The Harmonic Extension Technique with Applications to Optimal Stopping

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Contents

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Declarations

The work of Chapter 4 is the result of collaboration with Dr. Sigurd Assing and is in preparation for submission in the following paper:

1. Sigurd Assing & John Herman, *Extension Technique for Functions of Diffusion Operators: a stochastic approach* [4].

I declare that, to the best of my knowledge, all material contained in this thesis is my own original work, unless otherwise stated, cited or commonly known. No part of this thesis has been submitted for a degree at any other university.

Abstract

In this thesis we investigate the harmonic extension method first popularised by Caffarelli & Silvestre in [14] which allows the fractional Laplacian to be represented in terms of data retrieved from the solution u_f to a local PDE problem. We generalise this method to obtain local representations for a family of non-local operators $-\psi(-\mathcal{L}_x)$ where ψ is a complete Bernstein function and $\mathcal{L}_x$ is the generator of a diffusion semigroup on some Banach space using two different approaches; one based upon stochastic analysis and the other based upon semigroup theory. Underlying both of these approaches is the Krein correspondence which gives a one-to-one correspondence between complete Bernstein functions and a family of functions known as Krein strings. We study this correspondence and focus on a particular function φ_λ called the *the extension function* which provides the key to understanding the extension method.

As an application of this method, we show how an obstacle problem associated with the non-local operator $-\psi(-\mathcal{L}_x)$ can be studied via the techniques found in [9] which can usually only be applied to local problems. Under certain conditions placed on $\mathcal{L}_x$ and the obstacle G, we show that the solution V to this problem lies in the L^2-domain of the operator $-\psi(-\mathcal{L}_x)$. Furthermore, if ψ arises as the Laplace exponent of the inverse local time of a one-dimensional diffusion process, then we show that the solution will belong to the L^p-domain of the operator $-\psi(-\mathcal{L}_x)$ allowing us to prove a regularity result for V.

Chapter 1

Introduction

The aim of this thesis is to investigate a method for expressing a family of non-local operators in terms of data retrieved from a local PDE problem known as the harmonic extension technique. The earliest example of the harmonic extension technique is classical. If $f : \mathbb{R}^d \to \mathbb{R}$ is a smooth, bounded function and $u_f : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a solution to

$$\begin{cases} \Delta_x u_f(x, y) + \partial_y^2 u_f(x, y) = 0 & \text{for all } (x, y) \in \mathbb{R}^d \times (0, \infty), \\ u_f(x, 0) = f(x) & \text{for all } x \in \mathbb{R}^d, \end{cases} \tag{1.0.1}$$

(where Δ_x denotes the Laplace operator acting on the d-dimensional x-component of u), then we may express the square root of the Laplacian of f in terms of the Dirichlet-to-Neumann map of the function u_f:

$$-(-\Delta_x)^{1/2} f(x) = \partial_y u_f(x, 0),$$

for all $x \in \mathbb{R}^d$.

Naturally, this leads us to ask whether this classical example may be generalised to functions other than $\lambda \mapsto \lambda^{1/2}$. The first generalisation was provided by Caffarelli & Silvestre where this technique was first popularised in their 2007 paper [14]. In the paper, they consider two related elliptic equations. The first given by

$$\begin{cases} \Delta_x v_f + \frac{1-\alpha}{z} \partial_z v_f + \partial_z^2 v_f = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ v_f(x, 0) = f(x) & \text{for all } x \in \mathbb{R}^d, \end{cases} \tag{1.0.2}$$

which can be rewritten in divergence form by noting,

$$z^{\alpha-1}\nabla_{x,z}\cdot\left(z^{1-\alpha}\nabla_{x,z}v_f\right)=\Delta_x v_f+\tfrac{1-\alpha}{z}\partial_z v_f+\partial_z^2 v_f. \tag{1.0.3}$$

The second equation we can obtain by rescaling the z-coordinate, setting $y=c_\alpha z^\alpha$ where $\alpha\in(0,2)$ and $c_\alpha=2^{-\alpha}|\Gamma(-\frac{\alpha}{2})|/\Gamma(\frac{\alpha}{2})$ (we remark that this explicit constant c_α does not appear in [14], but rather in [46]). We then define a function $u_f:\mathbb{R}^d\times[0,\infty)\to\mathbb{R}$ by

$$u_f(x,y)=v_f(x,(\tfrac{y}{c_\alpha})^{1/\alpha}).$$

By applying the chain rule we find that

$$\begin{aligned}
&\partial_z v_f(x,z)=\alpha c_\alpha^{1/\alpha}y^{1-1/\alpha}\partial_y u_f(x,y),\\
&\partial_z^2 v_f(x,z)=\alpha^2 c_\alpha^{2/\alpha}y^{2-2/\alpha}\partial_y^2 u_f(x,y)+(\alpha(\alpha-1)c_\alpha^{2/\alpha}y^{1-2/\alpha})\partial_y u_f(x,y),
\end{aligned}$$

and as v_f solves (1.0.2) we see u_f solves

$$\begin{cases}\Delta_x u_f+\alpha^2 c_\alpha^{2/\alpha}y^{2-2/\alpha}\partial_y^2 u_f=0 & \text{in } \mathbb{R}^d\times(0,\infty),\\ u_f(x,0)=f(x) & \text{for all } x\in\mathbb{R}^d.\end{cases} \tag{1.0.4}$$

It is then proven that

$$-(-\Delta_x)^{\alpha/2}f(x)=\partial_y u_f(x,0)=\lim_{z\to 0}\frac{v_f(x,z)-v(x,0)}{c_\alpha z^\alpha}.$$

A key insight is that by taking the Fourier transform of (1.0.4), we obtain a family of ODEs indexed by $\xi\in\mathbb{R}^d$:

$$\begin{cases}-|\xi|^2\hat{u}_f(\xi,y)+\alpha^2 c_\alpha^{2/\alpha}y^{2-2/\alpha}\partial_y^2\hat{u}_f(\xi,y)=0 & \text{for all } (\xi,y)\in\mathbb{R}^d\times(0,\infty),\\ \hat{u}_f(\xi,0)=\hat{f}(\xi) & \text{for all } \xi\in\mathbb{R}^d.\end{cases} \tag{1.0.5}$$

To solve this equation, we note that for each $\lambda\geq 0$ there exists a unique solution $\varphi_\lambda^{(\alpha)}$ to the ODE

$$\alpha^2 c_\alpha^{2/\alpha}y^{2-2/\alpha}\phi''(y)=\lambda\phi(y),$$

which is non-negative, continuous and bounded on $[0,\infty)$ with $\phi(0)=1$. The explicit solution to this second order ODE is given by

$$\varphi_\lambda^{(\alpha)}(y) = \frac{2^{1-\alpha/2}}{\Gamma(\frac{\alpha}{2})}\Big(\frac{\lambda^{\alpha/2}y}{c_\alpha}\Big)^{1/2}K_{\alpha/2}\bigg(\Big(\frac{\lambda^{\alpha/2}y}{c_\alpha}\Big)^{1/\alpha}\bigg),$$

where $K_{\alpha/2}$ is the modified Bessel function of the second kind. This function allows us to write the solution u_f to (1.0.4) in terms of its Fourier transform:

$$\hat{u}_f(\xi, y) = \varphi_{|\xi|^2}^{(\alpha)}(y)\hat{f}(\xi).$$

As $\partial_y\varphi_\lambda^{(\alpha)}(0) = -\lambda^{\alpha/2}$, we see immediately that

$$\partial_y\hat{u}_f(\xi, 0) = -|\xi|^\alpha\hat{f}(\xi) \implies \partial_y u_f(\cdot, 0) = -(-\Delta_x)^{\alpha/2}f.$$

The function $\varphi^{(\alpha)}$, which we call the *extension function*, is key to understanding the extension method and the possible non-local operators which may be obtained via this method.

There have been several papers generalising the extension technique in various ways. The first of note was by Stinga & Torrea in [71], in which the technique is generalised to obtain a similar characterisation for operators of the form $-(-\mathcal{L}_x)^{\alpha/2}$ where $\mathcal{L}_x$ is a linear second-order partial differential operator which is non-positive, densely defined and self-adjoint in $L^2(\mathcal{X}, \mu)$. Their method uses the spectral theory for self-adjoint linear operators on Hilbert spaces which permits them to avoid using the Fourier transform which is of little use when the operator $\mathcal{L}_x$ is spatially inhomogeneous. Moreover, an alternative method called *the method of semigroups* is discussed in the expository article by Stinga [70]. In this article, several explicit formulas for v_f are given in terms of the semigroup associated with the Laplacian, an example being

$$v_f(x, z) = \frac{z^\alpha}{4^{\alpha/2}\Gamma(\frac{\alpha}{2})}\int_0^\infty e^{-z^2/(4t)}(e^{t\Delta_x}f)(x)\frac{\mathrm{d}t}{t^{1+\alpha/2}}.$$

This representation is advantageous in several ways. Heuristically, the formula suggests that the semigroup $(e^{t\Delta_x})_{t\geq 0}$ can be replaced by another semigroup $(P_t)_{t\geq 0}$ with generator $\mathcal{L}_x$ to give a corresponding formula for the harmonic extension associated with $-(-\mathcal{L}_x)^{\alpha/2}$ without requiring the operator $\mathcal{L}_x$ to be a self-adjoint operator on a Hilbert space. Another advantage is that if the semigroup $(P_t)_{t\geq 0}$ is given by a heat kernel, then we have a pointwise formula for the harmonic extension which is also unavailable when dealing with a general operator on a Hilbert space.

However, in almost all works on the extension technique, the authors restrict themselves to fractional powers. The only work we are aware of where functions ψ other than fractional powers were systematically treated is in Kwaśnicki & Mucha's recent paper [47]. In this paper, they consider a complete Bernstein function ψ and investigate the extension technique for operators $\psi(-\Delta_x)$ defined in $L^2(\mathbb{R}^d)$. The key result used in the paper is known as the Krein correspondence, a one-to-one mapping between a set of functions known as Krein strings and the set of Stieltjes functions, first proven by Krein in [45]. Although this one-to-one correspondence allows us to identify the most general possible operators we may obtain via the extension method, explicit examples of this correspondence are rare. Nonetheless, some useful practical examples are calculated in the papers [23, 24] in addition to the fractional power example. Although explicit examples are rare, the correspondence does have certain useful properties including a type of sequentially continuity property and the asymptotics of the Krein string are related to the asymptotics of the corresponding complete Bernstein function.

In addition to these purely analytic approaches, the extension technique can be studied probabilistically by considering the underlying stochastic processes associated with the elliptic equations (1.0.2) and (1.0.4). Indeed, the Caffarelli-Silvestre extension technique is related to subordination of a Brownian motion in $\mathbb{R}^d$ by the inverse local time at zero of an independent Bessel diffusion in $[0, \infty)$ as proven in the late 1960's by Molchanov & Ostrovskii [54]. They proved that the inverse local time at zero of a Bessel process of dimension $\delta = 2-\alpha$ where $\alpha \in (0, 2)$ is an $\frac{\alpha}{2}$-stable subordinator (and hence a Brownian motion in $\mathbb{R}^d$ independently subordinated by this inverse local time is a symmetric α-stable process). A useful way of visualising this procedure is by considering the $\mathbb{R}^d \times [0, \infty)$-valued process $((X_t, Y_t))_{t\geq 0}$ where $(X_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion and $(Y_t)_{t\geq 0}$ is an independent Bessel process. Then the trace process on $\{y = 0\}$ given by $(X_{T_t})_{t\geq 0}$, where $(T_t)_{t\geq 0}$ is the inverse local time at zero of $(Y_t)_{t\geq 0}$, is a symmetric α-stable process (see Figure 1). This probabilistic interpretation of the extension method in terms of the trace process was also studied by Kim, Song & Vondraček in [39].

The question of which subordinators may be obtained in this manner goes back to Itô & McKean [32] and a partial answer is given by the probabilistic interpretation of the Krein correspondence, independently studied by Knight [42] and Kotani & Watanabe [44]. Inverse local times of one-dimensional reflected diffusions have also been investigated by Pitman & Yor in several papers [57, 58] and recently by Chen & Wang in [20] where a type of perturbation of the Bessel diffusion is investigated. Additionally, trace processes have been studied extensively in the context

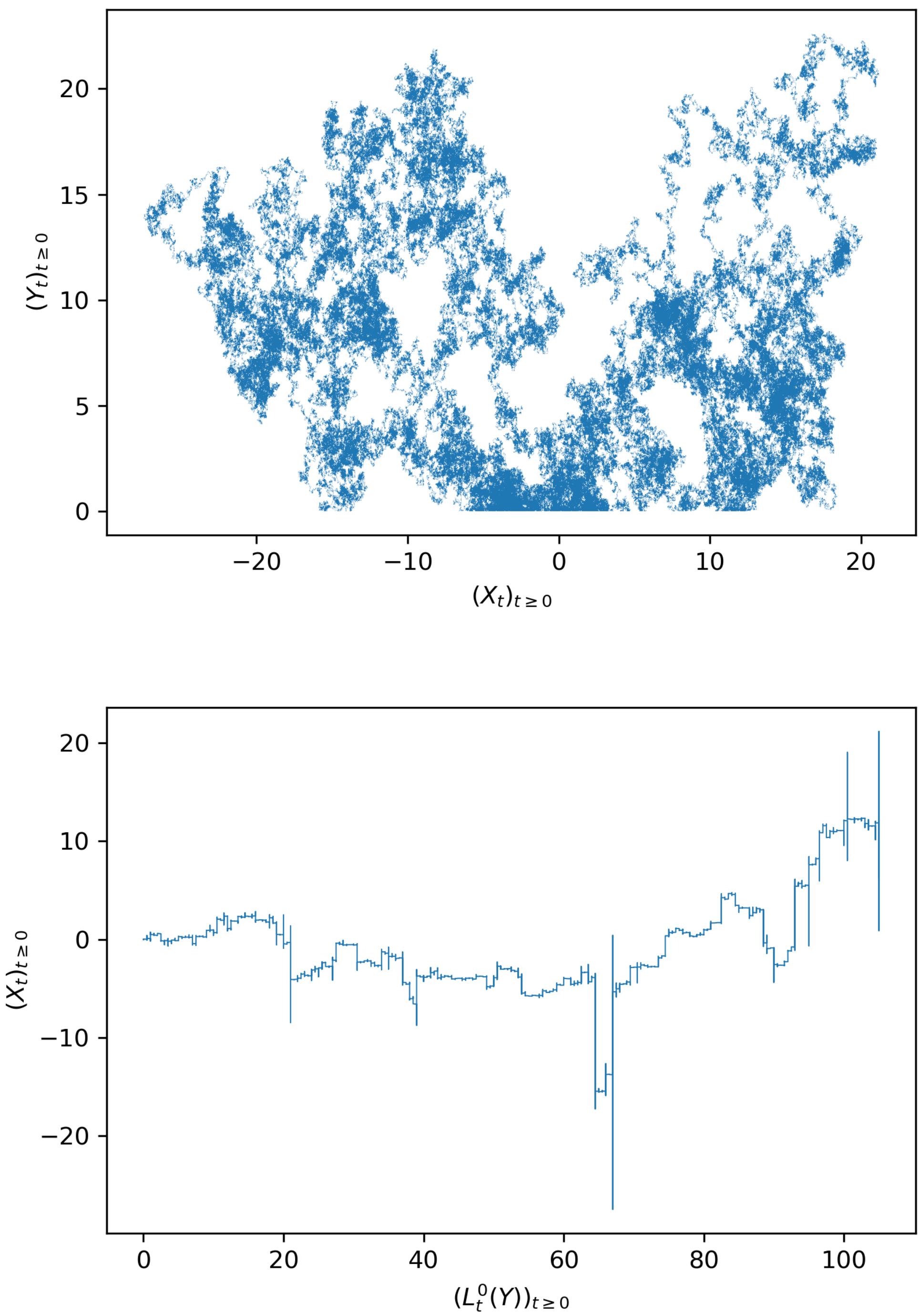


Figure 1.1: The first graph is a simulation of a sample path of a two-dimensional reflected Brownian motion, while the second graph shows the corresponding sample path of the trace process. For examples of Bessel processes of other dimensions, see Appendix A.1.

of Dirichlet forms (see [17, 18, 34]).

In this thesis, we extend the Caffarelli-Silvestre extension technique to obtain local representations of a larger family of non-local operators $-\psi(-\mathcal{L}_x)$ where ψ is a complete Bernstein function and $\mathcal{L}_x$ is a generator of a diffusion semigroup on some Banach space $\mathfrak{B}$. To do this we use two different methods, one based upon stochastic analysis and another based upon semigroup theory.

In the first approach, we assume $(X_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued diffusion process given by the solution to an SDE and ψ is a complete Bernstein function associated with a subordinator $(T_t)_{t\geq 0}$ which is given by the inverse local time at zero of a one-dimensional diffusion process $(Y_t)_{t\geq 0}$ in natural scale with speed measure $\tilde{m}$ (and corresponding Krein string m) which is independent of $(X_t)_{t\geq 0}$. Then for some $f : \mathbb{R}^d \to \mathbb{R}$, we assume that there is a solution to the elliptic PDE,

$$\begin{cases} \mathcal{L}_x u_f \times m(\mathrm{d}y) + \partial_y^2 u_f = 0 & \text{in } \mathbb{R}^d \times (0, l), \\ u_f(\cdot, 0) = f & \text{in } \mathbb{R}^d, \end{cases}$$

the precise definition of what we mean by a solution to this equation being given by Definition 4.3.1. We then prove that $u_f(X_t, Y_t)$ satisfies an Itô formula under relatively weak regularity conditions on u_f. Then by applying the time change $(T_t)_{t\geq 0}$ to this formula, we are able to show under certain assumptions the pointwise limit

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_x\big[f(X(T_t)) - f(x)\big] = \partial_y u_f(x, 0) + m_0 \mathcal{L}_x u_f(x, 0),$$

where the second term appears when the diffusion spends positive time at zero. Now if $(X_t)_{t\geq 0}$ is a Feller process with generator $\mathcal{L}_x$, then so is the subordinated diffusion process $(X_{T_t})_{t\geq 0}$ and its generator is given in some sense by the non-local operator $-\psi(-\mathcal{L}_x)$. As the pointwise limit should be related to the generator, the limit should provide a local characterisation for the operator $-\psi(-\mathcal{L}_x)$.

In the second approach, we begin with a semigroup $(P_t)_{t\geq 0}$ on some Banach space $\mathfrak{B}$ and a complete Bernstein function ψ corresponding to the inverse local time at zero of a gap diffusion $(Y_t)_{t\geq 0}$, a process constructed in a similar manner to a one-dimensional diffusion but where the speed measure does not necessarily have full support. If we assume $(Y_t)_{t\geq 0}$ spends no time at zero, then we may consider the family of bounded operators $\{\mathcal{H}_y\}_{y\in E_m}$ defined by the Bochner integral

$$\mathcal{H}_y f = \int_{[0,\infty)} (P_s f)\mathbb{P}_y[H_0 \in \mathrm{d}s],$$

where H_0 is the first hitting time of zero by $(Y_t)_{t\geq 0}$. Then under certain assumptions we can show $\partial_y \mathcal{H}_y\big|_{y=0} f = -\psi(-\mathcal{L}_x)f$ in some sense. Furthermore, when $\mathfrak{B}$ is a Hilbert space and $(P_t)_{t\geq 0}$ is a symmetric semigroup corresponding to a Hunt process $(X_t)_{t\geq 0}$, we show how the non-local Dirichlet form and corresponding Dirichlet space associated with the subordinated process $(X_{T_t})_{t\geq 0}$ is related to that of the pair process $((X_t, Y_t))_{t\geq 0}$.

As a tractable family of jump processes, subordinated diffusion processes (and in particular subordinated Brownian motion), have been studied extensively (see [40, 50, 59, 67, 75]). These processes have practical applications in the field of financial mathematics [21, 22, 29, 49] and numerous financial models given by subordinated Brownian motions are detailed in [73]. Therefore, a particular problem of interest in this field is the optimal stopping problem in which we are given an $\mathbb{R}^d$-valued Markov process $(S_t)_{t\geq 0}$, a gain function $G : \mathbb{R}^d \to \mathbb{R}$ and a non-negative interest rate function $R : \mathbb{R}^d \to \mathbb{R}$, and would like to study the value function $V : \mathbb{R}^d \to \mathbb{R}$ given by

$$V(x) = \sup_{\tau} \mathbb{E}_x \left[\exp\left(-\int_0^{\tau} R(S_s)\,\mathrm{d}s \right) G(S_\tau) \right],$$

where the supremum is taken over almost surely finite stopping times. Of particular interest in this thesis is the case where $(S_t)_{t\geq 0}$ is a subordinated diffusion process, certain examples of which have been studied in [48, 49, 50].

Analytically, the solution to the optimal stopping problem should be given by the solution to the free boundary problem

$$\begin{cases} V \geq G & \text{in } \mathbb{R}^d, \\ \mathcal{A}_x V - RV \leq 0 & \text{in } \mathbb{R}^d, \\ \mathcal{A}_x V - RV = 0 & \text{when } V > G, \end{cases} \tag{1.0.6}$$

where $\mathcal{A}_x$ is the infinitesimal generator of the process $(S_t)_{t\geq 0}$. When $(S_t)_{t\geq 0}$ is a diffusion process (and so $\mathcal{A}_x$ is a second-order differential operator), standard PDE techniques are well-suited to study this problem (see [8, 9]) and the value function may be computed using finite element or finite difference schemes. Unfortunately, these techniques can fail when the process has jumps and the infinitesimal generator is no longer a local operator. However in the case where $\mathcal{A}_x = -(-\Delta_x)^{\alpha/2}$, the free boundary problem has been studied extensively via the Caffarelli-Silvestre extension technique (see [6, 7, 15, 16]).

To conclude the thesis, we apply the extension technique to study the free boundary problem (1.0.6) when the operator $\mathcal{A}_x = -\psi(-\mathcal{L}_x)$ corresponds to the infinitesimal generator of a subordinated symmetric diffusion semigroup. In particular, we show how the methods found in [9] to study the solution to the local free boundary problem can be adapted to the non-local free boundary problem using the extension method.

Chapter 2

Preliminaries

We begin by establishing some of the background theory used throughout this thesis and notation used.

2.1 Bernstein Functions

We begin by recalling selected definitions and results related to Bernstein functions as detailed in [65]. Although this family of functions feature in various mathematical fields, their importance in this thesis is due to the fact that they appear as the Laplace exponent of subordinators. We recall the following definition.

Definition 2.1.1 (Laplace Transform)**.** *Let μ be a measure on $[0,\infty)$. Then the Laplace transform $\mathscr{L}(\mu;\lambda)$ is defined by*

$$\mathscr{L}(\mu;\lambda) = \int_{[0,\infty)} e^{-\lambda t}\mu(\mathrm{d}t),$$

whenever the integral converges.

A property that we use frequently in this thesis is that the Laplace transform allows us to convert problems involving convergence of measures into problems involving pointwise convergence. We recall that a sequence of locally finite measures $(\mu_n)_{n\in\mathbb{N}}$ on $[0,\infty)$ converges weakly (resp. vaguely) to a locally finite measure μ if for all $f \in C_b([0,\infty))$, the set of bounded continuous functions (resp. $f \in C_0([0,\infty))$, the set of continuous functions vanishing at infinity), we have $\int_{[0,\infty)} f\,\mathrm{d}\mu_n \to \int_{[0,\infty)} f\,\mathrm{d}\mu$. Then if we have a family of finite measures $(\mu_n)_{n\in\mathbb{N}}$, its weak limit (resp. vague limit) μ exists if and only if $\lim_{n\to\infty}\mathscr{L}(\mu_n;\lambda)$ exists for all $\lambda \geq 0$ (resp. $\lambda > 0$) in which case $\mathscr{L}(\mu;\lambda) = \lim_{n\to\infty}\mathscr{L}(\mu_n;\lambda)$.

To characterise the range of the Laplace transform, we require the following definition.

Definition 2.1.2 (Completely Monotone)**.** *A infinitely differentiable function $\phi : (0,\infty) \to \mathbb{R}$ is completely monotone if*

$$(-1)^n \phi^{(n)}(\lambda) \geq 0,$$

for all $n \in \mathbb{N}_0$ and $\lambda > 0$.

The importance of this definition is given by the following theorem due to Bernstein which allows us to characterise the range of the Laplace transform.

Theorem 2.1.3 (Bernstein)**.** *Let $\phi : (0,\infty) \to \mathbb{R}$ be a completely monotone function. Then there exists a unique measure μ on $[0,\infty)$ such that for all $\lambda > 0$,*

$$\phi(\lambda) = \mathscr{L}(\mu;\lambda) = \int_{[0,\infty)} e^{-\lambda t} \mu(\mathrm{d}t).$$

Conversely, whenever $\mathscr{L}(\mu;\lambda) < \infty$ for all $\lambda > 0$, $\lambda \mapsto \mathscr{L}(\mu;\lambda)$ is a completely monotone function.

Closely related to completely monotone functions is the set of Bernstein functions.

Definition 2.1.4 (Bernstein Function)**.** *An infinitely differentiable function $\psi : (0,\infty) \to \mathbb{R}$ is a Bernstein function if $\psi(\lambda) \geq 0$ for all $\lambda > 0$ and*

$$(-1)^{n-1} \psi^{(n)}(\lambda) \geq 0,$$

for all $n \in \mathbb{N}$ and $\lambda > 0$.

We note that a non-negative, infinitely differentiable function $\psi : (0,\infty) \to \mathbb{R}$ is Bernstein function if and only if ψ' is a completely monotone function. Furthermore, every Bernstein function admits a Lévy-Khintchine representation.

Theorem 2.1.5. *A function $\psi : (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if its admits the representation*

$$\psi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \nu(\mathrm{d}t),$$

where $a, b \geq 0$ and ν is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t) \nu(\mathrm{d}t) < \infty$. We call (a, b, ν) the Lévy triplet associated to ψ and ν the Lévy measure.

We shall be interested in the behaviour of a Bernstein function at infinity so we define the following indices first discussed in [12].

Definition 2.1.6 (Blumenthal-Getoor Indices)**.** *For a Bernstein function ψ, we define its lower and upper Blumenthal-Getoor indices by*

$$\underline{\mathrm{ind}}(\psi) = \sup\Big\{\rho > 0 : \lim_{\lambda\to\infty} \tfrac{\psi(\lambda)}{\lambda^\rho} = \infty\Big\} = \liminf_{\lambda\to\infty} \frac{\log\psi(\lambda)}{\log\lambda},$$
$$\overline{\mathrm{ind}}(\psi) = \inf\Big\{\rho > 0 : \lim_{\lambda\to\infty} \tfrac{\psi(\lambda)}{\lambda^\rho} = 0\Big\} = \limsup_{\lambda\to\infty} \frac{\log\psi(\lambda)}{\log\lambda}.$$

We also have the following useful representations for the indices:

$$\underline{\mathrm{ind}}(\psi) = \sup\Big\{\rho \le 1 : \int_1^\infty \frac{\lambda^{\rho-1}}{\psi(\lambda)}\,\mathrm{d}\lambda\Big\},$$
$$\overline{\mathrm{ind}}(\psi) = \inf\Big\{\rho > 0 : \int_0^1 y^\rho \nu(\mathrm{d}y) < \infty\Big\}.$$

Interestingly, pointwise convergence of Bernstein functions implies locally uniform convergence and this provides us information about convergence of the corresponding Lévy triplets.

Proposition 2.1.7. *Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of Bernstein functions such that $\lim_{n\to\infty}\psi_n(\lambda) = \psi(\lambda)$ exists for all $\lambda > 0$. Then ψ is a Bernstein function and for all $k \in \mathbb{N}_0$ the convergence $\lim_{n\to\infty}\psi_n^{(k)}(\lambda) = \psi^{(k)}(\lambda)$ is locally uniform for $\lambda > 0$. If (a_n, b_n, ν_n) and (a, b, ν) are the Lévy triplets for ψ_n and ψ respectively, then we have*

- $\lim_{n\to\infty}\nu_n = \nu$ *vaguely in* $(0,\infty)$*;* $\lim_{n\to\infty}\int_{(0,\infty)} f\,\mathrm{d}\nu_n = \int_{(0,\infty)} f\,\mathrm{d}\nu$ *for all* $f \in C_c((0,\infty))$*,*
- $a = \lim_{R\to\infty}\liminf_{n\to\infty}(a_n + \nu_n[R,\infty))$*,*
- $b = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\Big(b_n + \int_{(0,\epsilon)} t\nu_n(\mathrm{d}t)\Big)$*.*

In both formulae we may replace $\liminf_n$ *by* $\limsup_n$*.*

An important subclass of Bernstein functions is given by the family of complete Bernstein functions. This subclass is fundamental to this thesis due to their role in the Krein correspondence which shall be discussed in the next chapter.

Definition 2.1.8 (Complete Bernstein Function)**.** *A Bernstein function ψ is said to be a complete Bernstein function if its Lévy measure $\nu(\mathrm{d}t)$ has completely monotone density with respect to Lebesgue measure (which, abusing notation, we denote $\nu(t)$).*

An important property of this class of functions is that given any non-zero complete Bernstein function ψ, the function ψ^c defined by $\psi^c(\lambda) = \frac{\lambda}{\psi(\lambda)}$ is also a complete Bernstein function [65, Proposition 7.1]. Furthermore, there is a family of functions which are related to complete Bernstein functions are known as Stieltjes functions.

Definition 2.1.9 (Stieltjes function)**.** *A (non-negative) Stieltjes function is a function* $\mathfrak{h} : (0, \infty) \to [0, \infty)$ *which can be written in the form*

$$\mathfrak{h}(\lambda) = b + \int_{[0,\infty)} \frac{1}{\lambda + \eta} \sigma(\mathrm{d}\eta),$$

where $b \geq 0$ *is a non-negative constant and* σ *is a measure on* $[0, \infty)$ *such that* $\int_{[0,\infty)} \frac{1}{1+\eta} \sigma(\mathrm{d}\eta) < \infty$*.*

The connection to complete Bernstein functions is that a function ψ is a non-trivial complete Bernstein function if and only if $\frac{1}{\psi}$ is a non-trivial Stieltjes function [65, Theorem 7.3]. Therefore, in addition to the Lévy-Khintchine representation of a complete Bernstein function, we can use these properties to obtain the *Stieltjes representation* of ψ:

$$\psi(\lambda) = b^c \lambda + \int_{[0,\infty)} \frac{\lambda}{\lambda + \eta} \sigma^c(\mathrm{d}\eta),$$

as $\frac{1}{\psi^c(\lambda)} = b^c + \int_{[0,\infty)} \frac{1}{\lambda+\eta} \sigma^c(\mathrm{d}\eta)$ for some $b^c \geq 0$ and σ^c is a measure on $[0, \infty)$ such that $\int_{[0,\infty)} \frac{1}{1+\eta} \sigma^c(\mathrm{d}\eta) < \infty$.

2.1.1 Probabilistic Interpretation

As we have already alluded to, Bernstein functions can be interpreted probabilistically as the Laplace exponents of a family of increasing $[0, \infty]$-valued Lévy processes known as subordinators. Detailed properties of this family of processes can be found in [11].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a complete, right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $T = (T_t)_{t\geq 0}$ be a right-continuous, increasing, adapted process with values in $[0, \infty]$ where ∞ serves as a cemetery point for the process such that $T_0 = 0$ almost surely. We denote the lifetime of the process by $\zeta = \inf\{t \geq 0 : T_t = \infty\}$. T is then called a subordinator if it has independent and homogeneous increments on $[0, \zeta)$.

The law of a subordinator is specified by the Laplace transforms of its one-dimensional distributions. In fact, using the independence and homogeneity of the increments, the Laplace transform has form,

$$\mathbb{E}[\exp(-\lambda T_t)] = e^{-t\psi(\lambda)},$$

for some function $\psi : [0, \infty) \to [0, \infty)$ called the Laplace exponent of T. The connection between subordinators and Bernstein functions is given by the following theorem.

Theorem 2.1.10 (de Finetti, Lévy, Khintchine)**.** *If ψ is the Laplace exponent of a subordinator, then ψ is a Bernstein function. Conversely, if ψ is a Bernstein function, then ψ is the Laplace exponent of a subordinator.*

As we have seen, the Lévy-Khintchine decomposition of the subordinator is given by Theorem 2.1.5 in which case the values a and b correspond to the killing rate and drift coefficient respectively and ν is the Lévy (or jump) measure of the subordinator.

2.2 Semigroups & Infinitesimal Generators

Throughout this thesis, the theory of semigroups on Banach spaces is fundamental. We shall see the extension technique is intimately related to subordination of semigroups on Banach spaces. Let $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ be a Banach space and let $(\mathfrak{B}^*, \|\cdot\|_{\mathfrak{B}^*})$ denote it topological dual and $\langle f, \phi\rangle_{\mathfrak{B}}$ the dual pairing between $f \in \mathfrak{B}$ and $\phi \in \mathfrak{B}^*$.

Definition 2.2.1 (C_0-contraction semigroup)**.** *A semigroup is a family of bounded, linear operators $(P_t)_{t\geq 0}$ on $\mathfrak{B}$ satisfying*

- *$P_0 = I$, the identity mapping on $\mathfrak{B}$,*
- *(semigroup property) $P_sP_t = P_tP_s = P_{s+t}$ for all $s, t \geq 0$.*

A C_0-contraction semigroup is a semigroup which also satisfies

- *(strong continuity) $\lim_{t\to 0} \|P_tf - f\|_{\mathfrak{B}} = 0$ for all $f \in \mathfrak{B}$,*
- *(contraction property) $\|P_tf\|_{\mathfrak{B}} \leq \|f\|_{\mathfrak{B}}$ for all $f \in \mathfrak{B}$ and $t \geq 0$.*

An important family of C_0-contraction semigroups for probabilistic applications is given by Feller semigroups.

Definition 2.2.2 (Feller Semigroup)**.** *Let M be a locally compact, separable metric space and let $\mathfrak{B} = C_0(M)$ be the Banach space of continuous functions $f : M \to \mathbb{R}$ vanishing at infinity with uniform norm $\|\cdot\|_\infty$. A C_0-contraction semigroup $(P_t)_{t\geq 0}$*

on the Banach space $C_0(M)$ which is preserves positivity ($f \geq 0 \implies P_t f \geq 0$ for all $t \geq 0$) is called a Feller semigroup.

Furthermore, we say that a (homogeneous) Markov process $(X_t)_{t\geq 0}$ taking values in M is called *Feller process* if the semigroup $(P_t)_{t\geq 0}$ defined by $P_t f(x) = \mathbb{E}_x[f(X_t)]$ for $f \in C_0(M)$ is a Feller semigroup.

Definition 2.2.3 (Sub-Markovian)**.** *Let $(\mathcal{X}, \mathcal{B}, \mu)$ be a σ-finite measure space. A C_0-contraction semigroup on $L^p(\mathcal{X}, \mu)$ is called a sub-Markovian semigroup if $P_t f \geq 0$ μ-almost everywhere for any $f \geq 0$ μ-almost everywhere.*

Although it is often difficult to describe a semigroup explicitly, the semigroup property suggests a representation for the family of operators as the exponential of a (possibly unbounded) operator on a Banach space.

Definition 2.2.4 (Infinitesimal Generator)**.** *The (infinitesimal) generator of a C_0-contraction semigroup is the operator $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ defined by the strong limit,*

$$\mathcal{L}_x f = \lim_{t\to 0} \frac{P_t f - f}{t},$$

with domain

$$\mathrm{Dom}(\mathcal{L}_x) = \Big\{ f \in \mathfrak{B} : \lim_{t\to 0} \tfrac{P_t f - f}{t} \text{ exist as a strong limit} \Big\}.$$

The generator of a C_0-contraction semigroup is a densely defined, closed, linear operator which is dissipative;

$$\|\lambda f - \mathcal{L}_x f\|_{\mathfrak{B}} \geq \lambda \|f\|_{\mathfrak{B}},$$

for all $\lambda > 0$ and $f \in \mathrm{Dom}(\mathcal{L}_x)$. As $\mathcal{L}_x$ is closed, $\mathrm{Dom}(\mathcal{L}_x)$ is a Banach space endowed with the graph norm $\|f\|_{\mathrm{Dom}(\mathcal{L}_x)} = \|f\|_{\mathfrak{B}} + \|\mathcal{L}_x f\|_{\mathfrak{B}}$.

Proposition 2.2.5. *For any $t > 0$ and $f \in \mathfrak{B}$,*

$$P_t f - f = \int_0^t \mathcal{L}_x P_s f \, \mathrm{d}s,$$

and if $f \in \mathrm{Dom}(\mathcal{L}_x)$, $P_t f - f = \int_0^t P_s \mathcal{L}_x f \, \mathrm{d}s$ and so,

$$\|P_t f - f\|_{\mathfrak{B}} \leq \min\{t\|\mathcal{L}_x f\|_{\mathfrak{B}}, 2\|f\|_{\mathfrak{B}}\}.$$

Another way of describing a C_0-semigroup is via its resolvent.

Definition 2.2.6 (Resolvent). *For an infinitesimal generator* $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$, *let*

$$\rho(\mathcal{L}_x) = \{z \in \mathbb{C} : (zI - \mathcal{L}_x)^{-1} \text{ is a bounded, linear operator}\}$$

and $\sigma(\mathcal{L}_x) = \mathbb{C} \setminus \rho(\mathcal{L}_x)$. *For* $z \in \rho(\mathcal{L}_x)$,

$$R_z f = (zI - \mathcal{L}_x)^{-1} f.$$

For a C_0-semigroup $(P_t)_{t \geq 0}$, the resolvent is given by the Bochner integral

$$R_z f = \int_0^\infty e^{-zt} P_t f \, \mathrm{d}t,$$

for any $z \in \rho(\mathcal{L}_x)$ and $f \in \mathfrak{B}$ and $(0, \infty) \subset \rho(\mathcal{L}_x)$. The resolvent operators satisfy the resolvent estimate

$$\|R_z f\|_{\mathfrak{B}} \leq \frac{1}{\mathfrak{Re}(z)} \|f\|_{\mathfrak{B}},$$

for all $\mathfrak{Re}(z) > 0$ and $f \in \mathfrak{B}$.

A key result due to Bochner allows us to create a new C_0-semigroup from a given one via subordination [65, Proposition 13.1].

Proposition 2.2.7 (Bochner). *Let* $(P_t)_{t \geq 0}$ *be a* C_0*-contraction semigroup on the Banach space* $\mathfrak{B}$ *and let* $(T_t)_{t \geq 0}$ *be a subordinator on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with corresponding Bernstein function* ψ. *Then the Bochner integral*

$$P_t^\psi f = \int_{[0,\infty)} (P_s f) \mathbb{P}[T_t \in \mathrm{d}s]$$

defines again a C_0*-contraction semigroup on the Banach space* $\mathfrak{B}$ *called the* subordinate semigroup.

Heuristically, the semigroup property of $(P_t)_{t \geq 0}$ suggests that the semigroup can be thought of as the exponential of the operator $\mathcal{L}_x$, $(e^{t\mathcal{L}_x})_{t \geq 0}$. Formally substituting this representation into the Bochner integral representation for the subordinated semigroup we find,

$$P_t^\psi f = \int_{[0,\infty)} (e^{s\mathcal{L}_x}) \mathbb{P}[T_t \in \mathrm{d}s] = e^{-t\psi(-\mathcal{L}_x)} f,$$

which further suggests that the generator of the subordinated semigroup will be given by the operator $-\psi(-\mathcal{L}_x)$ in some sense. Indeed, in the case where $\mathcal{L}_x$ is a self-adjoint operator on a Hilbert space, then the functional calculus for self-adjoint operators allows us to make this idea rigorous. However, in the general Banach space setting, we only have the following partial result due to Phillips.

Theorem 2.2.8 (Phillips). *Let $(P_t)_{t\geq 0}$ be a C_0-contraction semigroup on the Banach space $\mathfrak{B}$ with generator $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ and let ψ be a Bernstein function with Lévy triplet (a, b, ν). Let $(P_t^\psi)_{t\geq 0}$ and $(\mathcal{L}_x^\psi, \mathrm{Dom}(\mathcal{L}_x^\psi))$ be the subordinated semigroup and its infinitesimal generator. Then, $\mathrm{Dom}(\mathcal{L}_x)$ is an operator core for $(\mathcal{L}_x^\psi, \mathrm{Dom}(\mathcal{L}_x^\psi))$ and for any $f \in \mathrm{Dom}(\mathcal{L}_x)$,*

$$\mathcal{L}_x^\psi f = -\psi(-\mathcal{L}_x) f = -af + b\mathcal{L}_x f + \int_{(0,\infty)} (P_s f - f)\nu(\mathrm{d}s),$$

where the integral is understood as a Bochner integral.

Due to this theorem, we denote the generator of the subordinated semigroup by $(-\psi(-\mathcal{L}_x), \mathrm{Dom}(-\psi(-\mathcal{L}_x)))$.

We can also extend this functional calculus to include Stieltjes functions. Let $(R_\lambda)_{\lambda>0}$ be the resolvent corresponding to the semigroup $(P_t)_{t\geq 0}$. Then we define $R_0 : \mathrm{Dom}(R_0) \to \mathfrak{B}$ by

$$\mathrm{Dom}(R_0) = \{f \in \mathfrak{B} : R_0 f = \lim_{\lambda\to 0} R_\lambda f \text{ exists in the strong sense}\}.$$

It is known that $\mathrm{Range}(R_0) \subset \mathrm{Dom}(\mathcal{L}_x)$ and we have $\mathcal{L}_x R_0 f = -f$ for all $f \in \mathrm{Dom}(R_0)$. Furthermore, R_0 is densely defined if and only if $\mathrm{Range}(\mathcal{L}_x)$ is dense in $\mathfrak{B}$. Given a Stieltjes function $\mathfrak{h}(\lambda) = b + \int_{[0,\infty)} \frac{1}{t+\lambda}\sigma(\mathrm{d}\lambda)$, we define

$$\mathfrak{h}(-\mathcal{L}_x) f = bf + \int_{[0,\infty)} R_t f \sigma(\mathrm{d}t),$$

for $f \in \mathrm{Range}(\mathcal{L}_x)$. We conclude with the following theorem which shows us that if $\mathfrak{h}$ is the Stieltjes function corresponding to a complete Bernstein function ψ, then $\mathfrak{h}(-\mathcal{L}_x)$ is the inverse operator of $-\psi(-\mathcal{L}_x)$.

Theorem 2.2.9. *Let $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ be the generator of a C_0-contraction semigroup on the Banach space $\mathfrak{B}$ such that the range $\mathrm{Range}(\mathcal{L}_x)$ is dense in $\mathfrak{B}$ and let ψ be a complete Bernstein functions. Then $\mathfrak{h}(\lambda) = \frac{1}{\psi(\lambda)}$ is a Stieltjes function and*

$$(-\psi(-\mathcal{L}_x))^{-1} f = \mathfrak{h}(-\mathcal{L}_x) f,$$

for all $f \in \mathrm{Range}(\mathcal{L}_x) \cap \mathrm{Dom}(-\psi(-\mathcal{L}_x))$.

2.2.1 Analytic Semigroups

In this subsection we suppose $\mathfrak{B}$ is a complex Banach space (or if $\mathfrak{B}$ is a real Banach space then we consider the complexification defined in [52, Appendix A]) and we assume that $\mathcal{L}_x : \mathrm{Dom}(\mathcal{L}_x) \subset \mathfrak{B} \to \mathfrak{B}$ (not necessarily densely defined) is a sectorial operator in the sense that there exists $\omega \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi)$ and $M > 0$ such that

$$\rho(\mathcal{L}_x) \supset S_{\theta,\omega} = \{z \in \mathbb{C} \setminus \{\omega\} : |\arg(z - \omega)| < \theta\},$$

(where $\arg(z) \in (-\pi, \pi]$ denotes the argument of the complex number z) and

$$\|R_z f\|_{\mathfrak{B}} \leq \frac{M}{|z - \omega|} \|f\|_{\mathfrak{B}},$$

for all $z \in S_{\theta,\omega}$ and $f \in \mathfrak{B}$. As the resolvent is non-empty, we know the operator is closed. It is possible to define a (not necessarily strongly continuous) semigroup $(P_t)_{t\geq 0}$ associated with the operator $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ which is analytic in the sense that the mapping $(0, \infty) \to L(\mathfrak{B}) : t \mapsto P_t$ is analytic.

Proposition 2.2.10. *Let $t > 0$ and $f \in \mathfrak{B}$. Then for an analytic semigroup $(P_t)_{t\geq 0}$ we know $P_t f \in \mathrm{Dom}(\mathcal{L}_x^k)$ for all $k \in \mathbb{N}$ and for each $k \in \mathbb{N}$ there is $M_k > 0$ such that*

$$\|\mathcal{L}_x^k P_t f\|_{\mathfrak{B}} \leq \frac{M_k}{t^k} \|f\|_{\mathfrak{B}}.$$

We now define some intermediate spaces between $\mathfrak{B}$ and $\mathrm{Dom}(\mathcal{L}_x)$.

Definition 2.2.11. *Let $\gamma \in (0, 1)$ and $p \in [1, \infty]$. Then define the Banach space*

$$D_{\mathcal{L}_x}(\gamma, p) = \left\{ f \in \mathfrak{B} : t \mapsto v(t) = \|t^{1-\gamma-1/p} \mathcal{L}_x T_t f\|_{\mathfrak{B}} \in L^p((0, 1)) \right\},$$

equipped with norm $\|f\|_{D_{\mathcal{L}_x}(\gamma,p)} = \|f\|_{\mathfrak{B}} + \|v\|_{L^p((0,1))}$ and let

$$D_{\mathcal{L}_x}(\gamma) = \{f \in D_{\mathcal{L}_x}(\gamma, \infty) : \lim_{t\to 0} t^{1-\gamma} \mathcal{L}_x T_t f\}.$$

Proposition 2.2.12. *Let $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ be the generator of an analytic C_0-contraction semigroup $(P_t)_{t\geq 0}$ on a Banach space $\mathfrak{B}$ and let ψ be a complete Bernstein function with $\underline{\mathrm{ind}}(\psi) = \gamma < 1$. Then,*

$$\mathrm{Dom}(-\psi(-\mathcal{L}_x)) \subset D_{\mathcal{L}_x}(\gamma, \infty).$$

Proof. This follows as $\mathrm{Dom}(-\psi(-\mathcal{L}_x)) \subset \mathrm{Dom}(-(-\mathcal{L}_x)^\gamma) \subset D_{\mathcal{L}_x}(\gamma,\infty)$ by [65, Corollary 13.36] and [52, Proposition 2.2.15]. ☐

For $k \in \mathbb{N}$ and $\gamma \in (0,1)$, let

$$C^{k+\gamma}(\mathbb{R}^d) = \Big\{ f \in C^k(\mathbb{R}^d) : [\partial^\alpha f]_{C^\gamma(\mathbb{R}^d)} < \infty \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| = k \Big\},$$

where $[\partial^\alpha f]_{C^\gamma(\mathbb{R}^d)} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\gamma}$ equipped with norm $\|f\|_{C^{k+\gamma}(\mathbb{R}^d)} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_\infty + \sum_{|\alpha|=k} [\partial^\alpha f]_{C^\gamma(\mathbb{R}^d)}$. We reference [33, 74] for the definitions and results on Besov spaces. Using results on real interpolation spaces we have the following corollary.

Corollary 2.2.13. *Let $(P_t^{(p)})_{t\geq 0}$ be a C_0-contraction semigroup on $L^p(\mathbb{R}^d)$ with generator $(\mathcal{L}_x^{(p)}, W^{2,p}(\mathbb{R}^d))$ and let ψ be a complete Bernstein function with $\underline{\mathrm{ind}}(\psi) = \gamma < 1$. Then $\mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)}) \subset B^{2\gamma}_{p,\infty}(\mathbb{R}^d) \subset C^{2\gamma - \frac{d}{p}}(\mathbb{R}^d)$.*

Proof. By [52, Proposition 2.2.2], $D_{\mathcal{L}_x^{(p)}}(\gamma,\infty)$ is equal to the real interpolation space $(L^p(\mathbb{R}^d), W^{2,p}(\mathbb{R}^d))_{\gamma,\infty}$ which itself is equal to $B^{2\gamma}_{p,\infty}(\mathbb{R}^d)$ by [74, Chapter 34] which is a subset of the Hölder space $C^{2\gamma-\frac{d}{p}}(\mathbb{R}^d)$ by [33, Corollary 3.11.13]. ☐

The main example of importance for this thesis is given in [52, Chapter 3.1]. We consider the second order differential operator

$$p(x,D) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} + c(x) I,$$

with real, uniformly continuous and bounded coefficients on $\mathbb{R}^d$. We assume further that the matrix $[a_{ij}]$ is symmetric and satisfies the uniform ellipticity condition.

Proposition 2.2.14. 1. *For $p \in (1,\infty)$ let $\mathfrak{B} = L^p(\mathbb{R}^d)$ and define $\mathrm{Dom}(\mathcal{L}_x^{(p)}) = W^{2,p}(\mathbb{R}^d)$ and let $\mathcal{L}_x^{(p)} = p(\cdot, D)$. Then $(\mathcal{L}_x^{(p)}, \mathrm{Dom}(\mathcal{L}_x^{(p)}))$ is sectorial.*

2. *Let $\mathfrak{B} = L^\infty(\mathbb{R}^d)$ and define*

$$\mathrm{Dom}(\mathcal{L}_x^{(\infty)}) = \{ f \in \cap_{p\geq 1} W^{2,p}_{loc}(\mathbb{R}^d) : f, p(\cdot,D) f \in L^\infty(\mathbb{R}^d) \}$$

with $\mathcal{L}_x^{(\infty)} : \mathrm{Dom}(\mathcal{L}_x^{(\infty)}) \to L^\infty(\mathbb{R}^d) : f \mapsto p(\cdot,D)f$. Then $\mathrm{Dom}(\mathcal{L}_x^{(\infty)})$ is continuously embedded in $C^{1+\gamma}(\mathbb{R}^d)$ for all $\gamma \in (0,1)$, the closure of $\mathrm{Dom}(\mathcal{L}_x^{(\infty)})$ in $L^\infty(\mathbb{R}^d)$ is the set of uniformly continuous functions in $\mathbb{R}^d$ and $(\mathcal{L}_x^{(\infty)}, \mathrm{Dom}(\mathcal{L}_x^{(\infty)}))$ is sectorial.

