq r^{\alpha} \lambda\right) \leq C \lambda^{\frac{\alpha-1}{4 \alpha-3}} e^{-c \lambda^{\frac{\alpha-1}{4 \alpha-3}}}
$$

for all $\ell \geq 1$. It follows that the sequence $\left(r^{-\alpha p}\left(\nu^{\ell}\left(B^{\ell}(\rho, r)\right)\right)^{p}\right)_{\ell \geq 1}$ is uniformly integrable for all $p \geq 1$ and so setting $C_{p}=\mathbf{E}\left[V^{p}\right]$ we deduce that

$$
r^{-\alpha p} \mathbf{E}\left[\left(\nu_{1}\left(B\left(\rho_{1}, r\right)\right)\right)^{p}\right] \rightarrow C_{p}
$$

for all $p \geq 1$.

### 5.4.2 Resistance bounds

We now turn to proving the resistance bounds. We use a version of the iterative procedure used to prove the volume bounds of Section 4.2.2, which we again index by a subcritical branching process, to count the number of sublooptrees intersecting the boundary of a ball of radius $r$. More formally, we will define another subtree $T_{\text {res }} \subset \mathcal{U}$, but this time selecting sublooptrees of large diameter, rather than of large volume, to form the offspring at each step. Since this argument is not given in Chapter 4, we write it more carefully.

Recall that in Section 3.2.3 we defined several notions of height of a compact stable looptree:
(i) We defined its $L^{W}$-Height to be the looptree distance from $\rho$ to $u_{H}$,
(ii) We defined its $L$-Height to be $\sup _{u \in \tilde{\mathcal{L}_{\alpha}}} d_{\tilde{\mathcal{L}_{\alpha}}}(\rho, u)$.
(iii) We defined its $L^{m}$-Height to be $\sup \tilde{X}_{s}^{\text {exc }}$, where $\tilde{X}^{\text {exc }}$ is the Lévy excursion coding $\tilde{\mathcal{L}_{\alpha}}$.

Note that $L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}}_{\alpha}\right) \geq L-\operatorname{Height}\left(\tilde{\mathcal{L}}_{\alpha}\right) \geq L^{W}-\operatorname{Height}\left(\tilde{\mathcal{L}}_{\alpha}\right)$.
The $L^{m}$-Height is $\mathbf{P}$-almost surely realised by a unique point in $\tilde{\mathcal{L}_{\alpha}}$, which we denote $u_{m}$. We refer to (the closure of) the set of loops coded by the ancestors of $u_{m}$ as the $m$-loopspine. In order to control the lengths of the loops on the $m$-spine we use Proposition 2.5.4, the absolute continuity relation (2.12), and the scaling relation $p_{t}(x)=t^{\frac{-1}{\alpha}} p_{1}\left(x t^{\frac{-1}{\alpha}}\right)$. In particular, on applying the Vervaat transform to
$X^{\text {exc }}$ to form the bridge $X^{\text {br }}$, the points $m_{\text {min }}, m_{\max } \in[0,1]$ respectively attaining the minimum and maximum of $X^{\text {br }}$ are uniform random variables on $[0,1]$ (they are not independent of each other, but we will only control them using a union bound, so this is not a problem).

Suppose that $A$ is a measurable event involving loop lengths and sublooptrees on the $m$-loopspine. If $m_{\min }, m_{\max }$ respectively denote the new locations of the minimum and maximum of $X^{\text {br }}$ obtained from applying the Vervaat transform to $X^{\text {exc }}$, we can therefore apply the absolute continuity relation (2.12) twice on the intervals $\left[0, m_{\max } \vee m_{\min }\right]$ and $\left[m_{\max } \wedge m_{\min }, 1\right]$ to control the probability of the event $A$. In particular, if $\left\{m_{\text {min }}, m_{\text {max }} \in\left(e^{-\tilde{c} \lambda^{p}}, 1-e^{-\tilde{c} \lambda^{p}}\right)\right\}$, we can therefore apply the absolute continuity relation (2.12) twice on the intervals $\left[0,1-e^{-\tilde{c} \lambda^{p}}\right]$ for both the Lévy bridge and also the time-reversed reflected bridge defined by $\tilde{X^{\mathrm{br}}}=X_{1-t}^{\mathrm{br}}$ to control the probability of the event $A$. Using subscripts to denote which law we are working under, we can then compare $\mathbb{P}_{X^{\operatorname{exc}}}(A)$ and $\mathbb{P}_{X}(A)$ as follows: suppose that $\mathbb{P}_{X}(A) \leq C e^{-c \lambda^{p}}$ for some $p>0$ and some constants $c, C \in(0, \infty)$. We can then write

$$
\begin{align*}
\mathbb{P}_{X^{\operatorname{exc}}}(A) & \leq \mathbb{P}\left(m_{\min } \notin\left(e^{-\tilde{c} \lambda^{p}}, 1-e^{-\tilde{c} \lambda^{p}}\right) \text { or } m_{\max } \notin\left(e^{-\tilde{c} \lambda^{p}}, 1-e^{-\tilde{c} \lambda^{p}}\right)\right) \\
& +\mathbb{E}_{X^{\operatorname{exc}}}\left[\mathbb{1}\{A\} \mathbb{1}\left\{m_{\min }, m_{\max } \in\left(e^{-\tilde{c} \lambda^{p}}, 1-e^{-\tilde{c} \lambda^{p}}\right)\right\}\right] \\
& \leq 4 e^{-\tilde{c} \lambda^{p}}+e^{\frac{\tilde{c}}{\alpha} \lambda^{p}} \frac{\left\|p_{1}\right\|_{\infty}}{p_{1}(0)} \mathbb{E}_{X}\left[\mathbb{1}\{A\} \mathbb{\mathbb { 1 }}\left\{m_{\min }, m_{\max } \in\left(e^{-\tilde{c} \lambda^{p}}, 1-e^{-\tilde{c} \lambda^{p}}\right)\right\}\right] \\
& \leq 4 e^{-\tilde{c} \lambda^{p}}+e^{\frac{\tilde{c}}{\alpha} \lambda^{p}} \frac{\left\|p_{1}\right\|_{\infty}}{p_{1}(0)} \mathbb{P}_{X}(A) \\
& \leq 4 e^{-\tilde{c} \lambda^{p}}+e^{\frac{\tilde{c}}{\alpha} \lambda^{p}} \frac{\left\|p_{1}\right\|_{\infty}}{p_{1}(0)} C e^{-c \lambda^{p}} . \tag{5.9}
\end{align*}
$$

Therefore, provided that we originally chose $\tilde{c}$ so that $\frac{\tilde{c}}{\alpha}<c$, we get that $\mathbb{P}_{X^{\operatorname{exc}}}(A) \leq C e^{-c \lambda^{p}}$ as well, just with slightly different values of the constants $c$ and $C$.

In what follows, we will therefore use the fact given in Proposition 2.5.4 that under the law of $X$, the jump sizes corresponding to the ancestors of a new maximum follow a size-biased version of the original Lévy measure (by reflection and time-reversal, the same result holds for a new backwards minimum). Additionally, [Ber92b, Corollary 1(iii)] combined with the Strong Markov property at times of hitting successive maxima implies that under the law of $X$, the sublooptrees grafted to the loopspine up to a given maximum are coded by the Itô excursion measure but precisely conditioned not to have $m$-height so large that it would create a new maximum of $X$ (as should be expected). We will not explicitly repeat the argument of (5.9) each time we make the comparison between $X$ and $X^{\text {exc }}$, and instead just
directly use the law given in Proposition 2.5.4 since we always obtain a bound of the form that can be dealt with as in (5.9).

We now define some terminology, in keeping with that used in Section 4.2.2 wherever possible.

Firstly, given $R>0$, we say that a loop on the m-loopspine is "good" if it has length at least $4 R$, and if the associated uniform random variable (that dictates the ratio of the two segments it splits into on either side of the loopspine) is in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$. We say the a loop is "goodish" if it just has length at least $4 R$. Additionally, for any $R>0$, and any (unconditioned) compact looptree $\tilde{\mathcal{L}_{\alpha}}$ (respectively any infinite looptree $\mathcal{L}_{\alpha}^{\infty}$ ), we let $I_{R}^{m}$ be the closure in $\tilde{\mathcal{L}_{\alpha}}$ (respectively $\mathcal{L}_{\alpha}^{\infty}$ ) of the union of all the loops in the m-loopspine (respectively infinite loopspine) that intersect $\tilde{B}(\tilde{\rho}, R)$ (respectively $B^{\infty}\left(\rho^{\infty}, R\right)$ ). Additionally, we let $\left|I_{R}^{m}\right|$ be the sum of the lengths of these loops.

We start by giving a technical lemma. Using the size-biased distribution for loop lengths on the $m$-loopspine, the proof is almost identical to that of Lemma 4.2.9. We have only included it since we refer to parts of it later in the prof of Proposition 5.4.4.

Lemma 5.4.7. (cf Lemma 4.2.9). For any $h>0, \lambda>1, R<\lambda^{-1-\frac{h}{\alpha-1}}$,

$$
\mathbf{P}\left(\left|I_{R}^{m}\right| \geq 3 R \lambda \left\lvert\, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}\right.\right) \leq C e^{-c \lambda^{h \wedge 1}}
$$

Proof. We use a similar strategy to Lemma 4.2.9. Indeed, we first condition on existence of a good loop in the m-loopspine. We then select the closest good loop to $\rho$. Given such a loop, the number of goodish loops between $\rho$ and the first good loop is stochastically dominated by $N-1$, where $N$ is a Geometric $\left(\frac{1}{2}\right)$ random variable. $\left|I_{R}^{m}\right|$ can then be upper bounded by the random variable

$$
\begin{equation*}
2 R N+\sum_{i=1}^{N} Q^{(i)} \tag{5.10}
\end{equation*}
$$

where $Q^{(i)}$ denotes the sum of the lengths of all the smaller loops on the m-loopspine that are between the $(i-1)^{\text {th }}$ and $i^{\text {th }}$ goodish loops, and the term $2 R N$ comes from selecting a segment of length at most $R$ in each direction round each of the goodish loops. Using the size-biased bound for the loop lengths, each $Q^{(i)}$ can be independently approximated by an $(\alpha-1)$-stable subordinator run up until an exponential time and conditioned not to have any jumps greater than $4 R$.

Since we model the loop lengths by a subordinator indexed by the m-spine
of the underlying tree, we upper bound the probability in question by:

$$
\begin{align*}
& \mathbf{P}\left(\left|I_{R}^{m}\right| \geq 3 R \lambda, T^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq R^{\alpha-1} \lambda^{h} \left\lvert\, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}\right.\right)  \tag{5.11}\\
& +\mathbf{P}\left(\left|I_{R}^{m}\right| \geq 3 R \lambda, T^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \leq R^{\alpha-1} \lambda^{h} \left\lvert\, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}\right.\right) .
\end{align*}
$$

The first of these terms can be upper bounded by $C e^{-c \lambda}$ using exactly the same arguments as in Lemma 4.2.9, the point being that if the m-spine in the underlying tree is long enough, then there is plenty of time for a good loop to occur in the corresponding subordinator. To summarise more concretely:

- The number of good loops on the m-loopspine is stochastically dominated by a Poisson $\left(\mathrm{c} \lambda^{h}\right)$ random variable, so $\mathbf{P}(\nexists$ a good loop $) \leq e^{-c \lambda^{h}}$.
- $N$ is Geometric $\left(\frac{1}{2}\right)$, so $\mathbf{P}(N \geq \lambda) \leq C e^{-c \lambda}$.
- $\mathbf{P}\left(\sum_{i=1}^{N} Q^{(i)} \geq R \lambda\right) \leq C e^{-c \lambda}$. Indeed, by Proposition 2.5.4 and independently for each $i$, we can model each term $Q^{(i)}$ by an $(\alpha-1)$-stable subordinator Sub ${ }^{(i)}$ with all jumps greater than $4 R$ removed, run up until a time $\mathcal{E}_{R} \sim \exp \left(c R^{\frac{-1}{\alpha-1}}\right)$. We also let $\mathrm{Sub}^{(i)^{\prime}}$ denote a rescaled version of $\mathrm{Sub}^{(i)}$, instead with all jumps greater than 4 removed, and let $\mathcal{E} \sim \exp (c)$. By rescaling $\operatorname{Sub}^{(i)}$ and choosing $\theta$ so that $\mathbf{E}\left[e^{\theta \text { Sub } b^{(i)^{\prime}}}\right]<\frac{3}{2}$ (which we can do by Lemma 4.2.2), we then have that

$$
\begin{align*}
\mathbf{P}\left(\sum_{i=1}^{N} Q^{(i)} \geq R \lambda\right) & =\sum_{n=1}^{\infty} \mathbf{P}\left(\sum_{i=1}^{N} \operatorname{Sub}_{\mathcal{E}}^{(i)^{\prime}} \geq \lambda \mid N=n\right) \mathbf{P}(N=n) \\
& \leq \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n} e^{-\theta \lambda}\left(\frac{1}{2}\right)^{n}  \tag{5.12}\\
& =C_{\theta} e^{-\theta \lambda} .
\end{align*}
$$

This deals with the first term in (5.11). If the m-spine is prohibitively short, then this logic cannot be applied, however we can remedy this by noting that if the $T^{m}$-Height is unusually small in relation to the $L^{m}$-Height, then this essentially forces the loop sizes to be large compared to what we would normally expect.

More concretely, in this case, let $M^{\prime}$ be the total number of goodish loops on the m-loopspine (i.e. the total number of loops of length at least $4 R$ ). Using the
subordinator representation of the loop lengths, we then have that

$$
\begin{aligned}
\mathbf{P}\left(M^{\prime} \leq \lambda,\right. & \left.T^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \leq R^{\alpha-1} \lambda^{h} \left\lvert\, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}\right.\right) \\
& \leq c \mathbf{P}\left(M^{\prime} \leq \lambda, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}, T^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \leq R^{\alpha-1} \lambda^{h}\right) \\
& \leq c \mathbf{P}\left(\left.\operatorname{Sub}_{R^{\alpha-1} \lambda^{h}} \geq \frac{1}{2}-4 R \lambda \right\rvert\, \text { no jumps of size at least } 4 R\right),
\end{aligned}
$$

where the third line follows by removing any jumps corresponding to goodish loops from Sub, and Sub is a subordinator with (time-dependent) jump measure

$$
C_{\alpha} \mathbb{1}_{\{[0,1]\}}(u) \mathbb{1}_{\left\{\left[0, H^{m}\right]\right\}}(t) l^{-\alpha} \operatorname{pen}\left(l, H^{m}, t\right) d u d t d l .
$$

Note that Sub is almost an $(\alpha-1)$-stable subordinator, but with the extra penalty against larger jumps. We therefore let Sub ${ }^{\alpha-1}$ denote an $(\alpha-1)$-stable subordinator. It follows that for any $k>0$, and any $t, x, y>0$ :

$$
\begin{aligned}
& \mathbf{P}\left(\text { Sub }_{t} \geq x \mid \text { no jumps of size at least } y\right) \\
& \leq \mathbf{P}\left(\text { Sub }_{t}^{\alpha-1} \geq x \mid \text { no jumps of size at least } y\right) \\
& =\mathbf{P}\left(\text { Sub }_{k^{\alpha-1}}^{\alpha-1} \geq k x \mid \text { no jumps of size at least } k y\right) .
\end{aligned}
$$

Taking $k=R^{-1} \lambda^{\frac{-h}{\alpha-1}}$, we therefore see that

$$
\begin{aligned}
& \mathbf{P}\left(M^{\prime} \leq \lambda, T^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \leq R^{\alpha-1} \lambda^{h} \left\lvert\, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}\right.\right) \\
& \quad \leq \mathbf{P}\left(\left.\operatorname{Sub}_{1}^{\alpha-1} \geq \frac{1}{2} R^{-1} \lambda^{\frac{-h}{\alpha-1}}-\lambda^{1-\frac{h}{\alpha-1}} \right\rvert\, \text { no jumps at least } 4 \lambda^{-\frac{h}{\alpha-1}}\right) \\
& \quad \leq \mathbf{E}\left[e^{\theta \operatorname{Sub}_{1}^{\alpha-1}}\right] e^{-\theta \lambda}
\end{aligned}
$$

for sufficiently small $\theta>0$, where the existence of the exponential moment in the last line follows from Remark 4.2.3, and we recall that $R<\lambda^{-1-\frac{h}{\alpha-1}}$ by assumption.

We can then proceed exactly as in the second and third bullet points above to deduce that the second term in (5.11) is upper bounded by $C e^{-c \lambda}$. This completes the proof.

Armed with this, we can prove the probabilistic resistance bound as follows.
Proof of Proposition 5.4.4. By scaling invariance of $\mathcal{L}_{\alpha}^{\infty}$, it is sufficient to prove the result for $r=1$.

Take $R=\lambda^{-2 t}$, for some positive constant $t$ that will be specified later. The aim will be to bound the cardinality of a set $A \subset \mathcal{L}_{\alpha}^{\infty}$ such that any path from $B^{\infty}\left(\rho^{\infty}, R\right)$ to $B^{\infty}\left(\rho^{\infty}, 1\right)^{c}$ must pass through at least one point in $A$. Do to this,
we will define a tree $T_{\text {res }} \subset \mathcal{U}$, obtained similarly to $T_{\text {vol }}$ in the box above, but with two important differences:

- Rather than decomposing along the W-loopspine in the second and subsequent steps, we decompose along the m-loopspine.
- Rather than reiterating around sublooptrees of larger mass, we reiterate around those with large $L$-Height: specifically, those that are grafted to the m-loopspine within distance $R$ of the root, and with $L^{m}$-Height at least $\frac{1}{2}$. We decompose along the m-loopspine rather than the loopspine to the point achieving the $L$-Height purely because it is more convenient to use absolute continuity and Proposition 2.5.4 to control loop lengths on the $m$-loopspine. However, an expression analogous to (4.19) should also be true in the case of this loopspine.

We will show that, with sufficiently high probability, the total progeny of $T_{\text {res }}$ is at most $\frac{1}{2} \lambda^{t}$, and that, on this event, we can pick a set $A$ of cardinality at most $\lambda^{2 t}$. In this case we are done: since $A$ is a cutset, we then have that

$$
\begin{equation*}
R_{\mathrm{eff}}^{\infty}\left(\rho^{\infty}, B^{\infty}\left(\rho^{\infty}, 1\right)^{c}\right) \geq R_{\mathrm{eff}}^{\infty}\left(\rho^{\infty}, A\right) \tag{5.13}
\end{equation*}
$$

and due to the underlying tree structure this latter quantity is lower bounded by the resistance of $2|A|$ edges connected in parallel, each of resistance $\lambda^{-2 t}$. More precisely:

$$
R_{\mathrm{eff}}^{\infty}\left(\rho^{\infty}, A\right) \geq\left(|A| \lambda^{2 t}\right)^{-1} \geq \frac{1}{2} \lambda^{-4 t}
$$

We will then optimise over $t$ to obtain the result.
To this end, we now turn to bounding $\left|T_{\text {res }}\right|$. As commented on page 118, the sequence of sublooptrees incident to the m-loopspine at a point in $I_{R}^{m}$ can be stochastically dominated by those coded by the classical Itô excursion measure along this segment, so the offspring distribution of a particular $u \in T_{\text {res }}$ will be Poisson $\left(\tilde{C}\left|I_{R}^{m, u}\right|\right)$, where $\tilde{C}=N\left(L^{m}\right.$-Height $\left.\geq \frac{1}{2}\right)$, and we have added an extra superscript $u$ to denote the dependence on $u$. By applying Lemma 5.4.7 with $h=(\alpha-1)(2 t-1)$, it then follows exactly as in Proposition 4.2.11 that, if $\hat{T}$ is a Galton-Watson tree with Poisson $\left(\tilde{C} \lambda^{-t}\right)$ offspring distribution, then

$$
\begin{aligned}
\mathbf{P}\left(\left|T_{\mathrm{res}}\right| \geq \lambda^{t}\right) & \leq \lambda^{t} \mathbf{P}\left(\left|I_{R}^{m}\right| \geq R \lambda^{t} \left\lvert\, L^{m}-\operatorname{Height}\left(\tilde{\mathcal{L}_{\alpha}}\right) \geq \frac{1}{2}\right.\right)+\mathbf{P}\left(|\hat{T}| \geq \frac{1}{2} \lambda^{t}\right) \\
& \leq C \lambda^{t} C e^{-c \lambda^{t(h \wedge 1)}}+C e^{-c \lambda^{t}}
\end{aligned}
$$

Assuming now that $\left|T_{\text {res }}\right|<\frac{1}{2} \lambda^{t}$, we claim that we can pick a set $A$ of cardinality at most $\lambda^{2 t}$. In fact, rather than just assuming that $\left|T_{\text {res }}\right|<\frac{1}{2} \lambda^{t}$, we can assume that all of the events we conditioned on in order to construct the event $\left\{\left|T_{\text {res }}\right|<\frac{1}{2} \lambda^{t}\right\}$ do indeed occur. In particular, we can assume that:
(i) For each $u \in T_{\text {res }}$, letting $N_{u}$ be the number of goodish loops on the m-loopspine between $\rho_{u}$ and the first good loop, we have that $N_{u}<\lambda^{t}$.
(ii) For each $u \in T_{\text {res }}$, letting $Q_{u}^{(i)}$ denote the sum of the length of the shorter loops between successive goodish loops on the m-loopspine,

$$
\sum_{i=1}^{N_{u}} Q_{u}^{(i)}<R \lambda^{t}=\lambda^{-t}
$$

(iii) $\left|T_{\text {res }}\right|<\frac{1}{2} \lambda^{t}$.

Assuming this, we now describe how we select the set $A$. This is illustrated in Figure 5.5 below which represents the m-loopspine of some $u \in T_{\text {res }}$. In particular, on this m-loopspine, we can pick two points on each of the goodish loops, and two points on the first good loop, to be in $A$. Moreover, these points can be chosen so that they are within distance $R+\lambda^{-t}$ of the "base point" of the loop (see Figure 5.5). If one of the goodish loops violates the condition that the length of its shorter segment is less than $R$, we can instead treat it as the first good loop.

From the assumptions above, we deduce the following:
(i) ${ }^{\prime}$ For all $u \in T_{\text {res }}$, the number of points of $A$ contained in $\mathcal{L}_{\alpha}^{(u)}$ is at most $2 N_{u}$ which by $(i)$ above is in turn at most $2 \lambda^{t}$.
(ii) $)^{\prime}|A| \leq\left|T_{\text {res }}\right| 2 \lambda^{t}=\lambda^{2 t}$.
(iii) $)^{\prime}$ Points in $A$ that are selected as points in the looptree corresponding to $u$ are within distance $\left|I_{R}^{m}\right|+\lambda^{-t}$ of $\rho_{u}$, i.e. distance $2 \lambda^{-t}$ of $\rho_{u}$.
$(\text { iv })^{\prime}$ All points in $A$ are within distance $\left|T_{\text {res }}\right| \lambda^{-t}+\lambda^{-t}$ of $\rho^{\infty}$, which is at most $\frac{1}{2}$ by (iii) above.
(v) ${ }^{\prime}$ Therefore, any sublooptree grafted to the m-loopspine of $\mathcal{L}_{\alpha}^{(u)}$ for some $u \in T_{\text {res }}$ that has $L$-Height less than $\frac{1}{2}$, will not intersect $B(\rho, 1)^{c}$. In other words, $A$ is really a cutset.

From the probabilistic bounds above, and since we set $h=(\alpha-1)(2 t-1)$, we therefore deduce that

$$
\begin{aligned}
\mathbf{P}\left(R_{\mathrm{eff}}^{\infty}\left(\rho^{\infty}, B^{\infty}\left(\rho^{\infty}, 1\right)^{c}\right) \leq \frac{1}{2} \lambda^{-4 t}\right) & \leq C \lambda^{t} C e^{-c \lambda^{t(h \wedge 1)}}+e^{-c \lambda^{t}} \\
& \leq C \lambda^{t} C e^{-c \lambda^{t(2 t-1)(\alpha-1)}}+C e^{-c \lambda^{t}}
\end{aligned}
$$

In particular, choosing $t>\frac{\alpha}{2(\alpha-1)}$, we obtain

$$
\mathbf{P}\left(R_{\mathrm{eff}}^{\infty}\left(\rho^{\infty}, B^{\infty}\left(\rho^{\infty}, 1\right)^{c}\right) \leq \frac{1}{2} \lambda^{-4 t}\right) \leq C e^{-c \lambda^{t}}
$$



Figure 5.5: How to select $A$. The red segment contains the portion of $B\left(\rho^{\infty}, R\right)$ intersecting the m-loopspine. NB This is a simplified diagram since no loops are actually adjacent.
or equivalently,

$$
\mathbf{P}\left(R_{\mathrm{eff}}^{\infty}\left(\rho^{\infty}, B^{\infty}\left(\rho^{\infty}, 1\right)^{c}\right) \leq \lambda^{-1}\right) \leq C e^{-c \lambda^{\frac{1}{4}}} .
$$

### 5.5 Random walk limits

### 5.5.1 Brownian motion and spectral dimension of $\mathcal{L}_{\alpha}^{\infty}$

As in the case of compact looptrees, the looptree convergence results can be used to give a collection of limit results for random walks and Brownian motion on sequences of looptrees. Before we do this, we have to show that $R^{\infty}$ is in fact a resistance metric, and that the resistance form associated with the metric space $\left(\mathcal{L}_{\alpha}^{\infty}, R^{\infty}\right)$ is regular, which implies that it is also a regular Dirichlet form on the space $L^{2}\left(\mathcal{L}_{\alpha}^{\infty}, \nu\right)$ and so is naturally associated with a stochastic process. This is done in the following two propositions.

Proposition 5.5.1. P-almost surely, $R^{\infty}$ is a resistance metric in the sense of Definition 2.4.1.

Proof. This follows from [Arc19, Proposition 4.4], in which we prove the same result for compact stable looptrees. In particular, any finite set of points $V$ in $\mathcal{L}_{\alpha}^{\infty}$ is contained in $B\left(\rho^{\infty}, r\right)$ for some $r>0$. Taking such an $r$, we then define $t_{g}(r)$ and $t_{d}(r)$ exactly as we did in the proof of Theorem 5.0.1; that is, we set

$$
t_{g}(r)=\inf \left\{s \geq 0: \Delta_{-s} \geq 4 r, \delta_{-s}^{\infty}\left(x_{-s, 0}^{\infty}\right) \geq r\right\}, t_{d}(r)=\inf \left\{s \geq 0: X_{s}^{\infty} \leq X_{-t_{g}(r)^{-}}^{\infty}\right\}
$$

As in previous proofs, it then follows that $B\left(\rho_{\infty}, r\right) \subset p^{\infty}\left(\left[-t_{g}(r), t_{d}(r)\right]\right)$, and
$p^{\infty}\left(-t_{g}(r)\right)=p^{\infty}\left(t_{d}(r)\right)$. Moreover, $p^{\infty}\left(\left[-t_{g}(r), t_{d}(r)\right]\right)$ codes a compact stable looptree, which, in keeping with earlier notation, we denote by $\mathcal{L}_{\alpha}(r)$. We endow it with a metric and a measure by restricting $R^{\infty}$ and $\nu^{\infty}$ to $\mathcal{L}_{\alpha}(r)$.

It then follows exactly as in [Arc19, Proposition 4.4] that $R^{\infty}$ restricted to $\mathcal{L}_{\alpha}(r)$ is a resistance metric on $\mathcal{L}_{\alpha}(r)$, and that we can therefore construct a weighted network with vertex set $V$ with matching effective resistance. The same network will therefore work for $\mathcal{L}_{\alpha}^{\infty}$.

Proposition 5.5.2. P-almost surely, the resistance form associated with the metric space $\left(\mathcal{L}_{\alpha}^{\infty}, R^{\infty}\right)$ is regular.

Proof. We let $\left(\mathcal{E}_{\infty}, \mathcal{F}_{\infty}\right)$ denote the resistance form on $\mathcal{L}_{\alpha}^{\infty}$ associated with the resistance metric $R^{\infty}$ as in (2.8). According to Definition 2.4.2, we need to show that for any $f \in C_{0}\left(\mathcal{L}_{\alpha}^{\infty}\right)$ and any $\varepsilon>0$, we can find $g^{\prime} \in \mathcal{F}_{\infty} \cap C_{0}\left(\mathcal{L}_{\alpha}^{\infty}\right)$ such that $\left\|f-g^{\prime}\right\|_{\infty} \leq \varepsilon$. The key point is that by cutting off the infinite loopspine of $\mathcal{L}_{\alpha}^{\infty}$ at an appropriate cutpoint, any such $f$ is also a compactly supported function on a compact stable looptree, and therefore approximable on this compact looptree, since all resistance forms on compact spaces are regular. Formally, we proceed as follows.

First, note that since $f$ is compactly supported, then its support must be contained in $B\left(\rho^{\infty}, r\right)$ for some $r>0$. Taking such an $r$, we then define $t_{g}(r)$ and $t_{d}(r)$ exactly as we did in the proof of Theorem 5.0.1; that is, we set

$$
t_{g}(r)=\inf \left\{s \geq 0: \Delta_{-s} \geq 4 r, \delta_{-s}^{\infty}\left(x_{-s, 0}^{\infty}\right) \geq r\right\}, t_{d}(r)=\inf \left\{s \geq 0: X_{s}^{\infty} \leq X_{-t_{g}(r)^{-}}^{\infty}\right\} .
$$

As in previous proofs, it then follows that $B\left(\rho_{\infty}, r\right) \subset p^{\infty}\left(\left[-t_{g}(r), t_{d}(r)\right]\right)$, and $v_{r}:=p^{\infty}\left(-t_{g}(r)\right)=p^{\infty}\left(t_{d}(r)\right)$. Moreover, $p^{\infty}\left(\left[-t_{g}(r), t_{d}(r)\right]\right)$ codes a compact stable looptree, which, in keeping with earlier notation, we denote by $\mathcal{L}_{\alpha}(r)$. We endow it with a metric and a measure by restricting $R^{\infty}$ and $\nu^{\infty}$ to $\mathcal{L}_{\alpha}(r)$, and denote the associated resistance form by $\left(\mathcal{E}_{r}, \mathcal{F}_{r}\right)$.

The key point is the following: by [Kig12, Theorem 8.4], and the one-toone correspondence given by (2.8) and its continuum extension on compact spaces, $\left(\mathcal{E}_{r}, \mathcal{F}_{r}\right)$ is obtained as the trace of $\left(\mathcal{E}_{\infty}, \mathcal{F}_{\infty}\right)$ on $\mathcal{L}_{\alpha}(r)$, and is such that for any $f \in \mathcal{F}_{r}, \mathcal{E}_{r}(f, f)=\mathcal{E}_{\infty}(h(f), h(f))$, where $h(f)$ is the unique harmonic extension of $f$ to $\mathcal{L}_{\alpha}^{\infty}$.

Now take $f \in \mathcal{F}_{\infty}$. Note that, necessarily, $f\left(v_{r}\right)=0$, since $f$ is continuous. Moreover, $v_{r}$ is a point on the infinite loopspine that cuts $\rho^{\infty}$ off from $\infty$. Arbitrarily, we now choose a new point $v_{r}^{\prime}$ on the loopspine, coded by a jump point of $X^{\infty}$, that also separates $\rho^{\infty}$ from $\infty$, but such that $R^{\infty}\left(\rho^{\infty}, v_{r}^{\prime}\right)>R^{\infty}\left(\rho^{\infty}, v_{r}\right)$. It follows that $v_{r}^{\prime}$ is coded by jump point of $X^{\infty}$ at a time that we denote by $-t_{g, 2}(r)$, where $t_{g, 2}(r)>t_{g}(r)$ and $-t_{g, 2}(r) \preceq 0$. For any $s$ with $-t_{g, 2}(r) \preceq s \prec-t_{g}(r)$, set $a_{s}=$
$\delta_{s}\left(x_{s, 0}^{\infty}\right)$, and $b_{s}=\Delta_{s}-\delta_{s}\left(x_{s, 0}^{\infty}\right)$, so that $a_{s}$ gives the length of the "shorter" segment of the corresponding loop in the loopspine, and $b_{s}$ gives the length of the "longer" segment (see Figure 5.6). Set

$$
d_{\min }=\sum_{-t_{g, 2}(r) \leq s \prec-t_{g}(r)} a_{s}, \quad d_{\max }=\sum_{-t_{g, 2}(r) \leq s \prec-t_{g}(r)} b_{s} .
$$

These are defined so that $d_{\text {min }}$ gives the looptree distance between $v_{r}$ and $v_{r}^{\prime}$, and $d_{\text {max }}$ gives the "longer distance" between them, which is the length of the path between them that traverses the longer side of all the loops in the loopspine that lie between $v_{r}$ and $v_{r}^{\prime}$ (see Figure 5.6).


Figure 5.6: Illustration of how we cut the infinite loopspine.
Additionally, let

$$
t_{d, 2}(r)=\inf \left\{s \geq 0: X_{s}^{\infty} \leq X_{-t_{g, 2}(r)^{-}}^{\infty}\right\} .
$$

Then $p^{\infty}\left(\left[-t_{g, 2}(r), t_{d, 2}(r)\right]\right)$ codes another compact stable looptree which we denote by $\mathcal{L}_{\alpha}(r)^{\prime}$, satisfying $\mathcal{L}_{\alpha}(r) \subset \mathcal{L}_{\alpha}(r)^{\prime} \subset \mathcal{L}_{\alpha}^{\infty}$.

Since $\mathcal{L}_{\alpha}(r)$ is compact, it follows that $\left(\mathcal{E}_{r}, \mathcal{F}_{r}\right)$ is regular, so there exists $g \in \mathcal{F}_{r} \cap C_{0}\left(\mathcal{L}_{\alpha}(r)\right)$ with $\left\|\left.f\right|_{\mathcal{L}_{\alpha}(r)}-g\right\|_{\infty} \leq \varepsilon$. We therefore define a function $g^{\prime} \in C_{0}\left(\mathcal{L}_{\alpha}^{\infty}\right)$ by setting $g^{\prime}=g$ on $\mathcal{L}_{\alpha}(r), g^{\prime}=0$ on $\mathcal{L}_{\alpha}^{\infty} \backslash \mathcal{L}_{\alpha}(r)^{\prime}$, and extending harmonically on $\mathcal{L}_{\alpha}(r)^{\prime} \backslash \mathcal{L}_{\alpha}(r)$.

Since $g$ approximates $\left.f\right|_{\mathcal{L}_{\alpha}(r)}$ in the supremum norm, it follows that $\left|g\left(v_{r}\right)\right| \leq$ $\varepsilon$, and moreover it then follows by the maximum principle for harmonic functions that $\left\|g_{\mathcal{L}_{\alpha}(r)^{\prime} \backslash \mathcal{L}_{\alpha}(r)}\right\|_{\infty} \leq \varepsilon$. Consequently, $\left\|f-g^{\prime}\right\|_{\infty} \leq \varepsilon$. It therefore just remains to show that $\mathcal{E}_{\infty}\left(g^{\prime}, g^{\prime}\right)<\infty$.

Let $\left(\mathcal{E}_{r}^{\prime}, \mathcal{F}_{r}^{\prime}\right)$ denote the restriction of $\left(\mathcal{E}_{\infty}, \mathcal{F}_{\infty}\right)$ to $\mathcal{L}_{\alpha}(r)^{\prime}$. Since the spaces $\mathcal{L}_{\alpha}(r), \mathcal{L}_{\alpha}(r)^{\prime} \backslash \mathcal{L}_{\alpha}(r)$ and $\mathcal{L}_{\alpha}^{\infty} \backslash \mathcal{L}_{\alpha}(r)^{\prime}$ are disjoint, and $g^{\prime}$ is the harmonic extension of $\left.g^{\prime}\right|_{\mathcal{L}_{\alpha}(r)^{\prime}}$ to $\mathcal{L}_{\alpha}^{\infty}$, it follows by bilinearity and from consistency properties of resistance
forms and their traces given in $[\operatorname{Kig} 12$, Section 8$]$ that

$$
\begin{equation*}
\mathcal{E}_{\infty}\left(g^{\prime}, g^{\prime}\right)=\mathcal{E}_{r}^{\prime}\left(\left.g^{\prime}\right|_{\mathcal{L}_{\alpha}(r)^{\prime}},\left.g^{\prime}\right|_{\mathcal{L}_{\alpha}(r)^{\prime}}\right) \tag{5.14}
\end{equation*}
$$

However, since $\mathcal{L}_{\alpha}(r)^{\prime}$ is simply a compact looptree, this is automatically finite.

As a result, we deduce that the resistance metric space is naturally associated with a Hunt process on $\left(\mathcal{L}_{\alpha}^{\infty}, R^{\infty}\right)$, which we call Brownian motion on $\mathcal{L}_{\alpha}^{\infty}$ and denote by $B^{\infty}$.

