Laws of Scarcity for a Finite Game:
Exact Bounds on Estimations

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Laws of scarcity for a ¯nite game; Exact bounds on estimations¤

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Summary. A \law of scarcity" is that scarceness is rewarded. We demonstrate laws of scarcity for cores and approximate cores of games. Furthermore, we show that equal treatment core payo® vectors satisfy a condition of cyclic monotonicity. Our results are developed for parameterized collections of games and exact bounds on the maximum possible deviation of approximate core payo® vectors from satisfying a law of scarcity are stated in terms of the parameters describing the games. We note that the parameters can, in principle, be estimated.

Keywords and Phrases: monotonicity, cooperative games, clubs, games with side payments (TU games), cyclic monotonicity, law of scarcity, law of demand, approximate cores, e®ective small groups, parameterized collections of games.

JEL Classi¯cation: C71, C78, D41

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1 Laws of scarcity, parameterized collections of games and equal treatment cores

This paper treats cooperative games with many players and provides some characterization results for approximate cores, outcomes that are stable against coalition formation. An advantage of the framework of cooperative games over detailed models of economies is that models of games can accommodate the entire spectrum of games derived from economies with only private goods to games derived from economies with pure public goods. Thus, it is of interest to determine conditions on games ensuring that they are `market-like' { that they satisfy analogues of well known properties of competitive economies. Important papers in this direction include Shubik [21], which introduced the study of large games as models of large private-goods economies, Shapley and Shubik [20], which demonstrated an equivalence between markets and totally balanced games, and Wooders ([26], [27]) demonstrating that games with many players are market games. Further motivation for the framework of cooperative games comes from Buchanan [2], who stressed the need for a general theory, including as extreme cases both purely private and purely public goods economies and the need for \a theory of clubs, a theory of cooperative membership."

The current paper employs the framework of parameterized collections of games and obtains Laws of Scarcity, analogues of the celebrated Laws of Demand and of Supply of general equilibrium theory. Roughly, the Law of Demand states that prices and quantities demanded change in the opposite directions while, with inputs signed negatively, the Law of Supply states that quantities demanded as inputs and produced as outputs change in the same direction as price changes.[1] In the framework of a cooperative game, supply and demand are not distinct concepts. Thus, following [26] we refer to our results for games as Laws of Scarcity. Roughly, our results state that, if almost all gains to collective activities can be realized by groups of players bounded in size, then numbers of players who are similar to each other and core payo®s respond in opposite directions. If player types are thought of a commodity types while payo®s to players are thought of as prices for commodities, our Laws of Scarcity are closely related to comparative statics results for general equilibrium models with quasi-linear utilities. As we discuss in a section relating our paper to the literature, our results extend the literature in several directions.

As in our prior papers on parameterized collections of games,[2] a game is described by certain parameters: (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly e®ective groups of players and their distance from exact e®ectiveness. An equal treatment payo® vector is de¯ned

[1] The Law of Demand therefore rules out \Gi®en goods" or treats compensated demands; see Mas-Colell, Whinston and Green [12], Sections 2.F and 4.C. This volume also provides a very clear exposition and further references.

[2] [6], [7], [8] and [28].

to be a payo® vector that assigns the same payo® to all players of the same approximate type. We show that equal treatment "-cores satisfy the property that numbers of players who are similar to each other and equal treatment "-core payo®s respond in nearly opposite directions; speci¯cally, we establish an exact upper bound on the extent to which equal treatment "-core payo®s may respond in the same direction and this bound will, under some conditions, be small. We actually demonstrate a stronger result { equal treatment "-core vectors and vectors of numbers of players of each approximate type satisfy cyclic monotonicity.[3] In addition to cyclic monotonicity, we demonstrate a closely related comparative statics result: When the relative size of a group of players who are all similar to each other increases, then equal treatment "-core payo®s to members of that group will not signi¯cantly increase and may decrease.