3. *Let $\mathfrak{B} = C_0(\mathbb{R}^d)$ and define*

$$\mathrm{Dom}(\mathcal{L}_x^{(0)}) = \{f \in \cap_{p\geq 1} W^{2,p}_{loc}(\mathbb{R}^d) : f, p(\cdot, D)f \in C_0(\mathbb{R}^d)\}$$

and let $\mathcal{L}_x^{(0)} : \mathrm{Dom}(\mathcal{L}_x^{(0)}) \to C_0(\mathbb{R}^d) : f \mapsto p(\cdot, D)f$. Then $\mathrm{Dom}(\mathcal{L}_x^{(0)})$ is dense in $C_0(\mathbb{R}^d)$ and $(\mathcal{L}_x^{(0)}, \mathrm{Dom}(\mathcal{L}_x^{(0)}))$ is sectorial.

In particular, when $\mathfrak{B} = L^\infty(\mathbb{R}^d)$ we have the following result.

Proposition 2.2.15. *Let $\gamma \in (0, 1)$ and let $(\mathcal{L}_x^{(\infty)}, \mathrm{Dom}(\mathcal{L}_x^{(\infty)}))$ be as defined in Example 2. Then,*

$$D_{\mathcal{L}_x^{(\infty)}}(\gamma, \infty) = \begin{cases} C^{2\gamma}(\mathbb{R}^d) & \text{if } \gamma \neq \frac{1}{2}, \\ \Lambda^1_* & \text{if } \gamma = \frac{1}{2}, \end{cases}$$

where Λ^1_* denotes the set of uniformly continuous and bounded functions f such that

$$[f]_{\Lambda^1_*} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x) - 2f(\frac{x+y}{2}) + f(y)|}{|x-y|} < \infty,$$

with norm $\|f\|_{\Lambda^1_*} = \|f\|_\infty + [f]_{\Lambda^1_*}$. Furthermore, the corresponding norms are equivalent.

Of course, this result will apply for functions in $C_0(\mathbb{R}^d)$. This allows us to prove the following analogue of [70, Chapter 7] to obtain Hölder estimates for the operator $-\psi(-\mathcal{L}_x)$.

Proposition 2.2.16. *Let $(P_t^{(0)})_{t\geq 0}$ be the Feller semigroup corresponding to $(\mathcal{L}_x^{(0)}, \mathrm{Dom}(\mathcal{L}_x^{(0)}))$ and let ψ be a complete Bernstein function such that $\overline{\mathrm{ind}}(\psi) = \gamma < 1$. Let $u = C^\beta(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ for some $\beta \in (0, 2]$.*

1. *If $0 < 2\gamma < \beta$ with $\beta - 2\gamma \neq 1$, then $\psi(-\mathcal{L}_x^{(0)})u \in C^{\beta-2\gamma}(\mathbb{R}^d)$ and*

$$\|\psi(-\mathcal{L}_x^{(0)})u\|_{C^{\beta-2\gamma}(\mathbb{R}^d)} \leq C\|u\|_{C^\beta(\mathbb{R}^d)}.$$

2. *If $0 < \beta < 2\gamma$ with $2\gamma - \beta \neq 1$, then $\psi(-\mathcal{L}_x^{(0)})u \in C^{1+\beta-2\gamma}(\mathbb{R}^d)$ and*

$$\|\psi(-\mathcal{L}_x^{(0)})u\|_{C^{1+\beta-2\gamma}(\mathbb{R}^d)} \leq C\|u\|_{C^{1+\beta}(\mathbb{R}^d)}.$$

Proof. We only prove the first case, the second case follows by the same reasoning. We first assume $\nu([1, \infty)) = 0$ and $a = 0$.

Let $u \in \mathrm{Dom}(\mathcal{L}_x^{(0)})$. Then by Proposition 2.2.15, it suffices to show that $\|\psi(-\mathcal{L}_x^{(0)})u\|_{\mathfrak{B}} \le C\|u\|_{C^\beta(\mathbb{R}^d)}$ where $\mathfrak{B} = D_{\mathcal{L}_x^{(\infty)}}(\frac{\beta-2\gamma}{2}, \infty)$. Now by Phillip's theorem we have,

$$-\partial_t P_t^{(0)}\psi(-\mathcal{L}_x^{(0)})u = \partial_t P_t^{(0)}\Big(\int_{(0,1)}\Big(\int_0^s \partial_r P_r^{(0)} u \,\mathrm{d}r\Big)\nu(s)\,\mathrm{d}s\Big)$$
$$= \int_{(0,1)}\Big(\int_0^s (\mathcal{L}_x^{(0)})^2 P_{t+r}^{(0)} u \,\mathrm{d}r\Big)\nu(s)\,\mathrm{d}s.$$

For $t \in (0,1)$ we know,

$$\int_{(0,t)}\Big(\int_0^s \Big\|t^{1-(\frac{\beta-2\gamma}{2})}(\mathcal{L}_x^{(0)})^2 P_{t+r}^{(0)} u\Big\|_\infty \mathrm{d}r\Big)\nu(s)\,\mathrm{d}s$$
$$\le \int_{(0,t)}\Big(\int_0^s \Big\|t^{2-\frac{\beta}{2}}(\mathcal{L}_x^{(0)})^2 P_{t+r}^{(0)} u\Big\|_\infty \mathrm{d}r\Big)t^{\gamma-1}\nu(s)\,\mathrm{d}s$$
$$\le \int_{(0,t)}\Big(\int_0^s [u]_{C^\beta(\mathbb{R}^d)}\,\mathrm{d}r\Big)s^{\gamma-1}\nu(s)\,\mathrm{d}s$$
$$= \Big(\int_{(0,t)} s^\gamma \nu(s)\,\mathrm{d}s\Big)[u]_{C^\beta(\mathbb{R}^d)},$$

and

$$\Big\|t^{1-(\frac{\beta-2\gamma}{2})}\int_{(t,1)}\partial_t P_t^{(0)}(P_s^{(0)}u - u)\nu(s)\,\mathrm{d}s\Big\|_\infty$$
$$\le 2\int_{(t,1)} t^{1-(\frac{\beta-2\gamma}{2})}\|\partial_t P_t^{(0)}u\|_\infty \nu(s)\,\mathrm{d}s$$
$$\le 2\int_{(t,1)} \|t^{1-\frac{\beta}{2}}\partial_t P_t^{(0)}u\|_\infty s^\gamma \nu(s)\,\mathrm{d}s$$
$$\le \Big(2\int_{(t,1)} s^\gamma\nu(s)\,\mathrm{d}s\Big)[u]_{C^\beta(\mathbb{R}^d)}.$$

Therefore,

$$\|t^{1-(\frac{\beta-2\gamma}{2})}\partial_t P_t^{(0)}\psi(-\mathcal{L}_x^{(0)})u\|_\infty \le 2\Big(\int_{(0,1)} s^\gamma\nu(s)\,\mathrm{d}s\Big)[u]_{C^\beta(\mathbb{R}^d)}.$$

To obtain the result for general complete Bernstein functions without drift, let the operator $\mathcal{J}$ be defined by

$$\mathcal{J}u = au + \int_{[1,\infty)}(P_s^{(0)}u - u)\nu(s)\,\mathrm{d}s.$$

Therefore,

$$\begin{aligned}
&\|t^{1-(\frac{\beta-2\gamma}{2})}\partial_t P_t^{(0)}\mathcal{J}u\|_\infty \\
&\qquad \le \|at^{1-(\frac{\beta-2\gamma}{2})}\partial_t P_t^{(0)}u\|_\infty + t^{1-(\frac{\beta-2\gamma}{2})}\int_{[1,\infty)}\|\partial_t P_{t+s}^{(0)}u - \partial_t P_t^{(0)}u)\|_\infty \nu(s)\,\mathrm{d}s \\
&\qquad \le t^\gamma\Big(a + 2\nu([1,\infty))\Big)\|t^{1-\frac{\beta}{2}}\partial_t P_t^{(0)}u\|_\infty \\
&\qquad \le C\Big(a + 2\nu([1,\infty))\Big)\|u\|_{C^\beta(\mathbb{R}^d)}.
\end{aligned}$$

Combining these two results, we have for a general complete Bernstein function ψ,

$$\| -\psi(-\mathcal{L}_x^{(0)})u\|_{C^{\beta-2\gamma}(\mathbb{R}^d)} \le C\|u\|_{C^\beta(\mathbb{R}^d)},$$

for any $u \in \mathrm{Dom}(\mathcal{L}_x^{(0)})$ and hence by density this result holds for all $u \in C^\beta(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. □

2.3 Dirichlet Form Theory

When we consider symmetric semigroups on a Hilbert space, it will be useful to apply certain results from Dirichlet form theory. As we will be concerned with symmetric Dirichlet forms, the material in this section can be found in [28].

Let $\mathfrak{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$. A symmetric form $\mathscr{E} : \mathrm{Dom}(\mathscr{E}) \times \mathrm{Dom}(\mathscr{E}) \to \mathbb{R}$ is a non-negative definite, symmetric, bilinear form which is densely defined on $\mathfrak{H}$. Given any symmetric form on $\mathfrak{H}$, we can define $\mathscr{E}_\lambda : \mathrm{Dom}(\mathscr{E}) \times \mathrm{Dom}(\mathscr{E}) \to \mathbb{R}$ by

$$\mathscr{E}_\lambda(\cdot,\cdot) = \lambda\langle\cdot,\cdot\rangle_{\mathfrak{H}} + \mathscr{E}(\cdot,\cdot).$$

The space $\mathrm{Dom}(\mathscr{E})$ with inner product $\mathscr{E}_1$ is a pre-Hilbert space and we say the symmetric form $\mathscr{E}$ is closed if $\mathrm{Dom}(\mathscr{E})$ is complete with respect to $\mathscr{E}_1$. We say that the symmetric form $\mathscr{E}$ is closable if for any $(u_n)_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathscr{E})$

$$\lim_{m,n\to\infty}\mathscr{E}(u_n - u_m, u_n - u_m) = 0, \lim_{n\to\infty}\|u_n\|_{\mathfrak{H}} = 0 \implies \lim_{n\to\infty}\mathscr{E}(u_n, u_n) = 0.$$

Importantly, there is a one-to-one correspondence between the family of closed symmetric forms $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ on $\mathfrak{H}$ and the family of non-positive definite self-adjoint operators $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$ on $\mathfrak{H}$ given by

$$\begin{cases} \mathrm{Dom}(\mathscr{E}) = \mathrm{Dom}(\sqrt{-\mathcal{L}}) \\ \mathscr{E}(u,v) = \langle \sqrt{-\mathcal{L}}u, \sqrt{-\mathcal{L}}v \rangle_{\mathfrak{H}}. \end{cases}$$

Now let $(\mathcal{X}, \mathcal{B}, \mu)$ be a σ-finite measure space and consider the real Hilbert space $\mathfrak{H} = L^2(\mathcal{X}, \mu; \mathbb{R}) = L^2(\mathcal{X}, \mu)$. We assume $\mathcal{X}$ is a locally compact, separable, metric space and μ is a positive Radon measure μ on $\mathcal{X}$ such that supp $\mu = \mathcal{X}$.

Definition 2.3.1 (Dirichlet Form)**.** *A symmetric form $\mathscr{E}$ on $L^2(\mathcal{X}, \mu)$ is Markovian if for all $\varepsilon > 0$ there exists a real function ϕ_ε with*

- *$\phi_\varepsilon(t) = t$ for all $t \in [0,1]$,*
- *$-\varepsilon \leq \phi_\varepsilon(t) \leq 1 + \varepsilon$, for all $t \in \mathbb{R}$,*
- *$0 \leq \phi_\varepsilon(t') - \phi_\varepsilon(t) \leq t' - t$ whenever $t' < t$,*

such that $u \in \mathrm{Dom}(\mathscr{E}) \implies \phi_\varepsilon(u) \in \mathrm{Dom}(\mathscr{E})$ and

$$\mathscr{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \leq \mathscr{E}(u,u).$$

A Dirichlet form is a closed, symmetric, Markovian form on $L^2(\mathcal{X}, \mu)$.

Given a symmetric form $\mathscr{E}$, a core is a subset $\mathcal{C} \subset \mathrm{Dom}(\mathscr{E}) \cap C_c(\mathcal{X})$ such that $\mathcal{C}$ is dense in $\mathrm{Dom}(\mathscr{E})$ with respect to the $\mathscr{E}_1$-norm and dense in $C_c(\mathcal{X})$ with respect to the uniform norm. $\mathscr{E}$ is said to be regular if $\mathscr{E}$ possesses a core.

For an μ-measurable function u, the support supp $u \cdot m$ of the measure $u(x)m(\mathrm{d}x)$ is denoted supp u and if $u \in C(\mathcal{X})$ then supp u is the closure of $\{x \in \mathcal{X} : u(x) \neq 0\}$. We say the symmetric form $\mathscr{E}$ is local if for any $u, v \in \mathrm{Dom}(\mathscr{E})$ such that supp u and supp v are disjoint compact sets then $\mathscr{E}(u,v) = 0$.

A useful property of Dirichlet forms is given by the following proposition.

Proposition 2.3.2. *Let $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ be a Dirichlet form on $L^2(\mathcal{X}, \mu)$. Then,*

- *$u, v \in \mathrm{Dom}(\mathscr{E}) \implies u \vee v, u \wedge v$ and $u \wedge 1 \in \mathrm{Dom}(\mathscr{E})$.*

We mention that there is a connection between Dirichlet forms and a family of Markov processes known as Hunt processes, a family of strong Markov processes which are quasi left-continuous with respect to the minimal completed admissible filtration (see [28] for detailed definition). We do not require the theory of Hunt processes for this thesis but mention that any Feller process is also a Hunt process.

It is well known that a Dirichlet form $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ generates a sub-Markovian semigroup of symmetric operators on $L^2(\mathcal{X}, \mu)$. It is also well known that for any regular Dirichlet form $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$, there exists a Hunt process $(X_t)_{t\geq 0}$. We note that the transition function $(p_t)_{t\geq 0}$ of the μ-symmetric Hunt process $(X_t)_{t\geq 0}$ on $\mathcal{X}$ uniquely determines a sub-Markovian semigroup $(P_t)_{t\geq 0}$ on $L^2(\mathcal{X}, \mu)$ and hence a Dirichlet form $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ on $L^2(\mathcal{X}, \mu)$. Moreover, $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ admits a diffusion process if and only if the form $\mathscr{E}$ is local.

We conclude this section with a result on the products of Dirichlet forms as proven in [56]. For any measure space $(\mathcal{X}, \mu)$ and any real Hilbert space on $\mathfrak{H}$ we denote by $L^2(\mathcal{X}, \mu; \mathfrak{H})$ the real L^2-space of $\mathfrak{H}$-valued functions on $\mathcal{X}$. Given linear spaces $\mathfrak{L}^{(i)}$ for $i = 1, 2$ of functions on $\mathcal{X}^{(i)}$ respectively, we denote by $\mathfrak{L}^{(1)} \otimes \mathfrak{L}^{(2)}$ the linear space generated by $\{u^1 \otimes u^2 : u^i \in \mathfrak{L}^{(i)}, i = 1, 2\}$ where $(u^1 \otimes u^2)(x_1, x_2) = u^1(x_1)u^2(x_2)$.

Theorem 2.3.3. *[56, Theorem 1.4] Let $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ be the Dirichlet form of the direct product of conservative Hunt processes $(X^{(i)}_t)_{t\geq 0}$ for $i = 1, 2$ associated with regular Dirichlet forms $(\mathscr{E}^{(i)}, \mathrm{Dom}(\mathscr{E}^i))$ on $L^2(\mathcal{X}^{(i)}, \mu^{(i)})$ respectively. Let $\mathcal{C}^{(i)}$ be any cores of $\mathscr{E}^{(i)}$, respectively. Then $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ possesses $\mathcal{C}^{(1)} \otimes \mathcal{C}^{(2)}$ as its core and admits the following expressions: for any $u \in \mathrm{Dom}(\mathscr{E})$,*

$$[\mathcal{X}^{(2)} \to \mathrm{Dom}(\mathscr{E}^{(1)}) : x_2 \mapsto u(\cdot, x_2)] \in L^2(\mathcal{X}^{(2)}, \mu^{(2)}; \mathrm{Dom}(\mathscr{E}^{(1)})), \tag{2.3.1}$$
$$[\mathcal{X}^{(1)} \to \mathrm{Dom}(\mathscr{E}^{(2)}) : x_1 \mapsto u(x_1, \cdot)] \in L^2(\mathcal{X}^{(1)}, \mu^{(1)}; \mathrm{Dom}(\mathscr{E}^{(2)})), \tag{2.3.2}$$

and

$$\mathscr{E}(u, u) = \int_{\mathcal{X}^{(2)}} \mathscr{E}^{(1)}(u(\cdot, x_2), u(\cdot, x_2))\mu^{(2)}(\mathrm{d}x_2) + \int_{\mathcal{X}^{(1)}} \mathscr{E}^{(2)}(u(x_1, \cdot), u(x_1, \cdot))\mu^{(1)}(\mathrm{d}x_1). \tag{2.3.3}$$

Furthermore, if $u = u^1 \otimes u^2$ and $v = v^1 \otimes v^2$ with $u^i, v^i \in \mathrm{Dom}(\mathscr{E}^{(i)})$ for $i = 1, 2$, then $u, v \in \mathrm{Dom}(\mathscr{E})$ and

$$\mathscr{E}(u, v) = \mathscr{E}^{(1)}(u^1, v^1)\langle u^2, v^2\rangle_{L^2(\mathcal{X}^{(2)}, \mu^{(2)})} + \mathscr{E}^{(2)}(u^2, v^2)\langle u^1, v^1\rangle_{L^2(\mathcal{X}^{(1)}, \mu^{(1)})}. \tag{2.3.4}$$

2.4 One-Dimensional Diffusions

In the stochastic approach to the extension method, the results on one-dimensional diffusions found in [5] are used extensively. We recall some of these results here and rewrite them for the special cases we shall focus on in this thesis.

In our setting, we consider a one-dimensional diffusion $(Y_t)_{t\geq 0}$ on a family of probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \{\mathbb{P}_y\}_{y\in I})$ taking values in the interval $I = [0, l] \cap \mathbb{R}$ where $l \in (0, \infty]$. For Borel set $A \subset I$, let

$$H_A(Y) = \inf\{t > 0 : Y_t \in A\},$$

and for any $y \in I$, define $H_y = H_{\{y\}}(Y)$, $H_{y+} = H_{(y,\infty)\cap I}(Y)$ and $H_{y-} = H_{(-\infty,y)\cap I}(Y)$. We recall the process is *regular* at a point $y \in I$ if

$$\mathbb{P}_y(\{H_{y+} = 0\}) = \mathbb{P}_y(\{H_{y-} = 0\}) = 1.$$

Otherwise we say the point y is *singular* and

- *left-singular* if $\mathbb{P}_y(\{H_{y+} = 0\}) = 0$,
- *right-singular* if $\mathbb{P}_y(\{H_{y-} = 0\}) = 0$,
- *absorbing* if $\mathbb{P}_y(\{H_{y+} = 0\}) = \mathbb{P}_y(\{H_{y-} = 0\}) = 0$.

We denote the set of left-singular points K_-, the set of right-singular points K_+ and the set of absorbing points $E = K_+ \cap K_-$.

We assume the diffusion $(Y_t)_{t\geq 0}$ has $R = (0, l)$ as the set of regular points and $0 \in K_+ \setminus E$ and if $l < \infty$, $l \in K_-$. Such a diffusion Y is uniquely determined by a scale function and a speed measure and its construction, given by [5, Theorem 6.5], is based on a random time change of a Wiener process. We restate the result for the special case we consider here where the process Y is in natural scale and has speed measure $\tilde{m}$ on $([0, l] \cap \mathbb{R}, \mathrm{Bor}([0, l] \cap \mathbb{R}))$.

Theorem 2.4.1. *Let $A \subset \mathbb{R}$ be a Borel set. We define a measure m (which we call the Krein string corresponding to the diffusion $(Y_t)_{t\geq 0}$) on $\mathbb{R}$ by,*

$$m(A) = \begin{cases} 2\tilde{m}(A \cap (0, l)) + \tilde{m}(A \cap \{0\}) & \text{if } l = \infty, \\ 2\tilde{m}(A \cap (0, l)) + \tilde{m}(A \cap \{0\}) + \tilde{m}(A \cap \{l\}) & \text{if } l < \infty, \end{cases} \tag{2.4.1}$$

Let $(W_t)_{t\geq 0}$ be a Wiener process on a family of filtered probability spaces $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\geq 0}, \{\mathbb{P}_y\}_{y\in\mathbb{R}})$ with shift operators $\Theta = (\theta_t)_{t\geq 0}$ and local time processes $\{(L_t^y(W))_{t\geq 0}\}_{y\in\mathbb{R}}$. We define $A = (A_t)_{t\geq 0}$,

$$A_t = \frac{1}{2} \int_{\mathbb{R}} L_t^y(W) m(\mathrm{d}y),$$

and

$$A_t^{-1} = \inf\{s \geq 0 : A_{s+} > t\}.$$

Then $(Y_t)_{t\geq 0}$ *given by* $Y_t = W_{A_t^{-1}}$ *is a continuous strong Markov process on the family of filtered probability spaces* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \{\mathbb{P}_y\}_{y\in I})$ *where* $\mathcal{F}_t = \mathcal{G}_{A_t^{-1}}$ *with shift operators* $\Theta \circ A^{-1}$ *whose set of regular points, left singular and right singular points is* R, K_- *and* K_+ *respectively.*

By [5, Proposition 5.34] the speed measure satisfies,

(S1) $\tilde{m}(K) < \infty$ for every compact $K \subset [0, l)$,

(S2) $\tilde{m}([0, l]) < \infty$ if $l < \infty$ is not absorbing,

(S3) $\tilde{m}(U) > 0$ for any open $U \subset (0, l)$,

and hence so does m. Furthermore, any measure satisfying (S1-3) admits a Lebesgue decomposition into the sum of two singular measures, one of which is absolutely continuous with respect to Lebesgue measure. By (S3), the density of the absolutely continuous measure can be chosen to be strictly positive. Without loss of generality, we can write the Lebesgue decomposition of the measure $\tilde{m}$ as

$$\tilde{m}(\mathrm{d}y) = \begin{cases} \frac{1}{2}b^{-2}(y)\,\mathrm{d}y + m_0\delta_0(\mathrm{d}y) + \frac{1}{2}n(\mathrm{d}y) & \text{if } l = \infty \text{ or } l < \infty \text{ is absorbing,} \\ \frac{1}{2}b^{-2}(y)\,\mathrm{d}y + m_0\delta_0(\mathrm{d}y) + \frac{1}{2}n(\mathrm{d}y) + m_l\delta_l(\mathrm{d}y) & \text{otherwise,} \end{cases} \tag{2.4.2}$$

where $b : [0, l] \cap \mathbb{R} \to \mathbb{R}$ is a measurable function, $m_0 = \tilde{m}(\{0\})$ and $m_l = \tilde{m}(\{l\})$, and the measure n satisfies $n([0, l] \cap \mathbb{R} \setminus \mathcal{N}) = 0$ for some Borel set $\mathcal{N} \subset (0, l)$ of Lebesgue measure zero. Note that, since b maps into $\mathbb{R}$, the density $b^{-2}(y)$ is indeed positive, for all $y \in [0, l] \cap \mathbb{R}$. Furthermore, for technical reasons, we set $b(y) = 0$ for all $y \in (\mathcal{N} \cup \{0, l\}) \cap \mathbb{R}$, where $b^{-2}(y) = \infty$ if $b(y) = 0$ as usual. We also note that as the measure $\tilde{m}$ satisfies (S1), b^{-2} is locally integrable in $[0, l)$.

With the above Lebesgue decomposition, [5, Theorem 7.9] yields that Y solves an SDE of type

$$\mathrm{d}Y_t = \sqrt{2}b(Y_t)\,\mathrm{d}B_t + \mathrm{d}L_t^0(Y) - \mathrm{d}L_t^l(Y), \tag{2.4.3}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion, $(L_t^y(Y))_{t\geq 0}$ stands for the symmetric local time process of Y at $y \in [0, l]$. By [5, Theorem 5.27], the local time process satisfies

$$\int_0^t \mathbf{1}_\Gamma(Y_s)\,\mathrm{d}s = \int_\Gamma L_t^y(Y)\tilde{m}(\mathrm{d}y)$$

for any Borel set $\Gamma \subset [0, l] \cap \mathbb{R}$ for all $t \geq 0$ almost surely.

Remark 2.4.2. *In this thesis, $\tilde{m}$ denotes the speed measure of the diffusion $(Y_t)_{t\geq 0}$ whereas m denotes the Krein string (as introduced for a more general class of processes in the following Chapter) corresponding to the diffusion.*

Chapter 3

Krein Strings and the Krein Correspondence

3.1 Introduction

Of key importance in this thesis is the connection between gap diffusions reflected at zero and a certain family of subordinators. The connection between these objects is provided by Krein's string theory which is discussed in detail in [26, 38, 44, 65]. We now include the relevant results required for this thesis.

Let $m(\mathrm{d}y)$ be a non-negative Borel measure on $[0, r]$ where $r \in (0, \infty]$. This measure is called an inextensible measure on $[0, r]$ if there exists a non-negative Radon measure (a Borel measure which is finite on compacta) $m'(\mathrm{d}y)$ on $[0, r)$ such that by extending $m'(\mathrm{d}y)$ to $[0, r]$ by setting $m'(\{r\}) = 0$ we have,

$$m(\mathrm{d}y) = \begin{cases} m'(\mathrm{d}y) & \text{if } r + m'([0, r)) = \infty, \\ m'(\mathrm{d}y) + \infty\delta_r(\mathrm{d}y) & \text{if } r + m'([0, r)) < \infty, \end{cases}$$

where δ_r is the Dirac delta measure at $y = r$.

Throughout, we assume $m([0, \delta)) > 0$ for all $\delta > 0$. The inextensible measure can be obtained as the Lebesgue-Stieltjes measure $\mathrm{d}m(y)$ of a non-decreasing, left-continuous function $m : [0, \infty) \to [0, \infty] : y \mapsto m([0, y))$ (abusing notation we use the same letter to denote the measure and the corresponding function). In the literature, this function is known as the *Krein string* associated with the inextensible measure, however, in this thesis we shall use the term Krein string to refer to either the measure $m(\mathrm{d}y)$ or the corresponding function $m(y)$. We denote by E_m the support of the measure m on $[0, r)$ and note that $0 \in E_m$ and let $l = \sup E_m \leq r$.

As $m'(\mathrm{d}y)$ is a σ-finite measure on $[0, r)$, it possesses a Lebesgue decomposition [27, Theorem 3.8] so there exists a Borel set $N \subset (0, r)$, a σ-finite Borel measure n on N and a locally integrable function $h : [0, r) \to [0, \infty]$ such that

$$m'(\mathrm{d}y) = h(y)\,\mathrm{d}y + m_0\delta_0(\mathrm{d}y) + n(\mathrm{d}y) + m_l\delta_l(\mathrm{d}y),$$

where $n((0, r) \setminus N) = \mathrm{Leb}(N) = 0$ and without loss of generality, we set

$$h(y) = \infty,$$

for $y \in N$. In analogy with the one-dimensional diffusion case, we define a measurable function $b : [0, r) \to [0, \infty]$ by $b = \frac{1}{\sqrt{h}}$ (with the convention that $b(y) = \infty$ if $h(y) = 0$ and $b(y) = 0$ if $h(y) = \infty$).

3.2 L^2-theory for Krein Strings

We now review the L^2-theory associated with the Krein's strings as detailed in [26]. The aim of this section is to define a non-positive self-adjoint operator $(\mathcal{G}_y, \mathrm{Dom}(\mathcal{G}_y))$ associated with the inextensible measure $m(\mathrm{d}y)$. We consider the (complex) Hilbert space $L^2([0, r), m(\mathrm{d}y))$ of $m(\mathrm{d}y)$-measurable functions $\phi : [0, r) \to \mathbb{C}$ such that

$$\|\phi\|_m^2 = \int_{[0,r)} |\phi(y)|^2 m(\mathrm{d}y) < \infty.$$

We now define the generalised second order differential operator $\mathcal{G}_y = \frac{\mathrm{d}^2}{\mathrm{d}m\,\mathrm{d}y}$ on certain domains of this Hilbert space. Let $\mathrm{Dom}_0(\mathcal{G}_y)$ be the set of functions $u : [0, r) \to \mathbb{C}$ such that there exists an $m(\mathrm{d}y)$-measurable function $g : [0, r) \to \mathbb{C}$ with

$$\begin{cases} \int_{[0,\tilde{y}]} |g(y)|^2 m(\mathrm{d}y) < \infty & \text{for all } \tilde{y} < l \text{ if } l + m[0, l) = \infty, \\ \int_{[0,l]} |g(y)|^2 m(\mathrm{d}y) < \infty & \text{if } l + m[0, l) < \infty, \end{cases}$$

such that,

$$u(y) = \alpha + \beta y + \int_0^y \int_{[0,\xi]} g(w) m(\mathrm{d}w)\,\mathrm{d}\xi.$$

In this case we write $\mathcal{G}_y u = g$, $u''(y) = g(y)m(\mathrm{d}y)$ or in the Lebesgue-Stieltjes form $\mathrm{d}u^+(y) = g(y)\,\mathrm{d}m(y)$. We note that every function in $\mathrm{Dom}_0(\mathcal{G}_y)$ is absolutely continuous, linear outside of E_m and has right and left derivatives

$$u^+(y) = \beta + \int_{[0,y]} g(w)m(\mathrm{d}w),$$
$$u^-(y) = \beta + \int_{[0,y)} g(w)m(\mathrm{d}w).$$

From the standpoint of $L^2([0,r),m)$, it does not make sense to prescribe f on $[0,r) \setminus E_m$. Nonetheless we find this 'broken line' characterisation is often useful as it allows us to define functions on all of $[0,r)$ while encoding the boundary behavior of the functions in $\partial E_m \cap (0,l]$ when E_m is disconnected, and hence, it is useful for defining the domain $\mathrm{Dom}(\mathcal{G}_y)$ of $\mathcal{G}_y$ in $L^2([0,r),m)$.

The operator $(\mathcal{G}_y, \mathrm{Dom}_0(\mathcal{G}_y))$ is now a 'local' operator which acts on functions which are sufficiently 'smooth' with respect to the measure m. For example, if $m(\mathrm{d}y) = h(y)\,\mathrm{d}y$ where $h > 0$ is continuous, then $f \in \mathrm{Dom}_0(\mathcal{G}_y)$ if and only if f' is absolutely continuous and f'' is locally L^2 in which case

$$\mathcal{G}_y f = \frac{1}{h(y)} f''(y).$$

On the other hand if m is singular with respect to Lebesgue measure, then for any element f to belong to $\mathrm{Dom}_0(\mathcal{G}_y)$, f^+ must be 'equally rough' to ensure $\frac{\mathrm{d}^2}{\mathrm{d}m\,\mathrm{d}y}$ is regular. The following calculations provide some insight into what form the operator $\mathcal{G}_y$ may take in the 'extremal' cases [26, Exercise 5.1.3].

Example 3.2.1 (Second Order Differential Operators)**.** *Let $\mathcal{A}_z = \frac{1}{2}\sigma^2(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mu(z)\frac{\mathrm{d}}{\mathrm{d}z}$ where $\sigma(z) > 0$ and μ are reasonably well behaved. Then there exists a change of variables $p : z \mapsto y$ such that $\mathcal{G}_y = \frac{1}{h(y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2} = \mathcal{A}_z$.*

Proof. Let $p(z) = \int_0^z \exp\big(-\int_0^\xi \frac{2\mu(\eta)}{\sigma^2(\eta)}\,\mathrm{d}\eta\big)\,\mathrm{d}\xi$ and define a new variable $y = p(z)$. Then we have,

$$p'(z) = \exp\Big(-\int_0^z \frac{2\mu(\eta)}{\sigma^2(\eta)}\,\mathrm{d}\eta\Big),$$
$$p''(z) = -\frac{2\mu(z)}{\sigma^2(z)}\exp\Big(-\int_0^z \frac{2\mu(\eta)}{\sigma^2(\eta)}\,\mathrm{d}\eta\Big),$$

so $\frac{\mathrm{d}}{\mathrm{d}y} = \frac{1}{p'(z)}\frac{\mathrm{d}}{\mathrm{d}z}$ and $\frac{\mathrm{d}^2}{\mathrm{d}y^2} = \frac{1}{p'(z)^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{p''(z)}{p'(z)^3}\frac{\mathrm{d}}{\mathrm{d}z}$ and hence,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}y^2} &= \exp\Big(\int_0^z \frac{4\mu(\eta)}{\sigma^2(\eta)}\,\mathrm{d}\eta\Big)\Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2\mu(z)}{\sigma^2(z)}\frac{\mathrm{d}}{\mathrm{d}z}\Big)\\ &= \exp\Big(\frac{8}{\sigma^2(z)}\int_0^z \frac{\mu(\eta)}{\sigma^2(\eta)}\,\mathrm{d}\eta\Big)\Big(\frac{1}{2}\sigma^2(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mu(z)\frac{\mathrm{d}}{\mathrm{d}z}\Big),\end{aligned}$$

so $h(y) = \exp\Big(\frac{8}{\sigma^2(p^{-1}(y))}\int_0^{p^{-1}(y)} \frac{\mu(\eta)}{\sigma^2(\eta)}\,\mathrm{d}\eta\Big)$. □

Example 3.2.2 (Point measure)**.** *Let $\mathcal{P} = \{y_0 = 0 < y_1 < \cdots\}$ be a partition of $[0,\infty)$ and let $m(\mathrm{d}y) = \sum_{n=0}^{\infty} m_n \delta_{y_n}(\mathrm{d}y)$ where $m_n > 0$ for all $n \geq 0$. Then*

$$\mathcal{G}_y f(y_k) = \frac{1}{m_k}\left(\frac{f(y_{k+1}) - f(y_k)}{y_{k+1} - y_k} - \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}}\right),$$

for $k \geq 1$ (although $\mathcal{G}_y f$ is still ambiguous at $y = 0$ and $y = l$).

Proof. Using the integral equations we find,

$$\begin{aligned}f(y_k) - f(y_{k-1}) &= \int_{[0,y_{k-1}]} (y_k - y_{k-1})\mathcal{G}_y f(\xi) m(\mathrm{d}\xi),\\ f(y_{k+1}) - f(y_k) &= \int_{[0,y_k]} (y_{k+1} - y_k)\mathcal{G}_y f(\xi) m(\mathrm{d}\xi),\end{aligned}$$

and so we have,

$$\left(\frac{f(y_{k+1}) - f(y_k)}{y_{k+1} - y_k} - \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}}\right) = \int_{(y_{k-1},y_k]} \mathcal{G}_y f(\xi) m(\mathrm{d}\xi) = m_k \mathcal{G}_y f(y_k).$$

□

We now restrict the operator $\mathcal{G}_y$ to a smaller domain $\mathrm{Dom}(\mathcal{G}_y) \subset \mathrm{Dom}_0(\mathcal{G}_y)$ by imposing boundary and integrability conditions. We define

$$\mathrm{Dom}_-(\mathcal{G}_y) = \{f \in \mathrm{Dom}_0(\mathcal{G}_y) : f^-(0) = 0\}.$$

At the right-end point, the situation is more complicated. We define

$$\mathrm{Dom}_+(\mathcal{G}_y) = \begin{cases} \{f \in \mathrm{Dom}_0(\mathcal{G}_y) : \|f\| + \|\mathcal{G}_y f\| < \infty\} \\ \qquad \text{if } l + m([0,l)) = \infty \text{ and } \int_{[0,r)} y^2 m(\mathrm{d}y) = \infty, \\ \{f \in \mathrm{Dom}_0(\mathcal{G}_y) : \|f\| + \|\mathcal{G}_y f\| < \infty, f^+(l-) = 0\} \\ \qquad \text{if } l + m([0,l)) = \infty \text{ and } \int_{[0,r)} y^2 m(\mathrm{d}y) < \infty, \\ \{f \in \mathrm{Dom}_0(\mathcal{G}_y) : \|f\| + \|\mathcal{G}_y f\| < \infty, (r-l)f^+(l) + f(l) = 0\} \\ \qquad \text{if } l + m([0,l)) < \infty. \end{cases}$$

We note in the second case, the assumption

$$\int_{[0,r)} y^2 m(\mathrm{d}y) < \infty \implies m([0,r)) < \infty,$$

so $l = r = \infty$ and so $f^+(l-) = 0 \implies \lim_{y\to\infty} f^+(y) = 0$. In the third case, we may interpret the condition as $f^+(l) = 0$ when $r = \infty$, otherwise this condition is the same as saying $f(r-) = 0$ when $r < \infty$. We may now state the main theorem of this section [26, Section 5.2].

Proposition 3.2.1. *The set* $\mathrm{Dom}(\mathcal{G}_y) = \mathrm{Dom}_+(\mathcal{G}_y) \cap \mathrm{Dom}_-(\mathcal{G}_y) \subset L^2([0,r), m)$ *is dense and* $(\mathcal{G}_y, \mathrm{Dom}(\mathcal{G}_y))$ *is a non-positive, self-adjoint operator and for all* $f \in \mathrm{Dom}(\mathcal{G}_y)$,

$$\langle f, \mathcal{G}_y f\rangle_{L^2(I,m)} = \begin{cases} -\int_0^l |f^+|^2 \,\mathrm{d}y \\ \qquad \textit{if } l + m([0,l)) = \infty, \\ -\int_0^r |f^+|^2 \,\mathrm{d}y = -\int_0^l |f^+|^2 \,\mathrm{d}y - (r-l)|f^+(l)|^2 \\ \qquad \textit{if } l + m([0,l)) < \infty \textit{ and } r < \infty, \\ -\int_0^l |f^+|^2 \,\mathrm{d}y \\ \qquad \textit{if } l + m([0,l)) < \infty \textit{ and } r = \infty. \end{cases}$$

As $(\mathcal{G}_y, \mathrm{Dom}(\mathcal{G}_y))$ is a non-positive, self-adjoint operator on $L^2([0,r), m)$, we can consider the corresponding Dirichlet form on $L^2([0,r), m)$. However, as $\operatorname{supp} m$ might not be equal to $[0,r)$, the results of Section 2.3 do not immediately apply. Nevertheless we shall revisit the Dirichlet form theory when we discuss the probabilistic interpretation of Krein strings.

3.2.1 Fundamental Functions

For an inextensible measure $m(\mathrm{d}y)$ on $[0,r]$, the following integral equations have unique solutions:

$$\Phi_z(y) = 1 + z\int_{[0,y]}(y-\xi)\Phi_z(\xi)m(\mathrm{d}\xi),$$
$$\Psi_z(y) = y + z\int_{[0,y]}(y-\xi)\Psi_z(\xi)m(\mathrm{d}\xi),$$

for all $y \in [0,r)$ and $z \in \mathbb{C}$. For each $y \in [0,r)$, $z \mapsto \Phi_z(y)$ and $z \mapsto \Psi_z(y)$ are analytic and real valued when $z \in \mathbb{R}$ (in which case we denote the variable λ instead of z). We call the pair $\{\Phi_z(y), \Psi_z(y)\}$ the system of fundamental functions for the inextensible measure $m(\mathrm{d}y)$. The following properties can be found in [26, 65].

Proposition 3.2.2 (Properties of the Fundamental Functions)**.** *The fundamental functions of the inextensible measure $m(\mathrm{d}y)$ satisfy the following properties:*

- *For all $\lambda > 0$, $y \mapsto \Phi_\lambda(y)$ and $y \mapsto \Psi_\lambda(y)$ are non-negative, absolutely continuous, linear outside of E_m and have left and right derivatives.*
- *For all $\lambda \geq 0$, $y \mapsto \Phi_\lambda(y)$ and $y \mapsto \Psi_\lambda(y)$ is increasing and convex.*
- *The Wronksian $w_\alpha = \Psi^+_\lambda(y)\Phi_\lambda(y) - \Psi_\lambda(y)\Phi^+_\lambda(y) = 1$ and so*

$$\Big(\frac{\Psi_\lambda(y)}{\Phi_\lambda(y)}\Big)^+ = \frac{1}{\Phi_\lambda(y)^2}.$$

- *For all $\lambda > 0$, $\Phi_\lambda \in \mathrm{Dom}_-(\mathcal{G}_y) \setminus \mathrm{Dom}_+(\mathcal{G}_y)$ and $\Psi_\lambda \in \mathrm{Dom}_+(\mathcal{G}_y) \setminus \mathrm{Dom}_-(\mathcal{G}_y)$.*

The key result for this thesis is known as the *Krein correspondence* which was first proven in [45], although we reference [38, 44, 65] for easier to find statements of the theorem. One formulation of the result can be given as follows.

Theorem 3.2.3 (Krein, Spectral Formulation)**.** *There is a one-to-one correspondence between Krein strings $m(\mathrm{d}y)$ on $[0,r)$ and complete Bernstein functions ψ. Furthermore, if $\{\Phi_\lambda(y), \Psi_\lambda(y)\}$ are the fundamental functions for m then,*

$$\lim_{y\to r}\frac{\Psi_\lambda(y)}{\Phi_\lambda(y)} = \int_0^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi = \frac{1}{\psi(\lambda)},$$

for $\lambda > 0$. The Stieltjes function $\mathfrak{h}(\lambda) = \frac{1}{\psi(\lambda)}$ is called the characteristic function of the Krein string m.

3.3 The Extension Function

In addition to the fundamental functions, we define the extension function associated to the inextensible measure $m(\mathrm{d}y)$ by

$$\varphi_\lambda(y) = \Phi_\lambda(y) - \psi(\lambda)\Psi_\lambda(y).$$

Proposition 3.3.1 (Properties of Extension Function)**.** *The function $\varphi_\lambda(y)$ satisfies the following properties:*

1. $\varphi_\lambda(y) = \psi(\lambda)\Phi_\lambda(y) \int_y^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi$*,*
2. $y \mapsto \varphi_\lambda(y)$ *is decreasing and convex for all* $\lambda \geq 0$*,*
3. $0 \leq \varphi_\lambda(y) \leq 1$ *for all* $y \in [0, r)$*,* $\lambda \geq 0$*.*

Proof. 1. By Theorem 3.2.2 and Theorem 3.2.3 we have,

$$\begin{aligned}
\varphi_\lambda(y) &= \psi(\lambda)\Big(- \Psi_\lambda(y) + \Phi_\lambda(y) \int_0^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi\Big) \\
&= \psi(\lambda)\Big(- \Phi_\lambda(y) \int_0^y \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi + \Phi_\lambda(y) \int_0^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi\Big) \\
&= \psi(\lambda)\Phi_\lambda(y) \int_y^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi.
\end{aligned}$$

2. To prove $y \mapsto \varphi_\lambda(y)$ is decreasing we note that $\Phi_\lambda^+(y) - \Phi_\lambda^+(y_0) = \int_{(y_0,y]} \lambda\Phi_\lambda(y)m(\mathrm{d}y) \geq 0$ for any $y \geq y_0$ and so,

$$\Phi_\lambda^+(y) \int_y^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi \leq \int_y^r \frac{\Phi_\lambda^+(\xi)}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi = -\int_y^r \Big(\frac{1}{\Phi_\lambda(\xi)}\Big)^+\,\mathrm{d}\xi \leq \frac{1}{\Phi_\lambda(y)}.$$

Therefore,

$$\begin{aligned}
\varphi_\lambda^+(y) &= \psi(\lambda)\Big(\Phi_\lambda(y) \int_y^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi\Big)^+ \\
&= \psi(\lambda)\Big(\Phi_\lambda^+(y) \int_y^r \frac{1}{\Phi_\lambda(\xi)^2}\,\mathrm{d}\xi - \frac{1}{\Phi_\lambda(y)}\Big) \\
&\leq 0.
\end{aligned}$$

Furthermore, it is clear that $\varphi_\lambda(y) \geq 0$ by Part 1 and so the distributional derivative $\varphi_\lambda''(y) = \lambda\varphi_\lambda(y)m(\mathrm{d}y)$ is a non-negative Radon measure and hence $y \mapsto \varphi_\lambda(y)$ is convex.

3. The upper bound follows as $\varphi_\lambda(0) = \Phi_\lambda(0) = 1$ and $y \mapsto \varphi_\lambda(y)$ is decreasing while the lower bound follows by Part 1.

$\square$

We also have the following variational characterisation for the extension function.

Theorem 3.3.2. *Suppose $m(\{0\}) = 0$. Then for all $\lambda > 0$, $\varphi_\lambda \in \mathrm{Dom}_+(\mathcal{G}_y)$. Let $f : [0, r) \to \mathbb{R}$ be absolutely continuous with $f(0) = 1$.*

i) If $l + m([0, l)) < \infty$ and $r = \infty$, then

$$\int_0^l f'(\tilde{y})^2 \,\mathrm{d}\tilde{y} + \int_{[0,l]} \lambda f(\tilde{y})^2 m(\mathrm{d}\tilde{y}) \geq \psi(\lambda).$$

ii) Otherwise assume $\lim_{y\to r} f(y) = 0$. Then,

$$\int_0^r f'(\tilde{y})^2 \,\mathrm{d}\tilde{y} + \int_{[0,r)} \lambda f(\tilde{y})^2 m(\mathrm{d}\tilde{y}) \geq \psi(\lambda).$$

In either case, equality holds if and only if $f = \varphi_\lambda$ m-a.e.

Proof. We split the proof into two cases:

1. $l + m([0, l)) = \infty$ or $r < \infty$:

   We begin by proving the equality for φ_λ. Clearly, $\varphi_\lambda \in \mathrm{Dom}_0(\mathcal{G}_y)$ so it remains to show the integrability and boundary conditions. Using integration by parts for Lebesgue-Stieltjes integrals we find for any $y < r$ that,

   $$\int_{(0,y]} \varphi_\lambda(\tilde{y}) \,\mathrm{d}\varphi_\lambda^+(\tilde{y}) + \int_{(0,y]} \varphi_\lambda^+(\tilde{y}) \,\mathrm{d}\varphi_\lambda(\tilde{y}) = \varphi_\lambda(y)\varphi_\lambda^+(y) - \varphi_\lambda(0)\varphi_\lambda^+(0).$$

   As $\mathrm{d}\varphi_\lambda^+(y) = \lambda\varphi_\lambda(y)m(\mathrm{d}y)$ and $\varphi_\lambda(y)$ is absolutely continuous, we have for almost every $y \in [0, r)$,

   $$\int_{[0,y]} \lambda\varphi_\lambda(\tilde{y})^2 m(\mathrm{d}\tilde{y}) + \int_0^y \varphi_\lambda'(\tilde{y})^2 \,\mathrm{d}\tilde{y} = \varphi_\lambda(y)\varphi_\lambda^+(y) + \psi(\lambda).$$

   Letting $y \to r$, we find $\varphi_\lambda(y)\varphi_\lambda^+(y) \to 0$ as $y \to r$. To see this we note if $r = \infty$, then by convexity we have,

   $$0 \leq -\varphi_\lambda^+(y) \leq \frac{\varphi_\lambda(0) - \varphi_\lambda(y)}{y} \leq \frac{1}{y} \to 0,$$

as $y \to \infty$. Otherwise,

$$\varphi_\lambda(y) = \psi(\lambda) \int_y^r \frac{\Phi_\lambda(y)}{\Phi_\lambda(\xi)^2} \, \mathrm{d}\xi \leq \psi(\lambda) \int_y^r \frac{1}{\Phi_\lambda(\xi)} \, \mathrm{d}\xi,$$

as $y \mapsto \Phi_\lambda(y)$ is increasing. By Cauchy-Schwarz,

$$\int_y^r \frac{1}{\Phi_\lambda(\xi)} \, \mathrm{d}\xi \leq \Big(\int_y^r \frac{1}{\Phi_\lambda(\xi)^2} \, \mathrm{d}\xi \Big)^{1/2} (r-y)^{1/2} \leq \sqrt{\mathfrak{h}(\lambda)} (r-y)^{1/2} \to 0,$$

as $y \to r$ so $\varphi_\lambda(r) = 0$ and as $0 \leq -\varphi_\lambda^+(r) \leq \frac{1}{r}$, $\varphi_\lambda(y)\varphi_\lambda^+(y) \to 0$ as $y \to r$. Therefore,

$$\int_0^r \varphi_\lambda'(\tilde{y})^2 \, \mathrm{d}\tilde{y} + \int_{[0,r)} \lambda \varphi_\lambda(\tilde{y})^2 m(\mathrm{d}\tilde{y}) = \psi(\lambda).$$

In particular, as $\mathcal{G}_y \varphi_\lambda = \lambda \varphi_\lambda$ and $\|\varphi_\lambda\|^2_{L^2([0,r),m)} \leq \frac{\psi(\lambda)}{\lambda} < \infty$,

$$\|\mathcal{G}_y \varphi_\lambda\|^2_{L^2([0,r),m)} \leq \lambda \psi(\lambda),$$

and so $\varphi_\lambda \in \mathrm{Dom}_+(\mathcal{G}_y)$ if $\int_{[0,l)} y^2 m(\mathrm{d}y) = \infty$. Otherwise if $\int_{[0,l)} y^2 m(\mathrm{d}y) < \infty$ it remains to show $\varphi_\lambda(l-) = 0$ which is equivalent to $\lim_{y\to\infty} \varphi_\lambda^+(y) = 0$ and follows by the same convexity argument as before. To prove the case when $l + m([0,l)) < \infty$ and $r < \infty$, we note that φ_λ is linear on (l,r) with $\varphi_\lambda(r) = 0$. As $\varphi_\lambda^+(y) = \varphi_\lambda^+(l)$ for all $y \in (l,r)$, we have

$$\frac{\varphi_\lambda(y) - \varphi_\lambda(l)}{y - l} = \varphi_\lambda^+(l) \implies \varphi_\lambda^+(l)(r-l) + \varphi_\lambda(l) = 0,$$

by taking $y \to r$.