### 5.5.2 Quenched results

We can apply Theorem 2.4.3 to the results of Theorems 5.0.1 and 5.0.2 to deduce convergence results for stochastic processes on the corresponding spaces. The only additional detail in the proofs of these results is that we have to check that the non-explosion condition at (2.9) is satisfied, i.e. that

$$
\lim _{r \rightarrow \infty} \liminf _{\ell \rightarrow \infty} R^{\ell}\left(\rho^{\ell}, B^{\ell}\left(\rho^{\ell}, r\right)^{c}\right)=\infty
$$

almost surely, where $R^{\ell}$ here denotes the resistance metric on $\mathcal{L}_{\alpha}^{\ell}$.

## Local limits

The local limit theorem of Theorem 5.0.1 immediately allows us to apply Theorem 2.4.3 to deduce that Brownian motion on $\mathcal{L}_{\alpha}^{\ell}$ converges in distribution to Brownian motion on $\mathcal{L}_{\alpha}^{\infty}$ as $\ell \rightarrow \infty$ on compact time intervals. Indeed, it follows from Theorem 2.2.4 and the Skorokhod Representation Theorem that there exists a probability space on which the convergence on Theorem 5.0.1 holds almost surely. Moreover, the explosion condition is satisfied as an immediate consequence of Proposition 5.4.4. In particular, the arguments used to prove Proposition 5.4.4 are also valid for compact stable looptrees, so we deduce that the resistance bounds of Proposition 5.4.4 almost surely hold along the sequence $\left(\mathcal{L}_{\alpha}^{\ell}\right)_{\ell \in \mathbb{N}}$.

Theorem 5.0.3 then follows by a direct application of Theorem 2.4.3.

## Scaling limits

We can also deduce similar results from Theorems 5.0.4, 5.3.3 and 5.3.4. In this case, the non-explosion condition is satisfied as a result of [BS15, Lemma 3.5], which says that for $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$, there exist $q, C \in(0, \infty)$ such that

$$
\begin{equation*}
\mathbf{P}\left(R_{\mathrm{eff}}\left(\rho, B(\rho, r)^{c}\right) \leq r \lambda^{-1}\right) \leq C \lambda^{-q} . \tag{5.15}
\end{equation*}
$$

In light of Proposition 5.4.4, we conjecture that there should in actual fact be exponential tail decay, but polynomial decay is sufficient for our purposes here. Indeed, to verify (2.9), we need to show that

$$
\lim _{r \rightarrow \infty} \liminf _{n \rightarrow \infty} n^{\frac{-1}{\alpha}} R_{\mathrm{eff}}\left(\rho, B\left(\rho, r n^{\frac{1}{\alpha}}\right)^{c}\right)=\infty
$$

$\mathbf{P}$-almost surely. This follows directly from applying a Borel-Cantelli argument along a suitable subsequence using the probabilistic bound (5.15). Moreover, the same applies for $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$ since $R_{\text {eff }}\left(\rho, \mathcal{B}_{r}\left(\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)\right)^{c}\right) \geq R_{\text {eff }}\left(\rho, B_{r-1}\left(\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)\right)^{c}\right)$.

Similarly, the result also holds for the two-type looptree $\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right)$, since $R_{\text {eff }}\left(\rho, \mathcal{B}_{r}\left(\overline{\operatorname{Loop}}\left(T_{\alpha}^{\infty}\right)\right)^{c}\right) \geq R_{\text {eff }}\left(\rho, B_{r-H e i g h t\left(T r e e\left(\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)^{r}\right)\right)}\left(\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)\right)^{c}\right)$, and also $r^{-1} \operatorname{Height}\left(\right.$ Tree $\left(\right.$ Loop $\left.\left.^{\prime}\left(T_{\alpha}^{\infty}\right)^{r}\right)\right) \rightarrow 0$ in probability, with exponential tail decay (as in Point 2 of the proof of Theorem 5.3.4), allowing further Borel-Cantelli arguments.

In all the different versions of infinite looptrees that we have considered, the Gromov-Hausdorff-Prohorov convergence holds with the uniform measure on vertices of the looptree, and the associated stochastic process is therefore a variable speed random walk.

In the case of $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$, all vertices have degree 4, so in this case the stochastic process is actually a constant speed random walk, with $\exp (4)$ waiting times at each vertex. However, by applying Kolmogorov's Maximal Inequality to the time index of this stochastic process (as in the proof of [Arc19, Theorem 1.1]) we can show that the waiting times average out sufficiently well over time so the scaling limit result will also hold for a simple random walk on $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$ (although sped up deterministically by a factor of 4).

Theorem 5.0.4 therefore follows by an immediate application of Theorem 2.4.3 to Proposition 5.3.2.

In the case of $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$, all internal vertices have degree 4, and all leaf vertices have degree 2 . This corresponds to the fact the the only significant difference between $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$ and $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$ is that in $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$ the loops corresponding to leaves are missing, and has the effect that (on average) the random walk waits twice as long at leaf vertices compared to internal vertices. This reflects the fact that the random walks on $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$ and $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$ can (almost, technically only after adding one extra vertex to the loop containing the root in Loop $\left(T_{\alpha}^{\infty}\right)$ ) be coupled so that they move identically at internal vertices, but so that a random walk on Loop $^{\prime}\left(T_{\alpha}^{\infty}\right)$ remains in its present position whenever the random walk on $\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)$ traverses a loop corresponding to a leaf vertex (note this can be traversed in either direction). It therefore makes sense that we should be taking a scaling limit of the variable speed random walk on $\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right)$, rather than the constant speed one.

We similarly have to take a variable speed random walk on $\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right)$, although there is not such a simple coupling in this case. In the next theorem, we
let $L_{\alpha}^{\infty, 1}=\operatorname{Loop}^{\prime}\left(T_{\alpha}^{\infty}\right), L_{\alpha}^{\infty, 2}=\operatorname{Loop}^{2}\left(T_{\alpha}^{\infty, 2}\right), Y^{\text {var,i }}$ denote a variable speed random walk on $L_{\alpha}^{\infty, i}$, and $\nu^{i}$ denote the measure giving mass 1 to each vertex. The nonexplosion condition is again satisfied by the same arguments as in Section 5.5.2 above. We then have the following analogues of Theorem 5.0.4.

Theorem 5.5.3. Take $i \in\{1,2\}$. There exists a probability space $\left(\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbf{P}^{\prime}\right)$ on which we can almost surely define a common metric space $\left(M, R_{M}\right)$ in which the spaces $\left(L_{\alpha}^{\infty, i}, a_{n}^{-1} \tilde{d}, n^{-1} \nu^{\prime}, \rho\right)$ and $\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)$ can all be isometrically embedded and such that

$$
\left(L_{\alpha}^{\infty, i}, a_{n}^{-1} \tilde{d}, n^{-1} \nu^{i}, \rho\right) \xrightarrow{(d)}\left(\mathcal{L}_{\alpha}^{\infty}, \tilde{d}^{\infty}, \nu^{\infty}, \rho^{\infty}\right)
$$

with respect to the Gromov-Hausdorff-vague topology, and the convergence specifically holds on the metric space $\left(M, R_{M}\right)$. Letting $Y^{v a r, i}$ and $B^{\infty}$ be as above, we have that

$$
\left(a_{n}^{-1} Y_{\left\lfloor n a_{n} t\right\rfloor}^{v a r, i}\right)_{t \geq 0} \xrightarrow{(d)}\left(B_{t}^{\infty}\right)_{t \geq 0}
$$

on the space $D\left(\mathbb{R}^{+}, M\right)$ as $n \rightarrow \infty$.
Remark 5.5.4. We could also prove other convergence results, for example by taking increasing sequences of increasingly rescaled discrete looptrees to approximate $\mathcal{L}_{\alpha}^{\infty}$, in some sense combining Theorems 5.0.1 and 4.1.6, and deduce similar convergence results for random walks, exactly as we did in the cases above. This corresponds to the diagonal line in Figure 5.1.

### 5.5.3 Heat kernel convergence and spectral dimension

To conclude, we now show how Theorem 5.0.1 can be applied to give results on the heat kernel of Brownian motion on compact stable looptrees. First, note that it follows from the scaling invariance of Proposition 5.1.1 that the annealed heat kernel for $\mathcal{L}_{\alpha}^{\infty}$ satisfies the scaling relation

$$
\begin{equation*}
\mathbf{E}\left[p_{t}^{\infty}(\rho, \rho)\right]=k^{\frac{\alpha}{\alpha+1}} \mathbf{E}\left[p_{k t}^{\infty}(\rho, \rho)\right] \tag{5.16}
\end{equation*}
$$

for any $k>0$. Similarly, if we let $p_{t}^{\ell}$ denote the transition density of Brownian motion on a looptree coded by an excursion of length $\ell$, we have that

$$
\mathbf{E}\left[p_{t}^{1}(\rho, \rho)\right]=k^{\frac{\alpha}{\alpha+1}} \mathbf{E}\left[p_{k t}^{k^{\frac{1}{\alpha+1}}}(\rho, \rho)\right]
$$

Setting $k=t^{-1}$ we see that

$$
t^{\frac{\alpha}{\alpha+1}} \mathbf{E}\left[p_{t}^{1}(\rho, \rho)\right]=\mathbf{E}\left[p_{1}^{t^{\frac{-1}{\alpha+1}}}(\rho, \rho)\right]
$$

Moreover, since we are in a resistance framework, it follows from [CH08, Theorem 2 and Proposition 14] that

$$
t^{\frac{\alpha}{\alpha+1}} p_{t}^{1}(\rho, \rho) \xrightarrow{(d)} p_{1}^{\infty}(\rho, \rho)
$$

as $t \downarrow 0$. To deduce that the corresponding expectations also converge, we just need to show that $\mathbf{E}\left[p_{1}^{\infty}(\rho, \rho)\right]$ is finite. However, since the transition density can be bounded by bounding the volume and resistance growth (by a continuum version of [KM08, Proposition 1.4], for example), the exponential tail decay of Propositions 5.4.3 and 5.4.4 also give an upper exponential tail decay for the transition density. We therefore deduce that $\mathbf{E}\left[p_{1}^{\infty}(\rho, \rho)\right]$ is finite, so we can apply similar arguments to those in the previous section to deduce that

$$
t^{\frac{\alpha}{\alpha+1}} \mathbf{E}\left[p_{t}^{1}(\rho, \rho)\right] \rightarrow \mathbf{E}\left[p_{1}^{\infty}(\rho, \rho)\right]
$$

as $t \rightarrow \infty$. This is stated as [Arc19, Theorem 1.8], where Brownian motion on $\mathcal{L}_{\alpha}$ is studied more closely.

Similarly, it also follows from [KM08, Theorem 1.5, Part II] (adapted to the continuum) that the heat kernel $p_{t}^{\infty}(\rho, \rho)$ almost surely experiences at most loglogarithmic fluctuations around a leading term of $t^{\frac{-\alpha}{\alpha+1}}$ as $t \uparrow \infty$ and as $t \downarrow 0$, and therefore that the quenched spectral dimension of $\mathcal{L}_{\alpha}$ is almost surely equal to $\frac{2 \alpha}{\alpha+1}$.

To establish the annealed spectral dimension, we take $k=t^{-1}$ in (5.16) to deduce that

$$
\mathbf{E}\left[p_{t}^{\infty}(\rho, \rho)\right]=t^{\frac{-\alpha}{\alpha+1}} \mathbf{E}\left[p_{1}^{\infty}(\rho, \rho)\right] .
$$

Since $\mathbf{E}\left[p_{1}^{\infty}(\rho, \rho)\right]$ is finite, this implies that the annealed spectral dimension is also equal to $\frac{2 \alpha}{\alpha+1}$. This concludes the proof of Theorem 5.0.6.

## Chapter 6

## Random Walks on Decorated Galton-Watson Trees

As outlined in the introduction, the purpose of this chapter is to consider random walks on a generalised decorated tree model. This time we work purely in the discrete setting and we will therefore consider limits as $t, r \rightarrow \infty$. As in previous chapters, we will take $\alpha \in(1,2]$ and assume that the underlying tree has critical offspring distribution satisfying $\xi([k, \infty)) \sim c k^{-\alpha}$ as $k \rightarrow \infty$ and in order to take the appropriate limits we will assume it is conditioned to survive, by which we mean it is constructed as Kesten's tree according to Definition 3.3.1. One could also add a slowly-varying function to the offspring distribution and carry this through all the computations, but we have omitted this for sake of clarity. We will denote this tree by $T_{\alpha}^{\infty}$.

Informally, we construct our decorated tree from $T_{\alpha}^{\infty}$ using the same procedure as for constructing looptrees, but rather than inserting a loop of length $n$ at a vertex $v$ of degree $n$, we insert a connected graph $G_{n}$ that has a $n$ boundary vertices. Similarly to the looptree construction, we uniquely identify each boundary vertex of $G_{n}$ with an edge incident to $v$, and then if $v \sim v^{\prime}$ in $T_{\alpha}^{\infty}$ we glue their corresponding graphs at the two vertices identified with this edge. $G_{n}$ may be random (e.g. an Erdös-Rényi graph on $n$ vertices) or deterministic (e.g. the complete graph on $n$ vertices); in the random case we sample independently for each vertex, conditional on the boundary size. We call the resulting structure $\mathcal{T}_{\alpha}^{\text {dec }}$, and we let the vertex of $\mathcal{T}_{\alpha}^{\text {dec }}$ that corresponds to the edge in $\mathcal{T}^{\text {dec }}$ joining $\rho$ to its leftmost child be the root of $\mathcal{T}_{\alpha}^{\text {dec }}$. See Figure 6.1 for an illustration for a finite tree.

In this chapter we consider a simple random walk on $\mathcal{T}_{\alpha}^{\text {dec }}$. Our aim is to establish the exponents governing the behaviour of the simple random walk, in terms of those for the underlying tree and those for the inserted graphs. We focus particularly on the displacement exponent, and the spectral dimension; as indicated


Figure 6.1: An example of a decorated tree, and its underlying Galton-Watson tree.
in the introduction, we will define these by the following limits, provided these exist.
(i) The spectral dimension, $d_{S}=-2 \lim _{n \rightarrow \infty} \frac{\log p_{2 n}(x, x)}{\log n}$,
(ii) The displacement exponent, $d_{\text {dis }}=\lim _{n \rightarrow \infty} \frac{\log \sup _{k \leq n} d\left(0, X_{k}\right)}{\log n}$.

As outlined in the introduction, here $p_{n}(x, y)=\frac{\mathbb{P}_{x}\left(X_{n}=y\right)}{\operatorname{deg} y}$ is the transition density of the simple random walk, and the limit in (i) does not depend on the choice of $x$. The quantity $d_{w}=\frac{1}{d_{\text {dis }}}$ is also known as the walk dimension, and can be naturally contrasted with the fractal dimension of $\mathcal{T}_{\alpha}^{\text {dec }}$, denoted $d_{f}$, given by

$$
d_{f}=\frac{\lim _{r \rightarrow \infty} \log (\operatorname{Vol}(B(x, r)))}{\log r} .
$$

We will see that, as is commonly the case for sufficiently homogeneous graphs, the relation

$$
d_{S}=\frac{2 d_{f}}{d_{w}}
$$

holds on $\mathcal{T}_{\alpha}^{\text {dec }}$. This will hold as a consequence of verifying the conditions of [KM08].
As outlined in Section 2.4.2, the values of these exponents depend on the resistance growth and volume growth of $\mathcal{T}^{\text {dec }}$, so we will have to make some assumptions on the growth of these quantities on the inserted graphs. Moreover, if we wish to understand how fast the random walk moves with respect to the graph metric (or some other metric different to the resistance metric), then we will have to make an assumption on this metric too.

Given $n \in \mathbb{N}$, we let $\left(U_{n}^{(1)}, U_{n}^{(2)}\right)$ denote a uniform pair of distinct points on the boundary of $G_{n}, d_{n}$ be the graph metric on $G_{n}$ (in fact the same results will hold for an arbitrary metric), and $R_{\text {eff }}$ denote effective resistance on $G_{n}$ when each
edge has weight 1 . We then set

$$
\begin{aligned}
d_{n}^{U} & =d_{n}\left(U_{n}^{(1)}, U_{n}^{(2)}\right), & & R_{n}^{U}=R_{\mathrm{eff}}\left(U_{n}^{(1)}, U_{n}^{(2)}\right), \\
\operatorname{Diam}\left(G_{n}\right) & =\sup _{x, y \in G_{n}} d_{n}(x, y), & & \operatorname{Diam}_{\mathrm{res}}\left(G_{n}\right)=\sup _{x, y \in G_{n}} R_{\mathrm{eff}}(x, y) .
\end{aligned}
$$

We also let $\operatorname{Vol}\left(G_{n}\right)$ denote the number of edges of $G_{n}$, or equivalently (the factor of 2 is not important) $\mu\left(G_{n}\right)$, where $\mu(x)=\operatorname{deg} x$ for all $x \in G_{n}$, as in Section 2.4.2 (although the volume growth results will also hold on an arbitrary measure on $G_{n}$ and extending to $\mathcal{T}_{\alpha}^{\text {dec }}$ by superposing the measures).

We will assume the following.

## Assumption 6.0.1.

(D) Metric growth. There exists $d \geq 1, k>0$ and constants $c, C \in(0, \infty)$ such that

$$
\mathbf{P}\left(d_{n}^{U} \geq n^{\frac{1}{d}}\right) \geq c, \quad \mathbf{P}\left(\operatorname{Diam}\left(G_{n}\right) \geq \lambda n^{\frac{1}{d}}\right) \leq C e^{-c \lambda^{k}}
$$

(R) Resistance growth. There exists $R \geq 1, k_{r}>0$ and constants $c, C \in(0, \infty)$ such that

$$
\mathbf{P}\left(R_{n}^{U} \geq n^{\frac{1}{R}}\right) \geq c, \quad \mathbf{P}\left(\operatorname{Diam}_{r e s}\left(G_{n}\right) \geq \lambda n^{\frac{1}{R}}\right) \leq C e^{-c \lambda^{k_{r}}}
$$

(V) Volume growth. There exists $v \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq n^{v}\right) \geq c, \quad \mathbf{P}\left(\operatorname{Vol}_{n}\left(G_{n}\right) \geq \lambda n^{v}\right)=O\left(\lambda^{\frac{-(\alpha+\varepsilon)}{v}}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{v}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{v}} \leq \mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq \lambda n^{v}\right) \leq C \lambda^{-m_{v}}
$$

Remark 6.0.2. (i) We believe that the requirements of stretched exponential tail decay in ( $D$ ) and ( $R$ ) above should not be strictly necessary and are endeavouring to weaken this assumption to something akin to ( $V$ ). At the moment the stretched exponential decay is necessary in order to prove a bound on the "decorated height" of a typical finite Galton-Watson tree, but the proof involves decoupling the effect from high degree vertices and large values of $\lambda$, which is unlikely to be optimal. In Section 6.8 we give a heuristic for a proof that wouldn't involve this, in which case we could write assumptions similar to that of $(V)$.
(ii) $(V)($ ii) corresponds to the case when the appropriate tails on the inserted graphs are heavier than the tails of the degree distribution of the underlying tree. As a result, graphs with large volume will not necessarily correspond to those inserted at vertices of large degree, so it is important to know the tail decay more precisely. If we instead $h a d m_{v}+\varepsilon$ and $m_{v}-\varepsilon$ as the exponents on $\lambda$ in point $(V)(i i)$, we would also end up with plus or minus $\varepsilon$ on the various exponents in Theorem 6.0.5.

The exponents $d, R$ and $v$ can be thought of as the internal exponents governing the behaviour of the inserted graphs, conditional on their boundary size. Necessarily $v>1$, and in planar cases where the boundary is itself connected we also have $d, R>1$.

The stretched exponential decay is not necessary to obtain the upper volume bounds that we will present below. In this case we can instead assume the following.

## Assumption 6.0.3.

( $\boldsymbol{D}^{\prime}$ ) Metric growth. There exists $d \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(d_{n}^{U} \geq n^{\frac{1}{d}}\right) \geq c, \quad \mathbf{P}\left(d_{n}^{U} \geq \lambda n^{\frac{1}{d}}\right)=O\left(\lambda^{-d(\alpha-1+\varepsilon)}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{d}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{d}} \leq \mathbf{P}\left(d_{n}^{U} \geq \lambda n^{\frac{1}{d}}\right) \leq C \lambda^{-m_{d}}
$$

( $\boldsymbol{R}^{\prime}$ ) Resistance growth. There exists $R \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(R_{n}^{U} \geq n^{\frac{1}{R}}\right) \geq c, \quad \mathbf{P}\left(R_{n}^{U} \geq \lambda n^{\frac{1}{R}}\right)=O\left(\lambda^{-R(\alpha-1+\varepsilon)}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{R}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{R}} \leq \mathbf{P}\left(R_{n}^{U} \geq \lambda n^{\frac{1}{R}}\right) \leq C \lambda^{-m_{R}}
$$

$\left(\boldsymbol{V}^{\prime}\right)$ Volume growth. There exists $v \geq 1$ such that EITHER:
(i) There exist constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq n^{v}\right) \geq c, \quad \mathbf{P}\left(\operatorname{Vol}_{n}\left(G_{n}\right) \geq \lambda n^{v}\right)=O\left(\lambda^{\frac{-(\alpha+\varepsilon)}{v}}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{v}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{v}} \leq \mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq \lambda n^{v}\right) \leq C \lambda^{-m_{v}} .
$$

Assumption ( $\mathrm{R}^{\prime}$ ) above is only necessary if one wants to understand volume growth with respect to the resistance metric.

In fact, as remarked above, the exponential tail decay is only used in the proof of one proposition which controls the decorated height of a decorated GaltonWatson tree. When proving the volume lower bounds, it is only necessary to control this when volumes are concentrated close to the leaves, since we then need to control distances all the way to the extremities of the underlying tree. It turns out that this is only the case when $v<\alpha$; otherwise, there is enough volume located in internal parts of the tree that we don't need to control these distances in order to catch enough volume close to the root, and we can instead assume the following.

## Assumption 6.0.4.

$\left(D^{\prime \prime}\right)$ Metric growth. There exists $d \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(d_{n}^{U} \geq n^{\frac{1}{d}}\right) \geq c, \quad \mathbf{P}\left(d_{n}^{U} \geq \lambda n^{\frac{1}{d}}\right)=O\left(\lambda^{-d(\alpha-1+\varepsilon)}\right),
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{d}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{d}} \leq \mathbf{P}\left(d_{n}^{U} \geq \lambda n^{\frac{1}{d}}\right) \leq C \lambda^{-m_{d}} .
$$

( $\boldsymbol{R}^{\prime \prime}$ ) Resistance growth. There exists $R \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(R_{n}^{U} \geq n^{\frac{1}{R}}\right) \geq c>0, \quad \mathbf{P}\left(R_{n}^{U} \geq \lambda n^{\frac{1}{R}}\right)=O\left(\lambda^{-R(\alpha-1+\varepsilon)}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{R}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{R}} \leq \mathbf{P}\left(R_{n}^{U} \geq \lambda n^{\frac{1}{R}}\right) \leq C \lambda^{-m_{R}} .
$$

( $V^{\prime \prime}$ ) Volume growth. There exists $v \geq \alpha$ such that EITHER:
(i) There exist constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq n^{v}\right) \geq c>0, \quad \mathbf{P}\left(\operatorname{Vol}_{n}\left(G_{n}\right) \geq \lambda n^{v}\right)=O\left(\lambda^{\frac{-(\alpha-1+\varepsilon)}{v}}\right),
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{v}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{v}} \leq \mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq \lambda n^{v}\right) \leq C \lambda^{-m_{v}} .
$$

The exponents $d, R$ and $v$ can be thought of as the internal exponents governing the behaviour of the inserted graphs, conditional on their boundary size. To understand the how these quantities behave for a graph inserted at a typical vertex of $T_{\alpha}^{\infty}$, we have to "compose" the bounds of Assumption 6.0.1 with the degree distribution of a typical vertex. From Definition 3.3.1, this depends on whether the vertex is on the backbone of $T_{\alpha}^{\infty}$ (we denote this by ' $s$ ', for spinal), or contained in a subtree (we denote this by ' $f$ ', for fragmental). Accordingly, we have to define two sets of exponents for each of $d, R$ and $v$. For the purposes of these definitions, we take $m_{d}, m_{R}$ and $m_{v}=\infty$ in the cases where we have not specified the specific exponent for the tail decay.

> We take $s_{\alpha}^{d}=d(\alpha-1) \wedge m_{d}, s_{\alpha}^{R}=R(\alpha-1) \wedge m_{R}$ and $s_{\alpha}^{v}=\frac{1}{v}(\alpha-1) \wedge m_{v}$.
> We take $f_{\alpha}^{d}=d \alpha \wedge m_{d}, f_{\alpha}^{R}=R \alpha \wedge m_{R}$ and $f_{\alpha}^{v}=\frac{\alpha}{v} \wedge m_{v}$.
> Later we also set $t_{\alpha}^{v}=\frac{f_{\alpha}^{\alpha} \wedge 1}{\alpha}, y_{\alpha}=\frac{f_{v}^{v} \alpha\left(\alpha-1 s_{\alpha}^{d}\right.}{s_{\alpha}^{d}(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)+\alpha f_{\alpha}^{v}}$.

We will see in Section 6.3 that, roughly speaking, for a vertex $v$ on the backbone of $T_{\alpha}^{\infty}$, and corresponding graph $G(v), \mathbf{P}(\operatorname{Diam}(G(v)) \geq x) \asymp x^{-s_{\alpha}^{d}}$ as $x \rightarrow \infty$, and similarly for the other exponents.

Given these exponents, we are now in a position to state the exponents of interest for the decorated tree. We start with the volume growth, which heuristically, can be understood by comparing with a ball in the tree. The volume of a ball of radius $r$ in the tree $T_{\alpha}^{\infty}$ is of order $r^{\frac{\alpha}{\alpha-1}}$ (this follows from [Duq09, Theorem 1.5], for example); informally this is because there are precisely $r$ backbone vertices within distance $r$ of the root, and between them, they have of order $r^{\frac{1}{\alpha-1}}$ offspring, so that there are approximately $r^{\frac{1}{\alpha-1}}$ subtrees grafted to the backbone within distance $r$ of the root. Amongst these subtrees, the largest one will have volume of order $r^{\frac{\alpha}{\alpha-1}}$ and height of order $r$, and due to the very heavy tails of the fragments, the volume of the whole ball will be on the same order as the volume of this maximal fragment. There are also other ways of understanding this exponent (e.g. see the arguments of [CK08] which involve decomposing at heights), but the one described here is most
insightful for understanding the proofs in this chapter.
One can also apply this logic in the decorated metric: to go distance $r$ along the decorated backbone in $\mathcal{T}_{\alpha}^{\text {dec }}$, one must sample approximately $r^{s_{\alpha}^{d} \wedge 1}$ backbone vertices in $T_{\alpha}^{\infty}$, resulting in $r^{\frac{s_{\alpha}^{d} \Lambda 1}{\alpha-1}}$ subtrees. Moreover, if a subtree has progeny $n$, then the volume of its decorated version will be approximately $n^{\frac{1}{f_{\alpha}^{\wedge 1}}}$, so that the volume of the largest subtree will be of order $r^{\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{f} \wedge 1\right)}}$.

This argument neglects to include volume contributions from graphs inserted at backbone vertices. In the case of considering volume growth of $T_{\alpha}^{\infty}$, this contribution is of order $r$ so is clearly insubstantial compared to that of the fragments. In fact, on a critical tree it is the case that asymptotically, almost all the mass is concentrated close to the leaves. However, on the decorated model, since the backbone vertices have (on average) the highest degrees in the tree, it is natural to expect that as the volume exponent $v$ increases, the balance between the volumes of graphs on the backbone and those in the fragments is shifted. In fact the critical point is when $v=\alpha$, and above this value the backbone contribution is of the same order as that of the fragments (the backbone contribution can never dominate the fragment contribution, though).

We therefore define the decorated volume exponent

$$
\begin{equation*}
d_{\alpha}^{\mathrm{dec}}=\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)} . \tag{6.1}
\end{equation*}
$$

We also let $\left.B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\text {dec }}, r\right)\right)$ denote a ball of radius $r$ around the root of $\mathcal{T}^{\text {dec }}$ with respect to the graph metric. The main volume growth theorem is as follows.

Theorem 6.0.5 (Volume growth). (i) Upper bound. Suppose that ( $D^{\prime}$ ), ( $V^{\top}$ ) hold. Then, $\mathbf{P}$-almost surely, for all $\varepsilon>0$ there exists $r_{0}(\varepsilon)<\infty$ such that for all $r \geq r_{0}(\varepsilon)$,

$$
\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}\left(\rho_{\alpha}^{d e c}, r\right)\right) \leq \begin{cases}r^{d_{\alpha}^{d e c}}(\log r)^{\frac{\alpha+\varepsilon}{\left(f_{\alpha}^{\alpha} \wedge 1\right)(\alpha-1)}} & \text { if } s_{\alpha}^{d}<1, \\ r^{d_{\alpha}^{d e c}}(\log r)^{1+\frac{\alpha+\varepsilon}{\left(f_{\alpha}^{d} \alpha 1\right)(\alpha-1)}} & \text { if } s_{\alpha}^{d}=1, \\ r^{d_{\alpha}^{d e c}}(\log r)^{\frac{\alpha+\varepsilon}{\left(f_{\alpha}^{\alpha} \wedge 1\right) y_{\alpha}}} & \text { if } s_{\alpha}^{d}>1 .\end{cases}
$$

(ii) Lower bound. Suppose that ( $D$ ), ( $V$ ) hold. Then, $\mathbf{P}$-almost surely, for all $\varepsilon>0$ there exists $r_{0}(\varepsilon)<\infty$ such that for all $r \geq r_{0}(\varepsilon)$,

$$
\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}\left(\rho_{\alpha}^{d e c}, r\right)\right) \geq r^{d_{\alpha}^{d e c}}(\log r)^{\frac{-(\alpha+\varepsilon)}{\alpha-1}-d_{\alpha}^{d e c}\left(1+\frac{1}{k}\right)}
$$

(iii) Improved lower bound. Suppose that $m_{v} \wedge 1 \geq \frac{\alpha}{v}$ and ( $\left.D^{\prime \prime}\right)$, ( $V^{\prime \prime}$ ) hold. Then,
$\mathbf{P}$-almost surely, for all $\varepsilon>0$ there exists $r_{0}(\varepsilon)<\infty$ such that for all $r \geq r_{0}(\varepsilon)$,

$$
\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}\left(\rho_{\alpha}^{d e c}, r\right)\right) \geq r^{d_{\alpha}^{d e c}}(\log r)^{\frac{-1}{s_{\alpha}^{v}}}
$$

These logarithmic fluctuations are not optimal in many cases, and in particular can often be improved to log-logarithmic when inserting deterministic graphs (although not always, $T_{\alpha}^{\infty}$ being an example where the upper fluctuations are genuinely logarithmic). However, although we make some comments on how the arguments can potentially be fine-tuned at appropriate parts of the proof, our emphasis here is on determining the correct leading term exponent for the volume growth, rather than the optimal fluctuations.

We have graphed the volume exponent in Figure 6 below. There are two cases for the graph, depending on which of $s_{\alpha}^{d}$ and $f_{\alpha}^{v}$ exceeds one "first". In both cases, there are up to three regimes. The case where both of the exponents exceed 1 can be thought of as the "tree regime": in this case the relevant tails on the inserted graphs are not heavy enough to impact the exponents, so we see the same exponent appearing as for an undecorated tree. The case where both of the exponents are less than 1 can be thought of as the "graph regime", and we lose the dependence on $\alpha$. This reflects the fact that as the offspring tails get heavier (i.e. as $\alpha \downarrow 1$ ), it is easier for a finite critical Galton-Watson tree to be "large" by having one vertex of macroscopic degree (cf [CK14, Proposition 3.6]), so that we essentially just see one macroscopic copy of the inserted graph in the decorated tree. In the case of a decorated Kesten's tree, we essentially just see a one-dimensional sequence of graphs glued along the backbone of $T_{\alpha}^{\infty}$. As $\alpha \uparrow 2$, however, the vertex degrees become more balanced, and the contribution from any one single vertex is less significant, so we can regain some tree structure and eventually recover it entirely once the distance and volumes across typical inserted graphs have finite expectation.

Depending on which of $s_{\alpha}^{d}$ and $f_{\alpha}^{v}$ tip over 1 first, we also see an intermediate regime where we "see" the effect of either volumes or distances in the inserted graphs, but not both.

We can use similar considerations to those discussed above to either establish the volume growth exponents with respect to the resistance metric, or otherwise add up resistance contributions along the backbone and along paths in subtrees to compare resistance to the graph distance. Again, there are two regimes depending on whether resistance across a typical spinal vertex has finite expectation or not: as result, we will also see a factor of $s_{\alpha}^{R} \wedge 1$ in the exponents below. We can then combine the resistance and volume estimates using results of [KM08] to identify the random walk exponents.


Figure 6.2: Different phases of the decorated volume exponent. NB, we do not necessarily see all three phases for one given model.

Accordingly, set

$$
\begin{aligned}
d_{\alpha}^{\mathrm{spec}} & =\frac{2 \alpha\left(s_{\alpha}^{R} \wedge 1\right)}{\alpha\left(s_{\alpha}^{R} \wedge 1\right)+(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)} \\
d_{\alpha}^{\mathrm{dis}} & =\frac{\left(s_{\alpha}^{R} \wedge 1\right)(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)+\alpha\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)}
\end{aligned}
$$

In what follows, we let assume that the decorated tree $\mathcal{T}_{\alpha}^{\text {dec }}$ is defined on the probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbf{P})$, and let $\mathbb{P}(\cdot)$ denote the law of a simple random walk on $\mathcal{T}_{\alpha}^{\text {dec }}$, started from the root. This is also a random variable on $(\boldsymbol{\Omega}, \mathcal{F}, \mathbf{P})$. In terms of the random walk results, we have the following.

Theorem 6.0.6 (Quenched random walk results). Under Assumption 6.0.1:
$\mathbb{P} \times \mathbf{P}$-almost surely,

$$
\begin{aligned}
d_{d i s}\left(\mathcal{T}_{\alpha}^{d e c}\right) & :=\lim _{n \rightarrow \infty} \frac{\log \sup _{k \leq n} d^{d e c}\left(\rho_{\alpha}^{d e c}, X_{k}\right)}{\log n}=d_{\alpha}^{d i s} \\
d_{s}\left(\mathcal{T}_{\alpha}^{d e c}\right) & :=-2 \lim _{n \rightarrow \infty} \frac{\log p_{2 n}\left(\rho_{\alpha}^{d e c}, \rho_{\alpha}^{d e c}\right)}{\log n}=d_{\alpha}^{s p e c}
\end{aligned}
$$

The annealed results follow similarly from [KM08, Proposition 1.4].
Theorem 6.0.7 (Annealed random walk results). Under Assumption 6.0.1,

$$
\begin{aligned}
d_{d i s}^{E}\left(\mathcal{T}_{\alpha}^{d e c}\right) & :=\lim _{n \rightarrow \infty} \frac{\log \mathbf{E}\left[\mathbb{E}\left[\sup _{k \leq n} d^{d e c}\left(\rho_{\alpha}^{d e c}, X_{k}\right)\right]\right]}{\log n}=d_{\alpha}^{d i s} \\
d_{s}^{E}\left(\mathcal{T}_{\alpha}^{d e c}\right) & :=-2 \lim _{n \rightarrow \infty} \frac{\log \mathbf{E}\left[p_{2 n}\left(\rho_{\alpha}^{d e c}, \rho_{\alpha}^{d e c}\right)\right]}{\log n} \geq d_{\alpha}^{\text {spec }}
\end{aligned}
$$

In general it is not possible to get an upper bound on the annealed spectral dimension, since this quantity is infinite on the underlying tree $T_{\alpha}^{\infty}$. This is because the expected volume of a unit ball is infinite in this case, as established by Croydon
and Kumagai [CK08]. This does not mean that it is always infinite in the decorated case (e.g. see [BS15, Theorem 1.2] for the corresponding result for discrete looptrees), but one must insert graphs that sufficiently "spread out" different branches of the tree. In particular, we will see in Section 5.5 that we can obtain an upper bound whenever the probabilistic tail decay for obtaining unusually large volumes is sufficiently strong.

We conclude the introduction by commenting briefly on the finite variance case of $\alpha=2$. It was shown in [KR20, Theorem 2] that the metric space scaling limit of any finite variance discrete looptree is the Brownian CRT, meaning that the loops do not persist in the scaling limit, building on [CHK15, Theorem 13] which applies when the offspring distribution has exponential tails. Assuming that $R, d \geq 1$ this would therefore also be the case for our decorated tree model (since then distances are stochastically no bigger than those in looptrees). If the offspring distribution has stretched exponential tail decay it will similarly be the case that the volumes of the inserted graphs will not have a tangible effect. However, if there is polynomial tail decay in the offspring distribution it is always possible to choose the volume exponent $v$ large enough that larger volumes persist in the scaling limit; in the same spirit, if $\alpha=2$ and we were to repeat the arguments of this chapter we expect that we would obtain a volume growth exponent

$$
d_{2}^{\mathrm{dec}}=\frac{2\left(s_{2}^{d} \wedge 1\right)}{\left(f_{2}^{v} \wedge 1\right)},
$$

with $s_{\alpha}^{d}$ and $f_{\alpha}^{v}$ defined as above. We have not pursued this line of enquiry in this chapter, but instead note that this kind of model would fall into the framework we briefly discuss later in Section 7.1.

