Invariant Brauer group of an abelian variety

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Abstract

We study a new object that can be attached to an abelian variety or a complex torus: the invariant Brauer group, as recently defined by Yang Cao. Over the field of complex numbers this is an elementary abelian 2-group with an explicit upper bound on the rank. We exhibit many cases in which the invariant Brauer group is zero, and construct complex abelian varieties in every dimension starting with 2, both simple and non-simple, with invariant Brauer group of order 2. We also address the situation in finite characteristic and over non-closed fields.

1 Introduction

The Brauer–Grothendieck group $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ is an important cohomological invariant of an algebraic variety X over a field k. It has applications in algebraic geometry (the rationality problem over algebraically closed fields) and in diophantine geometry (the Brauer–Manin obstruction to the local-to-global principles over global fields). If k is not separably closed, a more accessible part of $\mathrm{Br}(X)$ is the *algebraic Brauer group* $\mathrm{Br}_1(X)$ defined as the kernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{s}})$, where $X_{\mathrm{s}} = X \times_k k_{\mathrm{s}}$ and k_{s} is a separable closure of k.

If X is acted on by an algebraic k-group, it is natural to look for an "equivariant" version of the Brauer–Grothendieck group that takes into account the symmetries of X. Such a version was suggested recently by Y. Cao in the case of an action $m\colon G \times X \to X$ of a *connected* algebraic k-group G, see [C18, Déf. 3.1], [C, Déf. 1.1 (2)]. He defined the *invariant Brauer group* $\mathrm{Br}_G(X)$ as the subgroup of $\mathrm{Br}(X)$ consisting of the elements $x \in \mathrm{Br}(X)$ such that $m^*(x) \in \mathrm{Br}(G \times X)$ belongs to the subgroup $\pi_1^*\mathrm{Br}(G) + \pi_2^*\mathrm{Br}(X)$, where $\pi_1\colon G \times X \to G$ and $\pi_2\colon G \times X \to X$ are the natural projections. One crucial case is when G acts on itself by left translations; the resulting invariant Brauer group is denoted by $\mathrm{Br}_G(G)$.

Cao used the invariant Brauer group to study local-to-global principles, weak and strong approximation on varieties with an action of a connected linear group, see [C18, C20, C]. For a connected group G he proved that $\mathrm{Br}_G(G) = \mathrm{Br}_1(G)$ if G is *linear* and $\mathrm{char}(k) = 0$, see [C18, Lemme 3.6]. This naturally leads to a question raised by Cao in [C20, Exemple 4.1 (3)]: what happens when $G = A$ is an abelian variety? Namely, does the inclusion $\mathrm{Br}_A(A) \subset \mathrm{Br}_1(A)$ or the reverse inclusion $\mathrm{Br}_1(A) \subset \mathrm{Br}_A(A)$ hold for all abelian varieties?

In this paper we give a negative answer to the question of Cao by showing that neither inclusion holds in general. We aim to shed light on the properties of the somewhat mysterious invariant Brauer group of an abelian variety. The elusive nature of $\mathrm{Br}_A(A)$ becomes apparent already over an algebraically closed field k of characteristic zero. Let $A^\vee$ be the dual abelian variety of A, let $n \geq 2$ and let $e_{n,A}(x, y)$ be the Weil pairing $A[n] \times A^\vee[n] \to \mu_n$. It is known [SZ08, p. 492] that if n is odd, then every morphism of abelian varieties $u\colon A \to A^\vee$ that induces a symmetric map on n-torsion points $A[n] \to A^\vee[n]$, or, equivalently, such that $e_{n,A}(x, ux) = 0$ for any $x \in A[n]$, is congruent modulo n to a symmetric morphism $A \to A^\vee$. This fails if n is even. In fact, $\mathrm{Br}_A(A)$ precisely measures the failure of an element $\alpha \in \mathrm{Hom}(A, A^\vee)/2$ such that $e_{2,A}(x, \alpha x) = 0$ for any $x \in A[2]$ to come from a symmetric morphism $A \to A^\vee$. See Section 3, in particular, formula (9) and Remark 3.6.

Let $\mathrm{Br}_A(A)(p') \subset \mathrm{Br}_A(A)$ be the subgroup consisting of the elements not divisible by $\mathrm{char}(k)$. When k is algebraically closed of characteristic not equal to 2 we prove that $\mathrm{Br}_A(A)(p')$ is a finite elementary 2-group, which is invariant under isogenies of odd degrees. We give an upper bound for the order of $\mathrm{Br}_A(A)(p')$ in terms of the rank of the endomorphism algebra of A and the Picard number of A. Using different techniques we describe classes of abelian varieties with trivial invariant Brauer group.

Theorem 1.1 *Let A be an abelian variety over an algebraically closed field of characteristic different from* 2. *Then* $\mathrm{Br}_A(A)(p') \cong (\mathbb{Z}/2)^n$, *where*

$$n \leq \mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(A)) - \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}\,(A)).$$

Moreover, $\mathrm{Br}_A(A)(p') = 0$ *in the following cases:*

(i) *A is a product of elliptic curves;*

(ii) $\mathrm{char}(k) = 0$ *and A is isogenous to a power of a CM elliptic curve;*

(iii) $\mathrm{char}(k) = 0$ *and A is a simple abelian variety of CM type with complex multiplication by the ring of integers of a cyclotomic field;*

(iv) $\mathrm{char}(k) > 2$ *and A is a supersingular abelian variety.*

The upper bound is obtained in Proposition 3.14, part (i) is proved in Corollary 3.8, part (ii) is Corollary 3.11, part (iii) is Proposition 5.6, and part (iv) is Example 3.7. If E is an elliptic curve over any algebraically closed field, then $\mathrm{Br}(E) = 0$, so $\mathrm{Br}_A(A) \neq 0$ implies $\dim(A) \geq 2$.

The question of the non-triviality of $\mathrm{Br}_A(A)$ over an algebraically closed field turns out to be rather subtle. We address this question also for complex tori, for which the definition of the invariant Brauer group also makes sense.

Theorem 1.2 (i) *There exist simple, non-algebraisable complex tori in every dimension $g \geq 3$ with invariant Brauer group $\mathbb{Z}/2$.*

(ii) *There exist simple abelian varieties over $\mathbb{C}$ in every dimension $g \geq 3$ with invariant Brauer group $\mathbb{Z}/2$.*

(iii) *There exist abelian surfaces of CM type over $\mathbb{C}$, both simple and non-simple, with invariant Brauer group $\mathbb{Z}/2$.*

Part (i) is proved in Section 4.1, part (ii) is proved in Section 4.2, and part (iii) is proved in Sections 5.1 and 5.2. The non-simple abelian surface A with $\mathrm{Br}_A(A) \cong \mathbb{Z}/2$ that we construct in Section 5.1 carries a principal polarisation, see Remark 5.1.

Assume that the ground field k is an algebraically closed field of characteristic different from 2. Over k, the invariant Brauer group of a product of abelian varieties $A \times B$ is naturally isomorphic to $\mathrm{Br}_A(A) \oplus \mathrm{Br}_B(B)$ (up to p-torsion in characteristic p, see Corollary 3.8). We deduce that for any integer $n \geq 0$ there is a complex abelian variety A such that $\mathrm{Br}_A(A) \cong (\mathbb{Z}/2)^n$. It would be interesting to determine the smallest dimension $g = g(n)$ for which there exists a complex abelian variety A with invariant Brauer group $(\mathbb{Z}/2)^n$. For example, we prove that if $g = 2$, then $n \leq 2$, but we do not know if there is an abelian surface over $\mathbb{C}$ (or another algebraically closed field) with invariant Brauer group $(\mathbb{Z}/2)^2$.

The invariant Brauer group of an abelian variety over k is preserved by isogenies of odd degree (up to p-torsion in characteristic p, see Proposition 3.9). This implies that principally polarised g-dimensional abelian varieties A over $\mathbb{C}$ such that $\mathrm{Br}_A(A) \neq 0$ are dense in the complex topology of the moduli space $\mathcal{A}_g(\mathbb{C})$ for every $g \geq 2$, see Remark 3.13. However, for a complex abelian variety A which is general in the sense that $\mathrm{End}(A) \cong \mathbb{Z}$ we have $\mathrm{Br}_A(A) = 0$. The same holds for a general complex torus, that is, a complex torus A such that $\mathrm{End}(A) \cong \mathbb{Z}$ and $\mathrm{Hom}(A, A^\vee) = 0$.

Finally, in Section 6 we study the invariant Brauer group of an abelian variety A over a non-closed field k. Let $\mathrm{Br}_{a,A}(A)$ be the intersection of the following three subgroups of $\mathrm{Br}(A)$: the invariant subgroup $\mathrm{Br}_A(A)$, the algebraic subgroup $\mathrm{Br}_1(A)$ and the kernel of the evaluation map at the unit element of A.

Theorem 1.3 *Let A be an abelian variety over a field k. Then $\mathrm{Br}_{a,A}(A)$ is canonically isomorphic to the kernel of the composition of the natural maps*

$$\mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}})) \longrightarrow \mathrm{H}^1(k, \mathrm{NS}\,(A_{\mathrm{s}})) \longrightarrow \mathrm{H}^1(k, \mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee)).$$

This is proved in Proposition 6.2. Using this it is easy to construct an abelian surface for which $\mathrm{Br}_1(A)$ is not contained in $\mathrm{Br}_A(A)$, see Example 6.4.

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2 Definitions

Let k be a base field, let X be a k-scheme, let G be a group k-scheme, and let $m : G\times X \to X$ be a k-action of G on X. Write $\pi_1 : G\times X \to G$ and $\pi_2 : G\times X \to X$ for the natural projections. Let $e : \mathrm{Spec}(k) \to G$ be the identity element of G. It induces a homomorphism $e^* : \mathrm{Br}(G) \to \mathrm{Br}(k)$, and we define

$$\mathrm{Br}_e(G) \colon = \mathrm{Ker}[e^* : \mathrm{Br}(G) \to \mathrm{Br}(k)].$$

Recall that $\mathrm{Br}_0(X)$ denotes the image of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$; we have $\mathrm{Br}(G) = \mathrm{Br}_0(G) \oplus \mathrm{Br}_e(G)$. There are induced homomorphisms

$$m^* \colon \mathrm{Br}(X) \to \mathrm{Br}(G \times X), \quad \pi_1^* \colon \mathrm{Br}(G) \to \mathrm{Br}(G \times X), \quad \pi_2^* \colon \mathrm{Br}(X) \to \mathrm{Br}(G \times X).$$

The original Yang Cao's definition of the invariant Brauer group [C18, Déf. 3.1], [C, Déf. 1.1 (2)] is as follows:

$$\mathrm{Br}_G(X) \colon = \{x \in \mathrm{Br}(X) | m^*(x) - \pi_2^*(x) \in \pi_1^*\mathrm{Br}(G)\}.$$

It is clear that $\mathrm{Br}_0(X)$ is contained in $\mathrm{Br}_G(X)$.

The definition of $\mathrm{Br}_G(X)$ can be put in a more symmetric form.

Lemma 2.1 *We have* $\mathrm{Br}_G(X) = \{x \in \mathrm{Br}(X) | m^*(x) \in \pi_1^*\mathrm{Br}(G) + \pi_2^*\mathrm{Br}(X)\}$.

Proof. One inclusion is trivial. To prove the other inclusion take any x in $\mathrm{Br}(X)$ such that $m^*(x) = \pi_1^*(y) + \pi_2^*(z)$, where $y \in \mathrm{Br}(G)$ and $z \in \mathrm{Br}(X)$. Modifying y and z by an element from the image of $\mathrm{Br}(k)$ we can assume without loss of generality that $y \in \mathrm{Br}_e(G)$. Precomposing m with $i = (e, \mathrm{id}) \colon X \to G \times X$ gives the identity map on X. Hence $x = i^*(m^*(x)) = e^*(y) + z = z$, so that $m^*(x) - \pi_2^*(x) \in \pi_1^*\mathrm{Br}(G)$. □

In this paper we are interested in the case when $X = G$ and $m : G\times G \to G$ is the group law of G. It is clear that $\mathrm{Br}_G(G)$ is functorial in G: a morphism of group k-schemes $G \to H$ gives rise to a homomorphism of abelian groups $\mathrm{Br}_H(H) \to \mathrm{Br}_G(G)$.

Define

$$\mathrm{Br}_{e,G}(G) \colon = \mathrm{Br}_e(G) \cap \mathrm{Br}_G(G).$$

If k is such that $\mathrm{Br}(k) = 0$, e.g. k is separably closed, then $\mathrm{Br}_G(G) = \mathrm{Br}_{e,G}(G)$.

Proposition 2.2 *Let G be a group k-scheme with the group law $m : G \times G \to G$. Then we have* $\mathrm{Br}_G(G) = \mathrm{Br}_0(G) \oplus \mathrm{Br}_{e,G}(G)$ *and* $\mathrm{Br}_{e,G}(G) = \mathrm{Br}_e(G) \cap \mathrm{Ker}(\delta)$*, where*

$$\delta = m^* - \pi_1^* - \pi_2^* : \quad \mathrm{Br}(G) \longrightarrow \mathrm{Br}(G \times G).$$

Let $[-1] \colon G \to G$ be the morphism given by inversion in G. Then for every x in $\mathrm{Br}_{e,G}(G)$ we have $[-1]^*(x) = -x$.

Proof. The first claim is clear. Let $i_1 = (\mathrm{id}, e)$ and $i_2 = (e, \mathrm{id})$ be the obvious maps $G \to G \times G$. It is clear that on $\mathrm{Br}(G)$ the map $i_1^*\pi_1^* = i_1^*m^* = i_2^*\pi_2^* = i_2^*m^*$ is the identity map, whereas $i_1^*\pi_2^* = i_2^*\pi_1^*$ is the composition of $e^*\colon \mathrm{Br}(G) \to \mathrm{Br}(k)$ with the natural injective map $\mathrm{Br}(k) \to \mathrm{Br}(G)$. By Lemma 2.1, $x \in \mathrm{Br}_G(G)$ if and only if $m^*(x) = \pi_1^*(y) + \pi_2^*(z)$ for some $y, z \in \mathrm{Br}(G)$. Then $x = y + e^*(z) = e^*(y) + z$, which implies $e^*(x) = e^*(y) + e^*(z)$. When $x \in \mathrm{Br}_e(G)$, so that $e^*(y) + e^*(z) = 0$, we can replace y by $y - e^*(y)$ and replace z by $z - e^*(z)$ and hence assume without loss of generality that $y, z \in \mathrm{Br}_e(G)$. Then $x = y = z$ hence $\delta(x) = 0$. Since $\mathrm{Ker}(\delta)$ is clearly contained in $\mathrm{Br}_G(G)$, the second claim follows.