To prove the inequality, we now assume $f : [0,r) \to \mathbb{R}$ is absolutely continuous with $f(0) = 1$ and $f(r-) = 0$. Clearly, the result is trivial if $\int_0^r |f'(y)|^2 \, \mathrm{d}y = \infty$. Otherwise, let $g = f - \varphi_\lambda$ so by integration by parts we have,

$$\int_0^y \varphi_\lambda'(\tilde{y}) g'(\tilde{y}) \, \mathrm{d}\tilde{y} = \varphi_\lambda^+(y) g(y) - \varphi_\lambda^+(0) g(0) - \int_{(0,y]} \lambda \varphi_\lambda(\tilde{y}) g(\tilde{y}) m(\mathrm{d}\tilde{y}),$$

for any $y < r$. As we know $g(0) = 0$, $\varphi_\lambda^+(r-)$ is bounded and $\lim_{y \to r} g(y) = 0$, we may take a sequence $y \to r$ to obtain,

$$\int_0^r g'(\tilde{y}) \varphi_\lambda'(\tilde{y}) \, \mathrm{d}\tilde{y} + \int_{[0,r)} \lambda g(\tilde{y}) \varphi_\lambda(\tilde{y}) m(\mathrm{d}\tilde{y}) = 0.$$

Then,

$$
\begin{aligned}
&\int_0^r f'(y)^2 \,\mathrm{d}y + \int_{[0,r)} \lambda f(y)^2 m(\mathrm{d}y) \\
&\quad = \int_0^r (g'(y) + \varphi_\lambda'(y))^2 \,\mathrm{d}y + \int_{[0,r)} \lambda (g(y) + \varphi_\lambda(y))^2 m(\mathrm{d}y) \\
&\quad = \Big(\int_0^r g'(y)^2 \,\mathrm{d}y + \int_{[0,r)} \lambda g(y)^2 m(\mathrm{d}y) \Big) + \Big(\int_0^r \varphi_\lambda'(y)^2 \,\mathrm{d}y + \int_{[0,r)} \lambda \varphi_\lambda(y)^2 m(\mathrm{d}y) \Big) \\
&\qquad\quad + 2\Big(\int_0^r g'(y)\varphi_\lambda'(y) \,\mathrm{d}y + \int_{[0,r)} \lambda g(y)\varphi_\lambda(y) m(\mathrm{d}y) \Big) \\
&\quad > \psi(\lambda),
\end{aligned}
$$

as $\int_0^r g'(y)^2 \,\mathrm{d}y + \int_{[0,r)} \lambda g(y)^2 m(\mathrm{d}y) > 0$ when $f \neq \varphi_\lambda$.

2. $l + m([0,l)) < \infty$ and $r = \infty$:

   In this case,

$$
\psi(\lambda) = \lim_{y\to\infty} \frac{\Phi_\lambda(y)}{\Psi_\lambda(y)} = \lim_{y\to\infty} \frac{\Phi_\lambda(l) + \Phi^+(l)(y-l)}{\Psi_\lambda(l) + \Psi^+(l)(y-l)} = \frac{\Phi^+(l)}{\Psi^+(l)}.
$$

   Therefore,

$$
\varphi_\lambda^+(l) = \Phi^+(l) - \psi(\lambda)\Psi^+(l) = 0,
$$

   so φ_λ satisfies the boundary condition at l. Then arguing as before with $y < l$ we have,

$$
\int_0^y \varphi_\lambda'(\tilde{y})^2 \,\mathrm{d}\tilde{y} = \varphi_\lambda^+(y)\varphi_\lambda(y) + \psi(\lambda) - \int_{[0,y)} \lambda \varphi_\lambda(\tilde{y})^2 m(\mathrm{d}\tilde{y}),
$$

   so by taking $y \to l$ we have,

$$
\int_0^l \varphi_\lambda'(\tilde{y})^2 \,\mathrm{d}\tilde{y} + \int_{[0,l)} \lambda \varphi_\lambda(\tilde{y})^2 m(\mathrm{d}\tilde{y}) = \varphi_\lambda^-(l)\varphi_\lambda(l) + \psi(\lambda),
$$

   and by noting that $\varphi_\lambda^+(l) - \varphi_\lambda^-(y) = \lambda\varphi_\lambda(l)m(\{l\})$, we see that

$$
\int_0^l \varphi_\lambda'(\tilde{y})^2 \,\mathrm{d}\tilde{y} + \int_{[0,l]} \lambda \varphi_\lambda(\tilde{y})^2 m(\mathrm{d}\tilde{y}) = \psi(\lambda).
$$

   To prove the inequality, we now assume $f : [0,\infty) \to \mathbb{R}$ is absolutely continuous with $f(0) = 1$. Again, the result is trivial if $\int_0^l |f'(y)|^2 \,\mathrm{d}y = \infty$ and if

$\int_0^l |f'(y)|^2 \, \mathrm{d}y < \infty$, we note,

$$|f(l)| \leq |f(0)| + \int_0^l |f'(\tilde{y})| \, \mathrm{d}\tilde{y} \leq 1 + \Big(\int_0^l |f'(\tilde{y})|^2 \, \mathrm{d}\tilde{y} \Big)^{1/2} l^{1/2} < \infty.$$

With g as in the previous case, integration by parts for $y < l$ yields,

$$\int_0^y g'(\tilde{y}) \varphi_\lambda'(\tilde{y}) \, \mathrm{d}\tilde{y} = \varphi_\lambda^+(y) g(y) - \varphi_\lambda^+(0) g(0) - \int_{[0,y)} \lambda g(\tilde{y}) \varphi_\lambda(\tilde{y}) m(\mathrm{d}\tilde{y}).$$

By taking $y \to l$ and noting $\varphi_\lambda^+(l) g(l) = -\lambda \varphi_\lambda(l) g(l) m(\{l\})$ we have,

$$\int_0^l g'(\tilde{y}) \varphi_\lambda'(\tilde{y}) \, \mathrm{d}\tilde{y} + \int_{[0,l]} \lambda g(\tilde{y}) \varphi_\lambda(\tilde{y}) m(\mathrm{d}\tilde{y}) = 0.$$

Then the result follows using the same reasoning as before.

□

One property of interest for the extension function is that changing the mass of the speed measure at zero does not change the extension function.

Proposition 3.3.3. *Let $m(\mathrm{d}y)$ be a Krein string with $m_0 = m(\{0\})$ and let $m^0(\mathrm{d}y) = \mathbf{1}_{\{y>0\}} m(\mathrm{d}y)$ and let ψ, ψ^0 and $\varphi_\lambda(y), \varphi_\lambda^0(y)$ be the corresponding complete Bernstein functions and extension functions respectively. Then,*

$$\psi(\lambda) = m_0 \lambda + \psi^0(\lambda),$$

and

$$\varphi_\lambda(y) = \varphi_\lambda^0(y).$$

Proof. Let $\{\Phi_\lambda(y), \Psi_\lambda(y)\}$ (resp. $\{\Phi_\lambda^0(y), \Psi_\lambda^0(y)\}$) be the fundamental functions corresponding to $m(\mathrm{d}y)$ (resp. $m^0(\mathrm{d}y)$). We note that $\Psi_\lambda(y) = \Psi_\lambda^0(y)$ and

$$\Phi_\lambda(y) = \Phi_\lambda^0(y) + m_0 \lambda \Psi_\lambda^0(y).$$

Then we have that

$$\psi(\lambda) = \lim_{y \to r} \frac{\Phi_\lambda(y)}{\Psi_\lambda(y)} = \lim_{y \to r} \frac{\Phi_\lambda^0(y) + m_0 \lambda \Psi_\lambda^0(y)}{\Psi_\lambda^0(y)} = \psi^0(\lambda) + m_0 \lambda,$$

and

$$\begin{aligned}\varphi_\lambda(y) &= \left(\Phi_\lambda^0(y) + m_0\lambda\Psi_\lambda^0(y)\right) - (\psi^0(\lambda) + m_0\lambda)\Psi_\lambda^0(y)\\ &= \Phi_\lambda^0(y) - \psi^0(\lambda)\Psi_\lambda^0(y)\\ &= \varphi_\lambda^0(y).\end{aligned}$$

□

The following approximation result due to Kasahara [38] is useful for simulations as it allows us to approximate a Krein string numerically in order to find an approximation for the corresponding complete Bernstein function (see Appendix A.2).

Proposition 3.3.4. *For each $n \in \mathbb{N}_0$, let $m_n(y)$ be a Krein string and $\Phi_\lambda^n(y)$, $\varphi_\lambda^n(y)$ and ψ^n be the corresponding fundamental function, extension function and complete Bernstein function respectively. Then the following are equivalent:*

1. $\lim_{n\to\infty} m_n(y) = m_0(y)$ *for all continuity points $y < r$ of m_0 (i.e. each $y < r$ such that $m_0(\{y\}) = 0$).*
2. $\lim_{n\to\infty} \Phi_\lambda^n(y) = \Phi_\lambda^0(y)$ *for all $y \in [0, r)$ and $\lambda \geq 0$.*
3. $\lim_{n\to\infty} \psi^n(\lambda) \to \psi^0(\lambda)$ *for all $\lambda \geq 0$.*

We also have the following asymptotic property of the Krein correspondence which shows how the behaviour of the Krein string near zero gives us information about behaviour of the corresponding complete Bernstein function at infinity.

Proposition 3.3.5. *Let $L : (0,\infty) \to (0,\infty)$ be slowly varying function at ∞ (resp. 0) (i.e. $\lim_t \frac{L(at)}{L(t)} = 1$ for any $a > 0$) and for $\beta \in (0,1)$, let K_β be the slowly varying function such that $t \mapsto t^{1/\beta}K_\beta(t)$ is the inverse of $t \mapsto t^\beta L(t)$. then the following are equivalent:*

1. $m(y) \sim y^{1/\beta-1}K_\beta(y)$ *as $y \to \infty$ (resp. 0).*
2. $\psi(\lambda) \sim \lambda^\beta(D_\beta L(1/\lambda))^{-1}$ *as $\lambda \to 0$ (resp. ∞).*

where $D_\beta = (\beta(1-\beta))^{-\beta}\Gamma(1+\beta)\Gamma(1-\beta)^{-1}$ where $f \sim g$ if $\lim \frac{f}{g} = 1$.

We also prove the following comparison result which can be useful for finding the error of an approximation of a complete Bernstein function when simulating the Krein correspondence.

Proposition 3.3.6. *Let $m(\mathrm{d}y)$ be a Krein string on $[0,\infty)$ and for $r<\infty$ define on $[0,r]$, $m^{kill}(\mathrm{d}y)=\mathbf{1}_{[0,r)}(y)m(\mathrm{d}y)+\infty\cdot\delta_r(\mathrm{d}y)$ and $m^{ref}(\mathrm{d}y)=\mathbf{1}_{[0,r)}(y)m(\mathrm{d}y)$. Then for all $\lambda>0$,*

$$\psi_{kill}(\lambda)\geq\psi(\lambda)\geq\psi_{ref}(\lambda).$$

Proof. Let $\Phi_\lambda(y)$, $\Phi_\lambda^{kill}(y)$ and $\Phi_\lambda^{ref}(y)$ be the fundamental functions corresponding to m, m^{kill} and m^{ref}. Then for $y<r$, $\Phi(y)=\Phi^{kill}(y)=\Phi^{ref}(y)$. For $y\geq r$,

$$\begin{aligned}\Phi_\lambda^{ref}(y)&=1+\int_0^r\int_{[0,\xi]}\Phi_\lambda(w)m(\mathrm{d}w)\,\mathrm{d}\xi+\int_r^y\int_{[0,\xi]}(\Phi_\lambda(r)+\Phi_\lambda^+(r)w)m(\mathrm{d}w)\,\mathrm{d}\xi\\&\leq\Phi_\lambda(y).\end{aligned}$$

Therefore,

$$\int_0^r\frac{1}{\Phi_\lambda(y)^2}\,\mathrm{d}y\leq\int_0^\infty\frac{1}{\Phi_\lambda(y)^2}\,\mathrm{d}y\leq\int_0^\infty\frac{1}{\Phi^{ref}(y)^2}\,\mathrm{d}y,$$

and so $\psi^{kill}(\lambda)\geq\psi(\lambda)\geq\psi^{ref}(\lambda)$. ∎

3.4 Probabilistic Interpretation of the Krein Correspondence

The probabilistic counterpart of Krein strings are gap (or generalised) diffusions which are family of Markov processes obtained via a time change of a Wiener process. The construction of these processes is similar to that of the construction of a one-dimensional diffusion in natural scale.

Let $((W_t)_{t\geq0},\{\mathbb{P}_y\}_{y\in\mathbb{R}})$ be a one-dimensional standard Wiener process and for $y\in\mathbb{R}$, let $(L_t^y(W))_{t\geq0}$ be the local time process at $y\in\mathbb{R}$. Then for $r\in(0,\infty]$ and let $\zeta^W=\inf\{t>0:W_t=r\}$. Then we define,

$$A_t=\frac{1}{2}\int_{[0,r)}L^y_{t\wedge\zeta^W}(W)m(\mathrm{d}y).$$

It should be noted that the local time used in [44, 65] is half the local time used here (which corresponds to the local time defined in [5]). This measure is associated with the Krein string, not the speed measure in the language of diffusions which would be given by

$$\tilde{m}(\mathrm{d}y) = \begin{cases} m_0\delta_0(\mathrm{d}y) + \frac{1}{2}m(\mathrm{d}y)\mathbf{1}_{(0,l)} + m_l\delta_l(\mathrm{d}y) & \text{if } l < r, \\ m_0\delta_0(\mathrm{d}y) + \frac{1}{2}m(\mathrm{d}y)\mathbf{1}_{(0,l)} & \text{if } l = r. \end{cases}$$

Let $(A_t^{-1})_{t\geq 0}$ be the right-inverse of $(A_t)_{t\geq 0}$ given by

$$A_t^{-1} = \inf\{s \geq 0 : A_{s+} > t\},$$

with $A_t^{-1} = \infty$ if $\{s \geq 0 : A_{s+} > t\} = \emptyset$. Then the gap diffusion process $(Y_t)_{t\geq 0}$ associated to the speed measure $\tilde{m}(\mathrm{d}y)$ is given by

$$Y_t = \begin{cases} W_{A_t^{-1}} & \text{for } t < \zeta^Y, \\ \dagger & \text{for } t \geq \zeta^Y, \end{cases}$$

where $\zeta^Y = \lim_{t\to\infty} A_t = \frac{1}{2}\int_{[0,r)} L^y_{\zeta^W}(W)m(\mathrm{d}y)$ and the cemetery state, $\dagger$, is defined in the standard way. In fact, this process is an m-symmetric Hunt process on E_m which is associated with a regular Dirichlet form on $L^2(E_m, m)$ [28, Theorem 6.2.1].

For each $y \in E_m$, $(Y_t)_{t\geq 0}$ admits a local time process $(L_t^y(Y))_{t\geq 0}$ given by $L_t^y(Y) = L^y_{A_t^{-1}}(W)$ which satisfies

$$\int_0^t \mathbf{1}_\Gamma(Y_s)\,\mathrm{d}s = \int_\Gamma L_t^y(Y)\tilde{m}(\mathrm{d}y),$$

for all $t \geq 0$ almost surely for any Borel set $\Gamma \subset E_m$ and this equality specifies the local time of Y as used in [5], which based on the speed measure $\tilde{m}(\mathrm{d}y)$, not the Krein string $m(\mathrm{d}y)$.

Of particular interest in this thesis is the local time at zero, $L_t^0(Y)$, and the corresponding inverse local time at zero $(T_t)_{t\geq 0}$ defined by

$$T_t = \inf\{s > 0 : L_s^0(Y) > t\}.$$

This process is a (possibly killed) subordinator $(T_t)_{t\geq 0}$ (see [10, p.114, Theorem 8]). We may rewrite the spectral formulation of the Krein correspondence given by Theorem 3.2.3 in the following probabilistic formulation.

Theorem 3.4.1 (Krein, Probabilistic Formulation)**.** *Let $(Y_t)_{t\geq 0}$ be the gap diffusion corresponding to the Krein string $m(\mathrm{d}y)$ and let $(T_t)_{t\geq 0}$ be the subordinator obtained* *as the inverse local time at zero of* $(Y_t)_{t\geq 0}$*. Then the Laplace exponent* ψ*, given by*

$$\mathbb{E}_0[\exp(-\lambda T_t)] = \exp(-t\psi(\lambda)),$$

is a complete Bernstein function. Conversely, for any complete Bernstein function ψ*, there exists a unique gap diffusion such that* ψ *is the Laplace exponent of its inverse local time at zero.*

It is also known that the transition density of the gap diffusion of $(Y_t)_{t\geq 0}$ with respect to the Krein string m is given by,

$$p^Y(t, y_0, y_1) = \int_{[0,\infty)} e^{-\eta t}\Phi_\eta(y_0)\Phi_\eta(y_1)\sigma(\mathrm{d}\eta),$$

where σ is the spectral measure of the complete Bernstein function ψ and hence the Laplace transform of $p^Y(t, 0, 0)$ is given by,

$$\int_0^\infty e^{-\lambda t} p^Y(t, 0, 0)\,\mathrm{d}t = \mathfrak{h}(\lambda) = \frac{1}{\psi(\lambda)}.$$

This property leads to the following remark which indicates why we keep the Krein string as general as possible.

Remark 3.4.2. *Suppose* $(Y_t)_{t\geq 0}$ *is a one-dimensional diffusion in* $\mathbb{R}$ *with generator* $\mathcal{A}_y = \frac{1}{2}\sigma^2(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}$ *such that* $\eta^{-1} \leq \sigma(y) \leq \eta$ *for some* $\eta > 0$*. Then using Aronson estimates for the transition density of* $(Y_t)_{t\geq 0}$ *(see [3]), the transition density of the reflected diffusion* $(|Y_t|)_{t\geq 0}$ *satisfies*

$$p^{|Y|}(t, 0, 0) \asymp \frac{1}{\sqrt{t}}.$$

Then as $\int_0^\infty e^{-t\lambda} p^{|Y|}(t, 0, 0)\,\mathrm{d}t = \frac{1}{\psi(\lambda)} \asymp \frac{1}{\sqrt{\lambda}}$ *where* ψ *is the Laplace exponent of the local time at zero, we have*

$$\psi(\lambda) \asymp \sqrt{\lambda}.$$

3.4.1 The Dirichlet Form Corresponding to the Generalised Diffusion

As a gap diffusion $(Y_t)_{t\geq 0}$ is a m-symmetric Hunt process on E_m, we let $(\mathcal{A}_y, \mathrm{Dom}(\mathcal{A}_y))$ be the generator of its corresponding semigroup on $L^2(E_m, m)$ and let $(\mathscr{E}^Y, \mathrm{Dom}(\mathscr{E}^Y))$

be the corresponding regular Dirichlet form given by

$$\mathscr{E}^Y(f,f) = \left\langle \sqrt{-\mathcal{A}_y} f, \sqrt{-\mathcal{A}_y} f \right\rangle_{L^2(E_m,m)},$$

with domain $\mathrm{Dom}(\mathscr{E}^Y) = \mathrm{Dom}(\sqrt{-\mathcal{A}_y})$.

It is advantageous to consider this Dirichlet form on $L^2(E_m, m)$ rather than that on $L^2([0,r), m)$ as m has full support on E_m and so all the results of Section 2.3 are valid. However, we can connect this form with the operators defined in Section 3.2. Of course, if $E_m = [0,r)$ or $E_m = [0,l]$, as in the case where $(Y_t)_{t\geq 0}$ is a one-dimensional diffusion the results of this section are trivial. However, even in the case where m does not have full support, it can be useful to be able to characterise the abstract Dirichlet form $\mathscr{E}^Y$ via the more tractable quadratic form $\mathscr{D}$ given by,

$$\mathscr{D}(f,f) = \begin{cases} \int_0^l |f'|^2 \,\mathrm{d}y & \text{if } l + m([0,l)) = \infty \text{ or } l + m([0,l)) < \infty \text{ and } r = \infty, \\ \int_0^r |f'|^2 \,\mathrm{d}y & \text{if } l + m([0,l)) < \infty \text{ and } r < \infty, \end{cases}$$

as in the standard one-dimensional diffusion case. In this case we let

$$\mathrm{Dom}(\mathscr{D}) = \{f \in L^2([0,r), m) : f \text{ is absolutely continuous and } \mathscr{D}(f,f) < \infty\}.$$

Proposition 3.4.3. *Let $f \in \mathrm{Dom}(\mathscr{E}^Y)$. Then f is absolutely continuous on E_m. Furthermore, if we define the function $\overline{f} : [0,r) \to \mathbb{R}$ by*

$$\overline{f}(y) = \begin{cases} f(y) & \textit{for } y \in E_m, \\ \textit{linearly extended} & \textit{in } ([0,l] \cap [0,r)) \setminus E_m, \\ f(l)\left(\frac{r-y}{r-l}\right) & \textit{for } y \in (l,r) \textit{ if } l + m([0,l)) < \infty \textit{ and } r < \infty, \end{cases}$$

then $\overline{f} \in \mathrm{Dom}(\mathscr{D})$ and,

$$\mathscr{E}^Y(f,f) = \mathscr{D}(\overline{f}, \overline{f}).$$

Proof. Let $(G_\lambda)_{\lambda>0}$ be the resolvent corresponding to $(Y_t)_{t\geq 0}$ defined by

$$G_\lambda f(y) = \mathbb{E}_y\Big[\int_0^\infty e^{-\lambda t} f(Y_t)\,\mathrm{d}t\Big]$$

for $y \in E_m$ and $f \in C_b(E_m)$ and let $(R_\lambda)_{\lambda>0}$ be the $L^2([0,r), m)$-resolvent operator corresponding to the operator $\mathcal{G}_y$. Then by [65, Proposition 15.15], for all $g \in C_b(E_m) \cap L^2([0,r), m)$, $G_\lambda g = R_\lambda g$.

Let $(\widehat{G}_\lambda)_{\lambda>0}$ be the $L^2(E_m, m)$-resolvent of the Dirichlet form $\mathscr{E}^Y$. Then for any $g \in C_b(E_m) \cap L^2(E_m, m)$, $\widehat{G}_\lambda g = G_\lambda g$ and so $\widehat{G}_\lambda g = R_\lambda g$. As $\widehat{G}_\lambda$ is bounded in $L^2(E_m, m)$, the result holds for any $g \in L^2(E_m, m) \cap L^2([0, r), m) = L^2([0, r), m)$ by density. Therefore $\mathrm{Range}(\widehat{G}_\lambda) = \mathrm{Range}(R_\lambda)|_{E_m}$ and hence $g \in \mathrm{Dom}(\mathcal{A}_y)$ if and only if there exists $\hat{g} \in \mathrm{Dom}(\mathcal{G}_y)$ such that $\hat{g}|_{E_m} = g$ and by the resolvent equation we have

$$\mathcal{A}_y g = \mathcal{G}_y \hat{g}|_{E_m}.$$

By definition of $\mathcal{G}_y$, $\hat{g} = \overline{g}$ for on $[0, l] \cap [0, r)$ and by construction satisfies the boundary condition at l. Therefore $f, g \in \mathrm{Dom}(\mathcal{A}_y)$ implies that $\overline{f}, \overline{g} \in \mathrm{Dom}(\mathcal{G}_y)$ and so,

$$\mathscr{E}^Y(f, g) = \langle -\mathcal{A}_y f, g\rangle_{L^2(E_m, m)} = \langle -\mathcal{G}_y \overline{f}, \overline{g}\rangle_{L^2([0,r),m)} = \mathscr{D}(\overline{f}, \overline{g}).$$

Now let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathcal{A}_y)$ converge to $f \in \mathrm{Dom}(\mathscr{E}^Y)$ in $\mathscr{E}_1^Y$-norm. Then $\overline{f}_n \to \overline{f}$ in $L^2([0, r), m)$ and $(\overline{f}_n')_{n\in\mathbb{N}}$ converges to some $g \in L^2([0, r), \mathrm{d}y)$. We now show $\overline{f}$ is absolutely continuous and $g = \overline{f}'$ m-a.e. in E_m and $g = \overline{f}'$ $\mathrm{d}y$-a.e. in $[0, r) \setminus E_m$.

We adapt the proof for one-dimensional diffusions as in [28, Example 1.2.2] to gap diffusions. We note that for any $n \in \mathbb{N}$, $a, b \in [0, l] \cap [0, r)$ with $a < b$ and $m(E_m \cap [a, b]) > 0$,

$$\begin{aligned}
|\overline{f}_n(b) - \overline{f}_n(a)|^2 &= \Big| \int_a^b \overline{f}_n'(\xi)\, \mathrm{d}\xi \Big|^2 \\
&\leq \Big(\int_a^b |\overline{f}_n'(\xi)|\, \mathrm{d}\xi \Big)^2 \\
&\leq |b - a| \Big(\int_a^b |\overline{f}_n'(\xi)|^2\, \mathrm{d}\xi \Big) \\
&\leq |b - a| \sup_{n\in\mathbb{N}} \mathscr{D}(\overline{f}_n, \overline{f}_n),
\end{aligned}$$

by Cauchy-Schwarz and so $(\overline{f}_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous. Furthermore, as $\overline{f}_n \to \overline{f}$ m-almost everywhere, we may choose $y_0 \in [0, b] \cap E_m$ (noting that $m([0, b] \cap E_m) > 0$) such that $(\overline{f}_n(y_0))_{n\in\mathbb{N}}$ is convergent,

$$|\overline{f}_n(y)| \leq |\overline{f}_n(y_0)| + |\overline{f}_n(y) - \overline{f}_n(y_0)| \leq \sup_{n\in\mathbb{N}} |\overline{f}_n(y_0)| + \sqrt{|b - a| \sup_{n\in\mathbb{N}} \mathscr{D}(\overline{f}_n, \overline{f}_n)},$$

so $\overline{f}_n$ is uniformly bounded. By Azerla-Ascoli and passing to a subsequence we may assume $\overline{f}_n$ converges to a continuous function $\tilde{f}$ uniformly on each finite closed subinterval of $[0,r)$ and $\tilde{f} = \overline{f} = f$ m-a.e. on E_m.

Now on any subinterval of $J \subset [0,r) \setminus E_m$, $\overline{f}_n'$ is constant and hence $\overline{f}_n'' = 0$ in J so $\overline{f}_n'' \to 0$ uniformly on J. Therefore, for any $\phi \in C_c^\infty(J^\circ)$,

$$0 = \int_J \overline{f}_n''(x)\phi(x)\,\mathrm{d}x = -\int_J \overline{f}_n'(x)\phi'(x)\,\mathrm{d}x \to -\int_J g(x)\phi'(x)\,\mathrm{d}x,$$

Therefore, g is weakly differentiable on J and $g' = 0$ almost everywhere in J and hence g is constant on every subinterval of $[0,r) \setminus E_m$. Furthermore, for any $\phi \in C_c^\infty((0,r))$,

$$\begin{aligned}\int_0^r g(x)\phi(x)\,\mathrm{d}x &= \lim_{k\to\infty}\int_0^r \overline{f}_{n_k}'(x)\phi(x)\,\mathrm{d}x\\ &= -\lim_{k\to\infty}\int_0^r \overline{f}_{n_k}(x)\phi'(x)\,\mathrm{d}x\\ &= -\int_0^r \tilde{f}(x)\phi'(x)\,\mathrm{d}x,\end{aligned}$$

which implies that $\tilde{f}$ is absolutely continuous on $[0,r)$. Furthermore,

$$\mathscr{D}(\overline{f},\overline{f}) \le \liminf_{n\to\infty}\mathscr{D}(\overline{f}_n,\overline{f}_n) = \liminf_{n\to\infty}\mathscr{E}^Y(f_n,f_n) < \infty,$$

so $\overline{f} \in \mathrm{Dom}(\mathscr{D})$. □

Note, we do not prove that $(\mathscr{D}, \mathrm{Dom}(\mathscr{D}))$ is a Dirichlet form as in the standard one-dimensional diffusion case as we only know that the limit of any $\mathscr{D}_1$-Cauchy sequence is absolutely continuous m-a.e. which is not sufficient for absolute continuity $\mathrm{d}y$-a.e. as m does not necessarily have full support.

We obtain the following corollary immediately due to Theorem 3.3.2 and the previous proposition. We recall that $\mathrm{Dom}_{ext}(\mathscr{E}^Y)$ denotes the extended Dirichlet space as defined in [28], given by the family of m-measurable functions ϕ on E_m such that $|\phi| < \infty$ m-a.e. and there exists an $\mathscr{E}^Y$-Cauchy sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathrm{Dom}(\mathscr{E}^Y)$ such that $\lim_{n\to\infty}\phi_n = \phi$ m-a.e.. We have that $\mathrm{Dom}_{ext}(\mathscr{E}^Y)$ is a linear space containing $\mathrm{Dom}(\mathscr{E}^Y)$ and $\mathrm{Dom}(\mathscr{E}^Y) = L^2(E_m,m) \cap \mathrm{Dom}_{ext}(\mathscr{E}^Y)$.

Corollary 3.4.4. *Let $\lambda > 0$. Then for any $f \in \mathrm{Dom}_{ext}(\mathscr{E}^Y)$ with $f(0) = 1$,*

$$\mathscr{E}^Y(f,f) + \lambda\|f\|^2_{L^2(E_m,m)} \ge \psi(\lambda),$$

with equality if and only if $f = \varphi_\lambda|_{E_m}$.

Proof. Clearly, if $f \notin L^2(E_m, m)$ the inequality is trivial so it suffices to show the inequality for $f \in \mathrm{Dom}(\mathscr{E}^Y)$. As $(\mathscr{E}^Y, \mathrm{Dom}(\mathscr{E}^Y))$ is a regular Dirichlet form, we prove the result for all $f \in \mathcal{C}$ where $\mathcal{C}$ is a core for $\mathrm{Dom}(\mathscr{E}^Y)$. Now if $l = r$ then as supp $f \subset [0, r)$, $\lim_{E_m \ni y \to r} f(y) = 0$ and hence $\lim_{y \to r} \overline{f}(y) = 0$. Otherwise $l < r < \infty$ and by construction $\lim_{y \to r} \overline{f}(y) = 0$. Applying Theorem 3.3.2 we obtain for each case that

$$\mathscr{D}(\overline{f}, \overline{f}) + \lambda \|\overline{f}\|^2_{L^2([0,r),m)} \geq \psi(\lambda),$$

and hence

$$\mathscr{E}^Y(f, f) + \lambda \|f\|^2_{L^2(E_m,m)} \geq \psi(\lambda),$$

and so by density, the inequality holds for all $f \in \mathrm{Dom}(\mathscr{E}^Y)$. Furthermore, $\mathscr{E}^Y(f, f)+$ $\lambda \|f\|^2_{L^2(E_m,m)} = \psi(\lambda)$ if and only if

$$\mathscr{D}(\overline{f}, \overline{f}) + \lambda \|\overline{f}\|^2_{L^2([0,r),m)} = \psi(\lambda),$$

which occurs if and only if $\overline{f} = \varphi_\lambda \implies f = \varphi_\lambda|_{E_m}$. □

3.4.2 Probabilistic Interpretation of the Extension Function

An important aspect of the probabilistic interpretation of the Krein correspondence is the probabilistic interpretation of the extension function. For $y \in E_m$, let

$$H_y = \inf\{t > 0 : Y_t = y\}.$$

By [65, (15.40)] we have,

$$\mathbb{E}_y[e^{-\lambda H_0}] = \psi(\lambda)\Phi_\lambda(y) \int_y^r \frac{1}{\Phi_\lambda(\xi)^2} \, \mathrm{d}\xi = \varphi_\lambda(y),$$

so $\varphi_\lambda(y)$ is the Laplace transform of the measure $\mathbb{P}_y[H_0 \in \mathrm{d}t]$.

Proposition 3.4.5. $\mathbb{P}_0[H_0 \in \mathrm{d}t] = \delta_0(\mathrm{d}t)$, *the Dirac delta measure at zero. Suppose* $y \in E_m \cap (0, r)$ *and* $m((0, r)) \neq 0$. *Then* $\mathbb{P}_y[H_0 = 0] = 0$.

Proof. The first claim follows by inverting the Laplace transform as $\varphi_\lambda(0) = 1$. For the second claim we proceed by contradiction. Let $y \in E_m \cap (0, r)$. Then by the

Blumenthal zero-one law, $\mathbb{P}_y[H_0 = 0] \in \{0,1\}$. If $r < \infty$, then we know

$$\mathbb{P}_y[H_0 = 0] \leq \mathbb{P}_y[H_0 < \infty] = \frac{r-y}{r} < 1.$$

Otherwise, suppose $\mathbb{P}_y[H_0 = 0] = 1$ in which case $\mathbb{P}_y[H_0 \in \mathrm{d}t] = \delta_0(\mathrm{d}t)$ and hence the Laplace transform $\varphi_\lambda(y) = 1$ for all $\lambda \geq 0$. As the mapping $y \mapsto \varphi_\lambda(y)$ is non-increasing with $\varphi_\lambda(0) = 1$, $\varphi_\lambda(\xi) = 1$ for all $\lambda \geq 0$ and $\xi \in [0,y]$. Therefore $\varphi_\lambda^+(0) = 0$ so

$$\psi(\lambda) = m_0\lambda,$$

which occurs only when $m(\mathrm{d}y) = m_0\delta_0(\mathrm{d}y)$ which is a contradiction as $m((0,r)) \neq 0$. □

Furthermore, regularity of the function $y \mapsto \varphi_\lambda(y)$ gives us information about convergence of the distribution of the hitting time at zero in terms of the initial point y. In order investigate this we require information about the decay of the function $y \mapsto \varphi_\lambda(y)$. It is proven in [44] that for any $y \in E_m$,

$$\lim_{\lambda\uparrow\infty} \frac{\log(\varphi_\lambda(y))}{\sqrt{\lambda}} = -\int_0^y \sqrt{h(\xi)}\,\mathrm{d}\xi,$$

where h is density of the absolutely continuous part in the Lebesgue decomposition of the measure m. Therefore, for all $\varepsilon > 0$ there exists an $R > 0$ such that for $\lambda > R$,

$$\exp\Big(-\sqrt{\lambda}\big(\varepsilon + \int_0^y \sqrt{h(\xi)}\,\mathrm{d}\xi\big)\Big) \leq \varphi_\lambda(y) \leq \exp\Big(-\sqrt{\lambda}\big(-\varepsilon + \int_0^y \sqrt{h(\xi)}\,\mathrm{d}\xi\big)\Big).$$

In order to obtain exponential decay, we require the following assumption on the function h.

Assumption 3.4.6. *For all $y > 0$, $\int_0^y h(\xi)\,\mathrm{d}\xi > 0$.*

Under this assumption, for each $y > 0$ we may choose $\varepsilon = \delta\big(\int_0^y \sqrt{h(\xi)}\,\mathrm{d}\xi\big)$ for some $0 < \delta < 1$ and so for $\lambda > R > 1$

$$\varphi_\lambda(y) \leq e^{-k_y\sqrt{\lambda}},$$

where $k_y = (1-\delta)\int_0^y \sqrt{h(\xi)}\,\mathrm{d}\xi > 0$. This non-degeneracy condition allows us to prove the following property of the measures $\{\mathbb{P}_y[H_0 \in \mathrm{d}t]\}_{y\in E_m}$.

Lemma 3.4.7. *Let $\beta > 0$ and $\lambda \geq 0$ and assume the Krein string m satisfy Assumption 3.4.6 and let φ_λ be the corresponding extension function. Then for $y \in E_m \cap (0, r)$, the Laplace transform of the measure $\mu(\mathrm{d}t) = t^{-\beta}\mathbb{P}_y[H_0 \in \mathrm{d}t]$ is given by the Riemann-Liouville integral of $\varphi_\lambda(y)$:*

$$\int_{(0,\infty)} e^{-\lambda t} t^{-\beta}\mathbb{P}_y[H_0 \in \mathrm{d}t] = \big(J^\beta \varphi_\cdot(y)\big)(\lambda) = \frac{1}{\Gamma(\beta)}\int_\lambda^\infty (\sigma - \lambda)^{\beta-1}\varphi_\sigma(y)\,\mathrm{d}\sigma,$$

and $(J^\beta\varphi_\cdot(y))(\lambda) < \infty$ for all $\lambda \geq 0$.

Proof. We first prove the right-hand side is well-defined. By a change of variables, it suffices to show $\sigma \mapsto \sigma^{\beta-1}\varphi_{\sigma+\lambda}(y)$ is integrable on $(0,\infty)$ for any $\lambda \geq 0$. Choose $R > 1$ such that there is some $k_y > 0$ such that

$$\varphi_\eta(y) \leq e^{-k_y\sqrt{\eta}},$$

for $\eta > R$. For all $\beta > 0$, $\sigma^{\beta-1}$ is integrable on $(0, R]$ and so

$$\int_0^R |\sigma^{\beta-1}\varphi_{\sigma+\lambda}(y)|\,\mathrm{d}\sigma \leq \int_0^R \sigma^{\beta-1}\,\mathrm{d}\sigma < \infty,$$

as $0 \leq \varphi_\lambda(y) \leq 1$. Furthermore, using the exponential bound for φ,

$$\int_R^\infty |\sigma^{\beta-1}\varphi_{\sigma+\lambda}(y)|\,\mathrm{d}\sigma \leq \int_R^\infty \sigma^{\beta-1}e^{-k_y\sqrt{\sigma}}\,\mathrm{d}\sigma < \infty.$$

By calculating,

$$\begin{aligned}\frac{1}{\Gamma(\beta)}\int_\lambda^\infty (\sigma-\lambda)^{\beta-1}e^{-\sigma t}\,\mathrm{d}\sigma &= \frac{e^{-\lambda t}}{\Gamma(\beta)}\int_0^\infty \sigma^{\beta-1}e^{-\sigma t}\,\mathrm{d}\sigma \\ &= \frac{t^{-\beta}e^{-\lambda t}}{\Gamma(\beta)}\int_0^\infty \sigma^{\beta-1}e^{-\sigma}\,\mathrm{d}\sigma \\ &= t^{-\beta}e^{-\lambda t},\end{aligned}$$

we find,

$$\begin{aligned}\int_{(0,\infty)} e^{-\lambda t}t^{-\beta}\mathbb{P}_y[H_0 \in \mathrm{d}t] &= \int_{(0,\infty)} \Big(\frac{1}{\Gamma(\beta)}\int_\lambda^\infty (\sigma-\lambda)^{\beta-1}e^{-\sigma t}\,\mathrm{d}\sigma\Big)\mathbb{P}_y[H_0 \in \mathrm{d}t] \\ &= \frac{1}{\Gamma(\beta)}\int_\lambda^\infty (\sigma-\lambda)^{\beta-1}\varphi_\sigma(y)\,\mathrm{d}\sigma,\end{aligned}$$

completing the proof. □

We now use this lemma to prove the following useful corollary.

Corollary 3.4.8. *For all $\beta \geq 0$, $t^{-\beta}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \to t^{-\beta}\mathbb{P}_y[H_0 \in \mathrm{d}t]$ weakly as $n \to \infty$ for any sequence $(y_n)_{n\in\mathbb{N}} \subset E_m \cap (\frac{y}{2}, r)$ converging to some $y > 0$.*

Proof. For any $\lambda \geq 0$, $y \mapsto \varphi_\lambda(y)$ is a continuous function and so $\varphi_\lambda(y_n) \to \varphi_\lambda(y)$ for any $(y_n)_{n\in\mathbb{N}}$ converging to $y \geq 0$. As the Laplace transforms converge, $\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \to \mathbb{P}_y[H_0 \in \mathrm{d}t]$ weakly.

By assumption $y_n > \frac{y}{2}$, and so for all $\lambda \geq 0$, $\varphi_\lambda(\frac{y}{2}) \geq \varphi_\lambda(y_n)$ for all $n \in \mathbb{N}$. Therefore,

$$\int_0^\infty \sigma^{\beta-1}\varphi_{\sigma+\lambda}(\tfrac{y}{2})\,\mathrm{d}\sigma \leq \int_0^R \sigma^{\beta-1}\,\mathrm{d}\sigma + \int_R^\infty \sigma^{\beta-1}e^{-k_{y/2}\sqrt{\lambda+\sigma}}\,\mathrm{d}\sigma$$
$$< \infty,$$

and so by dominated convergence we have,

$$\begin{aligned}\big(J^\beta\varphi_\cdot(y_n)\big)(\lambda) &= \frac{1}{\Gamma(\beta)}\int_0^\infty \sigma^{\beta-1}\varphi_{\sigma+\lambda}(y_n)\,\mathrm{d}\sigma \\ &\to \frac{1}{\Gamma(\beta)}\int_0^\infty \sigma^{\beta-1}\varphi_{\sigma+\lambda}(y)\,\mathrm{d}\sigma = \big(J^\beta\varphi_\cdot(y)\big)(\lambda),\end{aligned}$$

as $n \to \infty$ and hence $t^{-\beta}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \to t^{-\beta}\mathbb{P}_y[H_0 \in \mathrm{d}t]$ weakly as $n \to \infty$. □

A final identity of interest allows us to describe the jump measure of ψ in terms of $\mathbb{P}_y[H_0 \in \mathrm{d}t]$ and $m(\mathrm{d}y)$. We note that for each $y \in E_m$, we may define (a possibly infinite) a measure on Borel subsets of $[0, \infty)$ by

$$\mu_y(A) = \int_{E_m} \mathbb{P}_y[H_0 \in A]m(\mathrm{d}y).$$

Proposition 3.4.9. *Let $\psi(\lambda) = m_0\lambda + \int_0^\infty(1 - e^{-\lambda t})\nu(t)\,\mathrm{d}t$ and m be in Krein correspondence and let φ_λ be the corresponding extension function. Then,*

$$\nu((t,\infty))\,\mathrm{d}t = \int_{E_m} \mathbb{P}_y[H_0 \in \mathrm{d}t]m(\mathrm{d}y).$$

Proof. Taking the Laplace transform of μ_y we see

$$\begin{aligned}\int_{[0,\infty)} e^{-\lambda t}\Big(\int_{E_m} \mathbb{P}_y[H_0 \in \mathrm{d}t]m(\mathrm{d}y)\Big) &= \int_{[0,\infty)} \varphi_\lambda(y)m(\mathrm{d}y) \\ &= \int_{[0,\infty)} \frac{1}{\lambda}\,\mathrm{d}\varphi_\lambda^+(y),\end{aligned}$$

noting that $\mathrm{d}\varphi_\lambda^+(\mathrm{d}y) = \lambda\varphi_\lambda(y)m(\mathrm{d}y)$. Therefore,

$$\begin{aligned}\int_{[0,\infty)} \Big(\int_0^\infty e^{-\lambda t}\,\mathrm{d}t\Big)\,\mathrm{d}\varphi_\lambda^+(y) &= -\int_0^\infty e^{-\lambda t}\varphi_\lambda^+(0)\,\mathrm{d}t\\ &= \int_0^\infty e^{-\lambda t}(\psi(\lambda) - m_0\lambda)\,\mathrm{d}t,\end{aligned}$$

as $\varphi_\lambda^+(y) \to 0$ as $y \to \infty$. Now as $\psi(\lambda) - m_0\lambda = \int_0^\infty \lambda e^{-\lambda s}\nu((s,\infty))\,\mathrm{d}s$ we have,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}(\psi(\lambda) - m_0\lambda)\,\mathrm{d}t &= \int_0^\infty\int_0^\infty \lambda e^{-\lambda(s+t)}\nu((s,\infty))\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_0^\infty e^{-\lambda s}\nu((s,\infty))\,\mathrm{d}s,\end{aligned}$$

so the Laplace transforms coincide and hence the measures are equal. □

3.5 Explicit Examples of the Krein Correspondence

In this chapter, we give some notable examples of the Krein correspondence which are of interest for applications. Although explicit pairs (ψ, m) which are in Krein correspondence are rare, there are several useful examples which we can state explicitly.

Other than the classical result that the distribution of the inverse local time of a reflected Brownian motion is a $\frac{1}{2}$-stable subordinator, the first explicit example of the Krein correspondence was proven by Molchanov & Ostrovskii [54] although we cite the exposition found in [46].

Example 3.5.1 (Stable subordinator)**.** *Let $\alpha \in (0,2)$ and define*

$$c_\alpha = 2^{-\alpha}|\Gamma(-\tfrac{\alpha}{2})|/\Gamma(\tfrac{\alpha}{2}),$$

and define the measure $m^{(\alpha)}(\mathrm{d}y) = \frac{1}{\alpha^2 c_\alpha^{2/\alpha}} y^{2/\alpha-2}\,\mathrm{d}y$. Then for $\lambda > 0$ the extension function is given by

$$\varphi_\lambda^{(\alpha)}(y) = \frac{2^{1-\alpha/2}}{\Gamma(\frac{\alpha}{2})}\Big(\frac{\lambda^{\alpha/2}y}{c_\alpha}\Big)^{1/2} K_{\alpha/2}\bigg(\Big(\frac{\lambda^{\alpha/2}y}{c_\alpha}\Big)^{1/\alpha}\bigg),$$

where $K_{\alpha/2}$ is the modified Bessel function of the second kind. The complete Bernstein function corresponding to this extension function is given by

$$\psi^{(\alpha)}(\lambda) = \lambda^{\alpha/2}.$$

Example 3.5.2 (Relativistic subordinator). *[23] Suppose $(Y_t)_{t\geq 0}$ is a one-dimensional reflected diffusion on probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \{\mathbb{P}_y\}_{y\in I})$ and let ψ be the Laplace exponent of the inverse local time at zero. We may define another one-dimensional diffusion $(Y_t^{(c)})_{t\geq 0}$ with laws $\{\mathbb{P}_y^{(c)}\}_{y\in I}$ given by,*

$$\left.\frac{\mathrm{d}\mathbb{P}_y^{(c)}}{\mathrm{d}\mathbb{P}_y}\right|_{\mathcal{F}_t} = \frac{\varphi_c(Y_t)}{\varphi_c(y)} \exp\Big(\psi(c) L_t^0(Y) - ct\Big),$$

for some $c > 0$. Then the inverse local time under $\mathbb{P}_0^{(c)}$ satisfies

$$\begin{aligned}\exp(-t\psi^{(c)}(\lambda)) &= \mathbb{E}_0^{(c)}[\exp(-\lambda T_t)] \\ &= \mathbb{E}_0\Big[\exp(-\lambda T_t)\exp\big(\psi(c)t - cT_t\big)\Big] \\ &= \exp\big(-t(\psi(\lambda+c) - \psi(c))\big).\end{aligned}$$

Example 3.5.3 (Point Mass Example). *[44] Let $\mathcal{P}_n = \{0 = y_0 < y_1 < \cdots < y_n = l\}$ be a partition of $[0, l]$ and let $m_i \in (0, \infty)$ for all $1 \leq i \leq n$. Then for some $r \geq l$ define a Krein string by $m(\mathrm{d}y) = \sum_{i=0}^n m_i \delta_{y_i}(\mathrm{d}y) + \infty\delta_r(\mathrm{d}y)$. The complete Bernstein function corresponding to m is given by the continued fraction*

$$\psi(\lambda) = m_0\lambda + \cfrac{1}{(y_1-y_0)+\cfrac{1}{m_1\lambda+\cfrac{1}{\ddots+\cfrac{1}{(y_n-y_{n-1})+\cfrac{1}{m_n\lambda+\frac{1}{(r-l)}}}}}},$$

where we assume $\frac{1}{r-l} = \infty$ if $l = r$ and $\frac{1}{r-l} = 0$ if $r = \infty$.

This final example is important for numerical simulations. For any Krein string $m(\mathrm{d}y)$ we may define an approximation of this measure by taking a sequence of partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ with mesh converging to zero and letting the Krein string m_n be defined by

$$m_n(\mathrm{d}y) = \sum_{i=0}^{n-1} m([y_i, y_{i+1}))\delta_{y_i}(\mathrm{d}y) + m(\{l\})\delta_l(\mathrm{d}y) + \infty\delta_r(\mathrm{d}y).$$

Then is it easy to see that $m_n \to m$ in the sense of Proposition 3.3.4 and hence $\psi_n(\lambda) \to \psi(\lambda)$ pointwise for all $\lambda \geq 0$ where ψ_n (resp. ψ) is in Krein correspondence with m_n (resp. m). This leads to possibility of numerically inverting the Laplace transform $\lambda \to e^{-t\psi(\lambda)}$ in order to find a numerical approximation of the probability distribution of T_t as detailed in [1].

3.5.1 Examples Related to Brownian Motion

The simplest example of the Krein correspondence is given by the Lebesgue measure $\mathrm{d}y$ on $[0, \infty)$ where the underlying equation is given by a simple constant coefficient ODE:

$$f''_\lambda(y) = \lambda f_\lambda(y).$$

This example is useful for seeing how changing the right-hand boundary condition changes the corresponding extension function and hence the complete Bernstein function. Probabilistically, this example corresponds to a reflected Brownian motion in $[0, \infty)$.

Example 3.5.1 (Reflected Brownian Motion)**.** *Given a Krein string $m(\mathrm{d}y) = \mathrm{d}y$ (Lebesgue measure) on $[0, \infty)$, the corresponding complete Bernstein function is given by $\psi_{cons}(\lambda) = \sqrt{\lambda}$.*

Proof. By solving the ODE $f''_\lambda(y) = \lambda f_\lambda(y)$ for each of the given initial conditions we see,

$$\begin{aligned}
\Phi_\lambda(y) &= \frac{1}{2}e^{y\sqrt{\lambda}} + \frac{1}{2}e^{-y\sqrt{\lambda}}, \\
\Psi_\lambda(y) &= \frac{1}{2\sqrt{\lambda}}e^{y\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-y\sqrt{\lambda}},
\end{aligned}$$

and therefore the complete Bernstein function corresponding to m is given by,

$$\psi_{cons}(\lambda) = \lim_{y\to\infty} \frac{\Phi_\lambda(y)}{\Psi_\lambda(y)} = \sqrt{\lambda}.$$

$\square$

By truncating the speed measure to $[0, l)$ and placing a point mass $m_l \in (0, \infty)$ at l we obtain the following example.