The conditions required on a game to obtain our results are that (i) each player has many close substitutes (a thickness condition) and (ii) almost all gains to collective activities can be realized by groups of players bounded in size (small group e®ectiveness - SGE). The ¯rst condition is frequently employed in economic theory. The second condition may appear to be restrictive, but in fact, if there are su±ciently many players of each type, then per capita boundedness (PCB) { ¯niteness of the supremum of average payo® { and SGE are equivalent.[4] Our results yield explicit bounds, in terms of the parameters describing the games, on the maximal deviation of equal treatment "-core payo®s from satisfying exact monotonicity. Moreover, our framework allows some latitude in the exact speci¯cation of approximate types. These two considerations suggest that in principle our results can be well applied to estimate the e®ects on equal treatment "-core payo®s of changes in the composition of the total player set. Note that all the bounds we obtain are exact, and depend on the parameters describing the games and on the " of the "-core.

For our results characterizing "-cores of games to be interesting, it is important that under some reasonably broad set of conditions, "-cores of large games are nonempty. Since Shapley and Shubik [19] showing nonemptiness of approximate cores of exchange economies with many players and quasi-linear utilities and Wooders [23], [24], showing nonemptiness of approximate cores of game with many players with and without side payments, there has been a number of further results. For parameterized collections of games, such results are demonstrated in [6], [7], [8] and [28]. The interest of our monotonicity results is further enhanced by results showing that approximate cores have the equal treatment property; in this regard, note that [26] shows that approximate cores of large games treat most similar players nearly equally. In research in progress, similar equal treatment results are demonstrated for

[3] Cyclic monotonicity relates to monotonicity in the same way as the Strong Axiom of Revealed Preference relates to the Weak Axiom of Revelaed Preference (see, for example, Richter [13], [14]).

[4] This is shown for \pregames" in [27], Theorem 4. Per capita boundedness and small group e®ectiveness were introduced into the study of large games in Wooders [24],[25] respectively.

parameterized collections of games.

In the next section we define parameterized collections of games. In Section 3, the results are presented. Section 4 consists of an example, applying our results to a matching model with hospitals and interns. Section 5 further relates the current paper to the literature and concludes the paper. In the Appendix we prove that the bounds cannot be tightened.

2 Cooperative games

Let $(N; v)$ be a pair consisting of a finite set N; called the player set, and a function v; called the characteristic function, from subsets of N to the non-negative real numbers with $v(\emptyset) = 0$: The pair $(N; v)$ is a game (with side payments or a TU game). Non-empty subsets of N are called coalitions or groups. A game $(N; v)$ is superadditive if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\{S^k\}$ of S. For the current paper we restrict our attention to superadditive games. This is slightly more restrictive than required, but simplifies notation and shortens the proof.

2.1 Parameterized collections of games

<u>$\delta -$ substitute partitions:</u> In our approach we approximate games with many players, all of whom may be distinct, by games with player types.

Let $(N; v)$ be a game and let $\delta \geq 0$ be a non-negative real number. Informally, a δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. That is, if players in a coalition are replaced by δ-substitutes, the payoff to that coalition changes by no more than δ per capita. Formally, given a partition $\{N[t] : t = 1, ..., T\}$ of N, a permutation τ of N is type consistent if, for any $i \in N$; $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A δ-substitute partition of N is a partition $\{N[t] : t = 1, ..., T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq \delta |S| :$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games, say $(N; v)$ and $(N; v')$, may have the same partitions into δ-substitutes but have no other relationship to each other (in contrast to games derived from a pregame).

<u>(δ,T)- type games.</u> The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game $(N;v)$ is a $(\delta;T)$-type game if there exists a T-member δ-substitute partition $\{N[t] : t = 1,\ldots,T\}$ of N. The set $N[t]$ is interpreted as an approximate type. Players in the same element of a δ-substitute partition are δ-substitutes. When $\delta = 0$, they are exact substitutes.

profiles. Profiles of player sets are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let $(N;v)$ be a game and let $\{N[t] : t = 1,\ldots,T\}$ be a partition of N into δ-substitutes. A profile relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$. Given $S \subset N$ the profile of S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

β-effective B-bounded groups: The following notion formulates the idea of small group effectiveness, SGE, in the context of parameterized collections of games. Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the per capita loss is no more than β.

