Statistics and growth in hyperbolic groups

by

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Contents

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Declarations

The main results of Chapters 4, 5 and 6 of this thesis are presented in the articles [11], [12] and [13] respectively. The first two have been accepted for publication in the Transactions of the American Mathematics Society and Mathematische Zeitschrift respectively. The third article [13] is still in preprint form and has been submitted to a journal. The work presented in [13] was produced in collaboration with R. Sharp.

I declare that the material in this thesis is, to the best of my knowledge, my own, except where otherwise indicated or cited in the text, or else where the material is widely known. This material has not been submitted for any other degree or qualification.

Chapter 1

Introduction

The work presented in this thesis is concerned with quantifying, in various different senses, how natural quantities associated to hyperbolic groups grow and distribute.

Hyperbolic groups were introduced by Gromov in his seminal work [29] and are a fundamental object of study in geometric group theory. Given a group G with generating set S, the word metric (with respect to S) assigns to a group element $g \in G$ its word length $|g|$, i.e. the length of the shortest word(s) that express g, with letters in $S \cup S^{-1}$. A hyperbolic group is a finitely generated group that, when equipped with the word metric for any finite generating set, satisfies an abstract geometrical condition that mimics a property of the hyperbolic plane. That is, geodesic triangles in the Cayley graph of G are 'thin'. This condition, although natural, seems at first to be somewhat superficial, yet the theory of hyperbolic groups is deep and interesting. For example, hyperbolic groups exhibit strong combinatorial properties and in particular have a solvable word problem: there exists an algorithm that decides whether two words (with letters in a fixed generating set) express the same group element. Furthermore, by the work of Cannon and Ghys and de le Harpe, hyperbolic groups are strongly Markov. That is, given a hyperbolic group G equipped with a finite generating set S, there exists a finite directed graph $\mathcal{G}$ that in some sense encodes the properties of G and S. Cannon proved that cocompact Kleinian groups are strongly Markov [10] and Ghys and de la Harpe showed that Cannon's approach worked for all hyperbolic groups [25].

Using this directed graph $\mathcal{G}$, we can associate a dynamical system $(\Sigma, \sigma : \Sigma \to \Sigma)$ to G. The system (Σ, σ) is known as a subshift of finite type and is a key object in symbolic dynamics. When $\mathcal{G}$ consists of a single connected component (i.e. given any vertices x, y in $\mathcal{G}$ there is a path from x to y and a path from y to x), then the ergodic properties of (Σ, σ) are well understood. For example the σ-invariant measures on Σ satisfy a variational principle. That is, there exists a unique σ-invariant probability measure that maximises the entropy (or randomness) of σ on Σ. We call this measure the measure of maximal entropy. Suppose for now that $\mathcal{G}$ consists of a single connected component.

A central theme of ergodic theory is to understand the growth and distributional behaviour of Birkhoff sums, which describe the behaviour of a function along the orbit of a point. Given a function $f : \Sigma \to \mathbb{R}$, the nth Birkhoff sum of f at $x \in \Sigma$ is given by $f^n(x) = f(x) + f(\sigma(x)) + \ldots + f(\sigma^{n-1}(x))$. The well-known Birkhoff Ergodic Theorem states that, since the measure of maximal entropy μ is ergodic with respect to σ, if $f \in L^1(\Sigma, \mu)$

$$\lim_{n\to\infty} \frac{f^n(x)}{n} = \int f \, d\mu$$

for μ almost every $x \in \Sigma$. It is then natural to ask if we can formulate a more precise description of how $f^n(x)$ grows as $n \to \infty$ for typical $x \in \Sigma$. The space Σ supports a collection of natural metrics and a result of Ratner [50] describes the distributional behaviour of f^n when f is Hölder with respect to one (and hence all) of these metrics. More specifically, if f is Hölder and not (up to a natural equivalence) a constant function, then as $n \to \infty$, the Birkhoff sums f^n with an appropriate normalisation, follow a non-degenerate normal distribution with respect to the measure of maximal entropy. This statistical result relies on the fact that $\mathcal{G}$ is a single connected component and in general, the graph $\mathcal{G}$ associated to a hyperbolic group may not have this property. In this thesis, one of the main difficulties is overcoming this issue.

Returning to our geometrical setting, suppose that G is a hyperbolic group equipped with a finite generating set and that $\varphi : G \to \mathbb{R}$ is a real valued function. An interesting and natural question to ask is the following: how does φ typically grow as we increase the word length of its input, i.e. how does φ distribute over the words of length n, $W_n = \{g \in G : |g| = n\}$, as $n \to \infty$? Furthermore, can we exploit the connection between G and (Σ, σ) to better understand how the values of φ distribute in $\mathbb{R}$?

One of the aims of this work is to answer these questions and to be able to precisely describe the asymptotic behaviour of real valued functions on hyperbolic groups. Our main results are split across three chapters (chapters 4, 5 and 6). In the first of these chapters we study the statistical behaviour of real valued functions on hyperbolic groups with respect to the sequence of uniform measures on W_n. A natural question that follows from this work is the following: how do these real valued functions grow as we travel along typical geodesic rays in the Gromov boundary of the group? We consider this question in Chaper 5. Lastly, in Chapter 6, we study the relative growth of normal subgroups of hyperbolic groups. In this introduction we will discuss these problems in more depth, beginning with the problems presented in Chapter 4.

Let G be a non-elementary hyperbolic group with a fixed finite generating set S. A non-elementary group is one that does not admit a finite index cyclic subgroup. Let $|g|$ denote the word length of $g \in G$ with respect to S and write $W_n = \{g \in G : |g| = n\}$. There has been significant interest in understanding how the images of elements of W_n, under natural real valued maps, such as group homomorphism or quasimorphisms, are distributed in $\mathbb{R}$. For example, Horsham and Sharp proved that when G is a free group (or surface group with presentation $G = \langle a_1, \ldots, a_g, b_1, \ldots, b_g | \prod_{i=1}^g [a_i, b_i] \rangle$ for $g \geq 2$) and S is the canonical free generating set for G and $\varphi : G \to \mathbb{R}$ is a sufficiently regular quasimorphism (i.e. a group homomorphism up to bounded error), then the normalized images

$$\{\varphi(g)/\sqrt{n} : g \in W_n\},$$

converge to a normal distribution as $n \to \infty$ [31], [32]. Other similar statistical results have been proved when G is a free group, [35], [49], [51].

In [9] Calegari and Fujiwara obtain a Gaussian limit law that holds for general non-elementary hyperbolic groups. They construct a sequence of measures ν_n on G, such that if $\varphi : G \to \mathbb{R}$ belongs to a class of functions, called bicombable functions, then there exists $\Lambda \in \mathbb{R}$ such that the distributions

$$\nu_n \left\{ g \in G : \frac{\varphi(g) - \Lambda n}{\sqrt{n}} \leq x \right\},$$

converge as $n \to \infty$ to a normal distribution. Calegari extends this result in his survey [8], showing that the above central limit theorem holds for a wider class of functions than bicombable functions.

The proof of these results rely on ideas and techniques from ergodic theory. In fact, these proofs follow a similar methodology that we briefly described above. That is, using that G is strongly Markov we associate to the pair G, φ (where G is a hyperbolic group and $\varphi : G \to \mathbb{R}$ is in the required class) a dynamical system $(\Sigma, \sigma : \Sigma \to \Sigma)$ and a suitable function $f : \Sigma \to \mathbb{R}$. The function f is chosen in such a way that the statistical behaviour of φ on G can be deduced from the statistical behaviour of f on (Σ, σ). Then, using techniques from ergodic theory, one can study the behaviour of f on Σ to deduce a central limit theorem for φ on G. In the result of Calegari and Fujiwara, the measures ν_n are supported on W_n and weight elements of W_n by a quantity depending on the system (Σ, σ). The system (Σ, σ) associated to G is not canonical and hence neither are the measures ν_n.

The above discussion leads to the following natural questions.

1. Does the result of Horsham and Sharp generalise to the case that G is an arbitrary non-elementary hyperbolic group?

2. In the result of Calegari and Fujiwara, can we replace the sequence ν_n with a sequence of measures that does not depend on (Σ, σ)? In particular, can we replace ν_n with the sequence of uniform measures on W_n?

In Chapter 4 we answer these questions in the affirmative. We also prove an averaging theorem, large deviation theorem, multidimensional central limit theorem and a local limit theorem.

To prove these results, we would like to employ the techniques described above. To do this we need we make assumptions on the real valued functions we consider. We are interested in the statistics of functions $\varphi : G \to \mathbb{R}$ that satisfy two conditions which we call Condition (1) and Condition (2). These conditions allow us to translate questions about φ on G to questions about a suitable function $f : \Sigma \to \mathbb{R}$. They are somewhat technical and so we defer their statement until Chapter 3. Intuitively, Condition (1) allows us to associate $f : \Sigma \to \mathbb{R}$ to φ and Condition (2) is a growth condition that we will use to deduce important properties of f. For now we note that there are many natural examples of functions satisfying these conditions, including group homomorphisms, some quasimorphisms and the displacement function associated to certain group actions on CAT(-1) spaces.

Our averaging theorem is analogous to the law of large numbers and in some sense is the most basic statistical result that we prove. Our central limit theorem is a more subtle result and as such requires an additional assumption to avoid degenerate cases. We simply need to assume that the function $\varphi(\cdot) - \Lambda|\cdot| : G \to \mathbb{R}$ is unbounded. Also, using Theorem 4.1.1, we quantify the rate of convergence associated to our central limit theorem. We show that the sequence of distributions that we consider converges uniformly to the Gaussian distribution at a $O\left(n^{-1/2}\right)$ rate. This is the so-called Berry-Esseen error term.

Suppose $\varphi : G \to \mathbb{R}$ satisfies Condition (1) and Condition (2) mentioned above. The results presented in Chapter 4 quantify, in some sense, how φ grows as we increase the word length of its input. A different way to measure this growth would be to study how φ grows along geodesic rays in the Gromov boundary of G, ∂G. There is a natural measure, the Patterson–Sullivan measure, which can be thought of as an extension to ∂G of the sequence of uniform measures on W_n. It is therefore natural to ask if we can quantify the growth rate of φ along Patterson–Sullivan typical geodesic rays in ∂G. We study this problem in Chapter 5 and will now discuss our results in more detail.

Let G be a non-elementary hyperbolic group and suppose that G acts cocompactly (i.e. the quotient under the action of G is compact) by isometries on a complete hyperbolic geodesic metric space (X, d). Fix a finite generating set S for G and an origin o for X. Let $C(G)$ denote the Cayley graph of G with respect to S and write ∂G for the Gromov boundary of G. By the Švarc-Milnor Lemma, there exists constants $C_1, C_2 > 0$ such that, for any infinite geodesic ray γ based at the identity in $C(G)$,

$$C_1 n \le d(o, \gamma_n o) \le C_2 n$$

for all $n \ge 1$. Here γ_n denotes the end point of γ after n steps. This inequality describes the coarse behaviour of the displacement function $g \mapsto d(o, go)$ along geodesic rays. It is then natural to ask whether we can describe more precisely how the displacement grows along typical geodesic rays in ∂G? The Patterson–Sullivan measure provides us with a natural way of quantifying typicality in this setting. We say that a property exhibited by elements of ∂G is typical if it holds on a full Patterson–Sullivan measure set.

Gekhtman, Taylor and Tiozzo asked the above question in a more general setting. They prove the following theorem in [24]. Let ν denote the Patterson–Sullivan measure obtained as the weak * limit

$$\lim_{n\to\infty} \frac{\sum_{|g|\le n} \lambda^{-|g|}\delta_g}{\sum_{|g|\le n} \lambda^{-|g|}},$$

where δ_g denotes the Dirac measure based at $g \in G$ and λ is the exponential growth rate of $\#W_n$. We write $[\gamma] \in \partial G$ for the element in ∂G that contains γ.

Proposition 1.0.1 (Theorem 1.3 [24]). *Suppose a hyperbolic group G has a non-elementary action by isometries on a separable, hyperbolic geodesic metric space X. Then, there is $L > 0$ such that for every $x \in X$ and ν almost every $[\widetilde{\gamma}] \in \partial G$,*

$$\lim_{n\to\infty} \frac{d_X(x, \gamma_n x)}{n} = L,$$

where γ is any geodesic ray in $[\widetilde{\gamma}]$.

To prove this, Gekhtman, Taylor and Tiozzo exploit the strongly Markov structure of G. That is, they use the fact that there exists a finite directed graph

$\mathcal{G}$ that in some sense encodes the key properties of G. They obtain the above theorem by studying random walks on the loop graph associated to $\mathcal{G}$. This is one way to exploit the structure provided by $\mathcal{G}$. However, we could instead use this strongly Markov structure in the way discussed above. That is, we can use the graph $\mathcal{G}$ to construct a dynamical system and then use our techniques from Chapter 4.

This discussion leads us to ask whether Proposition 1.0.1 remains true if we replace the displacement function with a different real valued function. Furthermore, can we formulate a more precise statement describing how these functions behave along geodesic rays? These are the questions that we consider in Chapter 5.

Lastly, in Chapter 6, we study the relative growth of normal subgroups of hyperbolic groups. Suppose G is equipped with a finite symmetric generating set. By a result of Coornaert [15], the growth rate of $\#W_n$ is purely exponential, i.e. there exist constants $\lambda > 1$ and $C_1, C_2 > 0$ such that

$$C_1\lambda^n \leq \#W_n \leq C_2\lambda^n$$

for all $n \geq 1$. Now suppose that N is a subgroup of G. An interesting question to ask is how $\#(W_n \cap N)$, which we call the relative growth of N, grows in comparison to $\#W_n$. A result of Gouëzel, Mathéus and Maucourant [26] states that if N has infinite index in G then

$$\lim_{n\to\infty} \frac{\#(W_n \cap N)}{\#W_n} = 0. \tag{1.0.1}$$

This is a subtle result that relies strongly on the hyperbolicity of G. If we suppose further that N is normal and the quotient G/N is isomorphic to $\mathbb{Z}^\nu$ for some $\nu \geq 1$, then we have access to more structure. With this additional information it seems reasonable to expect that we can describe the relative growth of N more precisely.

Pollicott and Sharp [46] studied this problem when G is the fundamental groups of a compact orientable surface of genus at least two and N is the commutator subgroup. Sharp [55] extended this to cover hyperbolic groups G that may be realised as convex cocompact groups of isometries of real hyperbolic space whose fundamental domain can be chosen to be a finite sided polyhedron R such that $\bigcup_{g\in G} \partial R$ is a union of geodesic hyperplanes, with generators given by the side pairings. The fundamental groups of compact surfaces were shown to satisfy this condition by Bowen and Series [6]. In addition, this class includes free groups on at least two generators and certain higher dimensional examples (see Bourdon's thesis [4]). In these cases, it was shown that there exists an integer $D \geq 1$ (related to the certain periodicities in the graph $\mathcal{G}$) such that, along the subsequence Dn, the relative growth $\#(W_{Dn} \cap N)$ grows asymptotically like $\lambda^{Dn}/(Dn)^{\nu/2}$, as $n \to \infty$. The aim of Chapter 6 is to extend this result so that it applies all non-elementary hyperbolic groups. This result has interesting consequences regarding relative growth series.

In Chapters 2 and 3 we introduce the preliminary materials needed for our proofs in the subsequent chapters. In Chapter 2 we will focus on the ideas and techniques that we will require from ergodic theory and more specifically thermodynamic formalism.

Chapter 2

Thermodynamic formalism and symbolic dynamics

2.1 General subshifts of finite type

Let A be a $k \times k$ matrix consisting of zeros and ones and let $A_{i,j}$ denote the (i,j)th entry of A. We can think of A as describing a finite directed graph on k vertices $1, \dots, k$ where vertex i is joined to vertex j by a directed edge if and only if $A_{i,j} = 1$. The subshift of finite type Σ_A associated to A is then the space of infinite paths in the graph described by A. More formally

$$\Sigma_A = \{(x_n)_{n=0}^{\infty} : x_n \in \{1, 2, \dots, k\}, A_{x_n, x_{n+1}} = 1, n \in \mathbb{Z}_{\geq 0}\}.$$

Given x in Σ_A we write x_n for the nth coordinate of x. We equip $\{1, 2, \dots, k\}$ with the discrete topology and use this to endow Σ_A with the Tychonov product topology. This topology is generated by sets of the form

$$[x_0, x_1, \dots, x_n] := \{y \in \Sigma_A : y_i = x_i \text{ for } 0 \leq i \leq n\}.$$

These are known as cylinder sets.

We define the shift map $\sigma : \Sigma_A \to \Sigma_A$ that sends $x \in \Sigma_A$ to $y = \sigma(x) \in \Sigma_A$ where $y_n = x_{n+1}$ for all $n \geq 0$ (i.e. σ shifts a sequence one index to the left and deletes the initial term).

The topology on Σ_A is metrizable. Fix $0 < \theta < 1$ and take $x, y \in \Sigma_A$. If $x_0 = y_0$ we set

$$d_\theta(x, y) = \theta^N,$$

where N is the largest positive integer such that $x_i = y_i$ for all $0 \leq i < N$. If $x_0 \neq y_0$ we set $d(x, y) = 1$ and if $x = y$, $d(x, y) = 0$. It is easy to see that d_θ is a metric and that it is compatible with the topology on Σ_A. It is also easy to see that (Σ_A, d_θ) is a compact metric space.

Let $C(\Sigma_A)$ denote the space of all continuous functions from Σ_A to $\mathbb{C}$. We now want to consider the vector space of complex valued functions on Σ_A that are Lipschitz with respect to d_θ.

Definition 2.1.1. Let

$$F_\theta = \{f : \Sigma_A \to \mathbb{C} : f \text{ is Lipschitz with respect to } d_\theta\}.$$

We will say that a function $f : \Sigma_A \to \mathbb{R}$ is Hölder if it belongs to F_θ for some $0 < \theta < 1$. Given $f \in F_\theta$, the nth variation of f is

$$\mathrm{var}_n(f) = \sup\{|f(x) - f(y)| : x_i = y_i \text{ for } |i| < n\}.$$

We use this to define

$$|f|_\theta = \sup\left\{\frac{\mathrm{var}_n(f)}{\theta^n} : n \geq 0\right\}$$

which is just the least Lipschitz constant for f. This does not define a norm on F_θ (in fact, $|\cdot|_\theta$ is a semi-norm) as it assigns 0 to all constant functions. However, we can easily modify $|\cdot|_\theta$ so that it becomes a norm.

Definition 2.1.2. We define a norm on F_θ by

$$\|f\|_\theta = |f|_\theta + |f|_\infty,$$

where $|\cdot|_\infty$ is the usual supremum norm.

We then have the following.

Proposition 2.1.3. *[5] When equipped with $\|\cdot\|_\theta$, F_θ becomes a Banach space.*

A central theme of ergodic theory is to understand the growth and distributional behaviour of Birkhoff sums: given $f \in F_\theta$, the n-th Birkhoff sum of f is the function

$$f^n(\cdot) = f(\cdot) + f(\sigma(\cdot)) + \ldots + f(\sigma^{n-1}(\cdot)).$$

Often, when trying to describe how the Birkhoff sums of a function $f \in F_\theta$ behave, there is a dichotomy that occurs depending on whether f is cohomologous to a constant or not.

Definition 2.1.4. We say that $f, g \in F_\theta$ are cohomologous (denoted by $f \sim g$) if there exists continuous $h : \Sigma_A \to \mathbb{C}$ such that

$$f = g + h \circ \sigma - h.$$

We say that $f \in F_\theta$ is cohomologous to a constant if $f \sim g$ where g is a constant function.

We now want to define topological entropy. This is a dynamical quantity which describes, in some sense, the randomness of a dynamical system. Consider a subshift of finite type (Σ_A, σ) and a σ-invariant probability measure μ on Σ_A. We equip Σ_A with the Borel σ-algebra described above. Now, given a finite measurable partition γ of Σ_A and a sub σ-algebra $\mathcal{C}$, we define the conditional information and the conditional entropy of γ given $\mathcal{C}$ as

$$I_\mu(\gamma|\mathcal{C}) = -\sum_{C \in \gamma} \chi_C \log \mu(C|\mathcal{C}) \quad \text{and} \quad H_\mu(\gamma|\mathcal{C}) = \int I_\mu(\gamma|\mathcal{C}) \, d\mu$$

respectively. Here, χ_C denotes the indicator function for $C \in \gamma$ and $\mu(C|\mathcal{C})$ denotes the conditional expectation $E_\mu(\chi_C|\mathcal{C})$. The entropies of σ with respect to μ and γ are defined as $h_\mu(\sigma, \gamma) = H_\mu(\gamma|\sigma^{-1}\mathcal{C})$ where $\mathcal{C}$ is the smallest σ-algebra containing $\bigcup_{j=0}^{\infty} \sigma^{-1}\gamma$. The entropy of σ with respect to μ is $h_\mu(\sigma) = \sup_\gamma h_\mu(\sigma, \gamma)$ where the supremum is taken over all finite measurable partitions. The topological entropy of (Σ_A, σ) is given by

$$h = \sup_\mu \{h_\mu(\sigma)\}$$

where the supremum is taken over all σ-invariant probability measures. As mentioned above, h describes the randomness of σ on Σ_A. Topological entropy will be useful for describing the growth of certain quantities. To see how, we need to define transfer operators.

Throughout this thesis there are many points at which we want to understand the growth of certain dynamical expressions. The key tools that will allow us to analyse these expressions are transfer operators.

Definition 2.1.5. Take $f \in F_\theta$, we define the transfer operator $L_f : C(\Sigma_A) \to C(\Sigma_A)$ by

$$(L_f w)(x) = \sum_{\sigma y = x} e^{f(y)} w(y).$$

The operator L_f preserves F_θ and the restriction $L_f : F_\theta \to F_\theta$ is a bounded linear operator [42]. Intuitively, $L_f w$ maps a point $x \in \Sigma_A$ to the sum of w

evaluated at the preimages $y \in \sigma^{-1}(x)$ each weighted by $e^{f(y)}$. We can directly calculate its iterates which are given by

$$(L_f^n w)(x) = \sum_{\sigma^n(y)=x} e^{f^n(y)} w(y).$$

To better understand the behaviour of transfer operators, we need to make additional assumptions on the subshift of finite type Σ_A which we consider. Specifically, we need to assume that the underlying matrix A is aperiodic or irreducible.

Definition 2.1.6. We say that a $k \times k$ zero-one matrix A is irreducible if given $i, j \in \{1, 2, \ldots, k\}$, there exists $n \in \mathbb{N}$ such that $(A^n)_{i,j} > 0$. If there exists $n \in \mathbb{N}$ such that $(A^n)_{i,j} > 0$ for all i, j, then A is aperiodic.

The following theorem provides us with information about the eigenvalues of irreducible and aperiodic matrices. Recall that an eigenvalue is simple if its corresponding (generalised) eigenspace is one-dimensional.

Theorem 2.1.7. ***(The Perron-Frobenius Theorem) [21].*** *Suppose that A is an irreducible matrix with non-negative entries. Then, A has a real, simple, maximal, positive eigenvalue λ. Furthermore, there exists an integer $p \geq 1$ (the period of A) such that A has eigenvalues $e^{2\pi ik/p}\lambda$ for $k = 0, \ldots, p-1$ and all other eigenvalues have absolute value strictly less than λ. Suppose that A has non-negative entries and is aperiodic. Then, A has spectrum as described above but for $p = 1$.*

Recall that a dynamical system (X, T) (i.e. where X is a metric space and $T : X \to X$ is a function) is said to be weak mixing if for all open $U, V \subset X$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $n > N$, $T^{-n}U \cap V \neq \emptyset$. If given any open $U, V \subset X$ there exists n such that $T^{-n}U \cap V \neq \emptyset$ then we say that our system is transitive. It is a simple exercise to show that A is aperiodic if and only if the system Σ_A is weak mixing mixing. Similarly, A is irreducible if and only if Σ_A is transitive.

2.2 Mixing subshifts

Throughout this subsection we will assume all subshifts are weak mixing. When Σ_A has this property, the transfer operators defined above exhibit a variety of useful properties. In particular, they exhibit a spectral gap and have a simple maximal eigenvalue with strictly positive eigenfunction. The following theorem will be useful.

Theorem 2.2.1 (Theorem 2.2 [42])**.** ***(The Ruelle-Perron-Frobenius Theorem).*** *Suppose $f : \Sigma_A \to \mathbb{R}$ belongs to F_θ and that A is aperiodic.*

1. *There is a simple maximal positive eigenvalue $\beta \in \mathbb{R}$ of $L_f : F_\theta \to F_\theta$, with a strictly positive eigenfunction $h \in F_\theta$.*

2. *The rest of the spectrum (excluding β) is contained in a disc of radius strictly less than β. (This is the aforementioned spectral gap property).*

3. *There is a unique probability measure μ on Σ_A such that $\int L_f v \, d\mu = \beta \int v \; d\mu$ for all $v \in F_\theta$.*

4. *Supposing h is normalised so that $\int h \, d\mu = 1$, we have that $\beta^{-n} L_f^n v \to h \int v \, d\mu$ uniformly for all $vC(\Sigma_A)$.*

Suppose the eigenfunction h from above satisfies $\int h \, d\mu = 1$ and define the measure $m = h\mu$. This measure is σ-invariant. Furthermore σ is strong mixing with respect to m and hence m is ergodic.

The following result provides an alternate characterisation for the lead eigenvalue β in Theorem 2.2.1.

Theorem 2.2.2 (Proposition 3.4 [42])**.** ***(The Variational Principle).*** *Take real valued $f \in F_\theta$. There exists a unique σ-invariant probability measure m, such that for all other σ-invariant measures m',*

$$h_\mu(\sigma) + \int f \, dm' < h_m(\sigma) + \int f \; dm.$$

Moreover, if μ and $h \in F_\theta$ are as in Theorem 2.2.1 and $\int h \, d\mu = 1$, then $m = h\mu$.

Definition 2.2.3. We define the pressure of a real valued function $f \in F_\theta$ to be

$$P(f) = \sup_\mu \left\{ h_\mu(\sigma) + \int f \, d\mu \right\},$$

where the supremum is taken over all σ-invariant probability measures. The measure that attains this supremum is known as the equilibrium state of f. We define the measure of maximal entropy to be the equilibrium state of the constant function $f = 0$.

An important result, proved by Ruelle [53], is that $e^{P(f)}$ is the simple maximal eigenvalue of L_f. Another useful fact is that $f, g \in F_\theta$ satisfy $f \sim g + c$ for some constant c if and only if f and g have the same equilibrium state.

So far we have mostly worked with transfer operators L_f associated to real valued $f \in F_\theta$. We now suppose that $f \in F_\theta$ is complex valued and so $f = u + iv$ for real valued $u, v \in F_\theta$. The following result describes the spectral properties of L_f.

Theorem 2.2.4 (Pollicott [44]). *Take f as above. Then, $\rho(L_f) \leq e^{P(u)}$ where $\rho(L_f)$ denotes the spectral radius of $L_f : F_\theta \to F_\theta$. Further, if L_f has an eigenvalue of modulus $e^{P(u)}$ then it is simple, unique and the remainder of the spectrum is contained in a disc of radius strictly smaller than $e^{P(u)}$. If L_f has no eigenvalues of modulus $e^{P(u)}$ then the spectral radius of L_f is strictly smaller than $e^{P(u)}$.*

We can extend the definition of pressure to complex valued functions $f \in F_\theta$ such that L_f has a simple maximal eigenvalue λ, with the rest of the spectrum of L_f contained in $\{z \in \mathbb{C} : |z| < |\lambda| - \epsilon\}$ for some $\epsilon > 0$. We define the pressure of f, $P(f)$, by $e^{P(f)} = \lambda$ which is defined modulo $2\pi i$. Note that we can take $P(f)$ to be real when f is real-valued. From now on, when we refer to the pressure function, we mean the 'extended version'.

We would like to understand how the simple maximal eigenvalues of L_f vary as we perturb the function $f \in F_\theta$. For this we require results from perturbation theory.

Theorem 2.2.5. *[34, Theorem 6.17] Let $B(V)$ denote the Banach algebra of bounded linear operators on a Banach space. Suppose T_0 has a simple isolated eigenvalue $\lambda(T_0)$ with corresponding eigenvector $v(T_0)$. Then, for any $\epsilon > 0$, there is $\delta > 0$ such that if $\|T - T_0\| < \delta$ then T has a simple isolated eigenvalue $\lambda(T)$ with corresponding eigenvector $v(T)$. Moreover*

- *the maps $T \mapsto \lambda(T)$ and $T \mapsto v(T)$ are analytic for $\|T - T_0\| < \delta$,*
- *if $\|T - T_0\| < \delta$, then $|\lambda(T) - \lambda(T_0)| < \epsilon$ and the part of the spectrum of T that does not include $\lambda(T)$ is contained in $\{z \in \mathbb{C} : |z - \lambda(T_0)| > \epsilon\}$.*

This theorem along with the fact that the map $f \mapsto L_f$ is continuous (in fact analytic [42]) implies that the domain on which the pressure function is defined is open. Theorem 2.2.5 can also be used to show that the pressure function is analytic in the following sense.