Let $\alpha\colon G \to G \times G$ be the morphism $\alpha(x) = (x, [-1]x)$. Then the composition $\alpha^* \circ \delta\colon \mathrm{Br}_e(G) \to \mathrm{Br}_e(G \times G) \to \mathrm{Br}_e(G)$ equals $-1 - [-1]^*$, giving the last claim. □

3 Abelian varieties over algebraically closed fields

3.1 Basic properties

Convention 3.1 In this section k is an algebraically closed field of $\mathrm{char}(k) \neq 2$. Let A be an abelian variety over k. Let $\mathrm{Br}(A)(p')$ be the subgroup of $\mathrm{Br}(A)$ consisting of the elements of order not divisible by $\mathrm{char}(k)$. Define

$$\mathrm{Br}_A(A)(p') = \mathrm{Br}(A)(p') \cap \mathrm{Br}_A(A).$$

The case of dimension 1 does not present any difficulty.

Remark 3.2 For an elliptic curve E over any algebraically closed field we have $\mathrm{Br}(E) = 0$, see [Gro68, III, Cor. 1.2]. In particular, $\mathrm{Br}_E(E) = 0$.

Using the Kummer exact sequence it is easy to show that the multiplication by n map $[n] : A \to A$ induces multiplication by n^2 on $\mathrm{Br}(A)(p')$, see [Ber72, Sect. 2]. In particular, $[-1] : A \to A$ acts on $\mathrm{Br}(A)(p')$ trivially. This fact and the last statement of Proposition 2.2 imply that $\mathrm{Br}_A(A)(p')$ is contained in $\mathrm{Br}_A(A)[2]$. By Proposition 2.2 we conclude that $\mathrm{Br}_A(A)(p')$ is the kernel of

$$\delta : \ \mathrm{Br}(A)[2] \longrightarrow \mathrm{Br}(A \times A)[2].$$

The Kummer exact sequence allows us to write this map as

$$\delta : \ \frac{\mathrm{H}^2(A, \mathbb{Z}/2)}{\mathrm{NS}\,(A)/2} \longrightarrow \frac{\mathrm{H}^2(A \times A, \mathbb{Z}/2)}{\mathrm{NS}\,(A \times A)/2}.$$

To compute this map we need to recall some standard facts about abelian varieties over an algebraically closed field.

For a commutative ring R and an R-module M we write $\wedge^2 M$ for the subgroup of $M^{\otimes 2} = M \otimes_R M$ generated by $x \otimes y - y \otimes x$ for all $x, y \in M$. We denote by ι the inclusion $\wedge^2 M \hookrightarrow M^{\otimes 2}$. This will be applied when R is $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Z}/2$.

Let $\sigma\colon A\times A\to A\times A$ be the involution that exchanges the factors. By abuse of notation we also denote by σ the maps induced by σ on cohomology groups of $A\times A$. Recall that $m\colon A\times A\to A$ is the multiplication map, and $\pi_1\colon A\times A\to A$ and $\pi_2\colon A\times A\to A$ are the natural projections. As usual, $A^\vee$ is the dual abelian variety of A. The dual morphism $m^\vee\colon A^\vee\to A^\vee\times A^\vee$ is the diagonal map.

Let $\mathrm{H}^i(-,\mathbb{Z}_2)$ be the 2-adic étale cohomology groups. (The following can be easily modified if A is a complex torus by considering classical cohomology groups $\mathrm{H}^i(-,\mathbb{Z})$ instead of étale cohomology.) Since A is an abelian variety, the cup-product

$$\cup_A\colon \mathrm{H}^1(A,\mathbb{Z}_2)\times\mathrm{H}^1(A,\mathbb{Z}_2)\longrightarrow\mathrm{H}^2(A,\mathbb{Z}_2)$$

gives rise to a canonical isomorphism of $\mathbb{Z}_2$-modules

$$\wedge^2\mathrm{H}^1(A,\mathbb{Z}_2)\cong\mathrm{H}^2(A,\mathbb{Z}_2).$$

The Künneth decomposition in degree 1 is

$$\mathrm{H}^1(A\times A,\mathbb{Z}_2)\cong\mathrm{H}^1(A,\mathbb{Z}_2)\oplus\mathrm{H}^1(A,\mathbb{Z}_2),\tag{1}$$

where the direct summands are embedded via the induced maps π_1^* and π_2^*. Precomposing m with $A\xrightarrow{\sim}A\times\{0\}\to A\times A$ gives the identity map, hence it is clear that $m^*:\mathrm{H}^1(A,\mathbb{Z}_2)\to\mathrm{H}^1(A\times A,\mathbb{Z}_2)$ sends x to $x\oplus x$. Taking the second exterior power in (1) we obtain the Künneth decomposition in degree 2:

$$\mathrm{H}^2(A\times A,\mathbb{Z}_2)\cong\mathrm{H}^2(A,\mathbb{Z}_2)\oplus\mathrm{H}^2(A,\mathbb{Z}_2)\oplus\left(\mathrm{H}^1(A,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\mathrm{H}^1(A,\mathbb{Z}_2)\right),\tag{2}$$

where the first factor is embedded via π_1^*, the second factor via π_2^*, and the third factor via the map that sends $x\otimes y$ to $\pi_1^*(x)\cup_{A\times A}\pi_2^*(y)$. The cup-product

$$\cup_{A\times A}:\quad \mathrm{H}^1(A\times A,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\mathrm{H}^1(A\times A,\mathbb{Z}_2)\longrightarrow\mathrm{H}^2(A\times A,\mathbb{Z}_2)$$

is skew-symmetric, hence σ^* sends $\pi_1^*(x)\cup_{A\times A}\pi_2^*(y)$ to

$$\pi_2^*(x)\cup_{A\times A}\pi_1^*(y)=-\pi_1^*(y)\cup_{A\times A}\pi_2^*(x).$$

It follows that σ acts on the third summand of (2) by sending $x\otimes y$ to $-y\otimes x$.

Taking the cup product of $m^*:\mathrm{H}^1(A,\mathbb{Z}_2)\to\mathrm{H}^1(A\times A,\mathbb{Z}_2)$ with itself, we find that in terms of (2) the map $m^*:\mathrm{H}^2(A,\mathbb{Z}_2)\to\mathrm{H}^2(A\times A,\mathbb{Z}_2)$ is $(\mathrm{id},\mathrm{id},\iota)$. Thus $\delta:\mathrm{H}^2(A,\mathbb{Z}_2)\to\mathrm{H}^2(A\times A,\mathbb{Z}_2)$ is the map $(0\oplus 0\oplus\iota)$.

Write $V=\mathrm{H}^1(A,\mathbb{Z}_2)$. This is a free $\mathbb{Z}_2$-module of rank $2g$, where $g=\dim(A)$. Using the Kummer sequence and the Weil pairing we obtain canonical isomorphisms

$$V/2\cong\mathrm{H}^1(A,\mathbb{Z}/2)\cong A^\vee[2]\cong\mathrm{Hom}(A[2],\mathbb{Z}/2).\tag{3}$$

We have $\wedge^2V=(V^{\otimes 2})^\sigma\cong\mathrm{H}^2(A,\mathbb{Z}_2)$.

Applying the canonical isomorphism $\mathrm{NS}\,(A) \cong \mathrm{Hom}(A, A^\vee)^{\mathrm{sym}}$ to $A \times A$ instead of A, we obtain

$$\mathrm{NS}\,(A \times A) = \mathrm{NS}\,(A) \oplus \mathrm{NS}\,(A) \oplus \mathrm{Hom}(A, A^\vee); \tag{4}$$

see also [SZ14, Prop. 1.7] or Lemma 6.1 below. The map $m^*\colon \mathrm{NS}\,(A) \to \mathrm{NS}\,(A \times A)$ is obtained from the map $\mathrm{Hom}(A, A^\vee) \to \mathrm{Hom}(A \times A, A^\vee \times A^\vee)$ that sends ϕ to $m^\vee \circ \phi \circ m$. Using that $m^\vee$ is the diagonal map, we deduce that in terms of (4) the map $\delta = m^* - \pi_1^* - \pi_2^*$ is the natural inclusion $\mathrm{Hom}(A, A^\vee)^{\mathrm{sym}} \hookrightarrow \mathrm{Hom}(A, A^\vee)$. The involution σ acts on $\mathrm{NS}\,(A \times A)$ by permuting the first two direct summands in (4) and preserving the third one, on which it acts by $\phi \mapsto \phi^\vee$. The first Chern class map sends each direct summand in (4) to the corresponding direct summand in (2). For the last summand the first Chern class gives a map

$$\mathrm{Hom}(A, A^\vee)/2 \hookrightarrow (V/2)^{\otimes 2} \cong \mathrm{Hom}(A[2], A^\vee[2]) \tag{5}$$

where the isomorphism $(V/2)^{\otimes 2} \cong \mathrm{Hom}(A[2], \mathbb{Z}/2) \otimes_{\mathbb{Z}} A^\vee[2] \cong \mathrm{Hom}(A[2], A^\vee[2])$ comes from (3).

Remark 3.3 At least when $\mathrm{char}(k) = 0$, this map is induced by the natural action of homomorphisms of abelian varieties on torsion points. (If 2 is replaced by an arbitrary positive integer, the analogous map is the negative of the action on torsion points.) This is proved by applying [OSZ, Lemma 2.6] to $A \times A$.

Putting together the descriptions of δ for the cohomology groups and the Néron–Severi groups we obtain that $\mathrm{Br}_A(A)(p')$ is the kernel of the natural map

$$\frac{\wedge^2(V/2)}{\mathrm{NS}\,(A)/2} \longrightarrow \frac{(V/2)^{\otimes 2}}{\mathrm{Hom}(A, A^\vee)/2}. \tag{6}$$

In order to understand the definition of this map, we recall that $\wedge^2(V/2)$ and $\mathrm{Hom}(A, A^\vee)/2$ are subgroups of $(V/2)^{\otimes 2}$, using the map (5) for $\mathrm{Hom}(A, A^\vee)/2$.

We remind the reader that $\wedge^2 V = (V^{\otimes 2})^\sigma$ and $\mathrm{NS}\,(A) = \mathrm{Hom}(A, A^\vee)^\sigma$. Write

$$L = \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Z}_2.$$

This is a primitive $\mathbb{Z}_2$-sublattice of $V^{\otimes 2}$ by the proof of [Tat66, Lemma 1]. The involution σ acts on L and we have

$$\mathrm{NS}\,(A) \otimes \mathbb{Z}_2 = L^\sigma = L \cap \wedge^2 V \subset V^{\otimes 2}.$$

In the following proposition, we write $(L/2) \cap \wedge^2(V/2)$ (respectively $(L/4) \cap \wedge^2(V/4)$) for the intersection in $(V/2)^{\otimes 2}$ (respectively, in $(V/4)^{\otimes 2}$).

Proposition 3.4 *We have* $\mathrm{Br}_A(A)(p') = \frac{(L/2)\ \cap\ \wedge^2(V/2)}{(L \cap \wedge^2 V)/2} = \frac{(L/2)\ \cap\ \wedge^2(V/2)}{((L/4) \cap \wedge^2(V/4))/2}$.

Proof. The first equality is clear, so let us prove that the image of $L \cap \wedge^2 V$ in $(L/2) \cap \wedge^2(V/2)$ is equal to the (a priori, larger) image of $(L/4) \cap \wedge^2(V/4)$. Take any $x \in (L/4) \cap \wedge^2(V/4)$ and lift it to $\tilde{x} \in L$. Clearly, $\tilde{x} + \sigma(\tilde{x}) \in L^\sigma = L \cap \wedge^2 V$. Since $\tilde{x}$ and $\sigma(\tilde{x})$ are congruent modulo 2, we can write $\tilde{x} + \sigma(\tilde{x}) = 2v$ for some $v \in L^\sigma$. Furthermore, $\tilde{x}$ and $\sigma(\tilde{x})$ are congruent modulo 4, so v is congruent to $\tilde{x}$ modulo 2, that is, $v \in L \cap \wedge^2 V$ is a lifting of the image of x in $(L/2) \cap \wedge^2(V/2)$. □

Proposition 3.5 *Let $\mathbb{Z}/2$ be the group generated by σ. Then $\mathrm{Br}_A(A)(p')$ is the kernel of*

$$\mathrm{H}^1(\mathbb{Z}/2, L) \longrightarrow \mathrm{H}^1(\mathbb{Z}/2, V^{\otimes 2}) = V/2 \cong (\mathbb{Z}/2)^{2g}, \tag{7}$$

where σ acts on $L = \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Z}_2$ by $\phi \mapsto \phi^\vee$ and on $V^{\otimes 2}$ by $x \otimes y \mapsto -y \otimes x$.

Proof. Using (6) we identify $\mathrm{Br}_A(A)(p')$ with the kernel of the natural map

$$\frac{\wedge^2(V/2)}{L^\sigma/2} \longrightarrow \frac{((V/2)^{\otimes 2})^\sigma}{(L/2)^\sigma}. \tag{8}$$

Alternatively, $\mathrm{Br}_A(A)(p')$ is the kernel of the natural map

$$\frac{(L/2)^\sigma}{L^\sigma/2} \longrightarrow \frac{((V/2)^{\otimes 2})^\sigma}{\wedge^2(V/2)} = \frac{S^2(V/2)}{\wedge^2(V/2)} \xleftarrow{\sim} V/2, \tag{9}$$

where the last map is given by $x \mapsto x \otimes x$.

For any abelian group M such that $M[2] = 0$, we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{[2]} M \longrightarrow M/2 \longrightarrow 0.$$

If M is acted on by $\mathbb{Z}/2$ with generator σ, then the associated long exact sequence in cohomology leads to a further exact sequence

$$0 \longrightarrow M^\sigma/2 \longrightarrow (M/2)^\sigma \longrightarrow \mathrm{H}^1(\mathbb{Z}/2, M) \longrightarrow 0.$$

This gives the interpretation in terms of $\mathrm{H}^1(\mathbb{Z}/2, -)$ in (7). □

Remark 3.6 This remark uses Remark 3.3 so we assume $\mathrm{char}(k) = 0$. Via the natural identification $V/2 \cong \mathrm{Hom}(A[2], \mathbb{Z}/2)$, the map in (9) sends $u \in (\mathrm{Hom}(A, A^\vee)/2)^\sigma$ to the linear form on $A[2]$ whose value on $x \in A[2]$ is $e_{2,A}(x, ux)$, where $e_{2,A}(x, y)$ is the Weil pairing $A[2] \times A^\vee[2] \to \mathbb{Z}/2$.

Example 3.7 Suppose that $\mathrm{char}(k) > 2$ and A is a *supersingular* abelian variety [LO98]. Then it is well known that $\mathrm{End}(A)$ is a free $\mathbb{Z}$-module of rank $(2g)^2$. Since A and $A^\vee$ are isogenous, $\mathrm{Hom}(A, A^\vee)$ is also a free $\mathbb{Z}$-module of rank $(2g)^2$. This implies that L is a free $\mathbb{Z}_2$-module of rank $(2g)^2$. Taking into account that L is a primitive submodule in $V^{\otimes 2}$, and the latter is also a free $\mathbb{Z}_2$-module of the same rank $(2g)^2$, we conclude that $L = V^{\otimes 2}$. This implies that (7) is the identity map, so by Proposition 3.5 we have $\mathrm{Br}_A(A)(p') = 0$ if A is supersingular.