Let β be a given non-negative real number, and let B be a given integer. A game $(N;v)$ has β-effective B-bounded groups if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $|S^k| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta |S|.$$

When $\beta = 0$, 0-effective B-bounded groups are called strictly effective B-bounded groups.

parametrized collections of games $\Gamma((\delta;T);(\beta;B))$. Let T and B be positive integers, let δ and β be non-negative real numbers. Define

$$\Gamma((\delta;T);(\beta;B))$$

to be the collection of all $(\delta;T)$-type games that have β-effective B-bounded groups.

2.2 Equal treatment ε-core

the core and ε-cores. Let $(N;v)$ be a game and let ε be a nonnegative real number. A payoff vector x is in the ε-core of $(N;v)$ if and only if $\sum_{a\in N} x_a \leq v(N)$ and $\sum_{a\in S} x_a \geq v(S) - \varepsilon|S|$ for all $S \subset N$. When $\varepsilon = 0$, the ε-core is the core.

the equal treatment ε-core. Given nonnegative real numbers ε and δ, we will define the equal treatment ε-core of a game $(N;v)$ relative to a δ-substitute partition $\{N[t]\}$ of the player set as the set of payoff vectors x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

With the definition of the equal treatment ε-core in hand, we can next address monotonicity properties and comparative statics for this concept. In the present paper we simply assume nonemptiness of the equal treatment ε-core of games. With SGE along with per capita boundedness, for $\varepsilon > 0$ this assumption is satisfied for all sufficiently large games in parameterized collections. Such a result appears in [7], [9]. Notice that we treat the equal treatment ε-core as a "stand-in" for the competitive equilibrium in the general context of the cooperative game theory. This motivates our use of the equal treatment ε-core and not the ε-core in the main subject of the present paper.

3 Laws of Scarcity

A technical lemma is required. For $x, y \in \mathbb{R}^T$, let $x \cdot y$ denote the scalar product of x and y, i.e. $x \cdot y := \sum_{t=1}^{T} x_t y_t$.

Lemma. Let $(N;v)$ be in $\Gamma((\delta;T);(\beta;B))$ and let $(S^1;v);(S^2;v)$ be subgames of $(N;v)$. Let $\{N[t]\}$ denote a partition of N into types and, for $k = 1;2;$ let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f^k_t \geq B$ for each k and each t. For each k; let $x^k \in \mathbb{R}^T$ represent a payoff vector in the equal treatment ε-core of $(S^k;v)$. Then

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

Proof: Since $(N;v)$ has β-effective B-bounded groups, there exists a partition $\{G^{1\ell}\}$ of S^1, such that $|G^{1\ell}| \leq B$ for any ℓ and $\sum v(G^{1\ell}) \geq v(S^1) - \beta\|f^1\|$. Let us denote the profiles of $G^{1\ell}$ by g^ℓ: Observe that $\sum g^\ell = f^1$.

Since $f^2_t \geq B$ for each t, it holds that $g^\ell \leq f^2$ for each ℓ. Therefore for each ℓ there exists a subset $G^{2\ell} \subset S^2$ with profile g^ℓ. Observe that since both $G^{1\ell}$ and $G^{2\ell}$ have profile g^ℓ, it holds that $|v(G^{1\ell}) - v(G^{2\ell})| \leq \delta \left\| g^\ell \right\|$. Since x^2 represents a payoff vector in the equal treatment ε-core of $(S^2;v)$ and $G^{2\ell} \subset S^2$ has profile g^ℓ; the total payoff $x^2 \cdot g^\ell$ cannot be improved on by the coalition $G^{2\ell}$ by more than $\varepsilon \left\| g^\ell \right\|$. Thus, for each set $G^{2\ell} \subset S^2$ with profile g^ℓ; it holds that $x^2 \cdot g^\ell \geq v(G^{2\ell}) - \varepsilon \left\| g^\ell \right\| \geq v(G^{1\ell}) - (\varepsilon + \delta) \left\| g^\ell \right\|$. Adding these inequalities we have $x^2 \cdot f^1 \geq \sum v(G^{1\ell}) - (\varepsilon + \delta)\|f^1\|$. It then follows that $x^2 \cdot f^1 \geq v(S^1) - (\varepsilon + \delta + \beta)\|f^1\|$.