Definition 2.2.6. [34] Let B be a complex Banach space and $U \subset \mathbb{C}$ an open subset of $\mathbb{C}$. A map $S : U \to B$ is said to be analytic if

$$l \circ S : U \to \mathbb{C}$$

is analytic for all $l \in B^*$. Here, B^* denotes the dual space to B.

If B_1, B_2 are complex Banach spaces with open $V \subset B_1$, then $T : V \to B_2$ is analytic if $T \circ S$ is analytic in its domain of definition for any analytic map $S : U \to B_1$ for $U \subset \mathbb{C}$ open.

Theorem 2.2.5 implies that there exists a complex neighbourhood of zero and a projection valued, analytic function $Q : U \to B(F_\theta)$, such that, for $s \in U$,

- $L_{g+sf}Q(s) = Q(s)L_{g+sf}$ and
- $L_{g+sf}w(s) = e^{P(g+sf)}w(s)$, where $w(s) = Q(s) \cdot 1$.

Using these expressions we can calculate derivatives of the pressure function. The following result is originally due to Ruelle.

Lemma 2.2.7 (Proposition 4.12 [42])**.** *Take real valued $f, g \in F_\theta$. Then,*

$$\frac{dP(g+sf)}{ds}\Big|_{s=0} = \int f\ dm, \quad \text{and}$$

$$\sigma^2 = \frac{d^2P(g+sf)}{ds^2}\Big|_{s=0} = \lim_{n\to\infty} \frac{1}{n} \int \left(f^n - \int f dm\right)^2 dm,$$

where m is the equilibrium state for g. Furthermore $\sigma^2 > 0$ if and only if f is not cohomologous to a constant.

We then obtain the following Taylor expansion for $P(g+sf)$ for s in a complex neighbourhood U of zero:

$$P(sf) = P(0) + s\int f d\mu + s^2\sigma^2/2 + s^3\psi(s), \tag{2.2.1}$$

where ψ is analytic in U.

Using this expansion it is possible to prove that the Birkhoff sums of functions $f \in F_\theta$ follow a central limit theorem. Let μ denote the measure of maximal entropy. If $f \in F_\theta$ for some $0 < \theta < 1$ and $\int f\ d\mu = 0$, then there exists $\sigma_f^2 \geq 0$ such that for $x \in \mathbb{R}$

$$\mu\left\{z \in \Sigma_A : \frac{f^n(z)}{\sqrt{n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{x} e^{-t^2/2\sigma_f^2}\ dt + O(n^{-1/2})$$

as $n \to \infty$ [14]. Furthermore, $\sigma_f^2 = 0$ if and only if f is cohomologous to a constant. The following lemma provides some useful characterisations of being cohomologous to a constant.

Proposition 2.2.8 (Livsic [39])**.** *A real valued function $f \in F_\theta$ is cohomologous to $C \in \mathbb{R}$ if and only if one (and hence both) of the following hold,*

1. *the set* $\{f^n(x) - nC : x \in \Sigma, n \in \mathbb{Z}_{>0}\}$ *is a bounded subset of* $\mathbb{R}$,

2. $f^n(x) - nC = 0$ *for all* $x \in \Sigma$ *and* $n \in \mathbb{Z}_{>0}$ *with* $\sigma^n(x) = x$.

2.3 Transitive subshifts

In this subsection we consider transitive subshifts that are not weak mixing. When this is the case, there exists a natural number $p > 1$ known as the period of A such that Σ_A has a unique p-cyclic disjoint decomposition

$$\Sigma_A = \bigsqcup_{k=0}^{p-1} \Sigma_{A_k}$$

where each $\Sigma_{A_k} \subset \Sigma_A$ is a finite union of length one cylinders. The shift map sends Σ_{A_j} to $\Sigma_{A_{j+1}}$ where $j, j+1$ are taken modulo p. Furthermore, for each j, $\sigma^p : \Sigma_{A_j} \to \Sigma_{A_j}$ is a mixing subshift. It follows from the results presented in the previous section that the transfer operator $L_0 : F_\theta \to F_\theta$ (i.e. where 0 denotes the zero valued constant function) has spectrum containing p simple maximal eigenvalues at $e^{2\pi i k/p} e^h$ for $k = 0, \ldots, p-1$. The rest of the spectrum is contained in the disk $\{z : |z| < e^h - \delta\}$ for some $\delta > 0$. The constant h is the topological entropy of (Σ, σ) and is obtained, as in the case when A is aperiodic, from the variational expression

$$h = \sup_m \{h_m(\sigma)\},$$

where the above supremum is taken over all σ-invariant probability measures. This supremum is attained uniquely by the measure of maximal entropy μ and this measure is ergodic with respect to σ.

As in the mixing case a central limit theorem holds for the normalised Birkhoff sums $n^{-1/2} f^n$ with respect to the measure of maximal entropy. As before this central limit theorem is non-degenerate if and only if f is not cohomologous to a constant. Proposition 2.2.8 also holds in this more general setting.

2.4 Non-transitive subshifts

In this thesis we will need to work with subshifts that are not transitive, i.e. we will consider Σ_A where A is not irreducible. Such subshifts will arise naturally when studying hyperbolic groups. More precisely, we will use that hyperbolic groups are strongly Markov: given a hyperbolic group G there exists a finite directed graph (and hence a subshift of finite type Σ_A) that encodes the properties of G.

When we associate a subshift Σ_A to a hyperbolic group, then A may not be irreducible and so the main results of the previous sections do not apply. For example, given real valued $f \in F_\theta$ the spectral properties of L_f may not be the same as those described in the Ruelle-Perron-Frobenius Theorem. It is also the case that there may not be a unique measure of maximal entropy for Σ_A. In order to obtain our main results we will need to overcome these issues. To do this we will exploit the geometrical and combinatorial properties of hyperbolic groups to learn more about the structure of the subshifts that we consider. This will allow us to apply results that hold for transitive subshifts.

To motivate the study of non-transitive subshifts we will now move on to our study of hyperbolic groups.

Chapter 3

Hyperbolic groups and the strongly Markov property

3.1 Hyperbolic groups

In this section we recall classical properties of hyperbolic groups. The concept of hyperbolicity was introduced by Gromov in his fundamental paper [29]. For a good account of the theory concerning hyperbolic groups, see [25].

Definition 3.1.1. Let (X, d) be a geodesic metric space. We say that X is hyperbolic if there exists a constant $\delta \geq 0$ such that given any geodesic triangle xyz in X, the side xy is contained in the union of the δ-neighbourhoods of the other two sides, yz and zx. A finitely generated group G is hyperbolic (in the sense of Gromov) if for any finite generating set S for G, the Cayley graph of G with respect to S is hyperbolic when equipped with the path metric.

A hyperbolic group is non-elementary if it is not virtually cyclic, i.e. it does not contain a finite index cyclic subgroup. In this work we are only interested in non-elementary hyperbolic groups. All groups labeled G are assumed to be non-elementary hyperbolic groups. Given an element $g \in G$, we use $|g|$ to denote the word length of g: the length of the shortest word(s) representing g with letters in $S \cup S^{-1}$. Let W_n denote the set consisting of group elements of word length n. We define the left and right word metrics on G as follows.

Definition 3.1.2. The left and right word metrics on G are

$$d_L(g, h) = |g^{-1}h| \quad \text{and} \quad d_R(g, h) = |gh^{-1}|$$

respectively.

We also define the Gromov product as follows.

Definition 3.1.3. Given $g, h \in G$ the Gromov product (g, h) of g and h is

$$(g,h) = \frac{1}{2}\left(|g| + |h| - |g^{-1}h|\right).$$

We will require some techniques from Patterson–Sullivan theory. We recall some basic facts about the boundaries of hyperbolic groups and the Patterson–Sullivan measure.

Let $C(G)$ denote the Cayley graph of G with respect to S. An infinite geodesic ray γ is an infinite path in $C(G)$ such that any finite sub-path of γ is a geodesic in $C(G)$. Given such a geodesic ray γ, let γ_n denote the element in G corresponding to the end point of γ after n steps. Two geodesic rays γ, γ' are said to be equivalent if $d_L(\gamma_n, \gamma_n')$ is bounded uniformly for $n \in \mathbb{Z}_{\geq 0}$. The Gromov boundary ∂G of G, is the set of equivalence classes of infinite geodesic rays in $C(G)$. The boundary ∂G supports a natural (metrizable) topology that can be seen as the extension of the topology on G given by the word metric. With this topology, $G \cup \partial G$ becomes the compactification of G (with the topology given by the word metric). Given a geodesic ray γ let $[\gamma] \in \partial G$ denote the equivalence class containing γ. The action of G extends to $G \cup \partial G$ by sending $[\gamma] \in \partial G$ to $[g\gamma] \in \partial G$.

A Patterson–Sullivan measure ν is a measure on $G \cup \partial G$ that is supported on ∂G. It is defined as a limit of a convergent subsequence of

$$\frac{\sum_{g \in G} \lambda^{-s|g|}\delta_g}{\sum_{g \in G} \lambda^{-s|g|}}$$

as s approaches 1 from above. All of the measures realised as the limit of one of these subsequences are equivalent to each other i.e. they have the same sets of measure 0. We can construct a specific measureν as the limit as $n \to \infty$, of the following sequence of measures

$$\frac{\sum_{|g| \leq n} \lambda^{-|g|}\delta_g}{\sum_{|g| \leq n} \lambda^{-|g|}}.$$

Here $\lambda = \limsup_{n\to\infty}(\#W_n)^{1/n}$ is the exponential growth rate of $\#W_n$ and δ_g denotes the Dirac measure based at $g \in G$. We will see later that the limit defining ν exists.The measure ν enjoys many useful properties and in particular is ergodic with respect to the action of G on ∂G.

Definition 3.1.4. A Borel measure μ on ∂G is ergodic if for any G-invariant Borel measurable subset E of ∂G, $\mu(E)$ is either 0 or 1.

For a comprehensive account of the above material, see [15] and [33].

Using the Patterson–Sullivan measure, Coornaert proved that the growth of $\#W_n$ is purely exponential [15].

Proposition 3.1.5 (Coornaert [15])**.** *There exists* $C_1, C_2 > 0$*,* $\lambda > 1$ *such that for all* $n \geq 1$*,*

$$C_1\lambda^n \leq \#W_n \leq C_2\lambda^n.$$

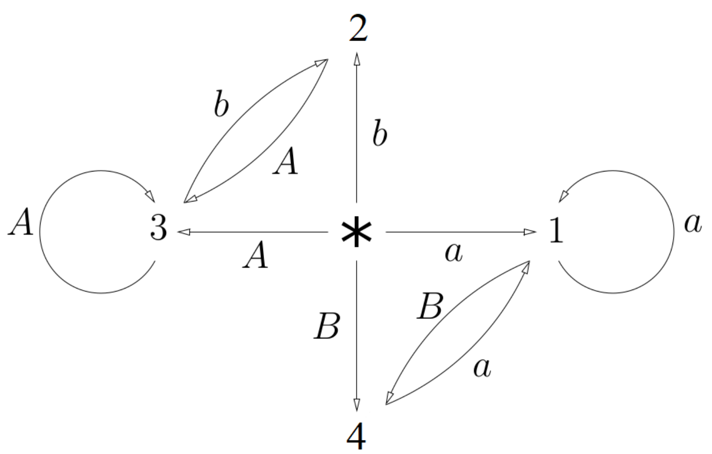
3.2 The strongly Markov property

Hyperbolic groups have interesting combinatorial properties. One of the reasons for this is their strongly Markov structure: a hyperbolic group can be represented by a finite directed graph with useful properties.

Definition 3.2.1. A group G is strongly Markov if given any symmetric generating set S for G, there exists a finite directed graph $\mathcal{G}$ with vertex set V and directed edge set $E \subset V \times V$ that exhibits the following properties: V contains a vertex $*$ and there exists a labelling $\rho : E \to S$ for which the following hold,

1. $(x, *)$ does not belong to E for any $x \in V$,
2. the map sending a path (starting at $*$) with concurrent edges $(*, x_0), (x_0, x_1), \ldots, (x_{n-1}, x_n)$ to the group element $\rho(*, x_0)\rho(x_0, x_1)\ldots\rho(x_{n-1}, x_n)$, is a bijection,
3. the above bijection preserves word length; if $|g| = n$, then the finite path corresponding to g consists of n edges.

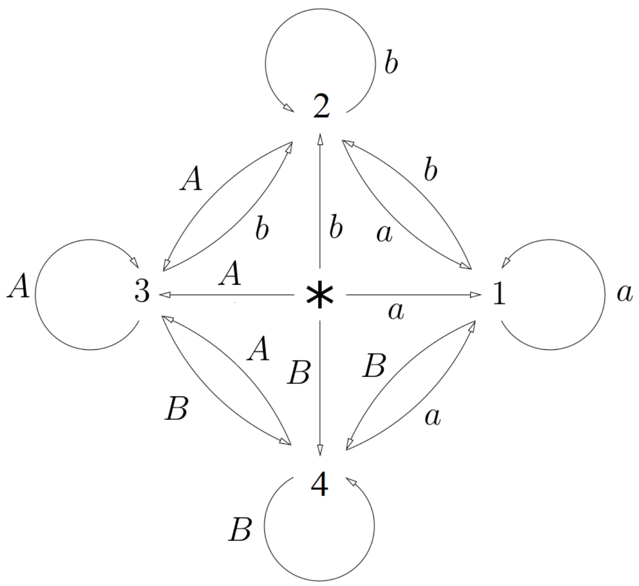
The following is an example of a directed graph (satisfying the properties in the above definition) associated to the group $\langle a, b | ab^2 \rangle$. This image is from the thesis of Horsham [31].



Here A and B denote the inverses of a and b respectively. The matrix describing this graph is

$$\begin{array}{c|ccccc} & * & 1 & 2 & 3 & 4 \\ \hline * & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 \end{array}$$

We can associate to the free group $\langle a, b\rangle$ a similar directed graph,



that is described by the matrix

$$\begin{array}{c|ccccc} & * & 1 & 2 & 3 & 4 \\ \hline * & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 4 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Given a directed graph $\mathcal{G}$ associated to a group, it will be convenient for us to augment $\mathcal{G}$ by adding an extra vertex, 0. We add directed edges from each vertex in $V\backslash\{*\}$ to 0 and also from 0 to itself. We extend the labelling ρ to these new edges by $\rho(x, 0) = e$ (the identity element in G) for all $x \in V \cup \{0\}\backslash\{*\}$.

Cannon proved that cocompact Kleinian groups are strongly Markov

[10]. Ghys and de la Harpe [25] showed that Cannon's approach worked for all hyperbolic groups. The augmentation method described above was first used by Lalley [38] to facilitate the use of thermodynamic formalism. More specifically Lalley introduced this augmentation so that group elements in G can be realised as infinite sequences, i.e. elements in a subshift of finite type.

Proposition 3.2.2 (Cannon [10], Ghys and de la Harpe [25])**.** *Any hyperbolic group is strongly Markov.*

Throughout the rest of this thesis, given a hyperbolic group G with generating set S, we use $\mathcal{G}$ to denote a directed graph associated to G and S via the strongly Markov property. We will always assume that such $\mathcal{G}$ has been augmented, to include the $*$ and 0 vertices, in the way described above. We note that G can admit infinitely many different graphs satisfying the properties in Definition 3.2.1.

This strongly Markov structure makes hyperbolic groups susceptible to analysis through the use of thermodynamic formalism and subshifts of finite type. Let G be a hyperbolic group with associated directed graph $\mathcal{G}$. Labelling the vertices of $\mathcal{G}$, $1, \ldots, k$, we can describe $\mathcal{G}$ by a $k \times k$ zero-one matrix A. We set the (i, j)th entry of A to be 1 if and only if there exists an edge from vertex i to vertex j. We call A the transition matrix associated to $\mathcal{G}$. We can then embed G into the shift space Σ_A via the function $i : G \to \Sigma_A$ defined by

$$i(g) = (*, x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots),$$

where $(*, x_0), (x_0, x_1), \ldots, (x_{n-2}, x_{n-1})$ is the unique shortest path in $\mathcal{G}$ corresponding to g and $|g| = n$. We use the notation $\dot{0}$ to denote the sequence in Σ_A consisting of only zeros.

Property (3) from Definition 3.2.1 implies that $\#W_n = \sum_{j \in V} A^n(*, j)$, i.e. the number of group elements in G of word length n is the same as the number of length n paths in $\mathcal{G} \backslash 0$ starting at $*$. Let B denote the matrix A with the columns and rows corresponding to the $*$ and 0 vertices removed.

Definition 3.2.3. Let $\mathcal{G}$, A and B be as above. We say that $\mathcal{G}$ is aperiodic (or irreducible) if B is aperiodic (or irreducible).

In general, it is possible that $\mathcal{G}$ is not irreducible. However, in certain cases, for example for surface groups with presentations $\langle a_1, \ldots, a_g, b_1, \ldots, b_g | \prod_{j=1}^{g} [a_j, b_j] \rangle$ (with $g \geq 2$) and free groups equipped with their canonical free generating sets, $\mathcal{G}$ can be chosen to be aperiodic [54]. When $\mathcal{G}$ is aperiodic, results from thermodynamic formalism apply more readily. One of the main difficulties in this work is overcoming the additional difficulties that arise in the case that $\mathcal{G}$ is neither aperiodic nor irreducible

Following [47], we can relabel the columns/rows of B to assume that B has the form

$$B = \begin{pmatrix} B_{1,1} & 0 & \dots & 0 \\ B_{2,1} & B_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \dots & B_{m,m} \end{pmatrix},$$

where the matrices $B_{i,i}$ are irreducible. The matrices $B_{i,i}$ are known as the irreducible components of B or $\mathcal{G}$. By property (3) in Definition 3.2.1 and Proposition 3.1.5, the spectral radius of each $B_{i,i}$ is bounded above by λ, the exponential growth rate of $\#W_n$. Furthermore, there must be at least one component that has λ as an eigenvalue (otherwise there would be $0 < \delta < \lambda$ for which $\#W_n = O((\lambda - \delta)^n)$).

Definition 3.2.4. We call an irreducible component maximal if its corresponding matrix has spectral radius λ.

An important property of $\mathcal{G}$ is the following.

Lemma 3.2.5. *[8, Lemma 3.4.2] Let $\mathcal{G}$ be a directed graph associated to G. The maximal components of $\mathcal{G}$ are disjoint. That is, there does not exist a path in $\mathcal{G}$ from one maximal component to another.*

Proof. Let B_1 and B_2 be maximal components and suppose there is a path P of length l from B_1 to B_2. Then for $n > l$, the number of length n paths in $\mathcal{G}$ would be at least

$$\sum_{r+s=n-l} B_1^r B_2^s, \tag{3.2.1}$$

where B_1^k denotes the number of length k paths contained in B_1 ending at the start vertex of P and B_2^k denotes the number of length k paths in B_2 starting at the end vertex of P. Quantity (3.2.1) grows like $n\lambda^n$ which implies that $\#W_n$ grows at least like $n\lambda^n$. This contradicts Proposition 3.1.5. $\square$

3.3 Properties of the Patterson–Sullivan measure

Our results rely on the work of Calegari and Fujiwara [9] that compares the Patterson–Sullivan measure ν to a natural measure μ on Σ_A. In this section we construct this measure and compare it to ν. To deduce our results we need to extend the work of Calegari and Fujiwara in [9] to obtain a deeper understanding of how the measures μ and ν compare.

Suppose that $\mathcal{G}$ has vertex set V. For $v \in \mathbb{R}^V$, define the function $p : \mathbb{R}^V \to \mathbb{R}^V$ by

$$p(v) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{A^k v}{\lambda^k}.$$

This function projects v to the eigenspace of A corresponding to the eigenvalue λ. Similarly, the function $r : \mathbb{R}^V \to \mathbb{R}^V$ defined by

$$r(v) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{(A^T)^k v}{\lambda^k}$$

projects v to the eigenspace of A^T corresponding to the eigenvalue λ. We need to know the rate of convergence associated to the limit defining p.

Lemma 3.3.1. *For $v \in \mathbb{R}^V$ we have that*

$$p(v) = \frac{1}{n} \sum_{k=0}^{n} \frac{A^k v}{\lambda^k} + O\left(\frac{1}{n}\right)$$

where the implied constant depends only on v.

Proof. Given $v \in \mathbb{R}^V$ we can write v as a linear combination of elements in a Jordan basis for A. Since maximal components are disjoint, if an eigenvalue x of A has absolute value λ, then there does not exist a Jordan chain of length strictly greater than one associated to x. A simple calculation then shows that if $\widetilde{v}$ belongs to the generalised eigenspace associated to the eigenvalue $x \neq \lambda$, then

$$\frac{1}{n} \sum_{k=0}^{n} \frac{A^k \widetilde{v}}{\lambda^k} = O\left(\frac{1}{n}\right).$$

The result follows. □

Let $\mathbf{1} \in \mathbb{R}^V$ denote the vector consisting of 1 in each coordinate and let v_* denote the vector consisting of a 1 in the coordinate corresponding to the $*$ vertex and zeros elsewhere. Using p and r, we define a measure μ on Σ_A via a stochastic matrix $N : \mathbb{R}^V \to \mathbb{R}^V$ and vertex distribution $\rho : V \to \mathbb{R}$. For a vector $v \in \mathbb{R}^V$, let v_j denote the coordinate of v corresponding to the vertex $j \in V$. The matrix N is defined as follows. Set

$$N_{i,j} = \frac{A_{i,j} p(\mathbf{1})_j}{\lambda p(\mathbf{1})_i}$$

if $p(\mathbf{1})_i \neq 0$ and $N_{i,i} = 1$, $N_{i,j} = 0$ (if $i \neq j$) when $p(\mathbf{1})_i = 0$. The vertex distribution ρ is defined by

$$\rho(j) = p(\mathbf{1})_j r(v_*)_j.$$

As for the usual construction of Markov measures, this defines a σ-invariant measure on Σ_A. We normalise this measure to obtain the probability measure μ. There is a nice description of μ in terms of thermodynamic formalism.

Proposition 3.3.2. *There exists $0 < \alpha_i < 1$ for $i = 1, \ldots, m$ with $\sum_{i=1}^m \alpha_i = 1$ such that*

$$\mu = \sum_{i=1}^{m} \alpha_i \mu_i, \tag{3.3.1}$$

where each μ_i is the measure of maximal entropy for the system (Σ_{B_i}, σ).

Proof. Choose a maximal component B_i. One can check that the vector obtained from restricting $p(\mathbf{1})$ or $r(v_*)$ to the vertices in B_i is a right or left eigenvector respectively for B_i (with eigenvalue λ). Then by comparing the construction of μ to Parry's construction of the measure of maximal entropy for a subshift of finite type [41], we see that the restriction of μ to the maximal component Σ_{B_i} is up to scaling, the measure of maximal entropy μ_i on this component. Furthermore, from the definitions of p and r, it is clear that μ assigns zero mass to the complement of the union of the maximal components. The result follows. $\square$

Let A' denote the matrix A with the row/column corresponding to the 0 vertex removed.

Definition 3.3.3. Define sets $Y, Y_1, \ldots, Y_m \subset \Sigma_{A'}$ by

$$Y = \{x \in \Sigma_{A'} : x_0 = *\},$$

$$Y_i = \{x \in Y : x \text{ eventually enters } B_i \text{ and never leaves}\}.$$

Let $h : Y \to \partial G$ be the natural map associated to the bijection defined in Definition 3.2.1. Given $y \in Y$, we use $h(y)_n$ to denote the nth step in the geodesic ray determined by y.

There is a measure $\widehat{\nu}$ on Y that pushes forward under h to the Patterson–Sullivan measure on ∂G. We denote the pushforward map by h_* so that $h_*\widehat{\nu} = \nu$.

The measure $\widehat{\nu}$ can be constructed as follows. Define, for $n \in \mathbb{Z}_{\geq 0}$,

$$\widehat{\nu}_n = \frac{\sum_{|g|\leq n} e^{-h|g|}\delta_{i(g)}}{\sum_{|g|\leq n} e^{-h|g|}}.$$

This is a sequence of measures living on Σ_A. Then $h_*\widehat{\nu}_n = \nu_n$ where

$$\nu_n = \frac{\sum_{|g|\leq n} e^{-h|g|}\delta_g}{\sum_{|g|\leq n} e^{-h|g|}}.$$

Now take a weak $*$ convergent subsequence $\widehat{\nu}_{n_k}$ of $\widehat{\nu}_n$ and suppose that $\widehat{\nu}_{n_k} \to \widehat{\nu}$ as $k \to \infty$. It is easy to see that $h : Y \to \partial G$ is continuous and hence

$$h_*\widehat{\nu} = \lim_{k\to\infty} h_*\widehat{\nu}_{n_k} = \lim_{k\to\infty} \nu_{n_k} = \nu.$$

We will now see that we can explicitly calculate the $\widehat{\nu}$ measure of cylinder sets. Given a finite path in $\mathcal{G}$ let $[y]$ denote the elements in $\Sigma_{A'}$ that have y as an initial segment.

Lemma 3.3.4. *Let y be a finite path in $\mathcal{G}$ starting at $*$. We have that*

$$\widehat{\nu}([y]) = \frac{p(\mathbf{1})_{v_y}}{p(\mathbf{1})_*}\lambda^{-|y|},$$

where $|y|$ is the length of y and v_y denotes the last vertex in y.

Proof. Since $[y]$ is both open and closed, $\widehat{\nu}[y] = \lim_{k\to\infty} \widehat{\nu}_{n_k}[y]$. However, for $n \geq |y|$,

$$\begin{aligned}\widehat{\nu}_n[y] &= \frac{\sum_{|g|\leq n} e^{-h|g|}\delta_g[y]}{\sum_{|g|\leq n} e^{-h|g|}} \\ &= \frac{\left(\frac{1}{n}\sum_{j=|y|}^{n} \frac{A^{j-|y|}(\mathbf{1})}{\lambda^{j-|y|}}\right)_{v_y} \lambda^{-|y|}}{\left(\frac{1}{n}\sum_{j=0}^{n} \frac{A^{j}(\mathbf{1})}{\lambda^{j}}\right)_*}\end{aligned}$$

and from the definition of p we see that this converges to the required expression. $\square$

The proof of Lemma 3.3.4 shows that $\widehat{\nu}_n \to \widehat{\nu}$ as $n \to \infty$, i.e. we need not look at convergence along a subsequence n_k. For $k \in \mathbb{Z}_{\geq 0}$, let $\sigma^k_*\widehat{\nu}$ denote the pushforward of $\widehat{\nu}$ under σ^k. The following lemma compares these pushforward measures to the measure μ.

Lemma 3.3.5. *For each $v \in V$ with $\mu[v] > 0$ and $k \in \mathbb{Z}_{\geq 0}$ there exists $\alpha_v^k \geq 0$ such that*

$$\sigma_*^k \widehat{\nu}|_{[v]} = \alpha_v^k \mu|_{[v]}.$$

There exists a length k path from $*$ to v if and only if $\alpha_v^k > 0$. If $\mu[v] = 0$ we define $\alpha_v^k = \widehat{\nu}(\sigma^{-k}[v])$ for all $k \in \mathbb{Z}_{\geq 0}$. Furthermore,

$$\frac{1}{n}\sum_{k=0}^{n} \alpha_v^k = \begin{cases} 1 + O(n^{-1}) & \text{if } \mu[v] > 0 \\ O(n^{-1}) & \text{if } \mu[v] = 0. \end{cases}$$

The implied constants can be taken to be independent of v.

Proof. This is a consequence of Lemma 3.3.1, the construction of $\widehat{\nu}$ and the proof of Lemma 4.22 in [9]. A simple calculation using the definition of $\widehat{\nu}$ shows the existence of α_v^k satisfying the first condition of the lemma. The convergence associated to the final statement is proved in Lemma 4.22 of [9]. By inspecting the proof of this lemma, we see that Lemma 3.3.1 quantifies the convergence as $O(n^{-1})$. □

It follows that

$$\frac{1}{n}\sum_{k=0}^{n} \sigma_*^k \widehat{\nu}$$

converges in the weak * topology to the measure μ. There is a much stronger relationship between $\widehat{\nu}$ and μ however. Given two measures, λ_1 and λ_2 on Σ_A, recall that their total variation $\|\lambda_1 - \lambda_2\|_{TV}$ is given by $\sup_{E \subset \Sigma_A} |\lambda_1(E) - \lambda_2(E)|$.

