ON FIBONACCI PARTITIONS

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To Carl Pomerance, a legend of number theory

ABSTRACT. We prove an exact formula for OEIS A000119, which counts partitions into distinct Fibonacci numbers. We also establish an exact formula for its mean value, and determine the asymptotic behaviour.

1. INTRODUCTION

For $n \in \mathbb{Z}_{\geqslant 0}$, let $R(n)$ be the number of solutions to

$$x_1 + \cdots + x_s = n,$$

where $s \in \mathbb{Z}_{\geqslant 0}$, and $x_1 < x_2 < \cdots < x_s$ are Fibonacci numbers. We call R the *Fibonacci partition function*, as it counts partitions into distinct Fibonacci numbers. Here $F_1 = 1$ and $F_2 = 1$ are not considered to be distinct, so $R(0) = R(1) = R(2) = 1$. The function has existed since the very first volume of the Fibonacci Quarterly in 1963, see [10], and its values comprise the sequence OEIS A000119. In 1968, Leonard Carlitz [5, Theorem 2] showed that

$$R(F_m) = \lfloor m/2 \rfloor \qquad (m = 2, 3, \ldots), \tag{1.1}$$

where $F_1 = F_2 = 1$, and $F_{m+2} = F_{m+1} + F_m$ for $m \in \mathbb{N}$. Many authors have investigated Fibonacci partitions, and the topic has received attention over several decades [2, 3, 5, 10, 12, 16, 18, 19, 21]. In general $R(n)$ behaves erratically.

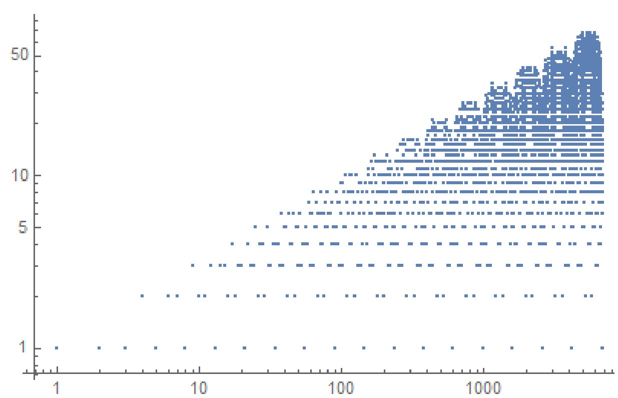

FIGURE 1. $R(n)$ against n for $n = 0, 1, \ldots, 6765$

Our first result is an exact formula for $R(n)$. Recall Zeckendorf's theorem [13, 15, 23], which asserts that each positive integer has a unique representation as a sum of non-consecutive Fibonacci numbers, called the *Zeckendorf expansion*.

Theorem 1.1. *Let*

$$H = F_{m_0} + F_{m_1} + \cdots + F_{m_k}$$

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be the Zeckendorf expansion of $H \in \mathbb{N}$*, where*

$$m_{i-1} - m_i \geqslant 2 \qquad (1 \leqslant i \leqslant k), \qquad m_k \geqslant 2.$$

Write

$$t_i = \left\lfloor \frac{m_{i-1} - m_i + 2}{2} \right\rfloor, \quad \varepsilon_i = 2t_i - 1 - m_{i-1} + m_i \qquad (1 \leqslant i \leqslant k).$$

Finally, let

$$a_0 = 1, \qquad a_1 = t_1, \qquad a_{\ell+1} = t_{\ell+1} a_\ell - \varepsilon_\ell a_{\ell-1} \quad (1 \leqslant \ell \leqslant k-1).$$

Then

$$R(H) = \begin{cases} a_k \lfloor m_k/2 \rfloor - \varepsilon_k a_{k-1}, & \text{if } k \geqslant 1 \\ \lfloor m_0/2 \rfloor, & \text{if } k = 0. \end{cases} \tag{1.2}$$

Note that

$$\varepsilon_i = \begin{cases} 1, & \text{if } m_{i-1} - m_i \text{ is even} \\ 0, & \text{if } m_{i-1} - m_i \text{ is odd} \end{cases} \qquad (1 \leqslant i \leqslant k).$$

Our formula (1.2) is extremely efficient in practice; we can compute $R(10^{100})$ in less than one second on a standard laptop computer, and *Mathematica* [22] code for this is provided in Appendix A. The time complexity of computing $R(H)$ is polylogarithmic in H: a crude upper bound is given by $O((\log H)^4)$. This is to say that the algorithm runs in polynomial time, since the input size has order of magnitude $\log H$.