Since x^1 represents a payoff vector in the equal treatment ε-core of $(S^1;v)$, $x^1 \cdot f^1$ is feasible for $(S^1;v)$, that is, $x^1 \cdot f^1 \leq v(S^1)$. Combining these inequalities we have $(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta)\|f^1\|$: ■

Now we can state and prove our main results.

3.1 Approximate cyclic monotonicity

We derive an exact bound on the amount by which an approximate core payoff vector for a given game can deviate from satisfying exact cyclic monotonicity. The bound depends on:

- δ, the extent to which players within each of T types may differ from being exact substitutes for each other;
- β, the maximal loss of per capita payoff from restricting effective coalitions to contain no more than B players; and
- ε, a measure of the extent to which the ε-core differs from the core.

Our result is stated both for absolute numbers and for proportions of players of each type. If exact cyclic monotonicity were satisfied, then the right hand sides of the equations (1) and (2) below could both be set equal to zero.

Proposition 1. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), \ldots, (S^K, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f^k_t \geq B$ for each k and each t. For each k, let $x^k \in \mathbb{R}^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + \ldots + (x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 + \ldots + f^K \right\| \quad (1)$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + \ldots + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq K(\varepsilon + \delta + \beta). \quad (2)$$

That is, the equal treatment ε-core correspondence approximately satisfies cyclic monotonicity both in terms of numbers of players of each type and percentages of players of each type.

Proof: From Lemma we have $(x^k - x^{k+1}) \cdot f^k \leq (\varepsilon + \delta + \beta) \left\| f^k \right\|$ for $k = 2, \ldots, K - 1$ and $(x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \left\| f^K \right\|$. Summing these inequalities we get (1).

Alternatively we have $(x^k - x^{k+1}) \cdot \frac{f^k}{\|f^k\|} \leq (\varepsilon + \delta + \beta)$ for $k = 1, \ldots, K - 1$ and $(x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain (2). ■

Remark. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 \right\|.$$

This form of monotonicity is typically called simply monotonicity or weak monotonicity. Note that weak monotonicity does not imply cyclic monotonicity.

Corollary. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2)\cdot(f^1 - f^2) \leq (\varepsilon + \delta + \beta)\left\|f^1 + f^2\right\| \text{ and } (x^1 - x^2)\cdot\left(\frac{f^1}{\|f^1\|} - \frac{f^2}{\|f^2\|}\right) \leq 2(\varepsilon + \delta + \beta).$$

That is, the equal treatment ε-core correspondence is approximately monotonic.

Note that the bound of Proposition 1 and its Corollary holds for any partition of the player set into δ-substitutes.

3.2 Comparative Statics

For $j = 1, \ldots, T$ let us define $e^j \in \mathbb{R}^T$ such that $e^j_l = 1$ for $l = j$ and 0 otherwise. Our comparative statics results relate to changes in the abundances of players of a particular type.

Proposition 2. Let $(N; v)$ be in $\Gamma((\delta; T); (\beta; B))$ and let $(S^1; v); (S^2; v)$ be subgames of $(N; v)$. Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f^k_t \geq B$ for each k and each t. For each k, let $x^k \in \mathbb{R}^T$ represent a payoff vector in the equal treatment ε-core of $(S^k; v)$. Then the following holds:

(A) If $f^2 = f^1 + me^j$ for some positive integer m (i.e., the second game has more players of approximate type j but the same numbers of players of other types) then

$$(x^2_j - x^1_j) \leq (\varepsilon + \delta + \beta)\frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} = (\varepsilon + \delta + \beta)\frac{2\|f^2\| - m}{m}.$$

(B) If $\frac{f^2}{\|f^2\|} = (1 - \mu)\frac{f^1}{\|f^1\|} + \mu e^j$ for some $\mu \in (0, 1)$ (i.e., the second game has proportionally more players of approximate type j but the same proportions between the numbers of players of other types) then

$$(x^2_j - x^1_j) \leq (\varepsilon + \delta + \beta)\frac{2 - \mu}{\mu}.$$

That is, approximately the equal treatment ε-core correspondence provides lower payoffs for players of a type that is more abundant.