For all the results in this chapter, we will assume that all the conditions of Assumption 6.0.1 hold, unless explicitly stated otherwise. The main exception to this is Section 6.4.1.

### 6.1 Definition of the model

Formally, we let $T_{\alpha}^{\infty}$ denote Kesten's tree with critical offspring distribution $\xi$ satisfying

$$
\begin{equation*}
\xi(k) \sim c k^{-\alpha} \tag{6.2}
\end{equation*}
$$

as $k \rightarrow \infty$, for some $\alpha \in(1,2]$, and let $d_{\alpha}^{\infty}$ denote the graph distance on $T_{\alpha}^{\infty}$. The results also hold on incorporating a slowly-varying function, but for sake of clarity we have not included this here, and just note that the slowly-varying function can
be carried through all the computations.
In line with the literature, we also let $p_{k}=\xi(k)$. By standard theory, (e.g. [BGT89, Chapter VIII]), if $\left(X_{i}\right)_{i=1}^{\infty}$ are i.i.d. distributed according to $\xi$, it follows that $a_{n}^{-1} \sum_{i=1}^{\infty} X_{i} \rightarrow S_{\alpha}$, where $S_{\alpha}$ is an $\alpha$-stable random variable and $a_{n}=(c|\Gamma(-\alpha)| n)^{\frac{1}{\alpha}}$.

To construct the decorated model, we will suppose that $G$ is a random graph with some pre-specified distribution, connected, and that $G_{n}$ denotes a copy of $G$ conditioned on having $n$ "boundary" vertices (for example, a tree with $n$ leaves, or a dissection of the $n$-gon). Informally, the decorated tree $\mathcal{T}_{\alpha}^{\text {dec }}$ is obtained inserting an independent copy of $G$ at every vertex, with boundary length equal to the number of edges incident to that vertex, and then gluing the inserted graphs along the tree structure of $T_{\alpha}^{\infty}$. See Figure 6.1.

More formally, we first sample $T_{\alpha}^{\infty}$ according to Definition 3.3.1 of Kesten's tree, and then independently sample a countable sequence $((G) v), b(v))_{v \in T_{\alpha}^{\infty}}$, where $G(v)$ is an independent copy of $G_{\operatorname{deg} v}$, and $b(v)$ is a uniform bijection from the boundary vertices of $G(v)$, to the edges of $T_{\alpha}^{\infty}$ that are incident to the vertex $v$. Given a vertex $x \in \bigcup_{v \in T_{\alpha}^{\infty}} G(v)$, if $x \in G(v)$ we say that $v=V_{T}(x)$. We then define an equivalence relation $\stackrel{e}{\sim}$ on the vertices of $\bigcup_{v \in T_{\alpha}^{\infty}} G(v)$ by saying that $x \stackrel{e}{\sim} y$ if and only if $b\left(V_{T}(x)\right)(x)=b\left(V_{T}(y)\right)(y)$, in other words that they are both in bijection with the same edge of $T_{\alpha}^{\infty}$.

We then set

$$
\mathcal{T}_{\alpha}^{\mathrm{dec}}=\bigcup_{v \in T_{\alpha}^{\infty}} G(v) / \stackrel{e}{\sim}
$$

If $\rho$ is the root of $T_{\alpha}^{\infty}$, we also define the root of $\mathcal{T}_{\alpha}^{\text {dec }}$ to be the vertex $x \in G(\rho)$ such that $b(\rho)(x)$ is equal to the edge joining $\rho$ to its first child, and denote this vertex by $\rho_{\alpha}^{\text {dec }}$.

We let $d_{g}^{\text {dec }}$ denote the graph distance on $\mathcal{T}_{\alpha}^{\text {dec }}$. If $\left[\left[V_{T}(x), V_{T}(y)\right]\right]=v_{0}, v_{1}, \ldots, v_{n}$ denotes the path of (internal) vertices between $V_{T}(x)$ and $V_{T}(y)$ in $T_{\alpha}^{\infty}$, this can be constructed by setting

$$
\begin{align*}
d_{g}^{\mathrm{dec}}(x, y)= & d_{G\left(v_{0}\right)}\left(x, b\left(v_{0}\right)^{-1}\left(v_{0} v_{1}\right)\right)+\sum_{1 \leq i \leq n-1} d_{G\left(v_{i}\right)}\left(b\left(v_{i}\right)^{-1}\left(v_{i-1} v_{i}\right), b\left(v_{i}\right)^{-1}\left(v_{i} v_{i+1}\right)\right) \\
& +d_{G\left(v_{n}\right)}\left(b\left(v_{n}\right)^{-1}\left(v_{n-1} v_{n}\right), y\right) \tag{6.3}
\end{align*}
$$

where $d_{G\left(v_{i}\right)}$ represents the graph distance on $G\left(v_{i}\right)$.
If $v \in T_{\alpha}^{\infty}$, let $T_{v}$ be the subtree of $T_{\alpha}^{\infty}$ rooted at $v$. This corresponds to a subgraph of $\mathcal{T}_{\alpha}^{\text {dec }}$ consisting of the graph $\bigcup_{u \in T_{v}} G(u)$. We denote this graph by $\mathcal{T}_{\alpha}^{\operatorname{dec}}(v)$.

We also endow the space $\mathcal{T}_{\alpha}^{\text {dec }}$ with a measure $\mu$ such that $\mu(x)=\operatorname{deg} x$ for
all $x$ in $\mathcal{T}_{\alpha}^{\text {dec }}$. The volume assumptions on the number of edges really correspond to assumptions on the measure $\mu$.

We will assume that $T_{\alpha}^{\infty}$ and the set $(G(v), b(v))_{v \in T_{\alpha}^{\infty}}$ are all defined on the probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbf{P})$. We denote the law of a random walk on $\mathcal{T}_{\alpha}^{\text {dec }}$ by $\mathbb{P}$ : this law is therefore also a random variable on the probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbf{P})$.

Notation. Throughout, $c$ and $C$ will denote constants bounded from above and below, but values may change on each appearance.

### 6.2 Technical lemmas

In this section we collect some technical lemmas regarding sums of stable random variables. These will be used regularly in the proofs.

Remark 6.2.1. Since we are not including slowly-varying functions in the tails of our random variables, this means that any "1-stable" random variable will fall into an infinite mean regime for the purposes of this chapter. In most of our proofs we have to deal with the infinite mean and finite mean regimes separately; however, since all of our phase transitions will ultimately be continuous, we don't expect that adding a slowly-varying function will change the result when any of the tail decay exponents are equal to -1 , but may just mean that the finite mean proof method is required instead.

Lemma 6.2.2. Let $\left(X_{i}\right)_{i=1}^{n}$ be i.i.d. such that $\mathbb{P}\left(X_{1}>x\right) \sim c x^{-\beta}$, for some $\beta \in$ $(0,1]$, and let $T^{(k)}=\inf \left\{i \geq 1: X_{i}>k\right\}$ (or equal to infinity if this set is empty). Set

$$
S^{(k)}=\sum_{i=1}^{T^{(k)}-1} X_{i}
$$

(i) If $\beta \in(0,1)$, then there exist $c, C \in(0, \infty)$ such that $\mathbb{P}\left(S^{(k)} \geq \lambda k\right) \leq C e^{-c \lambda}$.
(ii) If $\beta=1$, then there exist $c, C \in(0, \infty)$ such that $\mathbb{P}\left(S^{(k)} \geq \lambda k \log k\right) \leq C e^{-c \lambda}$.

Proof. (i) The proof is essentially the same as the argument for a similar result on [CK14, p. 25]. Let $S_{n}=\sum_{i=1}^{n} X_{i}, H_{0}=0$, and $H_{k}=\inf \left\{n \geq 0: S_{n}>k\right\}$ be the hitting time of $k$ for the random walk $\left(S_{n}\right)_{n \geq 1}$. Then, if $S^{(k)} \geq k \lambda$ it must be the case that $H_{k \lambda} \leq T^{(k)}$. Therefore, we can write

$$
\begin{aligned}
\mathbb{P}\left(S^{(k)} \geq k \lambda\right) & =\mathbb{P}\left(H_{k \lambda} \leq T^{(k)}\right) \\
& \leq \mathbb{P}\left(H_{2 k} \leq T^{(k)}, H_{4 k} \leq T^{(k)}, H_{6 k} \leq T^{(k)}, \ldots, H_{2\left\lfloor\frac{1}{2} \lambda\right\rfloor k} \leq T^{(k)}\right) \\
& \leq \prod_{i=1}^{2\left\lfloor\frac{1}{2} \lambda\right\rfloor} \mathbb{P}\left(H_{2 i k} \leq T^{(k)} \mid H_{2(i-1) k} \leq T^{(k)}\right)
\end{aligned}
$$

Since there are no jumps exceeding $k$ before time $T^{(k)}$, it follows that $S_{H_{2 i k}} \leq$ $(2 i+1) k$ for all $i \geq 1$. Moreover, since $T^{(k)}$ has a geometric distribution, it therefore follows from the memoryless property that for all $i \geq 1$ :

$$
\mathbb{P}\left(H_{2 i k} \leq T^{(k)} \mid H_{2(i-1) k} \leq T^{(k)}\right) \leq \mathbb{P}\left(H_{k} \leq T^{(k)}\right) .
$$

Therefore, the exponential decay will follow once we can show that $\mathbb{P}\left(H_{k} \leq T^{(k)}\right)$ can be bounded below 1 uniformly in $k$. To show this, we use Wald's equality to write:

$$
\mathbb{P}\left(H_{k} \leq T^{(k)}\right)=\mathbb{P}\left(S^{(k)} \geq k\right) \leq \mathbb{E}\left[S^{(k)}\right] k^{-1} \leq c \mathbb{E}\left[X_{1} \mid X_{1}<k\right] k^{\beta} k^{-1} \rightarrow 0
$$

as $k \rightarrow \infty$.
(ii) If $\beta=1$, we can use the same proof but we consider $\mathbb{P}\left(S^{(k)} \geq C \lambda k \log k\right)$ for a sufficiently large constant $C$, and use that $\mathbb{E}\left[X_{1} \mid X_{1}<k\right] \leq c \log k$ in the final line.

We will also need the following lemmas for the infinite mean case. The result should be standard; however we couldn't find a specific proof in the literature, so have provided one for completeness. It will be used at multiple times throughout this chapter.

Lemma 6.2.3. Let $\left(X_{i}\right)_{i \geq 1}$ be i.i.d. and such that $\mathbb{P}\left(X_{i} \geq x\right) \sim c x^{-\beta}$ for some $\beta \leq 1$. Let $S_{n}=\sum_{i=1}^{n} X_{i}$.
(i) If $\beta<1$, then there exists a constant $c^{\prime}<\infty$ such that for each $n \geq 2$,

$$
\mathbb{P}\left(S_{n} \geq n^{\frac{1}{\beta}} \lambda\right) \leq c^{\prime} \lambda^{-\beta}
$$

as $n \rightarrow \infty$.
(ii) If $\beta=1$, then there exists a constant $c^{\prime}<\infty$ such that for each $n \geq 2$,

$$
\mathbb{P}\left(S_{n} \geq \lambda n^{\frac{1}{\beta}} \log n\right) \leq c^{\prime} \lambda^{-\beta}
$$

as $n \rightarrow \infty$.
Proof. We give the proof in case ( $i$ ); the proof is the same in case (ii) and we just import an extra $\log$ term from the application of Lemma 6.2.2. The proof is no doubt standard and this particular formulation follows a similar strategy to the
analysis on [Dur10, Chapter 3, p.160]. Decompose $S_{n}$ as the sum $\hat{S}_{n}+\tilde{S}_{n}$, where

$$
\hat{S}_{n}=\sum_{i=1}^{n} X_{i} \mathbb{\mathbb { 1 }}\left\{X_{i} \geq n^{\frac{1}{\beta}}\right\}, \quad \tilde{S}_{n}=\sum_{i=1}^{n} X_{i} \mathbb{\mathbb { 1 }}\left\{X_{i}<n^{\frac{1}{\beta}}\right\} .
$$

We condition on the number of terms of $\hat{S}_{n}$, which is stochastically dominated by Binomial $\left(n, 2 c n^{-1}\right)$ for large enough $n$.

Given that $N_{n}:=\left|\left\{i \leq n: X_{i} \geq n^{\frac{1}{\beta}}\right\}\right|=m, \tilde{S}_{n}$ can be dealt with using Lemma 6.2.2: since we have $m+1$ copies of the sum considered there, it is necessary for one such copy to be at least $\frac{\lambda n^{\frac{1}{\beta}}}{2(m+1)}$ in order that $\tilde{S}_{n} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2}$. Similarly, to control $\hat{S}_{n}$, note that it is necessary that at least one term of $\hat{S}_{n}$ is at least $\frac{\lambda n^{\frac{1}{\beta}}}{2 m}$ in order that $\hat{S}_{n} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2}$. Moreover, $\mathbb{P}\left(\left.X_{1} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2 m} \right\rvert\, X_{1} \geq n^{\frac{1}{\beta}}\right) \leq 4^{\beta} m^{\beta} \lambda^{-\beta}$ for all sufficiently large $\lambda$. We can therefore write:

$$
\begin{aligned}
\mathbb{P}\left(S_{n} \geq \lambda n^{\frac{1}{\beta}}\right) \leq & \mathbb{P}\left(\hat{S}_{n} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2}\right)+\mathbb{P}\left(\tilde{S}_{n} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2}\right) \\
\leq & \sum_{m=0}^{n}\binom{n}{m}\left(2 c n^{-1}\right)^{m}\left(1-2 c n^{-1}\right)^{n-m} \\
& \times\left[\mathbb{P}\left(\left.\hat{S}_{n} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2} \right\rvert\, N_{n}=m\right)+\mathbb{P}\left(\left.\tilde{S}_{n} \geq \frac{\lambda n^{\frac{1}{\beta}}}{2} \right\rvert\, N_{n}=m\right)\right] \\
\leq & \sum_{m=0}^{n}\binom{n}{m}\left(2 c n^{-1}\right)^{m}\left(1-2 c n^{-1}\right)^{n-m}\left[m^{\beta+1} 4^{\beta} \lambda^{-\beta}+C m e^{\left.-\frac{\lambda}{2(m+1)}\right]}\right. \\
\leq & \sum_{m=0}^{n} \frac{1}{m!} e^{-2 c}\left(\frac{2 c}{1-2 c n^{-1}}\right)^{m}\left[C m^{\beta+1} \lambda^{-\beta}\right] \\
\leq & \lambda^{-\beta} \sum_{m=0}^{\infty} \frac{C m^{\beta+1}}{m!}\left(\frac{2 c}{1-2 c n^{-1}}\right)^{m} \\
\leq & C \lambda^{-\beta} .
\end{aligned}
$$

We also have a bound for the lower tails.
Lemma 6.2.4. Let $\left(X_{i}\right)_{i \geq 1}$ be i.i.d. and such that $\mathbb{P}\left(X_{i} \geq x\right) \sim c x^{-\beta}$ for some $\beta \leq 1$. Let $S_{n}=\sum_{i=1}^{n} X_{i}$. Then there exists a constant $c>0$ such that for each $n \geq 2$,

$$
\mathbb{P}\left(S_{n} \leq n^{\frac{1}{\beta}} \lambda^{-1}\right) \leq e^{-c \lambda^{\beta}}
$$

as $n \rightarrow \infty$.

Proof.

$$
\mathbb{P}\left(S_{n} \leq n^{\frac{1}{\beta}} \lambda^{-1}\right) \leq c^{\prime} \lambda^{-\beta} \leq \mathbb{P}\left(\nexists i \leq n: X_{i} \geq n^{\frac{1}{\beta}} \lambda^{-1}\right) \leq\left(1-n^{-1} \lambda^{\beta}\right)^{n} \leq e^{-\frac{1}{2} \lambda^{\beta}}
$$

for all sufficiently large $n$.
We also have the following tail bounds for the finite mean case.
Lemma 6.2.5. Let $\left(X^{(i)}\right)_{i=1}^{\infty}$ be i.i.d. non-negative such that $\mathbb{P}\left(X^{(1)}>x\right) \sim c x^{-\beta}$ as $x \rightarrow \infty$ for some $\beta>1$. Then, for all $\lambda>\mathbb{E}\left[X^{(1)}\right]+1$,

$$
\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} X^{(i)} \geq \lambda n\right) & =o\left(n^{-(\beta-1)}\right) \\
\mathbb{P}\left(\sum_{i=1}^{n} X^{(i)} \leq \lambda^{-1} n\right) & =o\left(n^{-(\beta-1)}\right)
\end{aligned}
$$

If the random variables are not independent, we still get that for any $\varepsilon>0$ there exists $c<\infty$ such that

$$
\mathbb{P}\left(\sum_{i=1}^{n} X^{(i)} \geq \lambda n\right)=c \lambda^{-(\beta-\varepsilon)} .
$$

Proof. In the independent case, this follows from applying [Pet75, Theorem 28] to the sum of recentred random variables.

If the random variables are not independent, we can apply Hölder's inequality with $p=\beta-\varepsilon$ to get that
$\mathbb{P}\left(\sum_{i=1}^{n} X^{(i)} \geq \lambda n\right) \leq \mathbb{E}\left[\left(\sum_{i=1}^{n} X^{(i)}\right)^{p}\right] n^{-p} \lambda^{-p} \leq n^{p} \mathbb{E}\left[\left(X^{(1)}\right)^{p}\right] n^{-p} \lambda^{-p} \leq c \lambda^{-(\beta-\varepsilon)}$.

Lemma 6.2.6. Let $X$ be a random variable, and suppose that $\mathbb{P}(X \geq x) \sim c x^{-\beta}$ as $x \rightarrow \infty$ for some $\beta<1$. Then there exists a constant $c \in(0, \infty)$ such that $1-\mathbb{E}\left[e^{-\theta X}\right] \sim c \theta^{\beta}$ as $\theta \rightarrow 0$.

Proof. This follows from the Tauberian theorem of [Kor04, Chapter IV, Theorem 8.2].

Lemma 6.2.7. Let $X$ be a random variable, and suppose that $\mathbb{P}(X \geq x) \sim c x^{-\beta}$ as $x \rightarrow 0$ for some $\beta>0$. Then there exists a constant $c \in(0, \infty)$ such that $\mathbb{E}\left[e^{-\theta X}\right] \sim c \theta^{-\beta}$ as $\theta \rightarrow \infty$.

Proof. This is a standard Tauberian theorem, for example see [Ber96, p.10].

Lemma 6.2.8. Let $\beta>0$, and let $X$ be a random variable with $\mathbb{P}(X>x) \sim c x^{-\beta}$ as $x \rightarrow \infty$, and let $f$ be a function of $X$ such that there exists $z>0$ and a function $p(\lambda)$ such that $\mathbb{P}\left(f(X) \geq \lambda n^{z} \mid X=n\right) \sim p(\lambda)$ for all sufficiently large $n$, and $p(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.
(i) If $z \leq 1$ and $f(x) \leq x$ for all $x$, then

$$
\mathbb{P}(f(X) \geq k) \lesssim \begin{cases}k^{-\frac{1}{z} \beta} & \text { if } p(\lambda)=o\left(\lambda^{\frac{-\beta}{z}}\right) \\ k^{-\frac{1}{z} \beta} \log k & \text { if } p(\lambda) \asymp \lambda^{\frac{-\beta}{z}}, \\ k^{-(\beta+m(1-z))} & \text { if } p(\lambda) \asymp \lambda^{-m}, m<\frac{\beta}{z} \\ k^{-\beta} & \text { if } p(\lambda) \text { decays sub-polynomially. }\end{cases}
$$

Additionally,

$$
\mathbb{P}(f(X) \geq k) \gtrsim \begin{cases}k^{-\frac{1}{z} \beta} & \text { if } p(\lambda)=o\left(\lambda^{\frac{-\beta}{z}}\right) \\ k^{-(\beta+m(1-z))} & \text { if } p(\lambda) \asymp \lambda^{-m}, m<\frac{\beta}{z} \\ k^{-\beta} p\left(k^{1-z}\right) & \text { if } p(\lambda) \text { decays sub-polynomially } .\end{cases}
$$

(ii) If there exists $j>0$ such that $\mathbb{P}(f(X) \geq k \mid X=j)>0$ for arbitrarily large $k>0$, then

$$
\mathbb{P}(f(X) \geq k) \lesssim \begin{cases}k^{-\frac{1}{z} \beta} & \text { if } p(\lambda)=o\left(\lambda^{\frac{-\beta}{z}}\right) \\ k^{-\frac{1}{z} \beta} \log k & \text { if } p(\lambda) \asymp \lambda^{\frac{-\beta}{z}} \\ k^{-m} & \text { if } p(\lambda) \asymp \lambda^{-m}, m<\frac{\beta}{z}\end{cases}
$$

In addition,

$$
\mathbb{P}(f(X) \geq k) \gtrsim \begin{cases}k^{-\frac{1}{z} \beta} & \text { if } p(\lambda)=o\left(\lambda^{\frac{-\beta}{z}}\right) \\ k^{-m} & \text { if } p(\lambda) \asymp \lambda^{-m}, m<\frac{\beta}{z}\end{cases}
$$

Proof. (i) This is just a computation. We first assume that $p(\lambda) \asymp \lambda^{-m}$. We can then write

$$
\begin{aligned}
\mathbb{P}(f(X) \geq k) & \leq \mathbb{P}\left(X \geq k^{\frac{1}{z}}\right)+\sum_{x=k}^{k^{\frac{1}{z}}} \mathbb{P}(X=x) \mathbb{P}\left(f(X) \geq k x^{-z} X^{z}\right) \\
& \leq c k^{-\frac{\beta}{z}}+\sum_{x=k}^{k^{\frac{1}{z}}} c x^{-(\beta+1)} k^{-m} x^{m z} \\
& \leq k^{-\frac{\beta}{z}}+k^{-m} \int_{k}^{k^{\frac{1}{z}}} x^{-(\beta+1)+m z} d x
\end{aligned}
$$

Computing the integral gives the stated upper bound. The proof is the same in the non-polynomial cases.

The lower bound is simpler - if $m \geq \frac{\beta}{z}$, we write:

$$
\mathbb{P}(f(X) \geq k) \geq \mathbb{P}\left(X \geq k^{\frac{1}{z}}\right) \mathbb{P}\left(f(X) \geq k \left\lvert\, X \geq k^{\frac{1}{z}}\right.\right) \geq c k^{\frac{-\beta}{z}} p(1) .
$$

If instead $m<\frac{\beta}{z}$ :

$$
\begin{aligned}
\mathbb{P}(f(X) \geq k) \geq \mathbb{P}(X \geq k) \mathbb{P}(f(X) \geq k \mid X \geq k) & \geq k^{-\beta} k^{-(1-z) m} \\
& =k^{-\beta-(1-z) m} .
\end{aligned}
$$

Again, the proof is the same as in the non-polynomial cases.
(ii) The proof is the same as above, except that the sums and integrals start at 1 , rather than $k$.

Remark 6.2.9. 1. In most cases we will consider we will in fact have (stretched) exponential tail decay for $p(\lambda)$.
2. The assumption of part ( $i$ ) of this theorem would be relevant in the case where $X$ is the boundary length of an inserted graph, and $f(X)$ denotes the two-point function for two points on the boundary. If $G$ is a planar graph and there is always an option to travel between two vertices directly round the boundary, then we would have a constraint of the form $f(X) \leq X$ and the distance cannot be arbitrarily large. For more general graphs, or in the case where $f(X)$ is instead the volume of $G$, then we are instead in the setting of part (ii).

### 6.3 Decorated bounds for finite Galton-Watson trees

In this section we prove some results on volume and distance on finite decorated Galton-Watson trees. Since $T_{\alpha}^{\infty}$ is constructed to have one infinite backbone to which many finite fringe Galton-Watson trees are grafted, these estimates will be crucial when we prove the volume and resistance bounds for $\mathcal{T}_{\alpha}^{\text {dec }}$ in Sections 6.4 and 6.5.

We will assume throughout that $\lambda$ is implicitly a function of $n$ (or of $x$ in Proposition 6.3.16), and that for all $\varepsilon>0, \lambda=\lambda_{n}=o\left(n^{\varepsilon}\right)\left(\right.$ or $\lambda=\lambda_{x}=o\left(x^{\varepsilon}\right)$ in Proposition 6.3.16).

### 6.3.1 Bounds for undecorated Galton-Watson trees

In this section, we assume that $T$ is a critical Galton-Watson tree with offspring distribution $\xi$, with $\xi$ satisfying (6.2). Note that $T$ is almost surely finite. The strategy will be to first give the relevant exponents for undecorated trees (these are already known), and then use Lemma 6.2.8 to compute the corresponding exponents for decorated trees.

## Progeny bounds

Theorem 6.3.1. There exists a constant $c \in(0, \infty)$ such that

$$
\mathbf{P}(|T| \geq k) \sim c k^{\frac{-1}{\alpha}}
$$

as $k \rightarrow \infty$.
Proof. This follows directly from [CK15, Proposition A.3(i)]. Alternatively, we can prove it directly: by the main theorem of [Dwa69],

$$
\mathbf{P}(|T| \geq k)=\sum_{n=k}^{\infty} \frac{1}{n} \mathbf{P}\left(\sum_{i=1}^{n} \xi^{(i)}=n-1\right)
$$

Since $\xi$ is aperiodic, the conditions of the local limit theorem on p. 236 of [GK54] are satisfied, and we therefore deduce that

$$
a_{n} \mathbf{P}\left(\sum_{i=1}^{n} \xi^{(i)}=k\right)-p_{\alpha}\left(\frac{k-n}{a_{n}}\right) \rightarrow 0
$$

uniformly in $k$, where $p_{\alpha}$ is the density of $Z_{\alpha}$. Since $p_{\alpha}$ is continuous, taking $k=n-1$ gives that for any $\varepsilon>0$,

$$
\frac{p_{\alpha}(0)-\varepsilon}{a_{n}} \leq \mathbf{P}\left(\sum_{i=1}^{n} \xi^{(i)}=n-1\right) \leq \frac{p_{\alpha}(0)+\varepsilon}{a_{n}}
$$

for all $n \geq N_{\varepsilon}$. Therefore, for any $k \geq N_{\varepsilon}$ we deduce that:

$$
\sum_{n=k}^{\infty} \frac{1}{n} \frac{p_{\alpha}(0)-\varepsilon}{a_{n}} \leq \mathbf{P}(|T| \geq k) \leq \sum_{n=k}^{\infty} \frac{1}{n} \frac{p_{\alpha}(0)+\varepsilon}{a_{n}}
$$

In particular, since necessarily $a_{n}=c_{\alpha} n^{\frac{1}{\alpha}}$ (see [BGT89, Section 8.3.2]), we deduce that

$$
\left(p_{\alpha}(0)-\varepsilon\right) \sum_{n=k}^{\infty} \frac{1}{n^{1+\frac{1}{\alpha}}} \leq \mathbf{P}(|T| \geq k) \leq\left(p_{\alpha}(0)+\varepsilon\right) \sum_{n=k}^{\infty} \frac{1}{n^{1+\frac{1}{\alpha}}}
$$

The result then follows since

$$
\sum_{n=k}^{\infty} \frac{1}{n^{1+\frac{1}{\alpha}}} \sim c k^{\frac{-1}{\alpha}}
$$

as $k \rightarrow \infty$.
Lemma 6.3.2. There exists a constant $q>0$ such that $1-\mathbf{E}\left[e^{-\theta|T|}\right] \sim q \theta^{\frac{1}{\alpha}}$ as $\theta \rightarrow 0$.

Proof. This is a direct consequence of Lemma 6.2.6, with $\beta=\frac{1}{\alpha}$.

## Height bounds and associated spinal decomposition

At various points, we will also need to perform spinal decompositions along various choices of spine in $T$. For this, the following results will be useful.

Lemma 6.3.3. As $n \rightarrow \infty$,

$$
\mathbf{P}(\operatorname{Height}(T) \geq n) \sim c n^{-\frac{1}{\alpha-1}} .
$$

Proof. This follows directly from [Sla68, Theorem 2], which gives the probability for an offspring distribution with $(1+\alpha)$-stable tails. In particular, rearranging equation (1.2) there and replacing $\alpha$ with $\alpha-1$ gives the result.

We also give the following (elementary) result. We include the proof for completeness.

Lemma 6.3.4. The function $h: \mathbb{N} \rightarrow[0,1], x \mapsto \mathbb{P}(H=x)$ is non-increasing in $x$.
Proof. Let $\xi^{(0)}$ denote the number of offspring of $\rho$, and conditional on $\left\{\xi^{(0)}=n\right\}$, let $T^{(i)}$ denote the subtree emanating from the $i^{t h}$ offspring of $\rho$. We then write:

$$
\begin{aligned}
\mathbf{P}(H=x+1) & =\sum_{n=0}^{\infty} \mathbf{P}\left(\xi^{(0)}=n\right) \mathbf{P}\left(H=x+1 \mid \xi^{(0)}=n\right) \\
& \leq \sum_{n=0}^{\infty} p_{n} \sum_{i=1}^{n} \mathbb{P}\left(\operatorname{Height}\left(T^{(i)}\right)=x\right) \mathbf{P}\left(\operatorname{Height}\left(T^{(j)}\right) \leq x \forall j \neq i\right) \\
& \leq \sum_{n=0}^{\infty} n p_{n} \mathbf{P}(H=x) \\
& =\mathbf{P}(H=x),
\end{aligned}
$$

where the final line follows since $\xi$ is critical.
At many points in this paper we will perform a spinal decomposition of a tree along its leftmost spine of maximal height. We denote the vertices of this spine
by $s_{0}, s_{1}, \ldots, s_{H}$ where $s_{0}=\rho$ and $s_{i}$ is the parent of $s_{i+1}$ for all $i<H$. It is well-known [GK99, Remark below Proposition 2.2] that the offspring distribution of spinal vertices is asymptotically size-biased as $H \rightarrow \infty$, but we will need the following precise formulation of this result.

Proposition 6.3.5. Take the notation as above, and let $\xi^{(n)}$ denote the number of offspring of $s_{n}$. Then, for every $C>0$ there exists a constant $c \in(0, \infty)$, uniform in $n$ and $k$, such that for all $n \in \mathbb{N}$ and all $k \leq C n^{\frac{1}{\alpha-1}}$ :

$$
\mathbf{P}\left(\xi^{(n)} \geq k \mid H \geq(1+\varepsilon) n\right) \geq c k^{-(\alpha-1)}
$$

Proof. We first prove a corresponding result for $\mathbf{P}\left(\xi^{(n)}=\tilde{k} \mid H \geq(1+\varepsilon) n\right)$, and then obtain the result by summing over $\tilde{k} \geq k$. In particular, letting

$$
P_{j}^{H}=p_{k} \mathbf{P}(H<m-n)^{j-1} \mathbf{P}(H<m+1-n)^{k-j} \frac{\mathbf{P}(H=m-n)}{\mathbf{P}(H=m)}
$$

it follows from [GK99, Lemma 2.1] and then Lemma 6.3.4 that

$$
\begin{aligned}
& \mathbf{P}\left(\xi^{(n)}=k \mid H \geq(1+\varepsilon) n\right) \\
& =\sum_{m \geq(1+\varepsilon) n} \mathbf{P}(H=m \mid H \geq(1+\varepsilon) n) \sum_{j=1}^{k} P_{j}^{H} \\
& \geq \sum_{m \geq(1+\varepsilon) n} \mathbf{P}(H=m \mid H \geq(1+\varepsilon) n) k p_{k} \mathbf{P}(H<\varepsilon n)^{k-1} \\
& =k p_{k} \mathbf{P}(H<\varepsilon n)^{k-1}
\end{aligned}
$$

Then, by Lemma 6.3.3, we know that $\mathbf{P}(\operatorname{Height}(T) \geq \varepsilon n) \leq 2(\varepsilon n)^{-\frac{1}{\alpha-1}}$, so if $k \leq$ $C n^{\frac{1}{\alpha-1}}$ then

$$
\mathbf{P}(H<\varepsilon n)^{k-1} \geq\left(1-2(\varepsilon n)^{-\frac{1}{\alpha-1}}\right)^{C n^{\frac{1}{\alpha-1}}} \geq e^{-3 C \varepsilon^{-\frac{1}{\alpha-1}}}
$$


To prove the result as stated, we then write

$$
\begin{aligned}
\mathbf{P}\left(\xi^{(n)} \geq k \mid H \geq(1+\varepsilon) n\right) & \geq \sum_{k \leq \tilde{k} \leq 2 k} \mathbf{P}\left(\xi^{(n)}=\tilde{k} \mid H \geq(1+\varepsilon) n\right) \\
& \geq \sum_{k \leq \tilde{k} \leq 2 k} e^{-3 C} k p_{k} \\
& \geq c k^{-(\alpha-1)} L(k) .
\end{aligned}
$$

We also note the following (unconditioned) probabilistic upper bound, which follows even more straightforwardly from [GK99, Lemma 2.1].

Proposition 6.3.6. Take the notation as above, and let $\xi^{(n)}$ denote the number of offspring of $s_{n}$. Then there exists $c<\infty$ such that

$$
\mathbf{P}\left(\xi^{(n)} \geq K\right) \leq c K^{-(\alpha-1)}
$$

Proof. Given $\xi^{(n)}=k$, let $T_{1}, T_{2}, \ldots, T_{k}$ denote the subtrees rooted at each of the offspring of $s_{n}$, listed in lexicographical order. Then

$$
\mathbf{P}\left(\xi^{(n)} \geq K\right) \leq \sum_{k \geq K} p_{k} \sum_{j=1}^{k} \mathbf{P}\left(\operatorname{Height}\left(T_{j}\right)=\sup _{i \leq k} \operatorname{Height}\left(T_{k}\right)\right) \leq \sum_{k \geq K} k p_{k} \leq c K^{-(\alpha-1)} .
$$

## Vertex degrees

We will also need the following result on the degree of a typical vertex.
Lemma 6.3.7. Let $T_{n}$ be a Galton-Watson tree with offspring distribution $\xi$ but conditioned to have $n$ vertices. Let $v_{U}$ be a uniformly chosen vertex of $T_{n}$. Then there exists a constant $c$ such that for all $n \geq 1$,

$$
\mathbf{P}\left(\operatorname{deg}\left(v_{U}\right) \geq k\right) \leq c k^{-\alpha} .
$$

Proof. Recall that the vertex degrees of the vertices of $T_{n}$ correspond to (one less than) each of the jump sizes of the Lukasiewicz path $W^{(n)}$, which is conditioned to first hit -1 at time $n$. Since $v_{U}$ is uniform amongst the vertices of $T_{n}$, its label in the lexicographical ordering is uniform amongst $\{1, \ldots, n\}$. Letting $U$ denote this label, it follows from the (discrete) Vervaat transform (e.g. see [Kor17, Proposition $10]$ ) that the $U^{\text {th }}$ cyclic shift of $W^{(n)}$, i.e. the random walk $\tilde{W}^{(n)}$ given by

$$
\tilde{W}^{(n)}(t)= \begin{cases}W^{(n)}(U+t)-W^{(n)}(U) & \text { if } U+t \leq n \\ W^{(n)}(U+t-n)-W^{(n)}(U) & \text { if } U+t>n\end{cases}
$$

is a random walk bridge from 0 to -1 , by which we mean that $\tilde{W}^{(n)}$ has the law of $W$ but conditioned on $W_{n}=-1$. In particular, $\tilde{W}^{(n)}$ has a density with respect to the unconditioned walk $W$, and moreover $\operatorname{deg} v_{U}$ is equal to $\tilde{W}^{(n)}(1)-1$. We deduce that

$$
\mathbf{P}\left(\operatorname{deg} v_{U} \geq k\right)=\mathbf{P}\left(\tilde{W}^{(n)}(1)>k\right)=\mathbf{E}\left[\mathbb{1}\left\{\tilde{W}^{(n)}(1)>k\right\} \frac{p_{n-1}\left(-\tilde{W}^{(n)}(1)-1\right)}{p_{n}(-1)}\right] .
$$

Moreover, by the local limit theorem on p. 236 of [GK54], we have that

$$
a_{n} \mathbf{P}\left(W_{n}=k\right)-\tilde{p}_{\alpha}\left(\frac{k}{a_{n}}\right) \rightarrow 0
$$

uniformly in $k$, where $\tilde{p}_{\alpha}$ is the density of a centred stable random variable $Y_{\alpha}$ (different to $Z_{\alpha}$ from before, which is almost surely non-negative), and such that $\tilde{p}_{\alpha}(0)=\left|\Gamma\left(\frac{-1}{\alpha}\right)\right|^{-1}>0$. It follows from this that the ratio $\frac{p_{n-1}\left(-\tilde{W}^{(n)}(1)-1\right)}{p_{n}(-1)}$ is bounded uniformly in $k$, which gives the result.