Proposition 3.3.6. *We have that,*

$$\left\| \frac{1}{n}\sum_{j=0}^{n} \sigma_*^j \widehat{\nu} - \mu \right\|_{TV} = O(n^{-1})$$

as $n \to \infty$.

Proof. For any $E \subset \Sigma_A$,

$$\begin{aligned} \left| \frac{1}{n}\sum_{j=0}^{n} \sigma_*^j \widehat{\nu}(E) - \mu(E) \right| &= \left| \frac{1}{n}\sum_{j=0}^{n} \sum_{v \in V} \left(\sigma_*^j \widehat{\nu}|_{[v]}(E) - \mu|_{[v]}(E) \right) \right| \\ &\leq \sum_{\substack{v \in V \\ \mu[v] > 0}} \left| \frac{1}{n}\sum_{j=0}^{n} \alpha_v^j - 1 \right| + \sum_{\substack{v \in V \\ \mu[v] = 0}} \left| \frac{1}{n}\sum_{j=0}^{n} \alpha_v^j \right|, \end{aligned}$$

where α_v^j are as defined in the previous lemma. Applying the previous lemma concludes the proof. $\square$

We will need the following definition and lemma later.

Definition 3.3.7. For each $j \in \mathbb{Z}_{\geq 0}$ let

$$A_j = \left(\sigma^{-j}\left(\bigcup_i \Sigma_{B_i}\right) \setminus \bigcup_{k=0}^{j-1} \sigma^{-k}\left(\bigcup_i \Sigma_{B_i}\right)\right) \cap Y.$$

Then, for each $n \in \mathbb{Z}_{\geq 0}$, define a measure $\widehat{\nu}_n$ on $\Sigma_{A'}$ by

$$\widehat{\nu}_n(E) = \widehat{\nu}\left(E \cap \bigcup_{j=0}^{n} A_j\right)$$

for $E \subset \Sigma_{A'}$.

Intuitively, each A_j consists of elements in $\Sigma_{A'}$ that correspond to a path in $\mathcal{G}$ that starts at $*$, enters a maximal component on exactly its jth step and then never leaves this component.

Lemma 3.3.8. *There exists $0 < \theta < 1$ such that $\|\widehat{\nu}_n - \widehat{\nu}\|_{TV} = O(\theta^n)$, as $n \to \infty$.*

Proof. We claim that

$$\widehat{\nu}\left(\bigcup_{j>n} A_j\right) \to 0$$

exponentially quickly as $n \to \infty$. To see this, note that the number of length n paths in $\mathcal{G}$ that start at $*$ and do not enter a maximal component is $O((\lambda-\delta)^n)$ for some $0 < \delta < \lambda$. Combining this observation with Lemma 3.3.4 implies that there exists $C > 0$ independent of j, n such that

$$\widehat{\nu}\left(\bigcup_{j>n} A_j\right) \leq C \sum_{j>n} \left(\frac{\lambda-\delta}{\lambda}\right)^j.$$

This proves the claim. Along with Lemma 3.3.4, this shows that $Y \setminus \cup_{i=1}^m Y_i$ can be written as a countable union of zero $\widehat{\nu}$ measure sets. Hence $\widehat{\nu}\left(Y \setminus \cup_{i=1}^m Y_i\right) = 0$ and for any $E \subset Y$,

$$\widehat{\nu}(E) - \widehat{\nu}_n(E) = \widehat{\nu}\left(E \cap \bigcup_{j>n} A_j\right) \leq \widehat{\nu}\left(\bigcup_{j>n} A_j\right).$$

Applying the claim a further time concludes the proof. $\square$

We end this section by observing that, for any $E \subset \cup_i \Sigma_{B_i}$,

$$\sigma_*^j \widehat{\nu}(E) = \sigma_*^j \widehat{\nu}_j(E). \tag{3.3.2}$$

3.4 Regularity of functions

In this section we discuss the regularity conditions required for functions to satisfy our theorems. Fix a generating set S for G. We are interested in functions $\varphi : G \to \mathbb{R}$ that satisfy two conditions, which we name Condition (1) and Condition (2).

Condition (1) There exists a directed graph $\mathcal{G}$ associated to G, S via the strongly Markov property with transition matrix A and a function $f \in F_\theta(\Sigma_A)$ (for some $0 < \theta < 1$) such that $\varphi(g) = f(x) + f(\sigma(x)) + \ldots + f(\sigma^{|g|-1}(x))$ for $g \in G$ and $x = i(g) \in \Sigma_A$.

Condition (2) φ is Lipschitz in the left and right word metrics on G.

We now discuss examples of functions that satisfy Condition (1). The first class we consider is of functions that satisfy the following Hölder condition.

Definition 3.4.1. We say that a map $\varphi : G \to \mathbb{R}$ is Hölder (for G and S) if for any fixed $a \in G$ there exists $C > 0$ and $0 < \theta < 1$ such that

$$|\Delta_a \varphi(g) - \Delta_a \varphi(h)| \leq C\theta^{(g,h)},$$

for any $g, h \in G$. Here $\Delta_a \varphi(g) = \varphi(ag) - \varphi(g)$.

Pollicott and Sharp proved that any function satisfying the above Hölder condition for G, S, satisfies Condition (1) (see Lemma 1 of [47]). In fact, they showed that for such functions, one can find an appropriate Hölder function $f : \Sigma_A \to \mathbb{R}$ given any graph $\mathcal{G}$ associated to G, S. We note that functions satisfying Definition 3.4.1 are always Lipschitz in the left word metric. This can be seen by setting h to be e, the identity of G, in Definition 3.4.1. When $h = e$, $\Delta_a \varphi(h) = \varphi(a) - \varphi(e)$ for any $a \in S$ and hence $|\varphi(ag) - \varphi(g)| \leq C + |\varphi(a)| + |\varphi(e)|$ for all $g \in G$. It is easy to see that this implies φ to be Lipschitz in the left word metric. Inspired by the work of Calegari and Fujiwara, we introduce the following class of functions.

Definition 3.4.2. Suppose S is symmetric. Given an element $g \in G$, there is a unique path of length $|g|$ in $\mathcal{G}$, starting at $*$, that is mapped to g under the bijection defined in part (3) of Definition 3.2.1. Let e_i^g belong to the edge set of $\mathcal{G}$ and let it denote the ith edge in the path corresponding to g. A map $\varphi : G \to \mathbb{R}$ is called edge combable (with respect to $\mathcal{G}$) if there exists a function $d\varphi$ from the edge set of $\mathcal{G}$ to $\mathbb{R}$ such that, for each $g \in G$,

$$\varphi(g) = \sum_{i=1}^{|g|} d\varphi(e_i^g).$$

We refer to $d\varphi$ as a (discrete) derivative of φ.

Remark 3.4.3. In [9] Calegari and Fujiwara define the class of combable functions. These functions are similar to edge combable functions except that a derivative $d\varphi$ is a function from the vertex set of $\mathcal{G}$ to $\mathbb{Z}$. The equation relating φ and $d\varphi$ is the same except the sum is taken over the vertices in the path corresponding to g. Given a combable function φ, one can consider φ as an edge combable function. To see this, take the derivative $d\varphi$ of φ (which is a function defined on the vertex set of $\mathcal{G}$) and define $d\varphi'$ on the edge set of $\mathcal{G}$ to send an directed edge to the value of $d\varphi$ evaluated at the end point of this edge. It is easy to see that φ can be considered an edge combable function with derivative $d\varphi'$. Therefore the set of edge combable functions contains the set of combable functions.

Remark 3.4.4. Suppose that φ is edge combable with respect to $\mathcal{G}$ and that $d\varphi$ is integer valued. Then we can find a different directed graph $\mathcal{G}'$ that satisfies the properties in Definition 3.2.1 and for which φ is combable. To see this, consider the following recoding of $\mathcal{G}$ to $\mathcal{G}'$. Define the vertex set for $\mathcal{G}'$ to be the edge set of $\mathcal{G}$ and say that two vertices u and v in $\mathcal{G}'$ are connected by a directed edge from u to v if the edges e, r in $\mathcal{G}$ corresponding to u, v are concurrent in $\mathcal{G}$. This process may introduce multiple $*$ vertices for $\mathcal{G}'$, however, we can simply identify these vertices to overcome this problem.

The above discussions imply that the class of edge combable functions includes combable functions and real valued homomorphisms.

Lemma 3.4.5. *Edge combable functions satisfy Condition* (1)*.*

Proof. Let φ be edge combable with derivative $d\varphi$. For $x = (x_n)_{n=0}^{\infty} \in \Sigma_A$, define

$$f(x) = \begin{cases} d\varphi((x_0, x_1)) & x_1 \neq 0 \\ 0 & x_1 = 0, \end{cases}$$

where ρ denotes the labelling map defined in Definition 3.2.1. Since f is constant on cylinders of length 2, $f \in F_\theta(\Sigma_A)$ for any $0 < \theta < 1$. To see that Condition

(1) is satisfied, note that

$$
\begin{aligned}
f^{|g|}(i(g)) &= \sum_{k=0}^{n-1} f(\sigma^k(*, y_0, \ldots, y_{n-1}, \dot{0})) \\
&= \sum_{i=1}^{n} d\varphi(e_i^g) \\
&= \varphi(g).
\end{aligned}
$$

□

We have now seen examples of functions that satisfy Condition (1). A large class of functions that satisfy Condition (2) are quasimorphisms.

Definition 3.4.6. A function $\varphi : G \to \mathbb{R}$ is a quasimorphism if there exists a constant $A > 0$ such that

$$|\varphi(gh) - \varphi(g) - \varphi(h)| \leq A$$

for all $g, h \in G$.

Quasimorphisms are a natural generalisation of homomorphisms. Indeed, quasimorphisms are homomorphisms up to a uniformly bounded error. Note that bounded functions are also quasimorphisms. A necessary and sufficient condition for a hyperbolic group G to admit a non-trivial real valued homomorphism is that the rank of the abelianisation of G must be greater than or equal to 1. This is because any homomorphism $\varphi : G \to \mathbb{R}$ factors through the abelianisation of G. Hence if a hyperbolic group has finite abelianisation it does not admit any non-trivial homomorphisms to $\mathbb{R}$. However, a result of Epstein and Fujiwara [18] shows that every hyperbolic group G admits uncountably many real valued quasimorphisms.

We now consider the class of functions satisfying both Condition (1) and Condition (2). In [9] Calegari and Fujiwara consider combable functions that are Lipschitz in the left and right words metrics on G. Furthermore, they prove that the class consisting of these functions is independent of the choice of symmetric S and $\mathcal{G}$ associated to G. Hence our results apply to all functions considered by Calegari and Fujiwara in [9]. In particular our results apply to Brooks counting quasimorphisms. We will now define and discuss Brooks counting quasimorphisms on free groups equipped with their canonical free generating sets. The general definition of a Brooks counting quasimorphism (for arbitrary hyperbolic groups and generating sets) is a bit more technical - see [7] for a general definition.

Suppose G is a free group equipped with its free generating set and $w \in G$ is an element of G. We define the Brooks counting quasimorphism φ_w associated to w as follows. Given $g \in G$ let $\phi_w : G \to \mathbb{R}$ be the function that counts the total number of copies of w that appear in the unique shortest word representing g. We define

$$\varphi_w = \phi_w - \phi_{w^{-1}}.$$

The difficulty in extending this definition to arbitrary hyperbolic groups is that there does not necessarily exist a unique shortest word expression for a given group element. In [9] the authors observe that Brook's counting quasimorphisms are not necessarily Hölder.

However, in [31], [32] Horsham and Sharp consider Hölder quasimorphisms. As discussed above, these functions satisfy Condition (1) and Condition (2). Hence our results also apply to these functions. The following is an example of a Hölder quasimorphism that is due to Barge and Ghys [1].

Example: Suppose G acts cocompactly by isometries on a simply connected Riemannian surface X with all sectional curvatures bounded above by -1. Write $M = X/G$. Given a smooth 1-form ω on M, we can lift ω to a G-invariant smooth 1-form $\widetilde{\omega}$ on X. Fix an origin $o \in X$ and define $\varphi : G \to \mathbb{R}$ by

$$\varphi(g) = \int_o^{go} \widetilde{\omega}.$$

Note that

$$\varphi(gh) - \varphi(g) - \varphi(h) = \int_{\partial T(g,h)} \widetilde{\omega} = \int_{T(g,h)} d\widetilde{\omega}$$

where $T(g,h)$ denotes the triangle in X with vertices o, go and gho. The last equality in the above follows from Stoke's Theorem. By compactness and hyperbolicity, the right hand side of the above is bounded uniformly in g, h. This proves that φ is a quasimorphism. In [43] Picaud proved that these quasimorphisms satisfy Condition (1).

There are many other examples of functions satisfying Conditions (1) and (2). See, for example, [1], [18] and [25]. As discussed in the introduction, the following examples are of particular interest to us.

Let (X, d) be a complete CAT(-1) geodesic metric space. A group G is said to act convex cocompactly on X if the quotient of the intersection of X and the convex hull (in X) of the limit set of G, is compact. Suppose G acts properly discontinuously, convex cocompactly by isometries on X. Fix a finite generating set for G and an origin o (in the convex hull of the limit set of G) for X.

Lemma 3.4.7. *In the setting described above, the displacement function $g \mapsto d(o, go)$ satisfies Condition (1) and Condition (2).*

Proof. The fact that the displacement satisfies Condition (1) is due to Pollicott and Sharp. This was proved in [48] (see Proposition 3) when G acts on a negatively curved manifold X. However, the only property of X required for the proof is the CAT(-1) property and hence the proof applies to our case also. Showing that Condition (2) is satisfied is a simple exercise. □

3.5 Spectral properties of transfer operators

3.5.1 Spectral description of certain transfer operators

Let G, S have associated directed graph $\mathcal{G}$ described by transition matrix A. To deduce our main results, we analyse the following weighted sum

$$\sum_{g \in W_n} e^{s\varphi(g)},$$

for small complex s as $n \to \infty$. We want to express this sum in terms of transfer operators. To form a useful expression, we exploit the structure of $\mathcal{G}$ and in particular, use the fact that maximal components are disjoint. We therefore consider transfer operators of a specific form. The aim of this section is to define and study these operators.

Definition 3.5.1. For $f \in F_\theta(\Sigma_A)$ define the transfer operator $L_{A,f} : F_\theta(\Sigma_A) \to F_\theta(\Sigma_A)$ by

$$L_{A,f}g(x) = \sum_{\substack{\sigma(y)=x \\ y \in \Sigma_A \setminus \{\dot{0}\}}} e^{f(y)} g(y).$$

Note that these transfer operators vary slightly from those previously defined, as we are excluding $\dot{0}$ as a possible preimage in the sum defining the operators. Pollicott and Sharp studied the spectral properties of these operators in [47].

Let B_i for $i = 1, \ldots, m$ denote the maximal components of A.

Definition 3.5.2. For each $i = 1, \ldots, m$, define a matrix C_i by,

$$C_i(u, v) = \begin{cases} 0 & \text{if } u \text{ or } v \text{ belong to a maximal component that is not } B_i, \\ A(u, v) & \text{otherwise.} \end{cases}$$

We define $L_{B_i,f}$ and $L_{C_i,f}$ analogously to $L_{A,f}$. Note that the operators $L_{B_i,f}$ are the same as the operators L_f acting on $F_\theta(\Sigma_{B_i})$ as given in Definition 2.2.

We want to understand the spectral properties of the operators $L_{C_i,sf}$ for small complex s. We analyse the operators in the case that $s = 0$ and then use perturbation theory to obtain our desired result. Suppose that λ is the exponential growth rate of $\#W_n$. It is well known that for each i, $L_{B_i,0}$ has the same simple maximal eigenvalues as B_i. These maximal eigenvalues have modulus λ since the B_i are maximal components. From our discussion in Section 2, λ is equal to e^h where h denotes the topological entropy of the system (Σ_{B_i}, σ). We want to show that $L_{C_i,0}$ has essentially the same spectrum as $L_{B_i,0}$.

Lemma 3.5.3. *Suppose each B_i has cyclic period p_i. Then, the operators $L_{C_i,0}$ are quasicompact, have spectra that consist of p_i finite multiplicity maximal eigenvalues at $e^{2\pi ik/p_i}e^h$ for $k = 0, 1, \ldots, p_i - 1$. The rest of the spectrum is contained in the disk $\{z : |z| < e^h - \delta\}$ for some $\delta > 0$.*

Proof. The proof is basically an application of Lemma 2 from [47]. Quasicompactness of the operators follows immediately. By relabelling the columns of C_i, we can rewrite each C_i in the form

$$C_i = \begin{pmatrix} C_{1,1} & 0 & \ldots & 0 \\ C_{2,1} & C_{2,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m,1} & C_{m,2} & \ldots & C_{m,m} \end{pmatrix}$$

where the $C_{j,j}$ correspond to irreducible components of $\mathcal{G}$. By construction all maximal components have corresponding matrix 0 except for the matrix corresponding to B_i. Let

$$P = \begin{pmatrix} C_{1,1} & 0 & 0 & \ldots & 0 \\ 0 & C_{2,2} & 0 & \ldots & 0 \\ 0 & 0 & C_{3,3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & C_{m,m} \end{pmatrix}$$

Lemma 2 in [47] states that the operators $L_{C_i,0}$ and $L_{P,0}$ have the same isolated eigenvalues. It is easy to see that the spectrum of $L_{P,0}$ consists of p_i finite multiplicity eigenvalues, $e^{2\pi ik/p_i}e^h$ for $k = 0, \ldots, p_i - 1$ and the rest of the spectrum is contained in $\{z : |z| < e^h - \delta\}$ for some $\delta > 0$. Quasicompactness of the $L_{C_i,0}$ now implies the result. $\square$

One can check that the finite multiplicity eigenvalues from the above lemma are in fact simple. Let B_i^* denote the matrices that describes the subgraph of $\mathcal{G}$ that contains the vertices in B_i, the 0 vertex and all edges between these vertices that are allowed by A. There are a few steps in showing that the eigenvalues in the above lemma are simple. We show that each of the following statements can be deduced from the previous one.

1. The maximal eigenvalue for $L_{B_i^*,0}$ is simple in the case that B_i is aperiodic.
2. The maximal eigenvalues for $L_{B_i^*,0}$ are simple in the case that B_i is irreducible.
3. The maximal eigenvalues for $L_{C_i,0}$ are simple when C_i is irreducible.

Statement (1) in the above is well known [35], [47]. We will show how to deduce (2) from (1) and (3) from (2).

Proof of (1) $\implies$ (2). Suppose that B_i is irreducible. Recall that there exists p_i, the period of B_i, such that Σ_{B_i} has p_i-cyclic decomposition

$$\Sigma_{B_i} = \bigsqcup_{k=0}^{p_i-1} \Sigma_{B_i^k}.$$

$L_{B_i,0}$ has spectrum containing maximal eigenvalues at $e^{2\pi ik/p_i}e^h$ for $k = 0, 1, \ldots, p_i - 1$. The rest of the spectrum is contained in a disk of radius strictly smaller than e^h.

The p_ith iterates of the transfer operators $L_{B_i,0}^{p_i}$ act as the direct sum of operators $L_{B_i^k,0}^{p_i}$ for $k = 0, \ldots, p_i - 1$ each acting on $F_\theta(\Sigma_{B_i^k})$ respectively. The analogous statement is true for the $L_{B_i^*,0}^{p_i}$. The following notation expresses this,

$$L_{B_i^*,0}^{p_i} = \left(L_{B_{i,0}^*,0}^{p_i}, L_{B_{i,1}^*,0}^{p_i}, \ldots, L_{B_{i,p_i-1}^*,0}^{p_i}\right),$$

$$L_{B_i,0}^{p_i} = \left(L_{B_i^0,0}^{p_i}, L_{B_i^1,0}^{p_i}, \ldots, L_{B_i^{p-1},0}^{p_i}\right).$$

Here, each $B_{i,k}^*$ corresponds to B_i^k with the 0 vertex (and all edges to the 0 vertex) added back in. We will continue to use the above notation through out the rest of this work.

Each $(\Sigma_{B_{i,k}^*}, \sigma^{p_i})$ is a subshift of finite type of the same form as the aperiodic case from (1). We know that, for each k, $L_{B_i^k,0}^{p_i}$ has simple maximal eigenvalue $e^{p_i h}$ and hence $L_{B_{i,k}^*,0}^{p_i}$ does also. From the definition of $L_{B_i^*,0}$ it is easy to see that the spectrum of $L_{B_i^*,0}$ consists of simple maximal eigenvalues at $e^{2\pi ik/p_i}e^h$ for $k = 0, 1, \ldots, p_i - 1$ and the rest of the spectrum is contained in the disk $\{z : |z| < e^h - \delta\}$ for some $\delta > 0$. This concludes the proof. $\square$

Proof of (2) $\implies$ (3). Suppose $g \in F_\theta(\Sigma_{B_i^*})$ is the eigenfunction for the eigenvalue $e^{2\pi ik/p_i}e^h$ for $L_{B_i^*,0}$. Let h be an eigenfunction corresponding to the eigenvalue $e^{2\pi ik/p_i}e^h$ for $L_{C_i,0}$. Suppose there exists $x \in \Sigma_{C_i}$ such that x_0 does not belong to B_i but there exists a path from x_0 into B_i. Then,

$$\begin{aligned} e^{p_i hn}|h(x)| &= |L_{C_i,0}^{p_i n}h(x)| \\ &\leq \sum_{\substack{\sigma^{p_i n}(y)=x \\ y\in\Sigma_{C_i}\setminus\{\dot{0}\}}} |h(y)| \\ &= \sum_{\substack{\sigma^{p_i n}(y)=x \\ y\in\Sigma_{C_i}\setminus\{\dot{0}\}: y_0,\ldots,y_{p_i n-1} \text{ are not in } B_i}} |h(y)|. \end{aligned}$$

However, the growth of the number of length n paths in $\mathcal{G}$, starting at $*$, that do not enter a maximal component, is $o(e^{hn})$. This implies that

$$e^{p_i hn}|h(x)| = o(e^{p_i hn}),$$

which forces $h(x) = 0$. Hence h is zero on

$$S := \{x \in \Sigma_{C_i} : x_0 \text{ is not in } B_i \text{ and there exists a path from } x_0 \text{ into } B_i \text{ in } \mathcal{G}\}.$$

We deduce that $h|_{\Sigma_{B_i^*}}$ is an eigenfunction for $L_{B_i^*,0}$. Now, suppose $L_{C_i,0}$ has another eigenfunction for the eigenvalue $e^{2\pi ik/p_i}e^h$. Then, by taking a linear combination of h and this new eigenfunction, we can assume that there exists an non-zero eigenfunction for $L_{B_i^*,0}$ that is zero on the set

$$\{x \in \Sigma_{C_i} : \text{there exists a path from } x_0 \text{ into } B_i \text{ in } \mathcal{G}\}.$$

However, by taking x such that $h(x) \neq 0$ and running the same growth argument as before, we see that any such eigenfunction cannot exists. Hence $L_{C_i,0}$ has algebraically simple eigenvalues at $e^{2\pi ik/p_i}e^h$ for $k = 0, \ldots, p_i - 1$.

To see geometric simplicity a similar argument can be applied. Suppose

$L_{C_i,0}$ has Jordan chain

$$\begin{aligned}
&g_{n-1} \\
&g_{n-2} = \left(L_{C_i,0} - e^{2\pi i k/p_i} e^h\right) g_{n-1} \\
&\vdots \\
&g = \left(L_{C_i,0} - e^{2\pi i k/p_i} e^h\right) g_1,
\end{aligned}$$

for $n \geq 3$. Then we see that there exists bounded linear operators $P_j(n)$ such that for each j and $n \in \mathbb{Z}_{\geq 0}$,

$$L_{C_i,0}^{p_i n} g_j = e^{np_i h} g_j + P_j(n) g_{j-1}.$$

By the same growth argument as before, if g_{j-1} is 0 on the set S, then g_j is also 0 on S. Hence, by induction, all the g_j are 0 on S. This implies that $g, g_1, \ldots, g_{n-1}$ restricts to a Jordan chain for $L_{B_i^*,0}$ which in turn implies $g|_{B_i^*} = 0$, a contradiction. This concludes the proof. $\square$

In summary we have shown.

Proposition 3.5.4. *Suppose each B_i has cyclic period p_i. Then, there exists $\delta > 0$ such that the operators $L_{C_i,0}$ have spectra that consist of p_i simple maximal eigenvalues at $e^{2\pi i k/p_i} e^h$ for $k = 0, 1, \ldots, p_i - 1$ and the rest of the spectrum is contained in $\{z : |z| < e^h - \delta\}$.*

We now study the perturbed operators $L_{C_i,sf}$. By Proposition 3.5.4, upper semi-continuity of the spectrum and Proposition 2.2.5, for all sufficiently small (complex) s, $L_{C_i,sf}$ has p_i simple maximal eigenvalues and exhibits a spectral gap to the rest of the spectrum. This gap is uniform for s in a small neighbourhood of the origin. Our aim is to show that, as we perturb $L_{C_i,0}$, these simple maximal eigenvalues vary in the same way. Specifically, we want to show that for small s, $L_{C_i,sf}$ has p_i simple maximal eigenvalues of the form $\lambda_s e^{2\pi i k/p_i}$ for $k = 0, \ldots, p_i - 1$, where $s \mapsto \lambda_s$ is analytic.

By Lemma 2 in [47], for sufficiently small s, the simple maximal eigenvalues of $L_{C_i,sf}$ are those of $L_{B_i,sf}$. Hence it suffices to study small perturbations of $L_{B_i,0}$. Suppose Σ_{B_i} has cyclic decomposition $\bigsqcup_{k=0}^{p_i-1} \Sigma_{B_i^k}$ as before.

We consider the p_i th iterate of $L_{B_i,0}$,

$$L_{B_i,sf}^{p_i} = \left(L_{B_i^0,sf}^{p_i}, L_{B_i^1,sf}^{p_i}, \ldots, L_{B_i^{p_i-1},sf}^{p_i}\right).$$

The systems $(\Sigma_{B_i^k}, \sigma^{p_i})$ are aperiodic subshifts and $L_{B_i^k,sf}^{p_i}$ acts as $L_{B_i^k,sf^{p_i}}$ on this system. Define $\tilde{f}^k : \Sigma_{B_i^k} \to \mathbb{R}$ by $\tilde{f}^k(x) = f^{p_i}(x)$. We can choose $\epsilon > 0$ such that for $|s| < \epsilon$ each of the $L_{B_i^k, sf^{p_i}}$ have a simple maximal eigenvalue $e^{P(s\tilde{f}^k)}$ and exhibit a spectral gap to the rest of the spectrum. Fix $|s| < \epsilon$. We deduce that the spectrum of $L_{B_i,sf}^{p_i}$ consists of a finite multiplicity maximal eigenvalue $\lambda := e^{P(s\tilde{f}^l)}$ for some $l \in \{0, \dots, p_i - 1\}$ and the rest of the spectrum is contained in a disk, centered at the origin, of radius strictly less than $|\lambda|$. It is easy to see that if x is in the spectrum of $L_{B_i,sf}^{p_i}$, then one of the p_ith roots of x must be in the spectrum of $L_{B_i,sf}$. Furthermore, each element in the spectrum of $L_{B_i,sf}$ is the p_ith root of an element in the spectrum of $L_{B_i,sf}^{p_i}$. By quasicompactness $L_{B_i,sf}$ has an eigenvalue that is a p_ith root of λ. Suppose g_0 is the associated eigenfunction. Note that g_0 restricted to $\Sigma_{B_i^k}$ is an eigenfunction for each k satisfying $L_{B_i^k,sf}^{p_i} g_0|_{B_i^k} = \lambda g_0|_{B_i^k}$. It follows from the definition of the transfer operator, that for each k, $g_0|_{B_i^k}$ is not identically zero (otherwise g_0 would be identically zero). We deduce that for all s sufficiently small, the eigenvalues $e^{P(s\tilde{f}^k)}$ agree for all k. It follows that for all s sufficiently small, the spectrum of $L_{B_i,sf}$ consists of p_i simple maximal eigenvalues of the form $e^{2\pi i k/p_i} e^{P(s\tilde{f}^0)/p_i}$ for $k = 0, 1, \dots, p_i - 1$ and the rest of the spectrum is contained in a disk of radius strictly less than the modulus of $e^{P(s\tilde{f}^0)/p_i} - \delta$, for some $\delta > 0$. To simplify notation we write $P_i(sf)$ to denote $P(s\tilde{f}^0)/p_i$. To summarise, we have shown the following.