Example 3.5.2 (Brownian Motion Elastically Reflected at l)**.** *Given a Krein string $m(\mathrm{d}y) = \mathbf{1}_{[0,l)}(y)\,\mathrm{d}y + m_l\delta_{y_l}(\mathrm{d}y)$ on $[0, \infty)$ where $l \in (0, \infty)$, the corresponding complete Bernstein function is given by*

$$\psi_{stick}(\lambda) = \sqrt{\lambda}\left(\frac{(1 + m_l\sqrt{\lambda})e^{l\sqrt{\lambda}} - (1 - m_l\sqrt{\lambda})e^{-l\sqrt{\lambda}}}{(1 + m_l\sqrt{\lambda})e^{l\sqrt{\lambda}} + (1 - m_l\sqrt{\lambda})e^{-l\sqrt{\lambda}}}\right).$$

Proof. For $y < r$, the fundamental functions are the same as in the previous example

$y \leq l$. However, for $y > l$ the measure is zero so we have,

$$\begin{aligned}\Phi_\lambda(y) &= \Phi_\lambda(0) + \Phi_\lambda^-(0)y + \lambda \int_{[0,l]} (y-\tilde{y})\Phi_\lambda(\tilde{y})m(\mathrm{d}\tilde{y}) \\ &= \Phi_\lambda(l) + \Phi_\lambda^+(l)(y-l),\end{aligned}$$

and similarly,

$$\Psi_\lambda(y) = \Psi_\lambda(l) + \Psi_\lambda^+(l)(y-l).$$

Therefore,

$$\lim_{y\to\infty} \frac{\Psi_\lambda(y)}{\Phi_\lambda(y)} = \frac{\Psi_\lambda^+(l)}{\Phi_\lambda^+(l)} = \frac{\Psi_\lambda^-(l) + \lambda m_l \Psi_\lambda(l)}{\Phi_\lambda^-(l) + \lambda m_l \Phi_\lambda(l)},$$

and so by calculating we find,

$$\begin{aligned}\Psi_\lambda^-(l) + \lambda m_l \Psi_\lambda(l) &= \Big(\frac{1 + m_l\sqrt{\lambda}}{2}\Big)e^{l\sqrt{\lambda}} + \Big(\frac{1 - m_l\sqrt{\lambda}}{2}\Big)e^{-l\sqrt{\lambda}}, \\ \Phi_\lambda^-(l) + \lambda m_l \Phi_\lambda(l) &= \Big(\frac{\sqrt{\lambda} + m_l\lambda}{2}\Big)e^{l\sqrt{\lambda}} + \Big(\frac{m_l\lambda - \sqrt{\lambda}}{2}\Big)e^{-l\sqrt{\lambda}}.\end{aligned}$$

Therefore,

$$\psi_{stick}(\lambda) = \sqrt{\lambda}\left(\frac{(1 + m_l\sqrt{\lambda})e^{l\sqrt{\lambda}} - (1 - m_l\sqrt{\lambda})e^{-l\sqrt{\lambda}}}{(1 + m_l\sqrt{\lambda})e^{l\sqrt{\lambda}} + (1 - m_l\sqrt{\lambda})e^{-l\sqrt{\lambda}}}\right).$$

□

By letting $m_l \to 0$ we obtain the complete Bernstein function for Brownian motion instantaneously reflected in $[0, l]$,

$$\psi_{ref}(\lambda) = \sqrt{\lambda}\left(\frac{e^{l\sqrt{\lambda}} - e^{-l\sqrt{\lambda}}}{e^{l\sqrt{\lambda}} + e^{-l\sqrt{\lambda}}}\right),$$

while if we let $m_l \to \infty$, we obtain the complete Bernstein function for a reflected Brownian motion killed at l,

$$\psi_{kill}(\lambda) = \sqrt{\lambda}\left(\frac{e^{l\sqrt{\lambda}} + e^{-l\sqrt{\lambda}}}{e^{l\sqrt{\lambda}} - e^{-l\sqrt{\lambda}}}\right).$$

3.6 A Note About Changes of Scale

So far, the gap-diffusion process $(Y_t)_{t\geq 0}$ has been in *natural scale* in the sense that for $v, w, y \in E_m$ such that $v < y < w$,

$$\mathbb{P}_y[H_w < H_v] = \frac{y-v}{w-v},$$

as proven in [65]. However, if we consider a general one-dimensional diffusion process $(Z_t)_{t\geq 0}$ with speed measure m_p and scale function p, we can put this process into natural scale by letting $Y_t = p(Z_t)$ (see [5]) and in certain situations it can be easier to deal with the process $(Z_t)_{t\geq 0}$.

If we restrict ourselves to the special case where the speed measure has no singular part in its Lebesgue decomposition, we may choose a particular scale function such that the scaled process has a divergence form generator with the same local time at zero as the unscaled process as in the Bessel process case. We assume $m(\mathrm{d}y) = b^{-2}(y)\,\mathrm{d}y$ for some function $b : [0, l) \to [0, \infty]$ with $b(y) \in (0, \infty)$ for almost all $y \in [0, l)$. To find this change of scale for general b, we require two technical lemmas. The first is due to Zareckii (we state the version found in [68]).

Lemma 3.6.1. *Let $f : [a, b] \to [c, d]$ be a strictly increasing function that maps $[a, b]$ onto $[c, d]$. Then the following hold:*

(i) f is absolutely continuous if and only if $\mathrm{Leb}(f(\{x : f'(x) = \infty\})) = 0$,

(ii) f^{-1} is absolutely continuous if and only if $\mathrm{Leb}(\{x : f'(x) = 0\}) = 0$.

The second lemma is a special case of [30, Corollary 20.5].

Lemma 3.6.2. *Let ϕ be a monotone, absolutely continuous function with domain $[a, b]$ and range $[\alpha, \beta]$. Then for any $f \in L^1([\alpha, \beta])$, we have $(f \circ \phi)|\phi'| \in L^1([a, b])$ and*

$$\int_\alpha^\beta f(y)\,\mathrm{d}y = \int_a^b f \circ \phi(x)|\phi'(x)|\,\mathrm{d}x.$$

Lemma 3.6.3. *Let $l_z = \int_0^l \frac{1}{b(\xi)}\,\mathrm{d}\xi \in (0, \infty]$. Then there exists an absolutely continuous function $p : [0, l_z) \to [0, l)$ such that,*

$$p'(z) = (b \circ p)(z), \qquad p(0) = 0. \tag{3.6.1}$$

Proof. As $m([0, y)) = \int_0^y b^{-2}(\xi)\,\mathrm{d}\xi < \infty$ for all $y < l$, by the Cauchy-Schwarz inequality we know that

$$\int_0^y b^{-1}(\xi)\,\mathrm{d}\xi \le y^{1/2}\Big(\int_0^y b^{-2}(\xi)\,\mathrm{d}\xi\Big)^{1/2} < \infty,$$

and so b^{-1} is locally integrable in $[0, l)$ and so we may define $q : [0, l) \to [0, l_z)$ by $q(y) = \int_0^y \frac{1}{b(\xi)}\,\mathrm{d}\xi$. Then q is absolutely continuous and strictly increasing with derivative $q'(y) = \frac{1}{b(y)} > 0$ almost everywhere. Let $p : [0, l_z) \to [0, l)$ denote the inverse of q. Now for any $a \in [0, l_z)$, q maps $[0, p(a)]$ onto $[0, a]$ and so by Lemma 3.6.1, p is absolutely continuous on $[0, a]$ as $q' = \frac{1}{b} > 0$ almost everywhere. Therefore, p is absolutely continuous on $[0, l_z)$ with locally integrable derivative p' almost everywhere.

To prove (3.6.1), we apply Lemma 3.6.2. Fix $z \in [0, l_z)$ so that q maps $[0, p(z)]$ onto $[0, z]$. For each $N \in \mathbb{N}$, $(b \circ p)\mathbf{1}_{\{(b\circ p)\le N\}} \in L^1([0, z])$ so we have

$$\begin{aligned}\int_0^z (b\circ p)(\tilde{z})\mathbf{1}_{\{(b\circ p)(\tilde{z})\le N\}}\,\mathrm{d}\tilde{z} &= \int_0^{p(z)} (b\circ p)(q(y))\mathbf{1}_{\{(b\circ p)(q(y))\le N\}}q'(y)\,\mathrm{d}y \\ &= \int_0^{p(z)} \mathbf{1}_{\{b(y)\le N\}}\,\mathrm{d}y.\end{aligned}$$

As $b < \infty$ almost everywhere we have,

$$p(z) = \int_0^z b\circ p(\tilde{z})\,\mathrm{d}\tilde{z},$$

by monotone convergence. □

If we assume b is continuously differentiable, then clearly the scale function p is twice continuously differentiable. Formally, if the process $(Y_t)_{t\ge0}$ is a associated with the generator $\frac{\mathrm{d}^2}{\mathrm{d}m\,\mathrm{d}y} = b^2(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}$, then the rescaled process $(Z_t)_{t\ge0} = (q(Y_t))_{t\ge0}$ will be associated with the generator by $\frac{\mathrm{d}}{\mathrm{d}m_p\,\mathrm{d}p} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{p''(z)}{p'(z)}\frac{\mathrm{d}}{\mathrm{d}z} = p'(z)\frac{\mathrm{d}}{\mathrm{d}z}(\frac{1}{p'(z)}\frac{\mathrm{d}}{\mathrm{d}z})$ which can be see by considering Example 3.2.1 with $\sigma^2(z) = 2$ and $\mu(z) = -\frac{p''(z)}{p'(z)}$. As this change of scale does not effect the local time of the corresponding process at zero, the inverse local times of the processes generated by these generators are equal. Furthermore, this change of scale allows us to rewrite certain equations associated with the $(Y_t)_{t\ge0}$ in divergence form.

Example 3.6.4. *Let $(Y_t)_{t\ge0}$ be the one-dimensional diffusion associated with the speed measure $m^{(\alpha)}(\mathrm{d}y) = \frac{1}{\alpha^2 c_\alpha^{2/\alpha}}y^{2/\alpha-2}\,\mathrm{d}y$ on $[0,\infty)$ for some $\alpha \in (0, 2)$. Then we* *can calculate*

$$q(y) = \int_0^y \frac{1}{\alpha c_\alpha^{1/\alpha}} \xi^{1/\alpha - 1}\,\mathrm{d}\xi = \Big(\frac{y}{c_\alpha}\Big)^{1/\alpha},$$

and so $y = p(z) = c_\alpha z^\alpha$*. Let* $(Z_t)_{t\geq 0}$ *be the rescaled process defined by* $Z_t = (\frac{Y_t}{c_\alpha})^{1/\alpha}$*. Then we can see this process is related to a Bessel process of dimension* $\delta = 2 - \alpha$ *as the generator corresponding to this process should be given in some sense by*

$$\alpha^2 c_\alpha^{2/\alpha} y^{2-2/\alpha} \frac{\mathrm{d}^2}{\mathrm{d}y^2} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} - \Big(\frac{p''(z)}{p'(z)}\Big)\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \Big(\frac{1-\alpha}{z}\Big)\frac{\mathrm{d}}{\mathrm{d}z},$$

applying the same calculations as found in the introduction. This representation provides some intuition about the properties of the trace process. As we can see, when $p'' > 0$ *(resp.* $p'' < 0$*) the drift term indicates that the diffusion is being "pulled to" (resp. "pushed away from") zero. This difference determines whether the subordinated process* $(X_{T_t})_{t\geq 0}$ *is of finite or infinite variation as in the case where* $(X_t)_{t\geq 0}$ *is an* $\mathbb{R}^d$*-valued Brownian motion and* $(T_t)_{t\geq 0}$ *is the inverse local time corresponding to the rescaled Bessel process* $(Y_t)_{t\geq 0}$*.*

Chapter 4

Stochastic Approach to the Harmonic Extension Technique

4.1 Introduction

We now introduce the first method to generalise the Caffarelli-Silvestre extension technique which is based upon stochastic analysis of one-dimensional diffusion processes as detailed in [5]. The probabilistic analogue of the extension technique was first proven by Molchanov & Ostrovskii in [54] where they considered the trace process of certain diffusion processes in $\mathbb{R}^d \times [0, \infty)$ but they did not make a connection between the generator of the trace process and the Neumann boundary condition of a solution to a PDE. This connection has been made in a stochastic setting by Hsu in [31] where he considered a uniformly elliptic diffusion taking values in a bounded domain of $\mathbb{R}^{d+1}$ with sufficiently smooth boundary. Formally, the connection is made by combining Itô's formula and a random time change given by the inverse local time at the boundary. The method can be illustrated by the simple example of Brownian motion in a half-space.

Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Brownian motion and let $(Y_t)_{t\geq 0}$ be an independent reflected Brownian motion in $[0, \infty)$ given by the SDE

$$\mathrm{d}Y_t = \mathrm{d}B_t + \mathrm{d}L^0_t(B).$$

Independence is assumed to ensure that the time-changed process $(X(T_t))_{t\geq 0}$ is a Markov process [64, Theorem 30.1]. Now if we suppose f is smooth and let

$$u_f(x, y) = \int_{\mathbb{R}^d} f(\tilde{x}) P(x - \tilde{x}, y) \,\mathrm{d}\tilde{x},$$

where $P(x,y) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{y}{(y^2+|x|^2)^{(d+1)/2}}$ is the Poisson kernel of an upper half-space, then u_f is the solution to

$$\begin{cases} \Delta_x u_f + \partial_y^2 u_f = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u_f(x,0) = f(x) & \text{for } x \in \mathbb{R}^d. \end{cases}$$

Formally applying Itô's formula to $u_f(X_t, Y_t)$ with $X_0 = x \in \mathbb{R}^d$ and $Y_0 = 0$ we should find

$$u_f(X_t,Y_t) - u_f(x,0) = \int_0^t \nabla_{(x,y)} u_f(X_s,Y_s)\,\mathrm{d}(X_s,B_s)^{\mathbf{T}} + \int_0^t \partial_y u_f(X_s,Y_s)\,\mathrm{d}L_t^0(B).$$

We note that this formula is not covered by the cases studied by Hsu but we shall see that it is a special case of Lemma 4.3.7 proven in this chapter. Time-changing this formula by the inverse local time at zero $(T_t)_{t\geq 0}$ and taking expectations we obtain,

$$\lim_{t\to 0} \frac{1}{t}\Big(\mathbb{E}_x[f(X_{T_t})] - f(x)\Big) = \partial_y u_f(x,0).$$

It was originally proven by Spitzer [69] that $(T_t)_{t\geq 0}$ is a subordinator with Laplace exponent $\psi(\lambda) = \sqrt{2\lambda}$ (and can be deduced from the examples in Section 3.5.1) and so the subordinated process $(X_{T_t})_{t\geq 0}$ is a symmetric, $\frac{1}{2}$-stable process and so this pointwise limit should be related its infinitesimal generator $-(-\Delta_x)^{1/2}$.

This example naturally leads us to consider the case where $(X_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued diffusion process given by an SDE and $(Y_t)_{t\geq 0}$ is a one-dimensional diffusion process in $[0,l] \cap [0,r)$, reflected at zero as constructed in Chapter 2. By the Krein correspondence, the inverse local time at zero of the diffusion $(Y_t)_{t\geq 0}$ will be given by a subordinator with a complete Bernstein function ψ as its Laplace exponent. If $\mathcal{L}_x$ denotes the second order elliptic operator associated with this diffusion process, then by following the approach in the Brownian example we should obtain a similar characterisation for the infinitesimal operator associated with the subordinated process in terms of the Dirichlet-to-Neumann map of an extension function u_f. By examining the Itô formula, this function should satisfy

$$\mathcal{L}_x u_f \times m(\mathrm{d}y) + \partial_y^2 u_f = 0,$$

in $\mathbb{R}^d \times (0,l)$ in some sense, where m is the Krein string corresponding to the diffusion $(Y_t)_{t\geq 0}$. In this general set-up the difficulty is due to the fact we do not know the regularity properties of this function and so we cannot immediately apply

Itô's formula.

4.2 Set Up

In this chapter, we let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued diffusion which is a global solution of the SDE

$$\mathrm{d}X_t^i = \sum_{k=1}^{p} \sigma_{ik}(X_t)\,\mathrm{d}B_t^k + a_i(X_t)\,\mathrm{d}t, \quad i = 1, \dots, d, \tag{4.2.1}$$

where $(B^1, \dots, B^p)^{\mathbf{T}}$ is a Brownian motion, $\sigma = (\sigma_{ik})$ is a $d \times p -$ matrix of functions on $\mathbb{R}^d$, and $a = (a_1, \dots, a_d)^{\mathbf{T}}$ is a vector field on $\mathbb{R}^d$. The infinitesimal operator associated with this diffusion formally reads

$$\mathcal{L}_x = \sum_{i=1}^{d} a_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^{\mathbf{T}})_{ij}(x)\, \frac{\partial^2}{\partial x_i \partial x_j}\,. \tag{4.2.2}$$

Assume that both coefficients a, σ are continuous but also satisfy some growth condition to allow for global solutions of (4.2.1).

Let $(Y_t)_{t\geq 0}$ be the diffusion as constructed in Theorem 2.4.1 with speed measure $\tilde{m}$ which we assume is independent of $(X_t)_{t\geq 0}$. In addition to the speed measure, we recall the Krein string corresponding to the one-dimensional diffusion given by

$$m(\mathrm{d}y) = \begin{cases} b^{-2}(y)\,\mathrm{d}y + m_0\delta_0(\mathrm{d}y) + n(\mathrm{d}y) & \text{if } l = \infty, \\ b^{-2}(y)\,\mathrm{d}y + m_0\delta_0(\mathrm{d}y) + n(\mathrm{d}y) + m_l\delta_l(\mathrm{d}y) & \text{if } l < \infty. \end{cases} \tag{4.2.3}$$

The constraints (S1-3) from Section 2.4 restrict the number of Krein strings we can encompass with the stochastic method. Clearly, (S1) is immediately satisfied by any Krein string. However, if $l < r$ then (S2) restricts us to Krein strings which are conservative so $r = \infty$. Property (S3) though, which is due to the wanted regularity of Y in $(0, l)$, requires the string to be strictly increasing on $(0, l)$. An example of such a process which is not included in our set-up is given by Example 3.2.2. By the assumption (S3), the density of the absolutely continuous measure can be chosen to be strictly positive.

Under the assumptions in Chapter 2, Section 2.4, we have that $l = r$, or $l < \infty$ and $r = \infty$. The condition $l = r$ means that the diffusion is either killed upon hitting $l < \infty$ or conservative with $l = \infty$, while the condition $l < \infty$ and $r = \infty$ means that the diffusion is conservative and reflected at l.

Recalling (2.4.3), we have that $(Y_t)_{t\geq 0}$ solves an SDE of type

$$\mathrm{d}Y_t = \sqrt{2}b(Y_t)\,\mathrm{d}B_t + \mathrm{d}L_t^0(Y) - \mathrm{d}L_t^l(Y).$$

We note that if $l = \infty$, $L_t^l(Y) = 0$ for all $t \geq 0$. Without loss of generality, we may assume $((B_t^1, \cdots, B_t^p))_{t\geq 0}$ and $(B_t)_t \geq 0$ are independent.

4.3 Elliptic PDE

We consider the elliptic PDE

$$\mathcal{L}_x u \times m(\mathrm{d}y) + \partial_y^2 u = 0 \quad \text{on} \quad \mathbb{R}^d \times (0, l), \tag{4.3.1}$$

where the product $\mathcal{L}_x u(x,y) \times m(\mathrm{d}y)$ is understood in the sense of distributions with respect to y, for any fixed x. We are looking for the following type of solutions:

Definition 4.3.1. A function $u : \mathbb{R}^d \times (0,l) \to \mathbb{R}$ such that

- $u(\cdot, y) \in C^2(\mathbb{R}^d)$, for any $y \in (0,l)$,
- $u(x,\cdot)$, $\partial_i u(x,\cdot)$, $\partial_{ij} u(x,\cdot)$, $1 \leq i,j \leq d$, are càdlàg functions[1] on $(0,l)$, for any $x \in \mathbb{R}^d$,

is said to be a *solution of (4.3.1)* if and only if

$$\int_{(0,l)} \mathcal{L}_x u(x,y) g(y) m(\mathrm{d}y) + \int_0^l u(x,y) g''(y)\,\mathrm{d}y = 0,$$

for any $x \in \mathbb{R}^d$, and any smooth function $g : (0,l) \to \mathbb{R}$ with compact support.

Of course, if u solves (4.3.1) in the sense of Definition 4.3.1, then $\partial_y^2 u(x,\cdot)$ is a locally finite signed Borel measure on $(0,l)$, for any $x \in \mathbb{R}^d$. Due to a result by Schwartz [25] which states that a distribution on $\mathbb{R}$ is a convex function if and only if its second derivative is a non-negative locally finite Borel measure, we know for each $x \in \mathbb{R}^d$, $y \mapsto u(x,y)$ is a difference of two convex functions. Therefore for each $x \in \mathbb{R}^d$, the partial derivatives $\partial_y^+ u(x,y)$ (from right) and $\partial_y^- u(x,y)$ (from left) exist, for all $y \in (0,l)$, and

$$\partial_y^2 u(x, (y_\star, y^\star]) = \partial_y^+ u(x, y^\star) - \partial_y^+ u(x, y_\star) = -\int_{(y_\star, y^\star]} \mathcal{L}_x u(x,y)\, m(\mathrm{d}y), \tag{4.3.2}$$

for all $y_\star, y^\star$ such that $0 < y_\star < y^\star < l$.

[1] In what follows, ∂_i and ∂_{ij} is short notation for $\partial/\partial x_i$ and $\partial^2/\partial x_i \partial x_j$, respectively.

Remark 4.3.2. Existence of solutions to (4.3.1) in the sense of Definition 4.3.1 implicitly requires the coefficients of the operator $\mathcal{L}_x$ to be 'good enough'. In this chapter, the only explicit assumptions on these coefficients is continuity. All other assumptions are made implicitly via properties of solutions to equations. First, we require existence of global solutions to the SDE (4.2.1), but all other implicit assumptions will be made via properties of solutions to the PDE (4.3.1). In the following chapter, we shall see a method to study the solution to this PDE in certain situations.

Our first goal is to establish a version of Itô's lemma for $u(X,Y)$, when u is a solution to (4.3.1). Solutions of (4.3.1) are jointly measurable, they are continuous in x, and also continuous in y (recall that $\partial_y^{\pm}u$ do exist), but they might <u>not</u> be jointly continuous. This suggests that more regularity than stated in Definition 4.3.1 would be needed for Itô's lemma to hold true. We should nonetheless try to keep assumptions on the regularity of u as *weak* as possible. Adding the following condition seems to be enough.

Assumption 4.3.3. The functions u, $\partial_i u$, $\partial_{ij}u$, $1 \le i,j \le d$, can be extended to locally bounded functions on $\mathbb{R}^d \times \Big([0,l] \cap \mathbb{R}\Big)$ by taking the limits

$$\lim_{y\downarrow 0} u(x,y), \quad \lim_{y\downarrow 0} \partial_i u(x,y), \quad \lim_{y\downarrow 0} \partial_{ij} u(x,y),$$

and

$$\lim_{y\uparrow l} u(x,y), \quad \lim_{y\uparrow l} \partial_i u(x,y), \quad \lim_{y\uparrow l} \partial_{ij} u(x,y),$$

at any fixed $x \in \mathbb{R}^d$, where the latter three limits are only taken when $l < \infty$.

Remark 4.3.4. Under the above assumption, the particular limit of $u(x,y)$ as $y \downarrow 0$ exists, which we will denote by $f(x)$. So $f : \mathbb{R}^d \to \mathbb{R}$ is a locally bounded measurable function which plays the role of a boundary condition for the solution u. In what follows, to emphasise this role, we will always write u_f instead of u when Assumption 4.3.3 is assumed. Using $u_f(x,0)$ as an alternative notation for the limit of $u_f(x,y)$, $y \downarrow 0$, leads to the wanted equality $u_f(x,0) = f(x)$, $x \in \mathbb{R}^d$. The other limits taken under Assumption 4.3.3 are going to be denoted by $\partial_i u_f(x,0)$, $\partial_{ij} u_f(x,0)$ and, when $l < \infty$, by $u_f(x,l-)$, $\partial_i u_f(x,l-)$, $\partial_{ij} u_f(x,l-)$, for all $x \in \mathbb{R}^d$, $1 \le i,j \le d$. The notations $\mathcal{L}_x u_f(\cdot,0)$ and $\mathcal{L}_x u_f(\cdot,l-)$ used further below would refer to these limits too.

Corollary 4.3.5. *Let u_f be the extension of a solution u to (4.3.1) satisfying Assumption 4.3.3. Then the limit $\partial_y^+ u_f(x,0) = \lim_{y\downarrow 0} \partial_y^+ u(x,0)$ exists, and this limit*

extends $\partial_y^+ u$ to a locally bounded function on $\mathbb{R}^d \times [0,l)$, which is càdlàg, for any fixed $x \in \mathbb{R}^d$.
If $l + m([0,l)) < \infty$, then the limit $\partial_y^- u_f(x, l-) = \lim_{y \uparrow l} \partial_y^+ u(x,y)$ also exists, extending $\partial_y^- u$ to a locally bounded function on $\mathbb{R}^d \times (0,l]$, which is càglàd in y, for any fixed $x \in \mathbb{R}^d$.

Remark 4.3.6. This corollary makes clear that, once all x-direction second order partial derivatives $\partial_{ij} u$, $1 \le i,j \le d$, are locally bounded on the interior $\mathbb{R}^d \times (0,l)$, then u, $\partial_i u$, $1 \le i \le d$, $\partial_y^{\pm} u$ are necessarily locally bounded on the interior $\mathbb{R}^d \times (0,l)$, too. The essence of Assumption 4.3.3 lies within the behaviour of the x-direction partial derivatives at the boundary.

Now, for fixed $x \in \mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P}_x)$ be a complete probability space big enough to carry all random variables $X, Y, (B_1, \ldots, B_r), B$, as described above, with X, Y starting at $X_0 = x$, $Y_0 = 0$, $\mathbb{P}_x$-almost surely respectively. Moreover, we choose a suitable[2] filtration, all processes are adapted to, and all stopping times refer to. The following stopping times,

$$\tau_r(Y) = \inf\{t \ge 0 : Y_t = r\},$$

with respect to $r \ge 0$, will frequently be used, with respect to Y, but also with respect to other one-dimensional processes, for example $\tau_r(|X|)$ with respect to $|X|$ etc. We note that this is indeed a stopping time as the processes $|X|$ and Y are continuous and adapted so we may apply [62, Lemma II.5.74.2].

Lemma 4.3.7. *Let u_f be a solution to (4.3.1) satisfying Assumption 4.3.3.*

(a) If $l = \infty$,

$$u_f(X_t, Y_t) - f(x) = \int_0^t \partial_y^+ u_f(X_s, 0) \, \mathrm{d}L_s^0(Y) + \int_0^t \mathcal{L}_x u_f(X_s, 0) \, \mathbf{1}_{\{0\}}(Y_s) \, \mathrm{d}s$$
$$+ \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, Y_s) \, \sigma_{ik}(X_s) \, \mathrm{d}B_s^k + \sqrt{2} \int_0^t \partial_y^+ u_f(X_s, Y_s) \, b(Y_s) \, \mathrm{d}B_s,$$

for all $t \ge 0$, a.s.

[2] That is, chosen according to the technical assumptions the results used throughout this paper rely upon.

(b) *If $l < \infty$, and Y is absorbing at $l < \infty$,*

$$
\begin{aligned}
&u_f(X_t, Y_t)\,\mathbf{1}_{\{t<\tau_l(Y)\}} + u_f(X_t, l-)\,\mathbf{1}_{\{t\geq\tau_l(Y)\}} - f(x) \\
&= \int_0^t \partial_y^+ u_f(X_s, 0)\,\mathrm{d}L_s^0(Y) \\
&+ \int_0^t \mathcal{L}_x u_f(X_s, 0)\,\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s + \int_0^t \mathcal{L}_x u_f(X_s, l-)\,\mathbf{1}_{\{s\geq\tau_l(Y)\}}\,\mathrm{d}s \\
&+ \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, Y_s)\,\sigma_{ik}(X_s)\,\mathbf{1}_{\{s<\tau_l(Y)\}}\,\mathrm{d}B_s^k \\
&+ \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, l-)\,\sigma_{ik}(X_s)\,\mathbf{1}_{\{s\geq\tau_l(Y)\}}\,\mathrm{d}B_s^k \\
&+ \sqrt{2}\int_0^t \partial_y^+ u_f(X_s, Y_s)\,b(Y_s)\,\mathbf{1}_{\{s<\tau_l(Y)\}}\,\mathrm{d}B_s,
\end{aligned}
$$

for all $t \geq 0$, a.s.

(c) *If $l < \infty$, and Y is not absorbing at l, and $\partial_y^- u_f(\cdot, l-)$ is a continuous function on $\mathbb{R}^d$,*

$$
\begin{aligned}
&u_f(X_t, Y_t)\,\mathbf{1}_{\{Y_t<l\}} + u_f(X_t, l-)\,\mathbf{1}_{\{l\}}(Y_t) - f(x) \\
&= \int_0^t \partial_y^+ u_f(X_s, 0)\,\mathrm{d}L_s^0(Y) - \int_0^t \partial_y^- u_f(X_s, l-)\,\mathrm{d}L_s^l(Y) \\
&+ \int_0^t \mathcal{L}_x u_f(X_s, 0)\,\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s + \int_0^t \mathcal{L}_x u_f(X_s, l-)\,\mathbf{1}_{\{l\}}(Y_s)\,\mathrm{d}s \\
&+ \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, Y_s)\,\sigma_{ik}(X_s)\,\mathbf{1}_{\{Y_s<l\}}\,\mathrm{d}B_s^k \\
&+ \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, l-)\,\sigma_{ik}(X_s)\,\mathbf{1}_{\{l\}}(Y_s)\,\mathrm{d}B_s^k \\
&+ \sqrt{2}\int_0^t \partial_y^+ u_f(X_s, Y_s)\,b(Y_s)\,\mathrm{d}B_s,
\end{aligned}
$$

for all $t \geq 0$, a.s.

Remark 4.3.1. *If $l < \infty$ is not absorbing, then $m([0, l]) < \infty$. Thus, the representation of $\partial_y^- u_f(\cdot, l-)$ given in Corollary 4.3.5 can be used to show the implication: if $\partial_y^- u_f(\cdot, l-)$ is a continuous function on $\mathbb{R}^d$, then $\partial_y^+ u(\cdot, y^*)$ would be one, too, for any interior value $y^* \in (0, l)$. Indeed, this implication follows by dominated convergence combining Assumption 4.3.3 and $u(\cdot, y) \in C^2(\mathbb{R}^d)$, $y \in (0, l)$. Vice versa, if $\partial_y^+ u_f(\cdot, y^*)$ was continuous, for some $y^* \in (0, l)$, then $\partial_y^- u_f(\cdot, l-)$ would be, too. All* *in all, stating $\partial_y^- u_f(\cdot, l-) \in C(\mathbb{R}^d)$ is equivalent to stating $\partial_y^+ u(\cdot, y) \in C(\mathbb{R}^d)$ for all $y \in (0, l)$. Moreover, by similar arguments, $\partial_y^+ u_f(\cdot, 0) \in C(\mathbb{R}^d)$ is also equivalent to $\partial_y^+ u(\cdot, y) \in C(\mathbb{R}^d)$, for all $y \in (0, l)$, and hence $\partial_y^+ u_f(\cdot, 0) \in C(\mathbb{R}^d)$ implies $\partial_y^- u_f(\cdot, l-) \in C(\mathbb{R}^d)$, in particular.*

Next, we observe that the pair of random variables (X, Y) describes a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ taking values in $\mathbb{R}^d \times \Big([0, l] \cap \mathbb{R}\Big)$. This process is associated with a so-called *trace process*, $(X_{T_t})_{t \geq 0}$, which is the trace of the process (X, Y) when touching the hyperplane $\{(x, 0) : x \in \mathbb{R}\} \subseteq \mathbb{R}^d \times \Big([0, l] \cap \mathbb{R}\Big)$, where $(T_t)_{t \geq 0}$ denotes the right-inverse of the symmetric local time $(L_t^0(Y))_{t \geq 0}$ of Y at zero:

$$T_t = \begin{cases} \inf\{s > 0 : L_s^0(Y) > t\} & \text{for } t < L_\infty^0(Y) \\ \infty & \text{for } t \geq L_\infty^0(Y), \end{cases}$$

where we define $L_\infty^0(Y) = \lim_{t \to \infty} L_t^0(Y)$. When $T_t = \infty$ for some $t > 0$, the process $(X_{T_t})_{t \geq 0}$ is killed with lifetime $L_\infty^0(Y)$. We denote its cemetery-state by $\dagger$, which is added to $\mathbb{R}^d$ in the usual way, and we also define $X(\infty) = \dagger$. Any function $f : \mathbb{R}^d \to \mathbb{R}$ is also considered a function $f : \mathbb{R}^d \cup \{\dagger\} \to \mathbb{R}$ by setting $f(\dagger) = 0$.

The purpose of Lemma 4.3.7 is to be able to work out the limit of $\frac{\mathbb{E}_x[f(X_{T_t}) - f(x)]}{t}$ when $t \downarrow 0$, for any fixed $x \in \mathbb{R}^d$, and any bounded function $f : \mathbb{R}^d \to \mathbb{R}$ of a certain degree of regularity. For the corresponding result, which is the main result of this paper, we have to differ between infinite and finite l, but also between l is absorbing and non-absorbing in the case of finite l. Moreover, as $\mathbb{P}_x[\{\tau_l(Y) < \infty\}] \in \{0, 1\}$, see [5, Lemma 2.9] for example, the case of l is absorbing splits into two further cases.

Lemma 4.3.2. *Choose a non-trivial Krein string m which is strictly increasing on $(0, l)$ with $l < \infty$, and let Y be the diffusion corresponding to m. Then,*

(i) Y is absorbing at l if and only if $m([0, l]) = \infty$,

(ii) $\tau_l(Y) = \infty$ a.s. if and only if $\int_{[0,l)} (l - y) m(\mathrm{d}y) = \infty$.

Theorem 4.3.3. *Choose a non-trivial Krein string m which is strictly increasing $(0, l)$, and select a PDE operator $\mathcal{L}_x$ according to (4.2.2) and Remark 4.3.2. Let u be a bounded solution to (4.3.1) satisfying Assumption 4.3.3, and suppose that the extension u_f satisfies $\partial_y^+ u_f(\cdot, 0) + m_0 \mathcal{L}_x u_f(\cdot, 0) \in C_b(\mathbb{R}^d)$. Consider one of the following cases:*

(a) $l = \infty$,

(b1) $l < \infty$, $m([0,l]) = \infty$ *but* $\int_{[0,l)}(l-y)m(\mathrm{d}y) < \infty$, $u_f(\cdot, l-) = 0$, *and the extension* $u_f : \mathbb{R}^d \times [0,l] \to \mathbb{R}$ *is jointly continuous at any* (x,l) *for any* $x \in \mathbb{R}^d$,

(b2) $l < \infty$, $\int_{[0,l)}(l-y)m(\mathrm{d}y) = \infty$ *and* $\sup_{x\in\mathbb{R}^d} |u(x, l-h)| \to 0$ *as* $h \downarrow 0$,

(c) $l < \infty$, $m([0,l]) < \infty$ *and* u_f *satisfies* $\partial_y^- u_f(\cdot, l-) = m_1 \mathcal{L}_x u_f(\cdot, l-) \in C(\mathbb{R}^d)$.

Then, for any $x \in \mathbb{R}^d$,

$$\lim_{t\downarrow 0} \frac{\mathbb{E}_x[f(X(T_t))] - f(x)}{t} = \partial_y^+ u_f(x,0) + m_0 \mathcal{L}_x u_f(x,0).$$

We recall that under the assumptions made, both processes X and Y are strong Markov processes, the process X due to [37, Theorem 5.4.20], and Y by Theorem 2.4.1. Of course, the generator of X is given by $\mathcal{L}_x$, being formally defined via (4.2.2), and a dense domain in some Banach space. In our setup, since the coefficients of the SDE (4.2.1) are supposed to be continuous, then X is a Feller process, and hence the natural choice of Banach space would be a space of continuous functions with a growth condition at infinity.

4.4 Proofs

*Proof of **Corollary 4.3.5**.* We first show that $\partial_y^+ u(\cdot, y^\star)$ is locally bounded on $\mathbb{R}^d$, for an arbitrary but fixed $y^\star \in (0,l)$.

Fix $y^\star \in (0,l)$, and assume the contrary. Then there exists a sequence $(x_n)_{n=1}^\infty \subseteq \mathbb{R}^d$, which converges to some $x \in \mathbb{R}^d$, such that $\sup_n |\partial_y^+ u(x_n, y^\star)| = \infty$. Without loss of generality, assume that

$$\forall\, R > 0\; \exists\, n_R\; \forall\, n \geq n_R : \; \partial_y^+ u(x_n, y^\star) \;\geq\; R.$$

Next, fix $y_\star \in (0, y^\star)$, and note that

$$\partial_y^+ u(x_n, y) \;=\; \partial_y^+ u(x_n, y^\star) + \int_{(y,y^\star]} \mathcal{L}_x u(x_n, y')\, m(\mathrm{d}y'),$$

for all $n \geq 0$, and all $y \in [y_\star, y^\star)$, is an easy consequence of (4.3.2). By Assumption 4.3.3, but also using $m(K) < +\infty$, for any compact subset $K \subseteq [0,l)$, as well as continuity of the coefficients of $\mathcal{L}_x$,

$$c(y_\star, y^\star) \overset{\text{def}}{=} \sup_{n\geq 0} \sup_{y\in[y_\star,y^\star)} \Big| \int_{(y,y^\star]} \mathcal{L}_x u(x_n, y')\, m(\mathrm{d}y') \Big| \;<\; \infty.$$

Then, for any $R > 0$,

$$\partial_y^+ u(x_n, y) \;\geq\; R - c(y_\star, y^\star), \quad \forall\, n \geq n_R,\ \forall\, y \in [y_\star, y^\star],$$

and hence

$$\begin{aligned} u(x_n, y_\star) \;&=\; u(x_n, y^\star) \;-\; \int_{y_\star}^{y^\star} \partial_y^+ u(x_n, y)\; \mathrm{d}y \\ &\leq\; u(x_n, y^\star) + [c(y_\star, y^\star) - R] \times (y^\star - y_\star), \quad \forall\, n \geq n_R. \end{aligned}$$

Of course, $\sup_{n\geq 0} |u(x_n, y^\star)| < +\infty$ since $u(\cdot, y^\star)$ is continuous and $x_n \to x \in \mathbb{R}^d$, $n \to \infty$, so that $\limsup_{n\to\infty} u(x_n, y_\star) = -\infty$, which contradicts the continuity of $u(\cdot, y_\star)$.

All in all $\partial_y^+ u(\cdot, y^\star)$ is indeed locally bounded on $\mathbb{R}^d$. Therefore,

$$\partial_y^+ u_f(x, 0) \stackrel{\text{def}}{=} \lim\nolimits_{y\downarrow 0} \left(\partial_y^+ u(x, y^\star) \;+\; \int_{(y, y^\star]} \mathcal{L}_x u(x, y')\, m(\mathrm{d}y') \right), \quad x \in \mathbb{R}^d,$$

defines a locally bounded function on $\mathbb{R}^d$, since $\mathcal{L}_x u_f$ is locally bounded on $\mathbb{R}^d \times [0, l)$. But this limit can be used to define the wanted extension because $\partial_y^+ u(x, 0) = \lim_{y\downarrow 0} \partial_y^+ u(x, y)$, for any $x \in \mathbb{R}^d$.

Obviously, for fixed $x \in \mathbb{R}^d$, the extended version of $\partial_y^+ u(x, \cdot)$ defined this way is right-continuous at zero, and thus it is càdlàg on $[0, l)$ because $u(x, \cdot)$ is difference of two convex functions on the interior $(0, l)$, finishing the proof of the corollary in the case of $\partial_y^+ u$.

In the case of $\partial_y^- u$, the wanted extension can be given by

$$\partial_y^- u(x, l-) \stackrel{\text{def}}{=} \lim\nolimits_{y\uparrow l} \left(\partial_y^+ u(x, y^\star) \;-\; \int_{(y^\star, y]} \mathcal{L}_x u(x, y')\, m(\mathrm{d}y') \right), \quad x \in \mathbb{R}^d,$$

though we omit the proof. □

*Proof of **Lemma 4.3.7**.* (a) To start with, we 'mollify' u_f introducing

$$u_f^\varepsilon(x, y) \stackrel{\text{def}}{=} \int_0^\infty \varrho_\varepsilon(y - y')\, u_f(x, y')\; \mathrm{d}y', \quad (x, y) \in \mathbb{R}^d \times [0, \infty), \quad \varepsilon > 0,$$

where $\varrho_\varepsilon(y) = \varrho(y/\varepsilon)/\varepsilon$, $y \in \mathbb{R}$, using a 'right-hand' mollifier $\varrho \in C^2(\mathbb{R})$ with compact support in $(-1, 0)$ satisfying $\varrho \geq 0$ and $\int_{\mathbb{R}} \varrho(y)\; \mathrm{d}y = 1$.

Note that, because of Assumption 4.3.3, on the one hand, and because the support of ϱ is bounded away from zero, on the other, the mollified solution u_f^ε is an element of $C^2(\mathbb{R}^d \times [0,\infty))$. Applying the Whitney extension theorem [76, Theorem I], we may extend u_f^ε from the closed space $\mathbb{R}^d \times [0,\infty)$ to a function $\overline{u_f^\varepsilon} \in C^2(\mathbb{R}^{d+1})$ such that $\partial^\alpha \overline{u_f^\varepsilon} = \partial^\alpha u_f^\varepsilon$ in $\mathbb{R}^d \times [0,\infty)$ for any multi-index $\alpha \in \mathbb{N}_0^{d+1}$ with $0 \leq |\alpha| \leq 2$.

Recall that u_f is not necessarily jointly continuous, so we do not know if $(u_f(X_t, Y_t))_{t\geq 0}$ is a continuous stochastic process. However,

$$u_f(X_t(\omega), Y_t(\omega)) \;=\; \lim_{\varepsilon\downarrow 0} u_f^\varepsilon(X_t(\omega), Y_t(\omega)), \quad \forall\,(\omega, t) \in \Omega \times [0,\infty),$$

where $(u_f^\varepsilon(X_t, Y_t))_{t\geq 0}$ can be seen to be an adapted, continuous stochastic process as (X, Y) is adapted and continuous and u_f^ε is jointly continuous. Therefore, the process $(u_f(X_t, Y_t))_{t\geq 0}$ is at least predictable and hence $(Z_t)_{t\geq 0}$ defined by

$$\begin{aligned}
Z_t \;&\overset{\text{def}}{=}\; u_f(X_t, Y_t) - f(x) \\
&- \int_0^t \partial_y^+ u_f(X_s, 0)\; \mathrm{d}L_s^0(Y) \;-\; \int_0^t \mathcal{L}_x u_f(X_s, 0)\, \mathbf{1}_{\{0\}}(Y_s)\; \mathrm{d}s \\
&- \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, Y_s)\, \sigma_{ik}(X_s)\; \mathrm{d}B_s^k \;-\; \sqrt{2}\int_0^t \partial_y^+ u_f(X_s, Y_s)\, b(Y_s)\; \mathrm{d}B_s
\end{aligned}$$

is predictable, too. Thus, using [35, Prop. I.2.18 b)] to prove part (a) of the lemma, it is sufficient to show that $\mathbb{P}_x(\{Z(t\wedge\tau) = 0\}) = 1$, for any $t \geq 0$ and any predictable stopping time τ. We recall that a stopping time τ is predictable if there is an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times such that almost surely, $\lim_{n\to\infty}\tau_n = \tau$ and $\tau_n < \tau$ for every $n \in \mathbb{N}$ on $\{\tau > 0\}$.

To be able to approximate events like $\{Z(t\wedge\tau) = 0\}$, denote the Euclidean ball of radius $R > 0$ by $\mathfrak{B}_R(0)$, introduce

$$\tau_R(X) \;=\; \inf\{t \geq 0 : X_t \notin \mathfrak{B}_R(0)\}, \quad \tau_R(Y) \;=\; \inf\{t \geq 0 : Y_t \geq R\},$$

and set

$$\tau_R = \tau_R(X) \wedge \tau_R(Y), \quad R > 0.$$

Next, the continuous local martingale $(\int_0^t b(Y_s)\; \mathrm{d}B_s)_{t\geq 0}$ can be $L^2(\mathbb{P}_x)$-localised by predictable stopping times $(S_N)_{N\in\mathbb{N}}$ given by

$$S_N = \inf\left\{ t > 0 : \int_0^t b^2(Y_s)\,\mathrm{d}s \geq N \right\},$$

for each $N \in \mathbb{N}$ noting that this random variable is equal to the first hitting time of $[N,\infty)$ for the quadratic variation process of $(\int_0^t b(Y_s)\,\mathrm{d}B_s)_{t\geq 0}$. As this quadratic variation process is continuous and adapted, these hitting times are indeed stopping times and each S_N can be seen to be predictable by considering the sequence of announcing stopping times $(S_{N-\frac{1}{k}})_{k\in\mathbb{N}}$. Furthermore, we see that the process $(\int_0^{t\wedge S_N} b(Y_s)\,\mathrm{d}B_s)_{t\geq 0}$ is clearly a local martingale and

$$\sup_t \mathbb{E}_x\Big[\Big\langle \int_0^{\cdot\wedge S_N} b(Y_s)\,\mathrm{d}B_s, \int_0^{\cdot\wedge S_N} b(Y_s)\,\mathrm{d}B_s\Big\rangle_t\Big] = \mathbb{E}_x\Big[\int_0^{S_N} b^2(Y_s)\,\mathrm{d}s\Big] = N,$$

so by [61, Prop. IV.1.23], $(\int_0^{t\wedge S_N} b(Y_s)\,\mathrm{d}B_s)_{t\geq 0}$ is an $L^2(\mathbb{P}_x)$-bounded martingale.

Hence $(\tau_N \wedge S_N)_{N\in\mathbb{N}}$ is a sequence of predictable stopping times such that $\tau_N \wedge S_N \uparrow \infty$ as $N \to \infty$. Consequently, showing $\mathbb{P}_x(\{Z(t\wedge\tau_N\wedge S_N\wedge\tau) = 0\}) = 1$, for any $t \geq 0$, $N \in \mathbb{N}$, and predictable stopping time τ, would be enough to prove. Also, by Assumption 4.3.3, the sequence $(\tau_N \wedge S_N)_{N\in\mathbb{N}}$, would be $L^2(\mathbb{P}_x)$-localising for all local martingales used in the definition of $(Z_t)_{t\geq 0}$.

So, fix $t \geq 0$, $N \geq 1$, and a predictable stopping time τ. Abbreviating $t\wedge\tau_N\wedge S_N\wedge\tau$ by t_N, we are going to show that $\mathbb{P}_x(\{Z(t_N) = 0\}) = 1$.

First, for any $\varepsilon > 0$ the classical version of Itô's lemma [63, IV.5.32.8] applied to the extended function $\overline{u_f^\varepsilon} : \mathbb{R}^{d+1} \to \mathbb{R}$ and noting that $(X, Y) \in \mathbb{R}^d \times [0,\infty)$ gives

$$\begin{aligned} u_f^\varepsilon(X_{t_N}, Y_{t_N}) - u_f^\varepsilon(x,0) \; - \int_0^{t_N} \partial_y u_f^\varepsilon(X_s,0)\; \mathrm{d}L_s^0(Y) \\ - \sum_{i=1}^d \sum_{k=1}^p \int_0^{t_N} \partial_i u_f^\varepsilon(X_s,Y_s)\,\sigma_{ik}(X_s)\; \mathrm{d}B_s^k \; - \sqrt{2}\int_0^{t_N} \partial_y u_f^\varepsilon(X_s,Y_s)\, b(Y_s)\; \mathrm{d}B_s \\ \overset{a.s.}{=} \int_0^{t_N} \mathcal{L}_x u_f^\varepsilon(X_s,Y_s)\; \mathrm{d}s + \int_0^{t_N} \partial_y^2 u_f^\varepsilon(X_s,Y_s)\, b^2(Y_s)\; \mathrm{d}s. \end{aligned} \tag{4.4.1}$$

We claim that, for any chosen sequence of ε-values converging to zero, there is a subsequence $(\varepsilon_n)_{n=1}^\infty$ such that the above equation's left-hand side almost surely converges to

$$\begin{aligned} u_f(X_{t_N}, Y_{t_N}) - f(x) \; - \int_0^{t_N} \partial_y^+ u_f(X_s,0)\; \mathrm{d}L_s^0(Y) \\ - \sum_{i=1}^d \sum_{k=1}^p \int_0^{t_N} \partial_i u_f(X_s,Y_s)\,\sigma_{ik}(X_s)\; \mathrm{d}B_s^k \; - \sqrt{2}\int_0^{t_N} \partial_y^+ u_f(X_s,Y_s)\, b(Y_s)\; \mathrm{d}B_s, \end{aligned}$$

when $n \to \infty$.

Indeed, the limit of $u_f^\varepsilon(X_{t_N}, Y_{t_N}) - u_f^\varepsilon(x,0)$ as $\varepsilon \downarrow 0$ is obvious, by Remark

4.3.4. Next, for $0 \le s \le t_N$, partial integration yields

$$\partial_y u_f^\varepsilon(X_s, Y_s) \;=\; \int_0^\infty \varrho_\varepsilon'(Y_s - y')\, u_f(X_s, y')\ \mathrm{d}y' \;=\; \int_0^\infty \varrho_\varepsilon(y')\, \partial_y^+ u_f(X_s, Y_s - y')\ \mathrm{d}y'$$

which converges to $\partial_y^+ u_f(X_s, Y_s)$, when $\varepsilon \downarrow 0$. Furthermore,

$$\sup_{0 \le s \le t_N} |\partial_y u_f^\varepsilon(X_s, Y_s)| \le \sup_{\substack{(x,y)\in \\ \mathfrak{B}_N(0)\times[0,N+\varepsilon]}} |\partial_y^+ u_f(x, y)| \underbrace{\int \varrho_\varepsilon(y')\ \mathrm{d}y'}_{=1}, \quad \forall\, \varepsilon > 0,$$

where, by Corollary 4.3.5, the supremum on the right-hand side is uniformly bounded in ε, for any chosen sequence of ε-values converging to zero. Therefore,

$$\begin{aligned}\int_0^{t_N} \partial_y u_f^\varepsilon(X_s, 0)\ \mathrm{d}L_s^0(Y) &= \int_0^{t_N} \partial_y u_f^\varepsilon(X_s, Y_s)\ \mathrm{d}L_s^0(Y) \\ &\xrightarrow{\varepsilon\downarrow 0} \int_0^{t_N} \partial_y^+ u_f(X_s, Y_s)\ \mathrm{d}L_s^0(Y) \;=\; \int_0^{t_N} \partial_y^+ u_f(X_s, 0)\ \mathrm{d}L_s^0(Y)\end{aligned}$$

follows by bounded convergence.

Lastly, we see that all stochastic integrals on the left-hand side of (4.4.1) converge duly in $L^2(\mathbb{P}_x)$, when $\varepsilon \downarrow 0$, proving the claim we made above. For $1 \le i, k \le d$ we have,

$$\begin{aligned}&\mathbb{E}_x\left[\Big(\int_0^{t_N} \partial_i u_f^\varepsilon(X_s, Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B_s^k - \int_0^{t_N} \partial_i u_f(X_s, Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B_s^k\Big)^2\right] \\ &= \mathbb{E}_x\left[\int_0^{t_N} [\partial_i u_f^\varepsilon(X_s, Y_s) - \partial_i u_f(X_s, Y_s)]^2 \sigma_{ik}^2(X_s)\,\mathrm{d}s\right] \\ &\le \Big(\sup_{x\in\mathfrak{B}_N(0)} \sigma_{ik}^2(x)\Big)\mathbb{E}_x\Big[\int_0^{t_N} [\partial_i u_f^\varepsilon(X_s, Y_s) - \partial_i u_f(X_s, Y_s)]^2\,\mathrm{d}s\Big].\end{aligned}$$

As σ_{ik} is continuous, this supremum is finite. Furthermore, we know that $\partial_i u_f^\varepsilon(X_s, Y_s)$ converges to $\partial_i u_f(X_s, Y_s)$ for each $s \in [0, t_N]$ and as $\partial_i u_f^\varepsilon$ and $\partial_i u_f$ are bounded on $\mathfrak{B}_N(0) \times [0, N]$ and $t_N \le t$, we may apply dominated convergence to see that the limit of the right-hand side is zero.