Corollary 3.8 *For abelian varieties A and B there is a canonical isomorphism*

$$\mathrm{Br}_{A\times B}(A\times B)(p') \cong \mathrm{Br}_A(A)(p') \oplus \mathrm{Br}_B(B)(p').$$

In particular, if A is the product of elliptic curves, then $\mathrm{Br}_A(A)(p') = 0$.

Proof. We have

$$\mathrm{Hom}(A\times B, A^\vee\times B^\vee) = \big(\mathrm{Hom}(A, A^\vee)\oplus\mathrm{Hom}(B, B^\vee)\big)\oplus\big(\mathrm{Hom}(A, B^\vee)\oplus\mathrm{Hom}(B, A^\vee)\big).$$

Here the second summand of the right hand side is a permutation $\mathbb{Z}/2$-module, hence it has trivial $\mathrm{H}^1(\mathbb{Z}/2, -)$ group. The displayed formula now follows from Proposition 3.5. The second statement follows from Remark 3.2. □

Proposition 3.9 *An isogeny of abelian varieties of odd degree $\phi : A \to B$ induces an isomorphism $\mathrm{Br}_A(A)(p') \cong \mathrm{Br}_B(B)(p')$.*

Proof. Let $m = \deg(\phi)$. There is an isogeny $\psi\colon B \to A$ such that

$$\psi\circ\phi = [m]_A, \qquad \phi\circ\psi = [m]_B.$$

By functoriality, ϕ and ψ induce homomorphisms of Brauer groups

$$\phi^*\colon \mathrm{Br}(B) \longrightarrow \mathrm{Br}(A), \qquad \psi^*\colon \mathrm{Br}(A) \longrightarrow \mathrm{Br}(B)$$

such that $\phi^*\psi^* = [m^2]_A$ and $\psi^*\phi^* = [m^2]_B$. We have

$$\phi^*(\mathrm{Br}_B(B)) \subset \mathrm{Br}_A(A), \qquad \psi^*(\mathrm{Br}_A(A)) \subset \mathrm{Br}_B(B).$$

Since m is odd and both $\mathrm{Br}_A(A)(p')$ and $\mathrm{Br}_B(B)(p')$ are annihilated by 2, the map ϕ^* induces an isomorphism $\mathrm{Br}_B(B)(p') \xrightarrow{\sim} \mathrm{Br}_A(A)(p')$. □

Remark 3.10 Recall [LO98, pp. 14–15] that if $\mathrm{char}(k) = p > 2$ and A is a *supersingular* abelian variety then there is a *minimal isogeny* $\phi\colon A \to B$ such that $\deg(\phi)$ is a power of p and B is a product of (supersingular) elliptic curves. Now Proposition 3.9 combined with Corollary 3.8 gives another proof that $\mathrm{Br}_A(A)(p') = 0$.

Corollary 3.11 *Suppose that $\mathrm{char}(k) = 0$. Let A be an abelian variety isogenous to a power of an elliptic curve with complex multiplication. Then $\mathrm{Br}_A(A) = 0$.*

Proof. By a result of C. Schoen [Sch92] in this case A is isomorphic to a product of simple abelian varieties, which necessarily are elliptic curves. We conclude by applying Corollary 3.8. □

Remark 3.12 If $\mathrm{End}(A)\otimes\mathbb{Q}$ is a direct sum of totally real fields, for example, when A is simple and has Albert type I, then σ acts trivially on $\mathrm{End}(A)$ hence also on $\mathrm{Hom}(A, A^\vee)$, so $\mathrm{Br}_A(A)(p') = 0$. In particular, for a general abelian variety (when $\mathrm{End}(A) = \mathbb{Z}$) we have $\mathrm{Br}_A(A)(p') = 0$. Similarly, for a general complex torus (when $\mathrm{Hom}(A, A^\vee) = 0$) we have $\mathrm{Br}_A(A) = 0$.

Remark 3.13 Let $\mathcal{A}_g$ be the coarse moduli space of principally polarised abelian varieties of dimension $g \geq 2$ over $\mathbb{C}$. The symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts transitively on the Siegel upper half-plane $\mathfrak{H}_g$, and we have $\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}(2g, \mathbb{Z}) \setminus \mathfrak{H}_g$.

Let (A, λ) and (B, μ) be complex principally polarised g-dimensional abelian varieties. One says that (A, λ) and (B, μ) are in the same *Hecke orbit* if there exist a g-dimensional complex abelian variety C and isogenies $\alpha : C \to A$ and $\beta : C \to B$ such that the polarisations $\alpha^*\lambda$ and $\beta^*\mu$ of C are equal [CO09, pp. 441–442]. If neither $\deg(\alpha)$ nor $\deg(\beta)$ is divisible by a prime p, then one says that (A, λ) and (B, μ) are in the same *prime-to-p Hecke orbit*. It follows from Proposition 3.9 that if (A, λ) and (B, μ) are in the same prime-to-2 Hecke orbit, then $\mathrm{Br}_A(A) \cong \mathrm{Br}_B(B)$.

It is known [Ch05, p. 92] that each prime-to-p Hecke orbit is dense in the complex topology of $\mathcal{A}_g(\mathbb{C})$. Indeed, consider the group scheme $G = \mathrm{Sp}(2g)$ over $\mathbb{Z}$. The connected semisimple algebraic group $G_{\mathbb{Q}}$ over $\mathbb{Q}$ is simply connected so it satisfies weak approximation [PR91, Prop. 7.9]. In particular, $G(\mathbb{Q})$ is dense in $G(\mathbb{R}) \times G(\mathbb{Q}_p)$, and since $G(\mathbb{Z}_p)$ is open in $G(\mathbb{Q}_p)$, we see that $G(\mathbb{Q}) \cap G(\mathbb{Z}_p)$ is dense in $G(\mathbb{R})$. But a prime-to-p Hecke orbit in $\mathcal{A}_g$ is the image of a $G(\mathbb{Q}) \cap G(\mathbb{Z}_p)$-orbit in $\mathfrak{H}_g$. (Moreover, a prime-to-p Hecke orbit is equidistributed in $\mathcal{A}_g$ with respect to the normalised Haar measure, as follows from equidistribution of Hecke points in $\mathrm{Sp}(2g, \mathbb{Z}) \setminus \mathrm{Sp}(2g, \mathbb{R})$, see [COU01, Thm. 1.6].)

Taking $p = 2$ and applying Corollary 3.8 together with Remark 5.1 below, we see that the set of (isomorphism classes of) principally polarised complex g-dimensional abelian varieties A such that $\mathrm{Br}_A(A) \neq 0$ is dense in the complex topology of $\mathcal{A}_g(\mathbb{C})$. The same applies to abelian varieties A with $\mathrm{Br}_A(A) = 0$.

3.2 Upper bound

Write $H = \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Q}$ and $H^- = \{\phi \in H | \phi^\vee = -\phi\}$. Note that the choice of a polarisation of A induces an isomorphism of $\mathbb{Q}$-vector spaces $H \cong \mathrm{End}(A) \otimes \mathbb{Q}$ under which $\phi \mapsto \phi^\vee$ corresponds to the Rosati involution, see [Mum74, Ch. 20].

The following proposition gives an upper bound for the size of $\mathrm{Br}_A(A)(p')$. The right hand side of this inequality is isogeny-invariant.

Proposition 3.14 $\dim_{\mathbb{F}_2}(\mathrm{Br}_A(A)(p')) \leq \dim_{\mathbb{Q}}(H^-) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(A)) - \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}\,(A))$.

Proof. From (9), we have

$$\mathrm{Br}_A(A)(p') \subset \frac{(L/2)^\sigma}{L^\sigma/2} \subset \frac{L/2}{L^\sigma/2}.$$

Since L^σ is primitive in L, we have

$$\dim_{\mathbb{F}_2}\left(\frac{L/2}{L^\sigma/2}\right) = \mathrm{rk}_{\mathbb{Z}_2}(L) - \mathrm{rk}_{\mathbb{Z}_2}(L^\sigma) = \dim_{\mathbb{Q}}(H) - \dim_{\mathbb{Q}}(H^\sigma) = \dim_{\mathbb{Q}}(H^-).$$

It is clear that $\dim_{\mathbb{Q}}(H^-) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(A)) - \mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}\,(A))$. $\square$

The following lemma is needed for the proof of Proposition 3.16 below. It is due to G. Shimura [Shi63, Prop. 15, p. 177], see also [Oort88, Subsection (4.1), p. 488].

Lemma 3.15 *Let A be a simple positive-dimensional abelian variety over an algebraically closed field k of characteristic 0. Suppose that A is of type III in Albert's classification, that is,* $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ *is a totally definite quaternion algebra over a totally real number field F. Then we have*

$$2\dim(A) = m \dim_{\mathbb{Q}}(\mathrm{End}^0(A)) = 4m[F:\mathbb{Q}] = 4m\,\mathrm{rk}(\mathrm{NS}\,(A))$$

for some integer $m \geq 2$.

Proof. Withous loss of generality we may assume that $k = \mathbb{C}$. Let us choose a polarisation on A. The associated Rosati involution is the standard involution on the quaternion F-algebra $\mathrm{End}^0(A)$, hence the space of symmetric elements of $\mathrm{End}^0(A)$ is $F \subset \mathrm{End}^0(A)$, see [Mum74, Ch. 21]. In particular, $\mathrm{rk}(\mathrm{NS}\,(A)) = [F:\mathbb{Q}]$.

It is known [Mum74, Ch. 21] that $\dim_{\mathbb{Q}}(\mathrm{End}^0(A))$ divides $2\dim(A)$, so $m \in \mathbb{Z}$, $m \geq 1$. If $m = 1$, then [Shi63, Prop. 15, p. 177] implies that A is isogenous to the square of a certain abelian variety. In particular, A is not simple, which contradicts our assumption. $\square$

Proposition 3.16 *Suppose that* $\mathrm{char}(k) = 0$*. Let the abelian variety A be isogenous to $A_1^{n_1} \times \cdots \times A_m^{n_m}$ where the A_i are simple and pairwise non-isogenous. Then* $\dim_{\mathbb{Q}}(H^-) \leq \max(n_i)\dim(A)$.

Proof. The endomorphism algebra of A is described by

$$\mathrm{End}^0(A) \cong \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(D_i) \tag{10}$$

where $D_i = \mathrm{End}^0(A_i)$ are division algebras.

Choose a polarisation on A which pulls back to a product of polarisations on the A_i. The associated Rosati involution preserves the decomposition (10) and acts on $\mathrm{Mat}_{n_i}(D_i)$ as the composition of matrix transpose with the entry-wise Rosati involution of D_i. Hence

$$\dim_{\mathbb{Q}}(H^-) = \sum_{i=1}^{m} \Big(\frac{n_i(n_i-1)}{2} \dim_{\mathbb{Q}}(D_i) + n_i \dim_{\mathbb{Q}}(D_i^-) \Big).$$

The division algebras which occur as the endomorphism algebras of simple abelian varieties were classified by Albert, see [Mum74, Ch. 21]. From this classification, we obtain the following bounds for the dimension.

Endomorphism type	Maximum value of $\dim_{\mathbb{Q}}(D_i)$	$\dim_{\mathbb{Q}}(D_i^-)$
I	$\dim(A_i)$	0
II	$2\dim(A_i)$	$\frac{1}{4}\dim_{\mathbb{Q}}(D_i)$
III	$\dim(A_i)$	$\frac{3}{4}\dim_{\mathbb{Q}}(D_i)$
IV	$2\dim(A_i)$	$\frac{1}{2}\dim_{\mathbb{Q}}(D_i)$

Most of this table is based on [Mum74, p. 202], noting that Mumford's η is equal to $1 - \dim_{\mathbb{Q}}(D_i^-)/\dim_{\mathbb{Q}}(D_i)$. The only entry in this table which is not taken from [Mum74, p. 202] is the maximum value of $\dim(D_i)$ for type III. In this case the desired result follows from Lemma 3.15, which gives us a better bound than [Mum74] for type III.

From the table, we deduce that

$$\dim_{\mathbb{Q}}(D_i) \leq 2\dim(A_i), \quad \dim_{\mathbb{Q}}(D_i^-) \leq \dim(A_i).$$

Hence

$$\frac{n_i(n_i-1)}{2}\dim_{\mathbb{Q}}(D_i) + n_i \dim_{\mathbb{Q}}(D_i^-) \leq n_i^2 \dim(A_i) = n_i \dim(A_i^{n_i}).$$

Summing this over the isotypic factors of A proves the proposition. □

Corollary 3.17 *Suppose that* $\mathrm{char}(k) = 0$*. If* A *is an abelian variety over* k *of dimension* 2*, then* $\dim_{\mathbb{F}_2}(\mathrm{Br}_A(A)) \leq 2$*.*

Proof. By Proposition 3.16, this is true whenever A is simple or is isogenous to a product of non-isogenous elliptic curves. It remains to check the case where A is isogenous to the square of an elliptic curve E.

If $\mathrm{End}(E) = \mathbb{Z}$, then $\mathrm{End}^0(A) \cong \mathrm{Mat}_2(\mathbb{Q})$ and the Rosati involution is transposition. So $\dim_{\mathbb{Q}}(H^-) = 1$.

If E has complex multiplication, then $\dim_{\mathbb{Q}}(H^-) = 4$ so we cannot deduce the corollary from Proposition 3.16. In this case $\mathrm{Br}_A(A) = 0$ by Corollary 3.11. □

3.3 Calculating $\mathrm{Br}_A(A)$ for complex tori

A *complex structure* on a real vector space W is an element $J \in \mathrm{End}(W)$ satisfying $J^2 = -1$. Giving a complex structure on W is equivalent to giving a complex vector space whose underlying real vector space is equal to W, by letting J represent the action of i on the complex vector space.

Let A be a complex torus of dimension g. By definition, $A = V/\Lambda$ where V is a complex vector space of dimension g and Λ is a discrete free abelian subgroup of V of rank $2g$. We can naturally identify Λ with $\mathrm{H}_1(A, \mathbb{Z})$. We also identify V with $\Lambda_{\mathbb{R}}$ equipped with a complex structure J.