Proof: (A): Applying Corollary we get $(x^2 - x^1)\cdot me^j \leq (\varepsilon + \delta + \beta)\|f^1 + f^2\|$. Since $\|f^2\| = \|f^1\| + m$, this inequality implies our first result.

(B): From Lemma we have $(1-\mu)(x^1 - x^2)\cdot\frac{f^1}{\|f^1\|} \leq (1-\mu)(\varepsilon+\delta+\beta)$ and similarly $(x^2 - x^1)\cdot\frac{f^2}{\|f^2\|} \leq (\varepsilon+\delta+\beta)$. Summing these inequalities we obtain $(x^2 - x^1)\cdot(\frac{f^2}{\|f^2\|} - (1-\mu)\frac{f^1}{\|f^1\|}) \leq (2-\mu)(\varepsilon+\delta+\beta)$. Thus we get that $(x^2 - x^1)\cdot\mu e^j \leq (2-\mu)(\varepsilon+\delta+\beta)$. This inequality implies our second result. ■

Obviously, again the bounds provided by the Proposition are independent of the specific partition of the player set into δ-substitutes. Note that all the bounds are exact; see Appendix.

4 Matching hospitals and interns; An example

Given the great importance of matching models (see, for example, Roth and Sotomayor [16] for an excellent study and numerous references to related papers), we present an application of our results to a model of matching interns and hospitals. Our example is highly stylized. For a more complete discussion of the matching interns and hospitals problem, we refer the reader to Roth [15].

The problem consists of the assignment of a set of interns $I = \{1, \ldots, i, \ldots, I\}$ to hospitals. The set of hospitals is $H = \{1, \ldots, h, \ldots, H\}$. The total player set N is given by $N = I \bigcup H$. Each hospital h has a preference ordering over the interns and a maximum number of interns $\overline{I}(h)$ that it wishes to employ. Interns also have preferences over hospitals. We'll assume $\overline{I}(h) \leq 9$ for all $h \in H$. This gives us a bound of 10 on the size of strictly effective groups ($\beta = 0$). For simplicity, we'll assume that both hospitals and interns can be ordered by the real numbers so that players with higher numbers in the ordering are more desirable. The rank held by a player will be referred to as the player's quality. More than one player may share the same rank in the ordering. In fact, we assume that the total payoff to a group consisting of a hospital and no more than nine interns is given by the sum of the rankings attached to the hospital and to the interns. Let us also assume that the rank assigned to any intern is between 0 and 1 and the rank assigned to any hospital is between 1 and 2. Thus, if the hospital is ranked 1.3 for example and is assigned 5 interns of quality .2 each, then the total payoff to that group is 2.3.

Since all interns have qualities in the interval $[0,1)$ and similarly, all hospitals have qualities in the interval $[1,2]$, given any positive real number $\delta = \frac{1}{n}$ for some positive integer n we can partition the interval $[0,2]$ into $2n$ intervals, $[0,\frac{1}{n}), \ldots, [\frac{j-1}{n}, \frac{j}{n}), \ldots, [\frac{2n-1}{n}, 2]$, each of measure $\frac{1}{n}$. Assume that if there is a player with rank in the jth interval, then there are at least 10 players with ranks in the same interval.

Given $\varepsilon \geq 0$, let x^1 represent a payoff vector in the ε-core that treats all interns with ranks in the same interval equally and all hospitals with ranks in the same interval equally (that is, x^1 is equal treatment relative to the given partition of the total player set into types). Let us now increase the abundance of some type of intern that appears in N with rank in the jth interval for some j. We could imagine, for example, that some university training medical students increases the number of type j interns by admitting more students from another country. Let x^2 represent an equal treatment payo® vector in the "-core after the increase in type j interns. It then holds, from result (A) of Proposition 2 that

$$(x^2_j \text{ ¡ } x^1_j) \cdot ('' + \frac{1}{n})\frac{kf^1 + f^2k}{kf^2 \text{ ¡ } f^1k}:$$

Of course this is not the most general application of our results { we could increase the proportions of players of one type by reducing the numbers of players of other types. Then part (B) of our Proposition could be applied.