Lemma 6.3.8. For all $k \geq 1$, there exist constants $c, C>0$ such that

$$
c k^{-1} \leq \mathbf{P}\left(\sup _{v \in T}(\operatorname{deg} v) \geq k\right) \leq C k^{-1}
$$

Proof. The proof is essentially the same as above: given $|T|=n$, and applying the Vervaat transform and absolute continuity relation on the first $\left\lfloor\frac{n}{2}\right\rfloor$ vertices as in the previous proof, we get that

$$
\begin{aligned}
& \mathbf{P}\left(\sup _{v \in T_{n}} \operatorname{deg} v \geq n^{\frac{-1}{\alpha}} \lambda\right) \geq 1-\left(1-c n^{-1} \lambda^{-\alpha}\right)^{\frac{n-1}{2}} \geq 1-e^{-c \lambda^{-\alpha}} \geq c \lambda^{-\alpha} \\
& \mathbf{P}\left(\sup _{v \in T_{n}} \operatorname{deg} v \geq n^{\frac{-1}{\alpha}} \lambda\right) \leq \sum_{v \in T_{n}} \mathbf{P}\left(\operatorname{deg} v \geq n^{\frac{-1}{\alpha}}\right)=n \mathbf{P}\left(\operatorname{deg} v_{U} \geq n^{\frac{-1}{\alpha}}\right) \leq c \lambda^{-\alpha} .
\end{aligned}
$$

Then, since $\mathbf{P}(|T| \geq n) \sim c n^{\frac{-1}{\alpha}}$, we can apply Lemma 6.2 .8 with $\beta=z=\frac{1}{\alpha}, m=\alpha$ to deduce the result.

### 6.3.2 Decorated height bounds

## Spinal decomposition

If $\mathcal{T}^{\text {dec }}$ is the decorated version of a finite Galton-Watson tree $T$, we define the decorated height of $\mathcal{T}^{\text {dec }}$ by

$$
\operatorname{Height}^{\mathrm{dec}}\left(\mathcal{T}^{\mathrm{dec}}\right)=\sup _{x \in \mathcal{T}^{\mathrm{dec}}} d_{g}^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, x\right)
$$

At some points in this chapter it will be necessary to decompose along a path achieving maximal decorated height, rather than the maximal tree height, and we give similar results for the offspring distribution along this spine below.

The path in $\mathcal{T}^{\text {dec }}$ joining $\rho_{\alpha}^{\text {dec }}$ to the point achieving maximal decorated height corresponds in a natural way to a path in $T$ joining $\rho$ to a leaf (if this point is not unique, we will take the leftmost path). Analogously with the notation above we call this path the decorated spine and denote this by $s_{0}^{\text {dec }}, s_{1}^{\text {dec }}, \ldots, s_{H}^{\text {dec }}$, where
$H^{\text {dec }}$ denotes the length of this decorated spine. We also let $\xi_{n}^{\text {dec }}$ denote the number of offspring of $s_{n}^{\text {dec }}$.

Note in particular that $H^{\text {dec }}$ gives the length of the decorated spine in the underlying tree, rather than the length with respect to the decorated metric, so $H^{\text {dec }} \leq H$.

The purpose of this section is to establish the tail decay of the decorated height of decorated trees. This is the point at which we use the stretched exponential decay of Assumption (D).

Proposition 6.3.9. Assume that Assumption (D) holds, and take $k$ as in the assumption. Then, if $d(\alpha-1) \neq 1$ :

$$
\mathbf{P}\left(\operatorname{Height}^{d e c}\left(\mathcal{T}^{d e c}\right) \geq n(\log n)^{1+\frac{1}{k}}\right) \leq n^{\frac{-\left(s_{\alpha}^{d} \wedge 1\right)}{\alpha-1}}
$$

If instead $d(\alpha-1)=1$ :

$$
\mathbf{P}\left(\text { Height }^{d e c}\left(\mathcal{T}^{d e c}\right) \geq n(\log n)^{2+\frac{1}{k}}\right) \leq n^{\frac{-\left(s_{\alpha}^{d} \wedge 1\right)}{\alpha-1}}
$$

Before giving the full proof, we give a lemma. This lemma show that it was important to condition on survival in Proposition 6.3.5 in order to get the size-biased lower bound on the offspring distribution of the Williams' spine.

Lemma 6.3.10. Set

$$
\begin{aligned}
\tilde{h}_{d} & =\sup _{v \in T} \sum_{\rho \preceq u \preceq v}(\operatorname{deg} u)^{\frac{1}{d}} \\
\tilde{v}_{h} & =\arg \max _{v \in T} \sum_{\rho \preceq u \preceq v}(\operatorname{deg} u)^{\frac{1}{d}} .
\end{aligned}
$$

(breaking ties by taking the vertex with the leftmost path). Then, if $d(\alpha-1) \neq 1$,

$$
\mathbf{P}\left(\tilde{h}_{d} \geq k\right) \leq c k^{\frac{-(d(\alpha-1) \wedge 1)}{\alpha-1}}
$$

If instead $d(\alpha-1)=1$,

$$
\mathbf{P}\left(\tilde{h}_{d} \geq k \log k\right) \leq c k^{\frac{-(d(\alpha-1) \wedge 1)}{\alpha-1}}=c k^{\frac{-1}{\alpha-1}}
$$

Proof. It is shown in the construction on [GK99, p.3] that the offspring distribution of each vertex on the Williams' spine of a Galton-Watson tree can be independently stochastically dominated by a size-biased version of the offspring distribution for that tree (this does not mean that the vertex degrees themselves are independent of each other, just that they can be independently stochastically dominated). This is achieved by building the Galton-Watson tree recursively, by starting at the tip of
the Williams' spine and working backwards towards the root, and the size-biased distribution corresponds to a heaviest tail one could possibly obtain during this process, and specifically arises when conditioning the tree to survive forever beyond the current vertex.

The same logic applies to the case when we are maximising $\sum_{\rho \preceq u \preceq v}(\operatorname{deg} u)^{\frac{1}{d}}$ along all paths in the tree: once $\tilde{v}_{h}$ is identified, we can work backwards from $\tilde{v}_{h}$ to construct the appropriate spine recursively, write down a similar expression to [GK99, Equation (1)] that determines the offspring distribution, which in turn implies that there exists a constant $c \leq \infty$ such that

$$
\begin{equation*}
\mathbf{P}\left(\operatorname{deg}\left(\tilde{s}_{i}\right) \geq k\right) \leq c k p_{k} \tag{6.4}
\end{equation*}
$$

for all $k \geq 0, i \leq \tilde{H}$. Moreover, the algorithm on [GK99, p.3] is insightful because at each step the only dependence on the previous steps is through the restriction that the heights of the subtrees sprouting from the non-spinal offspring of the next spinal vertex must not exceed the height of the Williams' spine, and the bound (6.4) holds for $\operatorname{deg}\left(\tilde{s}_{i}\right)$ regardless of the value of $\operatorname{deg}\left(\tilde{s}_{i+1}\right)$. Conditional on the value of $\tilde{H}$, we can therefore bound $\sum_{i=1}^{\tilde{H}}\left(\operatorname{deg} \tilde{s}_{i}\right)^{\frac{1}{d}}$ by a sum $\sum_{i=1}^{\tilde{H}} X_{i}$, where $X_{i}^{d}$ independently satisfy the tail bound of (6.4).

We will in fact condition on $H$ rather than $\tilde{H}$, since the law of $H$ is wellunderstood, and use the fact that $\tilde{H} \leq H$ by definition. More precisely, we will first show that for any $\varepsilon>0$, there exists a constant $c<\infty$ such

$$
\begin{equation*}
\mathbf{P}\left(\left.\sum_{i=0}^{\tilde{H}}\left(\operatorname{deg} \tilde{s}_{i}\right)^{\frac{1}{d}} \geq n^{\frac{1}{d(\alpha-1) \wedge 1}} \lambda \right\rvert\, H=n\right) \leq c \lambda^{-(\alpha d-\varepsilon)} \tag{6.5}
\end{equation*}
$$

for all $n \geq 1, \lambda \geq 1$. We will then apply Lemma 6.2 .8 by composing with the law of $H$.

To this end, we write the following:

$$
\begin{align*}
& \mathbf{P}\left(\left.\sum_{i=0}^{\tilde{H}}\left(\operatorname{deg} \tilde{s}_{i}\right)^{\frac{1}{d}} \geq n^{\frac{1}{d(\alpha-1) \wedge 1}} \lambda \right\rvert\, H=n\right) \\
& \leq \mathbf{P}\left(\left.|T| \geq n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon} \right\rvert\, H=n\right) \\
& +\mathbf{P}\left(\sup _{v \in T} \operatorname{deg} v \geq n^{\frac{1}{\alpha-1}} \lambda^{d-\varepsilon}\left|H=n,|T|<n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon}\right)\right.  \tag{6.6}\\
& +\mathbf{P}\left(\sum_{i=0}^{\tilde{H}}\left(\operatorname{deg} \tilde{s}_{i}\right)^{\frac{1}{d}} \geq n^{\frac{1}{d(\alpha-1) \wedge 1}} \lambda, \left.\sup _{v \in T} \operatorname{deg} v<n^{\frac{1}{\alpha-1}} \lambda^{d-\varepsilon} \right\rvert\, H=n\right)
\end{align*}
$$

We bound each of these terms separately. Firstly, by [Kor17, p.3],

$$
\begin{aligned}
\mathbf{P}\left(\left.|T| \geq n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon} \right\rvert\, H=n\right) & \leq \mathbf{P}\left(\left.|T| \geq n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon} \right\rvert\, H \in[n, 2 n]\right) \\
& =\frac{\mathbf{P}\left(|T| \geq n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon}\right) \mathbf{P}\left(H \in[n, 2 n]| | T \left\lvert\, \geq n^{\frac{\alpha}{\alpha-1}}\right.\right)}{\mathbb{P}(H \in[n, 2 n])} \\
& \leq \frac{C n^{\frac{-1}{\alpha-1}} \lambda^{-\varepsilon}}{n^{\frac{-1}{\alpha-1}}} e^{-c \lambda^{\varepsilon(1-\varepsilon)}}
\end{aligned}
$$

Secondly, by Lemma 6.3.7 and a union bound,

$$
\begin{aligned}
\mathbf{P}\left(\sup _{v \in T} \operatorname{deg} v \geq n^{\frac{1}{\alpha-1}} \lambda^{d-\varepsilon}\left|H=n,|T|<n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon}\right)\right. & \leq c n^{\frac{\alpha}{\alpha-1}} \lambda^{\varepsilon} n^{\frac{-\alpha}{\alpha-1}} \lambda^{-\alpha(d-\varepsilon)} \\
& \leq c \lambda^{-\alpha\left(d-\varepsilon^{\prime}\right)}
\end{aligned}
$$

Finally, to deal with the final probability, we use the bound at (6.4) and the fact that this holds independently for each of the vertices $\tilde{s}_{i}$ to deduce that we are in the setting of Lemma 6.2.2 with $\beta=d(\alpha-1)$.

If $d(\alpha-1) \leq 1$, then we can use Lemma 6.2 .2 with $k=n^{\frac{1}{d(\alpha-1)}} \lambda^{1-\varepsilon^{\prime}}$ to apply a Chernoff bound which gives

$$
\mathbf{P}\left(\sum_{i=0}^{\tilde{H}}\left(\operatorname{deg} \tilde{s}_{i}\right)^{\frac{1}{d}} \geq n^{\frac{1}{d(\alpha-1)}} \lambda, \left.\sup _{v \in T} \operatorname{deg} v<n^{\frac{1}{\alpha-1}} \lambda^{d-\varepsilon} \right\rvert\, H=n\right) \leq C e^{-c \lambda^{\varepsilon}}
$$

Therefore, in this case, we can take a union bound, and the worst decay is coming from the second term of (6.6), which gives the result of (6.5). We can therefore combine this with Lemma 6.3 .3 to plug into Lemma 6.2.8: taking $\beta=\frac{1}{\alpha-1}, z=$ $\frac{1}{d(\alpha-1) \wedge 1}$ and $m=\alpha d-\varepsilon$, we deduce that

$$
\mathbf{P}\left(\tilde{h}_{d} \geq k\right) \leq c k^{\frac{-(d(\alpha-1))}{\alpha-1}}
$$

whenever $d(\alpha-1) \leq 1$.
If instead $d(\alpha-1)=1$, we instead inherit the extra log term from Lemma 6.2.2 so we get that

$$
\mathbf{P}\left(\tilde{h}_{d} \geq k \log k\right) \leq c k^{\frac{-(d(\alpha-1))}{\alpha-1}}
$$

If instead $d(\alpha-1)>1$, we get from Lemma 6.2.5 that

$$
\mathbf{P}\left(\sum_{i=0}^{\tilde{H}}\left(\operatorname{deg} \tilde{s}_{i}\right)^{\frac{1}{d}} \geq n \lambda, \left.\sup _{v \in T} \operatorname{deg} v<n^{\frac{1}{\alpha-1}} \lambda^{d-\varepsilon} \right\rvert\, H=n\right) \leq C n^{-(d(\alpha-1)-1)}
$$

Since we are assuming that $\lambda=\lambda_{n}=o\left(n^{\varepsilon}\right)$ for all $\varepsilon>0$, this gives arbitrarily good
polynomial decay in $\lambda$, so we can again apply Lemma 6.2 .8 with $\beta=\frac{1}{\alpha-1}, z=1$ and $m=\frac{2}{\alpha-1}$ (say) to get the result.

Proof of Proposition 6.3.9. We prove the case $d(\alpha-1) \neq 1$; the proof in the case $d(\alpha-1)=1$ is the same, but we have to incorporate an extra log term to apply Lemma 6.3.10. Let $D^{\text {sup }}=\sup _{v \in T} \frac{\operatorname{Diam}(G(v))}{(\operatorname{deg} v)^{\frac{1}{d}}}$. Then

$$
\text { Height }{ }^{\operatorname{dec}}\left(\mathcal{T}^{\mathrm{dec}}\right) \leq D^{\mathrm{sup}} \sum_{i=1}^{H^{\mathrm{dec}}}\left(\operatorname{deg} s_{n}^{\mathrm{dec}}\right)^{\frac{1}{d}}
$$

so that

$$
\mathbf{P}\left(\text { Height }^{\text {dec }}\left(\mathcal{T}^{\text {dec }}\right) \geq n(\log n)^{1+\frac{1}{k}}\right) \leq \mathbf{P}\left(D^{\text {sup }} \geq(\log n)^{1+\frac{1}{k}}\right)+\mathbf{P}\left(\sum_{i=1}^{H^{\text {dec }}}\left(\operatorname{deg} s_{n}^{\text {dec }}\right)^{\frac{1}{d}} \geq n\right)
$$

It follows from Lemma 6.3.10 that $\mathbf{P}\left(\sum_{i=1}^{H^{\text {dec }}}\left(\operatorname{deg} s_{n}^{\operatorname{dec}}\right)^{\frac{1}{d}} \geq n\right) \leq n^{\frac{-\left(s_{\alpha}^{d} \wedge 1\right)}{\alpha-1}}$. To control the first term, note that it follows from a union bound that

$$
\mathbf{P}\left(D^{\text {sup }} \geq(K \log |T|)^{\frac{1}{k}} \lambda\right) \leq C|T|^{1-c K \lambda^{k}}
$$

Therefore, since $\mathbf{P}(\log |T| \geq k) \leq C e^{-\frac{1}{\alpha} k}$, we deduce that

$$
\begin{aligned}
\mathbf{P}\left(D^{\text {sup }} \geq x\right) \leq \mathbf{P}\left(\log |T| \geq x^{k}\right)+\int_{1}^{x^{k}} C e^{\frac{-u}{\alpha}} e^{-\frac{x}{u^{\frac{1}{k}}}} d u & \leq C e^{-c x^{\frac{k}{\alpha}}}+C x^{k} e^{-x^{\frac{k}{k+1}}} \\
& \leq C x^{k} e^{-x^{\frac{k}{k+1}}}
\end{aligned}
$$

so that $\mathbf{P}\left(D^{\text {sup }} \geq(K \log n)^{1+\frac{1}{k}}\right) \leq C(\log n)^{k+1} n^{-K}$. The result therefore follows on choosing some $K>\frac{s_{\alpha}^{d} \wedge 1}{\alpha-1}$.

We also have the following lower bound.
Proposition 6.3.11. Assume $\left(D^{\prime}\right)$. Then there exists a constant $c \in(0, \infty)$ such that

$$
\mathbf{P}\left(\text { Height }^{d e c}\left(\mathcal{T}^{d e c}\right) \geq n\right) \geq c n^{\frac{-\left(s_{\alpha}^{d} \wedge 1\right)}{\alpha-1}}
$$

Before we give the proof of the lower bound, we will need the following result on distances across the graphs corresponding to spinal vertices along the decorated spine.

Lemma 6.3.12. For an arbitrary vertex $s_{n}$ on the Williams' spine, let $d^{U}\left(s_{n}\right)$ denote the distance between the two vertices of $G\left(s_{n}\right)$ that correspond to edges joining $s_{n}$ to neighbouring spinal vertices (so these are two distinct uniform vertices in $\left.G\left(s_{n}\right)\right)$. Then, there exist $C, c, c^{\prime} \in(0, \infty)$ such that
(i) For all $k \geq 1$,

$$
\mathbf{P}\left(d^{U}\left(s_{n}\right) \geq k\right) C k^{-s_{\alpha}^{d}}
$$

(ii) For all $1 \leq k \leq c^{\prime} n^{\frac{1}{s_{\alpha}^{\alpha}}}$ :

$$
\mathbf{P}\left(d^{U}\left(s_{n}\right) \geq k \mid H \geq 2 n\right) \geq c k^{-s_{\alpha}^{d}}
$$

Proof. This follows from Proposition 6.3.5 and Lemma 6.2.8.
Remark 6.3.13. 1. The same bounds also hold for distances across vertices $s_{n}$ on the infinite backbone of $\mathcal{T}_{\alpha}^{\text {dec }}$.
2. Under ( $D$ ) we have that $s_{\alpha}^{d}=d(\alpha-1)$.

Proof of Proposition 6.3.11. Height ${ }^{\text {dec }}\left(\mathcal{T}^{\text {dec }}\right)$ stochastically dominates $\sum_{n=1}^{\frac{H}{2}} d^{U}\left(s_{n}\right)$, so we will bound this latter quantity.

To do this, first note that by Lemma 6.3.3, $\mathbf{P}\left(H \geq n^{s_{\alpha}^{d} \wedge 1}\right) \geq c n^{\frac{-\left(s_{\alpha}^{d} \wedge 1\right)}{\alpha-1}}$. Then, if $s_{\alpha}^{d} \leq 1$, we have by Lemma 6.3.12 that there exists a constant $c>0$ (deterministic), such that

$$
\mathbf{P}\left(\left.\sum_{n=1}^{\frac{H}{2}} d^{U}\left(s_{n}\right) \geq c n \right\rvert\, H \geq n^{s_{\alpha}^{d} \wedge 1}\right) \geq \mathbf{P}\left(\exists n \leq \frac{H}{2}: d^{U}\left(s_{n}\right) \geq c n \mid H \geq n^{s_{\alpha}^{d}}\right) \geq \frac{1}{2} .
$$

If $s_{\alpha}^{d}>1$ then the same result holds by the Law of Large Numbers. This proves the result.

### 6.3.3 Decorated volume bounds

We now give the asymptotics for the tail decay for the volume of $\mathcal{T}$ dec. Recall the fragmental volume exponent $f_{\alpha}^{v}$ defined by

$$
f_{\alpha}^{v}= \begin{cases}\frac{\alpha}{v} & \text { if } m_{v} \geq \frac{\alpha}{v} \\ m_{v} & \text { if } m_{v}<\frac{\alpha}{v}\end{cases}
$$

Recall also that $t_{\alpha}^{v}=\frac{f_{\alpha}^{v} \wedge 1}{\alpha}$.
Proposition 6.3.14. If $f_{\alpha}^{v} \neq 1$,

$$
\mathbf{P}\left(\operatorname{Vol}\left(\mathcal{T}^{d e c}\right) \geq x\right) \leq x^{\frac{-\left(f_{\alpha}^{v} \wedge 1\right)}{\alpha}}
$$

as $x \rightarrow \infty$. If $f_{\alpha}^{v}=1$,

$$
\mathbf{P}\left(\operatorname{Vol}\left(\mathcal{T}^{\text {dec }}\right) \geq x \log x\right) \leq x^{\frac{-\left(f_{\alpha}^{v} \propto 1\right)}{\alpha}}
$$

as $x \rightarrow \infty$.
Proof. Note that applying the Vervaat transform and then the absolute continuity allows us to treat the different vertices in $T$ as though they are independent. Therefore, if $f_{\alpha}^{v} \neq 1$ the result follows by two applications of Lemma 6.2.8: firstly, by similar arguments to those in Lemma 6.3.7, it follows that $\mathbf{P}(\operatorname{deg}(v) \geq k) \leq$ $c k^{-\alpha}$ for all $v \in T_{n}$, so taking $\beta=\alpha, z=v, m=m_{v}$ in Lemma 6.2.8(ii) gives $\mathbf{P}(\operatorname{Vol}(G(v)) \geq x) \leq x^{-f_{\alpha}^{v}}$.

Then, since $\operatorname{Vol}\left(\mathcal{T}_{n}^{\text {dec }}\right)=\sum_{v \in T_{n}} \operatorname{Vol}(G(v))$, we get from Lemma 6.2.3 that

$$
\mathbf{P}\left(\operatorname{Vol}\left(T_{n}\right) \geq \lambda n^{\frac{1}{f_{\alpha}^{v \wedge 1}}}\right) \leq \begin{cases}c \lambda^{-f_{\alpha}^{v}} & \text { if } f_{\alpha}^{v}<1, \\ c n^{-\left(f_{\alpha}^{v}-1\right)} & \text { if } f_{\alpha}^{v}>1 .\end{cases}
$$

Then, using Lemma 6.3.1, we can apply Lemma 6.2.8(ii) again with $\beta=\frac{1}{\alpha}, z=$ $\frac{1}{f_{\alpha}^{v} \wedge 1}, m=f_{\alpha}^{v}$ to deduce the result.

If instead $f_{\alpha}^{v}=1$, we have to incorporate the extra log term from Lemma 6.2.3, but the proof is the same on applying Lemma 6.2.8(ii) and setting $f(n)=$ $\frac{\operatorname{Vol}\left(T_{n}\right)}{\log n}$ there.

Corollary 6.3.15. There exists a constant $q>0$ such that, if $f_{\alpha}^{v} \neq 1$,

$$
1-\mathbf{E}\left[e^{-\theta \operatorname{Vol}(T)}\right] \sim q \theta^{\frac{f_{\alpha}^{v} \wedge 1}{\alpha}}
$$

as $\theta \rightarrow 0$. If $f_{\alpha}^{v}=1$,

$$
1-\mathbf{E}\left[e^{-\theta V o l(T)}\right] \geq q \theta^{\frac{f^{\nu} \wedge 1}{\alpha}} .
$$

Proof. This is a direct consequence of Lemma 6.2.6, with $\beta=\frac{f_{\alpha}^{v} \wedge 1}{\alpha}$.
We will also need the following proposition to relate the height of $T$ to the volume of $\mathcal{T}^{\text {dec }}$. Typically we will apply the result with $x$ being some power of the radius $r$.

Proposition 6.3.16. For any $q<1, \varepsilon>0$ there exist constants $c, C \in(0, \infty)$ such that for all $x>0, \lambda>1$,

$$
\begin{aligned}
& \mathbf{P}\left(\left.\operatorname{Height}(T) \leq c x^{\frac{\left(f_{\alpha}^{v} \wedge\right)(\alpha-1)}{\alpha}} \right\rvert\, \operatorname{Vol}\left(\mathcal{T}^{d e c}\right) \geq x \lambda\right) \\
& \leq \begin{cases}C e^{-c \lambda^{q-\varepsilon}}+c\left(x \lambda^{1-q}\right)^{-\left(f_{\alpha}^{v}-1\right)} & \text { if } f_{\alpha}^{v}>1, \\
c \lambda^{-f_{\alpha}^{v}(1-\varepsilon)} & \text { if } f_{\alpha}^{v}<1 .\end{cases}
\end{aligned}
$$

If $f_{\alpha}^{v}=1$, then
$\mathbf{P}\left(\left.\operatorname{Height}(T) \leq c\left(\frac{x}{\log x}\right)^{\frac{\left(f_{\alpha}^{v} \wedge 1\right)(\alpha-1)}{\alpha}} \right\rvert\, \operatorname{Vol}\left(\mathcal{T}^{d e c}\right) \geq x \lambda\right) \leq C e^{-c \lambda^{q-\varepsilon}}+c\left(x \lambda^{1-q}\right)^{-\left(f_{\alpha}^{v}-1\right)}$.
Proof. We need to consider the cases $f_{\alpha}^{v} \leq 1$ and $f_{\alpha}^{v}>1$ separately.
Case 1: $f_{\alpha}^{v}>1$. We again write $\operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right)=\sum_{v \in T} \operatorname{Vol}(G(v))$. As noted in the proof of the previous proposition, we have that $\mathbf{P}(\operatorname{Vol}(G(v)) \geq x) \leq x^{-f_{\alpha}^{v}}$ for all $v$ in $T$. Therefore, for $q \in(0,1)$, we have by Lemma 6.2 .5 that

$$
\mathbf{P}\left(|T| \leq x \lambda^{q} \mid \operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq x \lambda\right) \leq \mathbf{P}\left(\sum_{i=1}^{x \lambda^{q}} X^{(i)} \geq x \lambda\right) \leq c\left(x \lambda^{1-q}\right)^{-\left(f_{\alpha}^{v}-1\right)}
$$

where $\left(X^{(i)}\right)_{i=1}^{\infty}$ are i.i.d. and satisfying $\mathbf{P}\left(X^{(1)} \geq x\right) \leq c x^{-f_{\alpha}^{v}}$, and the final line follows from Lemma 6.2.5. It then follows from [Kor17, p.5] that for any $\varepsilon>0$, there exist $c, C \in(0, \infty)$ such that

$$
\mathbf{P}\left(\left.\operatorname{Height}(T) \leq x^{\frac{\alpha-1}{\alpha}}| | T \right\rvert\, \geq x \lambda^{q}\right) \leq C e^{-c \lambda^{q-\varepsilon}}
$$

Applying a union bound we therefore deduce that

$$
\mathbf{P}\left(\left.\operatorname{Height}(T) \leq c x^{\frac{\alpha-1}{\alpha}} \right\rvert\, \operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq x \lambda\right) \leq C e^{-c \lambda^{q-\varepsilon}}+c\left(x \lambda^{1-q}\right)^{-\left(f_{\alpha}^{v}-1\right)}
$$

Since we will assume that $\lambda=\lambda_{x}=o\left(x^{\varepsilon}\right)$, the second term gives arbitrarily high polynomial tail decay in $\lambda$.

Case 2: $f_{\alpha}^{v}<1$. The proof is essentially the same as Case 1 above, except that now $X^{(1)}$ has heavier tails, so that by Lemma 6.2.3(i):

$$
\mathbf{P}\left(|T| \leq x^{f_{\alpha}^{v}} \lambda^{q} \mid \operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq x \lambda\right) \leq \mathbf{P}\left(\sum_{i=1}^{x^{f_{\alpha}^{v} \lambda^{q}}} X^{(i)} \geq x \lambda\right) \leq c \lambda^{-\left(f_{\alpha}^{v}-q\right)}
$$

Then, [Kor17, p.5] gives that for any $\varepsilon>0$, there exist $c, C \in(0, \infty)$ such that

$$
\mathbf{P}\left(\left.\operatorname{Height}(T) \leq x^{\frac{f_{\alpha}^{v}(\alpha-1)}{\alpha}}| | T \right\rvert\, \geq x^{f_{\alpha}^{v}} \lambda^{q}\right) \leq C e^{-c \lambda^{q-\varepsilon}},
$$

so that this time a union bound with $q=2 \varepsilon$ gives

$$
\mathbf{P}\left(\left.\operatorname{Height}(T) \leq c x^{\frac{f_{\alpha}^{v}(\alpha-1)}{\alpha}} \right\rvert\, \operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq x \lambda\right) \leq c \lambda^{-f_{\alpha}^{v}(1-\varepsilon)}
$$

Case 3: $f_{\alpha}^{v}=1$. The proof is the same as Case 1, but we pick up an extra logarithmic term as usual.

### 6.4 Volume bounds for $\mathcal{T}_{\alpha}^{\text {dec }}$

In this section we prove volume bounds for $\mathcal{T}_{\alpha}^{\text {dec }}$. The proof for the upper bounds follow a similar strategy to those used for stable looptrees in Section 4.2.2, but more care is need to deal with the variability of the inserted graphs.

Recall that

$$
d_{\alpha}^{\mathrm{dec}}=\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)} .
$$

We will show that this is the correct volume growth exponent for $\mathcal{T}_{\alpha}^{\text {dec }}$.
We have stated the volume bounds with respect to the decorated metric as defined by (6.3), since these are of independent interest aside from determining the random walk exponents. However, the construction in (6.3) can be generalised to give alternative metrics by replacing $d_{G\left(v_{i}\right)}$ with an arbitrary metric on $G\left(v_{i}\right)$ and the proofs of the volume bounds are equally applicable in the general case. In particular, on replacing $s_{\alpha}^{d}$ with $s_{\alpha}^{R}$ we obtain the exponent for volume growth with respect to the effective resistance metric.

### 6.4.1 Upper bounds

For Section 6.4.1, we will assume that all conditions $\left(D^{\prime}\right)$ and $\left(V^{\prime}\right)$ of Assumption 6.0.3 hold.

The main result is as follows. Recall that $t_{\alpha}^{v}=\frac{f_{\alpha}^{v} \wedge 1}{\alpha}$, and

$$
y_{\alpha}=\frac{f_{\alpha}^{v} \alpha(\alpha-1) s_{\alpha}^{d}}{s_{\alpha}^{d}(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)+\alpha f_{\alpha}^{v}} .
$$

Proposition 6.4.1. Assume that conditions $\left(D^{\prime}\right)$ and $\left(V^{\prime}\right)$ of Assumption 6.0.3 hold. For any $\varepsilon>0, r, \lambda>1$, there exist constants $c, C \in(0, \infty)$ such that
(i) If $f_{\alpha}^{v} \neq 1$, then:

$$
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}(\rho, r)\right) \geq r^{\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}} \lambda\right) \leq \begin{cases}C \lambda^{-t_{\alpha}^{v}(\alpha-1)+\varepsilon}+C \lambda^{-s_{\alpha}^{v}+\varepsilon} & \text { if } s_{\alpha}^{d} \leq 1, \\ C \lambda^{-t_{\alpha}^{v} \cdot y_{\alpha}+\varepsilon}+C \lambda^{-s_{\alpha}^{v}+\varepsilon} & \text { if } s_{\alpha}^{d}>1 .\end{cases}
$$

(ii) If $f_{\alpha}^{v}=1$, then:

$$
\left.\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}\left(\rho_{\alpha}^{d e c}\right.}, r\right)\right) \geq \lambda r^{d_{\alpha}^{d e c}} \log r\right) \leq C \lambda^{-t_{\alpha}^{v}(\alpha-1-\varepsilon)} .
$$

Remark 6.4.2. 1. The tail decay here is not optimal. In all of the propositions in the rest of the section, the precise tail decay is not important for our purpose, other than that it is of polynomial form.
2. In the case $f_{\alpha}^{v}>1$ and $s_{\alpha}^{d}<1$ it is possible to tweak the proofs in the next section to get stretched exponential decay for the upper volume bounds, similarly to how we did for looptrees in Chapter 4, as long as there is some extra local control on volumes of small balls in the inserted graphs. This is certainly true in the case of discrete looptrees, which will be discussed in Section 6.7.2. For the sake of clarity, we have not pursued this here and just focus on establishing the main exponents. If $f_{\alpha}^{v} \leq 1$ the tail bound here in Proposition 6.3 .16 is otherwise a limiting factor in analysing Iterative Algorithm 2. Obtaining tighter decay would allow one to obtain more precise information on the appropriate volume gauge functions for $\mathcal{T}_{\alpha}^{\text {dec }}$. In particular, overall stretched exponential rather than polynomial tail decay for the likelihood of large volumes would indicate that there are at most log-logarithmic fluctuations in the volume, rather than logarithmic (as we saw for stable looptrees in Chapter 4).

The upper bound in Theorem 6.0.5(i) follows from this proposition by applying Borel-Cantelli along the subsequence $r_{n}=2^{n}$, just as we did for stable looptrees in Sections 4.2.1 and 4.2.2.

## Heuristics

Fix $r \geq 1$. Before starting the proof, we briefly outline the strategy, which has several steps.

1. Consider the vertices of the underlying tree along its backbone in sequential order of their distance from the root, and label them in order as $\rho=s_{0}, s_{1}, \ldots$.
2. We will make an appropriate choice of a vertex $s_{i}$ so that all of $B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)$ is completely contained within the inserted graphs corresponding to the segment of backbone from $\rho$ to $s_{i}$ and the decorated subtrees attached to these graphs.
3. We will first bound the quantity

$$
N_{r}^{f}=\sum_{j \leq i} \operatorname{deg}\left(s_{j}\right)
$$

which corresponds to the number of subtrees attached to the backbone within (decorated) distance $r$ of the root. More specifically, we will show that, w.h.p. as $r, \lambda \rightarrow \infty$ appropriately, $N_{r}^{f} \leq r^{\frac{s_{\alpha}^{d} \wedge 1}{\alpha-1}} \lambda$.
4. On this event, we proceed as follows. We first set

$$
\operatorname{Spine}_{r}=\bigcup_{j \leq i} G\left(s_{j}\right) \cap B_{\mathcal{T}_{\alpha}^{\operatorname{dec}}}(\rho, r), \quad \partial \operatorname{Spine}_{r}=\bigcup_{j \leq i} \partial G\left(s_{j}\right) \cap B_{\mathcal{T}_{d}^{\operatorname{dec}}}(\rho, r),
$$

where $\partial G(v)$ denotes the set of boundary vertices of $G(v)$. We then continue by analysing the tree structures of all the decorated subtrees that are grafted to a vertex $v \in \partial$ Spine $_{r}$. Recall that such a vertex $v$ corresponds to an edge of the underlying tree $T_{\alpha}^{\infty}$ joining a backbone vertex to one of its offspring. Let this offspring vertex be $w=w_{v}$. By Definition 3.3.1, $T_{w}$ is a critical Galton-Watson tree, again with offspring distribution $\xi$. We let $\mathcal{T}_{v}^{\text {dec }}$ denote the corresponding decorated subtree, including vertex $v$.

Given $p>0$, it is then the case that:

$$
\begin{align*}
\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right) \leq & \sum_{v \in \partial \operatorname{Spine}_{r}} \operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \leq r^{d_{\alpha}^{\text {dec }}} \lambda^{p}\right\} \\
& +\sum_{v \in \partial \text { Spine }_{r}} \operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(v, r) \cap \mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \geq r^{d_{\alpha}^{\text {dec }}} \lambda^{p}\right\} \\
& +\sum_{j \leq i} \operatorname{Vol}\left(G\left(s_{j}\right)\right), \tag{6.7}
\end{align*}
$$

5. The first term in the sum above can be controlled using the conditioning that $\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \leq r^{d_{\alpha}^{\text {dec }}} \lambda^{p}$. To control the second term, the key observation is that the number of terms in the sum is very small. In fact, the expected number terms less than one. Moreover, conditioned on $\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right)>r^{r_{\alpha}^{\text {dec }}} \lambda^{p}$, the local geometry of of $\mathcal{T}_{v}^{\text {dec }}$ looks very much like that of $\mathcal{T}_{\alpha}^{\text {dec }}$. This suggests that the natural thing to do is to repeat the decomposition described above on any subtree $\mathcal{T}_{v}^{\text {dec }}$ satisfying $\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right)>r^{d_{\alpha}^{\text {dec }}} \lambda^{p}$, by decomposing along its Williams' spine.
This allows us to write a decomposition analogous to (6.7) but instead for a large subtree $\mathcal{T}_{v}^{\text {dec }}$. Of course the same problem may again occur in that the second sum will be non-trivial, but once again we can solve this by reiterating around any subtrees appearing in the sum, and then repeat around further large subtrees as necessary. The hope is that there will not be too many large subtrees appearing throughout this process so that we will not have to reiterate too many times. To prove that this is indeed the case, we index the large subtrees by a branching process. More specifically, we define a tree $T_{\text {vol }}$, where the root "represents" the whole original tree $\mathcal{T}_{\alpha}^{\text {dec }}$. The offspring of the root then represent the large subtrees (i.e. of volume greater than $r^{d_{\alpha}^{\text {dec }}} \lambda^{p}$ ) grafted to $\mathrm{Spine}_{r}$, and so on. This branching process will have a subcritical
offspring distribution so that we can use a theorem of [Dwa69] to control its total progeny.
6. At the end of the process, it is then the case that $\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right)$ is upper bounded by

$$
\sum_{u \in T_{v o l}}\left(\operatorname{Vol}\left(\operatorname{Spine}_{r}(u)\right)+\sum_{v \in \partial \operatorname{Spine}_{r}(u)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \leq r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{p}\right\}\right)
$$

We make this more precise in the next subsection. The main argument following it then focuses on bounding $\left|T_{\text {vol }}\right|$, and typical terms of the form $\operatorname{Vol}\left(\operatorname{Spine}_{r}(u)\right)$ and $\sum_{v \in \partial \operatorname{Spine}_{r}(u)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \leq r^{d_{\alpha}^{\text {dec }}} \lambda^{p}\right\}$.