Let A' be another complex torus with $\Lambda' = \mathrm{H}_1(A', \mathbb{Z})$ and with associated complex structure $J' \in \mathrm{End}(\Lambda'_{\mathbb{R}})$. Then each homomorphism $A \to A'$ induces a homomorphism of homology groups $\Lambda \to \Lambda'$. Conversely, a homomorphism $f\colon \Lambda \to \Lambda'$

comes from a homomorphism of complex tori if and only if it intertwines the complex structures on $\Lambda_{\mathbb{R}}$ and $\Lambda'_{\mathbb{R}}$. In other words, there is a natural bijection

$$\mathrm{Hom}(A, A') = \{f \in \mathrm{Hom}(\Lambda, \Lambda') | f \circ J = J' \circ f\}. \tag{11}$$

The *dual complex torus* of A is $A^\vee = \Lambda_{\mathbb{R}}^\vee / \Lambda^\vee$, where $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ and the complex structure of $\Lambda_{\mathbb{R}}^\vee$ is given by $-J^*$ [Kem91, p. 5].

Write $\mathrm{Bi}(\Lambda)$ for the set of bilinear forms $\Lambda \times \Lambda \to \mathbb{Z}$ and write $\mathrm{Bi}_J(\Lambda)$ for the set of J-invariant bilinear forms, that is, bilinear forms $B \in \mathrm{Bi}(\Lambda)$ for which the induced form on $\Lambda_{\mathbb{R}}$ satisfies $B(x, y) = B(Jx, Jy)$ for all $x, y \in \Lambda_{\mathbb{R}}$. Because $J^2 = -1$, this is equivalent to saying

$$\mathrm{Bi}_J(\Lambda) = \{B \in \mathrm{Bi}(\Lambda) | B(Jx, y) = B(x, -Jy) \text{ for all } x, y \in \Lambda_{\mathbb{R}}\}. \tag{12}$$

There is a natural bijection

$$\mathrm{Hom}(\Lambda, \Lambda^\vee) \to \mathrm{Bi}(\Lambda)$$

which sends $f \in \mathrm{Hom}(\Lambda, \Lambda^\vee)$ to $B_f(x, y) = f(x)(y)$. By (12), we have $B_f \in \mathrm{Bi}_J(\Lambda)$ if and only if $f \circ J = -J^* \circ f$.

Hence, using (11) for $A' = A^\vee$, we obtain natural bijections

$$\mathrm{Hom}(A, A^\vee) = \{f \in \mathrm{Hom}(\Lambda, \Lambda^\vee) | f \circ J = -J^* \circ f\} = \mathrm{Bi}_J(\Lambda). \tag{13}$$

Under these bijections, the involution σ of $\mathrm{Hom}(A, A^\vee)$ given by $\phi \mapsto \phi^\vee$ corresponds to $-\tau$, where τ is the involution of $\mathrm{Bi}_J(\Lambda)$ which exchanges the arguments of a bilinear form. (This is due to the fact that canonical pairings $\Lambda \times \Lambda^\vee \to \mathbb{Z}$ and $\Lambda^\vee \times \Lambda \to \mathbb{Z}$, where we identified $(\Lambda^\vee)^\vee$ with Λ, differ by sign.) In particular, $\mathrm{NS}\,(A) = \mathrm{Hom}(A, A^\vee)^{\mathrm{sym}}$ corresponds to the alternating forms in $\mathrm{Bi}_J(\Lambda)$.

An element $\mathrm{Hom}(A, A^\vee)^{\mathrm{sym}}$ is a *polarisation* if and only if the corresponding alternating form $E \in \mathrm{Bi}_J(\Lambda)$ satisfies the following condition:

$$\text{the symmetric bilinear form } E(Jx, y) \text{ is positive definite.} \tag{14}$$

Note that $E(Jx, y)$ is the real part of the Hermitian form attached to E as in [Mum74, p. 19, Lemma]. Recall that a complex torus A is an abelian variety if and only if there exists a polarisation of A [Mum74, p. 35, Corollary]. A complex torus is said to be *non-algebraisable* if it does not possess a polarisation.

Lemma 3.18 *If* $\mathrm{Bi}_J(\Lambda)$ *contains a non-degenerate bilinear form (which is always true when* A *is an abelian variety), then*

$$\mathrm{rk}_{\mathbb{Z}}(\mathrm{Bi}_J(\Lambda)) = \mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(A)).$$

Proof. Since $\mathrm{Bi}_J(\Lambda)$ and $\mathrm{End}(A)$ are both free $\mathbb{Z}$-modules, it suffices to prove that $\dim_{\mathbb{Q}}(\mathrm{Bi}_J(\Lambda_{\mathbb{Q}})) = \dim_{\mathbb{Q}}(\mathrm{End}(A) \otimes \mathbb{Q})$.

Let B be a non-degenerate form in $\mathrm{Bi}_J(\Lambda)$. Then every bilinear form on $\Lambda_{\mathbb{Q}}$ can be written as $(x, y) \mapsto B(x, uy)$ for some $u \in \mathrm{End}(\Lambda_{\mathbb{Q}})$. The form $B(x, uy)$ is in $\mathrm{Bi}_J(\Lambda_{\mathbb{Q}})$ if and only if u commutes with J. The lemma is proved because $\mathrm{End}(A) \otimes \mathbb{Q} = \mathrm{End}_J(\Lambda_{\mathbb{Q}})$. $\square$

A bilinear form $B\colon \Lambda \times \Lambda \to \mathbb{Z}$ is said to be *even* if $B(x, x) \equiv 0 \bmod 2$ for all $x \in \Lambda$. Write $\mathrm{Bi}_J(\Lambda)^{\mathrm{sym,even}}$ for the set of symmetric, even forms in $\mathrm{Bi}_J(\Lambda)$.

Write $\Lambda_2 = \Lambda \otimes \mathbb{Z}_2$. Then $\mathrm{Bi}_J(\Lambda_2) = \mathrm{Bi}_J(\Lambda) \otimes \mathbb{Z}_2$ and $\mathrm{Bi}_J(\Lambda_2)^{\mathrm{sym,even}} = \mathrm{Bi}_J(\Lambda)^{\mathrm{sym,even}} \otimes \mathbb{Z}_2$.

Proposition 3.19 $\mathrm{Br}_A(A) \cong \mathrm{Bi}_J(\Lambda)^{\mathrm{sym,even}}/(1+\tau)\mathrm{Bi}_J(\Lambda)$.

Proof. This is a consequence of Proposition 3.5. Considering the action of $\mathbb{Z}/2$ on $L = \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Z}_2$ by σ, we have

$$\mathrm{H}^1(\mathbb{Z}/2, L) = L^{-\sigma}/(1-\sigma)L = \mathrm{Bi}_J(\Lambda_2)^{\mathrm{sym}}/(1+\tau)\mathrm{Bi}_J(\Lambda_2).$$

Note that $V = \mathrm{H}^1(A, \mathbb{Z}_2)$ can be canonically identified with $\Lambda_2^\vee$ and hence $V^{\otimes 2}$ can be identified with the set of bilinear maps $\Lambda_2 \times \Lambda_2 \to \mathbb{Z}_2$. The bijection (13) identifies the inclusion $L \subset V^{\otimes 2}$ with the inclusion $\mathrm{Bi}_J(\Lambda_2) \to V^{\otimes 2}$.

Consequently Proposition 3.5 tells us that $\mathrm{Br}_A(A)$ is isomorphic to the kernel of the natural map

$$\mathrm{Bi}_J(\Lambda_2)^{\mathrm{sym}}/(1+\tau)\mathrm{Bi}_J(\Lambda_2) \longrightarrow V/2$$

which sends $B \in \mathrm{Bi}_J(\Lambda_2)^{\mathrm{sym}}$ to the function $x \mapsto B(x, x) \bmod 2$. In other words, the kernel of this map consists precisely of the even forms.

Since

$$2\mathrm{Bi}_J(\Lambda)^{\mathrm{sym,even}} \subset 2\mathrm{Bi}_J(\Lambda)^{\mathrm{sym}} \subset (1+\tau)\mathrm{Bi}_J(\Lambda),$$

we have $\mathrm{Bi}_J(\Lambda_2)^{\mathrm{sym,even}}/(1+\tau)\mathrm{Bi}_J(\Lambda_2) = \mathrm{Bi}_J(\Lambda)^{\mathrm{sym,even}}/(1+\tau)\mathrm{Bi}_J(\Lambda)$. $\square$

In many of our examples, we will look at abelian varieties A for which $K = \mathrm{End}(A) \otimes \mathbb{Q}$ is a product of CM fields satisfying $\dim_{\mathbb{Q}}(K) = \dim(A)$. This includes simple abelian varieties of CM type. In this case, we can identify $\Lambda_{\mathbb{Q}}$ with K. Consequently, for each $\alpha \in K$, it makes sense to define a bilinear form

$$B_\alpha\colon \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}, \quad B_\alpha(x, y) = \mathrm{tr}_{K/\mathbb{Q}}(\alpha x \bar{y}).$$

Let $D_K(\Lambda) = \{\alpha \in K | B_\alpha(\Lambda \times \Lambda) \subset \mathbb{Z}\}$. Note that if Λ is a complex conjugation-invariant subring of K, then we can simplify the calculation further as in this case,

$$D_K(\Lambda) = \{\alpha \in K | \mathrm{tr}_{K/\mathbb{Q}}(\alpha x) \in \mathbb{Z} \text{ for all } x \in \Lambda\}. \tag{15}$$

The bilinear forms B_α are J-invariant, so thanks to Lemma 3.18 we have

$$\mathrm{Bi}_J(\Lambda) = \{B_\alpha | \alpha \in D_K(\Lambda)\}.$$

Note also that $\tau(B_\alpha) = B_{\iota(\alpha)}$ where ι denotes complex conjugation in K. Writing $D_K(\Lambda)^{\iota,\text{even}} = \{\alpha \in D_K(\Lambda) | B_\alpha \text{ is symmetric and even}\}$, we conclude from Proposition 3.19 that

$$\mathrm{Br}_A(A) \cong D_K(\Lambda)^{\iota,\text{even}}/(1+\iota)D_K(\Lambda). \tag{16}$$

4 Complex tori with non-zero invariant Brauer group

In this section we construct examples of simple complex tori of any dimension $g \geq 3$ with invariant Brauer group $\mathbb{Z}/2$. We show that these complex tori are simple by calculating their endomorphism algebras. We construct examples of two types:

(i) non-algebraisable complex tori with endomorphism ring $\mathbb{Z}$;

(ii) abelian varieties with endomorphism algebra an imaginary quadratic field.

Note that it follows from Proposition 3.14 that an abelian variety with endomorphism ring $\mathbb{Z}$ always has trivial invariant Brauer group, since an abelian variety has $\mathrm{rk}_{\mathbb{Z}}(\mathrm{NS}\ (A)) \geq 1$. Thus the examples (ii) have the smallest possible endomorphism algebras for abelian varieties with non-zero invariant Brauer group.

We use the following notation. The identity matrix of size n is denoted by I_n. We denote by $(1_m, -1_n)$ the diagonal matrix with m diagonal entries 1 and n diagonal entries -1, in this order. We write

$$\mathfrak{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We use the direct sum symbol $\oplus$ for the direct sum of matrices. For example, $\mathfrak{J}^{\oplus n}$ is the square matrix of size $2n$ which is the direct sum of n copies of $\mathfrak{J}$.

4.1 Complex tori with no non-trivial endomorphisms

The following theorem is the algebraic result which underlies our construction of non-algebraisable complex tori with endomorphism ring $\mathbb{Z}$ and invariant Brauer group $\mathbb{Z}/2$. The element J is the complex structure giving rise to the desired torus, and the symmetric bilinear form S is a generator for $\mathrm{Bi}_J(\Lambda)$ which we use in calculating the invariant Brauer group using Proposition 3.19. The precursors of this theorem are [Zar91, Lemma 2] and [OZ95, Lemma 2.5].

Theorem 4.1 *Let $g \geq 3$ and let Λ be a free abelian group of rank $2g$ with a non-degenerate symmetric bilinear form $S : \Lambda \times \Lambda \to \mathbb{Z}$. Let $\mathrm{SO}(\Lambda_{\mathbb{R}}, S) \subset \mathrm{End}(\Lambda_{\mathbb{R}})$ be the special orthogonal group defined by S. If the signature of S is congruent to $2g \bmod 4$, then there is an element $J \in \mathrm{SO}(\Lambda_{\mathbb{R}}, S)$ such that $J^2 = -1$ and the centraliser of J in $\mathrm{End}(\Lambda_{\mathbb{Q}})$ is $\mathbb{Q} \cdot \mathrm{Id}_\Lambda$.*

Proof. Let $\mathfrak{g} = \mathfrak{o}(\Lambda_{\mathbb{R}}, S)$ be the Lie algebra of the real Lie group $G = \mathrm{SO}(\Lambda_{\mathbb{R}}, S)$. Recall that

$$\mathfrak{g} = \{A \in \mathrm{End}(\Lambda_{\mathbb{R}}) | S(Ax, y) + S(x, Ay) = 0 \text{ for all } x, y \in \Lambda_{\mathbb{R}}\}.$$

It is well known that G can be given the structure of a *complete metric space* such that the group operations in G are continuous. For example, choosing a basis of the real vector space $\Lambda_{\mathbb{R}}$ we identify G with a closed subset of the algebra of matrices $\mathrm{Mat}_{2g}(\mathbb{R}) \simeq \mathbb{R}^{(2g)^2}$ with its usual Euclidean metric.

Let $C = \{J \in G | J^2 = -1\}$. This is the set of all complex structures which preserve the symmetric bilinear form S. Our strategy is to show that, for very general $J \in C$, $\mathrm{End}_J(\Lambda_{\mathbb{Q}})$ contains no elements other than $\mathbb{Q} \cdot \mathrm{Id}_{\Lambda}$.

The set C is non-empty. Indeed, there is a basis of $\Lambda_{\mathbb{R}}$ with respect to which S is given by the diagonal matrix $(1_m, -1_n)$. We have $m+n = 2g$ and $m-n \equiv 2g \bmod 4$ (by the condition on signature in the statement of the theorem), hence m and n are both even. Then the element of $\mathrm{End}(\Lambda_{\mathbb{R}})$ whose matrix in this basis is $\mathfrak{J}^{\oplus g}$ is in C, hence $C \neq \emptyset$.

We note that C is invariant under conjugations by the elements of G. We also note that $C \subset \mathfrak{g}$, because for any $J \in C$ and any $x, y \in \Lambda_{\mathbb{R}}$ we have

$$S(Jx, y) + S(x, Jy) = S(Jx, y) + S(J^{-1}x, y) = S(Jx, y) + S(-Jx, y) = 0.$$

Since C is a closed subset of G, it inherits the structure of a complete metric space. We shall show that, for each non-scalar endomorphism $u \in \mathrm{End}(\Lambda_{\mathbb{Q}})$, the closed subset $C_u = \{c \in C | cu = uc\}$ is nowhere dense in C. This suffices to prove the theorem. Indeed, by Baire's theorem a countable intersection of dense open subsets of a complete metric space is dense. Hence there exist elements $J \in C$ which are in the complement to all C_u for non-scalar $u \in \mathrm{End}(\Lambda_{\mathbb{Q}})$. Any such J commutes only with $\mathbb{Q} \cdot \mathrm{Id}_{\Lambda}$ in $\mathrm{End}(\Lambda_{\mathbb{Q}})$, as required.