Proposition 3.5.5. *There exists $\epsilon, \delta > 0$ such that for all $|s| < \epsilon$, $L_{C_i,sf}$ has p_i simple maximal eigenvalues $e^{2\pi i k/p_i} e^{P_i(sf)}$ for $k = 0, \dots, p_i - 1$, these are contained in the δ neighbourhood of $\{e^{2\pi i k/p_i} e^h : k = 0, \dots, p_i - 1\}$ and the rest of the spectrum is contained in the disk $|z| < e^h - 2\delta$.*

Let ϵ be as in the above proposition. We use $B(F_\theta(\Sigma_A))$ to denote the Banach algebra of bounded linear operators over Σ_A. Results from analytic perturbation theory (see Theorem 6.17 in [34]) imply that there exist analytic projection valued functions $Q_{i,k} : \{s \in \mathbb{C} : |s| < \epsilon\} \to B(F_\theta(\Sigma_A))$ such that $Q_{i,k}(s)$ projects a function in $F_\theta(\Sigma_{C_i})$ to the one-dimensional eigenspace associated to the simple maximal eigenvalue $e^{2\pi i k/p_i} e^{P_i(sf)}$ of the operator $L_{C_i,sf}$.

3.5.2 Comparing pressure across maximal components

In this section we show that, as we perturb the operators $L_{C_i,0}$, the simple maximal eigenvalues from Proposition 3.5.5 vary in a similar way. Specifically, we show that the quantities

$$\Lambda_i := \frac{dP_i(sf)}{ds}\Big|_{s=0} \text{ and } \sigma_i^2 := \frac{d^2 P_i(sf)}{ds^2}\Big|_{s=0}$$

are independent of the maximal component B_i.

To show that these quantities agree across components, we appeal to the work of Calegari and Fujiwara. We will use the argument presented in [8] and [9]. To apply this argument, we need the following technical lemma.

Lemma 3.5.6. *Suppose $r = (r_k)_{k=0}^{\infty} \in \Sigma_A$ with $r_0 = *$. Write $\tilde{r}_k \in G$ to denote the group element corresponding to the path $(*, r_1, \ldots, r_{k-1}, \dot{0})$ in $\mathcal{G}$ under the bijection from Definition 3.2.1. Then,*

$$f^n(\sigma^k(r)) = \varphi(\tilde{r}_{n+k}) - \varphi(\tilde{r}_k) + O(1),$$

where the above error term constant is independent of r, k and n.

Proof. Given $n, k \in \mathbb{Z}_{\geq 0}$ and $r \in \Sigma_A$, define $s_1, s_2, s_3 \in \Sigma_A$ by

$$s_1 = (*, r_1, \ldots, r_{k-1}, \dot{0}), \quad s_2 = (*, r_1, \ldots, r_{k+n-1}, \dot{0}), \quad s_3 = (r_k, r_{k+1}, \ldots, r_{k+n-1}, \dot{0}).$$

Then, by the Hölder property of f, there exists $C > 0$ independent of n, k and r, such that

$$|f^n(\sigma^k(r)) - f^n(s_3)| \leq C.$$

Then, note that $f^n(s_3) + f^k(s_2) = f^{n+k}(s_2)$ and also that there exists $C' > 0$ independent of n, k and r, such that

$$|f^k(s_2) - f^k(s_1)| \leq C'.$$

Finally, by Condition (1),

$$f^k(s_1) = \varphi(\tilde{r}_k) \text{ and } f^{k+n}(s_2) = \varphi(\tilde{r}_{k+n})$$

and so

$$\begin{aligned} f^n(\sigma^k(r)) &= f^n(s_3) + O(1) \\ &= f^{n+k}(s_2) - f^k(s_2) + O(1) \\ &= \varphi(\tilde{r}_{n+k}) - \varphi(\tilde{r}_k) + O(1), \end{aligned}$$

where the implied constant term is independent of n, k and r. $\square$

The main result of this section is the following. Recall that ν denotes the Patterson–Sullivan measure on ∂G.

Proposition 3.5.7. *The quantities, Λ_i and σ_i^2 do not depend on $i = 1, \ldots, m$.*

Proof. Let $Y \subset \Sigma_A$ and $h : Y \to \partial G$ be as defined in Definition 3.3.3. An important fact, on which this proof relies, is that if $\mu_i(E) > 0$ for some set $E \subset \Sigma_{B_i}$, then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\sigma_*^k \widehat{\nu}(E) > 0$. This property, which follows easily from Proposition 3.3.5, is the key ingredient that allows us to compare the $\widehat{\nu}$ measure with the μ measure.

The measure μ_i is ergodic with respect to σ on Σ_{B_i} and by the ergodic theorem, if $g \in L^1(\Sigma_{B_i}, \mu_i)$ then

$$\frac{1}{m} g^m(z) \to \int g \; d\mu_i,$$

as $m \to \infty$, for μ_i a.e $z \in \Sigma_{B_i}$. We define

$$F(n, x) = \left\{ r \in \Sigma_{B_i} : \frac{f^n(r) - \Lambda_i n}{\sqrt{n}} \leq x \right\}$$

and

$$\mu(z, m) = \frac{1}{m} \sum_{k=0}^{m} \delta_{\sigma^k z}.$$

Throughout the following it is helpful to keep the following expression in mind,

$$\int \mathbb{1}_{F(n,x)} \; d\mu(z, m) = \frac{1}{m} \# \left\{ 0 \leq j \leq m : \frac{f^n(\sigma^j(z)) - \Lambda_i n}{\sqrt{n}} \leq x \right\}$$

where $\mathbb{1}_{F(n,x)}$ denotes the indicator function for $F(n, x)$. To simplify our notation in the following, if $\sigma_i^2 = 0$, then we take

$$\frac{1}{\sqrt{2\pi}\sigma_i} \int_{-\infty}^{x} e^{-t^2/2\sigma_i^2} \; dt$$

to be the Heaviside function. The central limit theorem for subshifts of finite type [14] implies that there exists a set $N_i \subset \Sigma_{B_i}$ with $\mu_i(N_i) = 1$, such that for all $x \in \mathbb{R}$ and $z \in N_i$,

$$\begin{aligned} \lim_{n\to\infty} \lim_{m\to\infty} \int \mathbb{1}_{F(n,x)} \; d\mu(z, m) &= \lim_{n\to\infty} \mu_i(F(n, x)) \\ &= \frac{1}{\sqrt{2\pi}\sigma_i} \int_{-\infty}^{x} e^{-t^2/2\sigma_i^2} \; dt. \end{aligned}$$

We note that if $z \in \Sigma_A$ satisfies the above convergence, then any pre-image $y \in \sigma^{-1}(z)$ also satisfies the above convergence. Also, from the above discussion, there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\sigma_*^k \widehat{\nu}(N_i) > 0$. Combining these observations implies that there exists a set $E_i \subset Y$ of positive $\widehat{\nu}$ measure and for $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int \mathbb{1}_{F(n,x)}\, d\mu(y,m) = \frac{1}{\sqrt{2\pi}\sigma_i}\int_{-\infty}^{x} e^{-t^2/2\sigma_i^2}dt, \tag{3.5.1}$$

when $y \in E_i$. Hence, for each $i = 1,\dots,m$, $h(E_i) \subset \partial G$ has positive ν measure.

We define the set $S_i \subset \partial G$ to be the collection of elements in ∂G that have a corresponding infinite geodesic ray γ such that for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\limsup_{m\to\infty}\frac{1}{m}\#\left\{0 \le j \le m : \frac{\varphi(\gamma_{j+n}) - \varphi(\gamma_j) - \Lambda_i n}{\sqrt{n}} \le x\right\} = \frac{1}{\sqrt{2\pi}\sigma_i}\int_{-\infty}^{x} e^{-t^2/2\sigma_i^2}dt.$$

Since φ is Lipschitz in the left and right word metric, if γ_1, γ_2 are two geodesic rays with the same end point in ∂G, then γ_1 satisfies the above convergence if and only if γ_2 does. Further, as φ is Lipschitz in the right word metric S_i is G-invariant. See Lemma 4.3 in [9] for a more detailed explanation of these last two points.

This G invariance implies that, by the ergodicity of the action of G on ∂G with respect to ν, $\nu(S_i)$ either has full measure or zero measure. However, Lemma 3.5.6 and expression (6.2) imply that $h(E_i) \subset S_i$. To see this note that for $y \in E_i$,

$$\frac{1}{m}\#\left\{0 \le j \le m : \frac{\varphi(h(y)_{j+n}) - \varphi(h(y)_j) - \Lambda_i n}{\sqrt{n}} \le x\right\}$$

is equal to

$$\frac{1}{m}\#\left\{0 \le j \le m : \frac{f^n(\sigma^j(y)) + O(1) - \Lambda_i n}{\sqrt{n}} \le x\right\} = \int \mathbb{1}_{F(n,x+O(n^{-1/2}))}\, d\mu(y,m),$$

where the above error term arises from the application of Lemma 3.5.6. This error term does not affect the convergence exhibited in (6.2) and we deduce that $h(E_i) \subset S_i$. Since $\nu(h(E_i)) > 0$, S_i has full measure. It follows that the S_i coincide and hence that Λ_i and σ_i^2 do not depend on $i = 1,\dots,m$ as required. $\square$

From now on, we use the notation

$$\Lambda_\varphi := \frac{d}{ds}P_i(sf)\Big|_{s=0} \text{ and } \sigma_\varphi^2 := \frac{d^2}{ds^2}P_i(sf)\Big|_{s=0},$$

for any $i = 1,\dots,m$.

By the above discussion Λ_φ and σ_φ^2 are well defined i.e. independent of the choice of maximal component. Computing the Taylor expansion of each $P_i(sf)$ in a neighbourhood of zero gives the following.

Lemma 3.5.8. *There exists a neighbourhood U of 0 in $\mathbb{C}$ such that for $s \in U$ and for each $i = 1, \ldots, m$,*

$$P_i(sf) = h + \Lambda_\varphi s + \sigma_\varphi^2 s^2/2 + O(s^3) \tag{3.5.2}$$

as $s \to 0$.

3.6 Cohomology conditions

The aim of this section is to characterise the case that $\sigma_\varphi^2 = 0$. Let B_i be a maximal component with cyclic decomposition

$$\Sigma_{B_i} = \bigsqcup_{j=0}^{p_i-1} \Sigma_{B_i^j}.$$

Definition 3.6.1. Let B_i^G denote the elements in G that can be realised as a word corresponding to a path contained in the component B_i. Specifically, let ρ denote the labelling map from Definition 3.2.1, then B_i^G is the set,

$$\{g \in G : g = \rho(e_0)\rho(e_1)\ldots\rho(e_{n-1}) \text{ for some path with edges } e_0, \ldots, e_{n-1} \text{ in } B_i\}.$$

Recall that for small s, the spectral radius of the operator $L_{C_i,sf}$ is given by the modulus of $e^{P_i(sf)}$. Furthermore, $P_i(sf)$ denotes the quantity $P(s\tilde{f}^0)/p_i$ where $\tilde{f}^0$ is the function f^{p_i} restricted to $\Sigma_{B_i^0}$.

Lemma 3.6.2. *Suppose φ satisfies Condition (1) and Condition (2) with associated potential $f : \Sigma_A \to \mathbb{R}$. Let $(f^{p_i})^n(x)$ denote $f^{p_i}(x) + f^{p_i}(\sigma^{p_i}(x)) + \ldots + f^{p_i}(\sigma^{p_i(n-1)}(x))$. Then, the following are equivalent*

1. *$\sigma_\varphi^2 = 0$,*
2. *The function f^{p_i} on $(\Sigma_{B_i^0}, \sigma^{p_i})$ is cohomologous to a constant,*
3. *$\{(f^{p_i})^n(x) - np_i\Lambda_\varphi : x \in \Sigma_{B_i^0}, n \in \mathbb{Z}_{\geq 0}\}$ is bounded,*
4. *$\{(f^{p_i})^n(x) - np_i\Lambda_\varphi : x \in \Sigma_{B_i^j}, n \in \mathbb{Z}_{\geq 0}\}$ is bounded for $j = 0, 1, ..., p_i - 1$,*
5. *$\{f^n(x) - n\Lambda_\varphi : x \in \Sigma_{B_i}, n \in \mathbb{Z}_{\geq 0}\}$ is bounded,*
6. *$\{\varphi(g) - |g|\Lambda_\varphi : g \in B_i^G\}$ is bounded,*

7. $\{\varphi(g) - |g|\Lambda_\varphi : g \in G\}$ *is bounded.*

Proof. (1) $\iff$ (2) This is a standard result. See [42].
(2) $\iff$ (3) This is proved in [35], see Lemma 2.3.
(3) $\iff$ (4) This follows from the discussion leading up to Proposition 5.6.
(4) $\iff$ (5) This is a simple exercise.
(5) $\iff$ (6) Given $g \in B_i^G$, we can view g as a path contained in the component B_i. We can then extend this path on the left to a path that begins at the $*$ vertex and on the right so that it ends at the 0 vertex. Furthermore, there exists $L \in \mathbb{Z}_{\geq 0}$ such that we can always extend a group element in this way by adding at most L new vertices. This extended path corresponds to a group element $g' \in G$ and we have that, by Condition (2),

$$\varphi(g) = \varphi(g') + O(1),$$

where the implied constant is independent of g and g'. Then, using the embedding $i : G \to \Sigma_A$ we see that

$$\varphi(g) = f^{|g|}(\sigma^{|g'|-|g|}(i(g'))) + O(1),$$

where the implied constant is independent of g. Now choose any $x = (x_k)_{k=0}^{\infty} \in \Sigma_{B_i}$ for which $x_0, x_1, ..., x_{|g|}$ describes the path related to g. Then, by the Hölder condition on f,

$$\varphi(g) = f^{|g|}(x) + O(1),$$

where the implied constant is independent of g and our choice of x. This gives one of our desired implications. Running this argument backwards gives the other.
(6) $\iff$ (7) This is a consequence of hyperbolic groups being growth quasitight (see Definition 1.5 in [22]). By Lemma 4.6 of [26] there exists a finite set $M \subset G$ such that $MB_i^GM = G$ (see also Proposition 7.2 of [22]). The conclusion then follows easily from Condition (2). $\square$

Definition 3.6.3. We say that $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded if $\{\varphi(g) - |g|\Lambda_\varphi : g \in G\}$ is an unbounded subset of $\mathbb{R}$.

Remark 3.6.4. Lemma 7.2 characterises the degenerate case for Calegari and Fujiwara's central limit theorem [9]. This is because, as discussed earlier, the functions considered by Calegari and Fujiwara have an associated Hölder potential and the variance, σ_φ^2, associated to this potential agrees with the variance in Calegari and Fujiwara's central limit theorem.

We can use the positive variance conditions from Lemma 3.6.2 to deduce combinatorial and geometric properties of functions satisfying Condition (1) and Condition (2). The remainder of this section is dedicated to this end.

We begin by defining the following set

$$U = \{[\gamma] \in \partial G : \{\varphi(\gamma_n) - |\gamma_n|\Lambda_\varphi : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded}\}.$$

This set is a well defined because φ is Lipschitz in the left word metric on G. Given $[\gamma], [\gamma'] \in \partial G$, there exists $C > 0$ such that

$$|(\varphi(\gamma_n) - |\gamma_n|\Lambda_\varphi) - (\varphi(\gamma_n') - |\gamma_n'|\Lambda_\varphi)| = |\varphi(\gamma_n) - \varphi(\gamma_n')| + ||\gamma_n| - |\gamma_n'|| \leq C d_L(\gamma_n, \gamma_n').$$

If $[\gamma] = [\gamma']$, the right hand side of the above is bounded uniformly in n and hence $\{\varphi(\gamma_n) - |\gamma_n|\Lambda_\varphi : n \in \mathbb{Z}_{\geq 0}\}$ is bounded if and only if $\{\varphi(\gamma_n') - |\gamma_n'|\Lambda_\varphi : n \in \mathbb{Z}_{\geq 0}\}$ is bounded.

Definition 3.6.5. We say that φ is unbounded on the boundary if $\nu(U) > 0$.

Remark 3.6.6. As φ is Lipschitz in the right word metric, U is G-invariant. Therefore by the ergodicity of the action of G on ∂G with respect to ν, $\nu(U) = 0$ or 1. Hence the above definition can be equivalently stated by changing $\nu(U) > 0$ to $\nu(U) = 1$.

Proposition 3.6.7. *A function $\varphi : G \to \mathbb{R}$ satisfying Condition* (1) *and* (2) *is unbounded on the boundary if and only if $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded.*

Proof. It is clear that if φ is unbounded on the boundary then $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded

Conversely, suppose that $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded. Let $\nu, \widehat{\nu}, \mu$ and μ_i denote the measures defined in the proof of Proposition 6.2 and let $h : \Sigma_A \to \partial G$ denote the map defined in this proposition. Since $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded, f satisfies a non-degenerate central limit theorem on a maximal component B_i with respect to the measure of maximal entropy μ_i on that component, i.e., for $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu_i(G(n, y)) = \frac{1}{\sqrt{2\pi}\sigma_i} \int_y^\infty e^{-t^2/2\sigma_i^2}\, dt,$$

where

$$G(n, y) = \left\{ x \in \Sigma_{B_i} : \frac{f^n(x) - \Lambda_i n}{\sqrt{n}} \geq y \right\} \text{ and } \sigma_i^2 > 0.$$

Hence for any $y \in \mathbb{R}$,

$$\mu_i(\limsup_{n\to\infty} G(n,y)) = \mu_i\left(\bigcap_{n\geq 1}\bigcup_{j\geq n} G(j,y)\right)$$
$$\geq \limsup_{n\to\infty} \mu_i(G(n,y)) > 0.$$

Now fix $y > 0$ and note that

$$\mu\left\{x \in \Sigma_A : \{f^n(x) - n\Lambda_i : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded }\right\} \geq \mu_i(\limsup_{n\to\infty} G(n,y)) > 0.$$

As in the proof of Proposition 6.2, the relationship between $\widehat{\nu}$ and μ implies that

$$\widehat{\nu}\left\{x \in \Sigma_A : x_0 = * \text{ and } \{f^n(x) - n\Lambda_i : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded }\right\} > 0.$$

Then, by Condition (1) and the Hölder properties of f, for $x \in \Sigma_A$,

$$f^n(x) - n\Lambda_i = \varphi(g) - |g|\Lambda_i + O(1),$$

where g is the unique group element such that $i(g) = (*, x_0, ..., x_{n-1}, 0, 0, ...)$. The implied constant in the above is independent of x. Lastly, since $\widehat{\nu}$ pushes forward under $h : \Sigma_A \to \partial G$ to ν on ∂G,

$$\nu\left\{[\gamma] \in \partial G : \{\varphi(\gamma_n) - |\gamma_n|\Lambda_i : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded}\right\} > 0$$

and φ is unbounded on the boundary. $\square$

The following is a combinatorial condition that is equivalent to $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ being unbounded.

Definition 3.6.8. We say that φ is unbounded on a thick domain if whenever a subset, $H \subset G$ has the property that

$$\{\varphi(g) - |g|\Lambda_\varphi : g \in H\}$$

is bounded, then the asymptotic density of H with respect to W_n is zero, i.e.

$$\lim_{n\to\infty} \frac{\#(W_n \cap H)}{\#W_n} = 0.$$

Lemma 3.6.9. *A function satisfying Condition* (1) *and Condition* (2) *is unbounded on a thick domain if and only if* $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ *is unbounded.*

Proof. It is clear that if φ is unbounded on a thick domain, then $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded.

Conversely, suppose that $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded and that $H \subset G$ is such that $\{\varphi(g) - \Lambda_\varphi|g| : g \in H\}$ is bounded. There then exists real $M > 0$ such that,

$$\#(W_n \cap H) \leq \# \left\{ g \in W_n : \frac{\varphi(g) - n\Lambda_\varphi}{\sqrt{n}} \in \left[\frac{-M}{\sqrt{n}}, \frac{M}{\sqrt{n}} \right] \right\}$$

for all $n \geq 1$. Applying Theorem 4.1.2 then gives that, as $n \to \infty$,

$$\frac{\#(W_n \cap H)}{\#W_n} = \frac{1}{\sqrt{2\pi}\sigma_\varphi} \int_{-Mn^{-1/2}}^{Mn^{-1/2}} e^{-t^2/2\sigma_\varphi^2}\, dt + O(n^{-1/2}) = O(n^{-1/2}).$$

□

Remark 3.6.10. The proof of Lemma 7.9 shows that we can replace the limit in Definition 7.9 with a limit infimum without affecting the class of functions that are unbounded on a thick domain.

We will now provide a class of functions that satisfy our central limit theorem with positive variance.

Lemma 3.6.11. *If $\varphi : G \to \mathbb{R}$ is a non-trivial group homomorphism or an unbounded quasimorphism satisfying Condition (1), then $\sigma_\varphi^2 > 0$.*

Proof. From Theorem 4.1.1 and the equalities $|g| = |g^{-1}|$ and $\varphi(g) = -\varphi(g^{-1})$ that hold for all $g \in G$, we see that $\Lambda_\varphi = 0$. The result follows. □

Combining Propositions 7.8 and 7.9 gives the following result.

Corollary 3.6.12. *Suppose G is a non-elementary hyperbolic group and $\varphi : G \to \mathbb{R}$ satisfies Condition (1), Condition (2) and that $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded. Then the subset of ∂G consisting of (equivalence classes of) geodesic rays along which $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is unbounded, has full Patterson-Sullivan measure. Furthermore, if $\varphi(\cdot) - \Lambda_\varphi|\cdot|$ is bounded on $H \subset G$, then*

$$\frac{\#(W_n \cap H)}{\#W_n} = O\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$.

Chapter 4

Statistics in hyperbolic groups

4.1 Discussion and statement of results

In this section we study the statistical and distributional behaviour of real valued functions on hyperbolic groups that satisfy Condition (1) and Condition (2). We begin by stating our main results of this chapter.

Theorem 4.1.1 (Averaging Theorem)**.** *Let G be a non-elementary hyperbolic group equipped with a fixed generating set. Suppose that $\varphi : G \to \mathbb{R}$ satisfies Condition* (1) *and Condition* (2)*. Then, there exists $\Lambda \in \mathbb{R}$ such that*

$$\frac{1}{\#W_n} \sum_{g \in W_n} \frac{\varphi(g)}{n} = \Lambda + O\left(\frac{1}{n}\right)$$

as $n \to \infty$.

This result can be seen as an analogue of the law of large numbers, as it describes how $\varphi(g)/|g|$ averages over the sets W_n as $n \to \infty$. This leads us to ask if we can describe more precisely how φ averages over W_n, as $n \to \infty$. If we additionally assume that $\varphi(\cdot) - \Lambda|\cdot|$ is unbounded, then we obtain a central limit theorem for the normalised images

$$\left\{ \frac{\varphi(g) - n\Lambda}{\sqrt{n}} : g \in W_n \right\}.$$

Using Theorem 4.1.1 we deduce a Berry-Esseen error term.

Theorem 4.1.2 (Central Limit Theorem)**.** *Let G be a non-elementary hyperbolic group equipped with a finite generating set. Suppose that $\varphi : G \to \mathbb{R}$ satisfies Condition* (1) *and Condition* (2) *and that $\varphi(\cdot) - \Lambda|\cdot|$ is unbounded.*

Let Λ be the constant from Theorem 4.1.1. Then, there exists $\sigma^2 > 0$ such that

$$\frac{1}{\#W_n}\#\left\{g \in W_n : \ \frac{\varphi(g) - n\Lambda}{\sqrt{n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-t^2/2\sigma^2}\ dt + O\left(\frac{1}{\sqrt{n}}\right),$$

where the implied constant is independent of $x \in \mathbb{R}$.

We also prove the following large deviations result.

Theorem 4.1.3 (Large Deviation Theorem)**.** *Let G be a non-elementary hyperbolic group. Suppose that $\varphi : G \to \mathbb{R}$ satisfies Condition* (1) *and Condition* (2)*. Then, for any $\epsilon > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{1}{\#W_n}\#\left\{g \in W_n : \left|\frac{\varphi(g)}{n} - \Lambda\right| > \epsilon\right\}\right) < 0,$$

where Λ is as in Theorem 4.1.1.

We will show that Theorem 4.1.2 provides a positive answer to the two questions posed earlier in the introduction. Apart from answering these two questions, our motivation behind this work is to understand the statistics of the displacement function associated to group actions on CAT(-1) spaces. We are interested in answering the following question.

Let (X, d) be a complete CAT(-1) geodesic metric space and fix an origin for X. Suppose that a hyperbolic group G equipped with a finite generating set acts on X properly discontinuously, convex cocompactly by isometries. The Švarc-Milnor Lemma implies that there exists constants $C_0, C_1 > 0$ such that

$$C_0|g| \leq d(o, go) \leq C_1|g|$$

for all $g \in G$. We call the function $g \mapsto d(o, go)$ the displacement. The above inequality shows that word length and displacement are comparable quantities. This leads us to ask whether we can form a more refined comparison, on average, between them.

Recall that the displacement function satisfies Condition (1) and Condition (2). Theorems 4.1.1, 4.1.2 and 4.1.3 then apply and we obtain the following comparison results. Note that the fact $\Lambda > 0$ follows from the Svarc-Milnor Lemma.

Theorem 4.1.4. *Suppose a non-elementary hyperbolic group G acts convex cocompactly by isometries on a complete, geodesic, CAT(-1) metric space (X, d). Fix an origin $o \in X$ and a finite generating set for G. Then there exists $\Lambda > 0$*

such that

$$\frac{1}{\#W_n}\sum_{g\in W_n}\frac{d(o,go)}{n} = \Lambda + O\left(\frac{1}{n}\right).$$

Also, for any fixed $\epsilon > 0$,

$$\limsup_{n\to\infty}\ \frac{1}{n}\log\left(\frac{1}{\#W_n}\#\left\{g\in W_n : \left|\frac{d(o,go)}{n} - \Lambda\right| > \epsilon\right\}\right) < 0.$$

Furthermore, if $d(o,\cdot\, o) - \Lambda|\cdot|$ is unbounded unbounded, then there exists $\sigma^2 > 0$ such that

$$\frac{1}{\#W_n}\#\left\{g\in W_n :\ \frac{d(o,go)-n\Lambda}{\sqrt{n}} \le x\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-t^2/2\sigma^2}\ dt + O\left(\frac{1}{\sqrt{n}}\right),$$

where the implied constant is independent of $x \in \mathbb{R}$.

Remark 4.1.5. We note that similar results have been obtained by Gekhtman, Taylor and Tiozzo in [23] and [24].

(i) In [23], Gekhtman, Taylor and Tiozzo showed that

$$\frac{1}{\#W_n}\#\left\{g\in W_n : \left|\frac{d(o,go)}{n} - \Lambda\right| > \epsilon\right\} \to 0 \qquad (4.1.1)$$

with no estimate on the rate of convergence, for non-elementary actions (see Section 5 of [24] for a definition) of G on hyperbolic metric spaces. These actions are more general than convex cocompact actions. However, we have recently learned from these authors that the random walk results they used have been improved by Sunderland [57] and that this improvement, combined with the work in [24], gives exponential convergence in (4.1.1) at the level of generality considered in [24].

(ii) In [23], Gekhtman, Taylor and Tiozzo obtained a central limit theorem as above (but without an error term) in the special case where G is a free group or surface group.

After proving the above results, we generalise our method to the multi-dimensional setting with the aim of studying the statistics of the abelianisation homomorphism $\varphi : G \to G/[G,G]$. The abelianisation $G/[G,G]$ takes the form $\mathbb{Z}^k \oplus \text{Torsion}$ for some $k \ge 0$ and we are interested in how the image of G distributes in the non-torsion factor, $\mathbb{Z}^k$. We will assume that $k \ge 1$ and that we have fixed an isomorphism taking the non-torsion part of $G/[G,G]$ to $\mathbb{Z}^k$. We will refer to the induced homomorphism $\varphi : G \to \mathbb{Z}^k$ as the abelianisation homomorphism. Note that the components of this map are integer valued homomorphisms and so satisfy Condition (1) and Condition (2). This will allow us to apply the multidimensional analogues of the methods used to prove Theorems 4.1.1, 4.1.2 and 4.1.3. We prove that the abelianisation homomorphism satisfies a non-degenerate multidimensional central limit theorem.