Carlitz [5] had a recursive formula, but on attempting to produce a non-recursive formula found that "the general case is very complicated". Robbins [16] had a simpler recursive formula, leading to an algorithm used to produce some initial values of $R(H)$, but also did not write down a non-recursive formula. Theorem 1.1 builds upon Robbins's work. Weinstein [21] obtained a non-recursive expression, albeit a complicated one; the interested reader can see it by setting $t = 1$ in [21, Theorem 2.10] and unpacking the various definitions. The nicest formula that we could find in the literature is that of Berstel [3, Proposition 3.1], which is quite similar to Theorem 1.1. Berstel showed that

$$R(H) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lfloor d_0/2 \rfloor & \lceil d_0/2 \rceil \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ \lfloor d_k/2 \rfloor & \lceil d_k/2 \rceil \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where

$$d_i = m_i - m_{i+1} - 1 \quad (0 \leqslant i \leqslant k-1), \qquad d_k = m_k - 2.$$

Theorem 1.1, aside from being similar to Berstel's expression, follows readily from Robbins's recursion, so it is not our main result by any means.

We also study the mean value

$$M(H) := H^{-1} \sum_{n=0}^{H} R(n) \qquad (H \in \mathbb{N}),$$

or equivalently the summatory function

$$A(H) := \sum_{n=0}^{H} R(n) \qquad (H \in \mathbb{Z}).$$

Throughout, we adopt the convention that empty sums are 0, so that $A(H) = 0$ for $H < 0$. For $t \in \mathbb{N}$, let

$$f(t) = 1 + \frac{2(4^{t-1} - 1)}{3}. \tag{1.3}$$

We establish the following exact formula for $A(H)$.

Theorem 1.2. *Let $H \in \mathbb{N}$, and let the values of the t_i, ε_i and a_ℓ be as in Theorem 1.1. Further, write*

$$x_\ell = F_{m_\ell} + \cdots + F_{m_k} \quad (0 \leqslant \ell \leqslant k+1)$$

for the tail of the Zeckendorf expansion, noting that $x_{k+1} = 0$. Then for $\ell = 1, 2, \ldots, k$ we have

$$A(H) = a_\ell A(x_\ell) - \varepsilon_\ell a_{\ell-1} A(x_{\ell+1}) + \sum_{i \leqslant \ell} a_{i-1} f(t_i) 2^{m_{i-1}-2t_i}. \tag{1.4}$$

In particular

$$A(H) = \begin{cases} a_k \left\lfloor \frac{2^{m_k}}{6} + \frac{m_k+1}{2} \right\rfloor - \varepsilon_k a_{k-1} + \sum_{i \leqslant k} a_{i-1} f(t_i) 2^{m_{i-1}-2t_i}, & \text{if } k \geqslant 1 \\ \left\lfloor \frac{2^{m_0}}{6} + \frac{m_0+1}{2} \right\rfloor, & \text{if } k = 0. \end{cases} \tag{1.5}$$

This enables us to understand the asymptotic behaviour of $A(H)$ and $M(H)$. Put

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \lambda = \frac{\log 2}{\log \varphi} \approx 1.44,$$

and define

$$c_1 = \liminf_{H \to \infty} \frac{A(H)}{H^\lambda}, \qquad c_2 = \limsup_{H \to \infty} \frac{A(H)}{H^\lambda}.$$

We now present our main result.

Theorem 1.3 (Main Theorem)**.** *We have*

$$c_1 = 0.52534\ldots, \qquad c_2 = 0.54338\ldots,$$

and more precisely

$$0.525347 < c_1 < 0.525349, \qquad 0.5433878 < c_2 < 0.5433893. \tag{1.6}$$

It follows that $A(H) \asymp H^\lambda$, meaning that

$$0 < \liminf_{H \to \infty} \frac{A(H)}{H^\lambda} \leqslant \limsup_{H \to \infty} \frac{A(H)}{H^\lambda} < \infty.$$

Subject to hardware constraints, our method computes c_1 and c_2 to arbitrary precision.

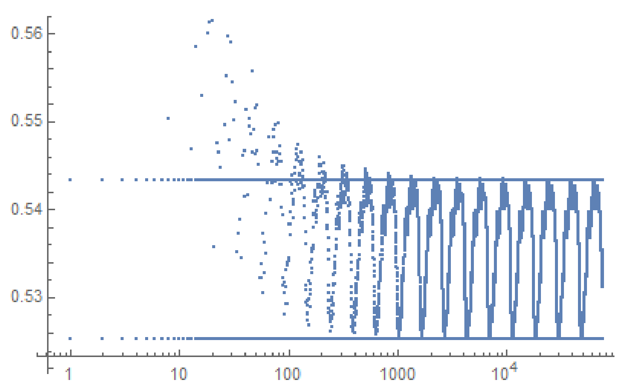

FIGURE 2. $A(H)/H^\lambda$ against H for $H = 0, 1, \ldots, 75025$, with horizontal lines at 0.525348 and 0.543388.