For the case of $\int_0^{t_N} \partial_y u_f^\varepsilon(X_s, Y_s)\, b(Y_s)\ \mathrm{d}B_s$ we note that as $t_N \le \tau_N \wedge S_N$, we obtain that

$$\begin{aligned}
&\mathbb{E}_x\left[\Big(\int_0^{t_N}\partial_y u_f^\varepsilon(X_s,Y_s)\,b(Y_s)\;\mathrm{d}B_s-\int_0^{t_N}\partial_y^+ u_f(X_s,Y_s)\,b(Y_s)\;\mathrm{d}B_s\Big)^2\right]\\
&=\mathbb{E}_x\left[\int_0^{t_N}[\,\partial_y u_f^\varepsilon(X_s,Y_s)-\partial_y^+ u_f(X_s,Y_s)\,]^2\,b^2(Y_s)\;\mathrm{d}s\right],
\end{aligned}$$

where $\mathbb{E}_x[\int_0^{t_N} b^2(Y_s)\;\mathrm{d}s\,]<\infty$. Then, again by dominated convergence, the ε-limit of the right-hand side can be taken with respect to the integrand, and this limit is zero, for all $s\in[0,t_N]$.

Eventually, for $\mathbb{P}_x(\{Z(t_N)=0\})=1$, it remains to show that

$$\int_0^{t_N}\mathcal{L}_x u_f^\varepsilon(X_s,Y_s)\;\mathrm{d}s+\int_0^{t_N}\partial_y^2 u_f^\varepsilon(X_s,Y_s)\,b^2(Y_s)\;\mathrm{d}s \tag{4.4.2}$$

almost surely converges to

$$\int_0^{t_N}\mathcal{L}_x u_f(X_s,0)\,\mathbf{1}_{\{0\}}(Y_s)\;\mathrm{d}s,$$

when $\varepsilon\downarrow 0$.

To see this, we recall the construction of Y given by Theorem 2.4.1. So, we assume that $Y_t=W(A_t^{-1})$, $t\geq 0$, where $(W_t)_{t\geq 0}$ is a standard one-dimensional Wiener process, given on $(\Omega,\mathcal{F},\mathbb{P}_x)$, and

$$A_t=\int_{[0,\infty)}\frac{1}{2}\,L_t^y(W)\,m(\mathrm{d}y),\quad t\geq 0,$$

where m is the Krein string corresponding to the speed measure $\tilde{m}$ as defined in Chapter 2.

Note that $A_\infty=+\infty$, a.s., as stated in [5, Lemma 6.15], and $t\mapsto A_t$ is continuous because the measure is finite on compact subsets of $[0,\infty)$ and hence $s=A_{A_s^{-1}}$, for all $s\geq 0$, a.s. Therefore, time change yields

$$\int_0^{t_N} \partial_y^2 u_f^\varepsilon(X_s, Y_s)\, b^2(Y_s)\ \mathrm{d}s \overset{a.s.}{=} \int_0^{A_{A_{t_N}^{-1}}} \partial_y^2 u_f^\varepsilon(X(A_{A_s^{-1}}), W(A_s^{-1}))\, b^2(W(A_s^{-1}))\ \mathrm{d}s$$

$$= \int_0^{A_{t_N}^{-1}} \partial_y^2 u_f^\varepsilon(X(A_s), W(s))\, b^2(W(s))\ \mathrm{d}A_s$$

$$\overset{a.s.}{=} \int_{[0,\infty)} \int_0^{A_{t_N}^{-1}} \partial_y^2 u_f^\varepsilon(X(A_s), y)\ \mathrm{d}L_s^y(W)\, b^2(y)\, [\frac{1}{2} b^{-2}(y)\ \mathrm{d}y + \frac{m_0}{2}\delta_0(\mathrm{d}y) + \frac{1}{2} n(\mathrm{d}y)]$$

$$= \int_0^\infty \int_0^{A_{t_N}^{-1}} \partial_y^2 u_f^\varepsilon(X(A_s), y)\ \mathrm{d}L_s^y(W)\, \mathbf{1}_{\{b^2>0\}}(y)\ \mathrm{d}y \times 1/2,$$

because $\mathcal{N} \cup \{0, l\} \subseteq \{b^2 = 0\}$. However, from Section 2.4 we know that b^{-2} is locally integrable on $[0, \infty)$ so the set $\{b^2 = 0\}$ has Lebesgue measure zero, and hence the last integral becomes

$$\int_0^\infty \int_0^{A_{t_N}^{-1}} \partial_y^2 u_f^\varepsilon(X(A_s), y)\ \mathrm{d}L_s^y(W)\ \mathrm{d}y \times 1/2$$

$$\overset{\varepsilon \downarrow 0}{\longrightarrow} -\int_{(0,\infty)} \int_0^{A_{t_N}^{-1}} \mathcal{L}_x u_f(X(A_s), y)\ \mathrm{d}L_s^y(W)\, m(\mathrm{d}y) \times 1/2,$$

using the PDE (4.3.1) to identify the limit of the measures $\partial_y^2 u_f^\varepsilon(X(A_s), y)\ \mathrm{d}y$, when $\varepsilon \downarrow 0$. Of course, the measure $m/2$ equals $\tilde{m}/2$ on $((0, \infty), \mathrm{Bor}((0, \infty)))$, so that

$$-\int_{[0,\infty)} \int_0^{A_{t_N}^{-1}} \mathcal{L}_x u_f(X(A_s), y)\ \mathrm{d}L_s^y(W) \mathbf{1}_{(0,\infty)}(y)\, \frac{m}{2}(\mathrm{d}y)$$

$$\overset{a.s.}{=} -\int_0^{A_{t_N}^{-1}} \mathcal{L}_x u_f(X(A_s), W(s)) \mathbf{1}_{(0,\infty)}(W(s))\ \mathrm{d}A_s \overset{a.s.}{=} -\int_0^{t_N} \mathcal{L}_x u_f(X_s, Y_s) \mathbf{1}_{(0,\infty)}(Y_s)\ \mathrm{d}s,$$

which almost surely is the ε-limit of the second summand in (4.4.2). Note the subtle point that the indicator $\mathbf{1}_{(0,\infty)}$ is due to the fact that the PDE only holds in the open half plane.

Finally, as $\int_0^{t_N} \mathcal{L}_x u_f(X_s, Y_s)\ \mathrm{d}s$ is the ε-limit of the first summand in (4.4.2), the ε-limit of (4.4.2) can almost surely be given by

$$\int_0^{t_N} \mathcal{L}_x u_f(X_s, Y_s) \mathbf{1}_{\{0\}}(Y_s)\ \mathrm{d}s = \int_0^{t_N} \mathcal{L}_x u_f(X_s, 0)\, \mathbf{1}_{\{0\}}(Y_s)\ \mathrm{d}s,$$

proving part (a) of the lemma.

(b) Fix t, τ_N, S_N, τ as in (a), but define $\tau = t \wedge \tau_N \wedge S_n \wedge \tau_N(L_\cdot^0(Y)) \wedge \tau$. It is again sufficient to show $\mathbb{P}_x(\{Z(t_N) = 0\}) = 1$, where

$$\begin{aligned}
Z_t =\,& u_f(X_t, Y_t)\, \mathbf{1}_{\{t<\tau_l(Y)\}} + u_f(X_t, l-)\, \mathbf{1}_{\{t\geq\tau_l(Y)\}} - f(x) \\
&- \int_0^t \partial_y^+ u_f(X_s, 0)\, \mathrm{d}L_s^0(Y) \\
&- \int_0^t \mathcal{L}_x u_f(X_s, 0)\, \mathbf{1}_{\{0\}}(Y_s)\, \mathrm{d}s - \int_0^t \mathcal{L}_x u_f(X_s, l-) \mathbf{1}_{\{s\geq\tau_l(Y)\}}\, \mathrm{d}s \\
&- \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, Y_s)\, \sigma_{ik}(X_s)\, \mathbf{1}_{\{s<\tau_l(Y)\}}\, \mathrm{d}B_s^k \\
&- \sum_{i=1}^d \sum_{k=1}^p \int_0^t \partial_i u_f(X_s, l-)\, \sigma_{ik}(X_s) \mathbf{1}_{\{s\geq\tau_l(Y)\}}\, \mathrm{d}B_s^k \\
&- \sqrt{2} \int_0^t \partial_y^+ u_f(X_s, Y_s)\, b(Y_s) \mathbf{1}_{\{s<\tau_l(Y)\}}\, \mathrm{d}B_s.
\end{aligned}$$

Observe that, different to the proof of (a), it is technically more demanding to work with the mollified version of $u_f(x, y)$, when y is close to $l < \infty$. We therefore choose $h \in (0, l)$ and build a function $u_{f,h}$ on the whole half-space $\mathbb{R}^d \times [0, \infty)$ by setting

$$u_{f,h}(x, y) = \begin{cases} u_f(x, y) & \text{for } y \in [0, l - \frac{h}{2}), \\ u_f(x, l - \frac{h}{2}) & \text{for } y \in [l - \frac{h}{2}, \infty). \end{cases}$$

Let $u_{f,h}^\varepsilon$ denote the mollified version of $u_{f,h}$ using the same mollifier ϱ as in the proof of (a), and let Y^h denote the process $(Y_{t\wedge\tau_{l-h}(Y)})_{t\geq 0}$. Note that for any such h, the stopping time $\tau_{l-h}(Y)$ is almost surely finite because $m([0, l - h]) < \infty$.

We first apply the classical version of Itô's lemma to $u_{f,h}^\varepsilon(X_{t_N}, Y_{t_N}^h)$ (or rather, the Whitney extension to $\mathbb{R}^{d+1}$ as in part a)) and let ε go to zero. We then prove our claim by letting h go to zero, too.

Recall that changing b on a set of Lebesgue measure zero would not change the law of Y, thus it would not change the law of Y^h either. Without loss of generality, we can therefore assume that $b(l-h) = 0$. As a consequence, Y^h satisfies the SDE,

$$\mathrm{d}Y_t^h = \sqrt{2} b(Y_t^h)\, \mathrm{d}B_t + \mathrm{d}L_t^0(Y^h),$$

and hence applying Itô's formula to $u_{f,h}^\varepsilon(X_{t_N}, Y_{t_N}^h)$ gives an equation identical to (4.4.1) with u_f^ε and Y replaced by $u_{f,h}^\varepsilon$ and Y^h, respectively.

Then, as in the proof of (a), for any sequence of ε-value converging to zero, there exists a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that, when $n \to \infty$, the left-hand side of this sequence converges almost surely to

$$\begin{cases} u_f(X_{t_N}, Y_{t_N\wedge\tau_{l-h}(Y)}) - f(x) \\ -\int_0^{t_N} \mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^+ u_f(X_s,0)\,\mathrm{d}L_s^0(Y) \\ -\sum_{i=1}^d\sum_{k=1}^p\int_0^{t_N}\partial_i u_f(X_s,Y_{s\wedge\tau_{l-h}(Y)})\sigma_{ik}(X_s)\,\mathrm{d}B_s^k \\ -\sqrt{2}\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^+ u_f(X_s,Y_s)b(Y_s)\,\mathrm{d}B_s \end{cases} \tag{4.4.3}$$

where we have used that $u_f = u_{f,h}$ on $\mathbb{R}^d\times[0,l-\frac{h}{2})$, that $L_s^0(Y^h)$ is constant, for $s\geq\tau_{l-h}(Y)$, and that $b(l-h)=0$.

The next step is to find the ε-limit of the right-hand side, i.e.

$$\int_0^{t_N}\mathcal{L}_x u_{f,h}^\varepsilon(X_s,Y_s^h)\,\mathrm{d}s + \int_0^{t_N}\partial_y^2 u_{f,h}^\varepsilon(X_s,Y_s^h)b^2(Y_s^h)\,\mathrm{d}s, \tag{4.4.4}$$

the second integral of which can be written

$$\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^2 u_{f,h}^\varepsilon(X_s,Y_s)b^2(Y_s)\mathbf{1}_{\{Y_s<l-h\}}\,\mathrm{d}s$$

since Y_h is absorbing at $l-h$, and $b(l-h)=0$. But $s=A_{A_s^{-1}}$, for all $s<\tau_{l-h}(Y)$, a.s., and hence, when $\varepsilon\downarrow 0$, the second integral converges almost surely to

$$-\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\mathcal{L}_x u_f(X_s,Y_s)\mathbf{1}_{(0,l-h]}(Y_s)\,\mathrm{d}s,$$

applying time change and partial integration as in the proof of (a).

By dominated convergence, when $\varepsilon\downarrow 0$, the first integral of (4.4.4) converges to

$$\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\mathcal{L}_x u_f(X_s,Y_s)\,\mathrm{d}s + \int_0^{t_N}\mathbf{1}_{\{s\geq\tau_{l-h}(Y)\}}\mathcal{L}_x u_f(X_s,l-h)\,\mathrm{d}s,$$

so that summing up gives the following almost sure limit of (4.4.4),

$$\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\mathcal{L}_x u_f(X_s,0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s + \int_0^{t_N}\mathbf{1}_{\{s\geq\tau_{l-h}(Y)\}}\mathcal{L}_x u_f(X_s,l-h)\,\mathrm{d}s,$$

when $\varepsilon\downarrow 0$.

Of course, being the limit of a left-hand and a right-hand side of Itô's formula, respectively, (4.4.3) almost surely equals the ε-limit of (4.4.4), and hence

$$
\begin{aligned}
0 &\overset{a.s.}{=} u_f(X_{t_N}, Y_{t_N})\mathbf{1}_{\{t_N<\tau_{l-h}(Y)\}} + u_f(X_{t_N}, l-h)\mathbf{1}_{\{t_N\geq\tau_{l-h}(Y)\}} - f(x)\\
&- \int_0^{t_N} \mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^+ u_f(X_s,0)\,\mathrm{d}L_s^0(Y)\\
&- \int_0^{t_N} \mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\mathcal{L}_x u_f(X_s,0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s - \int_0^{t_N} \mathbf{1}_{\{s\geq\tau_{l-h}(Y)}\mathcal{L}_x u_f(X_s,l-h)\,\mathrm{d}s\\
&- \sum_{i=1}^d\sum_{k=1}^p \int_0^{t_N} \mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_i u_f(X_s,Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B_s^k\\
&- \sum_{i=1}^d\sum_{k=1}^p \int_0^{t_N} \mathbf{1}_{\{s\geq\tau_{l-h}(Y)\}}\partial_i u_f(X_s,l-h)\sigma_{ik}(X_s)\,\mathrm{d}B_s^k\\
&- \sqrt{2}\int_0^{t_N} \mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^+ u_f(X_s,Y_s)b(Y_s)\,\mathrm{d}s.
\end{aligned}
\tag{4.4.5}
$$

Eventually, we choose a whole sequence of h-values converging to zero. Since countable many h-values still form a set of Lebesgue measure zero, $b(l-h)$ can be assume to be zero, for any h in this countable set, and we have to show that the h-limit of the above equation's right-hand side almost surely equals Z_{t_N}.

First, $\{\tau_{l-h}(Y) > t_N\} \uparrow \{\tau_l(Y) > t_N\}$, when $h \downarrow 0$, and Z_{t_N} equals the right-hand side of (4.4.5), on each $\{\tau_{l-h}(Y) > t_N\}$. Therefore, without loss of generality, we only show that the h-limit of this right-hand side almost surely equals Z_{t_N} under the assumption $\tau_l(Y)$ is finite.

Under this assumption, $\tau_l(Y) - \tau_{l-h}(Y) \to 0$, almost surely, when $h \downarrow 0$, and hence, using dominated convergence, all summands on the right-hand side of (4.4.5), except the last one, can be shown to convergence in $L^2(\mathbb{P}_x)$ to their Z_{t_N}-counterparts, when $h \downarrow 0$, in a straight forward way (recall that t_N satisfies $t_N \leq \tau_N(L_\cdot^0(Y))$, by definition).

Identifying the limit of the last summand is more involved because $\partial_y^+ u_f(\cdot,y)\mathbf{1}_{\{|\cdot|\leq N\}}$ may become unbounded, when $y \to l$. Recall that $\partial_y^+ u_f(\cdot,0)\mathbf{1}_{\{|\cdot|\leq N\}}$ is bounded by Corollary 4.3.5.

However, as the left-hand side of (4.4.5) is zero, if all other summands converge in $L^2(\mathbb{P}_x)$, then the last summand does too so that

$$
\lim_{h\downarrow 0}\mathbb{E}_x\Big[\Big(\int_0^{t_N} \mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^+ u_f(X_s,Y_s)b(Y_s)\,\mathrm{d}B_s\Big)^2\Big] < \infty.
$$

But,

$$
\begin{aligned}
\lim_{h\downarrow 0}&\mathbb{E}_x\Big[\Big(\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\partial_y^+u_f(X_s,Y_s)b(Y_s)\,\mathrm{d}B_s\Big)^2\Big]\\
&=\lim_{h\downarrow 0}\mathbb{E}_x\Big[\int_0^{t_N}\mathbf{1}_{\{s<\tau_{l-h}(Y)\}}\big(\partial_y^+u_f(X_s,Y_s)b(Y_s)\big)^2\,\mathrm{d}s\Big]\\
&=\mathbb{E}_x\Big[\int_0^{t}\mathbf{1}_{\{s<t_N\wedge\tau_{l-h}(Y)\}}\big(\partial_y^+u_f(X_s,Y_s)b(Y_s)\big)^2\,\mathrm{d}s\Big],
\end{aligned}
$$

where the last line follows by monotone convergence. The above justifies that

$$
\mathbf{1}_{[0,t_N\wedge\tau_l(Y))}\Big(\partial_y^+u_f(X_\cdot,Y_\cdot)b(Y_\cdot)\Big)\in L^2(\Omega,\times[0,t]),
$$

and hence

$$
\begin{aligned}
\lim_{h\downarrow 0}&\mathbb{E}_x\Big[\Big(\int_0^{t_N}\mathbf{1}_{[\tau_{l-h}(Y),\tau_l(Y))}(s)\partial_y^+u_f(X_s,Y_s)b(Y_s)\,\mathrm{d}B_s\Big)^2\Big]\\
&=\lim_{h\downarrow 0}\mathbb{E}_x\Big[\int_0^{t_N}\mathbf{1}_{[\tau_{l-h}(Y),\tau_l(Y))}(s)\big(\partial_y^+u_f(X_s,Y_s)b(Y_s)\big)^2\,\mathrm{d}s\Big]\\
&=0,
\end{aligned}
$$

by dominated convergence, proving part (b) of the lemma.

(c) Choose $h\in(0,l)$, and define $t_N,u_{f,h},u^{\varepsilon}_{f,h}$ as in proof of (b). Since Y is not absorbing at $l<\infty$, the local time $(L^l_t(Y))_{t\geq 0}$ does not vanish and so Y solves the SDE

$$
\mathrm{d}Y_t=\sqrt{2}b(Y_t)\,\mathrm{d}B_t+\mathrm{d}L^0_t(Y)-\mathrm{d}L^l_t(Y).
$$

Hence the classical version of Itô's lemma gives

$$
\begin{aligned}
&u^{\varepsilon}_{f,h}(X_{t_N},Y_{t_N})-u^{\varepsilon}_{f,h}(x,0)-\int_0^{t_N}\partial_y u^{\varepsilon}_{f,h}(X_s,0)\,\mathrm{d}L^0_s(Y)+\int_0^{t_N}\partial_y u^{\varepsilon}_{f,h}(X_s,l)\,\mathrm{d}L^l_s(Y)\\
&-\sum_{i=1}^{d}\sum_{k=1}^{p}\int_0^{t_N}\partial_i u^{\varepsilon}_{f,h}(X_s,Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B^k_s-\sqrt{2}\int_0^{t_N}\partial_y u^{\varepsilon}_{f,h}(X_s,Y_s)b(Y_s)\,\mathrm{d}B_s\\
&\qquad\overset{a.s.}{=}\int_0^{t_N}\mathcal{L}_x u^{\varepsilon}_{f,h}(X_s,Y_s)\,\mathrm{d}s+\int_0^{t_N}\partial_y^2 u^{\varepsilon}_f(X_s,Y_s)\,b^2(Y_s)\,\mathrm{d}s.
\end{aligned}
\tag{4.4.6}
$$

However, as $u^{\varepsilon}_{f,h}(x,\cdot)$ is constant on $[l-\frac{h}{2},\infty)$, for all $x\in\mathbb{R}^d$, the term involving $L^l_\cdot(Y)$ vanishes, so that the ε-limit of the above left-hand side almost surely equals

$$\begin{cases} u_{f,h}(X_{t_N}, Y_{t_N}) - f(x) \\ -\int_0^{t_N} \partial_y^+ u_f(X_s, 0)\,\mathrm{d}L_s^0(Y) \\ -\sum_{i=1}^d \sum_{k=1}^p \int_0^{t_N} \partial_i u_{f,h}(X_s, Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B_s^k \\ -\sqrt{2}\int_0^{t_N} \partial_y^+ u_{f,h}(X_s, Y_s) b(Y_s)\,\mathrm{d}B_s, \end{cases}$$

by the same arguments used in the proof of (a).

Furthermore, unlike the case (b) where l is absorbing, it now must hold that $m([0,l]) < \infty$ (see the beginning of Remark 4.3.1), and hence $s = A_{A_s^{-1}}$ for all $s \geq 0$, a.s. Therefore, the ε-limit of the second integral on the right-hand side of (4.4.6) can almost surely be given by

$$-\int_0^{t_N} \mathcal{L}_x u_f(X_s, Y_s)\mathbf{1}_{(0,l-h/2)}(Y_s)\,\mathrm{d}s + \int_0^{A_{t_N}^{-1}} \Delta\big[\partial_y^+ u_{f,h}(X_{A_s}, l - \tfrac{h}{2})\big]\,\mathrm{d}L_s^{l-\frac{h}{2}}(W) \times \frac{1}{2} \tag{4.4.7}$$

again applying time change and partial integration as in the proof of (a). Here, the 'artificial' jump of $\partial_y^+ u_{f,h}(X(A_s), \cdot)$ at $l - \frac{h}{2}$, which we created when extending u_f to half-space, equals

$$\Delta\big[\partial_y^+ u_{f,h}(X_{A_s}, l - \tfrac{h}{2})\big] = -\partial_y^- u_f(X(A_s), l - \tfrac{h}{2}). \tag{4.4.8}$$

All in all, letting ε go to zero on both sides of (4.4.6) yields

$$\begin{aligned} 0 \overset{a.s.}{=} \; & u_{f,h}(X_{t_N}, Y_{t_N})\mathbf{1}_{\{Y_{t_N} < l\}} + u_f(X_{t_N}, l - \tfrac{h}{2})\mathbf{1}_{\{Y_{t_N} = l\}} - f(x) \\ & - \int_0^{t_N} \partial_y^+ u_f(X_s, 0)\,\mathrm{d}L_s^0(Y) + \int_0^{A_{t_N}^{-1}} \partial_y^- u_f(X_{A_s}, l - \tfrac{h}{2})\,\mathrm{d}L_s^{l-\frac{h}{2}}(Y) \times \frac{1}{2} \\ & - \int_0^{t_N} \mathcal{L}_x u_f(X_s, 0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s - \int_0^{t_N} \mathcal{L}_x u_f(X_s, l - \tfrac{h}{2})\mathbf{1}_{\{Y_s \geq l - h/2\}}\,\mathrm{d}s \\ & - \sum_{i=1}^d \sum_{k=1}^p \int_0^{t_N} \mathbf{1}_{\{Y_s < l\}} \partial_i u_{f,h}(X_s, Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B_s^k \\ & - \sum_{i=1}^d \sum_{k=1}^p \int_0^{t_N} \mathbf{1}_{\{Y_s = l\}} \partial_i u_{f,h}(X_s, l - \tfrac{h}{2})\sigma_{ik}(X_s)\,\mathrm{d}B_s^k \\ & - \sqrt{2}\int_0^{t_N} \partial_y^+ u_{f,h}(X_s, Y_s) b(Y_s)\,\mathrm{d}s, \end{aligned}$$

and it needs to be shown that the h-limit of the above right-hand side almost surely coincides with Z_{t_N}, where Z_t is the case-(c)-version of what has been defined in the proofs of (a), (b).

By Corollary 4.3.5, $\partial_y^+ u_f$ always behaves well near the boundary of $\mathbb{R}^d \times (0,l)$ at zero. In case (c), Corollary 4.3.5 also implies that $\partial_y^+ u_f$ always behaves well near the boundary of $\mathbb{R}^d \times (0,l)$ at l. So, all terms except

$$\int_{[0,A_{t_N}^{-1}]} \partial_y^- u_f(X_{A_s}, l - \tfrac{h}{2})\, \mathrm{d}L_s^{l-\frac{h}{2}}(W) \times \frac{1}{2}$$

can be shown to converge almost surely or in $L^2(\mathbb{P}_x)$ to their Z_{t_N}-counterparts in a straight forward way, when $h \downarrow 0$.

Below, we verify that

$$\int_{[0,A_{t_N}^{-1}]} \partial_y^- u_f(X_{A_s}, l - \tfrac{h}{2})\, \mathrm{d}L_s^{l-\frac{h}{2}}(W) \times \frac{1}{2} \overset{a.s.}{\to} \int_0^{t_N} \partial_y^- u_f(X_s, l-)\, \mathrm{d}L_s^l(Y),$$

when $h \downarrow 0$, finishing the proof of part (c) and the lemma.

Since [5, Lemma 6.34, (i)] gives

$$\int_0^{t_N} \partial_y^- u_f(X_s, l-)\, \mathrm{d}L_s^l(Y) \overset{a.s.}{=} \int_0^{A_{t_N}^{-1}} \partial_y^- u_f(X_{A_s}, l-)\, \mathrm{d}L_s^l(W) \times \frac{1}{2},$$

we can almost surely bound

$$\left| \int_0^{A_{t_N}^{-1}} \partial_y^- u_f(X_{A_s}, l - \tfrac{h}{2})\, \mathrm{d}L^{l-\frac{h}{2}}(W) \times \frac{1}{2} - \int_0^{t_N} \partial_y^- u_f(X_s, l-)\, \mathrm{d}L_s^l(Y) \right|$$

by

$$\begin{aligned} &\frac{1}{2} \int_0^{A_{t_N}^{-1}} \left| \partial_y^- u_f(X_{A_s}, l - \tfrac{h}{2}) - \partial_y^- u_f(X_{A_s}, l-) \right| \mathrm{d}L_s^{l-\frac{h}{2}}(W) \\ &\quad + \frac{1}{2} \left| \int_0^{A_{t_N}^{-1}} \partial_y^- u_f(X_{A_s}, l-)\, \mathrm{d}L_s^{l-\frac{h}{2}}(W) - \int_0^{A_{t_N}^{-1}} \partial_y^- u_f(X_{A_s}, l-)\, \mathrm{d}L_s^l(W) \right|, \end{aligned}$$

and the task is to show that both summands vanish almost surely, when $h \downarrow 0$.

First, observe that $|X(A_s)| \le N$, for $s \le A_{t_N}^{-1}$ a.s., and that $0 < \frac{h}{2} < \frac{l}{2}$ by our choice of h. Therefore, the first summand is bounded by

$$\sup_{|x| \le N} \left| \partial_y^- u_f(x, l - \tfrac{h}{2}) - \partial_y^- u_f(x, l-) \right| \times \sup_{0 \le h' \le \frac{l}{2}} L_{A_{t_N}^{-1}}^{l-h'}(W).$$

We note that $\sup_{0\le h'\le \frac{l}{2}} L^{l-h'}_{A^{-1}_{t_N}}(W)$ is almost surely finite and by Corollary 4.3.5 that

$$\sup_{|x|\le N} \left|\partial_y^- u_f(x, l-\tfrac{h}{2}) - \partial_y^- u_f(x, l-)\right| \to 0,$$

as $h \downarrow 0$. Thus the limit of the first summand almost surely vanishes.

For the second summand, note that $A^{-1}_{t_N}$ is almost surely finite, and hence, for almost every $\omega \in \Omega$, and each $y \in \mathbb{R}$, $s \mapsto L^y_s(W)(\omega)$ can be considered a continuous distribution function of a finite measure $\nu_y(\omega)$ on $[0, A^{-1}_{t_N(\omega)}(\omega)]$, when h goes to zero. As a consequence, when $h \downarrow 0$, the second summand converges almost surely to zero, because $s \mapsto A_s$ is continuous, on the one hand, and because $x \mapsto \partial_y^-(x, l-)$ is by assumption a bounded continuous function, for $|x| \le N$, on the other. □

*Proof of **Lemma 4.3.2**.* Recall that by assumption $l < \infty$.

(i) It has already been pointed out at the beginning of Remark 4.3.1 that if Y is not absorbing at l then $m([0, l]) < \infty$. Vice versa, if $m([0, l]) < \infty$, then the diffusion Y constructed in Theorem 2.4.1 can never be absorbing at l, because $A_t < \infty$ almost surely for all $t \ge 0$. Thus, Y is not absorbing at l if and only if $m([0, l]) < \infty$, which is equivalent to the statement to be proven under (i)..

(ii) Again the construction given by Theorem 2.4.1, $\tau_l(Y) \overset{a.s.}{=} \infty$ if and only if $\lim_{t\uparrow\tau_l(W)} A_t \overset{a.s.}{=} \infty$, and it is a special case of [5, Proposition (A1.8)] that $\lim_{t\uparrow\tau_l(W)} A_t = \infty$ if and only if $\int_{[0,l)}(l-y)\tilde{m}(\mathrm{d}y) = \infty$ and hence if and only if $\int_{[0,l)}(l-y)m(\mathrm{d}y) = \infty$ so part (ii) of the lemma follows. □

*Proof of **Theorem 4.3.3**.* (a) Fix $x \in \mathbb{R}^d$, and choose an ar but small $t > 0$. Define the stopping times τ_N, S_N as at the beginning of the proof of Lemma 4.3.7, and set

$$t_N = T_t \wedge \tau_N \wedge S'_N,$$

where $S'_N = S_N \wedge \tau_N(Y)$. Then for fixed $N \ge 1$, Lemma 4.3.7 (a) yields

$$\begin{aligned}
&u_f(X_{t_N}, Y_{t_N}) - f(x)\\
&\quad \overset{a.s.}{=} \int_0^{t_N} \partial_y^+ u_f(X_s, 0)\,\mathrm{d}L^0_s(Y) + \int_0^{t_N} \mathcal{L}_x u_f(X_s, 0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s\\
&\quad + \sum_{i=1}^d \sum_{k=1}^p \int_0^{t_N} \partial_i u_f(X_s, Y_s)\sigma_{ik}(X_s)\,\mathrm{d}B^k_s + \sqrt{2}\int_0^{t_N} \partial_y^+ u_f(X_s, Y_s) b(Y_s)\,\mathrm{d}B_s;
\end{aligned}$$

and since $t_N \leq \tau_N \wedge S'_N$, all stochastic integral have zero expectation, so that

$$\lim_{N\uparrow\infty} \mathbb{E}_x[u_f(X_{t_N}, Y_{t_N}) - f(x)] = \lim_{N\uparrow\infty} \mathbb{E}_x\Big[\int_0^{t_N} \big(\partial_y^+ u_f(X_s,0) + m_0\mathcal{L}_x u_f(X_s,0)\big)\,\mathrm{d}L_s^0(Y)\Big],$$

where we also used

$$\int_0^{t_N} \mathcal{L}_x u_f(X_s,0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s \overset{a.s.}{=} \int_0^{t_N} m_0\mathcal{L}_x u_f(X_s,0)\,\mathrm{d}L_s^0(Y), \tag{4.4.9}$$

which is an easy consequence of [5, Theorem 5.27], when the scale function is the identity.

Now, recall that $Y_t = W(A_t^{-1})$ for $t \geq 0$ where A_t is given in the proof of Theorem 2.4.1. Then, since [5, Lemma 6.34 (i)] yields

$$L_t^0(Y) = \frac{1}{2}L^0_{A_t^{-1}}(W), \qquad t \geq 0, \text{ a.s.},$$

we have that

$$L_\infty^0(Y) \overset{a.s.}{=} \infty \text{ if and only if } A_\infty^{-1} \overset{a.s.}{=} \infty. \tag{4.4.10}$$

But, $A_\infty^{-1} \overset{a.s.}{=} \infty$ follows by the same arguments used to show that $s = A_{A_s^{-1}}$, for all $s \geq 0$ a.s., in the proof of part (a) of Lemma 4.3.7, so we do have $L_\infty^0(Y) \overset{a.s.}{=} \infty$, for part (a) of this proof.

As a consequence, since $\tau_N \wedge S'_N$ grows to infinity, $N \to \infty$, we have for almost every $\omega \in \Omega$ that there exists $N(\omega)$ such that $t_N(\omega) = T_t(\omega)$, for all $N \geq N(\omega)$, and hence

$$\lim_{N\uparrow\infty} \mathbb{E}_x[u_f(X_{t_N}, Y_{t_N}) - f(x)] = \mathbb{E}_x[f(X(T_t)) - f(x)],$$

by dominated convergence, even if f was not continuous.

On the other hand, since $t_N \leq T_t$, $N \geq 1$, and since $\partial_y^+ u_f(\cdot,0) + m_0\mathcal{L}_x u_f(\cdot,0)$ is bounded, again by dominated convergence,

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}_x\Big[\int_0^{t_N} \Big(\partial_y^+ u_f(X_s,0) + m_0\mathcal{L}_x u_f(X_s,0)\Big)\,\mathrm{d}L_s^0(Y)\Big] \\
&\quad = \mathbb{E}_x\Big[\int_0^{T_t} \Big(\partial_y^+ u_f(X_s,0) + m_0\mathcal{L}_x u_f(X_s,0)\Big)\,\mathrm{d}L_s^0(Y)\Big] \\
&\quad = \mathbb{E}_x\Big[\int_0^{t} \Big(\partial_y^+ u_f(X(T_s),0) + m_0\mathcal{L}_x u_f(X(T_s),0)\Big)\,\mathrm{d}s\Big].
\end{aligned}$$

All in all, to finish the proof of part (a) of the theorem, it remains to show that

$$\begin{aligned}&\lim_{t\downarrow 0}\mathbb{E}_x\Big[\frac{1}{t}\int_0^t \Big(\partial_y^+ u_f(X(T_s),0) + m_0\mathcal{L}_x u_f(X(T_s),0)\Big)\,\mathrm{d}s\Big]\\ &\quad = \partial_y^+ u_f(x,0) + m_0\mathcal{L}_x u_f(x,0),\end{aligned}$$

which follows by dominated convergence, because $\partial_y^+ u_f(\cdot,0)+m_0\mathcal{L}_x u_f(\cdot,0) \in C_b(\mathbb{R}^d)$.

(b) The proofs in the cases (b1) and (b2) are identical for a large part, and so we name this large part of the proof (b), and we only go into the differences between (b1) and (b2) at the end. By Lemma 4.3.2, the point of l will be absorbing in this case.

If $l < \infty$ and $(Y_t)_{t\geq 0}$ is absorbing at l, then $A_\infty^{-1} = \tau_l(W)$, by the construction of Y in Theorem 2.4.1. Of course, $\tau_l(W) < \infty$ a.s., and hence $L_\infty^0(Y) < \infty$, a.s., too, by (4.4.10). As a consequence, $\mathbb{P}_x(\{T_t = \infty\}) > 0$, for any $t > 0$, and this positive probability can be determined by Theorem 2.4.1. Indeed since $\mathbb{E}_x[e^{-\lambda T_t}\mathbf{1}_{\{T_t<\infty\}}] = \exp(-t\psi(\lambda))$ for $\lambda > 0$, we obtain

$$\mathbb{P}_x(\{T_t = \infty\}) = 1 - \exp(-t\psi(0)),$$

as $\lambda \to 0$.

Note that the above reasoning also implies that $\psi(0)$ has to be positive in case (b), though we obviously had $\psi(0) = 0$ in case (a).

Now, fix $x \in \mathbb{R}^d$, $t > 0$, and define t_N by in part (a), but using $S_N' = S_N \wedge \tau_{l-1/N}(Y^h)$, instead, where Y^h again denotes the process $(Y_{t\wedge\tau_{l-h}})_{t\geq 0}$, for some $h \in (0,l)$. Of course, if Y does hit l, then it would hit it after hitting $l-h$, that is

$$\mathbb{P}_x(\{t_N < \tau_l(Y)\}) = 1,$$

for all $N \geq 1$ though $\tau_{l-h}(Y)$ converges to $\tau_l(Y)$, when $h \downarrow 0$, whether $\tau_l(Y)$ is finite or not. Furthermore, if $l < \infty$ is absorbing then $L^0_{\tau_l(Y)}(Y)$ always equals $L^0_\infty(Y)$ almost surely, because Y can only be absorbed at $l > 0$ after hitting zero for the last time.

All in all, for fixed $N \geq 1$, Lemma 4.3.7 (b) yields

$$\begin{aligned}&u_f(X_{t_N}, Y_{t_N}) - f(x)\\ &\quad \overset{a.s.}{=} \int_0^{t_N} \partial_y^+ u_f(X_s,0)\,\mathrm{d}L_s^0(Y) + \int_0^{t_N} \mathcal{L}_x u_f(X_s,0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s + M_{t_N},\end{aligned}$$

where $(M_t)_{t\geq 0}$ denotes the sum of the stochastic integrals.

Taking expectations on both sides, we therefore obtain that

$$\lim_{N\uparrow\infty}\mathbb{E}_x\Big[u_f(X_{t_N},Y_{t_N})-f(x)\Big]=\lim_{N\uparrow\infty}\mathbb{E}_x\Big[\int_0^{t_N}\Big(\partial_y^+u_f(X_s,0)+m_0\mathcal{L}_xu_f(X_s,0)\Big)\,\mathrm{d}L_s^0(Y)\Big],$$

using the same reasoning as in part (a).

First, we deal with the limit of the above right-hand side. Time change yields

$$\begin{aligned}\int_0^{t_N}&\big(\partial_y^+u_f(X_s,0)+m_0\mathcal{L}_xu_f(X_s,0)\big)\,\mathrm{d}L_s^0(Y)\\&=\int_0^{L^0_{t_N}(Y)}\big(\partial_y^+u_f(X_{T_s},0)+m_0\mathcal{L}_xu_f(X_{T_s},0)\big)\,\mathrm{d}s,\end{aligned}$$

where t_N grows to $T_t\wedge\tau_{l-h}(Y)$, when $N\to\infty$. So, since $\partial_y^+u_f(\cdot,0)+m_0\mathcal{L}_xu_f(\cdot,0)$ is bounded, by dominated convergence, the limit of the right-hand side equals

$$\begin{aligned}&\mathbb{E}_x\Big[\int_0^{t\wedge L^0_{T_t\wedge\tau_{l-h}(Y)}}\big(\partial_y^+u_f(X(T_s),0)+m_0\mathcal{L}_xu_f(X(T_s),0)\big)\,\mathrm{d}s\mathbf{1}_{\{L^0_{\tau_{l-h}(Y)}(Y)<L^0_\infty(Y)\}}\Big]\\&\quad+\mathbb{E}_x\Big[\int_0^{t\wedge L^0_\infty}\big(\partial_y^+u_f(X(T_s),0)+m_0\mathcal{L}_xu_f(X(T_s),0)\big)\,\mathrm{d}s\mathbf{1}_{\{L^0_{\tau_{l-h}(Y)}(Y)\geq L^0_\infty(Y)\}}\Big],\end{aligned}$$

which converges to

$$\mathbb{E}_x\Big[\int_0^{t\wedge L^0_\infty(Y)}\Big(\partial_y^+u_f(X(T_s),0)+m_0\mathcal{L}_xu_f(X(T_s),0)\Big)\,\mathrm{d}s\Big],$$

which converges to

$$\mathbb{E}_x\Big[\int_0^{t\wedge L^0_\infty(Y)}\Big(\partial_y^+u_f(X(T_s),0)+m_0\mathcal{L}_xu_f(X(T_s),0)\Big)\,\mathrm{d}s\Big],$$

when $h\downarrow 0$.

Next, it is easy to see that, for almost every $\omega\in\{T_t<\tau_{l-h}(Y)\}$, there exists $N(\omega)$ such that $t_N(\omega)=T_t(\omega)$, for all $N\geq N(\omega)$. Also, since $m([0,l-h])<\infty$, we know that $\tau_{l-h}(Y)$ is almost surely finite, and therefore

$$\{T_t\geq\tau_{l-h}(Y)\}\overset{a.s.}{=}\{T_t>\tau_{l-h}(Y)\},$$

because the process Y cannot be at zero and $l-h$ at the same time. As a consequence, for almost every $\omega\in\{T_t\geq\tau_{l-h}(Y)\}$, there exists $N(\omega)$ such that

$t_N(\omega) = \tau_{l-h}(Y)(\omega)$, for all $N \geq N(\omega)$, and we obtain that

$$\begin{aligned}
&\lim_{N\uparrow\infty} \mathbb{E}_x\Big[u_f(X_{t_N}, Y_{t_N}) - f(x)\Big] \\
&\qquad = \mathbb{E}_x\Big[\big(f(X_{T_t}) - f(x)\big)\mathbf{1}_{\{T_t<\tau_{l-h}(Y)\}}\Big] \\
&\qquad + \mathbb{E}_x\Big[\big(u_f(X_{\tau_{l-h}(Y)}, l-h) - f(x)\big)\mathbf{1}_{\{T_t\geq\tau_{l-h}(Y)\}}\Big],
\end{aligned}$$

by dominated convergence only using boundedness of u_f, but not continuity.

Recall that $L^0_{\tau_l(Y)}(Y) \overset{a.s.}{=} L^0_\infty(Y)$, which implies $\mathbb{P}_x[\{T_t < \infty\} \setminus \{T_t < \tau_l(Y)\}] = 0$, and hence,

$$\begin{aligned}
\lim_{h\downarrow 0} \mathbb{E}_x\Big[\Big(f(X_{T_t}) - f(x)\Big)\mathbf{1}_{\{T_t<\tau_{l-h}(Y)\}}\Big] &= \lim_{h\downarrow 0} \mathbb{E}_x\Big[\Big(f(X_{T_t}) - f(x)\Big)\mathbf{1}_{\{T_t<\tau_{l-h}(Y)\}\cap\{T_t<\infty\}}\Big] \\
&= \mathbb{E}_x\Big[\Big(f(X_{T_t}) - f(x)\Big)\mathbf{1}_{\{T_t<\infty\}}\Big]
\end{aligned}$$

as well as

$$\lim_{h\downarrow 0} \mathbb{E}_x\Big[f(x)\mathbf{1}_{\{T_t\geq\tau_{l-h}(Y)\}}\Big] = \mathbb{E}_x\Big[f(x)\mathbf{1}_{\{T_t=\infty\}}\Big],$$

by dominated convergence.

Eventually, when treating the remaining limit

$$\lim_{h\downarrow 0} \mathbb{E}_x\Big[u_f(X_{\tau_{l-h}(Y)}, l-h)\mathbf{1}_{\{T_t\geq\tau_{l-h}(Y)\}}\Big], \tag{4.4.11}$$

we have to differ between the two cases (b1) and (b2), where $\tau_l(Y) < \infty$ a.s., in case (b1), by Lemma 4.3.2.

By dominated convergence, we only have to discuss $\lim_{h\downarrow 0} |u_f(X_{\tau_{l-h}(Y)}, l-h)|$, because if this limit vanishes almost surely then so does (4.4.11). However, $\lim_{h\downarrow 0} |u_f(X_{\tau_{l-h}(Y)}, l-h)|$ trivially vanishes, when applying the assumptions made in either (b1) or (b2).

On the whole, we have justified that

$$\mathbb{E}_x\Big[f(X_{T_t})\mathbf{1}_{\{T_t<\infty\}} f(x)\Big] = \mathbb{E}_x\Big[\int_0^{t\wedge L^0_\infty(Y)} \Big(\partial_y^+ u_f(X_{T_s}, 0) + m_0\mathcal{L}_x u_f(X_{T_s}, 0)\Big)\,\mathrm{d}s\Big],$$

where

$$\begin{aligned}
&\lim_{t\downarrow 0} \mathbb{E}_x\Big[\frac{1}{t}\int_0^{t\wedge L^0_\infty(Y)} \Big(\partial_y^+ u_f(X_{T_s}, 0) + m_0\mathcal{L}_x u_f(X_{T_s}, 0)\Big)\,\mathrm{d}s\Big] \\
&\qquad = \partial_y^+ u_f(x, 0) + m_0\mathcal{L}_x u_f(x, 0),
\end{aligned}$$

by the same arguments as in the proof of part (a), only taking into account that, when $t \downarrow 0$, for almost every $\omega \in \Omega$, it will eventually happen that $t < L^0_\infty(Y)(\omega)$.

Since $f(X_{T_t})\mathbf{1}_{\{T_t=\infty\}} = f(\dagger)\mathbf{1}_{\{T_t=\infty\}} = 0$, the proof is complete for both cases (b1) and (b2).

(c) By Lemma 4.3.2, the point $l < \infty$ is not absorbing and $m([0,l]) < \infty$ implies both $L^0_\infty(Y) \overset{a.s.}{=} \infty$ as well as $\psi(0) = 0$, as in the proof of part (a).

Now fix $x \in \mathbb{R}^d$, $t > 0$, and define t_N as in part (a) but using $S'_N = S_N$, instead - there is no localisation need with respect to Y because of Corollary 4.3.2. Since $\partial_y^- u_f(\cdot, l-)$ is continuous, Lemma 4.3.7 (c) yields

$$\begin{aligned}
&u_f(X_{t_N}, Y_{t_N})\mathbf{1}_{\{Y_{t_N}<l\}} + u_f(X_{t_N}, l-)\mathbf{1}_{\{l\}}(Y_{t_N}) - f(x) \\
&\quad\overset{a.s.}{=} \int_0^{t_N} \partial_y^+ u_f(X_s, 0)\,\mathrm{d}L^0_s(Y) + \int_0^{t_N} \mathcal{L}_x u_f(X_s, 0)\mathbf{1}_{\{0\}}(Y_s)\,\mathrm{d}s \\
&\quad - \int_0^{t_N} \partial_y^- u_f(X_s, l-)\,\mathrm{d}L^l_s(Y) + \int_0^{t_N} \mathcal{L}_x u_f(X_s, l-)\mathbf{1}_{\{l\}}(Y_s)\,\mathrm{d}s + M_{t_N},
\end{aligned}$$

where $(M_t)_{t\geq 0}$ denotes the sum of the stochastic integrals.

As in part (a) of the proof, for almost every $\omega \in \Omega$, there exists $N(\omega)$ such that $t_N(\omega) = T_t(\omega)$ for all $N \geq N(\omega)$, and hence

$$\lim_{N\uparrow\infty} u_f(X_{t_N}, l-)\mathbf{1}_{\{l\}}(Y_{t_N}) \overset{a.s.}{=} 0,$$

because Y cannot be at zero at l at the same time. Furthermore,

$$\begin{aligned}
&\int_0^{t_N} \partial_y^- u_f(X_s, l-)\,\mathrm{d}L^l_s(Y) - \int_0^{t_N} \mathcal{L}_x u_f(X_s, l-)\mathbf{1}_{\{l\}}(Y_s)\,\mathrm{d}s \\
&\qquad\overset{a.s.}{=} \int_0^{t_N} \Big(\partial_y^+ u_f(X_s, l-) - m_1 \mathcal{L}_x u_f(X_s, l-)\Big)\,\mathrm{d}L^l_s(Y),
\end{aligned}$$

and the last integral vanishes by assumption, in case (c).

All in all, we obtain that

$$\lim_{N\uparrow\infty} \mathbb{E}_x\Big[u_f(X_{t_N}, Y_{t_N}) - f(x)\Big] = \lim_{N\uparrow\infty} \mathbb{E}_x\Big[\int_0^{t_N} \Big(\partial_y^+ u_f(X_s, 0) + m_0 \mathcal{L}_x u_f(X_s, 0)\Big)\,\mathrm{d}L^0_s(Y)\Big],$$

and the rest is identical to the proof of part (a). □

Chapter 5

Analytic Approach to the Harmonic Extension Technique

5.1 Introduction

In this chapter we introduce an analytic method to study the harmonic extension technique. This approach is related to the *method of semigroups* first introduced by Stinga & Torrea in [71] (for a well written expository article about this method, see [70]) where they studied fractional powers of linear second order partial differential operators. Their starting point is the formula

$$(-\mathcal{L}_x)^{\alpha/2} f = \frac{1}{\Gamma(-\alpha/2)} \int_0^\infty (P_t f - f) \frac{1}{t^{1+\alpha/2}} \,\mathrm{d}t,$$

and in the paper they extend the extension technique to this family of operators. This approach is useful as we may obtain explicit formulas for the extension function u_f which can be used to obtain regularity results. However, their method relies heavily on explicit identities for fractional powers and solutions of Bessel equations which are unavailable for general complete Bernstein functions.

Alternatively, Kwaśnicki & Mucha investigate the extension technique for operators of form $\psi(-\Delta_x)$ where ψ is a complete Bernstein function [47]. Their approach relies heavily on using the Fourier transform which is not available spatially inhomogeneous operators such as diffusion operators with non-constant coefficients.

In this chapter, we generalise both methods to obtain a similar characterisation for operators $-\psi(-\mathcal{L}_x)$ where $\mathcal{L}_x$ is the generator of a C_0-contraction semigroup $(P_t)_{t\geq 0}$ on a Banach space $\mathfrak{B}$. Furthermore, this chapter shows how the method of semigroups is connected with the Fourier approach to the extension problem.

5.2 The Harmonic Extension

Let $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ be a Banach space and let $(\mathfrak{B}^*, \|\cdot\|_{\mathfrak{B}^*})$ be its topological dual where $\langle f, \phi \rangle$ denotes the dual pairing of $f \in \mathfrak{B}$ and $\phi \in \mathfrak{B}^*$. Let $(P_t)_{t\geq 0}$ a C_0-contraction semigroup in $\mathfrak{B}$ with infinitesimal generator $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$. In the stochastic approach, $(P_t)_{t\geq 0}$ would correspond to the diffusion semigroup associated with the process $(X_t)_{t\geq 0}$.

Let m be a Krein string on $[0, r)$ and let ψ be the complete Bernstein function in Krein correspondence with m and assume $m(\{0\}) = 0$. Let $(Y_t)_{t\geq 0}$ be the corresponding gap diffusion and $(T_t)_{t\geq 0}$ be the corresponding inverse local time at zero subordinator so that we have

$$e^{-\psi(\lambda)t} = \mathbb{E}_0[e^{-\lambda T_t}] = \int_{[0,\infty)} e^{-\lambda s}\mathbb{P}_0[T_t \in \mathrm{d}s].$$

As $m(\{0\}) = 0$, the extension function φ_λ satisfies $\varphi_\lambda^+(0) = -\psi(\lambda)$.