It is remarkable that our results apply so easily. For this simple sort of example, it is probably the case that a sharper result can be obtained. This is beyond the scope of our current paper, however. Research in progress considers whether sharper results are obtainable with assortative matching of the kind illustrated by this example { that is, where players can be ordered so that players with higher ranks in the orderings are superior in terms of their marginal contributions to coalitions.

Finally, the parameter values that we have used in this example were chosen for convenience and simplicity. In principle, these could be estimated and various questions addressed. For example, are payo®s to interns approximately competitive? Do non-market characteristics such as ethnic background or gender make signi¯cant di®erences to payo®s?

5 Relationships to the literature and conclusions

Our results may be viewed as a contribution to the literature on comparative statics properties of solutions of games. As noted by Crawford [3], the ¯rst suggestion of the sort of results obtained in this paper may be in Shapley [18], who showed that in a linear optimal-assignment problem the marginal product of a player on one side of a market weakly decreases when another player is added to that side of the market and weakly increases when a player is added to the other side of the market. Kelso and Crawford [5], building on the model of Crawford and Knoer [4], show that, for a many-to-one matching market with ¯rms and workers, adding one or more ¯rms to the market makes the ¯rm-optimal stable outcome weakly better for all workers and adding one or more workers makes the ¯rm-optimal stable outcome weakly better for all ¯rms. Crawford [3] extends these results to both sides of the market and to many-to-many matchings.[5] In contrast to this literature, our results are not restricted to matching markets and treat all outcomes in equal treatment "-cores. Moreover, we demonstrate cyclic monotonicity. Instead of the assumptions of \substitutability" of

[5] And also to pair-wise stable outcomes but this is apparently not so directly related to our paper.

Kelso and Crawford [5], however, we require our thickness condition and SGE. Unlike [5] and [3], our current results are limited to games with side payments { we plan to consider this limitation in future research.

Note that our results imply a certain continuity of comparative statics results with respect to changes in the descriptors of the total player set. In particular, the results are independent of the exact partition of players into approximate types. Speci¯cally, given a number T of approximate types and a measure of the required closeness of the approximation, subject to the condition that players of each type are approximate substitutes for each other, our results apply independently of exactly where the boundary lines between types are drawn. Suppose, for example, that we wished to partition candidates for positions as hospital interns into three categories { say \good," \ better" and \best." It may be that there is more than one way to partition the set of players into these categories while retaining the property that all players in each member of the partition are approximate substitutes for each other; the exact partition does not a®ect the results. Relating this feature of our work to general equilibrium theory, a ¯nite set of commodities is typically considered to be an approximation to the real-world situation that all units of each commodity may di®er. Descriptions of commodities are incomplete and a \commodity" is a group of objects that satisfy the description. For example, models of labor markets may have two types of workers, \skilled" and \unskilled" but no two workers (or two loaves of bread, or two oranges) may be exactly identical. In the di®erentiated commodities literature, results addressing this problem show that prices are continuous functions of attributes of commodities (cf., Mas-Colell [11]). Since our framework does not require a topology on the space of player types, continuity takes a di®erent but valid form and is more directly apparent.