The Fibonacci partition function behaves very differently to the usual partition function $p(n)$, for which there is a nice asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$$

going back to Hardy and Ramanujan [9], see also [1, §5], where the notation $f(m) \sim g(m)$ means that $\lim_{m\to\infty} f(m)/g(m) = 1$. One might indeed expect different behaviour, since the Fibonacci partition function involves distinct parts that grow exponentially. Our work shows that even the mean value $M(H)$ of the Fibonacci partition function does not have a 'nice' asymptotic formula, however we are able to describe the asymptotic behaviour fairly well.

The logarithmic average of $R(n)n^{1-\lambda}$, namely

$$B(H) := (\log H)^{-1} \sum_{n \leqslant H} \frac{R(n)}{n^\lambda} \qquad (H \geqslant 2),$$

might be better behaved. Breaking into ranges $I_m = (F_m, F_{m+1}]$, wherein

$$\frac{A(F_{m+1}) - A(F_m)}{F_{m+1}^\lambda} \leqslant \sum_{n \in I_m} \frac{R(n)}{n^\lambda} \leqslant \frac{A(F_{m+1}) - A(F_m)}{F_m^\lambda},$$

it follows from Theorem 1.3 that

$$B(H) \asymp 1.$$

Though $B(H)$ is not decreasing, it does exhibit a clear downward trend.

Conjecture 1.4. *There exists $B > 0$ such that*

$$B(H) \to B \qquad (H \to \infty).$$

We also invite the enthusiastic reader to consider:

1. Higher moments of the Fibonacci partition function
2. Lucas partitions [6, 11]
3. Partitions into distinct terms of a sequence $(\lfloor \tau^m \rfloor)_{m=1}^\infty$, where $\tau \in (1, 2)$ is fixed
4. Partitions into distinct terms of a Piatetski-Shapiro sequence $(\lfloor m^\tau \rfloor)_{m=1}^\infty$, where $\tau > 1$ is fixed, cf. for polynomials [7, 8]
5. Partitions into distinct Piatetski–Shapiro primes, cf. [14, 20].

Methods. We deduce Theorem 1.1 by iterating Robbins's recursion [16, Theorem 4]. For Theorem 1.2, we begin with the observation that $A(H)$ counts sets of distinct Fibonacci numbers whose sum is at most H. This enables us to prove a combinatorial recursion analogous to that of Robbins. By systematic applications of our recursion, we prove an exact formula for $A(H)$ in terms of the Zeckendorf expansion of H. Finally, for $m \in \mathbb{N}$ large, we subdivide $[F_m, F_{m+1}) \cap \mathbb{Z}$ into many discrete subintervals, according to the initial Zeckendorf digits. By estimating $A(H)$ at the endpoints of these subintervals, we are able to compute c_1 and c_2 to arbitrary precision, subject to hardware constraints. We used the software *Mathematica* [22] to perform the calculations, leading to Theorem 1.3.

Notation. We reiterate that empty sums are 0 throughout. We adopt the following standard asymptotic notations: if $f, g : \mathbb{N} \to \mathbb{R}_{>0}$, we write

$$f(m) \sim g(m) \qquad \text{if} \qquad \lim_{m\to\infty} \frac{f(m)}{g(m)} = 1,$$

$$f(m) = o(g(m)) \qquad \text{if} \qquad \lim_{m\to\infty} \frac{f(m)}{g(m)} = 0,$$

and

$$f(m) \asymp g(m) \qquad \text{if} \qquad 0 < \liminf_{m\to\infty} \frac{f(m)}{g(m)} \leqslant \limsup_{m\to\infty} \frac{f(m)}{g(m)} < \infty.$$

In words, the first notion is that f is asymptotic to g, the second notion is that f has a smaller asymptotic order of magnitude than g, and the third notion is that f and g have the same asymptotic order of magnitude.

Organisation. We prove Theorems 1.1, 1.2 and 1.3 in Sections 2, 3 and 4, respectively. The appendices contain the code that we used for the computations.

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2. An exact formula for Fibonacci partitions

In this section, we prove Theorem 1.1. With the notation of Theorem 1.1, Robbins [16, Theorem 4] established the following recursion.

Lemma 2.1 (Robbins)**.** *If $H \geqslant 2$ and $k \geqslant 1$ then*

$$R(H) = t_1 R(x_1) - \varepsilon_1 R(x_2).$$

Remark 2.2. The Fibonacci numbers satisfy the well-known identity

$$F_1 + \cdots + F_{m-2} = F_m - 1 \qquad (m \in \mathbb{N}), \tag{2.1}$$

the proof of which is a straightforward exercise in mathematical induction. Robbins's recursion stems from the fact that

$$R(H) = R(H - F_m) + R(H - F_{m-1}) - R(H - 2F_{m-1}) \qquad (m \geqslant 3,\ F_m \leqslant H < F_{m+1}),$$

wherein $R(x) = 0$ for $x < 0$. To understand this identity, observe from (2.1) that if H is a sum of distinct Fibonacci numbers then exactly one of F_m and F_{m-1} must be involved. There are $R(H - F_m)$ solutions involving F_m. Otherwise we have instead that F_{m-1} is involved, and we need to count Fibonacci solutions to

$$s - 1 \geqslant 0, \qquad x_1 < \ldots < x_{s-1}, \qquad x_1 + \cdots + x_{s-1} = H - F_{m-1}$$

with $x_{s-1} \neq F_{m-1}$, of which there are $R(H - F_{m-1}) - R(H - 2F_{m-1})$.