Theorem 4.1.6. *Let G be a non-elementary hyperbolic group equipped with a finite generating set S. Suppose that G has abelianisation $\mathbb{Z}^k \oplus Torsion$ for some $k \geq 1$ and that $\varphi : G \to \mathbb{Z}^k$ is the abelianisation homomorphism constructed in the way described above. Then there exists a symmetric, positive definite matrix $\Sigma \in M_k(\mathbb{R})$ such that for $A \subset \mathbb{R}^k$,*

$$\frac{1}{\#W_n} \# \left\{ g \in W_n : \frac{\varphi(g)}{\sqrt{n}} \in A \right\} \to \frac{1}{(2\pi \ \det(\Sigma))^{k/2}} \int_A e^{-\langle x, \Sigma x \rangle/2} \ dx$$

as $n \to \infty$.

This result generalises the work of Rivin, who, in [51], proves the above theorem for free groups.

In the last section, we consider a more subtle distributional result than those in the above theorems. That is, we prove local central limit theorems. To obtain these results we need to understand the arithmetic properties of the images of the functions $\varphi : G \to \mathbb{R}$ that we consider. To gain this understanding, we need to assume that φ satisfies some additional properties. Our methods and therefore results do not apply to all maps satisfying Condition (1) and Condition (2). We obtain the following local limit theorem for group homomorphisms to $\mathbb{R}$.

Theorem 4.1.7. *Suppose G is a non-elementary hyperbolic group equipped with a finite generating set. Let $\varphi : G \to \mathbb{R}$ be a group homomorphism that has a dense image in $\mathbb{R}$. Then, Theorem 4.1.2 holds and we obtain $\sigma > 0$ such that any $a, b \in \mathbb{R}$ with $a < b$,*

$$\frac{1}{\#W_n} \#\{g \in W_n : \ \varphi(g) \in [a,b]\} \sim \frac{b-a}{\sqrt{2\pi}\sigma\sqrt{n}}$$

as $n \to \infty$.

In [56] Sharp studies local limit theorems for homomorphisms $\varphi : G \to \mathbb{Z}$ (where G is a free group). In this work we are interested in the complementary case in which the image of $\varphi : G \to \mathbb{R}$ is dense in $\mathbb{R}$. We show that, in a natural sense, almost all group homomorphisms satisfy the hypotheses of Theorem 4.1.7. After proving this result we obtain a further local limit theorem for the displacement function associated to convex cocompact actions on pinched Hadamard surfaces. We defer the statement of this result until after the proof of Theorem 4.1.7.

We are nearly ready to prove these results. Before doing so we explain how we will make use of transfer operators in our proofs. We also establish the notation that we will use and make an observation that will allow us to simplify our analysis.

As mentioned previously, to prove our results, we need an understanding of the sums

$$\sum_{g \in W_n} e^{s\varphi(g)},$$

for small complex s, as $n \to \infty$. We now show how to express this quantity in terms of transfer operators. This expression highlights the link between the geometrical setting of φ on G and the dynamical setting of f on Σ_A. Let χ denote the indicator function for the set $\{(x_n)_{n=0}^{\infty} \in \Sigma_A : x_0 = *\}$.

Lemma 4.1.8. *There exists $\epsilon, \delta > 0$ such that for $|s| < \epsilon$, each $L_{C_i,sf}$ has spectrum as described in Proposition 3.5.5 and*

$$\sum_{g \in W_n} e^{s\varphi(g)} = \sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}) + O\left(e^{n(h-\delta)}\right),$$

where the implied constant is independent of $|s| < \epsilon$.

Proof. Note that

$$\sum_{g \in W_n} e^{s\varphi(g)} = \sum_{z} e^{sf^n(z)}. \tag{4.1.2}$$

where the second sum is taken over $\{z \in \Sigma_A : \sigma^n(z) = \dot{0}, z_0 = *, z_{n-1} \neq \dot{0}\}$. Hence, the quantity

$$\sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}),$$

expresses (4.1.2) up to overcounting contributions from elements belonging to

$$\{z \in \Sigma_A : \sigma^n(z) = \dot{0}, z_0 = *, z_{n-1} \neq 0 \text{ and the path corresponding to } z \text{ does not enter a maximal component}\}.$$

Since the cardinality of this set is $O(e^{n(h-\nu)})$ for some $\nu > 0$ and f is bounded, the result follows. $\square$

We now establish the notation that we will use throughout the remaining sections. Suppose G is equipped with a generating set S. Suppose G, S has associated directed graph $\mathcal{G}$ described by transition matrix A. Let W_n denote the elements in G of word length n and let $\#W_n$ denote the cardinality of W_n. Let B, B_i and C_i for $i = 1, \ldots, m$ denote the matrices defined in Chapter 3.5 and suppose that $\varphi : G \to \mathbb{R}$ is a function satisfying Condition (1) and Condition (2). Suppose φ has associated potential $f \in F_\theta(\Sigma_A)$. Let $L_{C_i,sf}$ denote the transfer operators and let $Q_{i,k}$ denote the projection valued operators previously defined. Denote by Q_i the projection

$$Q_i = \sum_{k=0}^{p_i-1} Q_{i,k}.$$

Let Λ_φ and σ_φ^2 be the quantities related to φ that were defined in Chapter 3.5.2.

Throughout our proofs, we use the notation established above. The following lemma will allow us to simplify our analysis.

Lemma 4.1.9. *Define $\gamma : G \to \mathbb{R}$ by $\gamma(g) = \varphi(g) - |g|\Lambda_\varphi$. Then γ satisfies Condition* (1) *and Condition* (2) *and the potential related to γ is $f - \Lambda_\varphi$. Furthermore*

$$\Lambda_\gamma = 0 \quad \textit{and} \quad \sigma_\gamma^2 = \sigma_\varphi^2.$$

Proof. It is easy to check that the word length function $g \mapsto |g|$ satisfies Conditions (1) and (2) with related potential given by the constant function with value 1. It follows that γ also satisfies Conditions (1) and (2) with potential $f - \Lambda_\varphi$. Using the notation established in Section 6, for any chosen maximal component with index i,

$$\Lambda_\gamma = \frac{d}{ds} P_i(s(f - \Lambda_\varphi))\Big|_{s=0} \quad \text{and} \quad \sigma_\gamma^2 = \frac{d^2}{ds^2} P_i(s(f - \Lambda_\varphi))\Big|_{s=0}.$$

For real s we have that,

$$P_i(s(f - \Lambda_\varphi)) = P_i(sf) - s\Lambda_\varphi,$$

from which the remainder of the lemma easily follows. $\square$

Assumption: The above lemma implies that, by swapping φ to γ, it suffices to prove Theorems 4.1.1, 4.1.2 and 4.1.3 under the assumption that $\Lambda_\varphi = 0$. We assume this throughout the remaining sections.

4.2 Averaging theorem

We define the following generating function.

Definition 4.2.1. Let

$$\eta(z,s) = \sum_{n=0}^{\infty} \frac{z^n}{n} \sum_{g \in W_n} e^{s\varphi(g)}.$$

We want to study the domain of analyticity for η.

Lemma 4.2.2. *We have that*

$$\eta(z,s) = \sum_{n=0}^{\infty} \frac{z^n}{n} \sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}) + \alpha(z,s),$$

for some function $\alpha(z,s)$ that is bi-analytic in $\{z : |z| < e^{-h+\delta}\} \times \{s : |s| < \epsilon\}$ for some $\epsilon, \delta > 0$.

Proof. Let $\epsilon, \delta > 0$ be as in Lemma 4.1.8. Using Lemma 4.1.8 we can write, for $|s| < \epsilon$,

$$\sum_{g \in W_n} e^{s\varphi(g)} = \sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}) + \omega_n(s),$$

where $\omega_n(s)$ is analytic in $|s| < \epsilon$ and $\omega_n(s) = O\left(e^{h(n-\delta)}\right)$. The implied constant is uniform in $|s| < \epsilon$. Define

$$\alpha(z,s) = \sum_{n=0}^{\infty} \frac{z^n}{n}\, \omega_n(s).$$

Clearly α satisfies the required identity for the lemma. Further, since the error term associated to ω_n is independent of s, for fixed $|s_0| < \epsilon$, $\alpha(z, s_0)$ is analytic in $\{z : |z| > e^{-h+\delta}\}$. Conversely, for fixed $|z_0| < e^{-\epsilon+\delta}$, $\alpha(z_0, s)$ is analytic in $|s| < \epsilon$. Hence, by Hartogs' Theorem (see Theorem 1.2.5 in [30]), $\alpha(z,s)$ satisfies the required analyticity condition. ☐

Let $\epsilon > 0$ be as in Lemma 4.1.8. By the Spectral Radius Theorem and Lemma 4.1.8, there exists $\delta' > 0$ such that

$$\left| \sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}) \right| = O\left(e^{n(h+\delta')}\right),$$

where the error term is independent of $|s| < \epsilon$. Lemma 3.5.5 and an application of Hartogs' Theorem then implies that η is bi-analytic in $\{|z| < e^{-h-\delta'}\} \times \{s :$

$|s| < \epsilon\}$. Taking the derivative of η with respect to s at $s = 0$ gives,

$$\frac{d}{ds}\eta(z,s)\bigg|_{s=0} = \sum_{n=0}^{\infty} \frac{z^n}{n} \sum_{g \in W_n} \varphi(g)$$

for all $|z|$ sufficiently small. Let $\epsilon, \delta > 0$ be as in Lemma 4.1.8. Recall that we have analytic projection valued functions $Q_{i,k}$ for the simple maximal eigenvalues of the transfer operators $L_{C_i,sf}$. For $|s| < \epsilon$, $Q_{i,k}(s)$ is the eigenprojection associated to the eigenvalue $e^{2\pi ik/p_i}e^{P_i(sf)}$ for $L_{C_i,sf}$. Using these projections we write

$$\sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}) = \sum_{i=1}^{m}\sum_{k=1}^{p_i} e^{2\pi ink/p_i}e^{nP_i(sf)}Q_{i,k}(s)\chi(\dot{0}) + O(e^{n(h-\delta)}), \qquad (4.2.1)$$

which is valid for $|s| < \epsilon$.

Using identity (4.2.1) we can apply the same argument as in the proof of Lemma 4.2.2, to the function

$$\sum_{n=0}^{\infty} \frac{z^n}{n} \sum_{i=1}^{m} L^n_{C_i,sf}\chi(\dot{0}),$$

to deduce the following.

Lemma 4.2.3.

$$\eta(z,s) = \sum_{n=0}^{\infty} \frac{z^n}{n} \sum_{i=1}^{m}\sum_{k=1}^{p_i} e^{2\pi ink/p_i}e^{nP_i(sf)}Q_{i,k}(s)\chi(\dot{0}) + \beta(z,s), \qquad (4.2.2)$$

for some $\beta(z,s)$ that is bi-analytic in $\{z : |z| < e^{-h+\epsilon}\} \times \{s : |s| < \delta\}$ for some $\epsilon, \delta > 0$.

We then turn our attention to the double sum in (4.2.2).

Lemma 4.2.4. *Define*

$$\psi_n(s) := \sum_{i=1}^{m}\sum_{k=1}^{p_i} e^{2\pi ink/p_i}e^{nP_i(sf)}Q_{i,k}(s)\chi(\dot{0}). \qquad (4.2.3)$$

Then, each ψ_n is analytic in a neighbourhood of 0 *and*

$$\psi_n'(0) = O\left(e^{nh}\right).$$

Proof. We recall that the projections $Q_{i,k}$ are analytic in a small neighbourhood of the origin. Hence the maps $s \mapsto Q_{i,k}(s)\chi(\dot{0})$ are analytic in a neighbourhood of the origin. Differentiating each ψ_n and using the Taylor expansions for the pressure (3.5.8) (recalling that $\Lambda_\varphi = 0$), gives the required result. $\square$

Taking the derivative of expression (4.2.2) with respect to s at $s = 0$ and then rearranging, we obtain

$$\sum_{n=0}^{\infty} z^n \left(-\frac{\psi_n'(0)}{n} + \sum_{g \in W_n} \frac{\varphi(g)}{n} \right) = \frac{d}{ds}\beta(z,s)\Big|_{s=0}. \tag{4.2.4}$$

The domain of bi-analyticity for β implies that the radius of convergence of the above series is strictly greater than e^{-h}.

We are now ready to prove our result.

Proof of Theorem 4.1.1. Equation (4.2.4) implies that

$$\sum_{g \in W_n} \frac{\varphi(g)}{n} = \frac{\psi_n'(0)}{n} + O(e^{n(h-\delta)})$$

for some $\delta > 0$.

Dividing the above identity by $\#W_n$ and then applying Proposition 3.1.5 and Lemma 4.2.4 implies that

$$\frac{1}{\#W_n} \sum_{g \in W_n} \frac{\varphi(g)}{n} = O\left(\frac{1}{n}\right)$$

as required. $\square$

4.3 Central limit theorem

We now move on to the proof of Theorem 4.1.2. Throughout this section, suppose that $\varphi(\cdot) - \Lambda|\cdot|$ is unbounded. By Lemma 3.6.2 we have that $\sigma_\varphi^2 > 0$. Recall that we want to study the convergence of the distributions

$$F_n(x) = \frac{1}{\#W_n} \# \left\{ g \in W_n : \frac{\varphi(g)}{\sqrt{n}} \leq x \right\}$$

as $n \to \infty$. A classical way of studying this convergence is to take the Fourier transforms $\widehat{F}_n : \mathbb{R} \to \mathbb{R}$ of each F_n and to apply a result from probability theory that gives a uniform bound on the difference $F_n - N$, where N is our desired normal distribution, in terms of the $\widehat{F}_n$. This is the approach we employ.

These Fourier transforms are given by

$$\widehat{F}_n(t) = \frac{1}{\#W_n} \sum_{g \in W_n} e^{it\varphi(g)n^{-1/2}}. \tag{4.3.1}$$

Lemma 4.3.1. *We have that, for the ϵ given in Proposition* 4.1.8*,*

$$\widehat{F}_n(t) = \frac{\sum_{i=1}^m L^n_{C_i, itfn^{-1/2}} \chi(\dot{0})}{\sum_{i=1}^m L^n_{C_i,0}\chi(\dot{0})} + o(1), \tag{4.3.2}$$

when $|t| < \epsilon\sqrt{n}$. The above error term is uniform in $|t| < \epsilon\sqrt{n}$.

Proof. Setting $s = itn^{-1/2}$ in Lemma 4.1.8 allows us to rewrite expression (4.3.1) as

$$\widehat{F}_n(t) = \frac{1}{\#W_n} \sum_{i=1}^m L^n_{C_i, itfn^{-1/2}} \chi(\dot{0}) + o(1).$$

Similarly, by setting $s = 0$ in Lemma 4.1.8, we have that

$$\#W_n \sim \sum_{i=1}^m L^n_{C_i,0}\chi(\dot{0}).$$

Combining these two identities proves the lemma. $\square$

To obtain our central limit theorem with Berry-Esseen error term we want to make use of an inequality similar to the well-known 'Basic Inequality'.

Proposition 4.3.2 (Basic Inequality [19] Lemma 2, Section XVI.3)**.** *Suppose that F is a probability distribution with vanishing expectation and Fourier transform $\widehat{F}$. Suppose that N is the normal distribution with mean* 0*, variance $\sigma^2 > 0$ and derivative N'. Suppose further that $F - N$ vanishes at $\pm\infty$. Then,*

$$\|F - N\|_\infty \le \frac{1}{\pi} \int_{-T}^{T} \frac{1}{|t|} \left| \widehat{F}(t) - e^{-\sigma^2 t^2/2} \right| dt + \frac{24\|N'\|_\infty}{\pi T},$$

where $T > 0$ is arbitrary.

As mentioned above, this inequality allows us to study the convergence rate of our central limit theorem via the Fourier transforms of our distributions. The standard 'Basic Inequality' applies to distributions with zero mean and for our purposes, we need a version of the inequality that applies to a sequence of distributions with varying means. We therefore amend the Basic Inequality to the following form.

Proposition 4.3.3. *Let H_n for $n \in \mathbb{Z}_{\geq 0}$ be a sequence of distributions with Fourier transforms $\widehat{H}_n$ and means E_n. Write N for the normal distribution with mean zero and variance $\sigma^2 > 0$ and suppose that $H_n - N$ vanishes at $\pm\infty$ for each $n \in \mathbb{Z}_{\geq 0}$. Suppose there exists a sequence of positive real numbers $T_n > 0$ and a constant $C > 0$ such that*

$$\int_{-T_n}^{T_n} |\widehat{H}_n(t)| \, dt \leq C,$$

for all $n \in \mathbb{Z}_{\geq 0}$. Then, there exists $K \geq 0$ such that

$$\|H_n - N\|_\infty \leq K \left(\int_{-T_n}^{T_n} \frac{1}{|t|} |\widehat{H}_n(t) - e^{-\sigma^2 t^2/2}| \, dt + \frac{1}{T_n} + |E_n| e^{|E_n T_n|} \right), \quad (4.3.3)$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. Consider the distributions $F_n(x) := H_n(x + E_n)$. These have mean zero. Hence, by Proposition 4.3.2

$$\|F_n - N\|_\infty \leq \frac{1}{\pi} \int_{-T_n}^{T_n} \frac{1}{|t|} |e^{-itE_n} \widehat{H}_n(t) - e^{-\sigma^2 t^2/2}| \, dt + \frac{24\|N'\|_\infty}{\pi T_n},$$

for all $n \in \mathbb{Z}_{\geq 0}$. We also have

$$\begin{aligned} \int_{-T_n}^{T_n} \frac{1}{|t|} |e^{-itE_n} \widehat{H}_n(t) - \widehat{H}_n(t)| \, dt &= \int_{-T_n}^{T_n} \frac{1}{|t|} |e^{-itE_n} - 1| |\widehat{H}_n(t)| \, dt \\ &\leq |E_n| e^{|T_n E_n|} \int_{-T_n}^{T_n} |\widehat{H}_n(t)| \, dt \\ &\leq C |E_n| e^{|T_n E_n|}, \end{aligned}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Now, define

$$M_n := \int_{-T_n}^{T_n} \frac{1}{|t|} |\widehat{H}_n(t) - e^{-\sigma^2 t^2/2}| \, dt + |E_n| e^{|T_n E_n|} + \frac{1}{T_n}.$$

From the above,

$$\|F_n - N\|_\infty = O(M_n).$$

We then observe that

$$\|H_n - F_n\|_\infty \leq \|N'\|_\infty |E_n| + 2M_n.$$

Lastly,

$$\|H_n - N\|_\infty \leq \|F_n - N\|_\infty + \|H_n - F_n\|_\infty = O(M_n + |E_n|),$$

where the implied error term does not depend on $n \in \mathbb{Z}_{\geq 0}$. This is precisely the required statement. $\square$

We could apply this result directly to our distributions F_n, however, the error term in expression (4.3.2) would lead to complications when comparing $\widehat{F}_n$ to $e^{-\sigma^2 t^2/2}$ in the right hand side of (4.3.3). Ideally, if we are to apply Proposition 4.3.3 to a sequence of distributions H_n, we would like an exact expression for each $\widehat{H}_n$ in terms of transfer operators. To achieve this, we consider, instead of F_n, the following sequence of distributions,

$$H_n(x) = \frac{1}{\#W_n + (m-1)\#N_n}\left(\#\left\{g \in W_n : \frac{\varphi(g)}{\sqrt{n}} \leq x\right\}\right.$$
$$\left. + (m-1)\#\left\{g \in N_n : \frac{\varphi(g)}{\sqrt{n}} \leq x\right\}\right),$$

where

$$N_n = \{g \in W_n : \text{ the path in } \mathcal{G} \text{ corresponding}$$
$$\text{to } g \text{ does not enter a maximal component}\}$$

and $\mathcal{G}$ has m maximal components.

Since $\#N_n = O\left(e^{n(h-\delta)}\right)$ for some $\delta > 0$, $\|F_n - H_n\|_\infty$ converges to zero exponentially quickly. Hence, to prove Theorem 4.1.2, it suffices to show the following.

Proposition 4.3.4. *We have that*

$$H_n(x) = \frac{1}{\sqrt{2\pi}\sigma_\varphi}\int_{-\infty}^{x} e^{-t^2/2\sigma_\varphi^2}\, dt + O\left(\frac{1}{\sqrt{n}}\right),$$

where the implied constant is independent of $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$.

We consider the distributions H_n, because each $\widehat{H}_n$ has an exact expression in terms of transfer operators.

Lemma 4.3.5. *For all $t \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$\widehat{H}_n(t) = \frac{\sum_{i=1}^m L^n_{C_i, itfn^{-1/2}}\chi(\dot{0})}{\sum_{i=1}^m L^n_{C_i,0}\chi(\dot{0})}.$$

Proof. We note that, for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$\sum_{g \in W_n} e^{it\varphi(g)n^{-1/2}} + (m-1)\sum_{g \in N_n} e^{it\varphi(g)n^{-1/2}} = \sum_{i=1}^m L^n_{C_i, itfn^{-1/2}}\chi(\dot{0}).$$

Using this expression and the same proof as Lemma 4.3.1 gives the required result. □

We want to apply Proposition 4.3.3 to the sequence H_n and a suitable sequence T_n. Our aim is to show that for any sufficiently small $\epsilon > 0$, Proposition 4.3.3 holds for the pair H_n and $T_n = \epsilon\sqrt{n}$.

Lemma 4.3.6. *For any fixed sufficiently small $\epsilon > 0$, there exists a constant $C > 0$ depending only on ϵ such that*

$$\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} |\widehat{H}_n(t)| \, dt \leq C,$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. Since $\sum_{i=1}^{m} L_{C_i,0}^n \chi(\dot{0}) = \Theta(e^{nh})$,

$$\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} |\widehat{H}_n(t)| \, dt = O\left(e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \sum_{i=1}^{m} |L_{C_i,itfn^{-1/2}}^n \chi(\dot{0})| \, dt\right).$$

Hence it suffices to show that for each $i = 1, \ldots, m$, if $\epsilon > 0$ is sufficiently small, then

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} |L_{C_i,itfn^{-1/2}}^n \chi(\dot{0})| \, dt = O(1), \tag{4.3.4}$$

where the implied constant is independent of $n \in \mathbb{Z}_{\geq 0}$.

Using the projections $Q_{i,k}$ and Q_i we can write for sufficiently small ϵ,

$$\begin{aligned} L_{C_i,itfn^{-1/2}}^n \chi(\dot{0}) = \sum_{k=0}^{p_i-1} & e^{nP_i(itfn^{-1/2})} e^{2\pi ikn/p_i} Q_{i,k}(itn^{-1/2}) \chi(\dot{0}) \\ & + L_{C_i,itfn^{-1/2}}^n (I - Q_i(itn^{-1/2})) \chi(\dot{0}). \end{aligned} \tag{4.3.5}$$

Substituting this expression into the left hand side of (4.3.4) implies that to prove (4.3.4) it suffices to show that

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \left| e^{nP_i(itfn^{-1/2})} Q_{i,k}(itn^{-1/2}) \chi(\dot{0}) \right| dt = O(1) \tag{4.3.6}$$

and

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \left| L_{C_i,itfn^{-1/2}}^n (I - Q_i(itn^{-1/2})) \chi(\dot{0}) \right| dt = O(1) \tag{4.3.7}$$

for each $i = 1, \ldots, m$, $k = 0, \ldots, p_i - 1$ and that these error terms are independent of n.

To prove (4.3.6), note that the Taylor expansion for the pressure (3.5.8) implies that if ϵ is sufficiently small, then for all $|t| < \epsilon\sqrt{n}$,

$$|e^{nP_i(itfn^{-1/2})-nh}| \leq e^{-\sigma_\varphi^2 t^2/4}.$$

Hence for fixed, sufficiently small ϵ,

$$e^{-nh}\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \left|e^{nP_i(itfn^{-1/2})}Q(itn^{-1/2})\chi(\dot{0})\right| dt = O\left(\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} e^{-\sigma^2t^2/4}dt\right) = O(1).$$

To prove (4.3.7), recall that, by Proposition 3.5.5, if ϵ is sufficiently small, then for fixed s with $|s| < \epsilon$, there exists $\delta' > 0$ such that

$$L^n_{C_i,sf}(I - Q_i(s))\chi(\dot{0}) = O\left(e^{n(h-\delta')}\right),$$

where the implied constant is independent of $n \in \mathbb{Z}_{\geq 0}$. Since the maps $s \mapsto L_s$ and $s \mapsto Q_i(s)$ for $i = 1, \ldots, m$ are continuous (in fact analytic), at the cost of reducing ϵ, we can find $\delta > 0$ and $K > 0$ such that

$$L^n_{C_i,sf}(I - Q_i(s))\chi(\dot{0}) \leq Ke^{n(h-\delta)},$$

for all $|s| < \epsilon$ and $n \in \mathbb{Z}_{\geq 0}$. Hence

$$L^n_{C_i,itfn^{-1/2}}(I - Q_i(itfn^{-1/2}))\chi(\dot{0}) = O\left(e^{n(h-\delta)}\right),$$

where the implied constant is independent of t and n with $|t| < \epsilon\sqrt{n}$. Substituting this expression into the left hand side of (4.3.6) gives the required decay rate. This concludes the proof. □

We have shown that Proposition 4.3.3 applies to the pair H_n and $T_n = \epsilon\sqrt{n}$ as long as $\epsilon > 0$ is sufficiently small. The bound (4.3.3) then provides us with a way of computing the decay rate of $\|H_n - N\|_\infty$, where N is the normal distribution with mean 0 and variance $\sigma_\varphi^2 > 0$. We now turn our attention to the terms in (4.3.3). We begin by studying the means E_n of the distributions H_n. These means are given by

$$\sqrt{n}E_n = \int \varphi(g)\ d\tilde{\mu}_n$$

where

$$\tilde{\mu}_n = \frac{1}{\#W_n + (m-1)\#N_n}\left(\sum_{g\in W_n}\delta_g + (m-1)\sum_{g\in N_n}\delta_g\right).$$

It follows easily from Theorem 4.1.1 that $E_n \to 0$ as $n \to \infty$. Further, we can quantify the rate of this convergence.

Proposition 4.3.7. *We have that*

$$E_n = O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. This is a simple application of Theorem 4.1.1. $\square$

We now study the decay rate of the first term in the right hand side of (4.3.3). Our aim is to prove the following.

Proposition 4.3.8. *For any fixed $\epsilon > 0$ sufficiently small,*

$$\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|}|\widehat{H}_n(t) - e^{-\sigma_\varphi^2 t^2/2}|\ dt = O\left(\frac{1}{\sqrt{n}}\right),$$

where the implied constant is independent of $n \in \mathbb{Z}_{\geq 0}$.

We will break the proof of this proposition into two lemmas. We begin by studying the following difference

$$\widehat{H}_n(t) - e^{-\sigma_\varphi^2 t^2/2} = \frac{\sum_{i=1}^m \left(L^n_{C_i, itn^{-1/2}f}\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2} L^n_{C_i,0}\chi(\dot{0})\right)}{\sum_{i=1}^m L^n_{C_i,0}\chi(\dot{0})}.$$

By Proposition 3.1.5 we can write

$$\left|\widehat{H}_n(t) - e^{-\sigma_\varphi^2 t^2/2}\right| \leq Ce^{-nh}\sum_{i=1}^m \left|L^n_{C_i, itn^{-1/2}f}\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2} L^n_{C_i,0}\chi(\dot{0})\right|,$$

where $C > 0$ is a constant independent of $n \in \mathbb{Z}_{\geq 0}$. Hence to prove Proposition 4.3.8 it suffices to show that for each $i = 1, \ldots, m$, if $\epsilon > 0$ is sufficiently small,

$$\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|}\left|L^n_{C_i, itn^{-1/2}f}\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2} L^n_{C_i,0}\chi(\dot{0})\right|\ dt = O\left(\frac{1}{\sqrt{n}}\right). \tag{4.3.8}$$

Substituting (4.3.5) into (4.3.8) we obtain (assuming that ϵ is sufficiently small),

$$e^{-nh}\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|}\left|L_{C_i, itn^{-1/2}f}\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2} L^n_{C_i,0}\chi(\dot{0})\right| dt \leq \mathrm{I}(\epsilon)^i_n + \mathrm{II}(\epsilon)^i_n,$$

where $\mathrm{I}(\epsilon)^i_n$, $\mathrm{II}(\epsilon)^i_n$ are given by

$$\sum_{k=0}^{p_i-1} e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|} \left| e^{nP_i(itfn^{-1/2})} Q_{i,k}(itn^{-1/2})\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2+nh} Q_{i,k}(0)\chi(\dot{0}) \right| dt,$$

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|} \left| L^n_{C_i,itn^{-1/2}f}(I - Q_i(itn^{-1/2}))\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2} L^n_{C_i,0}(I - Q_i(0))\chi(\dot{0}) \right| dt$$

respectively. We have therefore shown that to prove Proposition 4.3.8, it suffices to show that $\mathrm{I}(\epsilon)^i_n$ and $\mathrm{II}(\epsilon)^i_n$ decay at a $n^{-1/2}$ rate. The next two lemmas prove this.