Thus let $u \in \mathrm{End}(\Lambda_{\mathbb{Q}})$ be a non-scalar endomorphism. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of the Lie group G and let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of the Lie algebra $\mathfrak{g}$.

Suppose for contradiction that C_u has an interior point c, so there is an open neighbourhood U of $c \in C$ such that $U \subset C_u$. Let n be a positive integer. Recall that $C \subset \mathfrak{g}$. The function $G^n \to C$ sending $(g_1, \ldots, g_n)$ to

$$\mathrm{Ad}(g_n) \ldots \mathrm{Ad}(g_1)c = g_n \ldots g_1 c g_1^{-1} \ldots g_n^{-1}$$

is continuous, so there is an open neighbourhood G_0 of the unit element $e \in G$ such that $\mathrm{Ad}(g_n) \ldots \mathrm{Ad}(g_1)c \in U \subset C_u$ provided $g_i \in G_0$ for $i = 1, \ldots, n$. Hence $[\mathrm{Ad}(g_n) \ldots \mathrm{Ad}(g_1)c, u] = 0$. This implies

$$[\mathrm{ad}(x_n) \ldots \mathrm{ad}(x_1)c, u] = [[x_n, [x_{n-1}, \ldots [x_1, c] \ldots]], u] = 0$$

for any $x_i \in \mathfrak{g}$. This holds for any $n \geq 1$, hence the ideal of the Lie algebra $\mathfrak{g}$ generated by c commutes with u.

Recall that $\mathfrak{g} = \mathfrak{o}(\Lambda_{\mathbb{R}}, S)$ is an orthogonal Lie algebra of rank $2g$. Since the rank is at least 6, the Lie algebra $\mathfrak{o}(\Lambda_{\mathbb{R}}, S)$ is (absolutely) simple. Therefore this ideal must be all of $\mathfrak{g}$, thus $\mathfrak{g}$ commutes with u. However, the representation $\mathfrak{g} \to \mathfrak{gl}(\Lambda_{\mathbb{R}})$ is absolutely irreducible, thus u is scalar, which is a contradiction. $\square$

Theorem 4.2 *For every integer $g \geq 3$, there exists a non-algebraisable complex torus A of dimension g with* $\mathrm{End}(A) \cong \mathbb{Z}$ *and* $\mathrm{Br}_A(A) \cong \mathbb{Z}/2$.

Proof. Let Λ be a free abelian group of rank $2g$. Let S be a non-degenerate, primitive, even symmetric bilinear form $S : \Lambda \times \Lambda \to \mathbb{Z}$ of signature congruent to $2g$ mod 4, for example the form with matrix

$$2I_{2g-4} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 2}.$$

By Theorem 4.1, if $g \geq 3$, we can find a $J \in \mathrm{SO}(\Lambda_{\mathbb{R}}, S)$ such that $J^2 = -1$ and the centraliser of J in $\mathrm{End}(\Lambda_{\mathbb{Q}})$ is $\mathbb{Q} \cdot \mathrm{Id}_{\Lambda}$.

Let A be the complex torus $\Lambda_{\mathbb{R}}/\Lambda$, with complex structure J. Thanks to (11), $\mathrm{End}(A) = \mathrm{End}_J(\Lambda) \cong \mathbb{Z}$.

Since $J \in \mathrm{SO}(\Lambda_{\mathbb{R}}, S)$, we have $S \in \mathrm{Bi}_J(\Lambda)$. By Lemma 3.18, $\mathrm{Bi}_J(\Lambda) \otimes \mathbb{Q} = \mathbb{Q}S$. Since S is a primitive form, we conclude that $\mathrm{Bi}_J(\Lambda) = \mathbb{Z}S$.

Since S is symmetric, $(1 + \tau)\mathrm{Bi}_J(\Lambda) = 2\mathbb{Z}S$. Since also S is even, we conclude by Proposition 3.19 that $\mathrm{Br}_A(A) = \mathbb{Z}/2$.

Note that $\mathrm{Bi}_J(\Lambda)$ contains no alternating forms, so $\mathrm{NS}\,(A) = \emptyset$ and hence A is non-algebraisable. $\square$

4.2 Simple abelian varieties with non-zero invariant Brauer group

The construction of simple abelian varieties with invariant Brauer group $\mathbb{Z}/2$ follows a more sophisticated version of the strategy from Section 4.1. We begin with an algebraic result. In this theorem, J is the complex structure associated with our desired abelian variety A, S is an element of $\mathrm{Bi}_J(\Lambda)$ (indeed S is ultimately a representative of the non-zero class in $\mathrm{Br}_A(A)$) and J_0 is a generator of $\mathrm{End}(A)$.

Theorem 4.3 *Let $g \geq 3$ be a positive integer. Let Λ_1 and Λ_2 be free abelian groups of rank 2 and $2g - 2$, respectively. Let $S_i : \Lambda_i \times \Lambda_i \to \mathbb{Z}$ be a positive-definite symmetric bilinear form and let $J_i \in \mathrm{SO}(\Lambda_{i,\mathbb{R}}, S_i) \cap \mathrm{End}(\Lambda_i)$ be such that $J_i^2 = -\mathrm{Id}_{\Lambda_i}$, for $i = 1, 2$. Let S be the symmetric bilinear form on Λ which is the orthogonal direct sum of $-S_1$ and S_2. Let $J_0 = J_1 \oplus J_2 \in \mathrm{SO}(\Lambda_{\mathbb{R}}, S) \cap \mathrm{End}(\Lambda)$. Then there is an element $J \in \mathrm{SO}(\Lambda_{\mathbb{R}}, S)$ such that $J^2 = -1$, the centraliser of J in $\mathrm{End}(\Lambda_{\mathbb{Q}})$ is $\mathbb{Q}\,\mathrm{Id}_{\Lambda} + \mathbb{Q}J_0$, and the symmetric bilinear form $S(Jx, J_0 y)$ is positive-definite.*

Proof. Write $E(x,y) = S(x, J_0 y)$. Then E is an alternating form $\Lambda \times \Lambda \to \mathbb{Z}$ such that $E(x,y) = E(J_0 x, J_0 y)$.

Let G be the centraliser of J_0 in $\mathrm{SO}(\Lambda_\mathbb{R}, S)$. We turn $\Lambda_\mathbb{R}$ into a complex vector space V via the complex structure J_0. (This is not the complex structure we shall ultimately use to construct an abelian variety.) Then $G \subset \mathrm{GL}(V)$ is equal to the group of complex linear transformations preserving the Hermitian form $H(x,y) = S(x,y) + iE(x,y)$, that is, the unitary group $\mathrm{U}(V,H)$. As before, the real Lie group G can be given the structure of a complete metric space such that the group operations in G are continuous.

Let $\mathfrak{g} = \mathfrak{u}(V,H)$ be the Lie algebra of the real Lie group G. Recall that

$$\begin{aligned}\mathfrak{g} &= \{A \in \mathrm{End}_\mathbb{C}(\Lambda_\mathbb{R}) | H(Ax,y) + H(x,Ay) = 0 \text{ for all } x, y \in \Lambda_\mathbb{R}\} \\ &= \{A \in \mathrm{End}_\mathbb{R}(\Lambda_\mathbb{R}) | AJ_0 = J_0 A,\ S(Ax,y) + S(x,Ay) = 0 \text{ for all } x, y \in \Lambda_\mathbb{R}\}.\end{aligned}$$

Define C as the complement to $\{\pm J_0\}$ in

$$\{J \in \mathrm{SO}(\Lambda_\mathbb{R}, S) | J^2 = -1,\ JJ_0 = J_0 J\} = \{J \in \mathrm{U}(V,H) | J^2 = -1\}.$$

We exclude $\pm J_0$ here because they are isolated points in $\{J \in \mathrm{U}(V,H) | J^2 = -1\}$. This implies that C is a closed subset of G and therefore is a complete metric space, so we may use Baire's theorem as we did before.

For any $J \in C$ the bilinear form $S(Jx, J_0 y)$ is symmetric. Thus the set

$$C^+ = \{J \in C | S(Jx, J_0 x) > 0 \text{ for all } x \neq 0\}$$

is open in C. To show that $C^+ \neq \emptyset$ we exhibit an element of C^+. Namely, the linear transformation $J_0^\sharp = (-J_1) \oplus J_2$ preserves S, has determinant 1 and commutes with $J_0 = J_1 \oplus J_2$. The symmetric form $S(J_0^\sharp x, J_0 y)$ is the orthogonal direct sum of $S_1(x,y)$ and $S_2(x,y)$, hence is positive-definite. Thus $J_0^\sharp \in C^+$, so C^+ is a non-empty open subset of C.

As before, C is a subset of $\mathfrak{g}$ invariant under conjugation by the elements of G.

For $u \in \mathrm{End}(\Lambda_\mathbb{Q})$ let $C_u = \{c \in C | cu = uc\}$. We claim that if C_u has an interior point c, then $u \in \mathbb{Q}\,\mathrm{Id}_\Lambda + \mathbb{Q}J_0$. Indeed, arguing as in the proof of Theorem 4.1 we obtain that the ideal of $\mathfrak{g}$ generated by c commutes with u. The Lie algebra $\mathfrak{g}$ is the direct sum of its centre $Z(\mathfrak{g})$ and the simple Lie algebra $\mathfrak{su}(V,S)$. We have $Z(\mathfrak{g}) = i\mathbb{R}\mathrm{Id}_V = \mathbb{R}J_0$. The only $J \in \mathbb{R}J_0$ which satisfy $J^2 = -1$ are $\pm J_0$, which are not in C, so $c \notin Z(\mathfrak{g})$. Thus the ideal of $\mathfrak{g}$ generated by c contains $\mathfrak{su}(V,S)$. A direct calculation shows that the centraliser of $\mathfrak{su}(V,S)$ in $\mathrm{End}(\Lambda_\mathbb{R}) = \mathfrak{gl}(V) \oplus \mathfrak{gl}(V)\tau$, where τ is the complex conjugation, is $\mathbb{C}\,\mathrm{Id}_V \subset \mathfrak{gl}(V)$. (For this calculation it is crucial that $g \geq 3$.) Thus $u \in \mathbb{Q}\,\mathrm{Id}_\Lambda + \mathbb{Q}J_0$.

Let C_0 be the union of C_u for all $u \in \mathrm{End}(\Lambda_\mathbb{Q})$ such that $u \notin \mathbb{Q}\,\mathrm{Id}_\Lambda + \mathbb{Q}J_0$. By Baire's theorem, the complement to C_0 is dense in C. Hence we can find an element $J \in C^+$ outside of C_0. It has all the required properties. $\square$

Theorem 4.4 *For every integer $g \geq 3$, there exists a complex abelian variety A of dimension g with* $\mathrm{End}(A) \otimes \mathbb{Q} \cong \mathbb{Q}(i)$ *and* $\mathrm{Br}_A(A) \cong \mathbb{Z}/2$.

Proof. Let M be a symmetric, positive definite 2×2-matrix with entries in $\mathbb{Z}$ such that the diagonal entries are even. Assume that $M \neq nM'$, where $n \geq 2$ is an integer and M' has entries in $\mathbb{Z}$. For example, we can take

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let Λ be a free abelian group which is a direct sum $\Lambda_1 \oplus \Lambda_2$, where $\mathrm{rk}(\Lambda_1) = 2$ and $\mathrm{rk}(\Lambda_2) = 2g - 2$. Choose a $\mathbb{Z}$-basis in each Λ_i. Let S_1 be the symmetric bilinear form on Λ_1 with matrix $2I_2$. Let S_2 be the symmetric bilinear form on Λ_2 with matrix $M^{\oplus 2} \oplus 2I_{2g-6}$. It is clear that the orthogonal direct sum $S = (-S_1) \oplus S_2$ is non-degenerate, even and primitive. Let

$$J_0 = \mathfrak{J} \oplus \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \oplus \mathfrak{J}^{g-3}.$$

This is chosen to have the following properties: J_0 preserves the decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ and satisfies $J_0^2 = -1$ and $S(J_0x, J_0y) = S(x, y)$ for all $x, y \in \Lambda$. Hence by Theorem 4.3 there is an element $J \in \mathrm{SO}(\Lambda_\mathbb{R}, S)$ such that $J^2 = -1$, the centraliser of J in $\mathrm{End}(\Lambda_\mathbb{Q})$ is $\mathbb{Q}\,\mathrm{Id}_\Lambda + \mathbb{Q}J_0$, and the symmetric form $S(Jx, J_0y)$ is positive-definite.

Let A be the complex torus $\Lambda_\mathbb{R}/\Lambda$ associated with the complex structure J on $\Lambda_\mathbb{R}$. By (11), $\mathrm{End}(A) \otimes \mathbb{Q} = \mathrm{End}_J(\Lambda_\mathbb{Q}) = \mathbb{Q}\,\mathrm{Id}_\Lambda \oplus \mathbb{Q}J_0 \cong \mathbb{Q}(i)$.

Write $E(x, y) = S(x, J_0y)$. The matrix of E is

$$(-2\mathfrak{J}) \oplus \begin{pmatrix} 0 & -M \\ M & 0 \end{pmatrix} \oplus (2\mathfrak{J})^{g-3}.$$

We have $S, E \in \mathrm{Bi}_J(\Lambda)$. Thanks to Lemma 3.18, $\mathrm{Bi}_J(\Lambda) \otimes \mathbb{Q} = \mathbb{Q}S + \mathbb{Q}E$. Looking at the matrices of S and E we observe that the set of matrices in $\mathbb{Q}S + \mathbb{Q}E$ that have all entries in $\mathbb{Z}$ is precisely $\mathbb{Z}S + \mathbb{Z}E$. Hence $\mathrm{Bi}_J(\Lambda) = \mathbb{Z}S + \mathbb{Z}E$.

The form S is symmetric while E is anti-symmetric. Hence $(1+\tau)\mathrm{Bi}_J(\Lambda) = 2\mathbb{Z}S$. Since S is even, we conclude from Proposition 3.19 that $\mathrm{Br}_A(A) \cong \mathbb{Z}_2S/2\mathbb{Z}_2S \cong \mathbb{Z}/2$.

Finally $E(Jx, y) = S(Jx, J_0y)$ is positive definite. Hence by (14), $E \in \mathrm{Bi}_J(\Lambda)$ corresponds to a polarisation of A and so A is an abelian variety. $\square$

5 Complex multiplication

In this section we study the invariant Brauer group of complex abelian varieties of CM type. We construct two different examples of abelian surfaces of CM type with invariant Brauer group $\mathbb{Z}/2$: one is isogenous to a product of elliptic curves, the other is simple.