A similar idea will be deployed in the next section.

Let us now proceed in earnest with the proof of Theorem 1.1. We start by inducting on ℓ to show that if $\ell = 1, 2, \ldots, k$ then

$$R(H) = a_\ell R(x_\ell) - \varepsilon_\ell a_{\ell-1} R(x_{\ell+1}). \tag{2.2}$$

The base case $\ell = 1$ is Lemma 2.1. Now suppose $1 \leqslant \ell \leqslant k-1$, and that (2.2) holds. Then

$$\begin{aligned} R(H) &= a_\ell R(x_\ell) - \varepsilon_\ell a_{\ell-1} R(x_{\ell+1}) \\ &= a_\ell (t_{\ell+1} R(x_{\ell+1}) - \varepsilon_{\ell+1} R(x_{\ell+2})) - \varepsilon_\ell a_{\ell-1} R(x_{\ell+1}) \\ &= a_{\ell+1} R(x_{\ell+1}) - \varepsilon_{\ell+1} a_\ell R(x_{\ell+2}), \end{aligned}$$

which is (2.2) with $\ell + 1$ in place of ℓ. Thus, we have established (2.2) by induction.

For $k \geqslant 1$, applying (2.2) with $\ell = k$, and then applying (1.1) with $m = m_k$, gives

$$R(H) = a_k R(m_k) - \varepsilon_k a_{k-1} = a_k \lfloor m_k/2 \rfloor - \varepsilon_k a_{k-1}.$$

The $k = 0$ case of (1.2) is (1.1), which was already established by Carlitz [5, Theorem 2]. This completes the proof of Theorem 1.1.

3. The summatory function

In this section, we prove Theorem 1.2.

3.1. A combinatorial recursion.

Recall that the Fibonacci sequence enjoys the recursive relation $F_{m+1} = F_m + F_{m-1}$. Our starting point is the following recursion for the summatory function $A(H)$.

Lemma 3.1. *If $m \in \mathbb{Z}_{\geqslant 3}$ and $F_m \leqslant H < F_{m+1}$ then*

$$A(H) = A(H - F_m) + A(H - F_{m-1}) - A(H - 2F_{m-1}) + 2^{m-3}.$$

Proof. Observe that $A(H)$ counts tuples $(x_1, \ldots, x_s)$ of Fibonacci numbers such that

$$s \geqslant 0, \qquad x_1 < \cdots < x_s, \qquad x_1 + \cdots + x_s \leqslant H.$$

Note that $x_1, \ldots, x_s \in \{F_2, \ldots, F_m\}$, since $F_1 = F_2$. There are $A(H - F_m)$ such tuples for which $x_s = F_m$, since $H - F_m < F_m$. We remind the reader that $A(x) = 0$ whenever $x < 0$.

If $x_s = F_{m-1}$, then we have

$$x_1 + \cdots + x_{s-1} \leqslant H - F_{m-1} < F_m < 2F_{m-1}.$$

There would be $A(H - F_{m-1})$ solutions to this if x_{s-1} were allowed to equal F_{m-1}, but since $x_{s-1} < x_s$ this is forbidden, and we need to subtract $A(H - 2F_{m-1})$. Indeed, Fibonacci solutions to

$$s - 1 \geqslant 0, \qquad x_1 < \cdots < x_{s-1}, \qquad x_1 + \cdots + x_{s-1} \leqslant H - F_{m-1}$$

come in two types:

- $A(H - 2F_{m-1})$ of them have $x_{s-1} = F_{m-1}$;
- The other $A(H - F_{m-1}) - A(H - 2F_{m-1})$ of them have $x_{s-1} < F_{m-1}$.

When computing the contribution to $A(H)$ from this case $x_s = F_{m-1}$, we only want to count the second type, as we need to have $x_{s-1} < x_s$. The upshot is that there are

$$A(H - F_{m-1}) - A(H - 2F_{m-1})$$

valid tuples for which $x_s = F_{m-1}$.

Finally, if $x_1 < x_2 < \ldots < x_s \leqslant F_{m-2}$ are Fibonacci numbers, then we always have

$$x_1 + \cdots + x_s \leqslant F_2 + \cdots + F_{m-2} < F_m \leqslant H,$$

owing to (2.1). As there are 2^{m-3} subsets of $\{F_2, \ldots, F_{m-2}\}$, there are 2^{m-3} valid tuples for which $x_s \leqslant F_{m-2}$.