Main argument
Important: in what follows we will prove Proposition 6.4.1(i), i.e. we will assume that $f_{\alpha}^{v} \neq 1$. This is for the sake of clarity. If $f_{\alpha}^{v}=1$, we end up picking up an extra logarithm every time we apply Lemma 6.2.4(ii), but the arguments are otherwise identical; we give the details at the end of the subsection.

As described above, each vertex $v \in T_{\text {vol }}$ corresponds to a subtree of $T_{\alpha}^{\infty}$, obtained as a "large" fragment on performing a spinal decomposition of the subtree corresponding to its parent in $T_{\mathrm{vol}}$. Before giving the construction of $T_{\mathrm{vol}}$, we give a more precise definition of "large". Set $d_{\alpha}=\frac{s_{\alpha}^{d} \wedge 1}{\alpha-1}$, and define $v(r, \lambda)=r^{d_{\alpha}} \lambda$. We will show in Propositions 6.4.4 and 6.4.5 that $v(r, \lambda)$ gives a good upper bound for the total number of fragments obtained when performing a spinal decomposition of a "large" subtree up to distance $r$ from the root. Now take some $\kappa<(e-1)^{-1}$. We will condition on $N_{r}^{f} \leq v(r)$, so in order for the expected number of "large" fragments to be less than $\kappa$, we want to define a function $f(r, \lambda)$ so that

$$
v(r, \lambda) \mathbf{P}\left(\operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq f(r, \lambda)\right) \leq \kappa
$$

Recall from Proposition 6.3.14 that $\mathbf{P}(\operatorname{Vol}(T) \geq k) \leq k^{-t_{\alpha}^{v}}$ as $k \rightarrow \infty$. Clearly, $\mathbf{P}\left(\operatorname{Vol}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq k\right)$ is a decreasing function of $k$, decaying to zero as $k \rightarrow \infty$, so we can define its inverse $h$ by $h(p)=\inf \{x>0: \mathbf{P}(|T| \geq x) \leq p\}$, and set $f(r, \lambda)=$ $h\left(\kappa v(r, \lambda)^{-1}\right)$. Using the asymptotics for $v$ and the probabilistic tail decay, it follows that we can take

$$
\begin{equation*}
f(r, \lambda)=c_{\kappa} r^{\frac{d_{\alpha}^{\alpha}}{t_{\alpha}}} \lambda^{\frac{1}{t_{\alpha}^{v}}}=c_{\kappa} r^{d_{\alpha}^{\text {dec }}} \lambda^{\frac{1}{t_{\alpha}^{v}}} \tag{6.8}
\end{equation*}
$$

Accordingly, in the algorithm below we will take $f(r, \lambda)$ to be defined by
(6.8) and take $\operatorname{Vol}\left(\mathcal{T}^{\text {dec }}\right) \geq f(r, \lambda)$ as our definition of a "large" fragment.

We will assume throughout that for all $\varepsilon>0, \lambda=\lambda_{r}=o\left(r^{\varepsilon}\right)$ as $r \rightarrow \infty$.
Remark 6.4.3. As remarked in Remark 6.4.2, in the case $f_{\alpha}^{v}>1$ and $s_{\alpha}^{d}<1$ it is possible to tweak the proofs in the next section to get stretched exponential decay for the upper volume bounds, similarly to how we did for looptrees in Chapter 4, as long as there is some extra local control on volumes of small balls in the inserted graphs. This is the main motivation for defining Iterative Algorithm 2, since it provides a framework for doing this. Otherwise, we can instead condition on $\left|T_{\text {vol }}\right|=1$, which limits us to polynomial tail decay but makes the argument simpler.

## Iterative Algorithm 2

Start by defining $\emptyset$ to be the root of $T_{\text {vol }}$. This represents the initial tree $T_{\alpha}^{\infty}$. Given an element $x \in T_{\text {vol }}$, representing a subtree $T(x)$ of $T_{\alpha}^{\infty}$, or equivalently a decorated subtree $\mathcal{T}^{\text {dec }}(x)$ of $\mathcal{T}_{\alpha}^{\text {dec }}$, we proceed inductively as follows:

1. Consider a decomposition of $T(x)$ along either its infinite backbone (if $x=\emptyset)$, or otherwise its W -spine. Let $\operatorname{Spine}_{r}(x)$ consist of all vertices in $G(u)$ for some $u$ along this spine that fall within distance $r$ of the root of $\mathcal{T}_{\alpha}^{\text {dec }}(x)$, with respect to the metric $d_{g}^{\text {dec }}$.
2. Each $v \in \operatorname{Spine}_{r}(x)$ corresponds to an edge of $T(x)$ joining a backbone vertex to one of its offspring, which we denote by $w_{v}$. We let $T_{w_{v}}$ be the subtree of $T_{\alpha}^{\infty}$ rooted at $w_{v}$, and let $\mathcal{T}_{v}^{\text {dec }}$ denote the corresponding decorated subtree, including vertex $v$. For each $v \in \operatorname{Spine}_{r}(x)$, if $\mathcal{T}_{v}^{\text {dec }}$ has total volume at least $f(r, \lambda)$, then add a child to $x \in T_{\text {vol }}$ corresponding to the subtree $\mathcal{T}_{v}^{\text {dec }}$. Otherwise discard the subtree.
3. Repeat this process inductively to construct $T_{\mathrm{vol}}$ one generation at a time.
4. For each $x \in T_{\mathrm{vol}}$, set

$$
\begin{aligned}
F_{x} & =\sum_{v \in \partial \operatorname{Spine}_{r}(x)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \leq f(r, \lambda)\right\} \\
S_{x} & =\sum_{v \in \operatorname{Spine}_{r}(x)} \operatorname{Vol}(G(v))
\end{aligned}
$$

which respectively give the sum of the volumes of the smaller fragments discarded in step 2 above, and the volume contribution from the spine of $\mathcal{T}^{\mathrm{dec}}(x)$.

As explained above, on the event that $\left|T_{\text {vol }}\right|$ is finite, we then have that

$$
\begin{equation*}
\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\operatorname{dec}}}(\rho, r)\right) \leq \sum_{x \in T_{\mathrm{vol}}}\left(F_{x}+S_{x}\right) . \tag{6.9}
\end{equation*}
$$

In order to use this algorithm to bound $\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\text {dec }}, r\right)\right)$, we will need to control the following quantities:

1. The number of fragments obtained on decomposing a single large subtree as described above.
2. The quantity $\left|T_{\text {vol }}\right|$.
3. A typical term of the form $F_{x}$, for $x \in T_{\mathrm{vol}}$; in other words the sums of volumes of small fragments obtained on decomposing a single large subtree as described above.
4. A typical term of the form $S_{x}$, for $x \in T_{\mathrm{vol}}$; in other words the spinal volume of a single large subtree considered above.

We consider these one by one in the next subsections. At many points, this will involve adding up sums of random variables with exponents corresponding to those we introduced in Assumption 6.0.1 and Section 6.3. In many cases, the relevant exponent may be more than or less than 1 , so we will have to consider two regimes: one in which the sum follows law of large numbers type behaviour, and the other in which the tails are heavier and we see behaviour more like that of a stable subordinator. This will eventually give rise to several phase transitions in the value of $d_{\alpha}^{\text {dec }}$, which can also be surmised from its expression as

$$
d_{\alpha}^{\mathrm{dec}}=\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)} .
$$

## Controlling the number of fragments

We start with the case $s_{\alpha}^{d} \leq 1$. This result holds either along the infinite backbone of $\mathcal{T}_{\alpha}^{\text {dec }}$, or decomposing along the Williams' spine of a finite decorated Galton-Watson tree.

For $r>0$, we let $N_{r}^{f}$ denote the number of decorated subtrees that are grafted to the decorated backbone within decorated distance $r$ of $\rho_{\alpha}^{\text {dec }}$.

Proposition 6.4.4. Suppose that $s_{\alpha}^{d} \leq 1$. Suppose that we are decomposing along the infinite backbone of $T_{\alpha}^{\infty}$, or along the Williams' spine of a finite Galton-Watson tree $T$, as described in Iterative Algorithm 2. In the latter case, assume that this tree is conditioned on $\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right)>f(r, \lambda)=c_{\kappa} r^{d_{\alpha}^{d e c}} \lambda^{\frac{1}{t_{\alpha}}}$. Take any $\varepsilon>0$. Then, if $r$
is sufficiently large and $\lambda>1$, and if $f_{\alpha}^{v} \neq 1$, then

$$
\mathbf{P}\left(N_{r}^{f} \geq r^{\frac{s_{\alpha}^{d}}{\alpha-1}} \lambda\right) \leq C \lambda^{-(\alpha-1)+\varepsilon} .
$$

Proof. Take some $\varepsilon>0$, and set $N_{r}=r^{s_{\alpha}^{d}} \lambda^{\varepsilon}$. Consider the sequence of spinal vertices $s_{0}, s_{1}, \ldots, s_{N_{r}}$. We will show that, with high probability, Spine ${ }_{r}$ does not extend to graphs corresponding to spinal vertices beyond $s_{N_{r}}$, in which case $N_{r}^{f}$ is bounded by $\sum_{i=1}^{N_{r}} \operatorname{deg}\left(s_{i}\right)$. More precisely, first note by Proposition 6.3.16 that

$$
\mathbf{P}\left(\operatorname{Height}(T) \leq 2 r^{s_{\alpha}^{d}} \lambda^{\varepsilon}\right) \leq C \lambda^{\frac{-f_{\alpha}^{v}}{t_{\alpha}^{\nu}}} \leq C \lambda^{-\left(\alpha-1-\varepsilon^{\prime}\right)} .
$$

Then, given that $\operatorname{Height}(T)>2 r^{r_{\alpha}^{d}} \lambda^{\varepsilon}$, we know from Lemma 6.3.12(ii) that $\mathbf{P}\left(d^{U}\left(s_{i}\right) \geq k\right) \geq$ $c k^{-s_{\alpha}^{d}}$ for all $k \leq r \lambda^{\varepsilon^{\prime}}$, and all $i \leq r^{s_{\alpha}^{d}} \lambda^{\varepsilon}$. Therefore, setting $d_{\text {spine }}(j)=\sum_{i \preceq j} d^{U}\left(s_{i}\right)$

$$
\mathbf{P}\left(d_{\text {spine }}\left(N_{r}\right) \leq r\right) \leq \mathbf{P}\left(\nexists i \leq N_{r}: d^{U}\left(s_{i}\right) \geq r\right) \leq\left(1-c r^{-s_{\alpha}^{d}}\right)^{r^{s_{\alpha}^{d}} \lambda^{\varepsilon}} \leq e^{-c \lambda^{\varepsilon}} .
$$

Also, by Lemma 6.2.3,

$$
\mathbf{P}\left(\sum_{i=0}^{N_{r}} \operatorname{deg}\left(s_{i}\right) \geq r^{\frac{s_{\alpha}^{d}}{\alpha-1}} \lambda\right) \leq \mathbf{P}\left(N_{r}^{\frac{-1}{\alpha-1}} \sum_{i=0}^{N_{r}} \operatorname{deg}\left(s_{i}\right) \geq \lambda^{1-\varepsilon^{\prime}}\right) \leq C \lambda^{-\left(1-\varepsilon^{\prime}\right)(\alpha-1)}
$$

Combining these in a union bound, we deduce that

$$
\mathbf{P}\left(N_{r}^{f} \geq r^{\frac{s_{\alpha}^{d}}{\alpha-1}} \lambda\right) \leq C e^{-c \lambda^{\varepsilon \beta}}+C \lambda^{-\left(1-\varepsilon^{\prime}\right)(\alpha-1)} \leq C^{\prime} \lambda^{-\left(1-\varepsilon^{\prime}\right)(\alpha-1)} .
$$

We now turn to the case of $s_{\alpha}^{d}>1$. The proof of Proposition 6.4.4 is still valid in this case, but the exponent is no longer optimal.

Recall that $y_{\alpha}=\frac{f_{\alpha}^{v} \alpha(\alpha-1) s_{\alpha}^{d}}{s_{\alpha}^{d}(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)+\alpha f_{\alpha}^{v}}$.
Proposition 6.4.5. Take the setup as in Proposition 6.4.4, except suppose that $s_{\alpha}^{d}>1$. Then, for all sufficiently large $r$ and all $\lambda>1, \varepsilon>0$,

$$
\mathbf{P}\left(N_{r}^{f} \geq r^{\frac{1}{\alpha-1}} \lambda\right) \leq C \lambda^{-y_{\alpha}+\varepsilon} .
$$

Proof. The proof is very similar to the previous one, except that now we set $N_{r}=$ $r \lambda^{\varepsilon}$. Then, taking $z>0$ (to be chosen precisely later),

$$
\mathbf{P}\left(\operatorname{Height}(T) \leq 2 r^{s_{\alpha}^{d}} \lambda^{z}\right) \leq C \lambda^{-f_{\alpha}^{v}(1-\varepsilon)\left(\frac{1}{t_{\alpha}^{\tilde{\alpha}}}-\frac{z \alpha}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}\right)}
$$

by Proposition 6.3.16. This time however, the distances add up linearly along the
spine of $T$ : more precisely, by Lemma 6.3.12, there exists a constant $c \in(0, \infty)$ such that $\mathbf{P}\left(d^{U}\left(s_{i}\right) \geq k\right) \geq c k^{-s_{\alpha}^{d}}$ for all $k \leq r \lambda^{\frac{z}{s_{\alpha}^{d}}}$, and all $i \leq r^{s_{\alpha}^{d}} \lambda^{z}$. It therefore follows that $d_{\text {spine }}\left(N_{r}\right)$ "almost" stochastically dominates a sum of i.i.d. non-negative random variables with finite but positive mean. In particular, letting $\left(Y_{i}\right)_{i=1}^{N_{r}}$ be such a sequence, we have from Lemma 6.2 .5 that, provided we choose $A$ a large enough constant,

$$
\begin{aligned}
\mathbf{P}\left(d_{\text {spine }}\left(N_{r}\right)<A r\right) & =\mathbf{P}\left(\sum_{i=1}^{N_{r}} Y_{i}<A r\right)+\mathbf{P}\left(\exists i \leq N_{r}: Y_{i} \geq r \lambda^{\frac{z}{s_{\alpha}^{c}}}\right) \\
& \leq c\left(r \lambda^{\varepsilon}\right)^{-\left(s_{\alpha}^{d}-1\right)}+\lambda^{-(z-\varepsilon)}
\end{aligned}
$$

(Here the second term corresponds to "seeing the difference" between $d^{U}\left(s_{i}\right)$ and $Y-i$, for some $i$ ). Then, similarly to above, since $\left(\operatorname{deg}\left(v_{i}\right)\right)_{i=1}^{N_{r}}$ is a sequence with ( $\alpha-1$ )-stable upper tails, we again have by Lemma 6.2.3 that

$$
\mathbf{P}\left(\sum_{i \leq N_{r}} \operatorname{deg}\left(v_{i}\right) \geq r^{\frac{1}{\alpha-1}} \lambda\right) \leq c \lambda^{-(1-\tilde{\varepsilon})(\alpha-1)} .
$$

To optimise, we can therefore take $z=\frac{f_{\alpha}^{v} \alpha(\alpha-1)}{s_{\alpha}^{d}(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)+\alpha f_{\alpha}^{v}}$, and then apply a union bound to deduce the result.

Remark 6.4.6. By the same proof, the tail decay in Proposition 6.4.5 can clearly be improved if $f_{\alpha}^{v}>1$, since then we have much better tail decay in Proposition 6.3.16. Since there are already many subcases to keep track of, we haven't pursued this here. The precise exponent we obtained here in Proposition 6.4.5 is not particularly significant in itself, since this proof is probably not optimal anyway. We do this next.

The bounds of Propositions 6.4.4 and 6.4.5 allow us to control the progeny of $T_{\text {vol }}$, so that we can use the strategy outlined in Section 6.4.1 to bound the volume of a ball of radius $r$.

## Controlling $\left|T_{\text {vol }}\right|$

To bound the progeny of $T_{\mathrm{vol}}$, the key point is that, in light of Propositions 6.4.4, 6.4.5 and 6.3 .14 , the offspring distribution off $T_{\mathrm{vol}}$ is roughly $\operatorname{Binomial}\left(v(r, \lambda), \frac{\kappa}{v(r, \lambda)}\right)$. We will make this more precise shortly, but in this case, we can apply the following proposition to bound the progeny of $T_{\mathrm{vol}}$.

Proposition 6.4.7. Let $\tilde{T}$ be a Galton-Watson tree with Binomial( $n, \frac{\kappa}{n}$ ) offspring distribution, for some $\kappa<(e-1)^{-1}$. Then

$$
\mathbf{P}(|\tilde{T}| \geq k) \leq \frac{C}{k} e^{-c k}
$$

Proof. The main ingredient in this proof is the main theorem of Dwass from [Dwa69], that for a Galton-Watson tree with total progeny Prog and offspring distribution $\xi$, it holds that

$$
\mathbf{P}(\operatorname{Prog}=k)=\frac{1}{k} \mathbf{P}\left(\sum_{i=1}^{k} \xi^{(i)}=k-1\right)
$$

Applying this to our case, we deduce that for all $\theta>0$ :

$$
\begin{aligned}
\mathbf{P}(|\tilde{T}| \geq k)=\sum_{j \geq k} \frac{1}{j} \mathbf{P}\left(\sum_{i=1}^{j} \xi^{(i)}=j-1\right) & =\sum_{j \geq k} \frac{1}{j} \mathbf{P}\left(\operatorname{Binomial}\left(n j, \frac{\kappa}{n}\right)=j-1\right) \\
& \leq \sum_{j \geq k} \frac{1}{j} \mathbf{P}\left(\operatorname{Binomial}\left(n j, \frac{\kappa}{n}\right) \geq j-1\right) \\
& \leq \sum_{j \geq k} \frac{e^{\theta}}{j}\left(1+\left(e^{\theta}-1\right) \frac{\kappa}{n}\right)^{n j} e^{-\theta j} \\
& \leq \sum_{j \geq k} \frac{e^{\theta}}{j} \exp \left\{\left(\kappa\left(e^{\theta}-1\right)-\theta\right) j\right\}
\end{aligned}
$$

Taking $\theta=1$ and since $\kappa<(e-1)^{-1}$, we deduce that

$$
\mathbf{P}(\text { Prog } \geq k) \leq \frac{C}{k} \exp \{-c k\}
$$

Proposition 6.4.8. Take prog $>0$. Then, for any $\varepsilon>0$,

$$
\mathbf{P}\left(\left|T_{v o l}\right| \geq \lambda^{\text {prog }}\right) \leq\left\{\begin{array}{l}
C \lambda^{p r o g} \lambda^{-(\alpha-1-\varepsilon)} \text { if } s_{\alpha}^{d} \leq 1 \\
C \lambda^{p r o g} \lambda^{-\left(y_{\alpha}-\varepsilon\right)} \text { if } s_{\alpha}^{d}>1
\end{array}\right.
$$

Proof. Set $n=\lambda^{\text {prog }}$. The key point is that on performing a spinal decomposition of a discrete Galton-Watson tree $T$ along its infinite backbone or W -spine, the bounds of Propositions 6.4.4 and 6.4.5 hold for the spine of the resulting large fragments independently of whether they held for the spine of $T$. Therefore, we can first condition on the total number of fragments obtained from each of the first $k$ elements of $T_{\text {vol }}$ (say according to the lexicographical ordering) each being at most $r^{d_{\alpha}} \lambda$. This has probability at least $1-C k \lambda^{-(\alpha-1)+\varepsilon}$ if $s_{\alpha}^{d} \leq 1$, and probability at least $1-C k \lambda^{-y_{\alpha}+\varepsilon}$ if $s_{\alpha}^{d}>1$.

Conditional on this, the subtree of $T_{\text {vol }}$ restricted to its first $k$ elements is stochastically dominated by a Galton-Watson tree with offspring distribution as in Proposition 6.4.7, with $n=r^{d_{\alpha}} \lambda$. Applying the proposition, we deduce that with probability at least $1-\frac{C}{k} e^{-c k}$, the progeny of this subtree is strictly less than $k$. This implies that $T_{\text {vol }}$ also has progeny strictly less than $k$.

By applying a union bound and substituting $k=\lambda^{\text {prog }}$, we therefore deduce the result.

Remark 6.4.9. Since we are only aiming for polynomial tail decay in $\lambda$, we could instead tweak the powers of $\lambda$ in previous propositions and then condition on $\left|T_{\text {vol }}\right|=$ 1, which would be simpler. However, we can get tighter decay by reiterating, and this decay can also be improved to stretched exponential in the setting discussed in Remark 6.4.2.

## Controlling the volumes of small fragments

We now turn to bounding a quantity of the form

$$
F_{x}=\sum_{v \in \partial \operatorname{Spine}_{r}(x)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \leq f(r, \lambda)\right\}
$$

In keeping with earlier notation, we set $N_{r}^{f}(x)=\left|\partial \operatorname{Spine}_{r}(x)\right|$ for each $x \in$ $T_{\mathrm{vol}}$. We will condition on $N_{r}^{f}<r^{s_{\alpha \wedge 1}^{d}} \lambda$, and recall from Proposition 6.3 .14 that for $v \in \partial \operatorname{Spine}_{r}(x), \mathbf{P}\left(\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }} \geq x\right) \leq c^{\prime} x^{-t_{\alpha}^{v}}\right.$ as $x \rightarrow \infty$. We deduce that this expression falls into the framework of Lemma 6.2.2, with $\beta=t_{\alpha}^{v}$ and $k=f(r, \lambda)$.

The next proposition therefore follows directly from Lemma 6.2.2.
Proposition 6.4.10. There exists a deterministic constant $\tilde{K}<\infty$ such that:

$$
\begin{aligned}
& \mathbf{P}\left(\sum_{x \in T_{\text {vol }}} \sum_{v \in \partial S p i n e_{r}(x)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \leq f(r, \lambda)\right\} \geq \tilde{K} \lambda^{\text {prog }} f(r, \lambda)| | T_{v o l} \mid \leq \lambda^{\text {prog }}\right) \\
& \leq C e^{-c \lambda^{\text {prog }}} .
\end{aligned}
$$

Proof. For a given $x \in T_{\mathrm{vol}}$, the fragments obtained on performing a decomposition of $T(x)$ as described in Iterative Algorithm 2 are indexed by the set $\operatorname{Spine}_{r}(x)$. For a given $u$, all of the fragments are independent of each other, and their volumes satisfy the tail bound of Proposition 6.3.14. If we let $n_{x}$ denote the number of these fragments that are "large", the sum of the smaller fragments therefore falls into the framework of Lemma 6.2.2, except that we have $n_{x}+1$ independent copies of the sum considered there. More precisely, consider some arbitrary independent labelling of the vertices in $\partial \operatorname{Spine}_{r}(x)$ from 1 to $N_{r}^{f}(x)$, and label the corresponding decorated fragments $\left(T^{(i, \mathrm{dec})}\right)_{i=1}^{N_{r}^{f}(x)}$. Similarly to the notation of Lemma 6.2.2, let $k=f(r, \lambda)$, let $T^{(k, i)}$ be the label of the $i^{t h}$ "large" fragment for $i \leq n_{x}$, and set $T^{(k, 0)}=0, T^{\left(k, n_{x}+1\right)}=N_{r}^{f}(x)$. The sum of the smaller fragments can then be written
as:

$$
\sum_{v \in \partial \operatorname{Spine}_{r}(x)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \leq f(r, \lambda)\right\}=\sum_{i=0}^{n_{x}} \sum_{j=T^{(k, i)}+1}^{T^{(k, i+1)}} \operatorname{Vol}\left(T^{(i, \mathrm{dec})}\right),
$$

and each term of the form $\sum_{j=T^{(k, i)}+1}^{T^{(k, i+1)}} \operatorname{Vol}\left(T^{(i, \text { dec })}\right)$ is independently of the same form as that considered in Lemma 6.2.2. By Lemma 6.2.2, there exists a deterministic $K<\infty$ such that we can choose $\theta_{K}$ small enough that

$$
\mathbb{E}\left[\exp \left\{\theta_{K} f(r, \lambda)^{-1} \sum_{j=T^{(k, i)}+1}^{T^{(k, i+1)}} \operatorname{Vol}\left(T^{(i, \text { dec })}\right)\right\}\right] \leq e^{K}
$$

for all sufficiently large $r, \lambda$, and all $i \leq n_{x}$. Moreover,

$$
\sum_{x \in T_{\mathrm{vol}}}\left(n_{x}+1\right)=2 \operatorname{Vol}\left(T_{\mathrm{vol}}\right)-1 .
$$

Therefore, applying a Chernoff bound

$$
\begin{aligned}
& \mathbb{P}\left(\sum_{x \in T_{\text {vol }}} \sum_{v \in \partial \operatorname{Spine}_{r}(x)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right) \leq f(r, \lambda)\right\} \geq \tilde{K} \lambda^{\text {prog }} f(r, \lambda)| | T_{\text {vol }} \mid \leq \lambda^{\text {prog }}\right) \\
& =\mathbb{P}\left(\sum_{x \in T_{\text {vol }}} \sum_{i=0}^{n_{x}} f(r, \lambda)^{-1} \sum_{j=T^{(k, i)}+1}^{T^{(k, i+1)}} \operatorname{Vol}\left(T^{(i, \text { dec })}\right) \geq \tilde{K} \lambda^{\text {prog }}| | T_{\text {vol }} \mid \leq \lambda^{\text {prog }}\right) \\
& \leq \mathbb{E}\left[\exp \left\{\theta_{K} f(r, \lambda)^{-1} \sum_{j=1}^{T^{(k, 1)}} \operatorname{Vol}\left(T^{(i, \text { dec })}\right)\right\}\right]^{2 \lambda^{\text {prog }}} e^{-\theta_{K} \tilde{K} \lambda^{\text {prog }}} \\
& \leq e^{2 K \lambda^{\text {prog }}} e^{-\theta_{K} \tilde{K} \lambda^{\text {prog }}} .
\end{aligned}
$$

Therefore, $\tilde{K}=4 K \theta_{K}^{-1}$ will do the job.

## Controlling the spinal volume

Finally, we bound the spinal volume. Recall the spinal volume exponent defined on page 135 by

$$
s_{\alpha}^{v}= \begin{cases}\frac{1}{v}(\alpha-1) & \text { if } m_{v} \geq \frac{1}{v}(\alpha-1), \\ m_{v} & \text { if } m_{v}<\frac{1}{v}(\alpha-1) .\end{cases}
$$

The next result then follows by similar proofs to Propositions 6.4.4 and 6.4.5.
Proposition 6.4.11. Suppose that we are decomposing along the infinite backbone of $T_{\alpha}^{\infty}$, or along the Williams' spine of a finite Galton-Watson tree T, as described in Iterative Algorithm 2. In the latter case, assume that this tree is conditioned on
$\operatorname{Vol}\left(\mathcal{T}_{v}^{\text {dec }}\right)>f(r, \lambda)=c_{\kappa} r^{r_{\alpha}^{d e c}} \lambda^{\frac{1}{t_{\alpha}}}$. Take any $\varepsilon^{\prime}>0$ and any function $g:(0, \infty) \rightarrow$ $(0, \infty)$. Then, if $r$ is sufficiently large and $\lambda>1$,

$$
\left.\mathbf{P}\left(\operatorname{Vol}_{(\text {Spine }}^{r}\right) \geq r^{\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{\nu}}} \lambda\right) \leq \begin{cases}C \lambda^{-\left(s_{\alpha}^{v}-\varepsilon\right)} & \text { if } s_{\alpha}^{d} \leq 1, \\ C \lambda^{-\left(s_{\alpha}^{v}-\varepsilon\right)} \vee C \lambda^{-\left(y_{\alpha}-\varepsilon\right)} & \text { if } s_{\alpha}^{d}>1 .\end{cases}
$$

Proof. This follows identically to the proofs of Propositions 6.4.4 and 6.4.5, with $\alpha-1$ replaced by $s_{\alpha}^{v}$ (both are less than 1 ).

## Proof of Proposition 6.4.1(i)

Proof of Proposition 6.4.1. The key to the proof is (6.9) which says that

$$
\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right) \leq\left|T_{\text {vol }}\right| \sum_{x \in T_{\text {vol }}}\left(S_{x}+F_{x}\right) .
$$

Applying the previous propositions and a union bound, we therefore deduce that

$$
\begin{aligned}
& \left.\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right) \geq \lambda^{\text {prog }}(1+\tilde{K}) f(r, \lambda)\right)\right) \\
& \leq \mathbf{P}\left(\left|T_{\mathrm{vol}}\right| \geq \lambda^{\text {prog }}\right) \\
& +\mathbf{P}\left(\sum_{x \in T_{\text {vol }}} \sum_{v \in \partial \operatorname{Spine}_{r}(x)} \operatorname{Vol}\left(T_{w_{v}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(T_{w_{v}}\right) \leq f(r, \lambda)\right\} \geq \tilde{K} \lambda^{\text {prog }} f(r, \lambda)| | T_{\mathrm{vol}^{\prime}} \mid<\lambda^{\text {prog }}\right) \\
& +\mathbf{P}\left(\sum_{x \in T_{\mathrm{vol}}} \operatorname{Vol}\left(\operatorname{Spine}_{r}(x)\right) \geq \lambda^{\text {prog }} f(r, \lambda)| | T_{\mathrm{vol}} \mid<\lambda^{\text {prog }}\right) .
\end{aligned}
$$

Now recall from (6.8) that

$$
f(r, \lambda)=c_{\kappa} r^{\frac{d_{\alpha}}{t_{\alpha}}} \lambda^{\frac{1}{t_{\alpha}}}=c_{\kappa} r^{d_{\alpha}^{\text {dec }}} \lambda^{\frac{1}{t_{\alpha}}}
$$

We also showed in Proposition 6.4 .11 that, with high probability as $\lambda \rightarrow \infty$,

$$
\operatorname{Vol}\left(\text { Spine }_{r}\right) \leq r^{\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{\prime}}} \lambda .
$$

We also recall that

$$
f_{\alpha}^{v}=\left\{\begin{array}{ll}
\frac{\alpha}{v} & \text { if } m_{v} \geq \frac{\alpha}{v} \\
m_{v} & \text { if } m_{v}<\frac{\alpha}{v},
\end{array} \quad s_{\alpha}^{v}= \begin{cases}\frac{1}{v}(\alpha-1) & \text { if } m_{v} \geq \frac{1}{v}(\alpha-1), \\
m_{v} & \text { if } m_{v}<\frac{1}{v}(\alpha-1)\end{cases}\right.
$$

We deduce from this that $\frac{\alpha-1}{\alpha}\left(f_{\alpha}^{v} \wedge 1\right) \leq s_{\alpha}^{v}$ (with equality if and only if $m_{v} \wedge 1 \geq \frac{\alpha}{v}$ ). From the previous propositions we therefore have the following:

1. $\mathbf{P}\left(\left|T_{\mathrm{vol}}\right| \geq \lambda^{\text {prog }}\right) \leq\left\{\begin{array}{ll}C \lambda^{\text {prog }} \lambda^{-(\alpha-1-\varepsilon)} & \text { if } s_{\alpha}^{d} \leq 1, \\ C \lambda^{\text {prog }} \lambda^{-\left(y_{\alpha}-\varepsilon\right)} & \text { if } s_{\alpha}^{d}>1 .\end{array}\right.$.
2. Conditional on $\left|T_{\mathrm{vol}}\right| \leq \lambda^{p r o g}$,

$$
\mathbf{P}\left(\sum_{x \in T_{\text {vol }}} \sum_{v \in \operatorname{Spine}_{r}(x)} \operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \mathbb{1}\left\{\operatorname{Vol}\left(\mathcal{T}_{v}^{\mathrm{dec}}\right) \leq f(r, \lambda)\right\} \geq \tilde{K} \lambda^{\text {prog }} f(r, \lambda)\right)
$$

is upper bounded by $C e^{-c \lambda^{p r o g}}$.
3.

$$
\begin{aligned}
& \mathbf{P}\left(\sum_{x \in T_{\mathrm{vol}}} \operatorname{Vol}\left(\operatorname{Spine}_{r}(x)\right) \geq \lambda^{\text {prog }} f(r, \lambda)| | T_{\mathrm{vol}} \mid<\lambda^{\text {prog }}\right) \\
& \quad \leq \begin{cases}C \lambda^{\text {prog }} \lambda^{\frac{-s_{\alpha}^{v}}{t_{\alpha}^{v}}+\varepsilon} & \text { if } m_{v} \wedge 1 \geq \frac{\alpha}{v} \\
C \lambda^{\text {prog }^{-\left(d_{\alpha}^{\text {dec }}-\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{\nu}}\right)\left(s_{\alpha}^{v}-\varepsilon\right)} \lambda^{\frac{-s_{\alpha}^{v}}{t_{\alpha}^{\alpha}}}+\varepsilon} & \text { if } m_{v} \wedge 1<\frac{\alpha}{v}\end{cases}
\end{aligned}
$$

It is therefore optimal to take $\operatorname{prog}=\varepsilon$, and then replacing $\lambda^{\frac{1}{t_{\alpha}^{v}}}$ with $\lambda$ we obtain

$$
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\operatorname{dec}}}(\rho, r)\right) \geq r^{\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}} \lambda\right) \leq \begin{cases}C \lambda^{-t_{\alpha}^{v}(\alpha-1)+\varepsilon}+C \lambda^{-s_{\alpha}^{v}+\varepsilon} & \text { if } s_{\alpha}^{d} \leq 1 \\ C \lambda^{-t_{\alpha}^{v} \cdot y_{\alpha}+\varepsilon}+C \lambda^{-s_{\alpha}^{v}+\varepsilon} & \text { if } s_{\alpha}^{d}>1\end{cases}
$$

Since $s_{\alpha}^{v} \geq t_{\alpha}^{v}(\alpha-1)$, this gives the result.

## Proof of Proposition 6.4.1(ii)

In the case $f_{\alpha}^{v}=1$, we pick up an extra logarithm every time we apply Proposition 6.3.16, which we used to apply the Williams' decomposition on each subtree in $T_{\text {vol }}$. In this case, we instead take

$$
f(r, \lambda)=c_{\kappa} \lambda^{\frac{1}{t_{\alpha}^{v}}} r^{d_{\alpha}^{\text {dec }}} \log r
$$

Then $\operatorname{Vol}\left(\mathcal{T}^{\text {dec }}\right) \geq f(r, \lambda)$ implies that $|T| \geq r \lambda^{z}$, with sufficiently high probability, and we can otherwise continue as before.

### 6.4.2 Lower bounds

Recall that $d_{\alpha}^{\text {dec }}=\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}$. Our result for the volume lower bounds is the following.

Proposition 6.4.12. Assume that conditions $(D)$ and $(V)$ of Assumption 6.0.1 are satisfied. For any $\varepsilon>0$, there exist constants $c, C \in(0, \infty)$ such that for all $r>1, \lambda>1$,

$$
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}\left(\rho, r(\log r)^{1+\frac{1}{k}}\right)\right) \leq r^{d_{\alpha}^{d e c}} \lambda^{-1}\right) \leq C \lambda^{\frac{-(\alpha-1)}{\alpha}+\varepsilon}
$$

The lower bound in Theorem 6.0.5(ii) follows from this proposition by applying Borel-Cantelli along the subsequence $r_{n}=2^{n}$, just as we did for stable looptrees in Sections 4.2.1 and 4.2.2.

This bound can actually be improved to exponential tail decay in the case where the inserted graphs are deterministic: see Remark 6.4.15.

The first main weakness of this result is in the inclusion of the log term in the radius of the ball. This does not affect the overall volume estimates very much, since the polynomial tail decay indicates that one cannot rule out logarithmic fluctuations around the volume term, and the tail exponent of $\frac{\alpha-1}{\alpha}$ is unlikely to be optimal anyway. However, there is a question of whether it should really be there, in the sense of whether $\frac{r^{d_{\alpha}^{\text {dec }}}}{\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right)}$ is an $O(1)$ random variable, or whether an extra log term is really necessary for this.