5.1 Non-simple abelian surfaces of CM type

Let us first give a construction of abelian surfaces A isogenous to the product of two elliptic curves with CM and $\mathrm{Br}_A(A) \cong \mathbb{Z}/2$. The subtlety here is that, on the one hand, the two imaginary quadratic fields cannot be the same (Corollary 3.11) and, on the other hand, A cannot be a product of two elliptic curves (Corollary 3.8).

Let d_1 and d_2 be square-free negative integers such that $d_1 \equiv d_2 \equiv -1 \bmod 4$. Let

$$\Lambda_i = \mathbb{Z}[\sqrt{d_i}] = \{x + y\sqrt{d_i} | x, y \in \mathbb{Z}\}, \quad \text{for} \quad i = 1, 2.$$

Equip $\Lambda_{i,\mathbb{R}}$ with the obvious complex structure. Then $E_i = \Lambda_{i,\mathbb{R}}/\Lambda_i$ is an elliptic curve over $\mathbb{C}$ such that $\mathrm{End}(E_i) = \Lambda_i$.

Since 2 is ramified in $\mathbb{Q}(\sqrt{d_i})$, we have

$$\Lambda_i/2 = \mathbb{F}_2[\sqrt{d_i}] \cong \mathbb{F}_2[x]/(x^2)$$

for $i = 1, 2$, where the second isomorphism identifies x with $1 + \sqrt{d_i}$. Let p_1, p_2 denote the composed homomorphisms $\Lambda_i \to \Lambda_i/2 \to \mathbb{F}_2[x]/(x^2)$ and let

$$\Lambda = \{(x_1, x_2) \in \Lambda_1 \oplus \Lambda_2 | p_1(x_1) = p_2(x_2)\}.$$

Then Λ is a subring of $\Lambda_1 \oplus \Lambda_2$ and, as a lattice, Λ is generated by $(2, 0)$, $(1, 1)$, $(2\sqrt{d_1}, 0)$ and $(\sqrt{d_1}, \sqrt{d_2})$.

Let $A = \Lambda_\mathbb{R}/\Lambda$, where $\Lambda_\mathbb{R}$ is equipped with the complex structure J which is the direct sum of the complex structures on $\Lambda_{1,\mathbb{R}}$ and $\Lambda_{2,\mathbb{R}}$. Thus $A = (E_1 \times E_2)/G$, where G is the graph of the isomorphism of abelian groups $E_1[2] \xrightarrow{\sim} E_2[2]$ which is the same as $\Lambda_1/2 \xrightarrow{\sim} \mathbb{F}_2[x]/(x^2) \xrightarrow{\sim} \Lambda_2/2$.

Let $K = \mathbb{Q}(\sqrt{d_1}) \oplus \mathbb{Q}(\sqrt{d_2})$. If $d_1 \neq d_2$, then the elliptic curves E_1 and E_2 are not isogenous so $\mathrm{End}(A) \otimes \mathbb{Q} \cong K$. Since Λ is a conjugation-invariant subring of K, we can use (15) to calculate

$$D_K(\Lambda) = (\tfrac{1}{2}, 0)\mathbb{Z} + (\tfrac{1}{4}, -\tfrac{1}{4})\mathbb{Z} + (\tfrac{1}{2\sqrt{d_1}}, 0)\mathbb{Z} + (\tfrac{1}{4\sqrt{d_1}}, \tfrac{-1}{4\sqrt{d_2}})\mathbb{Z}.$$

Now $D_K(\Lambda)^\iota$ has a basis $e_1 = (\frac{1}{2}, 0)$ and $e_2 = (\frac{1}{4}, -\frac{1}{4})$. For $a, b \in \mathbb{Q}$, we can calculate

$$\begin{aligned}
B_{ae_1+be_2}((2,0),(2,0)) &= 4a + 2b, \\
B_{ae_1+be_2}((1,1),(1,1)) &= a, \\
B_{ae_1+be_2}((2\sqrt{d_1},0),(2\sqrt{d_1},0)) &= -4d_1a - 2d_1b, \\
B_{ae_1+be_2}((\sqrt{d_1},\sqrt{d_2}),(\sqrt{d_1},\sqrt{d_2})) &= -d_1a + \tfrac{1}{2}(d_2 - d_1)b.
\end{aligned}$$

Recalling that $d_1 \equiv d_2 \bmod 4$, we conclude that B_{e_1} is odd while B_{e_2} is even. Hence $D_K(\Lambda)^{\iota,\mathrm{even}} = 2\mathbb{Z}e_1 + \mathbb{Z}e_2$. Meanwhile $(1 + \iota)D_K(\Lambda) = 2\mathbb{Z}e_1 + 2\mathbb{Z}e_2$. Hence by (16), $\mathrm{Br}_A(A) = \mathbb{Z}/2$, generated by e_2.

Remark 5.1 The bilinear form $B_{(1/4\sqrt{d_1},1/4\sqrt{d_2})} \in \mathrm{Bi}_J(\Lambda)$ is alternating and unimodular and the associated Hermitian form is positive definite. Hence A has a principal polarisation. This shows that there are *principally polarised* abelian surfaces with non-trivial invariant Brauer group.

5.2 Simple abelian surfaces of CM type

We now give examples of simple abelian surfaces A of CM type with $\mathrm{Br}_A(A) \cong \mathbb{Z}/2$.

Let K be a quartic CM field which is not biquadratic over $\mathbb{Q}$, but such that $K_2 = K \otimes \mathbb{Q}_2$ is a biquadratic extension of $\mathbb{Q}_2$. (See below for an example of such a field.) Since K is a quartic CM field, it contains a real quadratic field $K_+ = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Q}$. Since $K_2/\mathbb{Q}_2$ is biquadratic, we can write $K_2 = \mathbb{Q}_2(\sqrt{d}, \sqrt{e})$ for some $e \in \mathbb{Q}_2$ (where $e \notin \mathbb{Q}^\times K_+^{\times 2}$ since $K/\mathbb{Q}$ is not biquadratic, and $e, e/d \notin \mathbb{Q}_2^{\times 2}$ since $\sqrt{e} \notin \mathbb{Q}_2(\sqrt{d})$).

Let Λ be a $\mathbb{Z}$-lattice in K such that

$$\Lambda_2 = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_2 = \mathbb{Z}_2 + \mathbb{Z}_2\sqrt{d} + \mathbb{Z}_2\sqrt{e} + \mathbb{Z}_2\sqrt{de}.$$

Let $A = K_{\mathbb{R}}/\Lambda$. Choosing $J \in K_{\mathbb{R}}$ such that $J^2 = -1$ gives A the structure of a complex torus. The argument of [Mum74, pp. 212–213] shows that A is polarisable and hence an abelian variety. (Unlike in [Mum74, pp. 212–213], we do not have $\Lambda = \mathcal{O}_K$. However $n\Lambda \subset \mathcal{O}_K$ for some positive integer n so we can replace the value α from [Mum74, p. 212] by $n^2\alpha$ to ensure that $\mathrm{tr}_{K/\mathbb{Q}}(\alpha x\bar{y}) \in \mathbb{Z}$ for all $x, y \in \Lambda$.)

Since $K/\mathbb{Q}$ is not biquadratic, K_+ is the only quadratic subfield of K. Hence K does not contain any imaginary quadratic fields, so every CM type for K is primitive. Hence A is a simple abelian variety with $\mathrm{End}(A) \otimes \mathbb{Q} = K$, by Shimura and Taniyama, see [Lan83, Thm. 3.5].

Write $D_K(\Lambda_2) = D_K(\Lambda) \otimes \mathbb{Z}_2$. Since Λ_2 is a subring of K_2, $\alpha \in D_K(\Lambda_2)$ if and only if $\mathrm{tr}_{K_2/\mathbb{Q}_2}(\alpha x) \in \mathbb{Z}_2$ for all $x \in \Lambda_2$, from which we obtain

$$D_K(\Lambda_2) = \frac{1}{4}\mathbb{Z}_2 + \frac{1}{4\sqrt{d}}\mathbb{Z}_2 + \frac{1}{4\sqrt{e}}\mathbb{Z}_2 + \frac{1}{4\sqrt{de}}\mathbb{Z}_2.$$

If $\alpha = a_1 + a_2\sqrt{d} + a_3\sqrt{e} + a_4\sqrt{de}$ $(a_1, a_2, a_3, a_4 \in \mathbb{Q}_2)$, then

$$B_\alpha(1,1) = 4a_1, \quad B_\alpha(\sqrt{d}, \sqrt{d}) = 4da_1,$$
$$B_\alpha(\sqrt{e}, \sqrt{e}) = -4ea_1, \quad B_\alpha(\sqrt{de}, \sqrt{de}) = -4dea_1.$$

Hence B_α is even if and only if $4a_1 \in 2\mathbb{Z}_2$ or in other words, $a_1 \in \frac{1}{2}\mathbb{Z}_2$. Consequently

$$D_K(\Lambda_2)^{\iota,\mathrm{even}} = \frac{1}{2}\mathbb{Z}_2 + \frac{1}{4\sqrt{d}}\mathbb{Z}_2$$

and

$$(1+\iota)D_K(\Lambda_2) = \frac{1}{2}\mathbb{Z}_2 + \frac{1}{2\sqrt{d}}\mathbb{Z}_2.$$

Hence by (16), $\mathrm{Br}_A(A) = \mathbb{Z}/2$, generated by $1/4\sqrt{d}$.

Lemma 5.2 *Let $K_+ = \mathbb{Q}(\sqrt{5})$ and $K = K_+(\sqrt{\delta})$ where $\delta = -30 + 8\sqrt{5}$. Then K is a quartic CM field, $K/\mathbb{Q}$ is not biquadratic, but $K \otimes \mathbb{Q}_2/\mathbb{Q}_2$ is biquadratic.*

Proof. We have $-30 \pm 8\sqrt{5} < 0$ so δ is a totally negative element of K_+. Hence K is a CM field.

If $K/\mathbb{Q}$ were biquadratic, then $\delta \in \mathbb{Q}^\times K_+^{\times 2}$ so $\mathrm{Nm}_{K_+/\mathbb{Q}}(\delta) \in \mathbb{Q}^{\times 2}$. But $\mathrm{Nm}_{K_+/\mathbb{Q}}(\delta) = 30^2 - 8^2 \times 5 = 580$ which is not a square.

Now we examine $K \otimes \mathbb{Q}_2$. First note that $K_+ \otimes \mathbb{Q}_2 = \mathbb{Q}_2(\sqrt{5})$ is the unramified quadratic extension of $\mathbb{Q}_2$. We have $\delta = 2(-15 + 4\sqrt{5})$ so the 2-adic valuation of δ is odd. Hence δ does not have a square root in $K_+ \otimes \mathbb{Q}_2$, so $K \otimes \mathbb{Q}_2$ is a field.

Finally $(2 + \sqrt{5})^2 = 9 + 4\sqrt{5}$. We have $-15 + 4\sqrt{5} \equiv 9 + 4\sqrt{5} \bmod 8$ so by Hensel's lemma, $-15 + 4\sqrt{5} = \zeta^2$ for some $\zeta \in \mathbb{Q}_2(\sqrt{5})$. Therefore

$$K \otimes \mathbb{Q}_2 = \mathbb{Q}_2(\sqrt{5})(\sqrt{2\zeta^2}) = \mathbb{Q}_2(\sqrt{5})(\sqrt{2})$$

is a biquadratic extension of $\mathbb{Q}_2$. $\square$

5.3 Simple abelian varieties of CM type

In this section we show that any abelian variety of CM type whose endomorphism ring is the ring of integers of a cyclotomic field has $\mathrm{Br}_A(A) = 0$.

Let K be a CM field. Write $[K : \mathbb{Q}] = 2g$ so that there is an isomorphism of $\mathbb{R}$-algebras $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{C}^g$. Let $\mathcal{O}_K$ be the ring of integers of K. Then $A = K_\mathbb{R}/\mathcal{O}_K$ is a real torus.

Suppose that $J \in K_\mathbb{R}$ is such that $J^2 = -1$ and J is not contained in a proper CM subfield of K (that is, the associated CM type is primitive). The action of J makes $K_\mathbb{R}$ a complex vector space so A is a complex torus such that $\mathrm{End}(A) = \mathcal{O}_K$. It is well known that A being a simple abelian variety is equivalent to the CM type being primitive [Mum74, Ch. 22].

Let $\mathcal{D}_K \subset \mathcal{O}_K$ be the different. Recall that the fractional ideal $\mathcal{D}_K^{-1} \subset K$ is the dual of $\mathcal{O}_K$ with respect to the bilinear form $\mathrm{tr}_{K/\mathbb{Q}}(xy)$. Hence by (15),

$$D_K(\mathcal{O}_K) = \mathcal{D}_K^{-1}.$$

Write $(\mathcal{D}_K^{-1})_+$ for the subgroup of all conjugation-invariant elements of D_K^{-1}.

Let us consider the particular case when $K = \mathbb{Q}(\zeta_n)$, where ζ_n is a primitive root of unity of degree $n \geq 3$. Since $\mathcal{O}_k = \mathbb{Z}[\zeta_n]$, we see that $\{\zeta_n^i | i = 0, \ldots, \varphi(n) - 1\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$. For any $x = \zeta_n^i$ we have $B_\alpha(x, x) = \mathrm{tr}_{K/\mathbb{Q}}(\alpha x \bar{x}) = \mathrm{tr}_{K/\mathbb{Q}}(\alpha)$. Thus $B_\alpha(x, y)$ is an even form if and only if $\mathrm{tr}_{K/\mathbb{Q}}(\alpha)$ is even. So in this case, from (16), we have

$$\mathrm{Br}_A(A) = \{\alpha \in (\mathcal{D}_K^{-1})_+ |\ \mathrm{tr}_{K/\mathbb{Q}}(\alpha) \equiv 0 \bmod 2\}/(1 + \iota)\mathcal{D}_K^{-1}.$$

Lemma 5.3 *Let $K = \mathbb{Q}(\zeta_n)$, where $n \geq 3$. If n is odd, then the different ideal of K is the principal ideal of $\mathcal{O}_K$ generated by*

$$n \prod_{\text{primes}\, p|n} \frac{1}{\zeta_p - \zeta_p^{-1}}.$$

If n is even, then the different ideal of K is generated by

$$\frac{n}{2} \prod_{\text{odd primes}\, p|n} \frac{1}{\zeta_p - \zeta_p^{-1}}.$$

Proof. If p is a prime and r is a positive integer, then the different of $\mathbb{Q}(\zeta_{p^r})$ is the ideal of $\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})}$ generated by $(1-\zeta_{p^r})^{p^{r-1}((p-1)r-1)}$, see [Swi01, Ch. 3, Thm. 27]. Using that $p\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})} = (1-\zeta_{p^r})^{p^{r-1}(p-1)}\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})}$ and $(1-\zeta_p)\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})} = (1-\zeta_{p^r})^{p^{r-1}}\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})}$ we obtain that the different is generated by $p^r(1-\zeta_p)^{-1}$.