Summing the contributions from the three cases completes the proof of the lemma. $\square$

Next, we provide a simple argument to show that

$$A(H) \asymp H^{\lambda}, \tag{3.1}$$

recalling our notational convention that this describes the asymptotic order of magnitude as $H \to +\infty$. Let $m \geqslant 4$ be an integer. If m is odd then, by Lemma 3.1, we have

$$\begin{aligned} A(F_m) &= 2^{m-3} + 1 + A(F_{m-2}) = \cdots \\ &= (2^{m-3} + 1) + (2^{m-5} + 1) + \cdots + (2^2 + 1) + A(F_3) \\ &= (1 + 2^2 + \cdots + 2^{m-3}) + (m+1)/2 \\ &= \frac{2^{m-1} - 1}{3} + \frac{m+1}{2} = \left\lfloor \frac{2^m}{6} + \frac{m+1}{2} \right\rfloor. \end{aligned}$$

Similarly, if $m \geqslant 4$ is even, we have

$$\begin{aligned} A(F_m) &= 2^{m-3} + 1 + A(F_{m-2}) = \cdots \\ &= (2^{m-3}+1) + (2^{m-5}+1) + \cdots + (2^1+1) + A(F_2) \\ &= (2^1 + 2^3 + \cdots + 2^{m-3}) + (m+2)/2 \\ &= \frac{2(2^{m-2}-1)}{3} + \frac{m+2}{2} = \left\lfloor \frac{2^m}{6} + \frac{m+1}{2} \right\rfloor. \end{aligned}$$

We can check directly that the conclusion also holds when $m = 2, 3$. Thus, we have

$$A(F_m) = \left\lfloor \frac{2^m}{6} + \frac{m+1}{2} \right\rfloor \qquad (m \geqslant 2) \tag{3.2}$$

and

$$A(F_m) \sim \frac{2^m}{6}. \tag{3.3}$$

Therefore

$$A(F_m) \sim cF_m^{\lambda}, \qquad c = \frac{1}{6}\sqrt{5}^{\lambda}.$$

Note from Binet's formula that

$$F_m\sqrt{5} \sim \varphi^m.$$

Hence, if $F_m \leqslant H < F_{m+1}$ then

$$\varphi^m(1+o(1)) = F_m\sqrt{5} \leqslant H\sqrt{5} < F_{m+1}\sqrt{5} = \varphi^{m+1}(1+o(1)),$$

and consequently

$$A(H) < A(F_{m+1}) = \frac{2^{m+1}}{6}(1+o(1)) \leqslant \frac{1}{3}(H\sqrt{5})^{\lambda}(1+o(1))$$

and

$$A(H) \geqslant A(F_m) = \frac{2^m}{6}(1+o(1)) \geqslant \frac{1}{12}(H\sqrt{5})^{\lambda}(1+o(1)).$$

These calculations furnish (3.1), in the stronger form

$$c/2 \leqslant c_1 \leqslant c_2 \leqslant 2c.$$

Example 3.2. By Lemma 3.1, as $m \to \infty$ we have

$$\begin{aligned} A(2F_{m-1}) &= A(2F_{m-1} - F_m) + A(F_{m-1}) - A(0) + 2^{m-3} \\ &= A(F_{m-3}) + A(F_{m-1}) + 2^{m-3} - 1 \\ &\sim \frac{11}{48}2^m \\ &\sim \frac{11}{24}(F_{m-1}\sqrt{5})^{\lambda} \\ &= \frac{11(\sqrt{5}/2)^{\lambda}}{24}(2F_{m-1})^{\lambda}, \end{aligned}$$

wherein $\frac{11(\sqrt{5}/2)^\lambda}{24} \approx 0.538$, and

$$\begin{aligned} A(F_m + F_{m-2}) &= A(F_{m-2}) + A(2F_{m-2}) - A(F_{m-4}) + 2^{m-3} \\ &\sim \frac{2^{m-2}}{6} + \frac{11}{48}2^{m-1} - \frac{2^{m-4}}{6} + 2^{m-3} \\ &\sim \frac{13}{48}2^m \\ &\sim \frac{13}{48}\left(\sqrt{5}\frac{F_m + F_{m-2}}{1+\varphi^{-2}}\right)^\lambda \\ &= \frac{13\sqrt{5}^\lambda}{48(1+\varphi^{-2})^\lambda}(F_m + F_{m-2})^\lambda \\ &= \frac{13\varphi^\lambda}{48}(F_m + F_{m-2})^\lambda \\ &= \frac{13}{24}(F_m + F_{m-2})^\lambda, \end{aligned}$$

wherein $\frac{13}{24} \approx 0.542$.

3.2. **An exact formula for the summatory function.** Recall (1.3). Applying Lemma 3.1 several times provides the following more elaborate recursion. Here we remind the reader once again that $A(H) = 0$ when $H < 0$.