Lemma 4.3.9. *For any fixed sufficiently small $\epsilon > 0$,*

$$\mathrm{I}(\epsilon)^i_n = O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. It suffices to show that for any fixed sufficiently small $\epsilon > 0$ and for all i, k, the quantity

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|} \left| e^{nP_i(itfn^{-1/2})} Q_{i,k}(itn^{-1/2})\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2+nh} Q_{i,k}(0)\chi(\dot{0}) \right| dt$$

is $O\left(n^{-1/2}\right)$. By the triangle inequality, this is a simple consequence of the following two estimates.

For any fixed sufficiently small $\epsilon > 0$,

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|} \left| e^{nP_i(itfn^{-1/2})} Q_{i,k}(itn^{-1/2})\chi(\dot{0}) - e^{nP_i(itfn^{-1/2})} Q_{i,k}(0)\chi(\dot{0}) \right| dt \tag{4.3.9}$$

and

$$e^{-nh} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|} \left| e^{nP_i(itfn^{-1/2})} Q_{i,k}(0)\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2+nh} Q_{i,k}(0)\chi(\dot{0}) \right| dt \tag{4.3.10}$$

are both $O\left(n^{-1/2}\right)$. To prove that (4.3.9) decays at an $O(n^{-1/2})$ rate, recall that for each i, k there exists bounded linear operators $\tilde{Q}_{i,k}$ such that

$$Q_{i,k}(t) = Q_{i,k}(0) + t\tilde{Q}_{i,k}(t)$$

for all t sufficiently small. Also, from the Taylor expansion for the pressure (3.5.8) (recall that we are assuming $\Lambda_\varphi = 0$), we can assume that ϵ is sufficiently small so that for $|t| < \epsilon\sqrt{n}$,

$$|e^{nP_i(itfn^{-1/2})-nh}| \leq e^{-\sigma_\varphi^2 t^2/4}.$$

Hence for fixed sufficiently small $\epsilon > 0$, there exists $C > 0$ such that

$$\begin{aligned} e^{-nh}\left|e^{nP_i(itfn^{-1/2})}Q_{i,k}(itn^{-1/2})\chi(\dot{0}) - e^{nP_i(itfn^{-1/2})}Q_{i,k}(0)\chi(\dot{0})\right| \\ = \frac{|t|}{\sqrt{n}}\left|\tilde{Q}_{i,k}\left(\frac{|t|}{\sqrt{n}}\right)\right|\left|e^{nP_i(tfn^{-1/2})-nh}\right| \\ \leq \frac{C|t|}{\sqrt{n}}e^{-\sigma_\varphi^2 t^2/4}, \end{aligned}$$

for all $|t| < \epsilon\sqrt{n}$. Substituting this inequality into (4.3.9) gives the result.

The required decay rate for (4.3.10) can be proved analogously to Theorem 1 in [14]. The proof is almost identical and hence we refer the reader to [14] for the proof.

Combining (4.3.9) and (4.3.10) concludes the proof of the lemma. □

Lemma 4.3.10. *For fixed small $\epsilon > 0$,*

$$\mathrm{II}(\epsilon)_n^i = O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Recall that by Lemma 2.2.5, $L^n_{C_i,0}(I - Q_i(0))\chi(\dot{0}) = O\left(e^{n(h-\delta)}\right)$ for some $\delta > 0$. Using this fact and the inequality $|e^z - 1| \leq |z|e^{|z|}$ it is easy to see that for any fixed sufficiently small ϵ,

$$e^{-nh}\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|}\left|L^n_{C_i,0}(I - Q_i(0))\chi(\dot{0}) - e^{-\sigma_\varphi^2 t^2/2}L^n_{C_i,0}(I - Q_i(0))\chi(\dot{0})\right| dt$$

is $O\left(n^{-1/2}\right)$.

Hence to conclude the proof of this lemma it suffices to show that for fixed small $\epsilon > 0$ and for all i,

$$e^{-nh}\int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|}\left|L^n_{C_i,itn^{-1/2}f}(I - Q(itn^{-1/2}))\chi(\dot{0}) - L^n_{C_i,0}(I - Q_{i,k}(0))\chi(\dot{0})\right| dt \tag{4.3.11}$$

is $O\left(n^{-1/2}\right)$.

To obtain the required decay rate for (4.3.11), we begin by defining operators $T_{i,n}(t)$ by

$$M^n L^n_{C_i,tf}(I - Q_i(t)) = M^n L^n_{C_i,0}(I - Q_i(0)) + T_{i,n}(t), \tag{4.3.12}$$

where M is the multiplication operator $Mg = e^{-h}g$. To simplify notation in the following, let L_t denote the operator $ML_{C_i,t}(I - Q_i(t))$. Note that the spectral radius of L_0 is strictly less than 1. As discussed earlier, we can find (at the cost of reducing ϵ), $0 < \rho < 1$ and $K > 0$ such that

$$\|L_s^n\| \leq K\rho^n$$

for all $|s| < \epsilon$ and $n \in \mathbb{Z}_{\geq 0}$.

An operator version of the Mean Value Theorem (see Theorem 3.2 of [3]) states that,

$$\|L_t^n - L_0^n\| \leq |t| \sup_{0<l<1} \|D(L_{tl}^n)\|,$$

where $D(L_t)$ denotes the derivative of an operator $s \mapsto L_s$ at t. Furthermore, applying the Leibniz rule yields

$$D(L_t^n) = \sum_{k=1}^{n} L_t^{n-k} \; DL_t \; L_t^{k-1}.$$

Hence, for fixed, small ϵ,

$$\begin{aligned}\|T_{i,n}(itn^{-1/2})\| = \|L^n_{itn^{-1/2}} - L_0^n\| &\leq |t|n^{-1/2} \sup_{0<l<1} \|D(L^n_{itn^{-1/2}l})\| \\ &\leq |t|n^{-1/2} Cn\rho^n \\ &= C|t|\sqrt{n}\rho^n,\end{aligned}$$

for some constant $C > 0$ independent of $|t| < \epsilon\sqrt{n}$.

Now note that

$$e^{-nh} \left| L^n_{C_i,itn^{-1/2}f}(I - Q(itn^{-1/2}))\chi(\dot{0}) - L^n_{C_i,0}(I - Q_{i,k}(0))\chi(\dot{0}) \right|$$

can be rewritten as

$$\left| T_{i,n}(itn^{-1/2})\chi(\dot{0}) \right|.$$

We see that for fixed, sufficiently small $\epsilon > 0$, there exists a constant $C > 0$ (independent of i, n and t) such that (4.3.11) is bounded above by

$$C \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{|t|} \sqrt{n}|t|\rho^n dt = 2C\epsilon n\rho^n.$$

This clearly satisfies the required decay rate for (4.3.11) and thus concludes the proof of the lemma. □

From these two lemmas, we deduce Proposition 4.3.8. We are now ready to prove our central limit theorem.

Proof of Theorem 4.1.2. By Lemma 4.3.6, Proposition 4.3.7 and Proposition 4.3.8, there exists $\epsilon > 0$ such that for $T_n = \epsilon\sqrt{n}$, the following hold.

1. The pair H_n, T_n satisfy the conditions required to apply Proposition 10.2, with N as the normal distribution with mean 0 and variance $\sigma_\varphi^2 > 0$.

2. $$\int_{-T_n}^{T_n} \frac{1}{|t|} |\widehat{H}_n(t) - e^{-\sigma_\varphi^2 t^2/2}| \, dt = O\left(\frac{1}{\sqrt{n}}\right).$$

3. $$|E_n| e^{|T_n E_n|} = O\left(\frac{1}{\sqrt{n}}\right).$$

Furthermore, the above implied error term constants are independent of $n \in \mathbb{Z}_{\geq 0}$. Proposition 4.3.3 then implies that

$$\|H_n - N\|_\infty = O\left(\frac{1}{\sqrt{n}}\right),$$

proving Proposition 4.3.4. As discussed in the paragraph preceding Proposition 10.3, this convergence implies that

$$\|F_n - N\|_\infty = O\left(\frac{1}{\sqrt{n}}\right)$$

as required. □

4.4 Large deviation theorem

In this section we prove our large deviation theorem for $\varphi : G \to \mathbb{R}$. As before, let f be the function associated to φ via Condition (1). We begin by defining the following sequence of measures on Σ_A,

$$\mu_n = \frac{1}{\#M_n} \sum_{z \in M_n} \delta_z,$$

where δ_x denotes the Dirac measure based at x and

$$M_n = \{z \in \Sigma_A : \sigma^n(z) = \dot{0}, z_0 = * \text{ and } z_{n-1} \neq 0\}.$$

We want to rephrase our large deviation result in terms of f and μ_n on Σ_A. A simple calculation gives that

$$\frac{1}{\#W_n}\#\left\{g\in W_n:\left|\frac{\varphi(g)}{n}\right|>\epsilon\right\}=\mu_n\left\{z\in\Sigma_A:\left|\frac{f^n(z)}{n}\right|>\epsilon\right\}.$$

Hence to prove Theorem 4.1.3, it suffices to show that for each $\epsilon>0$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n\left\{z\in\Sigma_A:\left|\frac{f^n(z)}{n}\right|>\epsilon\right\}<0.$$

We need the following lemma.

Lemma 4.4.1. *Fix $\epsilon>0$. Then, there exists $\rho>0$ and $k\in\{1,\ldots,m\}$ such that for fixed $t\in\mathbb{R}$ satisfying $0<t<\rho$,*

$$\int e^{tf^n(z)}d\mu_n=O\left(e^{-nh+nt\epsilon/2+nP_k(tf)}\right).$$

The implied constant depends on t and ϵ but not on n.

Proof. Let δ,ϵ be as in Lemma 4.1.8. Take $0<\rho<\epsilon$. For $0<t<\rho$ and for all i, $|P_i(tf)-h|<\delta$. By Lemma 4.1.8 we can write

$$\#W_n\int e^{tf^n(z)}d\mu_n=\sum_{i=1}^{m}L^n_{C_i,tf}\chi(\dot{0})+O(e^{n(h-\delta)}).$$

By the Spectral Radius Theorem we can take $C_t>0$, depending on t but not i, such that

$$\|L^n_{C_i,tf}\|\leq C_te^{n(P_i(tf)+\epsilon t/2)}$$

for all $n\in\mathbb{Z}_{\geq 0}$ and $i=1,\ldots,m$. Combining these observations gives that

$$\begin{aligned}\int e^{tf^n(z)}d\mu_n&=\frac{\sum_{i=1}^m L^n_{C_i,tf}\chi(\dot{0})}{\#W_n}+O(e^{-n\delta})\\&=O\left(e^{-nh+nt\epsilon/2}\sum_{i=1}^{m}e^{nP_i(tf)},e^{-n\delta}\right)\\&=O\left(e^{-nh+nt\epsilon/2}\sum_{i=1}^{m}e^{nP_i(tf)}\right).\end{aligned}$$

We now recall that, by Proposition 2.2.5, the maps $t\mapsto e^{P_i(tf)}$ for $i=1,\ldots,m$, are real analytic. Hence there exists $\xi>0$ and $k\in\{1,\ldots,m\}$ such that for all $0<t<\xi$,

$$\max_{i=1,\ldots,m}\left\{e^{P_i(tf)}\right\}=e^{P_k(tf)}.$$

By reducing ρ, if necessary, so that it is less that ξ, we see that for fixed

$0 < t < \rho$,

$$\int e^{tf^n(z)} d\mu_n = O\left(e^{-nh+nt\epsilon/2} \sum_{i=1}^{m} e^{nP_i(tf)}\right)$$
$$= O\left(e^{-nh+nt\epsilon/2+nP_k(tf)}\right),$$

as required. □

The same proof as the previous lemma gives the following.

Lemma 4.4.2. *Fix $\epsilon > 0$. Then, there exists $\rho' < 0$ and $k' \in \{1, \ldots, m\}$ such that for fixed $t \in \mathbb{R}$ satisfying $\rho' < t < 0$,*

$$\int e^{tf^n(z)} d\mu_n = O\left(e^{-nh-nt\epsilon/2+nP_{k'}(tf)}\right).$$

The implied constant depends on t and ϵ but not on n.

We are now ready to prove our large deviation theorem.

Proof of Theorem 4.1.3. Fix $\epsilon > 0$. Let ρ and k be those chosen in Lemma 4.4.1. Define $b(s) = -s\epsilon/2 - h + P_k(sf)$. Note that $b(0) = 0$ and

$$b'(0) = -\epsilon/2 + \frac{d}{ds} P_k(sf)\big|_{s=0} = -\epsilon/2 + \Lambda_\varphi = -\epsilon/2 < 0.$$

Hence we can choose $0 < t < \rho$ such that $b(t) < 0$. Fix t at this value, then,

$$\mu_n \left\{z \in \Sigma_A : \frac{f^n(z)}{n} > \epsilon\right\} \leq \int e^{t(f^n(z)-n\epsilon)} d\mu_n$$
$$= e^{-tn\epsilon} \int e^{tf^n(z)} d\mu_n$$
$$\leq \tilde{C}_t e^{-tn\epsilon-nh+tn\epsilon/2+nP_k(tf)}$$
$$= \tilde{C}_t e^{nb(t)},$$

where the second inequality in the above follows from Lemma 4.4.1 and $\tilde{C}_t$ is the constant associated to the error term from this lemma.

Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n \left\{z \in \Sigma_A : \frac{f^n(z)}{n} > \epsilon\right\} \leq b(t) < 0.$$

The inequality

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n \left\{z \in \Sigma_A : \frac{f^n(z)}{n} < -\epsilon\right\} < 0$$

can be proven in a similar way, this time using Lemma 4.4.2 instead of Lemma 4.4.1. By our earlier discussion, this concludes the proof. □

4.5 Multidimensional central limit theorem

In this section we prove Theorem 4.1.6. To do so, we generalise our current methods to the multidimensional setting. That is, we show that our methods apply to functions $\varphi : G \to \mathbb{R}^k$ that satisfy Condition (1) and Condition (2) componentwise. We begin by recalling the multidimensional central limit theorem for subshifts of finite type. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product.

Suppose Σ_M is an irreducible subshift of finite type and $f : \Sigma_M \to \mathbb{R}^k$ a function with components that belong to $F_\theta(\Sigma_M)$ for some $0 < \theta < 1$. Then, there exists a covariance matrix $\Sigma \in M_k(\mathbb{R})$ and $\Lambda \in \mathbb{R}^k$ such that for any $A \subset \mathbb{R}^k$,

$$\mu \left\{ x \in \Sigma_M : \frac{f^n(x) - n\Lambda}{\sqrt{n}} \in A \right\} \to \frac{1}{(2\pi \ \det(\Sigma))^{k/2}} \int_A e^{-\langle x, \Sigma x \rangle/2} \, dx$$

where μ is the measure of maximal entropy for (Σ_M, σ). Furthermore, the following are equivalent,

1. the above central limit theorem is non-degenerate,
2. Σ is positive definite,
3. $\langle t, f \rangle$ is not cohomologous to a constant for any $t \in \mathbb{R}^k \backslash \{0\}$,
4. for or each $t \in \mathbb{R}^k \backslash \{0\}$ the set $\{\langle t, (f^n(x) - n\Lambda)\rangle : x \in \Sigma_M, n \in \mathbb{Z}_{\geq 0}\}$ is unbounded.

Let $L_{\langle s, f \rangle}$ denote the transfer operator acting on $F_\theta(\Sigma_M)$ defined in Definition 2.2. Proposition 2.2.5 implies that for all sufficiently small $s \in \mathbb{C}^k$, the transfer operator $L_{\langle s, f \rangle}$ has p simple maximal eigenvalues of the form $e^{2\pi i j/p} e^{P(\langle s, f \rangle)}$ for $j = 1, \ldots, p$ where p is the period of M and $s \mapsto P(\langle s, f \rangle)$ is analytic in a neighbourhood of the origin. The constant Λ and covariance matrix Σ have entries

$$\Lambda_i = \frac{\partial}{\partial s_i}\bigg|_{s=0} P(\langle s, f \rangle) \ \text{ and } \ \Sigma_{i,j} = \frac{\partial^2}{\partial s_i \partial s_j}\bigg|_{s=0} P(\langle s, f \rangle)$$

for $i, j \in \{1, \ldots, k\}$ and where $s = (s_1, \ldots, s_k)$.

Using the same arguments as in Sections 5, we can deduce similar statements concerning the spectra of the operators $L_{C_i, \langle s, f \rangle}$.

Proposition 4.5.1. *There exists $\epsilon > 0$ such that for all $\|s\| < \epsilon$ the operators $L_{C_i,\langle s,f\rangle}$ for $i = 1, \dots, m$ each have p_i simple maximal eigenvalues $e^{2\pi ij/p_i}e^{P_i(\langle s,f\rangle)}$ for $j = 0, \dots, p_i - 1$, where each $s \mapsto e^{P_i(\langle s,f\rangle)}$ is analytic in $\|s\| < \epsilon$.*

Futhermore, the argument of Calegari and Fujiwara presented in Proposition 3.6.2 can be applied to compare the pressure functions $P_i(\langle s, f\rangle)$ for $i = 1, \dots, m$. The following result can be obtained using the same argument used to prove Proposition 3.6.2. The required modification to the proof is simple, we need only replace the use of the central limit theorem for subshifts of finite type with the multidimensional version stated above.

Proposition 4.5.2. *Given $\alpha, \beta \in \{1, \dots, k\}$ the quantities*

$$(\Lambda_\varphi)_\alpha := \frac{\partial}{\partial s_\alpha}\bigg|_{s=0} P_i(\langle s, f\rangle) \ \ \text{and} \ \ (\Sigma_\varphi)_{\alpha,\beta} := \frac{\partial^2}{\partial s_\alpha \partial s_\beta}\bigg|_{s=0} P_i(\langle s, f\rangle)$$

do not depend on the maximal component B_i. Furthermore, for each $i = 1, \dots, m$ and $\|s\| < \epsilon$,

$$P_i(\langle s, f\rangle) = h + \Lambda_\varphi s + \langle s, \Sigma_\varphi s\rangle + O(\|s\|^3)$$

as $s \to 0$.

We now turn our attention to the non-degeneracy criteria in the multidimensional setting. Lemma 3.6.2 can be easily generalised using the multidimensional criteria for degeneracy stated above. We obtain the following result.

Proposition 4.5.3. *Let Σ_φ be the covariance matrix defined above. Then Σ_φ is positive definite if and only if for each non-zero $t \in \mathbb{R}$, the function $\langle t, \varphi(\cdot) - \Lambda_\varphi|\cdot|\rangle : G \to \mathbb{R}$ is unbounded.*

We are now ready to prove a multidimensional central limit theorem.

Theorem 4.5.4. *Suppose $\varphi : G \to \mathbb{R}^k$ satisfies Condition (1) and Condition (2) componentwise. Then there exists $\Lambda_\varphi \in \mathbb{R}^k$ and a symmetric matrix $\Sigma_\varphi \in M_k(\mathbb{R})$ such that*

$$\frac{1}{\#W_n}\#\left\{g \in W_n : \frac{\varphi(g) - \Lambda_\varphi n}{\sqrt{n}} \in A\right\} \to \frac{1}{(2\pi \ \det(\Sigma_\varphi))^{k/2}} \int_A e^{-\langle x,\Sigma_\varphi x\rangle/2}\, dx$$

as $n \to \infty$. Furthermore, Σ_φ is positive definite if and only if for each non-zero $t \in \mathbb{R}^k$ the function $\langle t, \varphi(\cdot) - \Lambda_\varphi|\cdot|\rangle : G \to \mathbb{R}$ is unbounded.

Proof. We have already discussed the non-degeneracy criteria. We therefore just need to prove the central limit theorem. As in the previous sections, we may assume that $\Lambda_\varphi = 0$. It then suffices, by Lévy's Continuity Theorem, to show that we have the following pointwise convergence of the Fourier transform: for each $t \in \mathbb{R}^k$,

$$\widehat{F}_n(t) \to e^{-\langle t, \Sigma_\varphi t \rangle/2}$$

as $n \to \infty$, where

$$\widehat{F}_n(t) = \frac{1}{\# W_n} \sum_{g \in W_n} e^{i\langle t, \varphi \rangle n^{-1/2}}.$$

Using a multidimensional analogue of Lemma 4.1.8 (which can be proved in the same way as the one-dimensional version), we can write, for all $\|t\| n^{-1/2}$ sufficiently small,

$$\sum_{g \in W_n} e^{i\langle t, \varphi \rangle n^{-1/2}} = \sum_{i=1}^{m} L^n_{C_i, i\langle t, f \rangle n^{-1/2}} \chi(\dot{0}) + o(e^{nh})$$

as $n \to \infty$. Hence,

$$\widehat{F}_n(t) = \frac{\sum_{i=1}^m L^n_{C_i, i\langle t, f \rangle n^{-1/2}} \chi(\dot{0}) + o(e^{nh})}{\sum_{i=1}^m L^n_{C_i, 0} \chi(\dot{0}) + o(e^{nh})}$$

as $n \to \infty$. Using the projections $Q_{i,k}$ and Q_i for $i = 1, \ldots, m, k = 0, \ldots, p_i - 1$, we can write

$$\widehat{F}_n(t) = e^{-\langle t, \Sigma_\varphi t \rangle/2} \, G_n(t)$$

where

$$G_n(t) = \frac{\sum_{i=1}^m \sum_{k=0}^{p_i - 1} e^{nP(\langle itn^{-1/2}, f \rangle) + \langle t, \Sigma_\varphi t \rangle/2} e^{2\pi i k n/p_i} Q_{i,k}(itn^{-1/2}) \chi(\dot{0}) + o(1)}{\sum_{i=1}^m \sum_{k=0}^{p_i - 1} e^{2\pi i k n/p_i} Q_{i,k}(\chi)(\dot{0}) + o(1)}.$$

By the analyticity of the $Q_{i,k}$, for each $i = 1, \ldots, m$ and $k = 0, \ldots, p_i - 1$, $Q_{i,k}(t) = Q_{i,k}(0) + O(\|t\|)$. Also, using the Taylor expansions for the pressures from Proposition 4.5.2, for each $i = 1, \ldots, m$, $nP(\langle itn^{-1/2}, f \rangle) + \langle t, \Sigma_\varphi t \rangle/2 = O(n^{-1/2})$. Combining these facts gives that

$$G_n(t) = \frac{\sum_{i=1}^m \sum_{k=0}^{p_i - 1} e^{nP(\langle itn^{-1/2}, f \rangle) + \langle t, \Sigma_\varphi t \rangle/2} e^{2\pi i k n/p_i} Q_{i,k}(0) \chi(\dot{0}) + o(1)}{\sum_{i=1}^m \sum_{k=0}^{p_i - 1} e^{2\pi i k n/p_i} Q_{i,k}(0) \chi(\dot{0}) + o(1)}$$

and so for each $t \in \mathbb{R}$, $G_n(t) \to 1$ as $\to \infty$. Hence $\widehat{F}_n(t) \to e^{-\langle t, \Sigma_\varphi t \rangle/2}$ as $n \to \infty$ as required. $\square$

We can now deduce Theorem 4.1.6 as a corollary of the above result.

Suppose that the abelianisation of G is isomorphic to $\mathbb{Z}^k \oplus \text{Torsion}$ for some $k \geq 1$. Fix an isomorphism taking the non-torsion part of $G/[G,G]$ to $\mathbb{Z}^k$ and let $\varphi : G \to \mathbb{Z}^k$ be the induced homomorphism.

Proof of Theorem 4.1.6. To conclude the proof of Theorem 4.1.6 we need to show that $\Lambda_\varphi = 0$ and Σ_φ is positive definite. To see that $\Lambda_\varphi = 0$ note that for each $j = 1, \ldots, k$ the jth coordinate of Λ_φ is the mean Λ_{φ_j} of the homomorphism φ_j obtained by projecting φ to its jth coordinate. By Theorem 4.1.1 and a simple symmetry argument $\Lambda_{\varphi_j} = 0$ for all $j = 1, \ldots, k$ (see the proof of Lemma **??**). This concludes the first part of the proof. For the second part we need to show that $\langle t, \varphi \rangle$ is unbounded for any $t \in \mathbb{R}^k \backslash \{0\}$. Since φ is surjective onto $\mathbb{Z}^k$, the function $\psi_t : G \to \mathbb{R}$ defined by $\psi_t = \langle t, \varphi \rangle$ is a non-trivial group homomorphism for any $t \in \mathbb{R}^k \backslash \{0\}$. Hence by Lemma 7.11 the result follows. $\square$

Remark 4.5.5. The above proof applies to any surjective group homomorphism $\varphi : G \to \mathbb{Z}^k$.

4.6 Local limit theorem

In this section we prove our local limit theorem, Theorem 4.1.7. Suppose $\varphi : G \to \mathbb{R}$ is a group homomorphism satisfying the hypothesis of Theorem 4.1.7. As in the other sections, we want to study the function $f : \Sigma_A \to \mathbb{R}$ corresponding to φ. We begin by recalling the following definition.

Definition 4.6.1. We say that $f \in F_\theta$ is lattice if there exists $a, b \in \mathbb{R}$ such that

$$\{f^n(x) - an : x \in \Sigma_A, n \in \mathbb{Z}_{\geq 0} \text{ with } \sigma^n(x) = x\} \subseteq b\mathbb{Z}.$$

We want to prove that if f is related to φ via Condition (1), then the restriction of f to each maximal component is non-lattice. This will allow us to deduce important spectral properties for the transfer operators $L_{C_j, itf}$ where t is real. The aim of the next couple of lemmas is to prove this.

Lemma 4.6.2. *Suppose that there exists g_1, g_2, g_3 in G such that $\varphi(g_1), \varphi(g_2), \varphi(g_3)$ form a rationally independent triple. Then, for any $a, b \in \mathbb{R}$, $H_{a,b} = \varphi^{-1}(a\mathbb{Z} \oplus b\mathbb{Z})$ is an infinite index subgroup of G, i.e. $|H_{a,b} : G| = \infty$.*

Proof. For each $a, b \in \mathbb{R}$ there is $g \in G$ with $\varphi(g) \notin a\mathbb{Q} \oplus b\mathbb{Q}$. Indeed, if no such g exists then we can find $x_i, y_i \in \mathbb{Q}$ for $i = 1, 2, 3$ such that

$$\varphi(g_i) = ax_i + by_i \quad \text{for} \quad i = 1, 2, 3.$$

Eliminating a and b would imply that the $\varphi(g_i)$ are rationally dependent contrary to our assumption.

Now consider for $k, l \in \mathbb{Z}$ the cosets $g^k H_{a,b}, g^l H_{a,b}$ for $g \notin a\mathbb{Q} \oplus b\mathbb{Q}$. If these cosets coincide then $g^{k-l} \in H_{a,b}$, $(k-l)\varphi(g) \in a\mathbb{Z} \oplus b\mathbb{Z}$ and so $\varphi(g) \in a\mathbb{Q} \oplus b\mathbb{Q}$. This contradiction implies that $g^k H_{a,b}$ and $g^l H_{a,b}$ are distinct for $k \neq l$. Hence $|H_{a,b} : G| = \infty$ as required. $\square$

We now require the following result of Gouëzel, Mathèus and Maucourant.

Proposition 4.6.3 (Theorem 4.3 [26])**.** *Suppose G is a non-elementary hyperbolic group equipped with a finite generating set and $H < G$ is an infinite index subgroup of G. Then the density of H with respect to W_n is zero, i.e.*

$$\lim_{n\to\infty} \frac{\#(W_n \cap H)}{\#W_n} = 0.$$

Using this and the previous lemma we deduce the following.

Lemma 4.6.4. *For each $a, b \in \mathbb{R}$ there exist $D \in \mathbb{Z}_{\geq 0}$ such that*

$$\lim_{n\to\infty} \frac{1}{\#W_{Dn}} \#\{g \in W_{Dn} : \varphi(g) - aDn \in b\mathbb{Z}\} = 0.$$

Proof. Notice that

$$\{g \in G : \varphi(g) - a|g| \in b\mathbb{Z}\} \subseteq H_{a,b}.$$

Hence,

$$\#\{g \in W_n : \varphi(g) - an \in b\mathbb{Z}\} \leq \#(W_n \cap H_{a,b}).$$

If φ satisfies the hypotheses of Lemma 4.6.2, then we may then apply Proposition 4.6.3 to conclude that

$$\#(W_n \cap H_{a,b}) = o(\#W_n)$$

and the result follows.