Besides the matching literature, our results are related to prior results obtained within the context of a pregame, cf. [23], [26]. A pregame speci¯es a set of compact metric space of player types and a single worth function, assigning a worth to each ¯nite list of attributes (repetitions allowed). Since there is only one worth function, all games derived from a pregame are related and, given the attributes of the members of a coalition, the payo® to that coalition is independent of the total player set in which the coalition is embedded; widespread externalities are not allowed. In contrast, our results apply to given games and, as in the earlier results for matching models, there is no requisite topological structure on the space of players types. While our results for a given game hold for all games in a collection described by the same parameters, there are no necessary relationships between games. For example, consider the collection of games where two-player coalitions are e®ective and there are only two types of players. This collection includes two-sided assignment games, such as marriage games and buyer-seller games, and also games where any two-player coalition is e®ective. There appears to be no way in which one pregame can accommodate all the games in the collection. These considerations indicate that the framework of parameterized collections of games is signi¯cantly broader than that of a pregame.[6]

A major advantage of our approach over the prior approach using pregames is that, except for the special case of pregames satisfying strict small group e®ectiveness (or, in other words, `exhaustion of gains to scale by coalitions bounded in size') with a ¯nite number of exact types, the conditions used in the prior literature cannot be veri¯ed for any ¯nite game.[7] That is, since the conditions are stated on the worth function of the entire pregame, which includes speci¯cation of the worths of arbitrarily large groups, or on the closeness of the worth function to the limiting per capita utility function, it is not possible to determine whether the conditions are satis¯ed. In contrast, given any game, values of parameters describing that game can be computed.[8]

Another major advantage of our approach is that we provide exact bounds, in terms of the parameters describing a game, on the amounts by which equal treatment "-core payo® vectors can di®er from satisfying cyclic monotonicity. We are unaware of any comparable results in the literature. The prior literature does not indicate the sensitivity of the results to speci¯cations of bounds on group sizes and of types of players. Such an analysis is important for empirical testing since, in fact, few commodities are completely standardized. (This may be especially true in estimating hedonic prices.) Nor does the prior literature provide empirically testable conclusions on approximate monotonicity or comparative statics.

Numerous examples of games derived from pregames may lead one to expect our comparative statics result. Consider a glove game, for example where the payo® function can be written as $u(x;y) = \min f x;y g$. Suppose initially that the number of RH gloves, say $x;$ is equal to the number of LH gloves, y, and both x and y are greater than one. Then the equal-treatment core can be described by the set $f(p_x;p_y)\ 2\ R^2_+ : p_x + p_y = 1g$; each RH glove is assigned p_x and each LH glove is assigned p_y and a pair of gloves is assigned 1. Now increase the number of players with RH gloves. The equal treatment core is now described by $f(0;1)g$; each RH glove is assigned 0 and each LH glove is assigned 1.

In games with a ¯nite set of player types, de¯ning the core via linear programming also leads to a law of scarcity, quite immediately. Let $(N;v)$ be a game with a ¯nite number T of player types and with m_t players of type t, $t = 1;:::;T$: We take v as a mapping from subpro¯les s of m ($s\ 2\ Z^T_+$, $s\ \cdot\ m$). Then, following Wooders [23],

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[6] A short survey discussing parameterized collections of games and their relationships to pregames appears in [29].

[7] Strict small group efectiveness dictates that all gains to coalition formation can be realized by partitioning the total player set, no matter how large, into coalitions bounded in size. This condition was introduced in Wooders [23] (condition *) and, for NTU games, in Wooders [24], where it was called `minimum e±cient scale.'

[8] Since there may be many but a ¯nite number of coalitions, in fact determining the required sizes of ± and T; ¯ and B may be time-consuming but it is possible. In contrast, to verify that a pregame satis¯es SGE or PCB requires consideration of an in¯nite number of payo® sets or, even more demanding, a limiting set of equal treatment payo®s.

consider the following LP problem[9]:

$$\begin{aligned}&\text{minimize}_{p \geq 0}\ p \cdot m\\ &\text{subject to } p \cdot s \geq v(s) \text{ for all } s \leq m\end{aligned}$$

If the game has a nonempty core, then the solution p^* satisfies $v(m) = p^* \cdot m$. Now consider the same problem but with an increased number of players of type t in the objective function for some $t \in \{1, ..., T\}$. Assume that the same inequalities are the only constraints; this imposes a form of strict small group effectiveness on the game – only groups with profiles $s \leq m$ are effective. It is clear that the payoff to players of type t will not increase with the increase in the number of players of that type in the objective function since the constraint set has not changed – the payoff to type t can only decrease. This suggests some of the initial intuition underlying comparative statics results for games. Under conditions roughly equivalent to those of Wooders [23] – that all gains to coalition formation can be exhausted by coalitions bounded in size – a proof of the comparative statics result and weak monotonicity of core payoff vectors was provided in [17]. We provide a more comprehensive discussion of the literature in [10].