Lemma 3.3. *If $t \geqslant 2$, $m \geqslant 2t$, and $F_{m-2t+1} \leqslant x < F_{m-2t+3}$, then*

$$A(F_m + x) = tA(x) - A(x - F_{m-2t+2}) + f(t)2^{m-2t}.$$

Proof. For the base case $t = 2$ of our induction, for $m \geqslant 4$ and $F_{m-3} \leqslant x < F_{m-1}$ we have

$$\begin{aligned} A(F_m + x) &= A(x) + A(F_{m-2} + x) - A(x - F_{m-3}) + 2^{m-3} \\ &= 2A(x) - A(x - F_{m-2}) + 2^{m-4} + 2^{m-3} = 2A(x) - A(x - F_{m-2}) + \frac{3}{16}2^m \\ &= 2A(x) - A(x - F_{m-2}) + f(2)2^{m-4}. \end{aligned}$$

Now let $t \geqslant 3$, and suppose the result holds with $t - 1$ in place of t. Then for $m \geqslant 2t$ and $x \in [F_{m-2t+1}, F_{m-2t+3})$ we have

$$\begin{aligned} A(F_m + x) &= A(x) + A(F_{m-2} + x) + 2^{m-3} \\ &= tA(x) - A(x - F_{m-2-2(t-1)+2}) + \left(1 + \frac{2(4^{t-2} - 1)}{3}\right) 2^{m-2-2(t-1)} + 2^{m-3} \\ &= tA(x) - A(x - F_{m-2t+2}) + \left(1 + \frac{2^{2t-3} - 2}{3} + 2^{2t-3}\right) 2^{m-2t} \\ &= tA(x) - A(x - F_{m-2t+2}) + \left(1 + \frac{2(4^{t-1} - 1)}{3}\right) 2^{m-2t} \\ &= tA(x) - A(x - F_{m-2t+2}) + f(t)2^{m-2t}. \end{aligned}$$

□

The following immediate consequence is analogous to Lemma 2.1.

Corollary 3.4. *Let $t \geqslant 2$, $m \geqslant 2t$, and $F_{m-2t+1} \leqslant x < F_{m-2t+3}$. Set*

$$(\varepsilon, y) = \begin{cases} (1, x - F_{m-2t+2}), & \text{if } F_{m-2t+2} \leqslant x < F_{m-2t+3} \\ (0, x - F_{m-2t+1}), & \text{if } F_{m-2t+1} \leqslant x < F_{m-2t+2}. \end{cases}$$

Then

$$A(F_m + x) = tA(x) - \varepsilon A(y) + f(t)2^{m-2t}.$$

We now establish (1.4) for $1 \leqslant \ell \leqslant k$. For the base case $\ell = 1$ of our induction, we know from Corollary 3.4 that

$$A(H) = a_1 A(x_1) - \varepsilon_1 a_0 A(x_2) + a_0 f(t_1)2^{m_0 - 2t_1}.$$

Now suppose that for some $\ell \in \{1, 2, \ldots, k-1\}$ we have (1.4). Then

$$\begin{aligned} A(H) &= a_\ell(t_{\ell+1}A(x_{\ell+1}) - \varepsilon_{\ell+1}A(x_{\ell+2}) + f(t_{\ell+1})2^{m_\ell - 2t_{\ell+1}}) - \varepsilon_\ell a_{\ell-1}A(x_{\ell+1}) \\ &\qquad + \sum_{i \leqslant \ell} a_{i-1} f(t_i) 2^{m_{i-1} - 2t_1} \\ &= a_{\ell+1}A(x_{\ell+1}) - \varepsilon_{\ell+1} a_\ell A(x_{\ell+2}) + \sum_{i \leqslant \ell+1} a_{i-1} f(t_i) 2^{m_{i-1} - 2t_i}. \end{aligned}$$

We have proved (1.4) by induction on ℓ.

For $k \geqslant 1$, inserting (3.2) into the $\ell = k$ case of (1.4) yields (1.5). Meanwhile, the $k = 0$ case of (1.5) is precisely (3.2). This completes the proof of Theorem 1.2.

Example 3.5. Let m be large, and consider

$$H = F_m + F_{m-7} + F_{m-12} + F_{m-19}.$$

In this case

$$t_1 = 4, \quad t_2 = 3, \quad t_3 = 4, \qquad \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0.$$

Therefore

$$\begin{aligned} A(H) &= 48 \left\lfloor \frac{2^{m-19}}{6} + \frac{m-18}{2} \right\rfloor + f(4)2^{m-8} + 4f(3)2^{m-13} + 12f(4)2^{m-20} \\ &\sim (16 + 43 \times 2^{12} + 44 \times 2^7 + 12 \times 43)2^{m-20} = \frac{45573}{262144}2^m. \end{aligned}$$