The log term is arising due to an application of the bound in Proposition 6.3.9, which gives that

$$
\mathbf{P}\left(\text { Height }^{\text {dec }}\left(\mathcal{T}^{\text {dec }}\right) \geq n(\log n)^{1+\frac{1}{k}}\right) \leq n^{\frac{-(d(\alpha-1) \wedge 1)}{\alpha-1}}
$$

where $k$ is as in Assumption (D). This causes a problem in regimes where the mass of the decorated tree is concentrated in the leaves, since we must get some distance from the root in order to pick up enough mass. This typically occurs when we use "sparse" graphs to decorate the tree: since it is well known that the mass of undecorated trees is concentrated close to the leaves, we would expect this to remain true when we use sufficiently empty graphs as decoration (e.g. loops). However, if we insert graphs with a higher volume growth exponent, it is clear that volumes of graphs inserted at high-degree (i.e. non-leaf) vertices will start to become more proportionate to the total mass. In fact, the model undergoes a phase transition depending on the value of $v$ : if $v<\alpha$, the mass is still concentrated in the leaves, whereas if $v>\alpha$, the mass of graphs inserted at internal vertices will be comparable to the mass of graphs inserted at leaf vertices (this was also apparent in the proof of the upper bound on p.170). In this latter regime, we can therefore prove the following stronger result. The exponent here also does not depend on the tail decay in Assumption (D) (which is the second main weakness of Proposition 6.4.12), and the proof is very short.

Proposition 6.4.13. Suppose that $m_{v} \wedge 1 \geq \frac{\alpha}{v}$, and that $\left(D^{\prime \prime}\right),\left(V^{\prime \prime}\right)$ hold. Then,
for any $r, \lambda>1$, and all $\varepsilon>0$, there exist constants $c, C \in(0, \infty)$ such that $\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}\left(\rho_{\alpha}^{d e c}, r\right)\right) \leq r^{d_{\alpha}^{d e c}} \lambda^{-1}\right) \leq \begin{cases}c \lambda^{-\left(s_{\alpha}^{v}-\varepsilon^{\prime}\right)} & \text { if } s_{\alpha}^{d}<1 \\ \left(r^{r_{\alpha}^{v} \wedge 1} \lambda^{-s_{\alpha}^{v}(1-\varepsilon)}\right)^{-\left(s_{\alpha}^{d}-1\right)}+e^{-c \lambda^{\varepsilon}} & \text { if } s_{\alpha}^{d}>1 .\end{cases}$ If $s_{\alpha}^{d}=1$, we get that

$$
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{d e c}}\left(\rho_{\alpha}^{d e c}, r \log r\right)\right) \leq r^{d_{\alpha}^{d e c}} \lambda^{-1}\right) \leq c \lambda^{-\left(s_{\alpha}^{v}-\varepsilon^{\prime}\right)}
$$

Proof. Similarly to previous proofs, suppose for now that $s_{\alpha}^{d} \neq 1$, let $N_{r}=r^{s_{\alpha}^{d} \wedge 1} \lambda^{-s_{\alpha}^{v}(1-\varepsilon)}$, and let $\rho=s_{0}, s_{1}, \ldots, s_{N_{r}}$ denote the first $N_{r}$ vertices on the Williams' spine of $T_{\alpha}^{\infty}$. Then, by Lemmas 6.2.3 and 6.2.5,

$$
\mathbf{P}\left(\sum_{0=1}^{N_{r}} \operatorname{Diam}\left(G\left(s_{i}\right)\right) \geq r\right) \leq \begin{cases}c \lambda^{-\left(s_{\alpha}^{v}-\varepsilon^{\prime}\right)} & \text { if } s_{\alpha}^{d}<1 \\ \left(r_{\alpha}^{v} \wedge 1\right. \\ \left.\lambda^{-s_{\alpha}^{v}(1-\varepsilon)}\right)^{-\left(s_{\alpha}^{d}-1\right)} & \text { if } s_{\alpha}^{d}>1 .\end{cases}
$$

Also, note that, by Lemma 6.2.4,

$$
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)\right) \leq r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{-1}\right) \leq \mathbf{P}\left(\sum_{i=0}^{N_{r}} \operatorname{Vol}\left(G\left(s_{i}\right)\right) \leq N_{r}^{\frac{1}{s_{\alpha}^{s}}} \lambda^{-\varepsilon}\right) \leq e^{-c \lambda^{\varepsilon}}
$$

Then, conditional on $\sum_{i=1}^{N_{r}} \operatorname{Diam}\left(G\left(s_{i}\right)\right)<r$, it follows that

$$
\bigcup_{i=0}^{N_{r}} G\left(s_{i}\right) \subset B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right),
$$

so that

$$
\begin{aligned}
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\text {dec }}, r\right)\right) \leq r^{d_{\alpha}^{\text {dec }}} \lambda^{-1}\right) \leq \mathbf{P}( & \left.\sum_{0=1}^{N_{r}} \operatorname{Diam}\left(G\left(s_{i}\right)\right) \geq r\right) \\
& +\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\text {dec }}, r\right)\right) \leq r^{d_{\alpha}^{\text {dec }}} \lambda^{-1}\right),
\end{aligned}
$$

which gives the result.
As above, the lower bound in Theorem 6.0.5(iii) then follows from this proposition by applying Borel-Cantelli along the subsequence $r_{n}=2^{n}$.

The rest of this subsection is therefore devoted to proving Proposition 6.4.12. We will use a similar decomposition to the previous section, but this time we will not keep iterating around further fragments. The key observation is that $\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right)$
stochastically dominates the sum

$$
\sum_{i=1}^{\substack{\frac{r_{r}^{f}}{f}}} \operatorname{Vol}\left(B_{\mathcal{T}_{i}^{\mathrm{dec}}}\left(\rho_{i}^{\mathrm{dec}}, \frac{1}{2} r\right)\right)
$$

where $\left(\left(\mathcal{T}_{i}^{\text {dec }}, \rho_{i}\right)\right)_{i=1}^{N_{r}^{f}}{ }_{2}^{f}$ are i.i.d. decorated rooted finite Galton-Watson trees.
We will therefore have to bound both the number of fragments obtained on performing such a decomposition, as well as the volume of a typical fragment. The volumes can be controlled using the estimates of Section 6.3, and the number of fragments is dealt with in the following proposition.

Proposition 6.4.14. Recall that $d_{\alpha}=\frac{s_{\alpha}^{d} \wedge 1}{\alpha-1}$. For any $\varepsilon>0$ there exists $c_{\varepsilon} \in(0, \infty)$ such that, if $s_{\alpha}^{d} \neq 1$, then:

$$
\mathbf{P}\left(N_{\frac{r}{2}}^{f} \leq r^{d_{\alpha}} \lambda^{-1}\right) \leq \begin{cases}c \lambda^{-\left(\alpha-1-\varepsilon^{\prime}\right)} & \text { if } s_{\alpha}^{d}<1 \\ \left(r^{s_{\alpha}^{v} \wedge 1} \lambda^{-(\alpha-1)(1-\varepsilon)}\right)^{-\left(s_{\alpha}^{d}-1\right)}+e^{-c \lambda^{\varepsilon}} & \text { if } s_{\alpha}^{d}>1\end{cases}
$$

If $s_{\alpha}^{d}=1$, then

$$
\mathbf{P}\left(N_{\frac{r \log r}{2}}^{f} \leq r^{d_{\alpha}} \lambda^{-1}\right) \leq c \lambda^{-\left(\alpha-1-\varepsilon^{\prime}\right)}
$$

Proof. The proof is the same as that of Proposition 6.4.13, noting that we are instead summing a quantity with $(\alpha-1)$-stable tails rather than $s_{\alpha}^{v}$-stable tails in the final step.

Remark 6.4.15. In some cases, for example when inserting deterministic graphs rather than random ones, there may be some deterministic relationship between $r$ and $N_{\frac{r}{2}}^{f}$ that we can exploit to get better tail decay on the above event (such as exponential), or even a deterministic result, which will also allow us to improve the decay in Proposition 6.4.12. We will see in Section 6.7 that this is the case for trees and looptrees, for example.

Proof of Proposition 6.4.12. Assume for now that $s_{\alpha}^{d} \neq 1$. As noted above, we will use the fact that $\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\text {dec }}}(\rho, r)\right)$ stochastically dominates the sum

$$
\sum_{i=1}^{N_{\frac{r}{2}}^{f}} \operatorname{Vol}\left(B_{\mathcal{T}_{i}^{\mathrm{dec}}}\left(\rho_{i}^{\mathrm{dec}}, \frac{1}{2} r\right)\right)
$$

where $\left(\left(\mathcal{T}_{i}^{\text {dec }}, \rho_{i}\right)\right)_{i=1}^{N_{r}^{f}} \underset{2}{2}$ are i.i.d. decorated rooted finite Galton-Watson trees. By
independence, we can therefore write

$$
\begin{aligned}
& \mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\operatorname{dec}}}\left(\rho, r(\log r)^{1+\frac{1}{k}}\right)\right) \leq r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{-1}\right) \\
& \leq \mathbf{P}\left(\sum_{i=1}^{N_{r}^{f}} \underset{2}{2} \operatorname{Vol}\left(B_{\mathcal{T}_{i}^{\mathrm{dec}}}\left(\rho_{i}^{\mathrm{dec}}, \frac{1}{2} r(\log r)^{1+\frac{1}{k}}\right)\right) \leq r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{-1}\right) \\
& \leq \mathbf{E}\left[\mathbf{E}\left[\exp \left\{-\theta \operatorname{Vol}\left(B_{\mathcal{T}_{1}^{\operatorname{dec}}}\left(\rho, \frac{1}{2} r(\log r)^{1+\frac{1}{k}}\right)\right)\right\}\right]^{N_{\frac{r}{2}}^{f}}\right] e^{\theta r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{-1}} \\
& \leq \mathbf{E}\left[\left(\mathbf{E}\left[\exp \left\{-\theta \operatorname{Vol}\left(\mathcal{T}_{1}^{\mathrm{dec}}\right)\right\}\right]+\mathbf{P}\left(\operatorname{Height}{ }^{\operatorname{dec}}\left(\mathcal{T}_{1}^{\mathrm{dec}}\right) \geq \frac{1}{2} r(\log r)^{1+\frac{1}{k}}\right)\right)^{N_{\frac{r}{2}}^{f}}\right] e^{\theta r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{-1}}
\end{aligned}
$$



$$
\mathbb{E}\left[e^{-\theta \operatorname{Vol}\left(\mathcal{T}_{1}^{\mathrm{dec}}\right)}\right] \leq 1-\frac{q}{2} \theta^{t_{\alpha}^{v}}:=1-\frac{q}{2} r^{-d_{\alpha}} \lambda^{\frac{1}{\alpha}}
$$

as $r, \lambda \rightarrow \infty$, and by Proposition 6.3.9, $\mathbf{P}\left(\operatorname{Height}^{\operatorname{dec}}\left(\mathcal{T}_{1}^{\mathrm{dec}}\right) \geq \frac{1}{2} r(\log r)^{1+\frac{1}{k}}\right) \leq c r^{-d_{\alpha}}$ as $r \rightarrow \infty$.

Also, by Proposition 6.4.14 and Lemma 6.2.7, the rescaled variable $r^{-d_{\alpha}} N_{\frac{r}{2}}^{f}$ satisfies $\mathbf{E}\left[e^{\varphi r^{-d_{\alpha}} N_{\frac{r}{2}}^{f}}\right] \leq c \varphi^{-(\alpha-1-\varepsilon)}$ for all sufficiently large $\varphi$.

Therefore, for all sufficiently large $\lambda$, we deduce that

$$
\begin{aligned}
\mathbf{P}\left(\operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho, r(\log r)^{1+\frac{1}{k}}\right)\right) \leq r^{d_{\alpha}^{\mathrm{dec}}} \lambda^{-1}\right) & \leq e \mathbf{E}\left[\left(1-\frac{q}{2} r^{-d_{\alpha}} \lambda^{\frac{1}{\alpha}}+c r^{-d_{\alpha}}\right)^{N_{\frac{r}{2}}^{f}}\right] \\
& \leq e \mathbf{E}\left[\exp \left\{-\frac{q}{4} \lambda^{\frac{1}{\alpha}} r^{-d_{\alpha}} N_{\frac{r}{2}}^{f}\right\}\right] \\
& \leq c \lambda^{\frac{-(\alpha-1-\varepsilon)}{\alpha}} .
\end{aligned}
$$

If $s_{\alpha}^{d}=1$, we can replace $N_{\frac{r}{2}}^{f}$ with $N_{\frac{r \log r}{2}}^{f}$, and since we are considering a ball of radius $r(\log r)^{1+\frac{1}{k}}$ anyway, this does not affect the result.

Remark 6.4.16. 1. We can improve this a bit by considering $N_{r(\log r)^{1+\frac{1}{k}}}^{f}$ instead of $N_{\frac{r}{2}}^{f}$, but there will still be a log discrepancy.
2. In the case where we have either deterministic control on $N_{\frac{r}{2}}^{f}$, or stretched exponential tail decay in Proposition 6.4.14, we can also make these adjustments in the proof above to get stretched exponential tail decay.

### 6.5 Resistance on $\mathcal{T}_{\alpha}^{\text {dec }}$

In order to apply results of [KM08] about random walk exponents, we also need to understand resistance on $\mathcal{T}_{\alpha}^{\text {dec }}$.

In Assumption 6.0.1 we have assumed that the two-point function and diameters of the inserted graphs grow according to the same exponents. It would still be possible to get some kind of result if this was not the case, but we would need more information on the local geometry of the inserted graphs. When the two exponents are equal, it means that we are able to cut the decorated Williams' spine at an appropriate cut point such that the cut point is roughly distance $r$ from the root, and all vertices contained in the decorated graphs corresponding to ancestors of that cutpoint are also roughly within distance $r$ of the root. The same holds for the resistance distance, so this gives a concrete way to separate $B_{r e s}^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)$ from $B_{r e s}^{\text {dec }}\left(\rho_{\alpha}^{\text {dec }}, 2 r\right)^{c}$, for example. However, if the diameters of the inserted graphs grow differently to the two-point function, then we cannot separate balls just by exploiting the underlying tree structure.

Recall that the exponent $s_{\alpha}^{R}$ is defined so that, for a vertex $s_{i}$ on the infinite backbone of $T_{\alpha}^{\infty}$, there exist constants $c, c^{\prime}$ such that

$$
\begin{aligned}
\mathbf{P}\left(\operatorname{Diam}_{r e s}\left(G\left(s_{i}\right)\right)\right. & \geq r) \leq c r^{-s_{\alpha}^{R}} \\
\mathbf{P}\left(R^{U}\left(G\left(s_{i}\right)\right)\right. & \geq r) \geq c^{\prime} r^{-s_{\alpha}^{R}}
\end{aligned}
$$

as $r \rightarrow \infty$.
In what follows, we will use the subscript "res" to denote that distances are defined with respect to the resistance metric. For example, $B_{r e s}^{\text {dec }}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)$ refers to a ball of radius $r$ with respect to the effective resistance metric on $\mathcal{T}_{\alpha}^{\text {dec }}$, and $B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\text {dec }}, r\right)$ will refers to a ball with respect to the decorated graph metric $d_{g}^{\text {dec }}$.

The following bound is necessary in order to establish the random walk exponents.

Proposition 6.5.1. For all $\varepsilon>0$, there exists a constant $c<\infty$ such that for all $\lambda>1$,

$$
\mathbf{P}\left(R _ { e f f } \left(\rho_{\alpha}^{d e c}, B_{r e s}^{d e c}\left(\rho_{\alpha}^{d e c}, r(\log r)^{\left.\left.\left.1+\frac{1}{k_{r}}\right)^{c}\right) \leq r \lambda^{-1}\right) \leq c \lambda^{-\left(\frac{s_{\alpha}^{R} \wedge 1}{\alpha}-\varepsilon\right)} . . . ~}\right.\right.\right.
$$

Proof. Fix some $k, l>0$ (we will specify these precisely later). Similarly to previous proofs, we define a number $N_{r}$ that corresponds to the index of the vertex where we "cut off" the infinite backbone. This time, we take $N_{r}=r^{S_{\alpha}^{R} \wedge 1} \lambda^{-k}$, and first
observe that, by Lemma 6.2 .3 if $s_{\alpha}^{R}<1$ and by Lemma 6.2.5 if $s_{\alpha}^{R} \geq 1$,

$$
\mathbf{P}\left(\sum_{i \leq N_{r}} \operatorname{Diam}_{r e s}\left(s_{i}\right) \geq \frac{1}{2} r \log r\right) \leq \begin{cases}c \lambda^{-k} & \text { if } s_{\alpha}^{R}<1 \\ c\left(r \lambda^{-k}\right)^{s_{\alpha}^{R}-1} & \text { if } s_{\alpha}^{R} \geq 1\end{cases}
$$

(The $\log$ term is only really necessary if $s_{\alpha}^{R}=1$, but since we will also pick up a $\log$ term from Proposition 6.3.9, this doesn't make any difference here). Additionally, by Lemma 6.2 .4 if $s_{\alpha}^{R}<1$ and Lemma 6.2 .4 if $s_{\alpha}^{R}>1$,

$$
\mathbf{P}\left(\sum_{i \leq N_{r}} R^{U}\left(s_{i}\right) \leq r \lambda^{-k\left(s_{\alpha}^{R} \wedge 1\right)^{-1}-\varepsilon}\right) \leq \begin{cases}C e^{-c \lambda^{\varepsilon}} & \text { if } s_{\alpha}^{R} \geq 1  \tag{6.10}\\ c\left(r \lambda^{-k}\right)^{s_{\alpha}^{R}-1} & \text { if } s_{\alpha}^{R}>1\end{cases}
$$

Moreover, similarly to the proofs of Propositions 6.4.4 and 6.4.5, it follows from Lemma 6.2.3 that

$$
\mathbf{P}\left(\sum_{i \leq N_{r}} \operatorname{deg}\left(s_{i}\right) \geq r^{\frac{s_{\alpha}^{R} \wedge 1}{\alpha-1}} \lambda^{l-\frac{k}{\alpha-1}}\right) \leq c \lambda^{-l(\alpha-1)}
$$

The three probabilistic bounds above show that, with (quantified) high probability:

- all of the vertices contained in $\bigcup_{i \leq N_{r}} G\left(s_{i}\right)$ are within resistance distance $\frac{1}{2} r$ of the root;
- the number of subtrees joined to the decorated Williams' spine at a vertex contained in $\bigcup_{i \leq N_{r}} G\left(s_{i}\right)$ is at most $N_{r}^{\frac{1}{\alpha-1}} \lambda^{l}$;
- the vertex joining $G\left(s_{N_{r}-1}\right)$ to $G\left(s_{N_{r}}\right)$ is at resistance at least $N_{r}^{\left(s_{\alpha}^{R} \wedge 1\right)^{-1}} \lambda^{-\varepsilon}$ from the root.

Recall also that by Proposition 6.3.9 (with $s_{\alpha}^{R}$ instead of $s_{\alpha}^{d}$ ),

$$
\mathbf{P}\left(\text { Height }_{r e s}^{\mathrm{dec}}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq \frac{1}{2} r(\log r)^{1+\frac{1}{k_{r}}}\right) \leq c r^{\frac{-\left(s_{\alpha}^{R} \wedge 1\right)}{\alpha-1}}
$$

Moreover, conditionally on these three events all occurring, the number of fragments grafted to the backbone that intersect $B_{\text {res }}^{\text {dec }}\left(\rho_{\alpha}^{\text {dec }}, r\right)^{c}$ is stochastically dominated by a $\operatorname{Binomial}\left(r^{\frac{s_{\alpha}^{R} \wedge 1}{\alpha-1}} \lambda^{l-\frac{k}{\alpha-1}}, c r^{\frac{-\left(s_{\alpha}^{R} \wedge 1\right)}{\alpha-1}}\right)$ random variable - call this $M_{r}$. Then

$$
\begin{aligned}
\mathbf{P}\left(M_{r} \geq 1\right) & =1-\left(1-c r^{\frac{-\left(s_{\alpha}^{R} \wedge 1\right)}{\alpha-1}}\right)^{\frac{s_{\alpha}^{R} \wedge 1}{\alpha-1}} \lambda^{l-\frac{k}{\alpha-1}} \\
& \leq 1-\exp \left\{-2 c r^{\frac{-\left(s_{\alpha}^{R} \wedge 1\right)}{\alpha-1}} r^{\frac{s_{\alpha}^{R} \wedge 1}{\alpha-1}} \lambda^{l-\frac{k}{\alpha-1}}\right\} \\
& \leq c \lambda^{l-\frac{k}{\alpha-1}}
\end{aligned}
$$

Moreover, if $M_{r}=0$, then the vertex $v_{r}$ defined to be the vertex at which $G\left(s_{N_{r}}\right)$ and $G\left(s_{N_{r}+1}\right)$ intersect is a single point that separates the root from $B_{r e s}^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, r(\log r)^{1+\frac{1}{k_{r}}}\right)^{c}$, and itself is at distance at least $N_{r}^{\left(s_{\alpha}^{R} \wedge 1\right)^{-1}} \lambda^{-\varepsilon}$ from the root, so that

$$
\begin{aligned}
R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}}, B_{r e s}^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, r(\log r)^{1+\frac{1}{k_{r}}}\right)^{c}\right) & \geq N_{r}^{\left(s_{\alpha}^{R} \wedge 1\right)^{-1}} \lambda^{-\varepsilon} \\
& =r^{\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{R} \wedge 1\right)^{-1}} \lambda^{-k\left(s_{\alpha}^{R} \wedge 1\right)^{-1}} \lambda^{-\varepsilon} \\
& =\lambda^{-k\left(s_{\alpha}^{R} \wedge 1\right)^{-1}-\varepsilon} .
\end{aligned}
$$

Therefore, combining all the probabilistic bounds obtained with a union bound, and taking $l=\frac{k}{\alpha(\alpha-1)}$, we deduce that:

$$
\mathbf{P}\left(R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}}, B_{r e s}^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)^{c}\right) \leq r \lambda^{-k\left(s_{\alpha}^{R} \wedge 1\right)^{-1}-\varepsilon}\right) \leq c \lambda^{\frac{-k}{\alpha}}
$$

which is equivalent to the stated result.
Remark 6.5.2. 1. Instead taking $N_{r}=r^{s_{\alpha}^{d} \wedge 1} \lambda^{-k}$, the same proof gives

$$
\mathbf{P}\left(R_{e f f}\left(\rho_{\alpha}^{d e c}, B^{d e c}\left(\rho_{\alpha}^{d e c}, r(\log r)^{1+\frac{1}{k}}\right)^{c}\right) \leq r^{\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{R} \wedge 1}} \lambda^{-1}\right) \leq c \lambda^{-\left(\frac{s_{\alpha}^{R} \wedge 1}{\alpha}-\varepsilon\right)} .
$$

Applying Borel-Cantelli along the subsequence $r_{n}=2^{n}$ and shifting the log term over to the other side, we get that, $\mathbf{P}$-almost surely,

$$
R_{e f f}\left(\rho_{\alpha}^{d e c}, B^{d e c}\left(\rho_{\alpha}^{d e c}, r\right)\right) \geq\left(r(\log r)^{-\left(1+\frac{1}{k}\right)}\right)^{\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{R} \wedge 1}}(\log r)^{-\left(\frac{\alpha+\varepsilon)}{s_{\alpha}^{R} \wedge 1}\right.}
$$

for all sufficiently large $r$.
2. Similarly to the volume lower bounds, in some cases where we have better control over the inserted graphs, we may be able to improve this to exponential decay by introducing an interative process similarly to how we did in Section 6.4.1, but this time reiterating around fragments of large height rather than large volume. This was the strategy employed in Proposition 5.4.4 in the looptree case.

We will also need the following result in order to determine the random walk displacement exponent in terms of the intrinsic metric. To save space in the proposition below we write $B_{r}^{R}=B_{r e s}^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)$ and $B_{r}^{d}=B^{\mathrm{dec}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)$.

Proposition 6.5.3. $\mathbf{P}$-almost surely, there exists $r_{0}<\infty$ such that for all $r \geq r_{0}$ :

$$
\begin{aligned}
& B^{d} \\
& r^{\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)^{-1}(\log r)^{\frac{-(\alpha+\varepsilon)}{s_{\alpha}^{d} \wedge 1}}} \subset B_{r}^{R} \subset B_{r^{\left(s_{\alpha}^{d} \wedge 1\right)^{-1}\left(s_{\alpha}^{R} \wedge 1\right)}(\log r)^{\left(s_{\alpha}^{d} \wedge 1\right)^{-1}(\alpha+\varepsilon)}}^{d} \\
& B^{R} \\
& r^{\left(s_{\alpha}^{d} \wedge 1\right)\left(s_{\alpha}^{R} \wedge 1\right)^{-1}(\log r)^{\frac{-(\alpha+\varepsilon)}{s_{\alpha}^{R} \wedge 1}}} \subset B_{r}^{d} \subset B_{r^{\left(s_{\alpha}^{R} \wedge 1\right)^{-1}\left(s_{\alpha}^{d} \wedge 1\right)}(\log r)^{\left(s_{\alpha}^{R} \wedge 1\right)^{-1}(\alpha+\varepsilon)}}^{R}
\end{aligned}
$$

Proof. By replacing $R^{U}$ with $d^{U}$ in (6.10), the same proof as in the previous proposition shows that, with probability at least $1-\lambda^{-\left(\frac{s_{\alpha}^{d} \wedge 1}{\alpha}-\varepsilon\right)}$, there exists a single vertex at distance at least $r^{\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)^{-1}} \lambda^{-1}$ from the root, separating the root from $\left(B_{r}^{R}\right)^{c}$. Therefore, all vertices in $\left(B_{r}^{R}\right)^{c}$ lie at least distance $r^{\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)^{-1}} \lambda^{-1}$ from the root, so that $B_{r\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)^{-1} \lambda^{-1}}^{d} \subset B_{r}^{R}$. Therefore, setting $r_{n}=2^{n}, \lambda_{n}=\left(\frac{1}{2} \log r_{n}\right)^{\frac{\alpha+\varepsilon}{s_{\alpha}^{d} \wedge 1}}$, applying Borel Cantelli and using monotonicity similarly to how we did when proving the volume bounds for stable looptrees, we deduce that almost surely, there exists $r_{0}<\infty$ such that

$$
B_{r^{\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)^{-1}}(\log r)^{\frac{-(\alpha+\varepsilon)}{s_{\alpha}^{d} \wedge 1}}}^{d} \subset B_{r}^{R}
$$

for all $r \geq r_{0}$. By symmetry, we can also use the same argument to go in the other direction, and also deduce that

$$
B_{r^{\left(s_{\alpha}^{d} \wedge 1\right)\left(s_{\alpha}^{R} \wedge 1\right)^{-1}}(\log r)^{\frac{-(\alpha+\varepsilon)}{s_{\alpha}^{R} \wedge 1}}}^{R} \subset B_{r}^{d} .
$$

Note that, if $\tilde{r}=r^{x}(\log r)^{-y}$, then $\tilde{r}^{x^{-1}}(\log \tilde{r})^{x^{-1} y+\varepsilon} \geq r$ for all sufficiently large $r$, so that $B_{\tilde{r}}^{R} \subset B_{r}^{d}$ implies that $B_{\tilde{r}}^{R} \subset B_{\tilde{r}^{x}-1}^{d}(\log \tilde{r})^{x^{-1} y+\varepsilon}$ for all sufficiently large $r$. The second inclusion above therefore gives the result as stated.

### 6.6 Random walk exponents

The purpose of this section is to use the volume and resistance results of the previous sections to determine the exponents for a simple random walk on $\mathcal{T}_{\alpha}^{\text {dec }}$. To do this, we will apply results of [KM08].

To directly apply their results to get exponents for the decorated metric $d$, we would need to define deterministic functions $v$ and $r$ that govern the volume and resistance growth of the space, and for a given $\lambda>1$ define

$$
\begin{aligned}
J(\lambda)= & \left\{R \in[1, \infty]: \lambda^{-1} v(R) \leq \operatorname{Vol}\left(B_{R}^{d}\right) \leq \lambda v(R), R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}},\left(B_{R}^{d}\right)^{c}\right) \geq \lambda^{-1} r(R)\right\} \\
& \cap\left\{R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}}, y\right) \leq \lambda r\left(d\left(\rho_{\alpha}^{\mathrm{dec}}, y\right)\right) \forall y \in B_{R}^{d}\right\}
\end{aligned}
$$

and then show that $\mathbf{P}(R \in J(\lambda)) \rightarrow 1$ as $\lambda \rightarrow \infty$, uniformly in $R>1$ (cf [KM08,

Definition 1.1, Assumption 1.2(1)].
If we ignore the logarithmic discrepancies for a moment, by Propositions 6.4.1, 6.4 .12 and 6.5 .1 the appropriate volume function to take would be $v(R)=$ $R^{\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{d} \wedge 1\right)}}$, and the appropriate resistance function would be $r(R)=R^{\left(s_{\alpha}^{d} \wedge 1\right)\left(s_{\alpha}^{R} \wedge 1\right)^{-1}}$. However, we encounter some technical difficulties with the final condition in the definition of $J(\lambda)$, in that it requires $R_{\text {eff }}\left(\rho_{\alpha}^{\mathrm{dec}}, y\right) \leq \lambda r\left(d\left(\rho_{\alpha}^{\mathrm{dec}}, y\right)\right)$ for all $y \in B_{R}$.

For a general graph, it is usually only possible to achieve this kind of control uniformly when there is some deterministic relation between the resistance metric and the intrinsic metric, for example as is the case for random trees and looptrees. In our decorated tree setting, this is probably still achievable in the case when we decorate the tree with deterministic graphs, but in the case when the inserted graphs are random we anticipate that there will be genuine multiplicative fluctuations in the relationship between the resistance metric and the intrinsic metric (for example these could be on the order of $\log R$ on the ball of radius $R$ ), so it is not possible to bound $\mathbf{P}\left(R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}}, y\right) \leq \lambda d\left(\rho_{\alpha}^{\mathrm{dec}}, y\right) \forall y \in B_{R}^{d}\right)$ uniformly in $R$.

Therefore, we set $v(r)=r^{d_{\alpha}^{\text {dec }}}$, and choose $U$ and $L$ to be the exponents given in Theorem 6.0.5 so that, $\mathbf{P}$-almost surely

$$
v(r)(\log r)^{-L} \leq \operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)\right) \leq v(r)(\log r)^{U}
$$

for all sufficiently large $r$. We also set $r(R)=R^{\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{R} \wedge 1}}$, and use Remark 6.5.2 and Proposition 6.5.3 to choose $U_{R}, L_{R}$ so that

$$
\begin{aligned}
R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}},\left(B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)\right)^{c}\right) & \geq r(R)(\log R)^{-L_{R}} \\
R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}}, y\right) & \leq r\left(d\left(\rho_{\alpha}^{\mathrm{dec}}, y\right)\right)(\log R)^{U_{R}}
\end{aligned}
$$

for all $y \in B_{\mathcal{T}_{\alpha}^{\text {dec }}}\left(\rho_{\alpha}^{\text {dec }}, R\right)^{c}$, and for all sufficiently large $R$. We also define the set

$$
\begin{aligned}
\tilde{J}(\lambda)= & \left\{R \in[1, \infty]: \lambda^{-1} v(R)(\log R)^{-L} \leq \operatorname{Vol}\left(B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)\right) \leq \lambda v(R)(\log R)^{U}\right\} \\
& \cap\left\{R \in[1, \infty]: R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}},\left(B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)\right)^{c}\right) \geq \lambda^{-1} r(R)(\log R)^{-L_{R}}\right\} \\
& \cap\left\{R \in[1, \infty]: R_{\mathrm{eff}}\left(\rho_{\alpha}^{\mathrm{dec}}, y\right) \leq \lambda r\left(d\left(\rho_{\alpha}^{\mathrm{dec}}, y\right)\right)(\log R)^{\left.U_{R} \forall y \in B_{\mathcal{T}_{\alpha}^{\mathrm{dec}}}\left(\rho_{\alpha}^{\mathrm{dec}}, r\right)\right\}}\right.
\end{aligned}
$$

It then follows from Theorem 6.0.5, Remark 6.5.2 and Proposition 6.5.3 that $\mathbf{P}(R \in \tilde{J}(\lambda)) \rightarrow 1$ as $\lambda \rightarrow \infty$, uniformly in $R>1$.

If we instead had this result with $J(\lambda)$ in place of $\tilde{J}(\lambda)$, we could apply results of [KM08] to establish the long term displacement of the random walk up to constants. Given that we instead have to make do with $\tilde{J}(\lambda)$, it is not trivial to show that we can carry through these logarithmic corrections in the proofs of the results
of [KM08] to essentially just get logarithmic corrections on the final results, since some estimates in [KM08] require comparable upper and lower bounds on effective resistance.

However, these arguments were carried out very precisely in [Cro07], in which the results of [KM08] were simultaneously adapted to spaces with non-uniform volume and resistance fluctuations, and also to spaces in the continuum. Since we are not working in a continuum model in this chapter, we cannot directly apply the results of this paper; however, to incorporate the non-uniform fluctuations, we can write discrete analogues of all of the arguments, and we deduce that the log terms can indeed be carried through the computations.

Remark 6.6.1. Without the logarithmic discrepancies in Propositions 6.4.1, 6.4.12 and 6.5.1, we could circumvent this problem by instead using the results of [KM08] to estimate displacement with respect to the resistance metric, and then use Proposition 6.5.3 to account for the fluctuations and state the results in terms of the intrinsic metric. We would therefore define the set

$$
J^{R}(\lambda)=\left\{r \in[1, \infty]: \lambda^{-1} v^{R}(r) \leq \operatorname{Vol}\left(B_{r}^{R}\right) \leq \lambda v^{R}(r), R_{e f f}\left(\rho_{\alpha}^{d e c},\left(B_{r}^{R}\right)^{c}\right) \geq \lambda^{-1} r\right\}
$$

where $v^{R}(r)=r^{\frac{\alpha\left(s_{\alpha}^{R} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{*} \wedge 1\right)}}$, and hope to use Propositions 6.4.1, 6.4.12 and 6.5.1 to show that $\mathbf{P}\left(r \in J^{R}(\lambda)\right) \rightarrow 1$ as $\lambda \rightarrow \infty$, uniformly in $r>1$, which would allow us to directly apply [KM08, Proposition 1.3] to give exponents with respect to the resistance metric.

Ultimately we hope to do this, so for now we have not gone into all the details for incorporating the non-uniform fluctuations following the strategy of [Cro07].

Before doing so, we define the following exponents, for ease of notation:

1. The walk dimension, $e_{\alpha}=\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}+\frac{s_{\alpha}^{d} \wedge 1}{s_{\alpha}^{R} \wedge 1}$.
2. The transition density exponent, $k_{\alpha}=\frac{\alpha\left(s_{\alpha}^{R} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}\left(\frac{\alpha\left(s_{\alpha}^{R} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}+1\right)^{-1}$. Note that the spectral dimension is $2 k_{\alpha}$.
3. The displacement exponent, $D_{\alpha}=\frac{\left(s_{\alpha}^{R} \wedge 1\right)(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)+\alpha\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)}$.

The following proposition then follows from [KM08, Proposition 1.3] after carrying the log terms through the proofs using the techniques of [Cro07] (we do not give the details).

Proposition 6.6.2 (Probabilistic results w.r.t. intrinsic metric). Let $n, r \geq 1$.

Then, there exist (explicit) deterministic $\beta_{\alpha}^{(1)}, \beta_{\alpha}^{(2)}>0$ such that as $\theta \rightarrow \infty$,

$$
\begin{aligned}
& \mathbf{P}\left(\theta^{-1} \leq \frac{\mathbb{E}_{\rho_{\alpha}^{d e c}}\left[\tau_{r}^{d}\right]}{r^{e_{\alpha}}(\log r)^{-\left(\beta_{\alpha}^{(1)}+\beta_{\alpha}^{(2)}\right)}}\right) \rightarrow 1, \\
& \mathbf{P}\left(\frac{\mathbb{E}_{\rho_{\alpha}^{d e c}}\left[\tau_{r}^{d}\right]}{\left.r^{e_{\alpha}}(\log r)^{\beta_{\alpha}^{(1)}+\beta_{\alpha}^{(2)}} \leq \theta\right)} \rightarrow 1,\right. \\
& \mathbf{P}\left(\theta^{-1} \leq n^{k_{\alpha}} p_{2 n}\left(\rho_{\alpha}^{d e c}, \rho_{\alpha}^{d e c}\right) \leq \theta\right) \rightarrow 1, \\
& \mathbf{P} \times \mathbb{P}_{\rho_{\alpha}^{d e c}}\left(\frac{d^{d e c}\left(\rho_{\alpha}^{d e c}, X_{n}\right)}{\left.n^{D_{\alpha}}(\log n)^{\beta_{\alpha}^{(2)}} \leq \theta\right)} \rightarrow 1,\right. \\
& \mathbf{P} \times \mathbb{P}_{\rho_{\alpha}^{d e c}}\left(\theta^{-1} \leq \frac{1+d^{d e c}\left(\rho_{\alpha}^{d e c}, X_{n}\right)}{n^{D_{\alpha}}(\log n)^{-\beta_{\alpha}^{(2)}}}\right) \rightarrow 1 .
\end{aligned}
$$

As written, these exponents are not the most illuminating. Recall though that in most cases, we anticipate that $s_{\alpha}^{R}=R(\alpha-1), s_{\alpha}^{d}=d(\alpha-1), f_{\alpha}^{v}=\frac{\alpha}{v}$. In the most extreme cases in which these are all at most 1 , the exponents become :

$$
e_{\alpha}=r^{v d+\frac{d}{R}}, \quad k_{\alpha}=\frac{R v}{R v+1}, \quad D_{\alpha}=\frac{d}{R+v R^{2}} .
$$

This is the setting in which the local graph behaviour dominates the behaviour on the whole decorated tree, and we no longer see any dependence on $\alpha$. However, as $d$ and $R$ decrease, and $v$ increases, we see several phase transitions as the tails of the various volume and distance quantities become lighter. In particular we obtain the result of Theorem 6.0.6.