We recall the subordinated semigroup defined by the Bochner integral,

$$P_t^\psi f = \int_{[0,\infty)} (P_s f)\mathbb{P}_0[T_t \in \mathrm{d}s],$$

and we denote its generator by $(-\psi(-\mathcal{L}_x), \mathrm{Dom}(-\psi(-\mathcal{L}_x)))$.

Definition 5.2.1 (Harmonic Extension)**.** *For $y \in E_m$ and $f \in \mathfrak{B}$, define $\mathcal{H}_y f : \mathfrak{B} \to \mathfrak{B}$ by the Bochner integral,*

$$\mathcal{H}_y f = \int_{[0,\infty)} (P_t f)\mathbb{P}_y[H_0 \in \mathrm{d}t].$$

If M is metric space and $\mathfrak{B}$ is a Banach space of functions $f : M \to \mathbb{R}$, then for $(x, y) \in M \times E_m$, we define $u_f(x, y) = \mathcal{H}_y f(x)$.

We call this representation of the harmonic extension the *semigroup representation.* This expression is well-defined for all $y \in E_m$ as

$$\|\mathcal{H}_y f\|_{\mathfrak{B}} \leq \int_{[0,\infty)} \|P_t f\|_{\mathfrak{B}}\mathbb{P}_y[H_0 \in \mathrm{d}t] \leq \|f\|_{\mathfrak{B}}.$$

Example 5.2.2. *When underlying semigroup $(P_t)_{t\geq 0}$ is the heat semigroup on $L^2(\mathbb{R}^d)$, we may use Fourier analysis to study the harmonic extension. In this* *case, we see*

$$\widehat{P_t^{\psi}f}(\xi) = \int_{[0,\infty)} (\widehat{P_s f})(\xi)\mathbb{P}_0[T_t \in \mathrm{d}s] = \int_{[0,\infty)} e^{-s|\xi|^2}\hat{f}(\xi)\mathbb{P}_0[T_t \in \mathrm{d}s] = e^{-t\psi(|\xi|^2)}\hat{f}(\xi),$$

and similarly,

$$\widehat{\mathcal{H}_y f}(\xi) = \varphi_{|\xi|^2}(y)\hat{f}(\xi).$$

Therefore, provided $\int_{\mathbb{R}^d} \psi(|\xi|^2)^2|\hat{f}(\xi)|^2 \,\mathrm{d}\xi < \infty$,

$$\text{strong-}\lim_{y\downarrow 0} \frac{\widehat{\mathcal{H}_y f} - \hat{f}}{y} = \varphi^+_{|\xi|^2}(0)\hat{f}(\xi) = -\psi(|\xi|^2)\hat{f}(\xi),$$

and so by inverting the Fourier transform, $\text{strong-}\lim_{y\downarrow 0} \frac{\mathcal{H}_y f - f}{y} = -\psi(-\Delta_x)f$.

To see how the harmonic extension is connected to subordinated generator, we have the following theorem.

Theorem 5.2.3. *Let* $f \in \mathrm{Dom}(\mathcal{L}_x)$. *Then,*

$$\frac{\mathcal{H}_{y_n} f - f}{y_n} \rightharpoonup -\psi(-\mathcal{L}_x)f.$$

as $n \to \infty$ *for any sequence* $(y_n)_{n\in\mathbb{N}} \subset E_m \cap (0,r)$ *such that* $y_n \to 0$.

Proof. The proof of this theorem is similar to that of Phillip's theorem (see [65, Theorem 13.6]).

We first assume $\|P_t f\|_{\mathfrak{B}} \leq e^{-\epsilon t}\|f\|_{\mathfrak{B}}$ for some $\epsilon > 0$. As $m(\{0\}) = 0$, $m((0,\delta)) > 0$ for all $\delta > 0$ and so there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset E_m \cap (0,r)$ such that $y_n \to 0$ as $n \to \infty$. Let $(y_n)_{n\in\mathbb{N}}$ be any such sequence.

We note that for each $n \in \mathbb{N}$ we have,

$$\begin{aligned}\frac{1-\varphi_\lambda(y_n)}{y_n} &= \int_{[0,\infty)} (1-e^{-\lambda t})\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] + \frac{1}{y_n}\mathbb{P}_{y_n}[H_0 = \infty], \\ &= \int_{(0,\infty)} (1-e^{-\lambda t})\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] + \frac{1}{r},\end{aligned}$$

where we note that $\mathbb{P}_{y_n}[H_0 = 0] = 0$ as $y_n > 0$ by Proposition 3.4.5 and that $\mathbb{P}_{y_n}[H_0 = \infty] = ay_n$ where $a = \frac{1}{r}$. Therefore for each $n \in \mathbb{N}$ the mapping $\psi_n : \lambda \mapsto \frac{1-\varphi_\lambda(y_n)}{y_n}$ is a Bernstein function with Lévy triplet $(a, 0, \frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t])$ such that $\lim_{n\to\infty}\psi_n(\lambda) = \psi(\lambda)$ for all $\lambda > 0$. By Proposition 2.1.7 we have that

(i) $\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \to \nu(t)\,\mathrm{d}t$ vaguely in $(0,\infty)$ as $n \to \infty$,

(ii) $\lim_{C\to\infty}\lim_{n\to\infty}\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \geq C] = 0$,

(iii) $\lim_{c\to 0}\lim_{n\to\infty}\int_{(0,c)}\frac{t}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] = 0 (= b)$,

noting that as ψ is a complete Bernstein function, its Lévy measure is absolutely continuous and so all points in $(0,\infty)$ are continuity points for the measure so we may replace the $\liminf_{n\to\infty}$ by $\lim_{n\to\infty}$. By strong continuity of $(P_t)_{t\geq 0}$, $t \mapsto \langle P_t f, \phi\rangle$ is continuous for any $f \in \mathrm{Dom}(\mathcal{L}_x)$ and $\phi \in \mathfrak{B}^*$. Therefore,

$$\lim_{n\to\infty}\int_{(c,C)}\langle P_t f - f, \phi\rangle_{\mathfrak{B}}\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] = \int_c^C \langle P_t f - f, \phi\rangle_{\mathfrak{B}}\nu(t)\,\mathrm{d}t$$

as we have

$$|\langle P_t f - f, \phi\rangle_{\mathfrak{B}}| \leq \|P_t f - f\|_{\mathfrak{B}}\|\phi\|_{\mathfrak{B}^*} \leq \min\{t\|\mathcal{L}_x f\|_{\mathfrak{B}}, 2\|f\|_{\mathfrak{B}}\}\|\phi\|_{\mathfrak{B}^*},$$

for all $t > 0, f \in \mathfrak{B}$ and $t \mapsto \min\{t\|\mathcal{L}_x f\|_{\mathfrak{B}}, 2\|f\|_{\mathfrak{B}}\}$ is ν-integrable, we apply dominated convergence to obtain,

$$\lim_{c\to 0, C\to\infty}\lim_{n\to\infty}\int_{(c,C)}\langle P_t f - f, \phi\rangle_{\mathfrak{B}}\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] = \int_0^\infty \langle P_t f - f, \phi\rangle_{\mathfrak{B}}\nu(t)\,\mathrm{d}t.$$

Now for any $f \in \mathrm{Dom}(\mathcal{L}_x)$ and $c > 0$,

$$\begin{aligned}
&\int_{[0,c)}(P_t f - f)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&\quad = \int_{[0,c)}\Big(\int_0^t (P_s\mathcal{L}_x f)\,\mathrm{d}s\Big)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&\quad = \int_{[0,c)}\Big(\int_0^t (P_s\mathcal{L}_x f - \mathcal{L}_x f)\,\mathrm{d}s\Big)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] + \Big(\int_{[0,c)}\frac{t}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t]\Big)\mathcal{L}_x f.
\end{aligned}$$

By (iii), we have that $(\int_{[0,1)}\frac{t}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t])_{n\in\mathbb{N}}$ is a bounded sequence so

$$\begin{aligned}
\sup_{n\in\mathbb{N}}&\Big\|\int_{[0,c)}\Big(\int_0^t (P_s\mathcal{L}_x f - \mathcal{L}_x f)\,\mathrm{d}s\Big)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t]\Big\|_{\mathfrak{B}} \\
&\leq \sup_{s\leq c}\|P_s\mathcal{L}_x f - \mathcal{L}_x f\|_{\mathfrak{B}}\Big(\int_{[0,1)}\frac{t}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t]\Big) \\
&\to 0,
\end{aligned}$$

as $c \to 0$ and so

$$\text{strong-}\lim_{c\to 0}\lim_{n\to\infty}\int_{[0,c)}(P_t f - f)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] = 0.$$

Similarly,

$$\int_{[C,\infty)}(P_t f - f)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] = \int_{[C,\infty)}(P_t f)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] - \Big(\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \geq C]\Big)f.$$

By (ii), the sequence $(\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \geq C])_{n\in\mathbb{N}}$ is bounded and

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\Big\|\int_{[C,\infty)}(P_t f)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t]\Big\|_{\mathfrak{B}} &\leq \sup_{n\in\mathbb{N}}\int_{[C,\infty)}\|P_t f\|_{\mathfrak{B}}\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&\leq e^{-C\epsilon}\|f\|_{\mathfrak{B}}\Big(\sup_{n\in\mathbb{N}}\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \geq C]\Big) \\
&\to 0,
\end{aligned}$$

as $C \to \infty$ so,

$$\text{strong-}\lim_{c\downarrow 0}\lim_{n\to\infty}\int_{[C,\infty)}(P_t f - f)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] = 0.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\Big\langle\frac{\mathcal{H}_{y_n}f - f}{y_n}, \phi\Big\rangle &= \langle af, \phi\rangle + \lim_{n\to\infty}\int_{(0,\infty)}\langle P_t f - f, \phi\rangle\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&= \langle af, \phi\rangle + \lim_{c\to 0, C\to\infty}\lim_{n\to\infty}\Big(\int_{[0,c)}\cdots + \int_{(c,C)}\cdots + \int_{[C,\infty)}\cdots\Big) \\
&= \langle af, \phi\rangle + \int_0^\infty\langle P_t f - f, \phi\rangle\nu(t)\,\mathrm{d}t \\
&= \langle -\psi(-\mathcal{L}_x)f, \phi\rangle.
\end{aligned}$$

We now extend this to the case where $(P_t)_{t\geq 0}$ is a general C_0-contraction semigroup. Then the C_0-semigroup $(P_t^\epsilon)_{t\geq 0}$ defined by $P_t^\epsilon f = e^{-\epsilon t}P_t f$ satisfies the decay assumption and let $\mathcal{H}_y^\epsilon f$ be the corresponding harmonic extension. This semigroup has generator $(\mathcal{L}_x - \epsilon, \mathrm{Dom}(\mathcal{L}_x))$. We note

$$\begin{aligned}
\|\psi(-\mathcal{L}_x)f - &\psi(-\mathcal{L}_x + \epsilon)f\|_{\mathfrak{B}} \\
&= \left\| \Big(af + \int_0^\infty (P_t f - f)\nu(t)\,\mathrm{d}t\Big) - \Big(af + \int_0^\infty (e^{-\epsilon t}P_t f - f)\nu(t)\,\mathrm{d}t\Big) \right\|_{\mathfrak{B}} \\
&\leq \int_0^\infty (1 - e^{-\epsilon t})\|P_t f\|_{\mathfrak{B}}\nu(t)\,\mathrm{d}t \\
&\to 0,
\end{aligned}$$

as $\epsilon \to 0$. Furthermore,

$$\begin{aligned}
\lim_{n\to\infty} \left\| \Big(\tfrac{\mathcal{H}_{y_n} f - f}{y_n}\Big) - \Big(\tfrac{\mathcal{H}^\epsilon_{y_n} f - f}{y_n}\Big) \right\|_{\mathfrak{B}} &= \lim_{n\to\infty} \left\| \tfrac{\mathcal{H}_{y_n} f - \mathcal{H}^\epsilon_{y_n} f}{y_n} \right\|_{\mathfrak{B}} \\
&\leq \lim_{n\to\infty} \int_{(0,\infty)} \Big((1 - e^{-\epsilon t})\|P_t f\|_{\mathfrak{B}}\Big)\frac{1}{y_n}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&\leq \lim_{n\to\infty} \Big(\frac{1 - a y_n - \varphi_\epsilon(y_n)}{y_n}\Big)\|f\|_{\mathfrak{B}} \\
&\leq (\psi(\epsilon) - a)\|f\|_{\mathfrak{B}} \\
&\to 0,
\end{aligned}$$

as $\epsilon \to 0$. Therefore for any $\phi \in \mathfrak{B}^*$,

$$\begin{aligned}
\lim_{n\to\infty} &\left| \Big\langle -\psi(-\mathcal{L}_x)f - \Big(\tfrac{\mathcal{H}_{y_n} f - f}{y_n}\Big), \phi \Big\rangle \right| \\
&\leq \left| \Big\langle -\psi(-\mathcal{L}_x)f + \psi(-\mathcal{L}_x + \epsilon)f, \phi \Big\rangle \right| + \lim_{n\to\infty} \left| \Big\langle -\psi(-\mathcal{L}_x + \epsilon)f - \Big(\tfrac{\mathcal{H}^\epsilon_{y_n} f - f}{y_n}\Big), \phi \Big\rangle \right| \\
&\quad + \lim_{n\to\infty} \left| \Big\langle \Big(\tfrac{\mathcal{H}^\epsilon_{y_n} f - f}{y_n}\Big) - \Big(\tfrac{\mathcal{H}_{y_n} f - f}{y_n}\Big), \phi \Big\rangle \right| \\
&\leq \Big(\|\psi(-\mathcal{L}_x)f - \psi(-\mathcal{L}_x + \epsilon)f\|_{\mathfrak{B}} + (\psi(\epsilon) - a)\|f\|_{\mathfrak{B}}\Big)\|\phi\|_{\mathfrak{B}^*} \\
&\quad + \lim_{y\downarrow 0} \left| \Big\langle -\psi(-\mathcal{L}_x + \epsilon)f - \Big(\tfrac{\mathcal{H}^\epsilon_{y} f - f}{y}\Big), \phi \Big\rangle \right| \\
&\to 0,
\end{aligned}$$

as $\epsilon \to 0$. $\square$

By considering specific examples of Banach spaces we can use weak convergence to get other forms of convergence which are useful for applications.

Corollary 5.2.4. *Let M be a locally compact, separable, metric space. Suppose $(P_t)_{t\geq 0}$ is a Feller semigroup on $(C_0(M), \|\cdot\|_\infty)$. Then for all $f \in \mathrm{Dom}(\mathcal{L}_x)$ and* *any sequence* $(y_n)_{n\in\mathbb{N}} \subset E_m \cap (0,r)$ *such that* $y_n \to 0$ *we have,*

$$\sup_{n\in\mathbb{N}} \left\| \frac{\mathcal{H}_{y_n} f - f}{y_n} \right\|_\infty < \infty,$$

and hence for all $x \in M$*,*

$$\lim_{y\downarrow 0} \frac{\mathcal{H}_y f(x) - f(x)}{y} = -\psi(-\mathcal{L}_x) f(x).$$

Proof. This follows as weak convergence in $C_0(M)$ is equivalent to pointwise convergence and uniform boundedness of the approximating sequence. $\square$

We now prove that $\mathcal{H}_y f$ is a weak solution to the harmonic extension problem. We recall the adjoint of $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ is defined as the operator $(\mathcal{L}_x^*, \mathrm{Dom}(\mathcal{L}_x^*))$ where $\psi \in \mathrm{Dom}(\mathcal{L}_x^*) \subset \mathfrak{B}^*$ if there exists $\chi \in \mathfrak{B}^*$ such that $\langle \mathcal{L}_x f, \psi\rangle_{\mathfrak{B}} = \langle f, \chi\rangle_{\mathfrak{B}}$ for all $f \in \mathrm{Dom}(\mathcal{L}_x)$ in which case $\mathcal{L}_x^*\psi = \chi$.

Theorem 5.2.5. *Let* $f \in \mathfrak{B}$*. Then* $\mathcal{H}f$ *is a weak solution to the elliptic equation,*

$$\mathcal{L}_x \mathcal{H} f \times m(\mathrm{d}y) + \partial_y^2 \mathcal{H} f = 0, \tag{5.2.1}$$

in the sense that for any $\phi \in \mathrm{Dom}(\mathcal{G}_y) \cap \{\mathrm{supp}\ \phi \Subset (0,r)\}$*,*

$$\int_{(0,r)} \langle \mathcal{H}_y f, \mathcal{L}_x^* \psi \rangle_{\mathfrak{B}} \phi(y) m(\mathrm{d}y) + \int_0^r \langle \mathcal{H}_y f, \psi \rangle_{\mathfrak{B}} \,\mathrm{d}\phi^+(y) = 0, \tag{5.2.2}$$

for all $\psi \in \mathrm{Dom}(\mathcal{L}_x^*)$*.*

Proof. For each $\phi \in \mathrm{Dom}(\mathcal{G}_y) \cap \{\mathrm{supp}\ \phi \Subset (0,r)\}$, ϕ has right-derivative ϕ^+ and we denote by $\mathrm{d}\phi^+$ the Lebesgue-Stieltjes measure corresponding to ϕ^+ (noting that as $\phi \in \mathrm{Dom}(\mathcal{G}_y)$, this measure will be absolutely continuous with respect to the Krein string m). We define the measures on Borel subsets of $[0,\infty)$ by,

$$\mu_\phi^1(A) = \int_A \int_{(0,r)} \mathbb{P}_y[H_0 \le t] \,\mathrm{d}\phi^+(y)\,\mathrm{d}t,$$
$$\mu_\phi^2(A) = \int_{(0,r)} \phi(y) \mathbb{P}_y[H_0 \in A] m(\mathrm{d}y).$$

We now show these measures are equal by proving their Laplace transforms are equal. By taking the Laplace transform of $\mu^2_\phi(\mathrm{d}t)$ we find,

$$\begin{aligned}
\int_{[0,\infty)} e^{-\lambda t}\mu^2_\phi(\mathrm{d}t) &= \int_{(0,r)} \phi(y)\Big(\int_{[0,\infty)} e^{-\lambda t}\mathbb{P}_y[H_0 \in \mathrm{d}t]\Big)m(\mathrm{d}y) \\
&= \int_{(0,r)} \phi(y)\varphi_\lambda(y)m(\mathrm{d}y) \\
&= \int_{(0.r)} \phi(y)\frac{1}{\lambda}\,\mathrm{d}\varphi^+_\lambda(y),
\end{aligned}$$

where in the final line we have used that $\mathrm{d}\varphi^+_\lambda(y) = \lambda\varphi_\lambda(y)m(\mathrm{d}y)$ in $(0,r)$. Using the integration by parts formula for Lebesgue-Stieltjes integrals and using that supp $\phi \Subset (0,r)$ we see,

$$\begin{aligned}
\int_{(0,r)} \phi(y)\,\mathrm{d}\varphi^+_\lambda(y) + \int_{(0,r)} \varphi^+_\lambda(y)\,\mathrm{d}\phi(y) = 0, \\
\int_{(0,r)} \varphi_\lambda(y)\,\mathrm{d}\phi^+(y) + \int_{(0,r)} \phi^+(y)\,\mathrm{d}\varphi_\lambda(y) = 0.
\end{aligned}$$

As φ_λ and ϕ are absolutely continuous,

$$\int_{(0,r)} \varphi^+_\lambda(y)\,\mathrm{d}\phi(y) = \int_0^r \phi'(y)\varphi'_\lambda(y)\,\mathrm{d}y = \int_{(0,r)} \phi^+(y)\,\mathrm{d}\varphi_\lambda(y),$$

and so $\int_{(0,r)} \phi(y)\,\mathrm{d}\varphi^+_\lambda(y) = -\int_{(0,r)} \varphi_\lambda(y)\,\mathrm{d}\phi^+(y)$. Therefore,

$$\begin{aligned}
\int_{(0,r)} \phi(y)\frac{1}{\lambda}\,\mathrm{d}\varphi^+_\lambda(y) &= \int_{(0,r)} \frac{\varphi_\lambda(y)}{\lambda}\,\mathrm{d}\phi^+(y) \\
&= \int_{(0,r)} \Big(\int_{[0,\infty)} \frac{e^{-\lambda t}}{\lambda}\mathbb{P}_y[H_0 \in \mathrm{d}t]\Big)\,\mathrm{d}\phi^+(y) \\
&= \int_{(0,r)} \Big(\int_{[0,\infty)} \Big(\int_t^\infty e^{-\lambda u}\,\mathrm{d}u\Big)\mathbb{P}_y[H_0 \in \mathrm{d}t]\Big)\,\mathrm{d}\phi^+(y) \\
&= \int_{(0,r)} \Big(\int_0^\infty \mathbb{P}_y[H_0 \leq u]e^{-\lambda u}\,\mathrm{d}u\Big)\,\mathrm{d}\phi^+(y) \\
&= \int_0^\infty e^{-\lambda u}\mu^1_\phi(\mathrm{d}u).
\end{aligned}$$

As the Laplace transforms of the measures are equal, the corresponding measures are equal. Now by the properties of the Bochner integral,

$$\begin{aligned}
\big\langle \mathcal{H}_y f, \mathcal{L}_x^* \psi \big\rangle_{\mathfrak{B}^*} &= \int_{[0,\infty)} \big\langle P_t f, \mathcal{L}_x^* \psi \big\rangle_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t] \\
&= \int_{[0,\delta)} \big\langle P_t f, \mathcal{L}_x^* \psi \big\rangle_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t] + \int_{[\delta,\infty)} \big\langle \mathcal{L}_x P_t f, \psi \big\rangle_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t],
\end{aligned}$$

as $P_t f \in \mathrm{Dom}(\mathcal{L}_x)$ for all $t > 0$. Now,

$$\Big| \int_{[0,\delta)} \big\langle P_t f, \mathcal{L}_x^* \psi \big\rangle_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t] \Big| \leq \|f\|_{\mathfrak{B}} \|\mathcal{L}_x^* \psi\|_{\mathfrak{B}^*} \mathbb{P}_y[H_0 < \delta] \to 0,$$

as $\delta \to 0$ for any $y > 0$ and so,

$$\big\langle \mathcal{H}_y f, \mathcal{L}_x^* \psi \big\rangle_{\mathfrak{B}^*} = \int_{(0,\infty)} \big\langle \mathcal{L}_x P_t f, \psi \big\rangle_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t].$$

Therefore,

$$\begin{aligned}
\int_{(0,r)} \big\langle \mathcal{H}_y f, \mathcal{L}_x^* \psi \big\rangle_{\mathfrak{B}^*} \phi(y) m(\mathrm{d}y) &= \int_{(0,r)} \int_{(0,\infty)} \big\langle \mathcal{L}_x P_t f, \psi \big\rangle_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t] \phi(y) m(\mathrm{d}y) \\
&= \int_{(0,\infty)} \big\langle \tfrac{\mathrm{d}}{\mathrm{d}t} P_t f, \psi \big\rangle_{\mathfrak{B}} \Big(\int_{(0,r)} \mathbb{P}_y[H_0 \in \mathrm{d}t] \phi(y) m(\mathrm{d}y) \Big) \\
&= \int_{(0,\infty)} \tfrac{\mathrm{d}}{\mathrm{d}t} \big\langle P_t f, \psi \big\rangle_{\mathfrak{B}} \Big(\int_{(0,r)} \mathbb{P}_y[H_0 \leq t] \,\mathrm{d}\phi^+(y) \Big) \,\mathrm{d}t \\
&= -\int_{(0,r)} \int_{(0,\infty)} \big\langle P_t f, \psi \big\rangle_{\mathfrak{B}} \Big(\mathbb{P}_y[H_0 \in \mathrm{d}t] \,\mathrm{d}\phi^+(y) \Big) \\
&= -\int_{(0,r)} \big\langle \mathcal{H}_y f, \psi \big\rangle_{\mathfrak{B}} \,\mathrm{d}\phi^+(y),
\end{aligned}$$

completing the proof. □

5.2.1 Regularity of the Harmonic Extension

The semigroup representation allows us to obtain regularity results for the harmonic extension. Let $\mathcal{L}_x$ be given by (4.2.2) and assume the coefficients are uniformly bounded and continuous on $\mathbb{R}^d$. Then by Proposition 2.2.14, we can choose an extension of this operator to various Banach spaces so that the semigroup $(P_t)_{t\geq 0}$ associated with this operator is analytic. In particular, we consider the example given by Proposition 2.2.14, Example 3.

Proposition 5.2.6. *Suppose m satisfies Assumption 3.4.6. Then for all $y \in E_m \cap (0,r)$, $\mathcal{H}_y f \in \bigcap_{k\in\mathbb{N}} \mathrm{Dom}(\mathcal{L}_x^k)$.*

Proof. The semigroup $(P_t)_{t\geq 0}$ is analytic so for all $k \in \mathbb{N}$, there is $M_k > 0$ such that $\|\mathcal{L}_x P_t f\|_{\mathfrak{B}} \leq M_k t^{-k} \|f\|_{\mathfrak{B}}$. By Lemma 3.4.7 we know that

$$\int_{(0,\infty)} t^{-k} \mathbb{P}_y[H_0 \in \mathrm{d}t] < \infty,$$

and so $\int_{(0,\infty)} \|\mathcal{L}_x^k P_t f\|_{\mathfrak{B}} \mathbb{P}_y[H_0 \in \mathrm{d}t] < \infty$ and as the operator $\mathcal{L}^k$ is closed,

$$\mathcal{L}_x^k \mathcal{H}_y f = \int_{(0,\infty)} (\mathcal{L}_x^k P_t f) \mathbb{P}_y[H_0 \in \mathrm{d}t] \in \mathfrak{B},$$

and so $\mathcal{H}_y f \in \mathrm{Dom}(\mathcal{L}_x^k)$. □

In particular, we may consider the operator $\mathcal{L}_x$ defined on domains in Hölder spaces to obtain differentiability properties for the harmonic extension.

Lemma 5.2.7. *Let $f \in C^\gamma(\mathbb{R}^d)$. Suppose $(\sigma\sigma^{\mathbf{T}})_{ij}, a_i \in C^\gamma(\mathbb{R}^d)$ for all $1 \leq i, j \leq d$ for some $\gamma \in (0,1)$ and suppose that m satisfies Assumption 3.4.6.*

(i) For each $y \in (0,l) \cap E_m$, $u_f(\cdot, y) \in C^{2+\gamma}(\mathbb{R}^d)$.

(ii) For each $x \in \mathbb{R}^d$, $u_f(x,\cdot), \partial_i u_f(x,\cdot)$ and $\partial_{ij} u_f(x,\cdot)$ are continuous in $E_m \cap (0,l)$.

Hence u_f is a solution in the sense of Definition 4.3.1.

Proof. By the Schauder estimates for $\mathcal{L}_x$ [52, Theorem 3.1.15],

$$\|P_t f\|_{C^{2+\gamma}(\mathbb{R}^d)} \leq C\big(\|P_t f\|_\infty + \|\mathcal{L}_x P_t f\|_{C^\gamma(\mathbb{R}^d)}\big),$$

for any $t > 0$ for some $C > 0$ independent of $P_t f$. Therefore,

$$\begin{aligned}\|u_f(\cdot,y)\|_{C^{2+\gamma}(\mathbb{R}^d)} &\leq \int_{[0,\infty)} C\big(\|P_t f\|_\infty + \|\mathcal{L}_x P_t f\|_{C^\gamma(\mathbb{R}^d)}\big)\mathbb{P}_y[H_0 \in \mathrm{d}t] \\ &\leq C\Big(\|f\|_\infty + \int_{(0,\infty)} \frac{1}{t}\|f\|_{C^\gamma(\mathbb{R}^d)} \mathbb{P}_y[H_0 \in \mathrm{d}t]\Big) \\ &\leq C\|f\|_{C^\gamma(\mathbb{R}^d)},\end{aligned}$$

by Lemma 3.4.7.

For the second claim we apply Lemma 3.4.8. By [52, Corollary 3.1.16], we know that $t \mapsto t^\gamma \partial_i P_t f(x)$ and $t^\gamma \mapsto t^\gamma \partial_{ij} P_t f(x)$ are bounded, continuous functions for any $x \in \mathbb{R}^d$ and $1 \leq i, j \leq d$. Therefore, by weak convergence we have

$$u_f(x, y_n) = \int_{(0,\infty)} (P_t f)(x) \mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \to \int_{(0,\infty)} (P_t f)(x) \mathbb{P}_y[H_0 \in \mathrm{d}t] = u_f(x, y),$$

as $n \to \infty$. Furthermore $t^{-\gamma}\mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \to t^{-\gamma}\mathbb{P}_y[H_0 \in \mathrm{d}t]$ weakly as $n \to \infty$ so we see

$$\begin{aligned}
\partial_i u_f(x, y_n) &= \int_{(0,\infty)} (\partial_i P_t f)(x) \mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&= \int_{(0,\infty)} (t^\gamma \partial_i P_t f)(x) t^{-\gamma} \mathbb{P}_{y_n}[H_0 \in \mathrm{d}t] \\
&\to \int_{(0,\infty)} (\partial_i P_t f)(x) \mathbb{P}_y[H_0 \in \mathrm{d}t] \\
&= \partial_i u_f(x, y),
\end{aligned}$$

and by the same reasoning $\partial_{ij} u_f(x, y_n) \to \partial_{ij} u_f(x, y)$. The final statement follows by Proposition 5.2.5. □

5.3 Spectral Representation

In the situation when the underlying Banach space $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space, we may prove stronger results as we may characterise the domain of the $-\psi(-\mathcal{L}_x)$ more explicitly. Furthermore, we assume throughout this section that $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$ is a self-adjoint operator on $\mathfrak{H}$ so that we may use the spectral theory for self-adjoint operators on Hilbert spaces. The results in this section extend some of those found in [47] where the special case of operators $\psi(-\Delta_x)$ are considered but the methods are related.

By the multiplicative spectral representation for self-adjoint operators [60, Theorem VIII.4], there exists a finite measure space (Σ, ρ), a unitary operator $U : \mathfrak{H} \to L^2(\Sigma, \rho)$ (for $f \in \mathfrak{H}$, we denote $Uf = \hat{f}$), and a function $\eta : \Sigma \to \sigma(-\mathcal{L}_x)$ which is finite almost everywhere such that $f \in \mathrm{Dom}(\mathcal{L}_x)$ if and only if $\eta(\cdot)\hat{f} \in L^2(\Sigma, \rho)$ and

$$-\mathcal{L}_x f = U^{-1} \eta(\cdot) U f.$$

Furthermore, as

$$0 \leq \langle -\mathcal{L}_x f, f \rangle_{\mathfrak{H}} = \langle -U\mathcal{L}_x f, Uf \rangle_{L^2(\Sigma,\rho)} = \int_\Sigma \eta(\lambda) |\hat{f}(\lambda)|^2 \rho(\mathrm{d}\lambda),$$

we have that $\eta \geq 0$, ρ-almost everywhere. Similarly, the projection-valued measure form of the spectral theorem [60, Theorem VIII.6], there exists a unique resolution E of the identity on Borel subsets of the real line such that

$$\langle -\mathcal{L}_x f, g\rangle_{\mathfrak{H}} = \int_{[0,\infty)} \lambda \, \mathrm{d}E_{f,g}(\lambda),$$

for any $f \in \mathrm{Dom}(\mathcal{L}_x)$ and $g \in \mathfrak{H}$ and E is concentrated on the spectrum $\sigma(-\mathcal{L}_x)$ of $-\mathcal{L}_x$ (which is a subset of $[0,\infty)$). Then for a measurable function ϕ defined on $\sigma(-\mathcal{L}_x)$, we may formally define the operator

$$\phi(-\mathcal{L}_x)f = \int_{[0,\infty)} \phi(\lambda) \, \mathrm{d}E_{f,\cdot}(\lambda) = U^{-1}(\phi \circ \eta)Uf,$$

with domain

$$\begin{aligned}\mathrm{Dom}(\phi(-\mathcal{L}_x)) &= \Big\{ f \in \mathfrak{H} : \int_{[0,\infty)} |\phi(\lambda)|^2 \, \mathrm{d}E_{f,f}(\lambda) < \infty \Big\} \\ &= \Big\{ f \in \mathfrak{H} : \int_{\Sigma} |\phi \circ \eta(\lambda)|^2 |\hat{f}(\lambda)|^2 \rho(\mathrm{d}\lambda) < \infty \Big\}.\end{aligned}$$

In particular, we have

$$P_t f = e^{-t(-\mathcal{L}_x)} f,$$

for all $f \in \mathfrak{H}$.

Proposition 5.3.1. *For all $f \in \mathfrak{H}$ and $y \in [0, r)$,*

$$\mathcal{H}_y f = \int_{[0,\infty)} \varphi_\lambda(y) \, \mathrm{d}E_{f,\cdot}(\lambda) = \varphi_{(-\mathcal{L}_x)}(y) f$$

Proof. As $P_t f = \int_{[0,\infty)} e^{-\lambda t} \, \mathrm{d}E_{f,\cdot}(\lambda)$,

$$\begin{aligned}\int_{[0,\infty)} (P_t f)\mathbb{P}_y[H_0 \in \mathrm{d}t] &= \int_{[0,\infty)} \Big(\int_{[0,\infty)} e^{-\lambda t} \mathbb{P}_y[H_0 \in \mathrm{d}t] \Big) \, \mathrm{d}E_{f,\cdot}(\lambda) \\ &= \int_{[0,\infty)} \varphi_\lambda(y) \, \mathrm{d}E_{f,\cdot}(\lambda).\end{aligned}$$

We note the final expression is well-defined for all $f \in \mathfrak{H}$ as $0 \leq \varphi_\lambda(y) \leq 1$. $\square$

Using the spectral representation, we have the following generalisation of Theorem 4.2 in [47].

Theorem 5.3.2. *Let $f \in \mathfrak{H}$ and define $\mathcal{H}_y f$ as above. Then $f \in \mathrm{Dom}(\psi(-\mathcal{L}_x))$ if and only if* strong-$\lim_{y\downarrow 0} \frac{\mathcal{H}_y f - f}{y} \in \mathfrak{H}$. *In which case,*

$$-\psi(-\mathcal{L}_x)f = \lim_{y\downarrow 0} \frac{\mathcal{H}_y f - f}{y},$$

where the limit is taken in $\mathfrak{H}$.

Proof. As $y \mapsto \varphi_\lambda(y)$ is convex for all $\lambda \geq 0$,

$$\varphi_\lambda(y) \geq \varphi_\lambda(0) + \varphi_\lambda^+(0)y \implies 0 \leq \frac{1-\varphi_\lambda(y)}{y} \leq \psi(\lambda).$$

Therefore for any $f \in \mathrm{Dom}(\psi(-\mathcal{L}_x)) = \{f \in \mathfrak{H} : \int_{[0,\infty)} |\psi(\lambda)|^2 \,\mathrm{d}E_{f,f}(\lambda) < \infty\}$ and $g \in \mathfrak{H}$,

$$\int_{[0,\infty)} \Big(\frac{1-\varphi_\lambda(y)}{y}\Big) \,\mathrm{d}E_{f,g}(\lambda) \to \int_{[0,\infty)} \psi(\lambda)\,\mathrm{d}E_{f,g}(\lambda),$$

by dominated convergence so $\frac{\mathcal{H}_y f - f}{y} \rightharpoonup -\psi(-\mathcal{L}_x)f$ as $y \downarrow 0$. Furthermore, as $|\frac{1-\varphi_\lambda(y)}{y}|^2 \leq |\psi(\lambda)|^2$ we have,

$$\int_{[0,\infty)} \Big|\frac{1-\varphi_\lambda(y)}{y}\Big|^2 \,\mathrm{d}E_{f,f}(\lambda) \to \int_{[0,\infty)} |\psi(\lambda)|^2 \,\mathrm{d}E_{f,f}(\lambda),$$

as $y \downarrow 0$ so $\|\frac{\mathcal{H}_y f - f}{y}\|_{\mathfrak{H}} \to \| - \psi(-\mathcal{L}_x)f\|_{\mathfrak{H}}$ as $y \downarrow 0$ by dominated convergence. Therefore $\frac{\mathcal{H}_y f - f}{y} \to -\psi(-\mathcal{L}_x)f$ strongly as $y \downarrow 0$.

If $f \notin \mathrm{Dom}(-\psi(-\mathcal{L}_x))$, then by Fatou's lemma,

$$\infty = \int_{[0,\infty)} |\psi(\lambda)|^2 \,\mathrm{d}E_{f,f}(\lambda) \leq \liminf_{y\downarrow 0} \int_{[0,\infty)} \Big|\frac{1-\varphi_\lambda(y)}{y}\Big|^2 \,\mathrm{d}E_{f,f}(\lambda),$$

so $\liminf_{y\downarrow 0} \|\frac{\mathcal{H}_y f - f}{y}\|_{\mathfrak{H}} = \infty$ which implies that $\frac{\mathcal{H}_y f - f}{y}$ does not converge strongly as $y \downarrow 0$. □

5.4 Dirichlet Form Approach to the Extension Method

In the stochastic approach, we studied the connection between process $(X_{T_t})_{t\geq 0}$ and the product process $((X_t, Y_t))_{t\geq 0}$. If we suppose that the process $(X_t)_{t\geq 0}$ corresponds to a symmetric semigroup $(P_t)_{t\geq 0}$ on some Hilbert space $\mathfrak{H}$, then we know that the subordinated semigroup corresponding to the process $(X_{T_t})_{t\geq 0}$ will also symmetric on the same Hilbert space and we may study these semigroups via their

Dirichlet forms. Furthermore, we have seen any gap diffusion $(Y_t)_{t\geq 0}$ corresponds to a Dirichlet form on $L^2(E_m, m)$ indicating that we may study the Dirichlet form corresponding to $(P_t^{\psi})_{t\geq 0}$ via the Dirichlet form corresponding to the semigroup associated with the product process $((X_t, Y_t))_{t\geq 0}$ on some product of the Hilbert spaces $\mathfrak{H}$ and $L^2(E_m, m)$. We may this notion rigorous in this section.

We consider the special case where $\mathfrak{H} = L^2(\mathcal{X}, \mu)$ where $\mathcal{X}$ is a locally compact, separable, metric space and μ is a positive Radon measure on $\mathcal{X}$ such that $\operatorname{supp} \mu = \mathcal{X}$. Let $(P_t)_{t\geq 0}$ be a symmetric sub-Markovian semigroup on $L^2(\mathcal{X}, \mu)$ with self-adjoint generator $(\mathcal{L}_x, \mathrm{Dom}(\mathcal{L}_x))$. Using the same notation as in the previous section, we recall that the Dirichlet form associated with the generator $\mathcal{L}_x$ can be given in terms of the spectral resolution of $-\mathcal{L}_x$,

$$\mathscr{E}^X(f, f) = \langle \sqrt{-\mathcal{L}_x} f, \sqrt{-\mathcal{L}_x} f \rangle_{L^2(\mathcal{X},\mu)} = \int_{[0,\infty)} \lambda \, \mathrm{d}E_{f,f}(\lambda),$$

for any $f \in \mathrm{Dom}(\sqrt{-\mathcal{L}_x})$. We assume the Dirichlet form $(\mathscr{E}^X, \mathrm{Dom}(\mathscr{E}^X))$ is regular with core $\mathcal{C}^X$ and we assume $(X_t)_{t\geq 0}$ is the corresponding Hunt process associated with this Dirichlet form. In this thesis, we are most interested in the case when the Dirichlet form is local and so the corresponding process is a diffusion process.

Now let $(P_t^{\psi})_{t\geq 0}$ be the subordinated semigroup on $L^2(\mathcal{X}, m)$ which is also symmetric and sub-Markovian on $L^2(\mathcal{X}, \mu)$ with self-adjoint generator $-\psi(-\mathcal{L}_x)$ with domain $\mathrm{Dom}(-\psi(-\mathcal{L}_x))$ which as before can be defined in terms of the spectral resolution of $-\mathcal{L}_x$,

$$\mathscr{E}^{\psi}(f, f) = \langle \sqrt{\psi(-\mathcal{L}_x)} f, \sqrt{\psi(-\mathcal{L}_x)} f \rangle_{\mathfrak{H}} = \int_{[0,\infty)} \psi(\lambda) \, \mathrm{d}E_{f,f}(\lambda),$$

for any $f \in \mathrm{Dom}(\sqrt{\psi(-\mathcal{L}_x)})$.

Now let ψ and m be in Krein correspondence and let $(Y_t)_{t\geq 0}$ be the corresponding gap diffusion associated with m and let $(\mathrm{Dom}(\mathscr{E}^Y), \mathrm{Dom}(\mathscr{E}^Y))$ be the regular Dirichlet form on $L^2(E_m, m)$ with core $\mathcal{C}^Y$ corresponding to Y (possibly adjoining a cemetery state $\dagger$ to the state space in the usual way to ensure the Dirichlet form is conservative). By Theorem 2.3.3, the process $((X_t, Y_t))_{t\geq 0}$ is associated with a regular Dirichlet form $(\mathscr{E}^{X\times Y}, \mathrm{Dom}(\mathscr{E}^{X\times Y}))$ on $L^2(\mathcal{X} \times E_m, \mu \times m)$ with core $\mathcal{C}^X \otimes \mathcal{C}^Y$ and for any $u \in \mathrm{Dom}(\mathscr{E}^{X\times Y})$,

$$[E_m \to \mathrm{Dom}(\mathscr{E}^X) : y \mapsto u(\cdot, y)] \in L^2(E_m, m; \mathrm{Dom}(\mathscr{E}^X)),$$
$$[\mathcal{X} \to \mathrm{Dom}(\mathscr{E}^Y) : x \mapsto u(x, \cdot)] \in L^2(\mathcal{X}, \mu; \mathrm{Dom}(\mathscr{E}^Y)),$$

and

$$\mathscr{E}^{X\times Y}(u,u)=\int_{E_m}\mathscr{E}^X(u(\cdot,y),u(\cdot,y))m(\mathrm{d}y)+\int_{\mathcal{X}}\mathscr{E}^Y(u(x,\cdot),u(x,\cdot))\mu(\mathrm{d}x).$$

Although $\varphi_\lambda\in L^2(E_m,m)$ for all $\lambda>0$, it is not clear whether the extension function $u_f\in L^2(\mathcal{X}\times E_m,\mu\times m)$ as

$$\|u_f\|^2_{L^2(\mathcal{X}\times E_m,\mu\times E_m)}=\int_{[0,\infty)}\int_{E_m}|\varphi_\lambda(y)|^2m(\mathrm{d}y)\,\mathrm{d}E_{f,f}(\lambda),$$

so we would need to know whether $\lambda\mapsto\int_{E_m}|\varphi_\lambda(y)|^2m(\mathrm{d}y)$ was $\mathrm{d}E_{f,f}$-integrable for any $f\in L^2(\mathcal{X},\mu)$. For this reason we extend the domain of $\mathscr{E}^{X\times Y}$.

Let $\mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ be the set of locally $(\mu\times m)$-integrable functions u such that

i) $u(x,\cdot)\in\mathrm{Dom}_{ext}(\mathscr{E}^Y)$ for μ-a.e. $x\in\mathcal{X}$,

ii) $u(\cdot,y)\in\mathrm{Dom}(\mathscr{E}^X)$ for m-a.e. $y\in E_m$,

iii) $\mathscr{E}^{X\times Y}(u,u)<\infty$.

We see that $\mathrm{Dom}(\mathscr{E}^{X\times Y})\subset\mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$. Furthermore, by i) we know that $u(x,\cdot)$ is absolutely continuous for μ-a.e. $x\in\mathcal{X}$ and so we may define $\mathrm{Tr}_0u=u(\cdot,0)$.

We now arrive at the main results of this section which we use extensively in applications.

Theorem 5.4.1. *Let $f\in L^2(\mathcal{X},\mu)$ and define u_f as the harmonic extension $u_f=\varphi_{(-\mathcal{L}_x)}f$. Then $f\in\mathrm{Dom}(\mathscr{E}^\psi)$ if and only if $u_f\in\mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ and*

$$\mathscr{E}^\psi(f,f)=\mathscr{E}^{X\times Y}(u_f,u_f).$$

Proof. Let $f\in L^2(\mathcal{X},\mu)$ and let $U:L^2(\mathcal{X},\mu)\to L^2(\Sigma,\rho)$ be the unitary map associated with the operator $-\mathcal{L}_x$. We first see that for all $y\in E_m$, $u_f(\cdot,y)\in L^2(\mathcal{X},\mu)$ as

$$\|u_f(\cdot,y)\|^2_{L^2(\mathcal{X},\mu)}=\int_\Sigma|\varphi_{\eta(\xi)}(y)\hat{f}(\xi)|^2\rho(\mathrm{d}\xi)\le\int_\Sigma|\hat{f}(\xi)|^2\rho(\mathrm{d}\xi)=\|f\|^2_{L^2(\mathcal{X},\mu)}.$$

Then for any compact $C = C_x \times C_y \subset \mathcal{X} \times E_m$,

$$\begin{aligned}\int_{C_y}\int_{C_x} |u_f(x,y)|^2 \mu(\mathrm{d}x) m(\mathrm{d}y) &\leq \int_{C_y}\int_{\mathcal{X}} |u_f(x,y)|^2 \mu(\mathrm{d}x) m(\mathrm{d}y) \\ &\leq \|f\|_{L^2(\mathcal{X},\mu)} m(C_y) \\ &< \infty,\end{aligned}$$

so u_f is $(\mu \times m)$-locally integrable. For $f \in L^2(\mathcal{X}, \mu)$, we have that $f \in \mathrm{Dom}(\mathscr{E}^\psi)$ if and only if

$$\int_\Sigma \psi \circ \eta(\xi) |\hat{f}(\xi)|^2 \rho(\mathrm{d}\xi) < \infty,$$

so by Corollary 3.4.4,

$$\begin{aligned}\mathscr{E}^\psi(f,f) &= \int_\Sigma \psi \circ \eta(\xi) |\hat{f}(\xi)|^2 \rho(\mathrm{d}\xi) \\ &= \int_\Sigma \Big(\eta(\xi) \|\varphi_{\eta(\xi)}\|^2_{L^2(E_m,m)} + \mathscr{E}^Y(\varphi_{\eta(\xi)}, \varphi_{\eta(\xi)})\Big) |\hat{f}(\xi)|^2 \rho(\mathrm{d}\xi) \\ &= \int_{E_m}\int_\Sigma \eta(\xi) |\varphi_{\eta(\xi)}(y)\hat{f}(\xi)|^2 \rho(\mathrm{d}\xi) m(\mathrm{d}y) + \int_\Sigma \mathscr{E}^Y(\varphi_{\eta(\xi)}\hat{f}(\xi), \varphi_{\eta(\xi)}\hat{f}(\xi)) \rho(\mathrm{d}\xi) \\ &= \int_{E_m}\int_\Sigma \eta(\xi) |\hat{u}_f(\xi,y)|^2 \rho(\mathrm{d}\xi) m(\mathrm{d}y) + \int_\Sigma \mathscr{E}^Y(\hat{u}_f(\xi,\cdot), \hat{u}_f(\xi,\cdot)) \rho(\mathrm{d}\xi).\end{aligned}$$

Therefore we know,

$$\int_{E_m}\int_\Sigma \eta(\xi) |\hat{u}_f(\xi,y)|^2 \rho(\mathrm{d}\xi) m(\mathrm{d}y) = \int_{E_m} \mathscr{E}^X(u_f(\cdot,y), u_f(\cdot,y)) m(\mathrm{d}y) < \infty,$$

so $\mathscr{E}^X(u_f(\cdot,y), u_f(\cdot,y)) < \infty$ for m-a.e. $y \in E_m$. As $\|u_f(\cdot,y)\|_{L^2(\mathcal{X},\mu)} \leq \|f\|_{L^2(\mathcal{X},\mu)}$ for any $y \in E_m$, we have $u(\cdot,y) \in \mathrm{Dom}(\mathscr{E}^X)$ for m-a.e. $y \in E_m$.

Now as U and $\sqrt{-\mathcal{A}_y}$ commute on $L^2(\mathcal{X},\mu) \otimes \mathrm{Dom}(\mathscr{E}^Y)$, we see that the operators will also commute on $L^2(\mathcal{X},\mu) \otimes \mathrm{Dom}_{ext}(\mathscr{E}^Y)$. Therefore,

$$\int_\Sigma \mathscr{E}^Y(\hat{u}_f(\xi,\cdot), \hat{u}_f(\xi,\cdot)) \rho(\mathrm{d}\xi) = \int_{\mathcal{X}} \mathscr{E}^Y(u_f(x,\cdot), u_f(x,\cdot)) \mu(\mathrm{d}x),$$

and so $u_f(x,\cdot) \in \mathrm{Dom}_{ext}(\mathscr{E}^Y)$ for μ-a.e. $x \in \mathcal{X}$ and we have,

$$\mathscr{E}^\psi(f,f) = \int_{E_m} \mathscr{E}^X(u_f(\cdot,y), u_f(\cdot,y)) m(\mathrm{d}y) + \int_{\mathcal{X}} \mathscr{E}^Y(u_f(x,\cdot), u_f(x,\cdot)) \mu(\mathrm{d}x).$$

□

We also have the following generalisation of [47, Theorem 4.6].

Theorem 5.4.2. *For any* $v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ *such that* $v(\cdot,0) = f$ *we have,*

$$\mathscr{E}^{X\times Y}(v,v) \geq \mathscr{E}^{X\times Y}(u_f,u_f).$$

Moreover, the space $\mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) = \mathrm{Dom}_0(\mathscr{E}^{X\times Y}) \oplus \mathrm{Harm}$ *where,*

$$\mathrm{Dom}_0(\mathscr{E}^{X\times Y}) = \{v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) : v(\cdot,0) = 0\},$$
$$\mathrm{Harm} = \{\varphi_{(-\mathcal{L}_x)} f : f \in \mathrm{Dom}(\mathscr{E}^{\psi})\},$$

are orthogonal to each other with respect to $\mathscr{E}^{X\times Y}$.