Thus, as $m \to \infty$, we have

$$\frac{A(H)}{H^\lambda} \to \frac{45573}{262144} \left(\frac{\sqrt{5}}{1 + \varphi^{-7} + \varphi^{-12} + \varphi^{-19}} \right)^\lambda \approx 0.525352.$$

4. Subdivision

In this section, we prove Theorem 1.3. Let m be a large positive integer. We subdivide the discrete interval $[F_m, F_{m+1}) \cap \mathbb{Z}$ into subintervals according to the initial Zeckendorf digits, i.e. those F_{m_i} in the Zeckendorf expansion with $m - m_i \leqslant 27$, where 27 is an arbitrary cutoff amenable to practical computation with a sufficiently precise outcome. Explicitly, the left endpoints of these subintervals are given by

$$F_m + \sum_{i \leqslant \ell} F_{m-a_i}, \tag{4.1}$$

where

$$\ell \geqslant 0, \qquad a_1, a_2 - a_1, \ldots, a_\ell - a_{\ell-1} \geqslant 2, \qquad a_\ell \leqslant 27. \tag{4.2}$$

The right endpoints are the same, except that the final one is F_{m+1}.

The computer tells us that 317811 subintervals are produced in this way, but one can also deduce this by pure thought. Indeed, consider the function from these left endpoints to $\{0, 1, \ldots, 317810\}$ given by

$$F_m + \sum_{i \leqslant \ell} F_{m-a_i} \mapsto \sum_{i \leqslant \ell} F_{29-a_i}.$$

This is bijection, as the images are 0 and the Zeckendorf expansions of $1, 2, \ldots, F_{28} - 1 = 317810$. We can now write our discrete subintervals as

$$[p_j, p_{j+1}) \cap \mathbb{Z} \qquad (0 \leqslant j \leqslant 317810),$$

where $p_0 = F_m < p_1 < p_2 < \cdots < p_{317810}$ are the numbers of the form (4.1) subject to (4.2), and $p_{317811} = F_{m+1}$.

Using Theorem 1.2, we can show that

$$A(p_j) \sim v_j 2^m, \qquad p_j \sim w_j \varphi^m \qquad (0 \leqslant j \leqslant 317811)$$

as $m \to \infty$, for some computable values of v_j and w_j. Then

$$(1 + o(1))L_j \leqslant \frac{A(H)}{H^\lambda} \leqslant (1 + o(1))U_j \qquad (p_j \leqslant H < p_{j+1}),$$

where

$$L_j = \frac{v_j}{w_{j+1}^\lambda}, \quad U_j = \frac{v_{j+1}}{w_j^\lambda} \qquad (0 \leqslant j \leqslant 317810).$$

We carried out these computations using the software *Mathematica* [22]; the code is provided in Appendix B. Then

$$c_1 \geqslant \min_j L_j, \qquad c_2 \leqslant \max_j U_j.$$

The software also told us which subintervals attaining the least L_j and the greatest U_j, namely

$$j = 19401, \qquad (a_1, \ldots, a_\ell) = (7, 12, 18, 25)$$

and

$$j = 184839, \qquad (a_1, \ldots, a_\ell) = (3, 5, 8, 10, 12, 16, 18, 21, 23, 26),$$

respectively. Since

$$\frac{A(p_j)}{p_j^\lambda} \sim \frac{v_j}{w_j^\lambda},$$

we thereby also obtained an upper bound for c_1 and a lower bound for c_2. These calculations delivered (1.6), completing the proof of Theorem 1.3.

Appendix A. Code for $R(H)$

Lines 2–7 are *Rosetta code* [17], available for general use under the GNU Free Documentation License, version 1.2. The value of H in the first line can be changed.

```
H = 1234;
zeckendorf[0] = 0;
zeckendorf[n_Integer] :=
  10^(# - 1) + zeckendorf[n - Fibonacci[# + 1]] &@
   LengthWhile[
    Fibonacci /@
     Range[2, Ceiling@Log[GoldenRatio, n Sqrt@5]], # <= n &];
Z = IntegerDigits[zeckendorf[H]];
l = Total[Z];
X = ConstantArray[0, l];
t = 1;
If[l == 1, Floor[(Length[Z] + 1)/2],
```

```
 For[i = 1, i < Length[Z] + 1, i++,
  If[Z[[i]] == 1, X[[t]] = Length[Z] - i + 2; t++;,]
  ];
 T = ConstantArray[0, l - 1];
 Ep = ConstantArray[0, l - 1];
 For[i = 1, i < l, i++,
  T[[i]] = Floor[(X[[i]] - X[[i + 1]] + 2)/2];
  Ep[[i]] = 2 T[[i]] - 1 - X[[i]] + X[[i + 1]];
  ];
 a = ConstantArray[1, l];
 a[[2]] = T[[1]];
 For[i = 3, i < l + 1, i++,
  a[[i]] = T[[i - 1]] a[[i - 1]] - Ep[[i - 2]] a[[i - 2]]
  ];
 a[[l]]*Floor[X[[l]]/2] - a[[l - 1]]*Ep[[l - 1]]
 ]
```