With appropriate control, we can also get quenched and annealed results for these exponents. We give the quenched result first: again this follows from carrying the log terms through the proof of [KM08, Proposition 1.5] using arguments of [Cro07].

Theorem 6.6.3 (Quenched random walk results). Under Assumption 6.0.1, Palmost surely,
a) There exist constants $\beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \in(0, \infty)$ such that
(i) There exists $R<\infty$ such that $r^{e_{\alpha}}(\log r)^{-\beta_{2}} \leq \mathbb{E}_{\rho_{\alpha}^{d e c}}\left[\tau_{r}^{d}\right] \leq r^{e_{\alpha}}(\log r)^{\beta_{2}}$ for all $r \geq R$.
(ii) There exists $N<\infty$ such that $n^{-k_{\alpha}}(\log n)^{-\beta_{1}} \leq p_{2 n}\left(\rho_{\alpha}^{\text {dec }}, \rho_{\alpha}^{\text {dec }}\right) \leq n^{-k_{\alpha}}(\log n)^{\beta_{1}}$ for all $n \geq N$.
(iii) $\mathbb{P}$-almost surely, there exist $N, R<\infty$ such that, conditional on $X_{0}=\rho_{\alpha}^{\text {dec }}$,

$$
\begin{aligned}
r^{e_{\alpha}}(\log r)^{-\beta_{2}} & \leq \tau_{r}^{d} \rho_{\alpha}^{d e c} \leq r^{e_{\alpha}}(\log r)^{\beta_{2}} \quad \forall r \geq R \\
r^{D_{\alpha}}(\log r)^{-\beta_{2}} & \leq \sup _{k \leq n} d^{d e c}\left(\rho_{\alpha}^{d e c}, X_{k}\right) \leq r^{D_{\alpha}}(\log r)^{\beta_{2}} \quad \forall n \geq N
\end{aligned}
$$

b) $d_{s}\left(\mathcal{T}_{\alpha}^{d e c}\right):=-2 \lim _{n \rightarrow \infty} \frac{\log p_{2 n}\left(\rho_{\alpha}^{d e c}, \rho_{\alpha}^{d e c}\right)}{\log n}=2 k_{\alpha}$.
c) $\lim _{n \rightarrow \infty} \frac{\log \left(\mathbb{E}_{\rho_{\alpha}^{d e c}}\left[\tau_{r}^{d}\right]\right)}{\log r}=e_{\alpha}$.
d) Let $W_{n}=\left\{X_{0}, X_{1}, \ldots, X_{n}^{e x c}\right\}$, and let $S_{n}=\sum_{x \in W_{n}} \operatorname{deg} x$. Then $\mathbb{P}$-almost surely, $\lim _{n \rightarrow \infty} \frac{\log S_{n}}{\log n}=k_{\alpha}$.

The annealed results follow similarly from the proof of [KM08, Proposition 1.4], again requiring adaptation using the strategies of [Cro07]. In particular we obtain the result of Theorem 6.0.7.

Theorem 6.6.4 (Annealed random walk results). Under Assumption 6.0.1, we have that:
a) There exists constants $c_{1}, c_{2}$ such that $c_{1} r^{e_{\alpha}}(\log r)^{-\left(\beta_{\alpha}^{(1)}+\beta_{\alpha}^{(2)}\right)} \leq \mathbf{E}\left[\mathbb{E}_{\rho_{\alpha}^{\text {dec }}}\left[\tau_{r}^{d}\right]\right] \leq$ $c_{2} r^{e_{\alpha}}(\log r)^{\beta_{\alpha}^{(1)}+\beta_{\alpha}^{(2)}}$ for all $r \geq 1$.
b) There exists a constant $c_{3}>0$ such that $c_{3} n^{-k_{\alpha}}(\log n)^{-\beta_{1}} \leq \mathbf{E}\left[p_{2 n}\left(\rho_{\alpha}^{d e c}, \rho_{\alpha}^{\text {dec }}\right)\right]$ for all $n \geq 1$.
c) There exists a constant $c_{4}>0$ such that

$$
c_{4} r^{D_{\alpha}}(\log r)^{-\beta_{\alpha}^{(2)}} \leq \mathbf{E}\left[\mathbb{E}_{\rho_{\alpha}^{d e c}}\left[d^{d e c}\left(\rho_{\alpha}^{d e c}, X_{n}\right)\right]\right]
$$

d) If the tail decay of Proposition 6.4.12 is $O\left(\lambda^{-(1+\varepsilon)}\right)$, for some $\varepsilon>0$, then $\mathbf{E}\left[p_{2 n}\left(\rho_{\alpha}^{d e c}, \rho_{\alpha}^{d e c}\right)\right] \leq c_{3} n^{-k_{\alpha}}(\log n)^{-\beta_{1}}$ (see [KM08, Remark 1.6(1)]).

### 6.7 Examples

In terms of exponents, the main results of Section 5.5 are that we established that the following exponents are given as follows:

1. The volume growth exponent is equal to $\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}$.
2. The spectral dimension is equal to $\frac{2 \alpha\left(s_{\alpha}^{R} \wedge 1\right)}{\alpha\left(s_{\alpha}^{R} \wedge 1\right)+(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}$.
3. The displacement exponent is equal to $\frac{\left(s_{\alpha}^{R} \wedge 1\right)(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)+\alpha\left(s_{\alpha}^{R} \wedge 1\right)\left(s_{\alpha}^{d} \wedge 1\right)}$.

Below, we give the values of these exponents for several examples of interest.

| Inserted graph | Range | Volume | Spectral dim | Displacement |
| :---: | :---: | :---: | :---: | :---: |
| Star (tree) | all | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$ |
| Loop (looptree) | all | $\alpha$ | $\frac{2 \alpha}{\alpha+1}$ | $\frac{1}{\alpha+1}$ |
| $\beta$-stable trees | $\frac{\beta}{\beta-1} \geq \frac{1}{\alpha-1}$ | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$ |
|  | $\frac{\beta}{\beta-1}<\frac{1}{\alpha-1}$ | $\frac{\beta \alpha}{\beta-1}$ | $\frac{2 \beta \alpha}{\beta-1+\beta \alpha}$ | $\frac{\beta-1}{\beta-1+\beta \alpha}$ |
| Finite variance dissections | $\alpha \geq \frac{3}{2}$ | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$ |
|  | $\alpha \leq \frac{3}{2}$ | $2 \alpha$ | $\frac{4 \alpha}{2 \alpha+1}$ | $\frac{1}{2 \alpha+1}$ |
| $\beta$-stable dissections* | $\beta \geq \frac{1}{\alpha-1}$ | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$ |
|  | $\beta \leq \frac{1}{\alpha-1}$ | $\alpha \beta$ | $\frac{2 \alpha \beta}{\alpha \beta+1}$ | $\frac{1}{\alpha \beta+1}$ |
| Critical <br> Erdös-Rényi ( $n$ ) | $\alpha \geq \frac{3}{2}$ | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$. |
|  | $\alpha \leq \frac{3}{2}$ | $2 \alpha$ | $\frac{4 \alpha}{2 \alpha+1}$ | $\frac{1}{2 \alpha+1}$ |
| Critical Erdös-Rényi ( $n^{\beta}$ ) | $\alpha \geq \beta$ | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$ |
|  | $\frac{\beta+2}{2}<\alpha<\beta$ | $\underline{\beta}$ | $\frac{2 \beta}{2 \beta}$ | $\frac{\alpha-1}{}$ |
|  | $2 \leq \alpha \leq \beta$ | $\frac{\alpha-1}{}$ | $\frac{\beta+\alpha-1}{}$ | $\frac{\alpha-1+\beta}{}$ |
|  | $\alpha \leq \frac{\beta+2}{2}$ | 2 | $\frac{4}{3}$ | $\frac{1}{3}$ |
| Sierpinski triangle | $\alpha \geq \frac{\log 10-\log 3}{\log 2}$ | $\frac{\alpha}{\alpha-1}$ | $\frac{2 \alpha}{2 \alpha-1}$ | $\frac{\alpha-1}{2 \alpha-1}$ |
|  | $\frac{\log 3}{\log 2} \leq \alpha<\frac{\log 10-\log 3}{\log 2}$ | $\alpha$ | $\frac{2 \alpha \log 2}{\alpha \log 2+\log 5-\log 3}$ | $\frac{\log 2}{\text { 2 }}$ (og $2+\log 5$ |
|  |  |  | $\alpha \log 2+\log 5-\log 3$ | $\bar{\alpha} \log 2+\log 5-\log 3$ |
|  | $\alpha<\frac{\log 3}{\log 2}$ | $\frac{\log 3}{\log 2}$ | $\frac{2 \log 3}{\log 5}$ | $\frac{\log 2}{\log 5}$ |
| Complete graph | all | $\frac{2}{\alpha-1}$ | $\frac{4}{\alpha+1}$ | $\frac{\alpha-1}{\alpha+1}$ |

Table 6.1: Quenched exponents for the models considered below. *conjectural.

### 6.7.1 Trees

By inserting an appropriate "star" graph at every vertex, or simply repeating the arguments employed in the previous section directly for trees, we recover and improve some results for random walks on critical Galton-Watson trees with offspring distribution satisfying (6.2), conditioned to survive. We do not go into the details, but in this setting Assumption 6.0.1 is effectively satisfied with $d=R=\infty, v=1$ for $\lambda>2$ (though to make this rigorous, it easier just to repeat the arguments directly with these trees in mind). We therefore deduce from Proposition 6.4.1 that for any $\varepsilon>0$ there exists $C<\infty$ such that

$$
\mathbf{P}\left(\operatorname{Vol}\left(B_{T}(\rho, r)\right) \geq r^{\frac{\alpha}{\alpha-1}} \lambda\right) \leq C \lambda^{\frac{-(\alpha-1)}{\alpha}+\varepsilon}
$$

This is not as good as the result in [CK08, Proposition 2.2], where they prove polynomial tail decay bounded by $\lambda^{-(\alpha-1-\varepsilon)}$. This is not surprising since they are able to fine-tune their arguments specifically for trees in their paper, by using generating functions and decomposing at different heights of the tree. Moreover, corresponding results for volumes of stable trees [DW14, Theorem 1.2], which in theory could be obtained from probabilistic bounds in a similar way to our results for stable looptrees in Chapter 4, make it plausible that an exponent of $\alpha-1-\varepsilon$ is
in fact optimal.
However, for the lower volume bounds we are able to improve the polynomial tail decay of [CK08, Proposition 2.6] to stretched exponential. The result itself is not surprising, since corresponding volume results for stable trees [Duq12, Theorem 1.2] suggest that we should have tighter control on infimal volumes, and our approach involves a different decomposition to the one that formed the basis of [CK08].

To obtain the better bound, note that we are in the setting of Remark 6.4.15 where we do not need to control the randomness in the definition of $M_{r}$, and instead we can lower bound $N_{\frac{r}{2}}^{f}$ by the sum of the degrees of the first $\frac{1}{2} r$ vertices along the backbone. By directly composing the moment generating functions for the sum of the degrees and the sizes of the incident fragments, it is possible to show that

$$
\mathbf{P}\left(N_{\frac{r}{2}}^{f} \leq r^{\frac{1}{\alpha-1}} \lambda^{-q}\right) \leq C e^{-c \lambda^{\frac{\alpha-1}{\alpha}}}
$$

In line with analogous results on stable trees ([Duq12, Theorem 1.2] and [DW14, Theorem 1.1]), we conjecture that it should be possible to improve this upper bound to $C e^{-c \lambda^{\beta}}$ for any $\beta<\alpha-1$.

As a result of the bounds above, the quenched volume bounds we obtain are that, $\mathbf{P}$-almost surely,

$$
\limsup _{r \rightarrow \infty} \frac{\operatorname{Vol}\left(B_{T_{\alpha}^{\infty}}(\rho, r)\right)}{r^{\frac{\alpha}{\alpha-1}}(\log r)^{\frac{1+\varepsilon}{\alpha-1}}}=0, \quad \quad \liminf _{r \rightarrow \infty} \frac{\operatorname{Vol}\left(B_{T_{\alpha}^{\infty}}(\rho, r)\right)}{r^{\frac{\alpha}{\alpha-1}}(\log \log r)^{\frac{-\alpha}{\alpha-1}}}=\infty
$$

In terms of the random walk exponents, we recover the results of [CK08] that the spectral dimension of the walk is $\frac{2 \alpha}{2 \alpha-1}$, and the displacement exponent is $\frac{\alpha-1}{2 \alpha-1}$. For more detailed results, see [CK08] and [Kor17, Proposition 6].

### 6.7.2 Looptrees

By inserting deterministic loops at each vertex, we also recover the discrete looptree model that was considered more thoroughly in [BS15]. In this case, $d=R=v=1$. Moreover, $M_{r}$ is deterministically at least $\frac{1}{2} r$ in the proof of Proposition 6.4.14, so we can omit the final term in the final line in the proof of 6.4.12. We therefore deduce that for any $\varepsilon>0$, there exist constants $c, C \in(0, \infty)$ such that

$$
\begin{aligned}
\mathbf{P}\left(\operatorname{Vol}(B(\rho, r)) \geq r^{\alpha} \lambda\right) & \leq C \lambda^{\frac{1}{\alpha+1}} e^{-c \lambda^{\frac{\alpha-\varepsilon}{\alpha+1}}} \\
\mathbf{P}\left(\operatorname{Vol}(B(\rho, r)) \leq r^{\alpha} \lambda^{-1}\right) & \leq C e^{-c \lambda^{\frac{1}{\alpha}-\varepsilon}}
\end{aligned}
$$

This first result improves that of [BS15, Equation (3.41)] in which the authors obtained polynomial tail decay. This second bound agrees with that of [BS15, Equation (3.18)], which is not surprising since our approach for the volume lower
bound essentially boils down to the same method as that used for the looptree case in [BS15]. In terms of volume results, this means that

$$
\limsup _{r \rightarrow \infty} \frac{\operatorname{Vol}\left(B_{\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)}(\rho, r)\right)}{r^{\frac{\alpha}{\alpha-1}}(\log r)^{\frac{1}{\alpha-1}}}=0, \quad \liminf _{r \rightarrow \infty} \frac{\operatorname{Vol}\left(B_{\operatorname{Loop}\left(T_{\alpha}^{\infty}\right)}(\rho, r)\right)}{r^{\frac{\alpha}{\alpha-1}}(\log \log r)^{\frac{-\alpha^{2}}{\alpha-1}}}=\infty
$$

As previously shown in [BS15], we deduce that both the annealed and quenched spectral dimensions are $\frac{2 \alpha}{\alpha+1}$, and that the displacement exponent is $\frac{1}{\alpha+1}$. See [BS15] for more details.

### 6.7.3 Inserting trees

Since we have good control on volumes in trees, we could also insert a GaltonWatson tree conditioned to have $n$ leaves at each vertex of degree $n$. To establish the volume exponents in this case, note that since the number of edges of a tree is one less than the total number of vertices, we can use [Kor12, Proposition 1.6] to deduce that, if $T$ is an unconditioned Galton-Watson tree with offspring distribution $\hat{\xi}(x) \sim c x^{-\beta}$ as $x \rightarrow \infty$ for some $\beta \in(1,2]$, and $l(T)$ is its number of leaves, then (applying Lemma 6.3.2 and an LDP):

$$
\begin{aligned}
\mathbf{P}(l(T)=n,|T| \geq \lambda n) & =\sum_{p \geq \lambda n} \frac{1}{p} \mathbf{P}\left(W_{p-n}^{\prime}=n-1\right) \mathbf{P}\left(S_{p}=n\right) \\
& \leq \frac{1}{\lambda n} \sum_{p \geq \lambda n}\left(e^{-\lambda^{\beta}}+o\left(n^{\frac{-1}{\beta}}\right)\right) e^{-c p} \\
& \leq \frac{C}{\lambda n}\left(e^{-\lambda^{\beta}}+o\left(n^{\frac{-1}{\beta}}\right)\right) e^{-c \lambda n}
\end{aligned}
$$

where $W^{\prime}$ is a random walk started from zero with jump distribution $\eta(i)=\frac{p_{i+1}}{1-p_{0}}$ for $i \geq 1$, and $S_{p}$ is a sum of $p$ independent $\operatorname{Bernoulli}\left(p_{0}\right)$ random variables. Using also the asymptotic of [Kor12, Theorem 3.1(ii)] that $\mathbf{P}(l(T)=n) \sim c n^{-\left(1+\frac{1}{\beta}\right)}$ as $n \rightarrow \infty$, we deduce that

$$
\mathbf{P}(|T| \geq \lambda n \mid l(T)=n) \leq C e^{-\lambda^{\beta}}+C e^{-c \lambda n}=o\left(e^{-c \lambda}\right)
$$

Clearly also

$$
\mathbf{P}\left(|T| \leq \lambda^{-1} n \mid l(T)=n\right)=0
$$

for all $\lambda \geq 2, n \geq 1$, so we deduce that $v=1, s_{\alpha}^{v}=\alpha-1$, and $f_{\alpha}^{v}=\alpha$.
To bound $\mathbf{P}\left(\left.\operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}} \right\rvert\, l(T)=n\right)$, we first bound the quantity

$$
\mathbf{P}\left(l(T) \in[n, 2 n] \left\lvert\, \operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}}\right.\right)
$$

by decomposing along the Williams' spine, which we know has length at least
$\frac{1}{2} \lambda n^{1-\frac{1}{\beta}}$. By Proposition 6.3.5, we know that for any vertex $v$ on the Williams' spine within distance $\frac{1}{4} \lambda n^{1-\frac{1}{\beta}}$ of the root, there exists a constant $c$ such that for all $x \leq \lambda n^{\frac{1}{\beta}}, \mathbf{P}(\operatorname{deg} v \geq x) \geq c x^{-(\beta-1)}$, independently for each such $v$. Therefore, by coupling the degree of $v$ with an independent random variable $Y$ satisfying $\mathbf{P}(\operatorname{deg} v \geq x) \geq c x^{-(\beta-1)}$ for all $x \geq 0$, we get that

$$
\mathbf{E}\left[e^{-\theta(\operatorname{deg} v-2)}\right] \leq \mathbb{E}\left[e^{-\theta Y}\right]+\mathbf{P}\left(Y \geq \lambda n^{\frac{1}{\beta}}\right)
$$

where the latter term corresponds to a "worst-case" scenario on "seeing the difference" between $Y$ and $\operatorname{deg} v$. In particular, if $\theta=c n^{\frac{-1}{\beta}}$ then by Lemma 6.2.6

$$
\begin{equation*}
\mathbf{E}\left[e^{-\theta(\operatorname{deg} v-2)}\right] \leq 1-c^{\prime} n^{\frac{-(\beta-1)}{\beta}}+\lambda^{\frac{-1}{\beta-1}} n^{\frac{-(\beta-1)}{\beta}} \leq \exp \left\{-c^{\prime \prime} n^{\frac{-(\beta-1)}{\beta}}\right\}, \tag{6.11}
\end{equation*}
$$

Therefore, letting $v_{1}, v_{2}, \ldots v_{\frac{1}{4} \lambda n^{1-\frac{1}{8}}}$ denote the vertices on the Williams' spine within distance $\frac{1}{4} \lambda n^{1-\frac{1}{\beta}}$ of the root, $\left(T_{i}\right)_{i=1}^{\operatorname{deg} v_{j}-2}$ denote the subtrees emanating from all of the non-spinal offspring of vertex $v_{j}, l\left(T_{i}\right)$ denote the number of leaves in each $T_{i}$, using the asymptotic of [Kor12, Theorem 3.1(ii)] that $\mathbf{P}(l(T)=n) \sim c n^{-\left(1+\frac{1}{\beta}\right)}$ as $n \rightarrow \infty$, and then taking $\theta=n^{-1}$ and using (6.11) in the final line below we have that

$$
\begin{aligned}
\mathbf{P}\left(l(T) \leq 2 n \left\lvert\, \operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}}\right.\right) & \leq \mathbf{E}\left[\exp \left\{-\theta \sum_{i=1}^{N} l\left(T_{i}\right)\right\}\right] e^{2 \theta n} \\
& \leq \mathbf{E}\left[\exp \left\{-\theta \sum_{j=1}^{\frac{1}{4} \lambda n^{1-\frac{1}{\beta}}} \sum_{i=1}^{\operatorname{deg}\left(v_{j}\right)-2} l\left(T_{i}\right)\right\}\right] e^{2 \theta n} \\
& \leq \mathbf{E}\left[\mathbf{E}\left[e^{-\theta l\left(T_{i}\right)}\right]^{\operatorname{deg} v_{j}-2}\right]^{\frac{1}{4} \lambda n^{1-\frac{1}{\beta}}} e^{2 \theta n} \\
& \leq \mathbf{E}\left[e^{-c \theta^{\frac{1}{\beta}}\left(\operatorname{deg} v_{j}-2\right)}\right]^{\lambda n^{1-\frac{1}{\beta}}} e^{2 \theta n} \\
& \leq C e^{-c \lambda} .
\end{aligned}
$$

To recover the desired bound, we then use monotonicity to write

$$
\begin{aligned}
& \mathbf{P}\left(\left.\operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}} \right\rvert\, l(T)=n\right) \\
& \leq \mathbf{P}\left(\left.\operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}} \right\rvert\, l(T) \in[n, 2 n]\right) \\
& =\frac{\mathbf{P}\left(l(T) \in[n, 2 n] \left\lvert\, \operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}}\right.\right) \mathbf{P}\left(\operatorname{Diam}(T) \geq \lambda n^{1-\frac{1}{\beta}}\right)}{\mathbf{P}(l(T) \in[n, 2 n])} \\
& =\frac{C e^{-c \lambda} \lambda^{\frac{-1}{\beta-1}} n^{\frac{-1}{\beta}}}{n^{\frac{-1}{\beta}}} \\
& \leq C e^{-c \lambda},
\end{aligned}
$$

so that the remaining conditions of Assumption 6.0.1 are satisfied, with $R=d=$ $\frac{\beta}{\beta-1}$, so that $s_{\alpha}^{d}=s_{\alpha}^{R}=\frac{\beta(\alpha-1)}{\beta-1}$.

The volume growth exponent of $\frac{\alpha\left(s_{\alpha}^{d} \wedge 1\right)}{(\alpha-1)\left(f_{\alpha}^{v} \wedge 1\right)}$ is therefore given by

$$
d_{\alpha}^{\text {dec }}= \begin{cases}\frac{\beta \alpha}{\beta-1} & \text { if } \frac{\beta}{\beta-1}<\frac{1}{\alpha-1}, \\ \frac{\alpha}{\alpha-1} & \text { if } \frac{\beta}{\beta-1} \geq \frac{1}{\alpha-1} .\end{cases}
$$

In the first case, the exponent of $\frac{\beta}{\beta-1}$ comes from the volume growth of the inserted $\beta$-stable trees, and this is compounded by a factor of $\alpha$ coming from the effect of having lots of fragments attached to the backbone of the underlying tree $T_{\alpha}^{\infty}$. In the latter case, the inserted trees do not contain enough volume to have an effect and so we pick up a factor of $\frac{1}{\alpha-1}$ along the backbone of $T_{\alpha}^{\infty}$, and then a factor of $\alpha$ from considering all the fragments, just as if we were working directly with $T_{\alpha}^{\infty}$ as in Section 6.7.1.

The random walk exponents can be calculated using the results of Section 6.6 and are given in Table 6.1.

Note that we would expect the same results if we inserted a tree with $n$ vertices in total, rather than $n$ leaves, at a vertex of degree $n$, since the leaves asymptotically make up a constant proportion of the mass of the tree.

### 6.7.4 Outerplanar maps: inserting dissected polygons

Let $P_{n}$ be a convex polygon inscribed in the unit disc whose vertices correspond to the $n^{\text {th }}$ roots of unity. A dissection of $P_{n}$ is obtained from $P_{n}$ by inserting a collection of chords that make up distinct diagonals of $P_{n}$ : see Figure 6.3.

If $\mu$ is a critical probability measure on the set $\{0,2,3,4, \ldots\}$, we can define a Boltzmann measure on dissections of the $n$-gon, $\mathbb{P}_{n}^{\mu}$, by setting

$$
\mathbb{P}_{n}^{\mu}(D) \propto \prod_{F \in \operatorname{Faces}(D)} \mu_{\operatorname{deg} f-1}
$$



Figure 6.3: A dissection and its inscribed tree and looptree.

For convenience we will assume that the support of $\mu$ is the entirety of the set $\{0,2,3,4, \ldots\}$. The measure $\mathbb{P}_{n}^{\mu}$ is then well-defined under sensible assumptions on the tail of $\mu$.

Letting $D_{n}^{\mu}$ denote a random Boltzmann-dissection sampled according to $\mathbb{P}_{n}^{\mu}$, $D_{n}^{\mu}$ is now a natural candidate for decoration at a vertex of degree $n$ in $T_{\alpha}^{\infty}$. We will view $D_{n}^{\mu}$ as a metric space (rather than as an embedding in the plane) by giving each edge of $D_{n}^{\mu}$ length 1. To establish exponents for the diameter and two-point function of $D_{n}^{\mu}$, we will use a bijection between dissections of $P_{n}$ and trees with $n$ vertices, as illustrated in Figure 6.3. It is shown in [Kor14, Proposition 1.4] that, if $T_{n}$ is the tree obtained from $D_{n}$ in this way, then $T_{n}$ has the law of a Galton-Watson tree with offspring distribution $\mu$ and conditioned on having $n$ leaves.

If $T_{n}$ is the tree obtained from $D_{n}$ in this way, the main observation that will allow us to control the diameter and two-point function of $D_{n}^{\mu}$ is that $D_{n}^{\mu}$ looks a lot like $\operatorname{Loop}\left(T_{n}\right)$, as pointed out in [CK14, Section 4.3]. This can be seen from Figure 6.3.

In [CK14, Section 4.3], the authors prove a scaling limit result for $D_{n}^{\mu}$ by defining a correspondence between $D_{n}^{\mu}$ and $\operatorname{Loop}\left(T_{n}\right)$ to consist of all points $(a, x) \in$ $\operatorname{Loop}\left(T_{n}\right) \times D_{n}^{\mu}$ such that $a$ and $x$ are contained within a common edge in $D_{n}^{\mu}$. The authors then make the observation that if $(a, x)$ and $\left(a^{\prime}, x^{\prime}\right)$ are in correspondence, then

$$
\left|d_{D_{n}^{\mu}}\left(x, x^{\prime}\right)-d_{\operatorname{Loop}\left(T_{n}\right)}\left(a, a^{\prime}\right)\right| \leq \operatorname{Diam}\left(T_{n}\right) .
$$

This is because a geodesic $\gamma_{a, a^{\prime}}$ from $a$ to $a^{\prime}$ in $\operatorname{Loop}\left(T_{n}\right)$ is obtained by concatenating a series of subsets of the various loops that fall "between" $a$ and $a^{\prime}$, and similarly a geodesic $\Gamma_{x, x^{\prime}}$ from $x$ to $x^{\prime}$ in $D_{n}^{\mu}$ is obtained by concatenating a series of boundary segments of the faces that fall "between" $x$ and $x^{\prime}$. These loops and faces are naturally in correspondence, and such that that the contribution of a given loop
to the length of $\gamma_{a, a^{\prime}}$ differs from the contribution of the corresponding face to the length of $\Gamma_{x, x^{\prime}}$ by at most 1 (see Figure 6.3 for an illustration). Since the number of such loops/faces on a given path is bounded by $\operatorname{Diam}\left(T_{n}\right)$, we obtain the result.

We can also make a similar observation regarding resistance on $D_{n}^{\mu}$, in particular to compare $R_{\operatorname{Loop}\left(T_{n}\right)}\left(a, a^{\prime}\right)$ and $R_{D_{n}^{\mu}}\left(x, x^{\prime}\right)$ as above. The principle is similar, but now we have to take account of the fact that in $D_{n}^{\mu}$, neighbouring faces share edges, whereas in $\operatorname{Loop}\left(T_{n}\right)$, neighbouring loops only share vertices. However, since sharing entire edges rather than vertices can only reduce resistance, it should be clear by the same logic as above that

$$
R_{D_{n}^{\mu}}\left(x, x^{\prime}\right) \leq R_{\operatorname{Loop}\left(T_{n}\right)}\left(a, a^{\prime}\right)+\operatorname{Diam}\left(T_{n}\right)
$$

To prove a bound in the other direction, first let $F$ be a face that lies on the "path" from $x$ to $x^{\prime}$ in $D_{n}^{\mu}$. Let $e$ and $e^{\prime}$ be the edges of the boundary of $F$ that are respectively closest to $x$ and $x^{\prime}$ in $D_{n}^{\mu}$, in the sense that they are also on the boundary on the next face on the "path" from $x$ to $x^{\prime}$. Now consider the new graph obtained by contracting all such edges to a single point (i.e. by identifying their endpoints, or equivalently updating the edge length to zero). Call this graph $\overline{D_{n}^{\mu}} \cdot \overline{D_{n}^{\mu}}$ looks more like a looptree except that now it is possible to have more than two loops glued at the same vertex. In particular, $\overline{D_{n}^{\mu}}$ and $\operatorname{Loop}\left(T_{n}\right)$ will both have the same underlying tree structure, but the loops in $\overline{D_{n}^{\mu}}$ along with the appropriate loop segments will be shorter in $\overline{D_{n}^{\mu}}$, so that $R_{\overline{D_{n}^{\mu}}}\left(x, x^{\prime}\right) \leq R_{\operatorname{Loop}\left(T_{n}\right)}\left(a, a^{\prime}\right)$. However, by construction, the overall difference between the appropriate lengths in each loop can be at most 3 , so that

$$
R_{D_{n}^{\mu}}\left(x, x^{\prime}\right) \geq R_{\overline{D_{n}^{\mu}}}\left(x, x^{\prime}\right) \geq R_{\operatorname{Loop}\left(T_{n}\right)}\left(a, a^{\prime}\right)-3 \operatorname{Diam}\left(T_{n}\right)
$$

To control the volume of $D_{n}^{\mu}$, we also make the observation that the number of edges in $D_{n}^{\mu}$ is equal to the number of vertices of $T_{n}$. Also, since resistance and the shortest-path distance on looptrees can differ by at most a factor of 2 , we therefore have the following bounds. In what follows, we assume that $\mu$ has $\beta$-stable tails for some $\beta \in(1,2]$, in that $\mu(k) \sim k^{-(\beta-1)}$ as $k \rightarrow \infty$, or otherwise take $\beta=2$ if $\mu$ has finite variance. We can also recover analogous results in the finite variance case by
taking $\beta=2$ in what follows, even if the tails aren't precisely of the given form.

$$
\begin{aligned}
\mathbf{P}\left(\operatorname{Vol}\left(D_{n}^{\mu}\right) \geq n \lambda\right) & \leq \mathbf{P}\left(\left|T_{n}\right| \geq n \lambda \mid l\left(T_{n}\right)=n\right) \\
\mathbf{P}\left(\operatorname{Vol}\left(D_{n}^{\mu}\right) \leq n \lambda^{-1}\right) & \leq \mathbf{P}\left(\left|T_{n}\right| \leq n \lambda^{-1} \mid l\left(T_{n}\right)=n\right) \\
\mathbf{P}\left(\operatorname{Diam}\left(D_{n}^{\mu}\right) \geq n^{\frac{1}{\beta}} \lambda\right) & \leq \mathbf{P}\left(\left.\operatorname{Diam}\left(\operatorname{Loop}\left(T_{n}\right)\right)+\operatorname{Diam}\left(T_{n}\right) \geq n^{\frac{1}{\beta}} \lambda \right\rvert\, l\left(T_{n}\right)=n\right) \\
\mathbf{P}\left(\operatorname{Diam}\left(D_{n}^{\mu}\right) \leq n^{\frac{1}{\beta}} \lambda^{-1}\right) & \leq \mathbf{P}\left(\left.\operatorname{Diam}\left(\operatorname{Loop}\left(T_{n}\right)\right)-\operatorname{Diam}\left(T_{n}\right) \leq n^{\frac{1}{\beta}} \lambda^{-1} \right\rvert\, l\left(T_{n}\right)=n\right) \\
\mathbf{P}\left(\operatorname{Diam}_{r e s}\left(D_{n}^{\mu}\right) \geq n^{\frac{1}{\beta}} \lambda\right) & \leq \mathbf{P}\left(\left.\operatorname{Diam}\left(\operatorname{Loop}\left(T_{n}\right)\right)+\operatorname{Diam}\left(T_{n}\right) \geq n^{\frac{1}{\beta}} \lambda \right\rvert\, l\left(T_{n}\right)=n\right) \\
\mathbf{P}\left(\operatorname{Diam}\left(D_{n}^{\mu}\right) \leq \frac{1}{2} n^{\frac{1}{\beta}} \lambda^{-1}\right) & \leq \mathbf{P}\left(\left.\operatorname{Diam}\left(\operatorname{Loop}\left(T_{n}\right)\right)-\operatorname{Diam}\left(T_{n}\right) \leq n^{\frac{1}{\beta}} \lambda^{-1} \right\rvert\, l\left(T_{n}\right)=n\right) .
\end{aligned}
$$

In fact the control on $\operatorname{Diam}\left(T_{n}\right)$ is not strong enough to give stretched exponential decay, with

$$
\mathbf{P}\left(\left.\operatorname{Diam}\left(\operatorname{Loop}\left(T_{n}\right)\right) \geq n^{\frac{1}{\beta}} \lambda^{-1} \right\rvert\, l\left(T_{n}\right)=n\right) \asymp n^{-\beta}
$$

in the case of stable tails, although the decay is at least $o\left(\lambda^{-d(\alpha-1)}\right.$ in this case. We will only obtain stretched exponential decay when $\mu$ also has stretched exponential tails. In this case $d=R=2$, and $v=1$, so we get the results in the table. We also give conjectural results for the case of $\beta$-stable tails, which would hold if we could prove Conjecture 6.8.2.

We deduce that $d=R=\beta$, and $v=1$, so that

$$
s_{\alpha}^{d}=s_{\alpha}^{R}=\beta(\alpha-1), \quad f_{\alpha}^{v}=\alpha .
$$

which give the results in Table 6.1 when we make the appropriate substitutions.

### 6.7.5 Critical Erdös-Rényi

Motivated by the example of the IIC on the UIHPT, we can also consider a model where we insert a connected component of an Erdös-Rényi graph in the critical window, by which we mean the graph $G(n, p)$ such that $p=\frac{1}{n}+\frac{t}{n^{\frac{4}{3}}}$ for some $t>0$ (see e.g. [Gol20, Section 2] for an introduction to this model and the critical window).

It is well-known that, at criticality, a connected component of $G(n, p)$ looks roughly like a critical Galton-Watson tree with an $O(1)$ number of "surplus" edges. Heuristically, this can be explained as follows: let $\mathcal{C}$ denote a connected component of $G(n, p)$, and let $v_{0} \in \mathcal{C}$. We consider the "exploration tree" rooted at $v_{0}$, constructed as follows: first let $v_{0}$ be the root. Then consider all vertices connected to $v_{0}$ and let these form the next generation of the tree. The number of such vertices is Binomial $(n-1, p)$; denote this number $M_{1}$. Then, given a vertex $v_{1}$ in generation one, we can repeat this process to find all the new neighbours of $v_{1}$, and define these
to be the offspring of $v_{1}$ : the number of offspring is therefore $\operatorname{Binomial}\left(n-1-M_{1}, p\right)$. We can repeat this process inductively to explore the cluster in a depth-first way: this will produce a spanning tree of the cluster, and as long as the total number of vertices explored remains small compared to $n$, it is fairly accurate to approximate the offspring distribution of this tree by a $\operatorname{Binomial}(n-1, p)$ distribution. At any stage, there is a small probability that a given vertex $v$ also has some neighbours that correspond to vertices that have already been discovered, so that in order to reconstruct $\mathcal{C}$ from its spanning tree we must add a few extra edges.