6 Appendix.

We construct some sequences of games to demonstrates that all the bounds we obtained in our results are exact, that is, the bound cannot be decreased.

I). Let us concentrate first on the central case $\delta = \beta = 0$. Consider a game $(N; v)$ where any player can get only 1 unit or less in any coalition and there are no gains to forming coalitions. This game has strictly effective 1-bounded groups and all agents are identical. Formally, however, we may partition the set of players into many types. Thus $(N; v) \in \Gamma((0; \tau); (0; 1))$ for any integer τ; $1 \leq \tau \leq |N|$. Notice also that for any $\varepsilon \geq 0$ the ε-core of the game is nonempty and very simple: it includes all payoff vectors that are feasible and provide at least $1 - \varepsilon$ for each of the players. All the games that we are going to construct will be subgames of a game $(N; v)$.

a). For the bound in Lemma we can present even a single game with two payoffs vectors that realize this bound. Namely, let $\tau = 1$ (all players are of one type) and let us consider any two subgames $S^1; S^2$ with the same number of players and the equal treatment payoffs $x^1 = 1$ and $x^2 = 1 - \varepsilon$. Then $(x^1 - x^2) \cdot f^1 = \varepsilon \|f^1\|$.

b). For the bound in Proposition 1, for $K \leq |N|$ and some nonnegative integer $l \leq |N| - K$, let us consider $\tau = K$ and the subgroups $S^1; ...; S^K$ with the profiles $f^1; ...; f^K$ where $f^k_t = l + 1$ for $t = k$ and 1 otherwise. Let also consider payoff vectors

[9]The core has been described as an outcome of a linear programming problem since the seminal works of Gilles and Shapley. Wooders [23] introduces the linear programming formulation with player types.

x^k where $x^k_t = 1$ for $t = k$ and $1 - \varepsilon$ otherwise. Then $(x^i - x^j)\cdot f^i = \varepsilon l$ for any $i \neq j$. Hence

$$(x^1 - x^2)\cdot f^1 + (x^2 - x^3)\cdot f^2 + \dots + (x^K - x^1)\cdot f^K = \varepsilon l K = \varepsilon \left\| f^1 + f^2 + \dots + f^K \right\| \frac{l}{l + K}$$

$$\text{and } (x^1 - x^2)\cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3)\cdot \frac{f^2}{\|f^2\|} + \dots + (x^K - x^1)\cdot \frac{f^K}{\|f^K\|} = K\varepsilon \frac{l}{l + K}.$$

It is straightforward to verify that for any fixed K both our bounds in Proposition 2 can not be improved for sequences of games $(N; v)$; with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

c). For the bound in Proposition 2 it is enough to concentrate on (A) since it is a special case of the result (B). For $|N| \geq 2$ let us consider $\tau = 2$ and $l \leq |N| - 2$. Then consider the subgroups $S^1; S^2$ with the profiles $f^1 = (1; 1)$ and $f^2 = (l + 1; 1)$ and payoff vectors $x^1 = (1 - \varepsilon; 1)$ and $x^2 = (1; 1)$. Then

$$(x^2_1 - x^1_1) = \varepsilon = \varepsilon \frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} \frac{l}{l + 4}.$$

It follows that both our bounds in Proposition 2 can not be improved for sequences of games $(N; v)$; with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

II). It is easy to modify our example to allow for non-zero δ and β in a such a way that we will have the same profiles as in Part I, but will use the payoffs of $1 + \delta + \beta$ and $1 - \varepsilon$ instead of 1 and $1 - \varepsilon$. This will lead us to the appearance of $\varepsilon + \delta + \beta$ on the places of ε in all bound in Part I. We leave it as a simple exercise for the interested reader.

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