Appendix B. Code for $A(H)$

```
P = (1 + Sqrt[5])/2;
L = Log[2]/Log[P];
l = 27;
X = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2]}];
t = 1;
X[[2, 1]] = l;
For[i = 3, i < Fibonacci[l + 1] + 1, i++,
 If[X[[i - 1, t]] == l || X[[i - 1, t]] == l - 1,
  If[(t > 1 && X[[i - 1, t]] - X[[i - 1, t - 1]] == 2),
   t--;
   For[j = t, j > 0, j--,
    If[j == 1, t = j;
     For[k = 1, k < t, k++, X[[i, k]] = X[[i - 1, k]]];
     X[[i, t]] = X[[i - 1, t]] - 1; j = 0,
     If[X[[i - 1, j]] - X[[i - 1, j - 1]] != 2, t = j;
      For[k = 1, k < t, k++, X[[i, k]] = X[[i - 1, k]]];
      X[[i, t]] = X[[i - 1, t]] - 1; j = 0]
     ]
    ]
   ,
   X[[i]] = X[[i - 1]];
   X[[i, t]]--;
   ],
  t++;
  X[[i]] = X[[i - 1]];
  X[[i, t]] = l;
  ]
 ]
T = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2]}];
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 T[[i, 1]] = Floor[(X[[i, 1]] + 2)/2]
 ]
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 For[j = 2, j < Floor[l/2] + 1, j++,
  If[X[[i, j]] == 0, ,
   T[[i, j]] = Floor[(X[[i, j]] - X[[i, j - 1]] + 2)/2]
   ]
  ]
 ]
Ep = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2]}];
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 Ep[[i, 1]] = 2 T[[i, 1]] - 1 - X[[i, 1]]
 ]
```

```
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 For[j = 2, j < Floor[l/2] + 1, j++,
  If[X[[i, j]] == 0, ,
   Ep[[i, j]] = 2 T[[i, j]] - 1 - X[[i, j]] + X[[i, j - 1]]
   ]
  ]
 ]
a = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2] + 1}];
For[i = 1, i < Fibonacci[l + 1] + 1, i++, a[[i, 1]] = 1];
For[i = 2, i < Fibonacci[l + 1] + 1, i++, a[[i, 2]] = T[[i, 1]]];
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 For[j = 3, j < Floor[l/2] + 2, j++,
  If[X[[i, j - 1]] == 0, ,
   a[[i, j]] =
    T[[i, j - 1]] a[[i, j - 1]] - Ep[[i, j - 2]] a[[i, j - 2]]
   ]
  ]
 ]
f = Function[t, 1 + (2/3) (4^(t - 1) - 1)];
k = ConstantArray[0, {Fibonacci[l + 1] + 1}];
k[[1]] = (1/6);
k[[Fibonacci[l + 1] + 1]] = (1/3);
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 For[j = Floor[l/2], j > 0, j--,
  If[X[[i, j]] == 0, ,
   k[[i]] = (a[[i, j + 1]]/(6*2^(X[[i, j]]))) +
     Sum[(a[[i, l]] f[T[[i, l]]]/2^(X[[i, l - 1]] + 2 T[[i, l]])), {l,
        2, j}] + (a[[i, 1]] f[T[[i, 1]]]/2^(2 T[[i, 1]]));
   j = 0
   ]
  ]
 ]
p = ConstantArray[0, {Fibonacci[l + 1] + 1}];
p[[1]] = 1; p[[Fibonacci[l + 1] + 1]] = P;
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
 For[j = Floor[l/2], j > 0, j--,
  If[X[[i, j]] == 0, ,
   p[[i]] = 1 + Sum[P^(-X[[i, k]]), {k, 1, j}];
   j = 0
   ]
  ]
 ]
LU = ConstantArray[0, {Fibonacci[l + 1], 2}];
For[i = 1, i < Fibonacci[l + 1] + 1, i++,
 LU[[i, 1]] = k[[i]]*(Sqrt[5]/p[[i + 1]])^L;
 LU[[i, 2]] = k[[i + 1]]*(Sqrt[5]/p[[i]])^L
 ]
NumberForm[N[Min[LU]], 8]
NumberForm[N[Max[LU]], 8]
Position[LU, Min[LU]]
Position[LU, Max[LU]]

X[[19401]]
X[[184839]]
NumberForm[N[k[[19401]]*(Sqrt[5]/p[[19402]])^L], 8]
NumberForm[N[k[[19401]]*(Sqrt[5]/p[[19401]])^L], 8]
NumberForm[N[k[[184840]]*(Sqrt[5]/p[[184839]])^L], 8]
NumberForm[N[k[[184839]]*(Sqrt[5]/p[[184839]])^L], 8]
```

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