To avoid ambiguities, for this construction we will fix $t>0$, set $p_{n}=\frac{1}{n^{\frac{3}{2}}}+\frac{t}{n^{2}}$ and let $G_{n}$ have the law of the largest component of $G\left(n^{\frac{3}{2}}, p_{n}\right)$ conditioned on having $n$ vertices (by [Ald97, Corollary 2], $n$ is therefore on the natural scale to be the size of the largest cluster of $\left.G\left(n^{\frac{3}{2}}, p_{n}\right)\right)$. Using the tree viewpoint, we can relate the volume, two-point function and diameter of critical connected Erdös-Rényi graphs to give the following results.

Proposition 6.7.1. Take $G_{n}$ as above. Then there exist constants $c, c^{\prime} \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$ :
(i) $\mathbf{P}\left(\operatorname{Diam} G_{n} \geq \lambda \sqrt{n}\right) \leq e^{-c \lambda^{2}}$.
(ii) $\mathbf{P}\left(\operatorname{Diam}_{r e s} G_{n} \geq \lambda \sqrt{n}\right) \leq e^{-c \lambda^{2}}$.
(iii) $\mathbf{P}\left(d^{U}\left(G_{n}\right) \geq \sqrt{n}\right) \geq c>0$.
(iv) $\mathbf{P}\left(R^{U}\left(G_{n}\right) \geq \sqrt{n}\right) \geq c>0$.
(v) $\mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq \lambda n\right) \leq \mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq \lambda+n\right) \leq C e^{-c \lambda^{\frac{1}{3}}}$.
(vi) $\mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq n-1\right)=1$.

Proof. We just sketch the proof. For part (i) the result follows by repeating the proof of the height bound of [AB19, Theorem 1.1] (it does not quite follow directly since our exploration tree is not quite a critical Galton-Watson tree, but we are close enough that the proof still works, and being slightly subcritical is intuitively helpful for this bound anyway since this corresponds to more of a condensation regime). This also give part (ii) since the resistance is upper bounded by the graph distance. For part (iii), we first condition on having zero surplus, which has strictly positive probability in the limit. In this case the exploration tree is again close to a critical Galton-Watson tree (its law is not tilted, since if the surplus is zero we do not have to count the options for where we can add extra edges), and the resistance is equal to the graph distance. The result then follows since the offspring distribution of the tree is close to Poisson(1), which corresponds to a uniform labelled Galton-Watson
tree, and in this case the it is known that on rescaling by $\sqrt{n}$ the two-point function satisfies $\mathbf{P}\left(d^{U}\left(T_{n}\right)=\lfloor\lambda \sqrt{n}\rfloor\right) \sim \frac{\lambda}{\sqrt{2 n}} e^{-\frac{\lambda^{2}}{2}}$ as $n \rightarrow \infty$ (e.g. see [FDP06]).

To control the volume (i.e. number of edges) we control the surplus. To do this, note that given a vertex $v_{k}$ in the exploration tree, there can only be extra edges joining within the same generation, or to an adjacent generation (otherwise this disrupts the generation structure of the tree). Therefore, if $v_{k}$ is a vertex of the tree and $D_{m}$ denotes the $m^{t h}$ generation of the tree, we can introduce a Binomial $\left(\left|D_{\left|v_{k}\right|}\right|+\left|D_{\left|v_{k}\right|-1}\right|, p_{n}\right)$ random variable which we denote $S_{v_{k}}$, and the total surplus is upper bounded by summing these over all vertices in the tree. Then, again after taking care of the necessary details that our tree is not quite a critical GaltonWatson tree, we have by [ABDJ13, Theorem 1.1] that $\mathbf{P}\left(\sup _{m}\left|D_{m}\right| \geq \sqrt{n} \lambda^{p}\right) \leq \lambda^{2 p}$. Then, $\mathbf{P}\left(\operatorname{Binomial}\left(2 n^{\frac{3}{2}} \lambda^{p}, \frac{1}{n^{\frac{3}{2}}}+\frac{t}{n^{2}}\right) \geq \lambda\right) \leq c e^{-c \lambda^{1-p}}$. We take $p=\frac{1}{3}$. On the complement of these events, the surplus is less than $\lambda$, so $\operatorname{Vol}\left(G_{n}\right) \leq n+\lambda$, which gives part (iv).

Remark 6.7.2. The rescaled two point function for a uniform critical random tree in fact converges to a Rayleigh distribution. This is the same limit as that appearing in Pearson's problem [Pea05] on p.1.

We therefore deduce that $d=R=2, v=1$, and the fundamental exponents take the following values:

$$
f_{\alpha}^{v}=\alpha, \quad s_{\alpha}^{d}=s_{\alpha}^{R}=2(\alpha-1)
$$

Rather than forcing all vertices of the inserted critical graph to be boundary vertices, we could also consider inserting an independent copy of $G\left(n^{\beta}\right)$ at a vertex of degree $n$, for some $\beta \geq 1$ and uniformly choosing $n$ distinct vertices to be boundary vertices. In this case, it follows from Proposition 6.7.1 that $d=r=\frac{2}{\beta}$ and $v=\beta$, so that

$$
f_{\alpha}^{v}=\frac{\alpha}{\beta}, \quad s_{\alpha}^{d}=s_{\alpha}^{R}=\frac{2(\alpha-1)}{\beta} .
$$

Note that if $\beta>2$, the local geometry always dominates and we never see the tree geometry.

Remark 6.7.3. We have not written the details, but one would expect the same result on taking a critical configuration model in place of the Erdös-Rényi graph. We also anticipate that we could insert a $\beta$-stable graph, as considered in [GHS18, CKG20] and we would get the same results as for inserting $\beta$-stable trees by making similar arguments to the Erdös-Rényi example considered above.

### 6.7.6 Sierpinski triangle

In order to gain insight into the effect of inserting fractal-type graphs, one could also consider the exponents obtained when inserting a Sierpinski triangle. Letting $T_{n}^{\Delta}$ be the $n^{\text {th }}$ level approximation to the Sierpinski triangle as defined in [Bar98, Section 2] (also depicted in Figure 6.4), it is always the case that the boundary length of $T_{n}^{\Delta}$ is equal to $3 \cdot 2^{n}$ : therefore, if $m \in\left(3 \cdot 2^{n}, 3 \cdot 2^{n+1}\right)$, one would have to do appropriate "surgery" to the graph $G_{n+1}$ in order to define an appropriate version of "the Sierpinski triangle with boundary length $m$ ". We will not do this is explicitly here, and just give the appropriate volume bounds for the level $n$ approximation $T_{n}^{\Delta}$.


Figure 6.4: Sierpinski triangles
We can use the self-similarity of the Sierpinski triangle to study resistances, volumes and diameters of $T_{n}^{\Delta}$ as well, and give the (deterministic) results for these in Table 6.2. The effective resistance bound can be obtained using the $\Delta-Y$ transformation (e.g. [LP16, 2.3.III]): see also [Bar98, Section 2] for more explicit computations. Here we assume that 1 and 2 are the labels of two distinct extremal corners of $T_{n}^{\Delta}$.

| Boundary | Volume | $d(1,2)$ | $R_{\text {eff }}(1,2)$ | Diameter | Resistance diam |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $3 \cdot 2^{n}$ | $3^{n+1}$ | $2^{n}$ | $\frac{2}{3}\left(\frac{5}{3}\right)^{n}$ | $c_{3} 2^{n}$ | $c_{4}\left(\frac{5}{3}\right)^{n}$ |
| $m$ | $c_{1} m^{\log 3}$ | $\frac{1}{3} m$ | $c_{2} m^{\frac{\log 5-1053}{\log 2}} \log 2$ | $c_{3}^{\prime} m$ | $c_{4}^{\prime} m^{\frac{\log 5-\log 3}{\log 2}}$ |

Table 6.2: Volumes and distances in the $n^{\text {th }}$ level Sierpinski gasket, in terms of $m$ and $n$, where $m$ is the boundary length.

We can also (crudely) bound the diameters using the bound for the distances between extremal corners in the table: since to go from any point $x \in T_{n}^{\Delta}$ to any other point $y \in T_{n}^{\Delta}$ we have to pass through at most two triangles at a specific level $m \leq n$ (once on the "way up" from $x$, then once on the "way down" to $y$ ), we get that

$$
\operatorname{Diam}\left(T_{n}^{\Delta}\right) \leq 2 \frac{2}{3} \sum_{m=0}^{n} 2^{m} \leq c_{3} 2^{n}, \quad \operatorname{Diam}_{\mathrm{res}}\left(T_{n}^{\Delta}\right) \leq 2 \frac{2}{3} \sum_{m=0}^{n}\left(\frac{5}{3}\right)^{m} \leq c_{4}\left(\frac{5}{3}\right)^{n}
$$

Since the graphs are deterministic, the functions giving the polynomial tail decay in Asumption 6.0 .1 are all zero for sufficiently large $\lambda$, so we have that $d=1$, $R=\frac{\log 2}{\log 5-\log 3}, v=\frac{\log 3}{\log 2}$, and obtain the following exponents:

$$
s_{\alpha}^{d}=\alpha-1, \quad s_{\alpha}^{R}=\frac{(\alpha-1) \log 2}{\log 5-\log 3}, \quad f_{\alpha}^{v}=\frac{\alpha \log 2}{\log 3}
$$

We then obtain the results in Table 6.1 by substituting into the expressions given in Section 6.6.

We also note that the logarithmic terms appearing in the statements of Propositions 6.3.9, 6.4.12 and 6.5.1 would not need to be included, since the inserted graphs are deterministic.

### 6.7.7 Supercritical Erdös-Rényi or the complete graph

As well as critical Erdös-Rényi, one could also insert supercritical Erdös-Rényi graphs, as well as the complete graph. It is well-known that, if $G(n, p)$ is the Erdös-Rényi graph on $n$ vertices and $p=\frac{\lambda}{n}$ for some $\lambda>1$, then the largest connected component has order $n$ vertices, order $n^{2}$ edges, and diameter of order $\log n$ (e.g. see [RW10, Theorem 1.1]). The complete graph on $n$ vertices similarly has $n$ vertices, order $n^{2}$ edges, diameter 1 and resistance diameter of order $\frac{1}{n}$.

To fit these models into the framework of this chapter, we therefore effectively want to take $d=R=\infty$, and $v=2$. Some care is needed to check that we can really do this, but we can dominate $\log n$ by $n^{\delta}$ for some sufficiently small $\delta$, and this also gives us very good control on the tail decay required for Assumption 6.0.1 (D). Additionally, in sufficiently supercritical regimes resistance will actively stochastically decrease with $n$ which is clearly different to the assumptions of this chapter; however, since "most" vertices are of low degree it is clear that asymptotically resistance should still add up in the following way.

To define this model in the supercritical case, we let $G_{n}$ be the largest connected component of $G(C n, p)$, conditioned to have $n$ vertices, where $C \geq 1$ is an appropriately chosen constant. Alternatively, we can let $G_{n}$ be the complete graph on $n$ vertices.

Setting $d=R=\infty$ and $v=2$ therefore gives the results in the final line of Table 6.1.

### 6.8 Relaxing the assumptions

As remarked in the introduction, we believe that the stretched exponential decay in items ( $\mathrm{D)} \mathrm{and} \mathrm{(R)} \mathrm{of} \mathrm{Assumption} \mathrm{6.0.1} \mathrm{should} \mathrm{not} \mathrm{really} \mathrm{be} \mathrm{necessary} \mathrm{and} \mathrm{are}$ endeavouring to weaken this assumption. The only place this assumption is required
is to prove the decorated height bound in Proposition 6.3.9. In this section we present some heuristics for proving that

$$
\mathbf{P}\left(\operatorname{Height}^{\mathrm{dec}}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq n \log n\right) \leq c n^{-\left(\frac{f_{\alpha}^{d}}{\alpha} \wedge \frac{1}{\alpha-1}\right)}
$$

under the weaker assumption given below.
Firstly, recall that $f_{\alpha}^{d}=d \alpha \wedge m_{d}, s_{\alpha}^{d}=d(\alpha-1) \wedge m_{d}$, and we proved a complementary lower bound in Proposition 6.3.11, under the weaker assumption ( $D^{\prime}$ ), that

$$
\mathbf{P}\left(\text { Height }^{\operatorname{dec}}\left(\mathcal{T}^{\mathrm{dec}}\right) \geq n \log n\right) \geq c n^{-\left(\frac{s_{\alpha}^{d} \wedge 1}{\alpha-1}\right)}
$$

Note that

$$
\begin{aligned}
\frac{s_{\alpha}^{d} \wedge 1}{\alpha-1} & =d \wedge \frac{1}{\alpha-1} \wedge \frac{m_{d}}{\alpha-1} \\
\frac{f_{\alpha}^{d}}{\alpha} \wedge \frac{1}{\alpha-1}= & =d \wedge \frac{1}{\alpha-1} \wedge \frac{m_{d}}{\alpha}
\end{aligned}
$$

Therefore, the two exponents are always equal unless $m_{d}<d \alpha \wedge \frac{\alpha}{\alpha-1}$, which does not occur in the examples considered in Section 6.7, at least, and is certainly a weaker assumption than the stretched exponential decay in Assumption 6.0.1. Note that the case when $m_{d}<d \alpha \wedge \frac{\alpha}{\alpha-1}$ corresponds to the case where there is "more randomness" coming from the inserted graphs than the underlying tree structure, and therefore we would not expect to be able to get as much insight from the underlying Galton-Watson tree anyway.

We believe it should be possible to establish the tail decay under the following assumption (this was also given as Assumption 6.0.4 in the introduction to this chapter, but we restate it here for convenience).

Assumption 6.8.1 (same as Assumption 6.0.4).
$\left(\boldsymbol{D}^{\prime \prime}\right)$ Metric growth. There exists $d \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(d_{n}^{U} \geq n^{\frac{1}{d}}\right) \geq c>0, \quad \mathbf{P}\left(d_{n}^{U} \geq \lambda n^{\frac{1}{d}}\right)=O\left(\lambda^{-d(\alpha-1+\varepsilon)}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{d}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{d}} \leq \mathbf{P}\left(d_{n}^{U} \geq \lambda n^{\frac{1}{d}}\right) \leq C \lambda^{-m_{d}}
$$

( $\left.\boldsymbol{R}^{\prime \prime}\right)$ Resistance growth. There exists $R \geq 1$ such that EITHER:
(i) There exists constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(R_{n}^{U} \geq n^{\frac{1}{R}}\right) \geq c>0, \quad \mathbf{P}\left(R_{n}^{U} \geq \lambda n^{\frac{1}{R}}\right)=O\left(\lambda^{-R(\alpha-1+\varepsilon)}\right)
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{R}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{R}} \leq \mathbf{P}\left(R_{n}^{U} \geq \lambda n^{\frac{1}{R}}\right) \leq C \lambda^{-m_{R}} .
$$

( $V^{\prime \prime}$ ) Volume growth. There exists $v \geq \alpha$ such that EITHER:
(i) There exist constants $c, C \in(0, \infty)$ and $\varepsilon>0$ such that

$$
\mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq n^{v}\right) \geq c>0, \quad \mathbf{P}\left(\operatorname{Vol}_{n}\left(G_{n}\right) \geq \lambda n^{v}\right)=O\left(\lambda \frac{\left.\frac{-(\alpha-1+\varepsilon)}{v}\right)}{}\right.
$$

as $\lambda \rightarrow \infty$, uniformly in $n \geq 1$. OR:
(ii) There exists $m_{v}>0$ and constants $c, C \in(0, \infty)$ such that for all $n \geq 1, \lambda \geq 1$,

$$
c \lambda^{-m_{v}} \leq \mathbf{P}\left(\operatorname{Vol}\left(G_{n}\right) \geq \lambda n^{v}\right) \leq C \lambda^{-m_{v}} .
$$

Before outlining the heuristics for proving the result under this assumption, we briefly recall the setup of Section 6.3.

Let $T$ a Galton-Watson tree as in Section 6.3, and let $\mathcal{T}^{\text {dec }}$ be the rooted decorated tree obtained by replacing each vertex with an independent copy of $G$ with given boundary size, and fusing along the edges of $T$, exactly as described for $T_{\alpha}^{\infty}$ in earlier sections. Given such a construction, we define the decorated height of $\mathcal{T}_{\alpha}^{\text {dec by }}$

$$
\text { Height }^{\text {dec }}\left(\mathcal{T}_{\alpha}^{\text {dec }}\right)=\sup _{x \in \mathcal{T}_{\alpha}^{\text {dec }}} d_{\alpha}^{\text {dec }}\left(\rho_{\alpha}^{\text {dec }}, x\right)
$$

The path in $\mathcal{T}_{\alpha}^{\text {dec }}$ joining $\rho_{\alpha}^{\text {dec }}$ to the point achieving maximal decorated height corresponds in a natural way to a path in $T$ joining $\rho$ to a leaf (if this point is not unique, we will take the leftmost path). Analogously with the notation above we call this the decorated spine and denote this by $s_{1}^{\text {dec }}, \ldots, s_{H}^{\text {dec }}$, where $H^{\text {dec }}$ denotes the length of this decorated spine. We also let $\xi_{n}^{\text {dec }}$ denote the number of offspring of $s_{n}^{\text {dec }}$.

Note in particular that $H^{\text {dec }}$ gives the length of the decorated spine in the underlying tree, rather than the length with respect to the decorated metric, so $H^{\mathrm{dec}} \leq H$.

We provide heuristics for the following result (in fact we also think that the logarithmic correction should not be necessary in the second case below, but this is
a secondary issue).
Conjecture 6.8.2. There exists a constant $c \in(0, \infty)$ such that, if $f_{\alpha}^{d}<\frac{\alpha}{\alpha-1}$, then

$$
\mathbf{P}\left(\text { Height }^{d e c}\left(\mathcal{T}^{d e c}\right) \geq n\right) \leq c n^{-\left(\frac{f_{\alpha}^{d}}{\alpha} \wedge \frac{1}{\alpha-1}\right)}=c n^{\frac{-f_{\alpha}^{d}}{\alpha}}
$$

If instead $f_{\alpha}^{d} \geq \frac{\alpha}{\alpha-1}$, then

$$
\mathbf{P}\left(\text { Height }^{d e c}\left(\mathcal{T}^{d e c}\right) \geq n(\log n)^{\frac{\alpha}{f_{\alpha}^{d}}}\right) \leq c n^{-\left(\frac{f_{\alpha}^{d}}{\alpha} \wedge \frac{1}{\alpha-1}\right)}=n^{\frac{-1}{\alpha-1}}
$$

The following lemma can be proved rigorously, similar to the bound for the supremal volume in Proposition 6.3.14.

Lemma 6.8.3. There exists a constant $c<\infty$ such that

$$
\mathbf{P}\left(\sup _{v \in T} \operatorname{Diam} G(v) \geq x\right) \leq c x^{-\frac{f_{\alpha}^{d}}{\alpha}}
$$

To prove Conjecture 6.8.2, we would like to condition first on the height of the $T$, which should be comparable to the length of the decorated spine (and is an upper bound for it regardless), and then understand the diameter of the graphs inserted at typical vertices on this spine.

The proof of Lemma 6.8.3 relies on the fact that, for a typical (e.g. uniform) vertex $v \in T, \mathbf{P}(\operatorname{Diam} G(v) \geq x) \approx c x^{-f_{\alpha}^{d}}$. Therefore, if the height of the underlying tree is $h$, we expect the volume of the tree to be approximately $h^{\frac{\alpha}{\alpha-1}}$, and $\sup _{v \in T} \operatorname{Diam} G(v)$ will be approximately $h^{\frac{\alpha}{f_{\alpha}^{d}(\alpha-1)}}$, by a union bound.

On the other hand, if $v$ is a typical (e.g. uniform) vertex on the decorated spine, and $\mathbf{P}(\operatorname{Diam} G(v) \geq x) \approx c x^{-\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}}$, then again we expect the maximal diameter of a graph on this decorated spine to be roughly of order $h^{\frac{\alpha}{f_{\alpha}^{d}(\alpha-1)}}$, again using a union bound as an approximation (though of course this is not a lower bound). If the tail decay was heavier than $\mathrm{O}\left(x^{-\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}}\right)$, we would correspondingly expect the maximal diameter on the decorated spine to also be of higher order, which would contradict the fact that it is upper bounded by $\sup _{v \in T} \operatorname{Diam} G(v)$.

Moreover, just as the size-biased bound for the offspring distribution on the Williams' spine only holds up to a point depending on the length of the Williams' spine (cf Proposition 6.3.5), we would also expect that the bound $\mathbf{P}(\operatorname{Diam} G(v) \geq x) \approx$ $c x^{-\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}}$ for a typical vertex $v$ on the decorated spine also only holds up to the point where $x$ is of order $h^{\frac{\alpha}{f_{\alpha}^{d}(\alpha-1)}}$. Otherwise, the same union bound heuristics would give that

$$
\mathbf{P}\left(\left.\sup _{v \in T} \operatorname{Diam} G(v) \geq h^{\frac{\alpha}{f_{\alpha}^{d}(\alpha-1)}} \lambda \right\rvert\, H=h\right) \approx c \lambda^{-\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}}
$$

However, a rigorous application of Lemma 6.8.3 along with results of [Kor17] closely connecting the height and volume of a discrete Galton-Watson tree gives that

$$
\mathbf{P}\left(\left.\sup _{v \in T} \operatorname{Diam} G(v) \geq h^{\frac{\alpha}{f_{\alpha}^{d}(\alpha-1)}} \lambda \right\rvert\, H=h\right) \leq c \lambda^{-f_{\alpha}^{d}} .
$$

It is difficult to prove a precise bound for $\mathbf{P}(\operatorname{Diam} G(v) \geq x)$ for a typical vertex $v$ on the decorated spine since the degrees of vertices on the decorated spine are not independent. In any case, what we would really need to prove is the following conjecture. The point is that, assuming $\mathbf{P}\left(\operatorname{Diam} G\left(s_{i}^{\mathrm{dec}}\right) \geq x\right) \approx x^{\frac{-f_{\alpha}^{d}(\alpha-1)}{\alpha}}$ in some appropriate sense for a typical spinal vertex $s_{i}^{\text {dec }}$, then conditional on $H=h$ the decorated diameter should be approximately $h\left(\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha} \wedge 1\right)^{-1}$.

Conjecture 6.8.4. (i) If $\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}<1$ :

$$
\mathbf{P}\left(\left.\sum_{i=1}^{H^{d e c}} \operatorname{Diam} G\left(s_{i}^{d e c}\right) \geq h^{\left(\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha} \wedge 1\right)^{-1}} \lambda \right\rvert\, H=h\right) \leq c \lambda^{\frac{-f_{\alpha}^{d}}{\alpha}}
$$

(ii) If $\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}=1$ :

$$
\mathbf{P}\left(\left.\sum_{i=1}^{H^{d e c}} \operatorname{Diam} G\left(s_{i}^{d e c}\right) \geq \lambda h^{\left(\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha} \wedge 1\right)^{-1}} \log h \right\rvert\, H=h\right) \leq c \lambda^{\frac{-f_{\alpha}^{d}}{\alpha}} .
$$

(iii) If instead $\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha}>1$ :

$$
\mathbf{P}\left(\left.\sum_{i=1}^{H^{d e c}} \operatorname{Diam} G\left(s_{i}^{d e c}\right) \geq h^{\left(\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha} \wedge 1\right)^{-1}} \lambda \right\rvert\, H=h\right) \leq c \lambda^{\frac{-f_{\alpha}^{d}}{\alpha}} \log \lambda .
$$

Given this, Conjecture 6.8.2 would follow from applying Lemma 6.2.8 with $\beta=\frac{1}{\alpha-1}, z=\left(\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha} \wedge 1\right)^{-1}$ and $m=\frac{f_{\alpha}^{d}}{\alpha}$ (carrying the logarithmic term through the computation).

In order to prove Conjecture 6.8.4, we can first condition on the event that

$$
\left\{\nexists i \leq h: \operatorname{Diam} G\left(s_{i}^{\mathrm{dec}}\right) \geq h^{\left(\frac{f_{\alpha}^{d}(\alpha-1)}{\alpha} \wedge 1\right)^{-1}} \lambda^{\varepsilon}\right\}
$$

using [Kor17, Theorem 2] and Lemma 6.8.3 above (this step is rigorous).
Case $(i)$ corresponds to the case where the behaviour of the given sum is dominated by its largest term. In cases $(i)$ and (ii), although the terms of the sum are not quite independent, we believe (and have informal arguments supporting this)
that they can be coupled with and then stochastically dominated by independent random variables with the same tails, so that the result would follow from Lemma 6.2.2 (in fact we would get stretched exponential decay in $\lambda$ ).

One way to prove case (iii) would be to stochastically dominate it by case (ii), which just has the cost of the extra log term. To do it directly, we would need a better bound on the marginals of $\mathbf{P}\left(\operatorname{Diam} G\left(s_{i}^{\mathrm{dec}}\right) \geq x\right)$. By comparison with Lemma 6.8.3, we hope that decay of $x^{\frac{-f_{\alpha}^{d}}{\alpha}}$ would be achievable, although this would not be uniform in $h$, so the argument could be quite subtle. In this case, the stated result then follows from Markov's inequality: firstly we can compute that

$$
\mathbf{E}\left[\left(\operatorname{Diam} G\left(s_{i}^{\mathrm{dec}}\right)\right)^{p} \mid \sup _{i} \operatorname{Diam} G\left(s_{i}^{\mathrm{dec}}\right) \leq h \lambda^{\varepsilon}\right] \leq c \log h+\varepsilon \log \lambda
$$

and then Hölder's inequality with $p=\frac{f_{\alpha}^{d}}{\alpha}$ gives that $\mathbf{E}\left[\left(\sum_{i=1}^{H^{\text {dec }}} \operatorname{Diam} G\left(s_{i}^{\text {dec }}\right)\right)^{p}\right] \leq$ $h^{p}(\log h+\varepsilon \log \lambda)$.

However, it seems that a result of the form $\mathbf{P}\left(\operatorname{Diam} G\left(s_{i}^{\mathrm{dec}}\right) \geq x\right) \leq x^{\frac{-f_{\alpha}^{d}}{\alpha}}$ would not be uniform in $h$, which causes some difficulties. It seems that it may be possible to get a sufficient result by using the fact that we can both bound the tails for a typical term of the form Diam $G\left(s_{i}^{\text {dec }}\right)$ in terms of the tails for the decorated height, and also bound the tails for the decorated height in terms of the tails for a typical term of the form Diam $G\left(s_{i}^{\mathrm{dec}}\right)$. We can therefore start with a "bad" bound for one of these tails, and use these relationships to iteratively update this to a better bound, and repeat the process as many times as we like. Making this into a rigorous algorithm is currently work in progress.

It is also possible to get other bounds on the exponent for example using that the decorated height is upper bounded by $\sum_{i=0}^{H} \sup _{v \in G_{i}} \operatorname{Diam} G(v)$, where $G_{i}$ is generation $i$ in the tree, but we do not think this gives the right exponent so have not pursued this here.

## Chapter 7

## Outlook

In this chapter we comment briefly on future research directions leading on from the results of this thesis.

### 7.1 FIN diffusions on stable trees: trapping at nodes

Although this is not a planar graph, in the model of Chapter 6 one could also imagine inserting a copy of the complete graph $K_{n}$ at a vertex of degree $n$. By emulating the proof of the usual stable tree invariance principle [Duq03, Theorem 3.1], one imagines that it should be possible to show that, if $T_{n}$ is a critical Galton-Watson tree conditioned to have $n$ vertices, $T_{n}^{\text {com }}$ is formed from $T_{n}$ by inserting complete graphs at every vertex as described in Chapter $6, d_{n}$ is the graph distance on $T_{n}^{\mathrm{com}}$, and $\hat{\mu}_{n}(v)$ is the measure defined on $T_{n}$ by $\hat{\mu}_{n}(v)=\operatorname{deg}(v)$ for all $v \in T_{n}^{\text {com }}$, then, up to constants,

$$
\left(T_{n}, n^{1-\frac{1}{\alpha}} d_{n}, n^{\frac{-1}{2 \alpha}} \mu_{n}\right) \xrightarrow{(d)}\left(\mathcal{T}_{\alpha}, d, \hat{\mu}\right)
$$

in the Gromov-Hausdorff-Prohorov topology as $n \rightarrow \infty$, where $\hat{\mu}(t)=\Delta_{t}^{2}$ for $t \in$ $[0,1]$, and $\Delta_{1}=X_{t}^{\mathrm{exc}}-X_{t^{-}}^{\mathrm{exc}}$.

The measure $\hat{\mu}$ is singular with respect to the uniform volume measure $\mu$ on stable trees, which is supported on the leaves, and is instead reminiscent of that associated with Fontes-Isopi-Newman (FIN) diffusions. FIN diffusions were first introduced in [FIN02] in one dimension and provide a model for random walks subject to a polynomial trapping mechanism controlled by an appropriate FIN measure. FIN diffusions have since been studied on a wider class of graphs [CHK19] and the natural extension to trees involves trapping in the leaves. Mathematically this is achieved by weighting the measure $\mu$ by a random trapping factor, so that the resulting FIN measure is absolutely continuous with respect to $\mu$. This kind of diffusion arises as the scaling limit of the Bouchaud trap model on random trees, for example [CHK17]. In the model suggested above, one would instead get trapping
at hubs, which is qualitatively different behaviour. Moreover, it should be possible to adjust the measure to end up with the degree to some power other than 2 and obtain a wider class of diffusions.

The model of inserting the complete graph above behaves differently with respect to resistance since the resistance across $K_{n}$ behaves asymptotically like $\frac{1}{n}$ as $n \rightarrow \infty$, so the limiting resistance metric would be again have a density with respect to $d$ depending on the hub sizes. This would essentially just counteract some of the effect of the higher volume measure at hubs so it is perhaps more natural to just directly consider trees endowed with different volume measures but the usual notion of resistance. In this case we would expect there to be a balance between time spent at leaves and at hubs, similarly to the discrete model in Chapter 6, so that the limiting measure would instead be of the form $A \mu+B \hat{\mu}$.

One wonders whether there are natural trapping models that lead to this kind of behaviour and, if so, it would then be of interest to introduce this model more thoroughly and establish some of its basic properties.

### 7.2 Random walks on decorated Galton-Watson trees: the supercritical case

The work of this thesis concerns critical structures, but one could define similar decorated models on supercritical Galton-Watson trees. On supercritical structures, random walks are commonly superdiffusive and can display a rich array of behaviour; in particular, they sometimes have limiting speed: that is, $\frac{d\left(0, X_{n}\right)}{n}$ converges to a positive limit almost surely. For example, when considering a random walk on a supercritical Galton-Watson tree with mean $m>1$ one can add a bias parameter $\lambda$ which pushes the random walk towards the root, and it was shown in [LPP96] that there exists an explicit $\lambda_{c} \in(0, m)$ such that the speed is non-zero if and only if $\lambda \in\left(\lambda_{c}, m\right)$, and that the speed is unimodal in this region. At first it seems counterintuitive that adding a bias can initially increase the speed, but this is actually due to a trapping effect in the dead ends of the tree, which initially becomes less pronounced when the bias is increased.

It seems plausible that the introduction of extra decoration could similarly have the potential to increase or decrease the speed of a random walk. To construct a supercritical tree conditioned to survive one must replace the backbone of Kesten's tree with an entire supercritical tree with a related offspring distribution, and then attach smaller Galton-Watson trees ("dangling ends") to this core similarly to Kesten's critical construction. Adding decoration will therefore change the scaling exponents for the times spent in the dangling ends and the time spent moving through the core; it seems plausible that in some intermediate regimes, it could
have a greater effect on the exponent of the core so that the random walk spends proportionally more time moving through the core of the tree and the speed might therefore be positive.

### 7.3 Random walk on stable maps

As alluded to in the introduction, stable maps are a class of maps obtained when the face weights $\left(q_{k}\right)_{k \geq 1}$ are appropriately "critical" and satisfy $q_{k} \sim c k^{-\alpha}$ for some $\alpha \in(1,2)$ (see [Cur, Section 5.2] for more details of this model). They are of particular interest since they describe a range of statistical mechanics models on random planar maps. In the dense phase $\alpha \in\left(1, \frac{3}{2}\right)$, stable maps have been shown to have a decorated tree structure in which the underlying Galton-Watson tree has exponent $\left(\alpha-\frac{1}{2}\right)^{-1}$ [Ric18b, Theorem 1.2]. Random walks on stable maps have been considered in [CM19a] where the authors show a universal upper bound of $\frac{1}{3}$ on the displacement exponent. Their argument involves considering an appropriate set of cut-points and estimating the time required to pass through enough of these cutpoints along with the likelihood of a random walk actually visiting such a cut-point at any given time.

The use of cut-points in [CM19a] implicitly takes advantage of the tree structure; another approach might be to directly decompose as in Chapter 6 and try to estimate resistance across each component of the graph. This would also give quenched estimates, but estimating resistance is a difficult task. Another approach that would not necessarily require a resistance estimate might be to consider a subgraph that allows a sharpening of the Varopoulos-Carne bound used in [CM19a, Lemma 1], since this is not necessarily optimal on subdiffusive graphs, e.g. by choosing a denser subgraph.

### 7.4 Random walk on a critical percolation cluster

As outlined in the introduction, one example of particular interest falling into the dense stable map regime mentioned above is that of a critical percolation cluster on large uniform triangulations, which is conjectured to rescale to the $\frac{7}{6}$-stable map. A naive approach to study resistance across "large" components would be to emulate the logic applied by Richier [Ric18a] to study this model on the UIHPT (in some sense an infinite component); in particular, merely trying to "imitate the same picture" of the boundary of the critical cluster inside a loop, one might draw that shown in Figure 7.4.

There are several problems with doing this in practice, one of which is that to explore the interface in [Ric18a] one considers a specific boundary condition

(a) Critical cluster boundary, figure from [Ric18a].

(b) Imitation for a triangulation with large simple boundary, drawing the boundary of the open cluster only.
which ensures that one is already on the appropriate interface when one starts the exploration. In the case of percolating a finite loop however, it seems that an exploration process would initially be trapped inside a "smaller bubble" of the cluster which makes this trickier.

This issue is dealt with in the paper [GHS19a] where the authors explore the full set of percolation interfaces inside a percolated triangulation with open simple boundary condition by flipping the state of one boundary vertex which creates a starting point from which to start the exploration. This enables them to prove a strong result showing that the collection of full cluster boundaries converge to a conformal loop ensemble with parameter 6 in the scaling limit. In the case of exploring one critical cluster, it is not necessary to explore all of the interfaces since we are only interested in the part of the critical cluster that is connected to the boundary of the loop.

It is also worth noting that, at least in the case of site percolation, there is a duality between open and closed clusters at criticality since $p_{c}=\frac{1}{2}$. This is clearly affected by the boundary condition but is reflected in the result of Richier in that we see a white looptree in the middle of the cluster, and this has the same (probabilistic) structure as the black (half) looptree appearing along the boundary. This duality means that the number of open crossings directly "across" a large loop should be an order 1 random variable (since an open crossing impedes a closed one; this bias is more pronounced for bond percolation where $p_{c}<\frac{1}{2}$ on random triangulations [BCM19]). In other words, there is a good chance that the vertices contained in the critical cluster do not extend to the "centre" of the map.

If the picture in Figure 7.4 could be made rigorous then this gives rise to a fractal approximation by repeating this construction inside the resulting large faces, as indicated Figure 7.4. On deleting the dead-ends, this is then similar to a randomised diamond fractal (see [HK10, Alo19] and references therein for more on

(a) (Almost) critical bond percolation in the square conformally mapped to the unit disk, with blue boundary condition. Interfaces are in green. Image by Jason Miller.

(b) Imitation for a triangulation with large simple boundary, drawing the boundary of the open cluster only.

Figure 7.2: Comparison with a simulation.
diamond fractals) and therefore one might hope to study resistance across a large component by studying resistance across randomised diamond fractals. One would hope that the resistance has non-trivial polynomial scaling and therefore it should fall into the framework of Chapter 6.


Figure 7.3: Diamond fractal construction.

We also briefly remark that a random walk on a critical percolation cluster should be significantly easier to study on hyperbolic triangulations: in this setting, the cluster is conjectured to rescale to the Brownian CRT [Cur, Open question 12.12]. In this setting, it seems reasonable that the resistance metric should also rescale, so that one can recover a full scaling limit for the random walk as well as the cluster.

### 7.5 Spectral properties of decorated Galton-Watson trees

Another question one could investigate is whether one can prove spectral asymptotics similar to those presented in Section 4.5 for decorated Galton-Watson trees, saying on taking an appropriate scaling limit when conditioning the underlying tree to have $n$ vertices. Working in the discrete setting presents extra challenges anyway because it is more likely that there will be repeated eigenvalues, but when decomposing analogously to Proposition 4.5.2 there are also extra terms to consider since we cannot necessarily ignore the contributions of graphs inserted on the spine of the tree. This could allow for more interesting behaviour, however, since when iterating the result of Proposition 4.5 .2 we would have to keep track of an extra term at each level rather than just the "error term".

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