If p is odd, then $1+\zeta_p$ is a unit in $\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})}$ so the different is generated by

$$p^r(1-\zeta_p)^{-1}(1+\zeta_p)^{-1} = \frac{p^r}{\zeta_p - \zeta_p^{-1}}.$$

If $p = 2$, then $p^r(1-\zeta_p)^{-1} = p^r/2$. Thus the lemma holds whenever n is equal to a prime power p^r.

If m and m' are coprime positive integers, then the fields $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_{m'})$ are linearly disjoint with coprime discriminants, hence by [FT91, III.2.13, VI.1.14] we have $\mathcal{O}_{\mathbb{Q}(\zeta_{mm'})} = \mathcal{O}_{\mathbb{Q}(\zeta_m)} \otimes \mathcal{O}_{\mathbb{Q}(\zeta_{m'})}$. This implies that the different of K is the product of the differents of the subfields $\mathbb{Q}(\zeta_{p^r}) \subset K$, where r is the highest power of p dividing n. □

The following lemma is a version of [Ser77, Sec. 3.3, example 5] over $\mathbb{Z}$.

Lemma 5.4 *Let G be a group, let M be a $\mathbb{Z}[G]$-module which is free as a $\mathbb{Z}$-module, and let N be a free $\mathbb{Z}[G]$-module. Then $M \otimes_{\mathbb{Z}} N$ is a free $\mathbb{Z}[G]$-module.*

Proof. Let $\{m_i | i \in I\}$ be a $\mathbb{Z}$-basis for M, and let $\{n_j | j \in J\}$ be a $\mathbb{Z}[G]$-basis for N. For each $g \in G$, $\{gm_i | i \in I\}$ is also a $\mathbb{Z}$-basis for M. Consequently for each $g \in G$ and $j \in J$, $\{gm_i \otimes gn_j | i \in I\}$ is a $\mathbb{Z}$-basis for $M \otimes_{\mathbb{Z}} gn_j\mathbb{Z}$. Since $\{gn_j | g \in G, j \in J\}$ is a $\mathbb{Z}$-basis for N, we deduce that

$$\{gm_i \otimes gn_j | g \in G, i \in I, j \in J\}$$

is a $\mathbb{Z}$-basis for $M \otimes_{\mathbb{Z}} N$. This basis is G-stable so $M \otimes_{\mathbb{Z}} N$ is free as a $\mathbb{Z}[G]$-module. □

Lemma 5.5 *Let $n \geq 3$ be an integer which is not a power of 2. Then $\mathcal{O}_{\mathbb{Q}(\zeta_n)}$ is a free $\mathbb{Z}[\mathbb{Z}/2]$-module, where $\mathbb{Z}/2$ acts by complex conjugation.*

Proof. Let p be an odd prime and let r be a positive integer. Let

$$S = \{\zeta_{p^r}^j | 1 \leq j \leq p^{r-1}(p-1)/2\},$$
$$\overline{S} = \{\zeta_{p^r}^j | p^{r-1}(p+1)/2 \leq j \leq p^r - 1\}.$$

We claim that $S \cup \overline{S}$ is a basis of $\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})} = \mathbb{Z}[\zeta_{p^r}]$. Indeed, $|S \cup \overline{S}| = \varphi(p^r)$, so it is enough to show that every power of ζ_{p^r} is in the subgroup N of $\mathbb{Z}[\zeta_{p^r}]$ generated by $S \cup \overline{S}$. The set $S \cup \overline{S}$ contains $\zeta_{p^r}^{p^{r-1}a}$, where $a = 1, \ldots, p-1$, hence $1 \in N$. For $j = 1, \ldots, p^r - 1$ the relation $\sum_{a=0}^{p-1} \zeta_{p^r}^{j+p^{r-1}a} = 0$ shows that each power of ζ_{p^r} between $p^{r-1}(p-1)/2$ and $p^{r-1}(p+1)/2$ is also in N, so we are done. This proves the statement for $n = p^r$.

Now let $n \geq 3$ be an integer which is not a power of 2. Then we can write $n = p^r m$ where p is an odd prime, r is a positive integer, and $p \nmid m$. The fields $\mathbb{Q}(\zeta_{p^r})$ and $\mathbb{Q}(\zeta_m)$ are linearly disjoint with coprime discriminants, hence $\mathcal{O}_{\mathbb{Q}(\zeta_n)} = \mathcal{O}_{\mathbb{Q}(\zeta_{p^r})} \otimes \mathcal{O}_{\mathbb{Q}(\zeta_m)}$. We have proved that $\mathcal{O}_{\mathbb{Q}(\zeta_{p^r})}$ is a free $\mathbb{Z}[\mathbb{Z}/2]$-module, so we can conclude by applying Lemma 5.4. $\square$

Proposition 5.6 *Let $K = \mathbb{Q}(\zeta_n)$, where $n \geq 3$. Let A be an abelian variety attached to K as at the beginning of Section 5.3, that is, $A = K_{\mathbb{R}}/\mathcal{O}_K$ with a primitive CM type. Then $\mathrm{Br}_A(A) = 0$.*

Proof. Let us assume that n is not a power of 2. By Lemma 5.3 we have $\mathcal{D}_K^{-1} = \eta\mathcal{O}_K$ with η a rational multiple of $\prod_p(\zeta_p - \zeta_p^{-1})$, where p ranges over the odd prime factors of n. We note that $\bar{\eta} = \pm\eta$. If $\bar{\eta} = \eta$, then $\mathrm{Br}_A(A)$ is

$$(\mathcal{D}_K^{-1})_+^{\text{even}}/(1+\iota)\mathcal{D}_K^{-1} \subset (\mathcal{D}_K^{-1})_+/(1+\iota)\mathcal{D}_K^{-1} \cong (\mathcal{O}_K)_+/(1+\iota)\mathcal{O}_K = \widehat{\mathrm{H}}^0(\mathbb{Z}/2, \mathcal{O}_K).$$

By Lemma 5.5 the $\mathbb{Z}[\mathbb{Z}/2]$-module $\mathcal{O}_K$ is free, hence $\widehat{\mathrm{H}}^0(\mathbb{Z}/2, \mathcal{O}_K) = 0$.

If $\bar{\eta} = -\eta$, then $\mathrm{Br}_A(A)$ is

$$(\mathcal{D}_K^{-1})_+^{\text{even}}/(1+\iota)\mathcal{D}_K^{-1} \subset (\mathcal{D}_K^{-1})_+/(1+\iota)\mathcal{D}_K^{-1} \cong (\mathcal{O}_K)_-/(1-\iota)\mathcal{O}_K = \mathrm{H}^1(\mathbb{Z}/2, \mathcal{O}_K).$$

This is also zero by Lemma 5.5.

Now let $n = 2^m$, where $m \geq 2$. In this case $\mathcal{D}_K^{-1}$ is generated by 2^{1-m}, hence

$$(\mathcal{D}_K^{-1})_+/(1+\iota)\mathcal{D}_K^{-1} \cong (\mathcal{O}_K)_+/(1+\iota)\mathcal{O}_K = \widehat{\mathrm{H}}^0(\mathbb{Z}/2, \mathcal{O}_K).$$

Then $\mathcal{O}_K = \mathbb{Z}[\zeta_{2^m}]$ has a $\mathbb{Z}$-basis $\{1\} \cup \{i\} \cup R \cup \overline{R}$, where $R = \{\zeta_{2^m}^j | 1 \leq j \leq 2^{m-2} - 1\}$. The submodule spanned by $R \cup \overline{R}$ is a free $\mathbb{Z}[\mathbb{Z}/2]$-module so $\widehat{\mathrm{H}}^0(\mathbb{Z}R \oplus \mathbb{Z}\overline{R}) = 0$. Furthermore $\widehat{\mathrm{H}}^0(\mathbb{Z}/2, \mathbb{Z}i) = 0$. It follows that $\widehat{\mathrm{H}}^0(\mathbb{Z}/2, \mathcal{O}_K) \cong \mathbb{Z}/2$ generated by the class of 1. Thus $(\mathcal{D}_K^{-1})_+/(1+\iota)\mathcal{D}_K^{-1}$ is generated by the class of 2^{1-m}. We note that $\mathrm{tr}_{K/\mathbb{Q}}(2^{1-m}) = 1$, so 2^{1-m} is not contained in $(\mathcal{D}_K^{-1})^{\text{even}}$. Therefore, $\mathrm{Br}_A(A) = (\mathcal{D}_K^{-1})_+^{\text{even}}/(1+\iota)\mathcal{D}_K^{-1} = 0$. $\square$

Theorem 5.7 *Let $n \geq 3$ be an integer, let $K = \mathbb{Q}(\zeta_n)$ be the n-th cyclotomic field and let $\mathcal{O}_K$ be the ring of integers of K. Let $\varphi(n) := [K : \mathbb{Q}]$ and $g = \varphi(n)/2$. If A is a simple g-dimensional abelian variety of CM type over $\mathbb{C}$ with multiplication by $\mathcal{O}_K$, then $\mathrm{Br}_A(A) = 0$.*

Proof. We may assume that $A(\mathbb{C}) = K_{\mathbb{R}}/I$ where $I = \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$ is a non-zero ideal in $\mathcal{O}_K$. We claim that there is a non-zero *principal* ideal $J \subset I$ such that the (finite) quotient I/J has *odd* order. Then there is a simple complex abelian variety B of CM type with multiplication by $\mathcal{O}_K$ such that $J = \mathrm{H}_1(B(\mathbb{C}), \mathbb{Z})$, $B(\mathbb{C}) = K_{\mathbb{R}}/J$ and there is an isogeny $B \to A$ with kernel I/J. In particular, the degree of $B \to A$ is odd. Since A is simple, so is B, and thus B falls within the scope of Proposition 5.6. Hence $\mathrm{Br}_B(B) = 0$, and it follows from Proposition 3.9 that $\mathrm{Br}_A(A) = 0$.

Let us prove the existence of such an ideal J. According to [CR62, Thm. 18.20], there exists a non-zero ideal $M \subset \mathcal{O}_K$ such that M is coprime to the ideal $2\mathcal{O}_K$ and $J := IM$ is a principal ideal in $\mathcal{O}_K$. Clearly, $J \subset I$. On the other hand, according to [CR62, Cor. 18.24], the additive groups $\mathcal{O}_K/M$ and $I/IM = I/J$ are isomorphic. In particular, the order of $O_K/M \cong I/J$ is odd. □

6 Abelian varieties over non-closed fields

Let A be an abelian variety over a field k. Recall that the Picard scheme $\mathbf{Pic}_{A/k}$ is smooth. (Indeed, $\mathbf{Pic}_{A/k}$ is a group k-scheme which is smooth at e by [EGM, Ch. VI, Thm. 6.18] and hence is smooth.) Since A has a k-point, $\mathbf{Pic}_{A/k}$ represents the relative Picard functor $\mathrm{Pic}_{A/k}$ that sends a k-scheme T to $\mathrm{Pic}(A \times_k T)/\mathrm{Pic}(T)$, see [Kle05, Thm. 2.5]. Thus for any field extension $k \subset L$ we have $\mathbf{Pic}_{A/k}(L) = \mathrm{Pic}(A_L)$, where $A_L = A \times_k L$.

Let $\mathbf{Pic}^0_{A/k}$ be the connected component of e in $\mathbf{Pic}_{A/k}$. This is the dual abelian variety $A^\vee$, see [EGM, Ch. VI, (6.19)]. Write $\mathrm{Pic}^0(A_L) = \mathbf{Pic}^0_{A/k}(L)$. The Néron–Severi group of A_L is defined as $\mathrm{NS}\,(A_L) = \mathrm{Pic}(A_L)/\mathrm{Pic}^0(A_L)$. In particular, $\mathrm{NS}\,(\overline{A})$ is the group of connected components of $\mathbf{Pic}_{A/k} \times_k \bar{k}$.

The group k-scheme of connected components of $\mathbf{Pic}_{A/k}$ is étale [SGA3, VI_A, 5.5]. Thus the connected components of $\mathbf{Pic}_{A/k} \times_k \bar{k}$ are obtained from the connected components of $\mathbf{Pic}_{A/k} \times_k k_{\mathrm{s}}$ by base change from k_{s} to $\bar{k}$. The smoothness of $\mathbf{Pic}_{A/k}$ implies that each connected component of $\mathbf{Pic}_{A/k} \times_k k_{\mathrm{s}}$ is smooth, hence contains a k_{s}-point. It follows that the natural map $\mathrm{NS}\,(A_{\mathrm{s}}) \to \mathrm{NS}\,(\overline{A})$ is an isomorphism. See also [EGM, Ch. III, (3.27), (3.29)].

By a theorem of Chow, for abelian varieties A and B over k the natural map $\mathrm{Hom}(A_{\mathrm{s}}, B_{\mathrm{s}}) \to \mathrm{Hom}(\overline{A}, \overline{B})$ is an isomorphism (see [Con06, Thm. 3.19] for a modern proof).

Recall that a line bundle L on $\overline{A}$ defines a canonical map $\varphi_L \colon \overline{A} \to \overline{A}^\vee$, see [Mum74, Ch. 6, p. 60]. Let $\mathcal{P}$ be the Poincaré line bundle on $A \times A^\vee$. We write $\pi_j \colon A \times A \to A$ for the projection to the j-th factor, $j = 1, 2$. Let $i_1 \colon A \to A \times A$ be the map sending x to (x, e), and similarly $i_2(x) = (e, x)$.

The following lemma is essentially well known.

Lemma 6.1 (i) *There is a commutative diagram of abelian groups*

$$\begin{array}{ccccccc} \mathrm{Pic}(\overline{A}) & \oplus & \mathrm{Pic}(\overline{A}) & \oplus & \mathrm{Hom}(\overline{A},\overline{A}^\vee) & \xrightarrow{\sim} & \mathrm{Pic}(\overline{A}\times\overline{A}) \\ \uparrow & & \uparrow & & || & & \uparrow \\ \mathrm{Pic}(A_{\mathrm{s}}) & \oplus & \mathrm{Pic}(A_{\mathrm{s}}) & \oplus & \mathrm{Hom}(A_{\mathrm{s}},A_{\mathrm{s}}^\vee) & \xrightarrow{\sim} & \mathrm{Pic}(A_{\mathrm{s}}\times A_{\mathrm{s}}) \end{array}$$

where in each row the first two summands are embedded via π_1^* and π_2^*, the third summand is embedded via the map sending $\phi\in\mathrm{Hom}(\overline{A},\overline{A}^\vee)$ to $(\mathrm{id},\phi^*)\mathcal{P}$, and the vertical arrows are injective.