Otherwise, the image of φ is $c\mathbb{Z} \oplus d\mathbb{Z}$ for some rationally independent $c, d \in \mathbb{R}$. We can assume that $a\mathbb{Z} \cap (c\mathbb{Z} \oplus d\mathbb{Z})$ is non-empty, since if it is empty, $H_{a,b}$ has infinite index and we can apply the same argument used above. Fix $D \in \mathbb{Z}_{\geq 0}$ such that $aD \in a\mathbb{Z} \cap (c\mathbb{Z} \oplus d\mathbb{Z})$. Then note that

$$\{g \in W_{Dn} : \varphi(g) - aDn \in b\mathbb{Z}\} \subset \{g \in W_{Dn} : \widetilde{\varphi} \circ \varphi(g) = \widetilde{\varphi}(aDn)\}$$

where $\widetilde{\varphi} : c\mathbb{Z} \oplus d\mathbb{Z} \to c\mathbb{Z} \oplus d\mathbb{Z}/(b\mathbb{Z} \cap (c\mathbb{Z} \oplus d\mathbb{Z})) = K_b$ is the quotient homomorphism. We have that K_b is necessarily isomorphic to $\mathbb{Z} \oplus$ Torsion or $\mathbb{Z}^2$ depending on whether $b\mathbb{Z} \cap (c\mathbb{Z} \oplus d\mathbb{Z})$ is trivial. Let $\varphi' : G \to \mathbb{Z}$ be the composition $\psi \circ \widetilde{\varphi} \circ \varphi$ where $\psi : K_b \to \mathbb{Z}$ is a homomorphism that projects K_b to a $\mathbb{Z}$ factor. We then have that

$$\#\{g \in W_{Dn} : \varphi(g) - aDn \in b\mathbb{Z}\} \leq \#\{g \in W_{Dn} : \varphi'(g) - n(\psi \circ \widetilde{\varphi}(aD)) = 0\}.$$

and so we need to show that

$$\#\{g \in W_{Dn} : \varphi'(g) - n(\psi \circ \widetilde{\varphi}(aD)) = 0\} = o(\#W_{Dn}).$$

This follows from Corollary 3.6.12 if $\psi \circ \widetilde{\varphi}(aD) = 0$ and Theorem 4.1.3 if $\psi \circ \widetilde{\varphi}(aD) \neq 0$. This concludes the proof. □

We can now deduce the required properties of f.

Lemma 4.6.5. *For each maximal component B_j the restriction of f to Σ_{B_j}, f_j, is non-lattice.*

Proof. Suppose f_j is lattice. We can then find $a, b \in \mathbb{R}$ such that

$$\{f_j^n(x) - na : \sigma^n(x) = x, x \in \Sigma_{B_j}\} \subseteq b\mathbb{Z}.$$

Since $\#\{x \in \Sigma_{B_j} : \sigma^{np_j}(x) = x\}$ grows like λ^{np_j}, the correspondence between G and Σ_A implies (as in the proof of Proposition 3.6.11) that $\#\{g \in W_{np_j} : \varphi(g) - np_j a \in b\mathbb{Z}\} \geq C\lambda^{np_j}$ for some $C > 0$. We then have that, for any integer D,

$$\limsup_{n\to\infty} \frac{1}{\#W_{Dn}} \#\{g \in W_{Dn} : \varphi(g) - aDn \in b\mathbb{Z}\} > 0.$$

This contradicts Lemma 4.6.4. The result follows. □

Using this lemma, we deduce the following.

Proposition 4.6.6. *Suppose $\varphi : G \to \mathbb{R}$ is a group homomorphism satisfying the hypothesis of Theorem 4.1.7. Then for all $t \in \mathbb{R}\backslash\{0\}$ and each $j = 1, \ldots, m$, the spectral radius of $L_{C_j, itf}$ is strictly less than e^h.*

Proof. When C_j consists of a single connected component, it is well known that the spectral radius of $L_{C_j, itf}$ is less than or equal to e^h for all $t \in \mathbb{R}$. The non-lattice condition guarantees that for all $t \in \mathbb{R}\backslash\{0\}$ this inequality is strict [42]. When C_j is not a single component, C_j contains a component with spectral radius e^h and all other components have spectral radius strictly less than e^h.

We can then, by Lemma 2 from [47], apply the above result component-wise to deduce our result. $\square$

We are now ready to prove Theorem 4.1.7. Since our method follows that of [52], we will sketch the proof. We will highlight where our work is needed.

Proof of Theorem 4.1.7. We sketch a proof. Note that, by Corollary 3.6.11, $\sigma_\varphi^2 > 0$. Theorem 4.1.7 is concerned with the asymptotics of

$$\frac{1}{\#W_n} \sum_{g \in W_n} \chi_{[a,b]}(\varphi(g)),$$

where $\chi_{[a,b]}$ is the indicator function on $[a,b]$ for $a, b \in \mathbb{R}$. We first consider this expression when $\chi_{[a,b]}$ is replaced by a function $\phi_{[a,b]} : \mathbb{R} \to \mathbb{C}$ which is integrable and has Fourier transform $\widehat{\phi}_{[a,b]}$ that is compactly supported and satisfies $\widehat{\phi}_{[a,b]}(t) = \widehat{\phi}_{[a,b]}(0) + O(|t|)$. Using Fourier inversion we can write

$$\sum_{g \in W_n} \phi_{[a,b]}(\varphi(g)) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{g \in W_n} e^{it\varphi(g)} \widehat{\phi}_{[a,b]}(t) \; dt.$$

Then, using Lemma 4.1.8 and the same over-counting argument used to prove Theorem 4.1.2 (see Lemma 4.3.5), we can assume

$$\sum_{g \in W_n} e^{it\varphi(g)}$$

has an exact expression in terms of the transfer operators. We can then write,

$$\sum_{g \in W_n} \phi_{[a,b]}(\varphi(g)) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}_{[a,b]}(t) \sum_{j=1}^{m} L^n_{C_j, itf} \chi(\dot{0}) \; dt.$$

Then, using Proposition 4.6.6 and Lemma 4.1.8, we show that there exists $\epsilon > 0$ such that the lead terms describing the growth of this quantity are

$$\frac{1}{2\pi} \int_{[-\epsilon,\epsilon]} \widehat{\phi}_{[a,b]}(t) e^{2\pi ikn/p_j} e^{nP_j(itf)} Q_{j,k}(it) \chi(\dot{0}) \; dt$$

for all pairs j, k. We can then apply the arguments presented in [52] to show that, for each j, k this quantity grows asymptotically like

$$\frac{\int \phi_{[a,b]}(t) \; dt \quad e^{2\pi ikn/p_j} Q_{j,k}(0) \chi(\dot{0})}{\sqrt{2\pi}\sigma_\varphi \sqrt{n}} \, e^{nh}$$

where we have used that σ_φ is independent of the maximal component. We then normalise by $\#W_n$ and write $\#W_n$ in terms of transfer operators (see the proof of Theorem 4.5.4) to see that

$$\frac{1}{\#W_n}\sum_{g\in W_n}\phi_{[a,b]}(\varphi(g)) \sim \frac{\int \phi_{[a,b]}(t)\ dt}{\sqrt{2\pi}\sigma_\varphi\sqrt{n}}$$

as $n \to \infty$. Using a standard approximation argument we can remove the assumptions on $\phi_{[a,b]}$ and show that the above convergence holds when $\phi_{[a,b]}$ is replaced by any smooth positive function of compact support. We can then use a further standard approximation argument to deduce the same converges holds when we replace $\phi_{[a,b]}$ with $\chi_{[a,b]}$. This concludes the proof. $\square$

As mentioned previously, the hypothesis of Theorem 4.1.7 is satisfied, in some sense, by almost every homomorphism $\varphi : G \to \mathbb{R}$. We will now explain what we mean by this. Note that, since every homomorphism $\varphi : G \to \mathbb{R}$ factors through the abelianisation $G/[G,G]$ of G, any homomorphism is of the form $g \mapsto \langle \varphi_{ab}(g), v\rangle$ where $\varphi_{ab} : G \to \mathbb{Z}^k$ is the abelianisation homomorphism post-composed with the projection to the non-torsion factor of $G/[G,G]$, and v is a vector in $\mathbb{R}^k$. We can therefore naturally identify the space of homomorphisms $\mathrm{Hom}(G,\mathbb{R})$ with $\mathbb{R}^k$ where $k \in \mathbb{Z}$ is the rank of the abelianisation of G. As long as $k \geq 2$ then we can find homomorphisms in $\mathrm{Hom}(G,\mathbb{R})$ that satisfy our theorem, as these homomorphisms correspond to vectors $v \in \mathbb{R}^k$ that have two entries that form a rationally independent pair. Furthermore since rationally dependent pairs lie in a countable collection of planes of codimension at least 1 in $\mathbb{R}^k$ for $k \geq 2$, the set of vectors in $\mathbb{R}^k$ that correspond to homomorphisms that satisfy our theorem have complement in $\mathbb{R}^k$ that has Lebesgue measure zero. In this sense almost all homomorphisms satisfy the hypotheses of Theorem 4.1.7.

We now prove a local limit theorem for the displacement function associated to certain 'nice' actions. See [17] for the definitions of the objects used throughout the rest of this section. Our aim now is to prove the following.

Theorem 4.6.7. *Suppose that a fuchsian group G (equipped with a finite symmetric generating set) acts convex cocompactly on a pinched Hadamard surface X with origin $o \in X$. Then there exists $\sigma^2 > 0$ such that for $a, b \in \mathbb{R}$, $a < b$,*

$$\frac{1}{\#W_n}\#\left\{g \in W_n : d(o, go) - n\Lambda \in [a,b]\right\} \sim \frac{b-a}{\sqrt{2\pi}\sigma\sqrt{n}}$$

as $n \to \infty$.

Suppose for the rest of this section that G and X are as in the above theorem. It follows immediately that G is hyperbolic and that the displacement function satisfies Condition (1) and (2). We restrict our study to these actions because we have, in this setting, a good understanding of the length spectrum. Recall that the length spectrum for the action of G on X is the set of possible translation lengths, where, given $g \in G$, the translation length of g is

$$\tau(g) = \lim_{n \to \infty} \frac{d(o, g^n o)}{n}.$$

This limit exists by the triangle inequality and subadditivity. Let $r : \Sigma_A \to \mathbb{R}$ be the function related to the displacement function via Condition (1). We would like to use arguments involving the length spectrum to deduce non-lattice properties for r. The following definition and lemma allow us to do this.

Definition 4.6.8. Let v be a vertex in $\mathcal{G}$. The loop semi-group L_v associated to v is the semi-group consisting of group elements $g \in G$ that correspond (under the labeling ρ from Definition 3.3) to a loop in $\mathcal{G}$ starting (and also ending) at v.

This definition is taken from [23]. We then have the following.

Lemma 4.6.9. *The restriction $r : \Sigma_{B_j} \to \mathbb{R}$ is lattice if and only if there exists $a, b \in \mathbb{R}$ such that for each each vertex $v \in B_j$*

$$\{\tau(g) - a|g| : g \in L_v\} \subseteq b\mathbb{Z}.$$

Proof. Take $g \in L_v$. By the Hölder properties of r, we have that,

$$d(o, go) = r^{|g|}(x_g) + O(1)$$

where $x_g \in \Sigma_{B_j}$ is the periodic point obtained from repeating the loop corresponding to $g \in L_v$. The implied error is uniform in g. Applying this equality to g^n, using that $|g^n| = n|g|$ and then dividing by n and letting n tend to infinity shows that $\tau(g) = r^{|g|}(x_g)$. Substituting this expression into the non-lattice condition concludes the proof. □

We note that, if $r : \Sigma_{B_j} \to \mathbb{R}$ is non-lattice for any maximal component B_j, then by Lemma 3.6.2 the variance σ^2 associated to the displacement function is strictly positive. We are now ready to prove our result.

proof of Theorem 4.6.7. We can use the same method used above to prove Theorem 4.1.7. To apply our argument we need to show that the restrictions

$r : \Sigma_{B_j} \to \mathbb{R}$ are non-lattice. Once we have shown this, our result follows as before.

We begin by noting that, by Corollary 6.11 of [23], for a vertex v belonging to a maximal component, there exist independent hyperbolic elements $g, h \in L_v$ (i.e. g and h both have two fixed points in the boundary ∂X and these four fixed points are all distinct). We now consider for $n \in \mathbb{Z}_{\geq 0}$ the elements gh^n. These elements satisfy the following properties,

1. for all n sufficiently large gh^n is hyperbolic, and;
2. for all $n \in \mathbb{Z}_{\geq 0}$, $|gh^n| = |g| + n|h|$.

The first property is easy to verify and the second follows from the properties of the coding from Definition 3.3 . This second identity in the above implies that for any $a \in \mathbb{R}$,

$$e^{\tau(gh^n)-\tau(gh^{n-1})} = e^{(\tau(gh^n)-a|gh^n|)-(\tau(gh^{n-1})-a|gh^{n-1}|)+a} \tag{4.6.1}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Furthermore, it is known (see [17]) that

$$\lim_{n\to\infty} e^{\tau(gh^n)-\tau(gh^{n-1})} = e^{\tau(h)}. \tag{4.6.2}$$

We now suppose for contradiction that

$$\{\tau(g) - a|g| : g \in L_v\} \subseteq b\mathbb{Z}$$

for some $a, b \in \mathbb{R}$. By (4.6.1) we have that

$$e^{\tau(gh^n)-\tau(gh^{n-1})} \in e^{a+b\mathbb{Z}}$$

for all $n \in \mathbb{Z}_{\geq 0}$. The convergence in (4.6.2) then implies that for all n sufficiently large

$$\tau(gh^n) = \tau(gh^{n-1}) + \tau(h).$$

However, from the proof of Proposition 2.1 in [17], we can find arbitrarily large n such that that $\tau(gh^n) < \tau(gh^{n-1}) + \tau(h)$. This contradiction shows that the restrictions $r : \Sigma_{B_j} \to \mathbb{R}$ are non-lattice. By our above discussion, this concludes the proof. $\square$

Chapter 5

Growth along geodesic rays

5.1 Discussion and statement of results

In this chapter we study the growth of functions as we travel along Patterson–Sullivan typical geodesic rays in ∂G. Our first result is the following. Let ν denote the Patterson-Sullivan measure defined in Section 3.3.

Theorem 5.1.1. *Let G be a non-elementary hyperbolic group equipped with a finite generating set S. Suppose that $\varphi : G \to \mathbb{R}$ satisfies Condition* (1) *and Condition* (2)*. Then there exists $\Lambda \in \mathbb{R}$ such that for ν almost every $[\widetilde{\gamma}] \in \partial G$,*

$$\frac{\varphi(\gamma_n)}{n} = \Lambda + O\left(\frac{\sqrt{\log \log n}}{\sqrt{n}}\right),$$

for any γ belonging to $[\widetilde{\gamma}]$. The implied error constant depends only on γ.

Remark 5.1.2. When φ is the displacement function associated to a convex cocompact group action on a CAT(-1) metric space, we recover a special case of Proposition 1.0.1 with an improved error term. We note that the non-elementary actions to which Proposition 1.0.1 applies are more general than convex cocompact.

This shows that, along typical elements of ∂G, a function φ satisfying the hypotheses of Theorem 1.2 grows asymptotically like Λn. We can then ask if it is possible to describe more precisely how φ grows along elements of ∂G. To achieve this, we need to impose an additional assumption on φ to ensure that $\varphi(\cdot) - |\cdot|\Lambda$ grows along typical geodesic rays. Specifically, we need that the set

$$\{[\gamma] \in \partial G : \{\varphi(\gamma_n) - n\Lambda : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded}\}$$

is non-empty. The fact that this set is well-defined will follow from Condition

(2). Surprisingly, this is the only additional hypothesis we need in order to obtain the following, more precise description of how φ grows.

Theorem 5.1.3. *Let G be a non-elementary hyperbolic group equipped with a finite generating set S. Fix a bounded subset H of the vertex set of the Cayley graph of G. Suppose $\varphi : G \to \mathbb{R}$ satisfies Condition* (1) *and Condition* (2) *and that Λ is the quantity defined in Theorem* 1.2. *Then, if the set*

$$\{[\gamma] \in \partial G : \{\varphi(\gamma_n) - n\Lambda : n \in \mathbb{Z}_{\geq 0}\} \textit{ is unbounded}\}$$

is non-empty, there exists $\sigma^2 > 0$ such that for $x \in \mathbb{R}$,

$$\nu(\mathcal{A}_n(x)) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-t^2/2\sigma^2} \, dt + O(n^{-1/4}),$$

as $n \to \infty$, where

$$\mathcal{A}_n(x) = \left\{ [\widetilde{\gamma}] \in \partial G : \textit{ for all } \gamma \in [\widetilde{\gamma}] \textit{ with } \gamma_0 \in H, \frac{\varphi(\gamma_n) - n\Lambda}{\sqrt{n}} \leq x \right\}.$$

The implied constant is uniform in $x \in \mathbb{R}$.

Remark 5.1.4. The reason that we ask for $\gamma_0 \in H$ is due to the following fact. For ν almost every $[\widetilde{\gamma}] \in \partial G$ and every $n \geq 1$, we can find $\gamma \in [\widetilde{\gamma}]$ for which $\varphi(\gamma_n) - n\Lambda$ is arbitrarily large. Therefore without this assumption, $\mathcal{A}_n$ would have zero ν measure for all $n \in \mathbb{Z}_{\geq 0}$.

Corollary 3.6.11 shows that real-valued group homomorphisms satisfy the hypotheses of Theorem 5.1.3.

5.2 Proofs of results

Suppose that $\varphi : G \to \mathbb{R}$ satisfies Condition (1) and Condition (2) and let $f : \Sigma_A \to \mathbb{R}$ be the function related to φ via Condition (1). Fix a bounded subset $H \subset C(G)$ of the vertex set of the Cayley graph $C(G)$ (i.e. $\sup_{g \in H}\{|g|\} < \infty$).

We begin by proving that Theorem 5.1.1 holds without an error term, i.e.we show that there exists $\Lambda \in \mathbb{R}$ for which the set

$$\mathcal{U}_\Lambda = \left\{ [\gamma] \in \partial G : \lim_{n \to \infty} \frac{\varphi(\gamma_n)}{n} = \Lambda \right\},$$

is well-defined and has full ν measure.

Lemma 5.2.1. *For any $\Lambda \in \mathbb{R}$ the set $\mathcal{U}_\Lambda$ is well-defined and G-invariant.*

Proof. Since φ is Lipschitz in the right word metric, if $[\gamma] \in \partial G$ and $g \in G$, then there exists $C > 0$ for which

$$|\varphi(\gamma_n) - \varphi(g\gamma_n)| \leq C$$

uniformly for $n \in \mathbb{Z}_{\geq 0}$. Hence

$$\lim_{n\to\infty} \frac{\varphi(\gamma_n)}{n} = \Lambda \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{\varphi(g\gamma_n)}{n} = \Lambda.$$

This proves G-invariance. The proof that $\mathcal{U}_\Lambda$ is well-defined follows the same argument, this time using that φ is Lipschitz in the left word metric. $\square$

We are now ready to prove Theorem 5.1.1 without the associated error term.

Proof of Theorem 5.1.1 without error term. Since the action of G on ∂G is ergodic with respect to ν, it suffices, by Lemma 5.2.1, to prove that there exists Λ for which $\mathcal{U}_\Lambda$ has positive ν measure. Consider a maximal component B_i. By the ergodic theorem, $\mu(E_\Lambda) > 0$, where

$$E_\Lambda = \left\{ y \in \Sigma_{B_i} : \frac{f^n(y)}{n} \to \Lambda \text{ as } n \to \infty \right\}$$

and $\Lambda = \int_{\Sigma_{B_i}} f \, d\mu_i$. Hence by Proposition 3.3.6 there exists $k \in \mathbb{Z}_{\geq 0}$ for which $\sigma_*^k \widehat{\nu}(E_\Lambda) > 0$. We now note that if $y \in E_\Lambda$ and $x \in \bigcup_{n\geq 0} \sigma^{-n}(\{y\})$ then

$$\lim_{n\to\infty} \frac{f^n(x)}{n} \to \Lambda$$

as $n \to \infty$. Hence,

$$\widehat{\nu} \left\{ y \in Y : \frac{f^n(y)}{n} \to \Lambda \text{ as } n \to \infty \right\} \geq \sigma_*^k \widehat{\nu}(E_\Lambda) > 0.$$

By Condition (1), for $y \in Y$, $f^n(y) = \varphi(h(y)_n) + O(1)$ where the implied constant is independent of both n and y. Combining this with the fact that $h_* \widehat{\nu} = \nu$ implies that $\nu(\mathcal{U}_\Lambda) > 0$ and thus concludes the proof. $\square$

We will now improve this result by including a proof of the error term.

proof of Theorem 5.1.1. Note that if φ has the property that the set

$$\{[\gamma] \in \partial G : \{\varphi(\gamma_n) - n\Lambda : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded}\}$$

is empty, then the error term associated to the convergence in Theorem 5.1.1 is $O(n^{-1})$. Hence to prove our result, we can assume that this set is non-empty. By Proposition 3.6.2 this implies that for any fixed maximal component Σ_{B_i} the function f restricted to Σ_{B_i} is not cohomologous to a constant. With this knowledge, we re-run the proof of our theorem without the error term, but replace the set $\mathcal{U}_\Lambda$ with the set

$$\widehat{\mathcal{U}}_\Lambda = \left\{[\gamma] \in \partial G : \frac{\varphi(\gamma_n)}{n} = \Lambda + O\left(\frac{\sqrt{\log\log n}}{\sqrt{n}}\right)\right\}.$$

As before this set is G-invariant and well defined. We can then follow the same proof as before but replace the use of the ergodic theorem for f on Σ_{B_i} with an application of the law of the iterated logarithm. We can do this because of the assumption that f is not cohomologous to a constant. We obtain that $\mu(\widehat{E}_\Lambda) > 0$, where

$$\widehat{E}_\Lambda = \left\{y \in \Sigma_{B_i} : \frac{f^n(y)}{n} = \Lambda + O\left(\frac{\sqrt{\log\log n}}{\sqrt{n}}\right)\right\}.$$

One can then check that the same algebraic manipulations used in our previous proof allow us to deduce that $\nu(\widehat{\mathcal{U}}_\Lambda) > 0$. This concludes the proof. $\square$

We now move on to the proof of Theorem 5.1.3. By replacing $\varphi(\cdot)$ with $\varphi(\cdot) - \Lambda|\cdot|$ and $f(\cdot)$ with $f(\cdot) - \Lambda$, it suffices to prove Theorem 5.1.3 under the assumption that $\Lambda = 0$. We will assume this from now on.

The intuition behind our proof of Theorem 5.1.3 is the following. By Proposition 3.3.6, μ is obtained from averaging the pushforwards of $\widehat{\nu}$. If we could therefore, in some sense, reverse this averaging and express $\widehat{\nu}$ in terms of μ, then we could use our knowledge of μ to learn about $\widehat{\nu}$. The relationship between these measures is particularly nice and allows us carry out such a procedure.

Recall that we want to study the convergence of the following distributions.

Definition 5.2.2. Define, for $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$,

$$R_n(x) = \nu\left\{[\widetilde{\gamma}] \in \partial G : \text{ for all } \gamma \in [\widetilde{\gamma}] \text{ with } \gamma_0 \in H, \frac{\varphi(\gamma_n)}{\sqrt{n}} \leq x\right\}$$

and

$$N(x,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{x} e^{-t^2/2\sigma}\, dt.$$

We want to prove that there exists $\sigma^2 \geq 0$ for which

$$\|R_n(x) - N(x,\sigma)\|_\infty = O(n^{-1/4})$$

as $n \to \infty$. To simplify notation we will express this as $R_n = N(\sigma) + O(n^{-1/4})$. We will use the following fact multiple times.

Lemma 5.2.3. *Let $F_n, H_n : \mathbb{R} \to \mathbb{R}$ be sequences of distributions and suppose that k_n, l_n are sequences of integers with $k_n \to \infty$ and $l_n \to \infty$ as $n \to \infty$. Suppose further that there exists a constant $C > 0$ independent of n and x such that*

$$H_n(x - Cl_n^{-1}) \leq F_n(x) \leq H_n(x + Cl_n^{-1}),$$

for all n, x. Then, if $H_n = N(\sigma) + O(k_n^{-1})$, we have that $F_n = N(\sigma) + O(k_n^{-1}, l_n^{-1})$.

Proof. This is a simple consequence of the fact that the derivative of $N(\sigma)$ is uniformly bounded. $\square$

Our aim is to construct a sequence of distributions on Y with respect to $\widehat{\nu}$ from which we can gain an understanding of the R_n. The following two lemmas are the first step in achieving this. The first lemma is an easy consequence of the hyperbolicity of G and so we exclude the proof.

Lemma 5.2.4. *There exists $C > 0$ such that*

$$\sup_{\substack{\gamma,\gamma' \in [\widetilde{\gamma}] \\ \gamma_0,\gamma_0' \in H}} \sup_{n \in \mathbb{Z}_{\geq 0}} \{d_L(\gamma_n, \gamma_n')\} < C$$

uniformly for $[\widetilde{\gamma}] \in \partial G$.

Using this lemma we obtain the following.

Lemma 5.2.5. *Define, for $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$,*

$$\widetilde{R}_n(x) = \nu \left\{ [\widetilde{\gamma}] \in \partial G : \text{ for some } \gamma \in [\widetilde{\gamma}] \text{ with } \gamma_0 \in H, \frac{\varphi(\gamma_n)}{\sqrt{n}} \leq x \right\}.$$

Then, if $\widetilde{R}_n = N(\sigma) + O(n^{-1/4})$, we have that $R_n = N(\sigma) + O(n^{-1/4})$.

Proof. Clearly $R_n(x) \leq \widetilde{R}_n(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$. Also, by the previous lemma and the fact that φ is Lipschitz in the d_L metric, there exists $C > 0$ independent of x and n such that

$$\widetilde{R}_n(x - Cn^{-1/2}) \leq R_n(x),$$

for all x, n. Combining these two bounds and applying Lemma 5.2.3 concludes the proof. $\square$

The previous two lemmas show that, without loss of generality, we may assume that the identity element of G belongs to H. We will assume this from now on. We can now construct distributions on Y from which we can deduce the convergence of R_n. Recall that given $y \in Y$, $h(y)_n$ for $n \in \mathbb{Z}_{\geq 0}$ denotes the nth group element in the geodesic ray determined by y.

Definition 5.2.6. Define distributions

$$H_n(x) = \widehat{\nu}\left\{ y \in \bigcup_i Y_i : \frac{\varphi(h(y)_n)}{\sqrt{n}} \leq x \right\}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$.

The following lemma shows that to prove Theorem 5.1.3, it suffices to prove the analogous statement for the distributions H_n.

Lemma 5.2.7. *If* $H_n = N(\sigma) + O(n^{-1/4})$ *then* $R_n = N(\sigma) + O(n^{-1/4})$.

Proof. It is proven in [8] that h is surjective, see Lemma 3.5.1. Hence there exists $K > 0$ independent of n, x such that

$$\begin{aligned} H_n(x) &\leq \widehat{\nu}\left(h^{-1}\left\{ [\widetilde{\gamma}] \in \partial G : \text{ for some } \gamma \in [\widetilde{\gamma}] \text{ with } \gamma_0 \in H, \frac{\varphi(\gamma_n)}{\sqrt{n}} \leq x \right\}\right) \\ &\leq H_n(x + Kn^{-1/2}), \end{aligned}$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$. Since $h_*\widehat{\nu} = \nu$,

$$\widehat{\nu}\left(h^{-1}\left\{ [\widetilde{\gamma}] \in \partial G : \text{ for some } \gamma \in [\widetilde{\gamma}] \text{ with } \gamma_0 \in H, \frac{\varphi(\gamma_n)}{\sqrt{n}} \leq x \right\}\right) = \widetilde{R}_n(x)$$

and applying Lemmas 5.2.3 and 5.2.4 completes the proof. $\square$

The next step is to study the H_n. We do this by constructing distributions on $\cup_i \Sigma_{B_i}$ with respect to μ and then, by relating μ to $\widehat{\nu}$, use these to understand the H_n distributions. To simplify notation, we define, for $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$E_n(x) = \left\{ y \in \bigcup_i Y_i : \frac{f^n(y)}{\sqrt{n}} \leq x \right\} \subset Y.$$

The following lemma along with Proposition 3.3.6 will allow us to compare the $\widehat{\nu}$ and μ measures.

Lemma 5.2.8. *For any sequence of integers k_n such that $k_n \to \infty$ as $n \to \infty$,*

$$\frac{1}{k_n} \sum_{j=0}^{k_n} \widehat{\nu}_j(E_n(x)) = \widehat{\nu}(E_n(x)) + O(k_n^{-1}),$$

where the implied constant is independent of n, x.