Proof. Let $v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ such that $v(\cdot,0) = f$. Then for fixed $\xi \in \Sigma$ such that $\hat{f}(\xi) \neq 0$, let

$$\phi(y) = \frac{\hat{v}(\xi,y)}{\hat{f}(\xi)}.$$

As $v(x,\cdot) \in \mathrm{Dom}_{ext}(\mathscr{E}^{Y})$ for μ-a.e. $x \in \mathcal{X}$, we know $\hat{v}(\xi,\cdot) \in \mathrm{Dom}_{ext}(\mathscr{E}^{Y})$ for ρ-a.e. $\xi \in \Sigma$ and so $y \mapsto \phi(y) \in \mathrm{Dom}_{ext}(\mathscr{E}^{Y})$ and $\phi(0) = 1$. Therefore, ϕ satisfies the conditions of Corollary 3.4.4 for any $\xi \in \Sigma$ such that $\eta(\xi) > 0$,

$$\int_{E_m} \eta(\xi)|\phi(y)|^2 m(\mathrm{d}y) + \mathscr{E}^{Y}(\phi,\phi) \geq \psi \circ \eta(\xi).$$

Furthermore, the inequality holds when $\eta(\xi) = 0$ and so,

$$\int_{E_m} \eta(\xi)|\hat{v}(\xi,y)|^2 m(\mathrm{d}y) + \mathscr{E}^{Y}(\hat{v}(\xi,y),\hat{v}(\xi,y))\,\mathrm{d}y \geq \psi \circ \eta(\xi)|\hat{f}(\xi)|^2,$$

noting the above inequality holds trivially for $\xi \in \Sigma$ such that $\hat{f}(\xi) = 0$. Hence,

$$\int_{E_m}\int_{\Sigma} \eta(\xi)|\hat{v}(\xi,y)|^2 \rho(\mathrm{d}\lambda) m(\mathrm{d}y) + \int_{\Sigma} \mathscr{E}^{Y}(\hat{v}(\xi,y),\hat{v}(\xi,y))\rho(\mathrm{d}\xi) \geq \int_{\Sigma} \psi \circ \eta(\xi)|\hat{f}(\xi)|^2 \rho(\mathrm{d}\xi).$$

Therefore,

$$\mathscr{E}^{X\times Y}(v,v) \geq \mathscr{E}^{\psi}(f,f) = \mathscr{E}^{X\times Y}(u_f,u_f).$$

The second part of the theorem follows the same reasoning as in [47] which we repeat here. Clearly, $u_f = \varphi_{(-\mathcal{L}_x)} f \in \mathrm{Harm}$ and $v - \varphi_{(-\mathcal{L}_x)} f \in \mathrm{Dom}_0(\mathscr{E}^{X\times Y})$ so the direct sum result holds. For $h > 0$, $u \in \mathrm{Harm}$ and $v \in \mathrm{Dom}_0(\mathscr{E}^{X\times Y})$, we have

$\mathscr{E}^{X\times Y}(u \pm hv, u \pm hv) \geq \mathscr{E}^{X\times Y}(u,u)$ and so

$$\pm 2h\mathscr{E}^{X\times Y}(u,v) + h^2\mathscr{E}^{X\times Y}(v,v) \geq 0 \implies \pm\mathscr{E}^{X\times Y}(u,v) \geq 0,$$

and so $\mathscr{E}^{X\times Y}(u,v) = 0$. $\square$

Remark 5.4.3. *Using the parallelogram law for inner products, we have for any* $f, g \in \mathrm{Dom}(\mathscr{E}^{\psi})$,

$$\begin{aligned}
\mathscr{E}^{\psi}(f,g) &= \frac{1}{4}[\mathscr{E}^{\psi}(f+g, f+g) - \mathscr{E}^{\psi}(f-g, f-g)] \\
&= \frac{1}{4}[\mathscr{E}^{X\times Y}(u_{(f+g)}, u_{(f+g)}) - \mathscr{E}^{\psi}(u_{(f-g)}, u_{(f-g)})] \\
&= \frac{1}{4}[\mathscr{E}^{X\times Y}(u_f + u_g, u_f + u_g) - \mathscr{E}^{\psi}(u_f - u_g, u_f - u_g)] \\
&= \mathscr{E}^{X\times Y}(u_f, u_g),
\end{aligned}$$

where we note $u_{(f+g)} = \mathcal{H}_y(f+g) = \mathcal{H}_y f + \mathcal{H}_y g = u_f + u_g$. *Similarly, let* $u, v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ *with* $\mathrm{Tr}_0 u = f$ *and* $\mathrm{Tr}_0 v = g$, *so* $u = u_f + u_0$ *where* $u_f = \varphi_{(-\mathcal{L}_x)} f$ *and* $u_0 = u - u_f$ *(and similarly for* v*). Then,*

$$\begin{aligned}
\mathscr{E}^{X\times Y}(u,v) &= \mathscr{E}^{X\times Y}(u_f + u_0, v_g + v_0), \\
&= \mathscr{E}^{X\times Y}(u_f, v_g) + \mathscr{E}^{X\times Y}(u_0, v_g) + \mathscr{E}^{X\times Y}(u_f, v_0) + \mathscr{E}^{X\times Y}(u_0, v_0), \\
&= \mathscr{E}^{\psi}(f,g) + \mathscr{E}^{X\times Y}(u_0, v_0),
\end{aligned}$$

and so in particular, $\mathscr{E}^{X\times Y}(u_f, v) = \mathscr{E}^{X\times Y}(u_f, v_g) + \mathscr{E}^{X\times Y}(u_f, v_0) = \mathscr{E}^{\psi}(f,g)$ *as* $\mathscr{E}^{X\times Y}(u_f, v_0) = 0$.

Chapter 6

Application to Problems in Optimal Stopping

6.1 Introduction

We now apply the harmonic extension technique to study the optimal stopping problem for a subordinated diffusion process. If we suppose $(X_t)_{t\geq 0}$ is a diffusion process in $\mathbb{R}^d$ and $(T_t)_{t\geq 0}$ is an inverse local time process of a gap diffusion, we are interested in the value function $V : \mathbb{R}^d \to \mathbb{R}$ defined by

$$V(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-\int_0^\tau R(X_{T_s})\,\mathrm{d}s} G(X_{T_\tau}) \right]$$

for a given gain function $G : \mathbb{R}^d \to \mathbb{R}$ and interest function $R : \mathbb{R}^d \to \mathbb{R}$ where the supremum is taken over all almost surely finite stopping times. This probabilistic problem is related to the analytic obstacle problem. By considering the Itô formula corresponding to $(X_{T_t})_{t\geq 0}$, the value function above should solve the following free boundary problem,

$$\begin{cases} V(x) \geq G(x) & \text{for } x \in \mathbb{R}^d \\ -\psi(-\mathcal{L}_x)V(x) - R(x)V(x) = 0 & \text{for } \{x \in \mathbb{R}^d : V(x) > G(x)\} \\ -\psi(-\mathcal{L}_x)V(x) - R(x)V(x) \leq 0 & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{6.1.1}$$

where $-\psi(-\mathcal{L}_x)$ is the non-local operator associated with the subordinated diffusion process.

As we have seen, the extension method allows us to represent the non-local operator $-\psi(-\mathcal{L}_x)$ via the Dirichlet-to-Neumann map of a local PDE problem. This allows us to formally rewrite the nonlocal obstacle problem in terms a thin obstacle problem related to the measure m associated with ψ. Formally, $V : \mathbb{R}^d \to \mathbb{R}$ should be given by $u_V(\cdot, 0)$ where $u_V : \mathbb{R}^d \times ([0, l] \cap [0, r)) \to \mathbb{R}$ is the solution to one of the following problems:

If $l + m([0, l)) = \infty$ and $\int_{[0,l)} y^2 m(\mathrm{d}y) = \infty$,

$$\begin{cases} u_V(x,0) \geq G(x) & \text{for } x \in \mathbb{R}^d \\ \mathcal{L}_x u_V \times m(\mathrm{d}y) + \partial_y^2 u_V = 0 & \text{in } \mathbb{R}^d \times (0,l) \\ \partial_y u_V(x,0) - R(x) u_V(x,0) = 0 & \text{for } \{x : u_V(x,0) > G(x)\} \\ \partial_y u_V(x,0) - R(x) u_V(x,0) \leq 0 & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{6.1.2}$$

if $l + m([0, l)) = \infty$ and $\int_{[0,l)} y^2 m(\mathrm{d}y) < \infty$,

$$\begin{cases} u_V(x,0) \geq G(x) & \text{for } x \in \mathbb{R}^d \\ \mathcal{L}_x u_V \times m(\mathrm{d}y) + \partial_y^2 u_V = 0 & \text{in } \mathbb{R}^d \times (0,l) \\ \partial_y u_V(x,0) - R(x) u_V(x,0) = 0 & \text{for } \{x : u_V(x,0) > G(x)\} \\ \partial_y u_V(x,0) - R(x) u_V(x,0) \leq 0 & \text{for } x \in \mathbb{R}^d \\ \partial_y u_V(x,l) = 0 & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{6.1.3}$$

and if $l + m([0, l)) < \infty$,

$$\begin{cases} u_V(x,0) \geq G(x) & \text{for } x \in \mathbb{R}^d \\ \mathcal{L}_x u_V \times m(\mathrm{d}y) + \partial_y^2 u_V = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ \partial_y u_V(x,0) - R(x) u_V(x,0) = 0 & \text{for } \{x : u_V(x,0) > G(x)\} \\ \partial_y u_V(x,0) - R(x) u_V(x,0) \leq 0 & \text{for } x \in \mathbb{R}^d \\ (r-l)\partial_y u_V(x,l) + u_V(x,l) = 0 & \text{for } x \in \mathbb{R}^d. \end{cases} \tag{6.1.4}$$

As in the previous chapter, the key to understanding the PDE in the y-direction is the extension function φ_λ and the boundary conditions here are in analogy with those found in the definition of $\mathrm{Dom}_+(\mathcal{G}_y)$.

Although we prove existence and uniqueness to the variational inequality associated with the non-local obstacle problem using similar methods to those found in [77], the local characterisation of the value function allows us to actually prove that the value function belongs to domain of the generator in certain situations. Furthermore, numerical techniques such as finite difference schemes may be applied to the local problem to compute the value function whereas these techniques fail when applied to the non-local problem.

6.2 Abstract Existence to the Non-local Variational Inequality

We begin by proving existence and uniqueness for the nonlocal obstacle problem via similar abstract methods to those found in [77] originally proven in [55].

Let $\mathcal{X}$ be a locally compact, separable, metric space with Radon measure μ on $\mathcal{X}$ such that supp $m = \mathcal{X}$ and suppose the Banach space $\mathfrak{B} = L^p(\mathcal{X}, \mu)$ for some $p \in (1, \infty)$ or $\mathfrak{B} = C_0(\mathcal{X})$. Omitting the index, we consider the subordinated semigroup $(P_t^{\psi})_{t\geq 0}$ with generator $(-\psi(-\mathcal{L}_x), \mathrm{Dom}(-\psi(-\mathcal{L}_x)))$ on this Banach space.

We recall by standard perturbation theory [43, Theorem 1.9.2], the operator $-\psi(-\mathcal{L}_x)-(R-r_0)$ with domain $\mathrm{Dom}(-\psi(-\mathcal{L}_x))$ generates another sub-Markovian semigroup $(Q_t^{\psi})_{t\geq 0}$. We recall the resolvent $(R_\lambda^Q)_{\lambda>0}$ associated with $(Q_t^{\psi})_{t\geq 0}$ given by

$$R_\lambda^Q f = \int_0^\infty e^{-\lambda t} Q_t^{\psi} f \,\mathrm{d}t,$$

for any $f \in \mathfrak{B}$ and satisfies $\|R_\lambda^Q f\|_{\mathfrak{B}} \leq \frac{1}{\lambda}\|f\|_{\mathfrak{B}}$. Furthermore, for any $f \in \mathfrak{B}$,

$$(\lambda + r_0 - (-\psi(-\mathcal{L}_x) - R))R_\lambda^Q f = f$$

In order to find a solution to the obstacle problem (6.1.1), we begin by considering the penalised problem:

$$\begin{cases} \text{Let } G \in \mathfrak{B}. \text{ For each } \varepsilon > 0 \text{ find } V_\varepsilon \in \mathfrak{B} \text{ such that} \\ -\psi(-\mathcal{L}_x)V_\varepsilon - RV_\varepsilon + \frac{1}{\varepsilon}(V_\varepsilon - G)^- = 0. \end{cases} \tag{6.2.1}$$

As $(V_\varepsilon - G)^- = -V_\varepsilon + (V_\varepsilon \vee G)$ we see,

$$-\psi(-\mathcal{L}_x)V_\varepsilon - (R - r_0)V_\varepsilon - (r_0 + \tfrac{1}{\varepsilon})V_\varepsilon = -\tfrac{1}{\varepsilon}(V_\varepsilon \vee G),$$

and so the solution to (6.2.1) is given by

$$V_\varepsilon = R^Q_{r_0+\frac{1}{\varepsilon}}(\tfrac{1}{\varepsilon}(V_\varepsilon \vee G)).$$

To prove existence and uniqueness, we apply a fixed point argument. We define a map $I : \mathfrak{B} \to \mathfrak{B}$ by

$$I_\varepsilon(h) = R^Q_{r_0+\frac{1}{\varepsilon}}(\tfrac{1}{\varepsilon}(h \vee G)),$$

and so we have

$$\|I_\varepsilon(h_1) - I_\varepsilon(h_2)\|_{\mathfrak{B}} \leq \frac{1}{\varepsilon}\frac{1}{r_0 + \frac{1}{\varepsilon}}\|(h_1 \vee G) - (h_2 \vee G)\|_{\mathfrak{B}} \leq \frac{1}{r_0\varepsilon + 1}\|h_1 - h_2\|_{\mathfrak{B}},$$

and so I_ε is a contraction mapping and therefore there exists a unique fixed point $V_\varepsilon \in \mathfrak{B}$. Furthermore, $V_\varepsilon \in \operatorname{im}(R^Q_{r_0+1/\varepsilon}) = \operatorname{Dom}(-\psi(-\mathcal{L}_x))$.

We now consider $\mathfrak{B} = \mathfrak{H} = L^2(\mathcal{X}, \mu)$ and let $(\mathscr{E}^\psi, \operatorname{Dom}(\mathscr{E}^\psi))$ denote the Dirichlet form associated with $(P^\psi_t)_{t\geq 0}$. We wish find a solution to the variational inequality

$$\begin{cases} \text{Let } G \in \operatorname{Dom}(\mathscr{E}^\psi). \text{ Find } V \in \operatorname{Dom}(\mathscr{E}^\psi) \text{ such that} \\ \mathscr{E}^\psi(V, U - V) + \langle RV, U - V\rangle_{L^2(\mathcal{X},\mu)} \geq 0 \\ \text{for all } U \in K, \text{ where } K = \{U \in \operatorname{Dom}(\mathscr{E}^\psi) : U \geq G\}. \end{cases} \tag{6.2.2}$$

K is closed and convex so by standard theory of variational inequalities [41], this problem has a unique solution $V \in \operatorname{Dom}(\mathscr{E}^\psi)$ which is given by

$$\min_{v\in K} \mathscr{I}(v),$$

where the functional $\mathscr{I}$ is given by

$$\mathscr{I}(v) = \mathscr{E}^\psi(v, v) + \langle Rv, v\rangle_{L^2(\mathcal{X},\mu)}.$$

It is useful to identify V as the limit of V_ε as $\varepsilon \to 0$. It is clear that V_ε is a solution to

$$\mathscr{E}^\psi(V_\varepsilon, U) + \langle RV_\varepsilon, U\rangle_{L^2(\mathcal{X},\mu)} + \frac{1}{\varepsilon}\langle (V_\varepsilon - G)^-, U\rangle_{L^2(\mathcal{X},\mu)} = 0,$$

for all $U \in \operatorname{Dom}(\mathscr{E}^\psi)$. We can identify this as the minimiser of the following functional:

$$\mathscr{I}^\varepsilon(V_\varepsilon) = \min_{v\in\operatorname{Dom}(\mathscr{E}^\psi)} \mathscr{I}^\varepsilon(v),$$

where,

$$\mathscr{I}^{\varepsilon}(v) = \mathscr{E}^{\psi}(v,v) + \langle Rv, v\rangle_{L^2(\mathcal{X},\mu)} + \frac{1}{\varepsilon}\langle (v-G)^-, (v-G)^-\rangle_{L^2(\mathcal{X},\mu)}.$$

To see this we note for any $U \in \mathrm{Dom}(\mathscr{E}^{\psi})$ and $t > 0$, $\mathscr{I}^{\varepsilon}(V_\varepsilon + tU) \geq \mathscr{I}^{\varepsilon}(V_\varepsilon)$ so we have,

$$\begin{aligned}
0 &\leq \frac{1}{t}\Big(\mathscr{I}^{\varepsilon}(V_\varepsilon + tU) - \mathscr{I}^{\varepsilon}(V_\varepsilon)\Big) \\
&= 2\Big(\mathscr{E}^{\psi}(V_\varepsilon, U) + \langle RV_\varepsilon, U\rangle_{L^2(\mathcal{X},\mu)}\Big) + t\mathscr{I}^{\varepsilon}(U,U) \\
&+ \frac{1}{\varepsilon t}\Big(\Big\langle (V_\varepsilon - G + tU)^-, (V_\varepsilon - G + tU)^-\Big\rangle_{L^2(\mathcal{X},\mu)} - \Big\langle (V_\varepsilon - G)^-, (V_\varepsilon - G)^-\Big\rangle_{L^2(\mathcal{X},\mu)}\Big).
\end{aligned}$$

We know,

$$\begin{aligned}
&\frac{1}{\varepsilon t}\Big(\Big\langle (V_\varepsilon - G + tU)^-, (V_\varepsilon - G + tU)^-\Big\rangle_{L^2(\mathcal{X},\mu)} - \Big\langle (V_\varepsilon - G)^-, (V_\varepsilon - G)^-\Big\rangle_{L^2(\mathcal{X},\mu)}\Big) \\
&\quad = \frac{1}{\varepsilon t}\Big(\Big\langle (V_\varepsilon - G + tU)^-, V_\varepsilon - G + tU\Big\rangle_{L^2(\mathcal{X},\mu)} - \Big\langle (V_\varepsilon - G)^-, V_\varepsilon - G\Big\rangle_{L^2(\mathcal{X},\mu)}\Big) \\
&\quad = \frac{1}{\varepsilon t}\Big\langle (V_\varepsilon - G + tU)^- - (V_\varepsilon - G)^-, V_\varepsilon - G\Big\rangle_{L^2(\mathcal{X},\mu)} + \frac{1}{\varepsilon}\Big\langle (V_\varepsilon - G + tU)^-, U\Big\rangle_{L^2(\mathcal{X},\mu)} \\
&\quad \to \frac{1}{\varepsilon}\Big\langle -U\mathbf{1}_{(-\infty,0)}(V_\varepsilon - G), V_\varepsilon - G\Big\rangle_{L^2(\mathcal{X},\mu)} + \frac{1}{\varepsilon}\Big\langle (V_\varepsilon - G)^-, U\Big\rangle_{L^2(\mathcal{X},\mu)} \\
&\quad = \frac{2}{\varepsilon}\Big\langle (V_\varepsilon - G)^-, U\Big\rangle_{L^2(\mathcal{X},\mu)},
\end{aligned}$$

as $t \to 0$ as

$$\begin{aligned}
\Big\langle -U\mathbf{1}_{(-\infty,0)}(V_\varepsilon - G), V_\varepsilon - G\Big\rangle_{L^2(\mathcal{X},\mu)} &= \Big\langle -(V_\varepsilon - G)\mathbf{1}_{(-\infty,0)}(V_\varepsilon - G), U\Big\rangle_{L^2(\mathcal{X},\mu)} \\
&= \Big\langle (V_\varepsilon - G)^-, U\Big\rangle_{L^2(\mathcal{X},\mu)}.
\end{aligned}$$

Therefore we may show $(V_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $\mathrm{Dom}(\mathscr{E}^{\psi})$:

$$\mathscr{I}(V_\varepsilon) \leq \mathscr{I}^{\varepsilon}(V_\varepsilon) \leq \mathscr{I}^{\varepsilon}(V) = \mathscr{I}(V),$$

therefore there is a weakly convergent subsequence converging to some $V_0 \in \mathrm{Dom}(\mathscr{E}^{\psi})$. In particular, we have

$$\frac{1}{\varepsilon}\langle (V_\varepsilon - G)^-, (V_\varepsilon - G)^-\rangle_{L^2(\mathcal{X},\mu)} \leq \mathscr{I}(V) \implies \|(V_\varepsilon - G)^-\|_{L^2(\mathcal{X},\mu)} \leq \sqrt{\varepsilon}\sqrt{\mathscr{I}(V)} \to 0,$$

as $\varepsilon \to 0$ so $(V_\varepsilon - G)^- \to 0$ as $\varepsilon \to 0$ in $L^2(\mathcal{X}, \mu)$. By Fatou's lemma, we have that

$$\|(V_0 - G)^-\|_{L^2(\mathcal{X},\mu)} \leq \liminf_{\varepsilon \to 0} \|(V_\varepsilon - G)^-\|_{L^2(\mathcal{X},\mu)} = 0,$$

so $(V_0 - G)^- = 0$ and hence $V_0 \in K$. Furthermore as a norm on a Hilbert space is weakly lower-semicontinuous,

$$\mathscr{I}(V_0) \leq \liminf_{\varepsilon \to 0} \mathscr{I}(V_\varepsilon) \leq \mathscr{I}(V),$$

so $V_0 = V$.

Although we only have weak convergence of a subsequence of $(V_\varepsilon)_{\varepsilon>0}$ to V, we can strengthen this to obtain strong convergence in $\mathrm{Dom}(\mathscr{E}^\psi)$. We note

$$\begin{aligned}
\limsup_{\varepsilon \to 0} & [\mathscr{E}^\psi(V_\varepsilon - V, V_\varepsilon - V) + \langle R(V_\varepsilon - V), (V_\varepsilon - V)\rangle_{L^2(\mathcal{X},\mu)}] \\
&= \limsup_{\varepsilon \to 0} [\mathscr{E}^\psi(V_\varepsilon, V_\varepsilon) + \langle RV_\varepsilon, V_\varepsilon\rangle_{L^2(\mathcal{X},\mu)} - 2\mathscr{E}^\psi(V_\varepsilon, V) - 2\langle RV_\varepsilon, V\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad\qquad + \mathscr{E}^\psi(V, V) + \langle RV, V\rangle_{L^2(\mathcal{X},\mu)}] \\
&= \limsup_{\varepsilon \to 0} [\mathscr{I}(V_\varepsilon)] - \mathscr{I}(V) \\
&\leq 0,
\end{aligned}$$

and so $V_\varepsilon \to V$ in $\mathrm{Dom}(\mathscr{E}^\psi)$.

Another property we may prove without resorting to the local representation is that the value function is bounded whenever the gain function is bounded.

Proposition 6.2.1. *Let $V \in \mathrm{Dom}(\mathscr{E}^\psi)$ be the solution to (6.2.2). If $G \in L^2(\mathcal{X}, \mu) \cap L^\infty(\mathcal{X}, \mu)$ and $G \geq 0$, then $V \in L^\infty(\mathcal{X}, \mu)$ (and hence $V \in L^p(\mathcal{X}, \mu)$ for $p \in (2, \infty)$).*

Proof. Let $g_0 = \|G\|_{L^\infty(\mathcal{X},\mu)} > 0$. Then $V \wedge g_0 \in \mathrm{Dom}(\mathscr{E}^\psi)$ and $V \wedge g_0 \geq G$ so $V \wedge g_0 \in K$. Furthermore as $\mathscr{E}^\psi(V \wedge 1, V \wedge 1) \leq \mathscr{E}^\psi(V, V)$,

$$\begin{aligned}
\mathscr{E}^\psi(V \wedge g_0, V \wedge g_0) &= g_0^2 \mathscr{E}^\psi((\tfrac{V}{g_0}) \wedge 1, (\tfrac{V}{g_0}) \wedge 1) \\
&\leq g_0^2 \mathscr{E}^\psi((\tfrac{V}{g_0}), (\tfrac{V}{g_0})) \\
&\leq \mathscr{E}^\psi(V, V).
\end{aligned}$$

As $\langle R(V \wedge g_0), (V \wedge g_0)\rangle_{L^2(\mathcal{X},\mu)} \leq \langle RV, V\rangle_{L^2(\mathcal{X},\mu)}$, we have $\mathscr{I}(V \wedge g_0) \leq \mathscr{I}(V)$ so $V = V \wedge g_0$. Therefore, V is bounded. □

6.2.1 Identification with the Optimal Stopping Problem

We conclude this section by proving that the solution V to the variational inequality is the solution to the optimal stopping problem. Suppose $G \in L^2(\mathcal{X}, \mu) \cap C_0(\mathcal{X})$. Then we see that the fixed point $V_\varepsilon \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(2)}) \cap \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(\infty)}) \subset C_0(\mathcal{X})$ and so

$$V_\varepsilon(X_{T_t}) - V_\varepsilon(x) + \int_0^t \psi(-\mathcal{L}_x)V_\varepsilon(X_{T_s})\,\mathrm{d}s,$$

is a martingale. Let $K_t = \exp\big(-\int_0^t R(X_{T_s})\,\mathrm{d}s\big)$ which is clearly a finite variation process adapted to the filtration generated by $(X_{T_t})_{t\geq 0}$ which satisfies

$$\mathrm{d}K_t = -R(X_{T_t})K_t\,\mathrm{d}t.$$

Therefore, by the integration by parts formula for semimartingales,

$$K_tV_\varepsilon(X_{T_t}) - V_\varepsilon(x) - \int_0^t V_\varepsilon(X_{T_s})\,\mathrm{d}K_t + \int_0^t \psi(-\mathcal{L}_x)V_\varepsilon(X_{T_s})K_s\,\mathrm{d}s$$

is a martingale and therefore for any stopping time τ and $T < \infty$,

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\Big(-\int_0^{\tau\wedge T} R(X_{T_s})\,\mathrm{d}s\Big)V_\varepsilon(X_{T_{\tau\wedge T}})\right] - V_\varepsilon(x)\\
&\quad= \mathbb{E}_x\left[\Big(-\psi(-\mathcal{L}_x)V_\varepsilon(X_{T_{\tau\wedge T}}) - R(X_{T_{\tau\wedge T}})V_\varepsilon(X_{T_{\tau\wedge T}})\Big)\exp\Big(-\int_0^{\tau\wedge T} R(X_{T_s})\,\mathrm{d}s\Big)\right]\\
&\quad= \mathbb{E}_x\left[-\frac{1}{\varepsilon}\Big(V_\varepsilon(X_{T_{\tau\wedge T}}) - G(X_{T_{\tau\wedge T}})\Big)^{-}\exp\Big(-\int_0^{\tau\wedge T} R(X_{T_s})\,\mathrm{d}s\Big)\right]\\
&\quad\leq 0.
\end{aligned}$$

Therefore by letting $\varepsilon \to 0$ and $T \to \infty$ we have,

$$\mathbb{E}_x\left[\exp\Big(-\int_0^{\tau} R(X_{T_s})\,\mathrm{d}s\Big)V(X_{T_\tau})\right] - V(x) \leq 0,$$

and so $\sup_\tau \mathbb{E}_x[\exp(-\int_0^\tau R(X_{T_s})\,\mathrm{d}s)G(X(T_\tau))] \leq V(x)$ as $V \geq G$ where the supremum is taken over almost surely finite stopping times. Now let $\tau_\varepsilon = \inf\{t \geq 0 :$

$V_\varepsilon(X(T_s)) \leq G(X(T_s))\}$. Then for any $T < \infty$, $(V_\varepsilon(X_{T_{\tau_\varepsilon \wedge T}}) - G(X_{T_{\tau_\varepsilon \wedge T}}))^- = 0$,

$$V_\varepsilon(x) = \mathbb{E}_x\bigg[\exp\Big(-\int_0^{\tau_\varepsilon \wedge T} R(X_{T_s})\,\mathrm{d}s\Big) V_\varepsilon(X_{T_{\tau_\varepsilon \wedge T}}))\bigg].$$

By Fatou's lemma applied as $T \to \infty$,

$$\begin{aligned} V_\varepsilon(x) &\leq \mathbb{E}_x\bigg[\exp\Big(-\int_0^{\tau_\varepsilon} R(X_{T_s})\,\mathrm{d}s\Big) V_\varepsilon(X_{T_{\tau_\varepsilon}}))\bigg] \\ &\leq \mathbb{E}_x\bigg[\exp\Big(-\int_0^{\tau_\varepsilon} R(X_{T_s})\,\mathrm{d}s\Big) G(X_{T_{\tau_\varepsilon}})\bigg], \end{aligned}$$

where the final line follows by right-continuity of $t \mapsto X(T_t)$. All in all we have

$$V(x) = \sup_\tau \mathbb{E}_x\bigg[\exp\Big(-\int_0^{\tau} R(X_{T_s})\,\mathrm{d}s\Big) G(X_{T_\tau})\bigg].$$

6.3 Identification with the Local Problem

We now identify the above non-local variational problems with the local counterparts via the extension method. The local penalised problem is given by,

$$\begin{cases} \text{For each } \varepsilon > 0 \text{ find } u_\varepsilon \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) \text{ such that for all } v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) \\ \mathscr{E}^{X\times Y}(u_\varepsilon, v) + \langle Ru_\varepsilon(\cdot,0), v(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)} + \frac{1}{\varepsilon}\langle (u_\varepsilon(\cdot,0) - G)^-, v(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)} = 0, \end{cases} \tag{6.3.1}$$

and the local obstacle problem is given by,

$$\begin{cases} \text{Find } u_V \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) \text{ such that for all } v \in \widehat{K} = \{v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) : v(\cdot,0) \geq G\} \\ \mathscr{E}^{X\times Y}(u_V, v - u_V) + \langle Ru_V(\cdot,0), v(\cdot,0) - u_V(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)} \geq 0. \end{cases} \tag{6.3.2}$$

We now shall prove this local obstacle problem is equivalent to the nonlocal obstacle problem (6.2.2).

Lemma 6.3.1. *A function V (resp. V_ε) $\in \mathrm{Dom}(\mathscr{E}^\psi)$ is a solution to (6.2.2) (resp. (6.2.1)) if and only if $u_V = \varphi_{(-\mathcal{L}_x)} V \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ (resp. $u_\varepsilon = \varphi_{(-\mathcal{L}_x)} V_\varepsilon \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$) is a solution to (6.3.2) (resp. 6.3.1).*

Proof. If $V \in \mathrm{Dom}(\mathscr{E}^\psi)$ is a solution to (6.2.2), then for any $f \in K$

$$\mathscr{E}^\psi(V,V) + \langle RV, V\rangle_{L^2(\mathcal{X},\mu)} \leq \mathscr{E}^\psi(f,f) + \langle Rf, f\rangle_{L^2(\mathcal{X},\mu)}.$$

Let u_V denote the harmonic extension of V and let $v \in \widehat{K}$. By Theorem 5.4.2, $v = u_f + w$ where u_f denotes the harmonic extension for some $f \in \mathrm{Dom}(\mathscr{E}^\psi)$ and $w \in \mathrm{Dom}_0(\mathscr{E}^{X\times Y})$. As $v(\cdot,0) = u_f(\cdot,0) = f \geq G$, $f \in K$. Therefore,

$$\begin{aligned}\mathscr{E}^{X\times Y}(v,v) &= \mathscr{E}^{X\times Y}(u_f,u_f) + 2\mathscr{E}^{X\times Y}(u_f,w) + \mathscr{E}^{X\times Y}(w,w)\\ &\geq \mathscr{E}^{X\times Y}(u_f,u_f).\end{aligned}$$

By Theorem 5.4.1, the harmonic extension $u_V \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ and

$$\begin{aligned}&\mathscr{E}^{X\times Y}(u_V,u_V) + \langle Ru_V(\cdot,0), u_V(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)}\\ &\quad\leq \mathscr{E}^{X\times Y}(u_f,u_f) + \langle Ru_f(\cdot,0), u_f(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)}\\ &\quad\leq \mathscr{E}^{X\times Y}(v,v) + \langle Rv(\cdot,0), v(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)},\end{aligned}$$

so u_V is a solution to (6.3.2).

Similarly, if u_V is a a solution to (6.3.2), then by letting $v = u_f$ for some $f \in K$,

$$\mathscr{E}^{X\times Y}(u_V,u_V) + \langle Ru_V(\cdot,0), u_V(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)} \leq \mathscr{E}^{X\times Y}(u_f,u_f) + \langle Ru_f(\cdot,0), u_f(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)},$$

which by Theorem 5.4.1 is equivalent to

$$\mathscr{E}^\psi(V,V) + \langle RV, V\rangle_{L^2(\mathcal{X},\mu)} \leq \mathscr{E}^\psi(f,f) + \langle f, f\rangle_{L^2(\mathcal{X},\mu)},$$

and so V is a solution to (6.2.2). □

Using this local chacaterisation, we can easily show a monotonicity property of the value function.

Proposition 6.3.2. *Let $G_1, G_2 \in \mathrm{Dom}(\mathscr{E}^\psi)$ and $G_1 \leq G_2$ and let u_1 and u_2 be solutions to the local variational inequality (6.3.2) corresponding to the gain functions G_1 and G_2 respectively. Then $V_1 \leq V_2$ almost everywhere where $V_i = \mathrm{Tr}_0 u_i$ for $i = 1, 2$.*

Proof. As $G_1 \leq G_2$, we have $K_2 \subset K_1$ for $K_i = \{v \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y}) : v(\cdot,0) \geq G_i\}$.

Let $v = u_2 + (u_2 - u_1)^- \in K_2$ so

$$\mathscr{E}^{X\times Y}(u_2, (u_2 - u_1)^-) + \langle Ru_2(\cdot, 0), (u_2(\cdot, 0) - u_1(\cdot, 0))^- \rangle_{L^2(\mathcal{X},\mu)} \geq 0.$$

Let $v = u_1 - (u_2 - u_1)^-$. If $u_1(x) \leq u_2(x)$, then $(u_2(x) - u_1(x))^- = 0$ almost everywhere so $v(x) = u_1(x)$ almost everywhere whereas when $u_2(x) < u_1(x)$, $(u_2(x) - u_1(x))^- = -u_2(x) + u_1(x)$ so $v(x) = u_2(x)$ and therefore $v(\cdot, 0) \geq G_1$. Therefore,

$$\mathscr{E}^{X\times Y}(u_1, -(u_2 - u_1)^-) + \langle Ru_1(\cdot, 0), -(u_2(\cdot, 0) - u_1(\cdot, 0))^- \rangle_{L^2(\mathcal{X},\mu)} \geq 0.$$

Therefore,

$$\mathscr{E}^{X\times Y}(u_2 - u_1, (u_2 - u_1)^-) + \langle R(u_2(\cdot, 0) - u_1(\cdot, 0)), (u_2(\cdot, 0) - u_1(\cdot, 0))^- \rangle_{L^2(\mathcal{X},\mu)} \geq 0.$$

As $\mathscr{E}^{X\times Y}$ is a local Dirichlet form, we know

$$\begin{aligned}&\mathscr{E}^{X\times Y}((u_2 - u_1)^-, (u_2 - u_1)^-)\\ &\qquad + \langle R(u_2(\cdot, 0) - u_1(\cdot, 0))^-, (u_2(\cdot, 0) - u_1(\cdot, 0))^- \rangle_{L^2(\mathcal{X},\mu)} \leq 0.\end{aligned}$$

so $\|(u_2(\cdot, 0) - u_1(\cdot, 0))^-\|_{L^2(\mathcal{X},\mu)} = 0$ and so $(u_2(\cdot, 0) - u_1(\cdot, 0))^- = 0$. □

We recall that $W \in \mathrm{Dom}(\mathscr{E}^\psi)$ is a supersolution if

$$\mathscr{E}^\psi(W, \phi) + \langle RW, \phi \rangle_{L^2(\mathcal{X},\mu)} \geq 0 \text{ for all } \phi \in \mathrm{Dom}(\mathscr{E}^\psi), \phi \geq 0.$$

Proposition 6.3.3. *Let V be a solution to (6.2.2) and suppose that $W \in \mathrm{Dom}(\mathscr{E}^\psi)$ is a supersolution satisfying $W \geq G$. Then*

$$V \leq W.$$

Proof. Let $u_W \in \mathrm{Dom}(\mathscr{E}^{X\times Y})$ be the harmonic extension of $W \in \mathrm{Dom}(\mathscr{E}^\psi)$ and let $v \in \mathrm{Dom}(\mathscr{E}^{X\times Y})$ such that $v \geq 0$. Then,

$$\mathscr{E}^{X\times Y}(u_W, v) + \langle Ru_W(\cdot, 0), v(\cdot, 0) \rangle_{L^2(\mathcal{X},\mu)} = \mathscr{E}^\psi(W, \mathrm{Tr}_0 v) + \langle RW, \mathrm{Tr}_0 v \rangle_{L^2(\mathcal{X},\mu)} \geq 0,$$

as $v \geq 0 \implies \mathrm{Tr}_0 v \geq 0$. Then set $v = u_V \wedge u_W \in \widehat{K}$ as $V \wedge W \geq G$. Therefore,

$$\mathscr{E}^{X\times Y}(u_V, v - u_V) + \langle Ru_V(\cdot, 0), v(\cdot, 0) - u_V(\cdot, 0) \rangle_{L^2(\mathcal{X},\mu)} \geq 0.$$

Furthermore, as $V \geq V \wedge W$, we know $v - u_V \leq 0$ as so,

$$\mathscr{E}^{X\times Y}(u_W, v - u_W) + \langle Ru_W(\cdot,0), v(\cdot,0) - u_W(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)} \leq 0.$$

Therefore as $v - u_V = -(u_V - u_W)^+$,

$$\begin{aligned}
0 &\geq \mathscr{E}^{X\times Y}(u_W - u_V, v - u_V) + \langle R(u_W(\cdot,0) - u_V(\cdot,0)), v(\cdot,0) - u_V(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)} \\
&= -\mathscr{E}^{X\times Y}(u_W - u_V, (u_V - u_W)^+) \\
&\qquad - \langle R(u_W(\cdot,0) - u_V(\cdot,0)), (u_V(\cdot,0) - u_W(\cdot,0))^+\rangle_{L^2(\mathcal{X},\mu)} \\
&= \mathscr{E}^{X\times Y}((u_V - u_W)^+, (u_V - u_W)^+) \\
&\qquad + \langle R(u_V(\cdot,0) - u_W(\cdot,0))^+, (u_V(\cdot,0) - u_W(\cdot,0))^+\rangle_{L^2(\mathcal{X},\mu)},
\end{aligned}$$

therefore $(u_V - u_W)^+ = 0$ and hence $u_V \leq u_W \implies V \leq W$. □

6.4 Regularity of the Value Function via the Local Characterisation

6.4.1 Set Up

In this section we suppose $\mathcal{X} \subset \mathbb{R}^d$ and let $(X_t)_{t\geq 0}$ be a diffusion process taking values in $\mathcal{X}$. For any $f \in C_0(\mathcal{X})$, we define the family of operators $(P_t^{(0)})_{t\geq 0}$ by

$$P_t^{(0)} f(x) = \mathbb{E}_x[f(X_t)],$$

and assume this is a Feller semigroup with generator $(\mathcal{L}_x^{(0)}, \mathrm{Dom}(\mathcal{L}_x^{(0)}))$. Furthermore, for each $p \in (1,\infty)$, we assume $P_t^{(0)}|_{L^p(\mathcal{X},\mu)\cap C_0(\mathcal{X})}$ extends to a sub-Markovian semigroup $(P_t^{(p)})_{t\geq 0}$ on $L^p(\mathcal{X},\mu)$ with generator $(\mathcal{L}_x^{(p)}, \mathrm{Dom}(\mathcal{L}_x^{(p)}))$. We note for any $f \in \mathrm{Dom}(\mathcal{L}_x^{(p)}) \cap \mathrm{Dom}(\mathcal{L}_x^{(q)})$, $\mathcal{L}_x^{(p)} f = \mathcal{L}_x^{(q)} f$ so in such cases we may omit the index. Furthermore, we assume $(P_t^{(2)})_{t\geq 0}$ is symmetric and $(\mathscr{E}^X, \mathrm{Dom}(\mathscr{E}^X))$ is the corresponding local Dirichlet form.

Example 6.4.1. *[28, 72] Let $\mathcal{X} = \mathbb{R}^d$ and μ be Lebesgue measure on $\mathbb{R}^d$. Let $p(x,D) = \nabla_x \cdot (\Gamma(x)\nabla_x)$ where $\{\Gamma_{ij}\}_{1\leq i,j\leq d}$ is a family of Borel functions on $\mathbb{R}^d$ satisfying $\Gamma_{ij} = \Gamma_{ji}$ for all $1 \leq i,j \leq d$ and*

$$\delta \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d \Gamma_{ij}(x)\xi_i\xi_j \leq \frac{1}{\delta}\sum_{i=1}^d \xi^2,$$

for all $\xi, x \in \mathbb{R}^d$ for some $\delta \in (0,1)$. Then $\mathscr{E}^X$ given by

$$\mathscr{E}^X(u,v) = \int_{\mathbb{R}^d} \Gamma(x)\nabla_x u(x) \cdot \nabla v(x)\,\mathrm{d}x$$

with $\mathrm{Dom}(\mathscr{E}^X) = W^{1,2}(\mathbb{R}^d)$ defines a regular Dirichlet form.

If we assume that $\Gamma_{ij} \in C_b^1(\mathbb{R}^d)$ for all $1 \leq i,j \leq d$, then the extension of $p(\cdot, D)$ to $W^{2,2}(\mathbb{R}^d)$ is the self-adjoint generator of the $L^2(\mathbb{R}^d)$-semigroup $(P_t^{(2)})_{t\geq 0}$ corresponding to $\mathscr{E}^X$. Furthermore, for each $p \in (2,\infty)$, the restriction of this semigroup to $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ (resp. $L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$) extends to a sub-Markovian semigroup $(P_t^{(p)})_{t\geq 0}$ (resp. Feller semigroup $(P_t^{(0)})_{t\geq 0}$) on $L^p(\mathbb{R}^d)$ (resp. $C_0(\mathbb{R}^d)$) with generator given by the extension of $p(\cdot, D)$ to $W^{2,p}(\mathbb{R}^d)$ (resp. closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the graph norm).

We also note that as Γ is continuously differentiable, we may rewrite the operator $p(\cdot, D)$ in non-divergence form so that the results for analytic semigroups from Chapter 2 apply.

Now let $\psi : (0,\infty) \to \mathbb{R}$ be a complete Bernstein function given by

$$\psi(\lambda) = \frac{1}{r} + \int_0^\infty (1 - e^{-\lambda t})\nu(t)\,\mathrm{d}t,$$

corresponding to a subordinator $(T_t)_{t\geq 0}$ and let $(Y_t)_{t\geq 0}$ be the corresponding gap diffusion given by the Krein correspondence. Furthermore, let $(\mathscr{E}^Y, \mathrm{Dom}(\mathscr{E}^Y))$ be the regular Dirichlet form in $L^2(E_m, m)$. For each $p \in \{0\} \cup (1,\infty)$, let $(P_t^{\psi,(p)})_{t\geq 0}$ be the subordinated semigroup given by

$$P_t^{\psi,(p)} f = \int_{[0,\infty)} (P_s^{(p)} f)\mathbb{P}_0[T_t \in \mathrm{d}s],$$

for $f \in L^p(\mathcal{X}, \mu)$ or $f \in C_0(\mathcal{X})$ with generator $(-\psi(-\mathcal{L}_x)^{(p)}, \mathrm{Dom}(\psi(-\mathcal{L}_x)^{(p)}))$.

We note that if $G \in L^2(\mathcal{X}, \mu) \cap C_0(\mathcal{X}) \subset \cap_{p\in[2,\infty]} L^p(\mathcal{X}, \mu)$ and $R \in L^\infty(\mathcal{X})$ such that $R \geq r_0$ μ-almost everywhere for some $r_0 > 0$, then we know $V_\varepsilon \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)})$ for all $p \in \{0\} \cup [2,\infty)$ as V_ε satisfies the resolvent equation.

Then we may use the local variational inequality to prove regularity results for the non-local variational inequality. As the Dirichlet form $(\mathscr{E}^{X\times Y}, \mathrm{Dom}(\mathscr{E}^{X\times Y}))$ is local,

$$\mathscr{E}^{X\times Y}(v^-, v^-) = \mathscr{E}^{X\times Y}(v, v^-),$$

which is not true for $\mathscr{E}^\psi$. This fact permits us to prove regularity of the function

V using similar techniques to those used in the local situation as detailed in [9, Chapter 3].

Theorem 6.4.2. *Let $V \in \mathrm{Dom}(\mathscr{E}^\psi)$ be a solution to (6.2.2). If $G \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(2)})$ then $V \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(2)})$.*

Proof. Let $u_G = \varphi_{(-\mathcal{L}_x)} G \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ and as $V_\varepsilon \in \mathrm{Dom}(\mathscr{E}^\psi)$, $u_\varepsilon \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$. As $u_\varepsilon, u_G \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$, we know $(u_\varepsilon - u_G)^- \in \mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$. Therefore,

$$\begin{aligned}
&\mathscr{E}^{X\times Y}((u_\varepsilon - u_G)^-, (u_\varepsilon - u_G)^-) + \langle R(u_\varepsilon(\cdot,0) - u_G(\cdot,0))^-, (u_\varepsilon(\cdot,0) - u_G(\cdot,0))^-\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad + \frac{1}{\varepsilon}\langle (u_\varepsilon(\cdot,0) - u_G(\cdot,0))^-, (u_\varepsilon(\cdot,0) - u_G(\cdot,0))^-\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad = \mathscr{E}^{X\times Y}((u_\varepsilon - u_G), (u_\varepsilon - u_G)^-) + \langle R(u_\varepsilon(\cdot,0) - G), (u_\varepsilon(\cdot,0) - G)^-\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad\quad + \frac{1}{\varepsilon}\langle (u_\varepsilon(\cdot,0) - G)^-, (u_\varepsilon(\cdot,0) - G)^-\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad = -\mathscr{E}^{X\times Y}(u_G, (u_\varepsilon - u_G)^-) - \langle RG, (u_\varepsilon(\cdot,0) - G)^-\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad = -\mathscr{E}^{\psi}(G, (u_\varepsilon(\cdot,0) - G)^-) - \langle RG, (u_\varepsilon(\cdot,0) - G)^-\rangle_{L^2(\mathcal{X},\mu)} \\
&\qquad = \langle -\psi(-\mathcal{L}_x)G, (V_\varepsilon - G)^-\rangle_{L^2(\mathcal{X},\mu)} - \langle RG, (V_\varepsilon - G)^-\rangle_{L^2(\mathcal{X},\mu)},
\end{aligned}$$

noting that $\mathscr{E}^{X\times Y}(u_G, (u_\varepsilon - u_G)^-) = \mathscr{E}^\psi(G, (u_\varepsilon(\cdot,0) - G)^-)$ due to Remark 5.4.3. Therefore,

$$\frac{1}{\varepsilon}\|(u_\varepsilon(\cdot,0) - u_G(\cdot,0))^-\|^2_{L^2(\mathcal{X},\mu)} \leq \langle -\psi(-\mathcal{L}_x)G - RG, (u_\varepsilon(\cdot,0) - G)^-\rangle_{L^2(\mathcal{X},\mu)},$$

and hence,

$$\frac{1}{\varepsilon}\|(V_\varepsilon - G)^-\|_{L^2(\mathcal{X},\mu)} \leq C\| - \psi(-\mathcal{L}_x)G - RG\|_{L^2(\mathcal{X},\mu)}.$$

Therefore,

$$\| - \psi(-\mathcal{L}_x)V_\varepsilon\|_{L^2(\mathcal{X},\mu)} \leq \|RV_\varepsilon\|_{L^2(\mathcal{X},\mu)} + \frac{1}{\varepsilon}\|(V_\varepsilon - G)^-\|_{L^2(\mathcal{X},\mu)} \leq C,$$

so there is a weakly convergent subsequence in $\mathrm{Dom}(\psi(-\mathcal{L}_x)^{(2)})$ and its limit $V \in \mathrm{Dom}(\psi(-\mathcal{L}_x)^{(2)})$. □

If we impose additional assumptions on the Dirichlet form, we can prove the value function is in the domain of the $L^p(\mathcal{X},\mu)$-generator of the subordinated semigroup. We now consider the special case where the Dirichlet form is given by

$$\mathscr{E}^X(u,v) = \int_{\mathcal{X}} (\Gamma(x)\nabla_x u)\cdot(\nabla_x v)\mu(\mathrm{d}x),$$

with domain $\mathrm{Dom}(\mathscr{E}^X)$ consisting of $u \in L^2(\mathcal{X},\mu)$ which are weakly differentiable and $\mathscr{E}^X(u,u) < \infty$. We also assume that $E_m = [0,l] \cap [0,r)$ (so m corresponds to a diffusion) so that the Dirichlet form $(\mathscr{E}^Y, \mathrm{Dom}(\mathscr{E}^Y))$ is given by $(\mathscr{D}, \mathrm{Dom}(\mathscr{D}))$.