(ii) *There is a commutative diagram of abelian groups*

$$\begin{array}{ccccccccc} \mathrm{Pic}(\overline{A}) & \twoheadrightarrow & \mathrm{NS}\,(\overline{A}) & \xrightarrow{\sim} & \mathrm{Hom}(\overline{A},\overline{A}^\vee)^{\mathrm{sym}} & \hookrightarrow & \mathrm{Hom}(\overline{A},\overline{A}^\vee) & \hookrightarrow & \mathrm{Pic}(\overline{A}\times\overline{A}) \\ \uparrow & & || & & || & & || & & \uparrow \\ \mathrm{Pic}(A_{\mathrm{s}}) & \twoheadrightarrow & \mathrm{NS}\,(A_{\mathrm{s}}) & \xrightarrow{\sim} & \mathrm{Hom}(A_{\mathrm{s}},A_{\mathrm{s}}^\vee)^{\mathrm{sym}} & \hookrightarrow & \mathrm{Hom}(A_{\mathrm{s}},A_{\mathrm{s}}^\vee) & \hookrightarrow & \mathrm{Pic}(A_{\mathrm{s}}\times A_{\mathrm{s}}) \end{array}$$

where in each row the composed map is $\delta=m^*-\pi_1^*-\pi_2^*$ and the second arrow is $L\mapsto\varphi_L$.

Proof. (i) Injectivity of the vertical arrows follows from the fact that $\mathbf{Pic}_{A/k}$ represents the relative Picard functor. The fact that the horizontal maps are isomorphisms is a consequence of the obvious properties of the maps π_1, π_2, i_1, i_2, and the universal property of the Poincaré line bundle, cf. [SZ14, Prop. 1.7].

(ii) Let $L\in\mathrm{Pic}(\overline{A})$. If $L\in\mathrm{Pic}^0(\overline{A})$, then $\delta(L)=m^*L\otimes p_1^*L^{-1}\otimes p_2^*L^{-1}=0$ by the basic theory of abelian varieties, so δ factors through $\mathrm{NS}\,(\overline{A})$. Since $i_1^*\delta=i_2^*\delta=0$, we see that δ maps $\mathrm{Pic}(\overline{A})$ to $\mathrm{Ker}(i_1^*)\cap\mathrm{Ker}(i_2^*)$, which by part (i) is identified with $\mathrm{Hom}(\overline{A},\overline{A}^\vee)$ by the map sending $\phi\in\mathrm{Hom}(\overline{A},\overline{A}^\vee)$ to $(\mathrm{id},\phi^*)\mathcal{P}$. By [Mum74, Ch. 8] there is a canonical isomorphism

$$(\mathrm{id},\varphi_L^*)\mathcal{P}=m^*L\otimes p_1^*L^{-1}\otimes p_2^*L^{-1}.$$

The same arguments apply to the bottom row. □

Recall that $e\colon\mathrm{Spec}(k)\to A$ denotes the unit element of A. In Section 2 we defined $\mathrm{Br}_e(A)=\mathrm{Ker}[e^*\colon\mathrm{Br}(A)\to\mathrm{Br}(k)]$. Let

$$\mathrm{Br}_a(A):=\mathrm{Br}_e(A)\cap\mathrm{Br}_1(A),\qquad \mathrm{Br}_{a,A}(A):=\mathrm{Br}_e(A)\cap\mathrm{Br}_1(A)\cap\mathrm{Br}_A(A).$$

Let $\bar{k}$ be an algebraic closure of k, and let k_{s} be the separable closure of k in $\bar{k}$, with Galois group $\Gamma=\mathrm{Gal}(k_{\mathrm{s}}/k)$. Write $\overline{A}=A\times_k\bar{k}$ and $A_{\mathrm{s}}=A\times_k k_{\mathrm{s}}$. The Leray spectral sequence [Mil80, Thm. III.1.18 (a)]

$$\mathrm{H}^i(k,\mathrm{H}^j(A_{\mathrm{s}},\mathbb{G}_m))\Rightarrow\mathrm{H}^{i+j}(A,\mathbb{G}_m) \tag{17}$$

gives rise to an exact sequence

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(A) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}})) \longrightarrow \mathrm{H}^3(k, \mathbb{G}_m) \longrightarrow \mathrm{H}^3(A, \mathbb{G}_m).$$

The first arrow composed with e^* is the identity map on $\mathrm{Br}(k)$. Similarly, the composition of the last arrow with the map induced by e is the identity map on $\mathrm{H}^3(k, \mathbb{G}_m)$. Thus we obtain canonical isomorphisms

$$\mathrm{Br}_a(A) \cong \mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}})) \cong \mathrm{Br}_1(A)/\mathrm{Br}_0(A).$$

Proposition 6.2 *Let A be an abelian variety over a field k. Then $\mathrm{Br}_{a,A}(A)$ is canonically isomorphic to the kernel of the composition*

$$\mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}})) \longrightarrow \mathrm{H}^1(k, \mathrm{NS}\,(A_{\mathrm{s}})) \longrightarrow \mathrm{H}^1(k, \mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee)),$$

where the second arrow is induced by the map $L \mapsto \varphi_L$.

Proof. Due to the isomorphism $\mathrm{Br}_a(A) \cong \mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}}))$ and Proposition 2.2, the group $\mathrm{Br}_{a,A}(A)$ is the kernel of the map induced by

$$\delta = m^* - \pi_1^* - \pi_2^* \ : \ \mathrm{Pic}(A_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(A_{\mathrm{s}} \times A_{\mathrm{s}}) \tag{18}$$

on the first Galois cohomology groups $\mathrm{H}^1(k, -)$. By Lemma 6.1 the map (18) factors through the map of Γ-modules $\mathrm{NS}\,(A_{\mathrm{s}}) \to \mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee)$, where $\mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee)$ is a direct summand of the Γ-module $\mathrm{Pic}(A_{\mathrm{s}} \times A_{\mathrm{s}})$. □

M. Stoll [Sto07, §7, p. 378] denoted by $\mathrm{Br}_{1/2}(A)$ the subgroup of $\mathrm{Br}_1(A)$ consisting of the elements whose image in $\mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}}))$ comes from $\mathrm{H}^1(k, \mathrm{Pic}^0(A_{\mathrm{s}}))$. Define

$$\mathrm{Br}_{e,1/2}(A) := \mathrm{Br}_e(A) \cap \mathrm{Br}_{1/2}(A);$$

this group is canonically isomorphic to

$$\mathrm{Im}\left[\mathrm{H}^1(k, \mathrm{Pic}^0(A_{\mathrm{s}})) \to \mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}}))\right] = \mathrm{Ker}\left[\mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}})) \to \mathrm{H}^1(k, \mathrm{NS}\,(A_{\mathrm{s}}))\right].$$

Corollary 6.3 *Let A be an abelian variety over a field k. Then $\mathrm{Br}_{e,1/2}(A)$ is contained in $\mathrm{Br}_{a,A}(A)$, and we have*

$$\mathrm{Br}_{a,A}(A)/\mathrm{Br}_{e,1/2}(A) \subset \mathrm{H}^1(k, \mathrm{NS}\,(A_{\mathrm{s}}))[2].$$

If $\mathrm{NS}\,(A_{\mathrm{s}})$ is a trivial Γ-module, then $\mathrm{Br}_{a,A}(A) = \mathrm{Br}_{e,1/2}(A)$.

Proof. The first inclusion is immediate from Proposition 6.2. Recall that $\mathrm{NS}\,(A_{\mathrm{s}})$ is the subgroup of $\mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee)$ given by the condition $\phi = \phi^\vee$. The composition of the inclusion $\mathrm{NS}\,(A_{\mathrm{s}}) \hookrightarrow \mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee)$ with the map $\mathrm{Hom}(A_{\mathrm{s}}, A_{\mathrm{s}}^\vee) \to \mathrm{NS}\,(A_{\mathrm{s}})$ sending ϕ to $\phi + \phi^\vee$ is multiplication by 2. This gives the second inclusion. The last statement follows since $\mathrm{H}^1(k, \mathbb{Z}) = 0$ for the trivial Γ-module $\mathbb{Z}$. □

Example 6.4 Now it is easy to construct an abelian variety A such that $\mathrm{Br}_1(A)$ is not a subgroup of $\mathrm{Br}_A(A)$. Let E be the elliptic curve $y^2 = x^3 - x$ over $k = \mathbb{R}$, and let $A = E \times E$. We have direct sum decompositions of Γ-modules

$$\mathrm{Pic}(\overline{A}) = \mathrm{Pic}(\overline{E}) \oplus \mathrm{Pic}(\overline{E}) \oplus \mathrm{End}(\overline{E}),$$

$$\mathrm{NS}\,(\overline{A}) = \mathrm{NS}\,(\overline{E}) \oplus \mathrm{NS}\,(\overline{E}) \oplus \mathrm{End}(\overline{E}),$$

and isomorphisms of Γ-modules $\mathrm{NS}\,(\overline{E}) \cong \mathbb{Z}$ (with trivial Γ-action) and $\mathrm{End}(\overline{E}) \cong \mathbb{Z}[\sqrt{-1}]$, where the generator of $\Gamma \cong \mathbb{Z}/2$ acts as complex conjugation. In particular, $\mathrm{H}^1(k, \mathrm{End}(\overline{E})) \cong \mathbb{Z}/2$ is a subgroup of $\mathrm{H}^1(k, \mathrm{Pic}(\overline{A}))$ which maps isomorphically onto $\mathrm{H}^1(k, \mathrm{NS}\,(\overline{A}))$. Next, the Γ-module $\mathrm{Hom}(\overline{A}, \overline{A}^\vee) \cong \mathrm{End}(\overline{A})$ is the algebra of (2×2)-matrices over $\mathbb{Z}[\sqrt{-1}]$. The Rosati involution associated to the canonical principal polarisation of A acts on $\mathrm{Mat}_2(\mathbb{Z}[\sqrt{-1}])$ as the composition of transposition and complex conjugation. Hence the injective image of $\mathrm{NS}\,(\overline{A})$ consists of matrices with diagonal entries in $\mathbb{Z}$ such that the two non-diagonal entries are conjugate. The Γ-submodule $\mathrm{Mat}_2(\mathbb{Z}[\sqrt{-1}])$ is the direct sum of $\mathrm{NS}\,(\overline{A})$ and the subgroup of upper-triangular matrices with purely imaginary diagonal entries. Hence $\mathrm{NS}\,(\overline{A})$ is a direct summand of the Γ-module $\mathrm{Hom}(\overline{A}, \overline{A}^\vee)$. Thus we have an element of order 2 in $\mathrm{H}^1(k, \mathrm{Pic}(\overline{A}))$ with non-zero image in $\mathrm{H}^1(k, \mathrm{Hom}(\overline{A}, \overline{A}^\vee))$ under the map of Proposition 6.2, hence $\mathrm{Br}_1(A)$ is not contained in $\mathrm{Br}_A(A)$. Note also that in this case we have $\mathrm{Br}_{a,A}(A) = \mathrm{Br}_{e,1/2}(A)$, hence the inclusion of $\mathrm{Br}_{a,A}(A)/\mathrm{Br}_{e,1/2}(A)$ into $\mathrm{H}^1(k, \mathrm{NS}\,(\overline{A}))[2]$ is strict.

The following statement concerns $\mathrm{Br}_{e,A}(A) = \mathrm{Br}_e(A) \cap \mathrm{Br}_A(A)$, which contains $\mathrm{Br}_{a,A}(A)$ as a subgroup. We are grateful to the referee for pointing it out to us.

Proposition 6.5 *Let A be an abelian variety over a field k. Then the group*

$$\mathrm{Br}_{e,A}(A)(p')/\mathrm{Br}_{e,1/2}(A)(p')$$

has exponent 2.

Proof. Every element of $\mathrm{Br}_{e,A}(A)(p')/\mathrm{Br}_{e,1/2}(A)(p')$ lifts to an element of $\mathrm{Br}_e(A)[n]$ for some positive integer n not divisible by $\mathrm{char}(k)$.

The natural injective map $\mu_n \to \mathbb{G}_m$ transforms the spectral sequence

$$\mathrm{H}^i(k, \mathrm{H}^j(A_{\mathrm{s}}, \mu_n)) \Rightarrow \mathrm{H}^{i+j}(A, \mu_n).$$

into the spectral sequence (17). Thus the exact sequences of low degree terms of these spectral sequences give rise to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(k, \mathrm{H}^1(A_{\mathrm{s}}, \mu_n)) \arrow[r] \arrow[d] & \mathrm{H}^2_e(A, \mu_n) \arrow[r] \arrow[d] & \mathrm{H}^2(A_{\mathrm{s}}, \mu_n) \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(k, \mathrm{Pic}(A_{\mathrm{s}}))[n] \arrow[r] & \mathrm{Br}_e(A)[n] \arrow[r] & \mathrm{Br}(A_{\mathrm{s}})
\end{tikzcd}$$

Here the subscript e means the kernel of the map induced by $e\colon \mathrm{Spec}(k) \to A$. By the Kummer sequence for A_{s}, we have canonial isomorphisms

$$\mathrm{H}^1(A_{\mathrm{s}}, \mu_n) \cong \mathrm{Pic}^0(A_{\mathrm{s}})[n] \cong A^\vee[n].$$

Taking Galois cohomology gives a surjective map

$$\mathrm{H}^1(k, \mathrm{H}^1(A_{\mathrm{s}}, \mu_n)) \twoheadrightarrow \mathrm{H}^1(k, \mathrm{Pic}^0(A_{\mathrm{s}}))[n]. \tag{19}$$

Since $\mathrm{H}^1(A_{\mathrm{s}}, \mu_n)$ is a torsion group, its image in $\mathrm{Pic}(A_{\mathrm{s}})$ is contained in $\mathrm{Pic}^0(A_{\mathrm{s}})$, because $\mathrm{NS}\,(A_{\mathrm{s}})$ is torsion-free. Thus the left vertical map in the diagram factors through (19).

By the Kummer sequence for A, the middle vertical map is surjective. It follows that the quotient of $\mathrm{Br}_e(A)[n]$ by $\mathrm{Br}_{e,1/2}(A)[n]$ is a quotient of a subgroup of $\mathrm{H}^2(A_{\mathrm{s}}, \mu_n)$. From the canonical isomorphism

$$\mathrm{H}^2(A_{\mathrm{s}}, \mathbb{Z}/n) = \wedge^2 \mathrm{H}^1(A_{\mathrm{s}}, \mathbb{Z}/n)$$

we see that $[-1]$ acts trivially on $\mathrm{H}^2(A_{\mathrm{s}}, \mu_n)$, hence also on $\mathrm{Br}_e(A)[n]/\mathrm{Br}_{e,1/2}(A)[n]$.

In particular, $[-1]$ acts on $\mathrm{Br}_{e,A}(A)(p')/\mathrm{Br}_{e,1/2}(A)(p')$ trivially. However, as follows from the last claim of Proposition 2.2, $[-1]$ acts on this group at -1. This implies our statement. □

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