Proof. By Lemma 3.3.8 there exists $0 < \theta < 1$ such that for each $j \in \mathbb{Z}_{\geq 0}$,

$$\widehat{\nu}_j(E_n(x)) = \widehat{\nu}(E_n(x)) + O(\theta^j),$$

where the implied constant is independent of j, n and x. Taking the average of $\widehat{\nu}_1(E_n(x)), \ldots, \widehat{\nu}_{k_n}(E_n(x))$ and letting $n \to \infty$ gives the result. $\square$

We now describe how f distributes over Σ_A with respect to the measure μ. Along with the previous lemma, this will allow us to deduce the convergence of the H_n distributions.

Proposition 5.2.9. *There exists $\sigma^2 \geq 0$ such that for each $x \in \mathbb{R}$,*

$$\mu \left\{ y \in \bigcup_i \Sigma_{B_i} : \frac{f^n(y)}{\sqrt{n}} \leq x \right\} = N(x, \sigma) + O(n^{-1/2})$$

as $n \to \infty$ and the above error term is uniform in $x \in \mathbb{R}$. Furthermore, $\sigma^2 > 0$ if and only if

$$\{[\gamma] \in \partial G : \{\varphi(\gamma_n) : n \in \mathbb{Z}_{\geq 0}\} \text{ is unbounded}\}$$

is non-empty.

Proof. By Proposition 3.3.2, the measure μ is a weighted sum of the measures of maximal entropy μ_i on each maximal component B_i. We obtain a central limit theorem, with mean Λ_i and variance σ_i, for μ_i and f on each Σ_{B_i}. Proposition 3.5.7 shows that Λ_i and σ_i do not depend on the maximal component B_i (and by assumption $\Lambda_i = 0$ for each $i = 1, \ldots, m$). From this and the Berry-Esseen Theorem for subshifts of finite type [14] we obtain the desired central limit theorem, with error term, for μ and f. The criteria for positive variance follows from Lemma 3.6.2 and Proposition 3.6.2. $\square$

We are now ready to prove Theorem 5.1.3.

Proof of Theorem 5.1.3. By Lemma 5.2.7 it suffices to prove that for $x \in \mathbb{R}$

$$H_n(x) = N(x, \sigma) + O(n^{-1/4})$$

as $n \to \infty$.

We begin by applying Proposition 3.3.6 and Proposition 5.2.9 to deduce that for any integer valued sequence k_n, with $k_n \to \infty$ as $n \to \infty$,

$$\frac{1}{k_n}\sum_{j=0}^{k_n} \sigma_*^j \widehat{\nu}\left\{ y \in \bigcup_i \Sigma_{B_i} : \frac{f^n(y)}{\sqrt{n}} \leq x \right\} = N(x,\sigma) + O(k_n^{-1}, n^{-1/2}), \qquad (5.2.1)$$

as $n \to \infty$, uniformly for $x \in \mathbb{R}$. We then define, for $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$,

$$C_n^{\pm}(x) = \left\{ y \in \bigcup_i \Sigma_{B_i} : \frac{f^n(y)}{\sqrt{n}} \leq x \pm \frac{2k_n|f|_\infty}{\sqrt{n}} \right\}.$$

If we suppose further that $k_n = o(\sqrt{n})$, then expression (5.2.1) implies that

$$\frac{1}{k_n}\sum_{j=0}^{k_n} \sigma_*^j \widehat{\nu}(C_n^{\pm}(x)) = N(x,\sigma) + O(k_n n^{-1/2}, k_n^{-1}). \qquad (5.2.2)$$

We now note that, by inclusion,

$$\sigma_*^j \widehat{\nu}_j(C_n^{-}(x)) \leq \widehat{\nu}_j(E_n(x)) \leq \sigma_*^j \widehat{\nu}_j(C_n^{+}(x)) \qquad (5.2.3)$$

for all n, $j \leq k_n$ and x. Recall that, by (3.3.2), $\sigma_*^j \widehat{\nu}(C_n^{\pm}(x)) = \sigma_*^j \widehat{\nu}_j(C_n^{\pm}(x))$ for all n, x. Hence, if we choose $k_n = \lfloor n^{1/4} \rfloor$, then (5.2.2) along with inequality (5.2.3) imply that

$$\frac{1}{k_n}\sum_{j=0}^{k_n} \widehat{\nu}_j(E_n(x)) = N(x,\sigma) + O(n^{-1/4})$$

and so by Lemma 5.2.8,

$$\widehat{\nu}(E_n(x)) = N(x,\sigma) + O(n^{-1/4}).$$

Lastly, using Lemma 5.2.3 and the fact that, for $y \in Y$, $f^n(y_n) = \varphi(h(y)_n) + O(1)$, it is easy to see that

$$H_n(x) = \widehat{\nu}(E_n(x)) + O(n^{-1/2}) = N(x,\sigma) + O(n^{-1/4}),$$

concluding the proof. □

Remark 5.2.10. The $O(n^{-1/4})$ error term arises due to the fact that ν is supported on Y whereas μ is supported $\cup_i \Sigma_{B_i}$. To pass the central limit theorem in Proposition 5.2.9 to one for ν and Y, we need to compare the values f takes on Y to the values f takes on $\cup_i \Sigma_{B_i}$. This comparison introduces an error term that can be seen explicitly as the $2k_n|f|_\infty n^{-1/2}$ terms in the sets $C_n^{\pm}(x)$. In the case that A is aperiodic (or irreducible) this term is no longer needed since for any $y \in Y$, $\sigma(y)$ belongs to the only (necessarily maximal) component.

In [6], Bowen and Series provide a geometrical condition for Fuchsian groups that guarantees the existence of a coding Σ_A described by an aperiodic matrix. This condition is satisfied by the fundamental groups of compact hyperbolic surfaces (i.e. surface groups) equipped with the generating set consisting of the side pairings for the standard fundamental domain. Free groups equipped with their canonical generating set also satisfy this condition. The above remark then implies the following.

Corollary 5.2.11. *If G and $\varphi : G \to \mathbb{R}$ satisfy the hypotheses of Theorem 5.1.3 and G is a free group or surface group equipped with the generating set described above, then the error term in Theorem 5.1.3 can be improved to $O(n^{-1/2})$.*

Remark 5.2.12. It seems plausible that the optimal error term in Theorem 5.1.3 is $O(n^{-1/2})$. The author has not pursued this however.

Chapter 6

Relative growth of normal subgroups

6.1 Discussion of results

In this chapter we study the relative growth of subgroups of hyperbolic groups. In particular we focus of normal subgroups that form a free abelian quotient. Our main theorem is the following.

Theorem 6.1.1. *Let G be a non-elementary hyperbolic group equipped with a finite generating set and let $N \lhd G$ be a normal subgroup with $G/N \cong \mathbb{Z}^\nu$ for some $\nu \geq 1$. Then*

$$\#(W_n \cap N) = O\left(\frac{\lambda^n}{n^{\nu/2}}\right)$$

as $n \to \infty$. Furthermore, there exists $D \in \mathbb{Z}_{\geq 0}$ and $C > 0$ such that

$$\#(W_{Dn} \cap N) \sim \frac{C\lambda^{Dn}}{(Dn)^{\nu/2}}$$

as $n \to \infty$.

This result has the following immediate corollary.

Corollary 6.1.2. *Let G be a non-elementary hyperbolic group equipped with a finite generating set and let $N \lhd G$ be a normal subgroup such that the abelianisation of G/N has rank $\nu \geq 1$. Then*

$$\#(W_n \cap N) = O\left(\frac{\lambda^n}{n^{\nu/2}}\right)$$

as $n \to \infty$.

Proof. Write the abelianisation of G/N as $\mathbb{Z}^\nu \times F$, where F is finite. There are then natural surjective homomorphisms $\phi : G \to G/N$ and $\psi : G/N \to \mathbb{Z}^\nu$. Set $\phi_0 = \psi \circ \phi$ and $N_0 = \ker \phi_0$. Then $N \subset N_0$. Furthermore, by Theorem 6.1.1, $\#(W_n \cap N_0) = O(\lambda^n n^{-\nu/2})$, giving the required estimate. □

Remark 6.1.3. The relative growth in Corollary 6.1.2 may occur at a slower exponential rate. Indeed, Coulon, Dal'Bo and Sambusetti recently showed that $\#(W_n \cap N) = O(\lambda_0^n)$, for some $0 < \lambda_0 < \lambda$ precisely when G/N is *not* amenable [16]. In fact, their result does not require normality of the subgroup, in which case amenability is replaced by co-amenability of N in G, i.e. that the G-action on the coset space G/N is amenable.

To prove Theorem 6.1.1, we would like to employ the strategy used by Sharp in [55]. However, there are significant technical obstacles which we need to overcome in order to use this method. We summarise these below.
(i) Firstly, as mentioned above, in [55] there are strong restrictions on the hyperbolic groups and their generating sets. This makes it much easier to study the relative growth quantity $\#(W_n \cap N)$. We need to find a new approach that works for general non-elementary hyperbolic groups, that will allow us to express $\#(W_n \cap N)$ in terms of quantities which we can analyse.
(ii) Secondly, we need a good understanding of how real valued group homomorphisms on hyperbolic groups grow as we increase the word length of the input. Our work from Chapter 4 allows us to deduce the required properties of these homomorphisms.

We end this section with a discussion of relative growth series. We define the relative growth series for N in G (with respect to the given generators) to be the power series

$$\sum_{n=0}^{\infty} \#(W_n \cap N) z^n.$$

When $N = G$, this is the standard growth series and, for hyperbolic groups, is well-known to be the series of a rational function [10], [25]. The requirement that a power series be rational imposes a strong constraint on the coefficients: if $\sum_{n=0}^{\infty} a_n z^n$ is rational then there are complex numbers $\xi_1, \ldots, \xi_m$ and polynomials $P_1, \ldots, P_m$ such that

$$a_n = \sum_{j=1}^{m} P_j(n) \xi_j^n$$

(Theorem IV.9 of [20]). Comparing with the asymptotic in Theorem 6.1.1, we see that $\#(W_n \cap N)$ does not satisfy this constraint. Thus we obtain the following.

Corollary 6.1.4. *Suppose G is a non-elementary hyperbolic group equipped with a finite generating set. Let $N \triangleleft G$ be a normal subgroup with $G/N \cong \mathbb{Z}^\nu$ for some $\nu \geq 1$. Then, the relative growth series*

$$\sum_{n=1}^{\infty} \#(W_n \cap N) z^n$$

is not the series of a rational function.

Remark 6.1.5. (i) The first result of this type is due to Grigorchuk, who showed that the relative growth series is not rational when G is the free group on two generators and N is the commutator subgroup (see [28]). A similar result was obtained for the fundamental groups of compact surfaces of genus ≥ 2 in [46] and this was extended to a wider class of hyperbolic groups in [55].
(ii) We note that, as Corollary 6.1.4 requires the asymptototic along a subsequence in Theorem 6.1.1, it does not apply to general infinite index subgroups of hyperbolic groups. In fact, Grigorchuk showed that if N is a finite index subgroup of a free group than its relative growth series is rational [27].

6.2 Proof of Theorem 6.1.1

Suppose G is a non-elementary hyperbolic group and N a normal subgroup satisfying the hypothesis of Theorem 6.1.1. Let $\varphi : G \to \mathbb{Z}^\nu$ denote the quotient homomorphism. In this section it is easier to work with weighted matrices than transfer operators. We define the following matrices.

Definition 6.2.1. For each $j = 1, \ldots, m$, define a matrix C_j by,

$$C_j(u,v) = \begin{cases} 0 & \text{if } u \text{ or } v \text{ belong to a maximal component that is not } B_j, \\ A(u,v) & \text{otherwise.} \end{cases}$$

Now suppose that $\varphi : G \to G/N \cong \mathbb{Z}^\nu$ is the quotient homomorphism. We define a function $f : \Sigma_A \to \mathbb{Z}^\nu$ by

$$f((x_n)_{n=0}^\infty) = \varphi(\rho(x_0, x_1)),$$

where ρ is the labelling map from Definition 3.2.1. Since $f((x_n)_{n=0}^\infty)$ depends only on the first two coordinates of $(x_n)_{n=0}^\infty$, we can consider f as a map from the directed edge set of $\mathcal{G}$ to $\mathbb{R}$. We then have that $\varphi(g) = f(*, x_1) + f(x_1, x_2) +$

$\cdots + f(x_{|g|-1}, x_{|g|})$ where $(*, x_1), \ldots, (x_{|g|-1}, x_{|g|})$ is the unique path associated to g by Property (2) of Definition 3.2.1. Using f, we weight the matrices C_j componentwise and define, for $t \in \mathbb{R}^\nu$,

$$C_j(t)(u, v) = e^{2\pi i \langle t, f(u,v) \rangle} C_j(u, v).$$

We define the matrices $B_j(t)$ analogously.

To study the relative growth of N we would like to express $\#(W_n \cap N)$ in terms of the matrices $C_j(t)$. Using the orthogonality identity

$$\int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} e^{2\pi i \langle t, \varphi(g) \rangle} \, dt = \begin{cases} 1 & \text{if } \varphi(g) = 0 \\ 0 & \text{otherwise} \end{cases}$$

we can write

$$\#(W_n \cap N) = \sum_{|g|=n} \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} e^{2\pi i \langle t, \varphi(g) \rangle} \, dt = \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} \sum_{|g|=n} e^{2\pi i \langle t, \varphi(g) \rangle} \, dt.$$

The following result will allow us to rewrite $\#(W_n \cap N)$ in terms of the matrices C_j. Let v_* be the vector in $\mathbb{R}^V$ with a one in the coordinate corresponding to the $*$ vertex and zeros elsewhere. Also, let $\mathbf{1} \in \mathbb{R}^\nu$ be the vector with a one in each coordinate.

Lemma 6.2.2. *There exists $\epsilon > 0$ such that for all $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$*

$$\sum_{|g|=n} e^{2\pi i \langle t, \varphi(g) \rangle} = \sum_{j=1}^{m} \langle C_j^n(t) v_*, \mathbf{1} \rangle + O((\lambda - \epsilon)^n)$$

as $n \to \infty$. The implied constant is independent of t.

Proof. Using the correspondence between G and Σ_A, we can write

$$\left| \sum_{|g|=n} e^{2\pi i \langle t, \varphi(g) \rangle} - \sum_{j=1}^{m} \langle C_j^n(t) v_*, \mathbf{1} \rangle \right| = (m-1) \left| \sum_{g \in M_n} e^{2\pi i \langle t, \varphi(g) \rangle} \right| \leq (m-1) \, \#M_n,$$

where M_n consists of the elements in G of word length n whose corresponding path in $\mathcal{G}$ does not enter a maximal component. It is clear that $\#M_n = O((\lambda - \epsilon)^n)$ for some $\epsilon > 0$ and so the result follows. $\square$

Using this lemma, we see that

$$\#(W_n \cap N) = \sum_{j=1}^{m} \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} \langle C_j^n(t)v_*, \mathbf{1}\rangle \; dt + O((\lambda - \epsilon)^n).$$

Hence to study the relative growth of N would like to understand the spectral behaviour of the $C_j(t)$ for $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$. From their definitions, it is clear that the matrices C_j each have p_j simple maximal eigenvalues of modulus λ and the rest of the spectrum is contained in a disk of radius strictly smaller than $\lambda - \epsilon$, for some $\epsilon > 0$. We shall be interested in the values of t for which the operators $C_j(t)$ have spectral radius λ. These values of t are characterised by the following lemma.

Lemma 6.2.3. *For any $t \in \mathbb{R}^\nu$, the operator $C_j(t)$ has spectral radius at most λ. Furthermore, $C_j(t)$ has spectral radius exactly λ if and only if it has p_i simple maximal eigenvalues of the form $e^{2\pi i\theta}e^{2\pi ik/p_i}\lambda$ for $k = 0, \dots, p_i - 1$ and some $\theta \in \mathbb{R}$. This occurs if and only if $B_j(t) = e^{2\pi i\theta}MB_jM^{-1}$ where M is a diagonal matrix with modulus one diagonal entries. Furthermore, when $C_j(t)$ has p_i simple maximal eigenvalues of modulus λ, the rest of the spectrum is contained in a disk of radius strictly less than λ.*

Proof. When C_j consists of a single component (ignoring the $*$ vertex) and so is the same as B_j, this is Wielandt's Theorem [21]. When this is not the case, we can write the spectrum of $C_j(t)$ as a union of the spectra of the irreducible components making up $C_j(t)$. By definition, each C_j has one component B_j with spectral radius λ and all other components have spectral radius strictly less than λ. Therefore applying Wielandt's Theorem to each component gives the required result. □

We now follow the method presented in [55]. Let $f_j = f|_{\Sigma_{B_i}}$ for $j = 1, \dots, m$. If a sequence $\gamma = (x_0, x_1, \dots, x_n)$ is such that $B_j(x_i, x_{i+1}) = 1$ for $i = 0, \dots, n$ and $x_0 = x_n$, then we call γ a cycle and define its length as $l(\gamma) = n$. Let $\mathcal{C}_j$ be the collection of all such cycles and note that the length of any cycle in $\mathcal{C}_j$ is a multiple of p_j. Given a cycle $\gamma \in \mathcal{C}_j$, we define its f_j-weight to be

$$w_{f_j}(\gamma) = f_j(x_0, x_1) + \dots + f_j(x_{n-1}, x_n).$$

Let Γ_j be the subgroup of $\mathbb{Z}^\nu$ generated by $\{w_{f_j}(\gamma) : \gamma \in \mathcal{C}_j\}$. We define Δ_j to be the following subgroup of Γ_j,

$$\Delta_j = \{w_{f_j}(\gamma) - w_{f_j}(\gamma') : \gamma, \gamma' \in \mathcal{C}_j \text{ and } l(\gamma) = l(\gamma')\}.$$

(This is a version of Krieger's Δ-group [37]. For a proof that it is a group, see page 892 of [56].) We now choose two cycles $\gamma, \gamma' \in \mathcal{C}_j$ such that $l(\gamma) - l(\gamma') = p_j$ and set $c_j = w_{f_j}(\gamma) - w_{f_j}(\gamma')$. Applying the results of [40] to the aperiodic shift $(\Sigma_{B_j}, \sigma^{p_j})$, we see that the group Γ_j/Δ_j is cyclic and is generated by the element $c_j + \Delta_j$. Our aim is to show that this group has finite order. To do so, we will use a result of Marcus and Tuncel. For each $j = 1, \ldots, m$, let E_j denote the directed edge set for the graph with transition matrix B_j. Write V_j for the analogously defined vertex sets. We say that a function $g : E_j \to \mathbb{R}$ is cohomologous to a constant if there exists $C \in \mathbb{R}$ and $h : V_j \to \mathbb{R}$ such that $g(x, y) = C + h(y) - h(x)$ for all $(x, y) \in E_j$.

Lemma 6.2.4 ([40])**.** *If $\langle t, f_j^{p_j} \rangle$ is not cohomologous to a constant for any non-zero $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, then Γ_j/Δ_j has finite order.*

It is clear that, for $t \in \mathbb{R}^\nu$, $\langle t, f_j^{p_j} \rangle$ is cohomologous to a constant if and only if $\langle t, f_j \rangle$ is cohomologous to constant. Using ideas from [11], we will show that the hypothesis of the above lemma is satisfied for each $j = 1, \ldots, m$.

Lemma 6.2.5. *For non-zero $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$ and for all $j = 1, \ldots, m$, $\langle t, f_j \rangle$ is not cohomologous to a constant.*

Proof. We begin by noting that, since φ is surjective, for any $t \in \mathbb{R}^\nu \backslash \{0\}$ the function $\psi_t := \langle t, \varphi \rangle : G \to \mathbb{R}$ is a non-trivial group homomorphism. Theorem 6.1.1 implies that if $\langle t, f_j \rangle$ (for any $j \in \{1, \ldots, m\}$) is cohomologous to a constant, then that constant is given by

$$\lim_{n \to \infty} \frac{1}{\# W_n} \sum_{|g|=n} \frac{\psi_t(g)}{n}.$$

By symmetry this limit is zero. Then Corollary 3.6.12 shows that ψ_t is not cohomologous to 0 as required. $\square$

Remark 6.2.6. Since the above proof relies on the zero density result of Gouëzel, Mathéus and Maucourant [26], quantifying the decay rate in (1.0.1) requires a priori knowledge of the convergence to zero.

Let $D_j = |\Gamma_j/\Delta_j|$ for $j = 1, \ldots, m$. From the above discussion, we know that each D_j is finite. We also note that Lemma 6.2.5 shows that $\mathrm{rank}_{\mathbb{Z}}(\Gamma_j) = \nu$ and so $|\mathbb{Z}^\nu/\Gamma_j|$ is finite for each $j = 1, \ldots, m$. Combining this with all of the above work, allows us to state the following result that describes the spectral behaviour of the $C_j(t)$ as t varies. We use the notation $\varrho(M)$ to denote the spectral radius of a matrix M.

Proposition 6.2.7. *For $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, define $\chi_t \in \widehat{\mathbb{Z}^\nu}$ by $\chi_t(x) = e^{2\pi i\langle t,x\rangle}$. Then we have that*

$$\{\chi_t : \varrho(C_j(t)) = \lambda\} = \Delta_{f_j}^{\perp},$$

where $\Delta_{f_j}^{\perp} = \{\chi \in \widehat{\mathbb{Z}^\nu} : \chi(\Delta_{f_j}) = 1\}$. Furthermore, when $\chi_t \in \Delta_{f_j}^{\perp}$, $C_j(t)$ has p_j simple maximal eigenvalues of the form $e^{2\pi i\theta}e^{2\pi ik/p_j}\lambda$ for some $\theta \in \mathbb{R}$ and $k = 0, \ldots, p_j - 1$.

Proof. This is essentially Proposition 4 from [55] which is derived from work in [45]. However, here we need to consider the non-aperiodic matrices $C_j(t)$. To deduce this more general statement, we can apply Proposition 4 from [55] to the maximal component associated to the matrix $C_j^{p_j}(t)$. This is justified since this maximal component is aperiodic. To conclude the proof, we note that the part of the spectrum of $C_j(t)$ coming from $B_j(t)$ is invariant under the rotation $z \mapsto ze^{2\pi i/p_j}$. □

Proposition 6.2.7 implies that there exist $D_j < \infty$ values of t for which the spectral radius of $C_j(t)$ is maximal and equal to λ. Denote these values by $t = 0, t_1^j \ldots, t_{D_j-1}^j$. When t takes one of these values, $C_j(t)$ has p_j simple maximal eigenvalues of the form $e^{2\pi i\theta}e^{2\pi ik/p_j}\lambda$ for $k = 0, \ldots, p_j - 1$ and for some $\theta \in \mathbb{R}$. We now choose, for each $j = 1, \ldots, m$, a neighbourhood U_0^j of zero and define $U_r^j = U_0^j + t_k^j$ for $k = 0, \ldots, D_j - 1$. Results from perturbation theory guarantee that, as long as each U_0^j is sufficiently small, there exists $\epsilon > 0$ such that the following hold for each $j = 1, \ldots, m$.

1. If $t \in \bigcup_{r=0}^{D_j-1} U_r^j$, then the matrices $C_j(t)$ each have p_j simple, maximal eigenvalues of the form $\lambda_j(t)e^{2\pi ik/p_j}$ for $k = 0, \ldots, p_j - 1$, where $t \to \lambda_j(t)$ is analytic and independent of $k = 0, \ldots, p_j - 1$.

2. Let $M_\nu(\mathbb{C})$ denote the vector space of $\nu \times \nu$ complex matrices. For each $j = 1, \ldots, m$ and $k = 0, \ldots, p_j - 1$, there exists an analytic matrix-valued function $Q_{j,k} : \bigcup_{r=0}^{D_j-1} U_r^j \to M_\nu(\mathbb{C})$, where $Q_{j,k}(t)$ is the eigenprojection onto the eigenspace associated to the eigenvalue $\lambda_j(t)e^{2\pi ik/p_j}$ of the matrix $C_j(t)$.

3. If $t \in (\mathbb{R}^\nu/\mathbb{Z}^\nu)\backslash\bigcup_{r=0}^{Dj-1} U_r^j$ then the spectral radius of each $C_j(t)$ is bounded uniformly above by $\lambda - \epsilon$.

Using this description of the spectrum, we can write

$$\#(W_n \cap N) = \sum_{j=1}^{m} \sum_{r=0}^{D_j-1} \sum_{k=0}^{p_j-1} \int_{U_r^j} \lambda_j(t)e^{2\pi ikn/p_j} \langle Q_{j,k}(t)v_*, \mathbf{1}\rangle \, dt + O((\lambda - \epsilon)^n),$$

for some $\epsilon > 0$. Hence there exists constants $c^j_{r,k} = \langle Q_{j,k}(t^j_r)v_*, \mathbf{1}\rangle$, for $r = 0, \ldots, D_j - 1$ and $k = 0, \ldots, p_j - 1$, such that $\#(W_n \cap N)$ is equal to

$$\sum_{j=1}^{m} \left(\sum_{r=0}^{D_j-1} \sum_{k=0}^{p_j-1} e^{2\pi i n(r/D_j + k/p_j)} c^j_{r,k} \right) \int_{U^j_0} \lambda_j(t)^n \left(1 + O(\|t\|)\right) dt + O((\lambda - \epsilon)^n). \tag{6.2.1}$$

The asymptotics of each

$$a^j_n := \int_{U^j_0} \lambda_j(t)^n \left(1 + O(\|t\|)\right) dt$$

were studied in [45], where it was shown that, for each $j = 1, \ldots, m$, there exists $\tau_j > 0$ such that

$$a^j_n \sim \frac{\tau_j \lambda^n}{n^{\nu/2}} \tag{6.2.2}$$

as $n \to \infty$. Applying this along the subsequence Dn, where D is given by the product of all the $p_1, \ldots, p_m$ and $D_1, \ldots, D_m$, we see that

$$\#(W_{Dn} \cap N) = \frac{\widetilde{C} \lambda^{Dn}}{(Dn)^{\nu/2}} + o\left(\frac{\lambda^{Dn}}{(Dn)^{\nu/2}} \right) \tag{6.2.3}$$

as $n \to \infty$, where

$$\widetilde{C} = \sum_{j=1}^{m} \tau_j \left(\sum_{r=0}^{D_j-1} \sum_{k=0}^{p_j-1} c^j_{r,k} \right).$$

It is clear that $\widetilde{C} \in \mathbb{R}_{\geq 0}$. However, for (6.2.3) to be a useful asymptotic expression, we would like that $\widetilde{C}$ is strictly positive. We now show that this is always the case.

Lemma 6.2.8. *We necessarily have that $\widetilde{C} > 0$.*

Proof. Fix $j \in \{1, \ldots, m\}$ and recall that for any loop $\gamma = (x_0, \ldots, x_{Dn}) \in \mathcal{C}_j$ with $w_{f_j}(\gamma) = 0$, the group element $g_\gamma = \rho(x_0, x_1)\rho(x_1, x_2) \ldots \rho(x_{Dn-1}, x_{Dn})$ belongs to the kernel of φ (or, equivalently, to N) and furthermore, g_γ has word length Dn. Also, for any two distinct loops $\gamma, \gamma' \in \mathcal{C}_j$, we have $g_\gamma \neq g_{\gamma'}$ whenever γ and γ' have the same initial vertex. Combining these observations and applying the pigeonhole principle gives that

$$\#(W_{Dn} \cap N) \geq (\#V_j)^{-1} \#\{\gamma \in \mathcal{C}_j : l(\gamma) = Dn,\, w_{f_j}(\gamma) = 0\}$$

for all $n \geq 1$. Pollicott and Sharp proved in [45] that

$$\#\{\gamma \in \mathcal{C}_j : l(\gamma) = Dn,\, w_{f_j}(\gamma) = 0\} \sim \frac{K\lambda^{Dn}}{(Dn)^{\nu/2}}$$

as $n \to \infty$ for some $K > 0$. Hence

$$\widetilde{C} = \limsup_{n\to\infty} \frac{(Dn)^{\nu/2}\#(W_{Dn} \cap N)}{\lambda^{Dn}} \geq K(\#V_j)^{-1} > 0,$$

as required. □

We can now conclude the proof of our main result.

Proof of Theorem 6.1.1. Combining (6.2.1) and (6.2.2) implies that

$$\#(W_n \cap N) = O\left(\sum_{j=1}^{m} \int_{U_0^j} \lambda_j(t)^n \left(1 + O(\|t\|)\right) dt\right) = O\left(\frac{\lambda^n}{n^{\nu/2}}\right)$$

which proves the first part of Theorem 6.1.1. The second part follows from (6.2.3) and the fact that $\widetilde{C} > 0$. □

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