Theorem 6.4.3. *Let $p \in (2,\infty)$ and suppose $G \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(2)}) \cap \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)})$ and is bounded. Then,*

$$V \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)}).$$

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ (so $q \in (1,2)$) and set $\phi = (u_\varepsilon - u_G)$. Using the semigroup representation for each $y \in E_m$ and using that $V_\varepsilon \in L^\infty(\mathcal{X},\mu)$,

$$\|\phi(\cdot,y)\|_{L^\infty(\mathcal{X},\mu)} \leq \int_{[0,\infty)} \|P_t^{(0)}(V_\varepsilon - G)\|_{L^\infty(\mathcal{X},\mu)} \mathbb{P}_y[H_0 \in \mathrm{d}t] \leq \|V_\varepsilon - G\|_{L^\infty(\mathcal{X},\mu)} < \infty.$$

As $\phi(\cdot,y) \in L^p(\mathcal{X},\mu)$, $(\phi^-(\cdot,y))^{p-1} \in L^q(\mathcal{X},\mu) \cap L^\infty(\mathcal{X},\mu) \subset L^2(\mathcal{X},\mu)$. As $\phi^-(\cdot,y) \in \mathrm{Dom}(\mathscr{E}^X)$ for m-a.e. $y \in E_m$ we know $(\phi^-(\cdot,y))^{p-1}$ is weakly differentiable for m-a.e. $y \in E_m$ and as

$$\nabla_x[(\phi^-(\cdot,y))^{p-1}] = (p-1)(\phi^-(\cdot,y))^{p-2}\nabla_x(\phi^-)(\cdot,y),$$

we have

$$\begin{aligned}
&\mathscr{E}^X((\phi^-(\cdot,y))^{p-1}, (\phi^-(\cdot,y))^{p-1}) \\
&\quad = (p-1)^2 \int_{\mathcal{X}} (\phi^-(x,y))^{2p-4} (\Gamma(x)\nabla_x\phi^-)(x,y) \cdot (\nabla_x\phi^-)(x,y)\mu(\mathrm{d}x) \\
&\quad \leq C \int_{\mathcal{X}} (\Gamma(x)\nabla_x\phi^-)(x,y) \cdot (\nabla_x\phi^-)(x,y)\mu(\mathrm{d}x) \\
&\quad < \infty,
\end{aligned}$$

so $(\phi^-(\cdot,y))^{p-1} \in \mathrm{Dom}(\mathscr{E}^X)$ for each $y \in E_m$ and we have,

$$\begin{aligned}
&\mathscr{E}^X(\phi(\cdot,y), (\phi^-(\cdot,y))^{p-1}) \\
&\quad = \mathscr{E}^X(\phi^-(\cdot,y), (\phi^-(\cdot,y))^{p-1}) \\
&\quad = (p-1) \int_{\mathcal{X}} |\phi^-(x,y)|^{p-2} \big(\Gamma(x)\nabla_x\phi^-\big)(x,y)) \cdot (\nabla_x\phi^-)(x,y)\mu(\mathrm{d}x) \\
&\quad \geq 0.
\end{aligned}$$

For μ-a.e. $x \in \mathcal{X}$, $\phi^-(x,\cdot) \in \mathrm{Dom}(\mathscr{E}^Y) = \mathrm{Dom}(\mathscr{D})$ so $(\phi^-(x,\cdot))^{p-1}$ is absolutely continuous and

$$\begin{aligned}
&\mathscr{D}((\phi^-(x,\cdot))^{p-1},(\phi^-(x,\cdot))^{p-1})\\
&\quad=(p-1)^2\int_0^l(\phi^-(x,y))^{2p-4}|(\partial_y\phi^-)(x,y)|^2\,\mathrm{d}y\\
&\quad<\infty,
\end{aligned}$$

(and if $l+m([0,l))<\infty$ and $r<\infty$, $(p-1)^2(\phi^-(x,l))^{2p-4}|\partial_y\phi^-(x,l)|^2<\infty$) so $(\phi^-(x,\cdot))^{p-1}\in\mathrm{Dom}(\mathscr{E}^Y)$ μ-a.e. $x\in\mathcal{X}$ and

$$\mathscr{E}^Y(\phi^-(x,\cdot),(\phi^-(x,\cdot))^{p-1})=(p-1)\int_0^l|\phi^-(x,y)|^{p-2}|(\partial_y\phi^-)(x,y)|^2\,\mathrm{d}y\geq 0,$$

(and if $l+m([0,l))<\infty$ and $r<\infty$, $(p-1)(\phi^-(x,l))^{p-2}|\partial_y\phi^-(x,l)|^2\geq 0$). Therefore, $(\phi^-)^{p-1}\in\mathrm{Dom}_{loc}(\mathscr{E}^{X\times Y})$ and

$$\begin{aligned}
&\mathscr{E}^{X\times Y}(\phi,(\phi^-)^{p-1})\\
&\quad=\mathscr{E}^{X\times Y}(\phi^-,(\phi^-)^{p-1})\\
&\quad=\int_{E_m}\mathscr{E}^X(\phi^-(\cdot,y),(\phi^-(\cdot,y))^{p-1})m(\mathrm{d}y)+\int_{\mathcal{X}}\mathscr{E}^Y(\phi^-(x,\cdot),(\phi^-(x,\cdot))^{p-1})\mu(\mathrm{d}x)\\
&\quad\geq 0.
\end{aligned}$$

As u_ε satisfies (6.3.1), we have

$$\begin{aligned}
&\mathscr{E}^{X\times Y}(u_\varepsilon-u_G,((u_\varepsilon-u_G)^-)^{p-1})+\langle R(u_\varepsilon(\cdot,0)-u_G(\cdot,0)),((u_\varepsilon-u_G)^-)^{p-1}(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)}\\
&\quad+\frac{1}{\varepsilon}\langle(u_\varepsilon(\cdot,0)-u_G(\cdot,0))^-,((u_\varepsilon-u_G)^-)^{p-1}(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)}\\
&\quad=-\mathscr{E}^{X\times Y}(u_G,((u_\varepsilon-u_G)^-)^{p-1})-\langle Ru_G(\cdot,0),((u_\varepsilon-u_G)^-)^{p-1}(\cdot,0)\rangle_{L^2(\mathcal{X},\mu)}\\
&\quad=\langle-\psi(-\mathcal{L}_x)G-RG,((V_\varepsilon-G)^-)^{p-1}\rangle_{L^2(\mathcal{X},\mu)},
\end{aligned}$$

where the final equality follows by Remark 5.4.3. Thus,

$$\begin{aligned}
\frac{1}{\varepsilon}\|(V_\varepsilon-G)^-\|^p_{L^p(\mathcal{X},\mu)}&\leq\langle\psi(-\mathcal{L}_x)G+RG,((V_\varepsilon-G)^-)^{p-1})\rangle_{L^2(\mathcal{X},\mu)}\\
&\leq\|-\psi(-\mathcal{L}_x)G-RG\|_{L^p(\mathcal{X},\mu)}\|(V_\varepsilon-G)^-\|^{p/q}_{L^p(\mathcal{X},\mu)},
\end{aligned}$$

by Hölder's inequality. As $p-\frac{p}{q}=1$ we have,

$$\|-\psi(-\mathcal{L}_x)V_\varepsilon-RV_\varepsilon\|_{L^p(\mathcal{X},\mu)}=\frac{1}{\varepsilon}\|(V_\varepsilon-G)^-\|_{L^p(\mathcal{X},\mu)}\leq\|-\psi(-\mathcal{L}_x)G-RG\|_{L^p(\mathcal{X},\mu)},$$

and so there is a weakly convergent subsequence of $(V_\varepsilon)_{\varepsilon>0}$ in $\mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)})$ and so its limit $V \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)})$. $\square$

This theorem can be interpreted as a sort of regularity result for the function V. In the local analogue of this theorem where the operator $-\psi(-\mathcal{L}_x)$ is replaced a second order elliptic differential operator (see [9, p.206-7]), it is shown that $V \in W^{2,p}(\mathcal{X}) = \mathrm{Dom}(\mathcal{L}_x^{(p)})$ and so by Sobolev embedding, $V \in C^{1,\gamma}(\overline{\mathcal{X}})$ for all $\gamma < 1$. Therefore, to obtain similar regularity results for the non-local problem, we need similar Sobolev embedding-type results for the spaces $\mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)})$. By considering Example 6.4.1, we may apply the results for analytic operators from Chapter 2 to obtain the following corollary.

Corollary 6.4.4. *Suppose the conditions of Example 6.4.1 are satisfied and suppose there is $\beta \in (0,1)$ such that the Krein string satisfies $m(y) \asymp y^{1/\beta-1}$ as $y \to 0$. Then $V \in C^{2\beta-\frac{d}{p}}(\mathbb{R}^d)$ for all $p \geq 2$.*

Proof. By Proposition 3.3.5, the corresponding complete Bernstein function satisfies $\psi(\lambda) \asymp \lambda^\beta$ as $\lambda \to \infty$ and hence $\underline{\mathrm{ind}}(\psi) \leq \beta < 1$. As $V \in \mathrm{Dom}(-\psi(-\mathcal{L}_x)^{(p)})$ for any $p \geq 2$, the result follows by Corollary 2.2.13. $\square$

Chapter 7

Further Areas of Interest

To conclude this thesis, we indicate some further areas of interest.

7.1 Local Representations for Time-Fractional Problems

Although we have only considered non-local operators that correspond to subordinated diffusions, other types of non-local equations can be considered from the perspective of the Krein correspondence. For example, if we consider the time-fractional diffusion,

$$\begin{cases} D^{\beta}_{0+*}u(t,x) = \frac{1}{2}\Delta_x u(t,x) & \text{for } (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = f(x) & \text{for } x \in \mathbb{R}^d, \end{cases}$$

where D^{β}_{0+*} denotes the Caputo fractional derivative of order $\beta \in (0,1)$,

$$D^{\beta}_{0+*}f(t) = \frac{1}{\Gamma(-\beta)} \int_0^t \frac{f(t-s) - f(t)}{s^{1+\beta}} \, \mathrm{d}s + \frac{(f(t) - f(0))}{\Gamma(1-\beta)} \int_t^\infty \frac{1}{s^{1+\beta}} \, \mathrm{d}s,$$

then u has stochastic representation

$$u(t,x) = \mathbb{E}[f(B^x_{E_t})],$$

where $(B^x_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion started at $x \in \mathbb{R}^d$ and $E_t = \inf\{s > 0 : T^{(\beta)}_s > t\}$ where $(T^{(\beta)}_t)_{t\geq 0}$ a β-stable subordinator independent of $(B_t)_{t\geq 0}$ (see [19]). However, we have seen that $(E_t)_{t\geq 0} = (L^0_t(Y^{(\beta)}))_{t\geq 0}$ where $(Y^{(\beta)}_t)_{t\geq 0}$ is a Bessel diffusion.

For complete Bernstein functions η and ψ corresponding to gap diffusions $(Y^{\eta}_t)_{t\geq 0}$ and $(Y^{\psi}_t)_{t\geq 0}$ respectively. If we assume $\eta(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\nu(t) \, \mathrm{d}t$, we can define the operator D^{ν}_{0+*} in a similar way to the Caputo case,

$$D^{\nu}_{0+*}f(t) = \int_0^t (f(t-s) - f(t))\nu(s)\,\mathrm{d}s + (f(0) - f(t))\int_t^{\infty} \nu(s)\,\mathrm{d}s.$$

Then an interesting area for further study would be whether the anomalous diffusions equation,

$$\begin{cases} D^{\nu}_{0+*}u(t,x) = -\psi(-\mathcal{L}_x)u(t,x) & \text{for } (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = f(x) & \text{for } x \in \mathbb{R}^d, \end{cases}$$

associated with the process $((-E^{\eta}_t, X(T^{\psi}_t)))_{t\geq 0}$ could be studied via an extension method.

7.2 Further Regularity of the Value Function

In the fractional Laplacian case, the next step towards proving optimal regularity of the value function is to note that if V satisfies the fractional obstacle problem, then formally the function $W = -(-\Delta_x)^{\alpha/2}V$, should satisfy

$$\begin{cases} -(-\Delta_x)^{1-\alpha/2}W = \Delta_x G & \text{in } \{V = G\}, \\ W = RV & \text{in } \{V > G\}, \end{cases}$$

and so provided with have regularity for this equation, we shall have regularity for V. There are numerous technicalities which need to be dealt with in order to show this (see [66]).

In the general case we can extend this heuristic reasoning by noting that $\lambda^{1-\alpha/2}$ is the conjugate complete Bernstein function of $\lambda^{\alpha/2}$. So for any complete Bernstein function ψ, the conjugate Bernstein function $\psi^c(\lambda) = \frac{\lambda}{\psi(\lambda)}$ is a also a complete Bernstein function and hence there exists a measure m^c in Krein correspondence with ψ^c. Furthermore, by setting $W = -\psi(-\mathcal{L}_x)V$, we can see formally that

$$\begin{cases} -\psi^c(-\mathcal{L}_x)W = -\mathcal{L}_x G & \text{in } \{V = G\}, \\ W = RV & \text{in } \{V > G\}, \end{cases}$$

and so it is natural to ask whether the same method for regularity for fractional obstacle problem from [66] can be adapted to this more general situation.

Appendix A

Numerical Simulations

In this appendix we provide Python simulations of some of the examples found in this thesis [1].

A.1 Bessel Process Example

The first program simulates the sample paths of a one-dimensional Brownian motion $(X_t)_{t\geq 0}$, a rescaled Bessel process $(Y_t)_{t\geq 0}$ and its corresponding local time at zero allowing us to visualise how changing the dimension of the Bessel process effects the corresponding trace process. The program relies on the simulation algorithm found in [53] which notes that the values of a squared Bessel process $(Q_0, Q_1 \dots, Q_n)$ at times $0 = t_0 < t_1 < \cdots < t_n$ can be simulated by first simulating a Poisson random variable

$$P_n \sim \mathrm{Pois}\Big(\frac{Q_{n-1}}{2(t_n - t_{n-1})}\Big),$$

and then simulating Q_n as a Gamma distributed random variable,

$$Q_n \sim \Gamma\Big(P_n + \frac{\delta}{2}, \frac{1}{2(t_n - t_{n-1})}\Big).$$

By setting $Y_n = c_\alpha Q_n^{(2-\delta)/2}$, we can approximate the local time of the Bessel process by

$$L^0_{t_i}(Y) \approx \frac{1}{m^{(2-\delta)}([0,\delta])} \sum_{i=0}^{i} \mathbf{1}_{[0,\delta]}(Y_{i-1})(t_i - t_{i-1}),$$

[1]The code found in this appendix can be downloaded at `https://github.com/JA-H/Krein-Correspondence-Simulations`

where $m^{(2-\delta)}(\mathrm{d}y)$ is the measure from Example 3.5.1. By simulating a Brownian motion $(X_t)_{t\geq 0}$, we can plot the trace process $(X_{T_t})_{t\geq 0}$ by plotting $((L_t^0(Y), X_t))_{t\geq 0}$. In the following images we simulate two rescaled Bessel process sample paths, one of dimension $\delta = 0.8$ and the other of dimension $\delta = 1.2$ and see the difference in behaviour of the subordinated process. It should be noted that as the dimension approaches 2, it becomes more computationally expensive to simulate the local time at zero as the Bessel process visits zero less frequently whereas when the dimension approaches 0, we have to decrease the size of time step in order to capture the non-local nature of the subordinated process.

A.1.1 Python Code

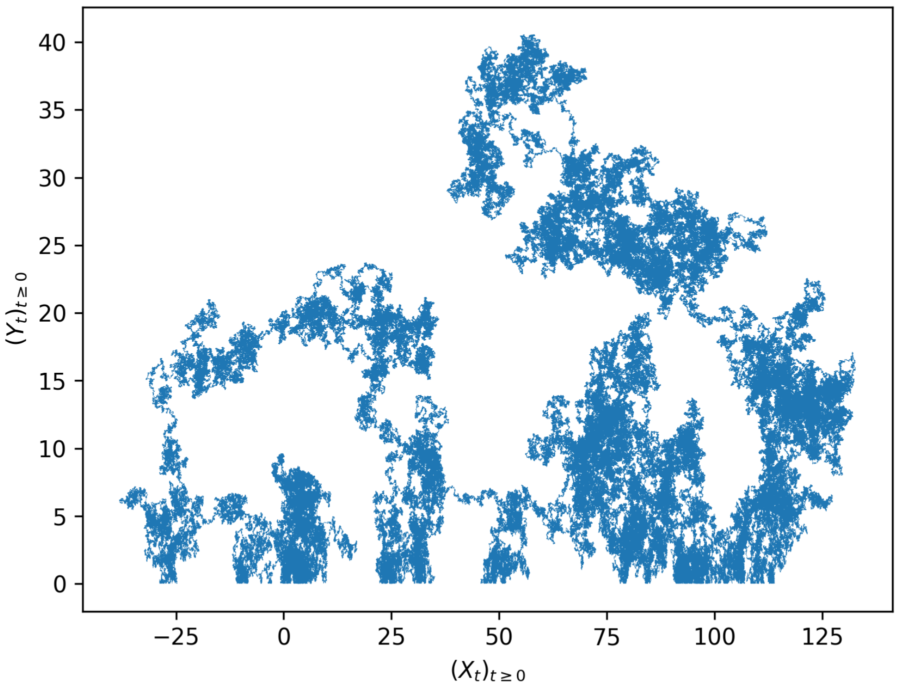
```
import matplotlib.pyplot as plt
import numpy as np
from scipy.special import gamma

class Bessel:
    def __init__(self, T, dt, dim):
        self.T = T #Length of time interval
        self.dt = dt #Length of time step
        self.Num = round(T/dt) #Number of time steps
        self.dim = dim #Dimension of the Bessel process

        self.alpha = 2.0 - dim #Alpha corresponding to the dimension of the Bessel process
        self.c_alpha = ( (2.0**(-self.alpha) )*np.absolute( gamma(-self.alpha/2.0) ) )/gamma(self.alpha/2.0) \
                  #Constant that appears in corresponding speed measure of the rescaled Bessel process

    def Squared_Bessel_Process(self):
        """In order to simulate a Bessel process, we use the algorithm \
            found in 'Makarov, & Glew. Exact simulation of Bessel diffusions'. """
        Q = np.zeros(self.Num) # Memory for Squared Bessel process
        error = 1E-10

        for i in range(self.Num - 1):
            if Q[i] <= error:
                Q[i + 1] = np.random.gamma(self.dim/2.0, 2.0*self.dt )
            else:
                Y = np.random.poisson( Q[i]/(2.0*self.dt)  )
```



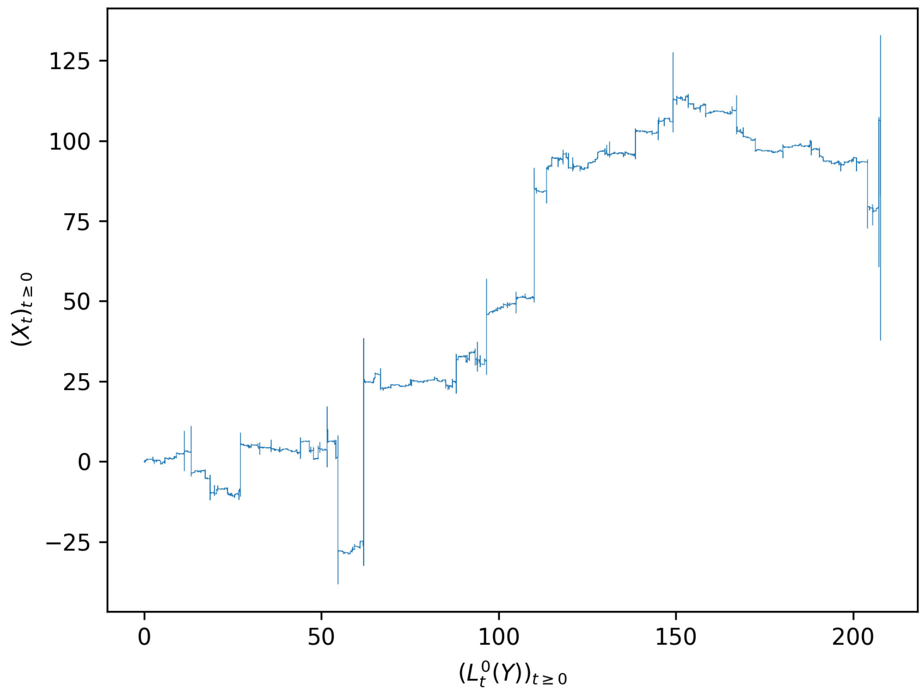


Figure A.1: $\delta = 1.2$

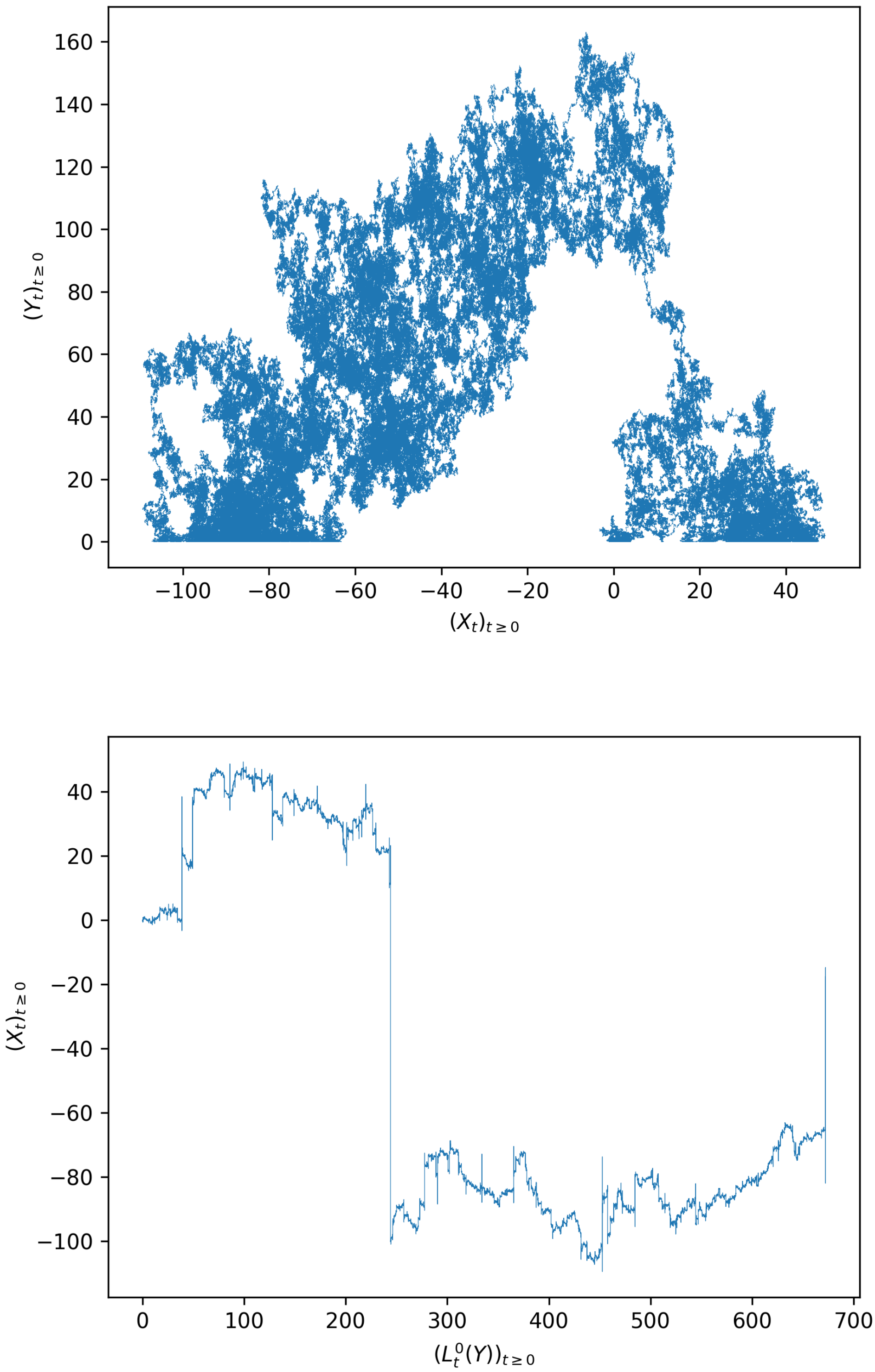


Figure A.2: $\delta = 0.8$

```
                Q[i + 1] = np.random.gamma( Y + self.dim/2.0, 2.0*self.dt )
        return Q

    def Bessel_Process(self):
        return np.sqrt( self.Squared_Bessel_Process() )

    def Rescaled_Bessel_Process(self):
        return (self.c_alpha)*(self.Bessel_Process() )**(self.alpha)

    def Local_Time(self, delta, Y):
        """This function takes a time T, a time increment dt, a rescaled Bessel \
            Process Y with dimension dim and a small value delta and returns the \
            approximate local time of the sample path."""
        alpha = self.alpha
        dim = self.dim
        c_alpha = self.c_alpha

        m_delta = ( 1.0/( dim*alpha*(c_alpha)**(2.0/alpha) ) )*delta**( dim/alpha )

        Lt = np.zeros(self.Num)

        for i in range(self.Num - 1):
            if 0.0 <= Y[i] <= delta:
                Lt[i+1] = (1.0/m_delta)*self.dt
            else:
                Lt[i+1] = 0.0

        Lt = np.cumsum(Lt)

        return Lt

def Brownian_Motion(T, dt):
    """This function takes a time T and a time increment dt and returns an \
        array of the values of the X_t process at these time increments."""

    N = round(T/dt) # Number of time-steps
    X = np.random.standard_normal(size = N)
    X = np.cumsum(X)*np.sqrt(dt)
    return X

```

```python
def main(T, dt, dim, delta):
    X = Brownian_Motion(T, dt)

    Bes = Bessel(T, dt, dim)

    Y = Bes.Rescaled_Bessel_Process()
    L = Bes.Local_Time(delta, Y)

    #Pair Process Plot
    plt.plot(X, Y, linewidth=0.1)
    plt.xlabel(r'$(X_t)_{t \geq 0}$')
    plt.ylabel(r'$(Y_t)_{t \geq 0}$')
    plt.savefig('./Plot_of_Pair_Process_alpha=' + str(Bes.alpha) + '.png', dpi=300)
    plt.show()
    plt.close()

    #Subordinated Process Plot
    plt.plot(L, X, linewidth=0.3)
    plt.xlabel(r'$(L^0_t(Y))_{t \geq 0}$')
    plt.ylabel(r'$(X_t)_{t \geq 0}$')
    plt.savefig('./Plot_of_Trace_Process_alpha='+ str(Bes.alpha) + '.png', dpi=300)
    plt.show()
    plt.close()

    return 0

main(T = 10.0**4, dt = 1E-3, dim = 0.8, delta = 1E-3)
main(T = 10.0**4, dt = 1E-3, dim = 1.2, delta = 5E-2)
```

A.2 Simulating the Krein Correspondence

In this program, we develop a class to simulate the Krein correspondence in the special case where the Krein string is given as a weighted sum of Dirac delta measures:

$$m(\mathrm{d}y) = \sum_{i=0}^{N} m_i \delta_{y_i}(\mathrm{d}y),$$

where $0 = y_0 < y_1 < \cdots < y_N = r < \infty$, $m_i > 0$ for all $0 \leq i \leq N$ and $m_N = \infty$ (for simplicity, we assume there is killing at r although the methods in this section can be adapted to other boundary behaviours). As we have noted, this example is useful as we may approximate any Krein string by a Krein string of this form as detailed in Example 3.5.3.

We can calculate the extension function explicitly corresponding to this Krein string by solving the corresponding difference equation. By considering Example 3.5.3, we see that the extension function φ_λ satisfies

$$-\varphi_\lambda(y_{i-1}) + \left(1 + m_i\lambda(y_i - y_{i-1}) + \left(\frac{y_i - y_{i-1}}{y_{i+1} - y_i}\right)\right)\varphi_\lambda(y_i) - \left(\frac{y_i - y_{i-1}}{y_{i+1} - y_i}\right)\varphi_\lambda(y_{i+1}) = 0$$

for $1 \leq i \leq N-1$ with $\varphi_\lambda(0) = 1$ and $\varphi_\lambda(r) = 0$. By translating this into a matrix equation, we can calculate $\{\varphi_\lambda(y_i), 0 \leq i \leq N\}$ by solving the corresponding linear system.

We may also calculate the complete Bernstein function corresponding to m in two different ways. In the first way, we exploit the explicit representation for the corresponding complete Bernstein function as a continued fraction. In order to calculate this continued fraction representation, we solve the recurrence formulas

$$A_i = b_i A_{i-1} + A_{i-2}, \qquad B_i = b_i B_{i-1} + B_{i-2},$$

where for each fixed $\lambda > 0$,

$$b_i = \begin{cases} m_i\lambda & \text{for even } i, \\ y_{i/2} - y_{i/2-1} & \text{for odd } i, \end{cases}$$

with initial conditions $A_{-1} = 1$, $A_0 = m_0\lambda$, $B_{-1} = 0$ and $B_0 = 1$ so that we have

$$\frac{A_{2N-1}}{B_{2N-1}} = m_0\lambda + \cfrac{1}{(y_1-y_0)+\cfrac{1}{m_1\lambda+\cfrac{1}{\ddots+\cfrac{1}{y_N-y_{N-1}}}}}.$$

The second way uses the fact we calculate the extension function explicitly and so we can use the fact that $\psi(\lambda) = m_0\lambda + (\frac{1-\varphi_\lambda(y_1)}{y_1})$. As we can calculate the complete Bernstein function corresponding to m, it is possible to calculate the distribution of the corresponding subordinator T_t at a fixed time $t > 0$ by numerically inverting the Laplace transform of the function $\lambda \mapsto e^{-t\psi(\lambda)}$ (see [1]). To do this, we use the mpmath library for Python [36]. However, this method does not seem practical for most simulation purposes as numerically inverting the Laplace transform is very computationally expensive.

To test the implementation of the Krein correspondence we consider the Krein string given by $m(\mathrm{d}y) = \mathbf{1}_{[0,r)}(y)\,\mathrm{d}y + \infty\delta_r(\mathrm{d}y)$ with corresponding extension function

$$\varphi_\lambda(y) = \frac{1}{1-e^{-2r\sqrt{\lambda}}}\left(e^{-y\sqrt{\lambda}} - e^{y\sqrt{\lambda}-2r\sqrt{\lambda}}\right), \tag{A.2.1}$$

and complete Bernstein function

$$\psi(\lambda) = \sqrt{\lambda}\left(\frac{1+e^{-2r\sqrt{\lambda}}}{1-e^{-2r\sqrt{\lambda}}}\right). \tag{A.2.2}$$

To approximate this speed measure, we consider a partition of $[0, r]$ given by $\mathcal{P}_n = \{\frac{hr}{n} : 0 \le h \le n\}$ and define $m_n(\mathrm{d}y) = \frac{r}{n}\sum_{k=0}^{n-1}\delta_{hr/n}(\mathrm{d}y) + \infty\delta_r(\mathrm{d}y)$.

A.2.1 Python Code

We first program a class for the test case.

```
import numpy as np
import matplotlib.pyplot as plt

class Krein_Brownian_Killed:
    """
    This class contains the formulas for the extension function and \
        Laplace exponent of a reflected Brownian motion in [0, R] \
        killed upon hitting R.
    """
    def __init__(self, R, error = 1E-10):
```

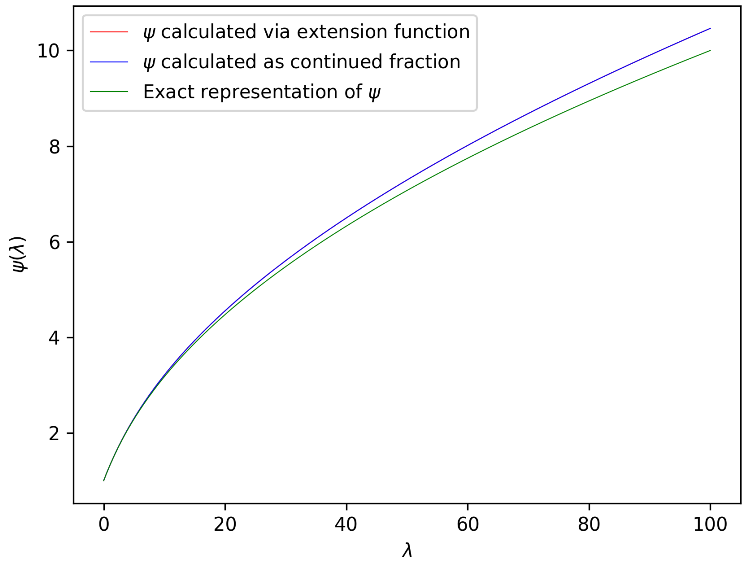


Figure A.3: A comparison between the explicit formula for ψ and the corresponding approximations. Note that the finite difference scheme and the continued fraction method lead to the same result.

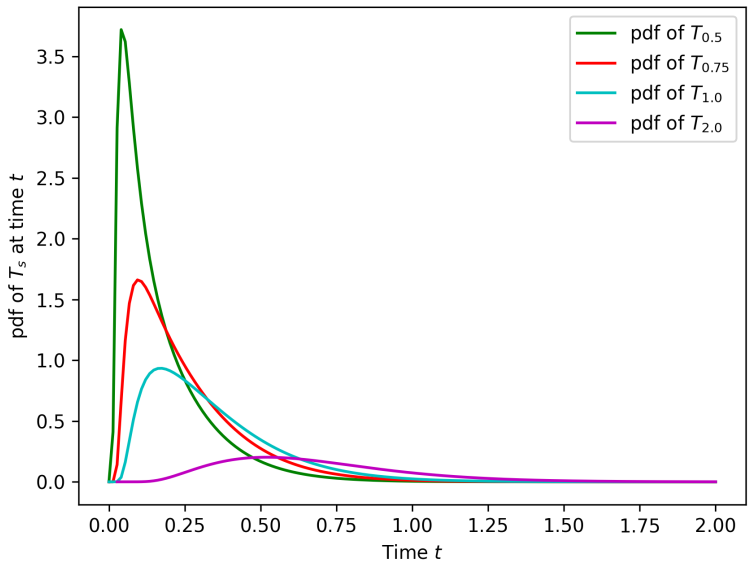


Figure A.4: The probability density function of T_t corresponding to the point measure for $t = 0.5, 0.75, 1.0$ and 2.0.

```
        self.R = R
        self.error = error

    def Extension_Function(self, xi, y):
        if xi <= self.error:
            return 1.0
        else:
            return ( np.exp(-y*np.sqrt(xi)) - np.exp( y*np.sqrt(xi) - 2.0*self.R*np.sqrt(xi) )  )/(1.0 - np.exp(-2.0*self.R*np.sqrt(xi)) )

    def Laplace_Exponent(self, xi):
        if xi <= self.error:
            return 1.0/self.R
        else:
            return np.sqrt(xi)*(1 + np.exp(-2.0*self.R*np.sqrt(xi) ))/(1 - np.exp(-2.0*self.R*np.sqrt(xi) ))
```

```
import numpy as np
import matplotlib.pyplot as plt
import mpmath as mpm
import time

import Krein_Brownian_Killed_Class as BMKill

class Krein_Corr:
    """This class simulates the Krein correspondence for a Krein string given by a sum of \
        weighted Dirac measures, which may be used to approximate any given Krein string. \
        To initialise the class, we require two numpy arrays, y and m, where y is a partition \
        of an interval [0, R] (with y[0] = 0 and y[-1] = R) and m corresponds to the Dirac point \
        measure on this partition. For simplicity, we assume that (in the notation of the thesis) \
        that L = R < infty so the corresponding gap diffusion is killed upon hitting R.
    """

    def __init__(self, y, m):
        self.y = y # Points where the Krein string is defined
        self.m = m # Krein string which we assume is given by \sum_{y_i \in y} m_i\delta_{y_i}(dy)
        self.R = y[-1] #Endpoint which we assuime is killing
        self.drift_coeff = m[0] # This is forced to be positive due to
```

```
 the form of the Krein string

    def Extension_Func(self, xi):
        """
        We use the finite difference approximation of the BVP problem associated with the \
            extension function. This BVP is given by, \
            f''(y) = xi f(y)m(diff y), f(0) = 1, f(R) = 0, \
            for fixed xi in [0, infty).
        """
        y, m = self.y, self.m
        N = y.size

        #RHS of equation defining the Dirichlet boundary condition
        b = np.zeros(N)
        b[0] = 1.0

        #LHS matrix of the difference equation
        A = np.zeros( (N, N) )
        A[0, 0], A[N-1, N-1] = 1, 1

        for i in range(1, N - 1):
            A[i, i - 1] = -1.0
            A[i, i] = (( y[i] - y[i - 1] )/( y[i + 1] - y[i] )) + xi*m[i]*( y[i] - y[i - 1] ) + 1.0
            A[i, i+1] = -(( y[i] - y[i - 1] )/( y[i + 1] - y[i] ))

        varphi = np.linalg.solve(A, b)

        return varphi

    def Laplace_Exponent(self, xi, method = "CtdFrac"):
        """
        In this function we calculate the Laplace exponent at xi associated with the Krein string m via two \
            different methods: first by calculating the derivative at zero of extension function associated \
            with m, the second by directly calculating the continued fraction representation of the complete \
            Bernstein function. We set the default method to be the continued fraction method as the \
            extension method is much slower due to the matrix computations imvolved.
        """
        if method == "FinDiff":
            phi_approx = self.Extension_Func(xi)
```

```
            return (1.0 - phi_approx[1])/self.y[1] + self.m[0]*xi

        elif method == "CtdFrac":
            m, y = self.m, self.y

            A = np.array([1.0, m[0]*xi ])
            B = np.array([ 0.0, 1.0 ])

            for i in range(1, y.size - 1):
                # Convergents
                A_2 = ( y[i]-y[i-1] )*A[1] + A[0]
                A_3 = m[i]*xi*A_2 + A[1]
                A = np.array( [A_2, A_3] )

                B_2 = ( y[i]-y[i-1] )*B[1] + B[0]
                B_3 = m[i]*xi*B_2 + B[1]
                B = np.array( [B_2, B_3] )

                # Renormalisation every 10 iterations
                if i % 10 == 0:
                    A = A/B[1]
                    B = B/B[1]

            psi_xi = ( (y[-1] - y[-2])*A[1] + A[0] )/( (y[-1] - y[-2])*B[1] + B[0] )
            return psi_xi

    def Subordinator_pdf(self, t, T, N):
        """
        In this function, we employ mpmath library to invert the Laplace transform of exp(-t*psi)\
            where psi is the Laplace exponent numerically, giving the pdf of T_t.
        """
        def Laplace_Trans_of_Sub(eta):
            Log_Lap_of_Sub = self.Laplace_Exponent( eta , method = "CtdFrac" )
            mpmLaplace = mpm.convert( Log_Lap_of_Sub )
            return mpm.exp( -t*mpmLaplace )

        times = np.linspace(0.0, T, N)
        sub_dist = np.zeros(N)
        for i in range(N):
            try:
                sub_dist[i] = mpm.invertlaplace( Laplace_Trans_of_Sub, times[i], method = 'talbot' )
```

```python
            except ZeroDivisionError:
                continue

        return sub_dist

def main():
    #Defining the approxiamtion of BM in [0, 1.0] killed upon hitting 1.0.
    N = int(1E2)
    R = 1.0
    y = np.linspace(0.0, R, N)
    m = (R/N)*np.ones(N)

    BM_Example = Krein_Corr(y, m)
    BM_Actual = BMKill.Krein_Brownian_Killed(R)

    xi_N = int(1E4)
    xi_max = 100.0
    xi_values = np.linspace(0.0, xi_max, xi_N)

    phi_exact = np.zeros(xi_N)
    phi_approx = np.zeros(xi_N)
    phi_formula = np.zeros(xi_N)

    tic = time.perf_counter()
    for i in range(xi_N):
        phi_approx[i] = BM_Example.Laplace_Exponent(xi_values[i], "FinDiff")
    toc = time.perf_counter()
    print(f"'FinDiff' took {toc - tic:0.2f} seconds")

    tic = time.perf_counter()
    for i in range(xi_N):
        phi_exact[i] = BM_Example.Laplace_Exponent(xi_values[i], "CtdFrac")
    toc = time.perf_counter()
    print(f"'CtdFrac' took {toc - tic:0.2f} seconds")

    tic = time.perf_counter()
    for i in range(xi_N):
        phi_formula[i] = BM_Actual.Laplace_Exponent(xi_values[i])
    toc = time.perf_counter()
    print(f"Exact formula took {toc - tic:0.2f} seconds")

    plt.plot(xi_values, phi_approx, 'r-', label=r"$\psi$ calculated via extension function", linewidth = 0.5)
```

```
    plt.plot(xi_values, phi_exact, 'b-', label=r"$\psi$ calculated as continued fraction", linewidth = 0.5)
    plt.plot(xi_values, phi_formula,'g-', label=r"Exact representation of $\psi$", linewidth = 0.5)
    plt.xlabel(r"$\lambda$")
    plt.ylabel(r"$\psi(\lambda)$")
    plt.legend()
    plt.savefig("CBF_comparison.png", dpi = 300)
    plt.close()


    T = 2.0
    N_t = 150
    times = np.linspace(0.0, T, N_t)

    tic = time.perf_counter()
    sub_05 = BM_Example.Subordinator_pdf(0.5, T, N_t)
    print("Laplace transform to find pdf of T_{0.5} complete.")

    sub_075 = BM_Example.Subordinator_pdf(0.75, T, N_t)
    print("Laplace transform to find pdf of T_{0.75} complete.")

    sub_1 = BM_Example.Subordinator_pdf(1.0, T, N_t)
    print("Laplace transform to find pdf of T_{1.0} complete.")

    sub_2 = BM_Example.Subordinator_pdf(2.0, T, N_t)
    print("Laplace transform to find pdf of T_{2.0} complete.")

    toc = time.perf_counter()
    print(f"Laplace transforms in {toc - tic:0.4f} seconds")

    plt.plot(times, sub_05, "g-", label=r"pdf of $T_{0.5}$")
    plt.plot(times, sub_075, "r-", label=r"pdf of $T_{0.75}$")
    plt.plot(times, sub_1, "c-", label=r"pdf of $T_{1.0}$")
    plt.plot(times, sub_2, "m-", label=r"pdf of $T_{2.0}$")
    plt.xlabel(r"Time $t$")
    plt.ylabel(r"pdf of $T_s$ at time $t$")
    plt.legend()
    plt.savefig("subordinator_pdfs.png", dpi = 300)
    plt.show()
    plt.close

    return 0

main()
```

A.3 Numerically Solving the Extension Problem

To conclude, we numerically solve a special case of the extension method using the FEniCS programming environment for differential equations [2, 51].

We consider the special case where $\mathcal{L}_x = \partial_x^2$ defined in $L^2([0,\pi])$ with domain $\mathrm{Dom}(\partial_x^2) = W^{2,2}([0,\pi]) \cap W_0^{1,2}([0,\pi])$. The advantage of considering this particular case is that we know that the eigenvectors of the operator $-\partial_x^2$ are $x \mapsto \sin(nx)$ for $n \in \mathbb{N}$ with corresponding eigenvalues n^2 and these eigenvectors form a basis of $L^2([0,\pi])$ by Sturm-Liouville theory [13, Theorem 8.22].

Let $f \in L^2([0,\pi])$ have series representation $f = \sum_{k=1}^{\infty} f_k \sin(kx)$ and let ψ be a complete Bernstein function. Then $f \in \mathrm{Dom}(-\psi(-\partial_x^2))$ if $\sum_{k=1}^{\infty} |\psi(k^2)|^2 f_k^2 < \infty$ in which case

$$-\psi(-\partial_x^2) f = -\sum_{k=1}^{\infty} \psi(k^2) f_k \sin(kx).$$

In this simulation, we give an example of how to solve the equation

$$\begin{cases} -\psi(-\partial_x^2) f = -g & \text{in } (0,\pi), \\ f(x) = 0 & \text{for } x = 0 \text{ or } x = \pi, \end{cases} \tag{A.3.1}$$

numerically using the extension method for the special case where ψ is given by A.2.2 with $r = \pi$ and $g = \sin(x) + 3\sin(3x) + 10\sin(10x)$. We note that the solution to this equation is given explicitly by

$$f(x) = \frac{\sin(x)}{\psi(1^2)} + \frac{3\sin(3x)}{\psi(3^2)} + \frac{10\sin(10x)}{\psi(10^2)}. \tag{A.3.2}$$

As we have seen, the extension problem associated with (A.3.1) is given by

$$\begin{cases} \partial_x^2 u_f + \partial_y^2 u_f = 0 & \text{in } (0,\pi) \times (0,\pi), \\ u_f(x,y) = 0 & \text{for } x = 0,\ x = \pi \text{ or } y = \pi, \\ \partial_y u_f(x,0) = -g & \text{for } x \in (0,\pi), \end{cases} \tag{A.3.3}$$

the explicit solution to this problem being given by

$$u_f(x,y) = \frac{\varphi_{1^2}(y)\sin(x)}{\psi(1^2)} + \frac{3\varphi_{3^2}(y)\sin(3x)}{\psi(3^2)} + \frac{10\varphi_{10^2}(y)\sin(10x)}{\psi(10^2)},$$

where φ_λ is the extension function given by (A.2.1). We solve this numerically using the FEniCS library.

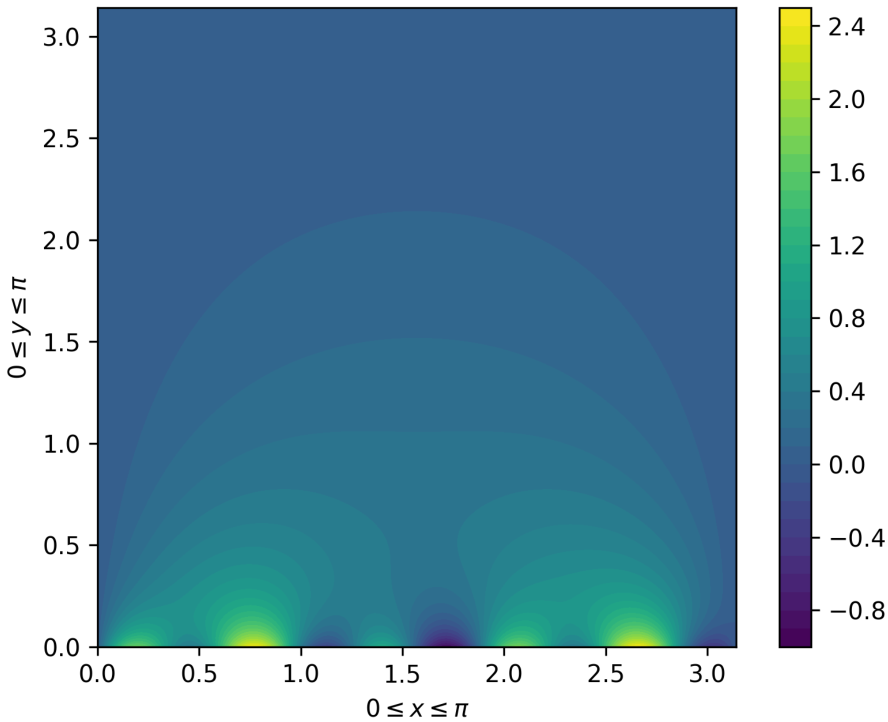


Figure A.5: A colour plot of the approximate solution to the extension method.

A.3.1 Python Code

```
import numpy as np
import fenics as pde
import matplotlib.pyplot as plt

from Krein_Brownian_Killed_Class import *

# PDE domain is (x_0, x_1)x(0, R) with N_x (resp. N_y) points in x (
    resp. y)
x_0, x_1 = 0.0, np.pi
R = np.pi
N_x, N_y = 150, 150

test = Krein_Brownian_Killed(R)

#Creating our mesh and test function space
mesh = pde.RectangleMesh( pde.Point(x_0, 0.0), pde.Point(x_1, R), N_x,
     N_y )
```

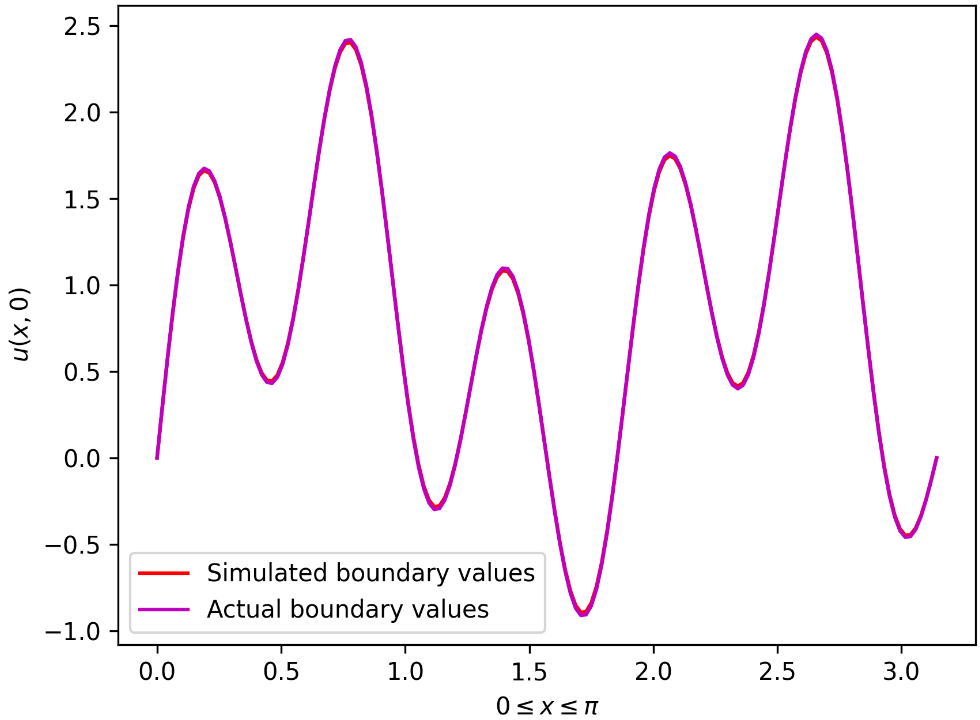


Figure A.6: A comparison between the values of $u_f(\cdot, 0)$ given by the finite element approximation and the explicit formula given above.

```
V = pde.FunctionSpace(mesh, "Lagrange", 1 )

#We have a zero Dirichlet boundary around the boundary except at x =
    0.
tol = 1E-14

def Outer_Boundary(x, on_boundary):
    return (on_boundary) and (x[0] <= x_0 + tol or x[0] >= x_1 - tol
    or x[1] >= R - tol)

#We set this to be a zero boundary condition on the boundary defined
    above.
u0 = pde.Constant( 0.0 )
bc = pde.DirichletBC(V, u0, Outer_Boundary)

#To solve -psi(-d^2)u = f, we define u'(x, 0) = g written in C++
g_str = 'sin(x[0]) + 3.0*sin(3.0*x[0]) + 10.0*sin(10.0*x[0])'

#Solving as in Poisson problem
u = pde.TrialFunction(V)
v = pde.TestFunction(V)
g = pde.Expression( g_str, element = V.ufl_element() )
a = pde.inner( pde.grad(u), pde.grad(v) )*pde.dx
L = g*v*pde.ds

u = pde.Function(V)
pde.solve( a == L, u, bc )
u.set_allow_extrapolation(True)

#Plot of 2D solution
p = pde.plot(u)
vtkfile = pde.File("test_extension.pvd")
vtkfile << u

plt.colorbar(p)
plt.plot()
plt.xlabel(r"$0 \leq x \leq \pi$")
plt.ylabel(r"$0 \leq y \leq \pi$")
plt.savefig("pde.png", dpi = 300)
plt.close()

#Simulated u(x, 0) vs. Actual u(x, 0)
x_bound = np.linspace(x_0, x_1, N_x )
u_bound_val = np.array( [u(x, 0.0) for x in x_bound ] )
act_val = (1.0/test.Laplace_Exponent(1.0**2.0))*np.sin( x_bound ) \
    + (3.0/test.Laplace_Exponent(3.0**2.0))*np.sin( 3.0*x_bound ) \
```

```
    + (10.0/test.Laplace_Exponent(10.0**2.0))*np.sin( 10.0*x_bound )

#Plot of u(x, 0)
plt.plot( x_bound, u_bound_val, "-r", label="Simulated boundary values
    ")
plt.plot( x_bound, act_val, "-m", label="Actual boundary values" )
plt.xlabel(r"$0 \leq x \leq \pi$")
plt.ylabel(r"$u(x, 0)$")
plt.legend()
plt.savefig("boundary_values_of_pde.png", dpi = 300)
plt.show()
plt.close()
```

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