Infinite dimensional degree theory and Ramer's finite co-dimensional differential forms

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Abstract

Infinite dimensional degree theory, especially for Fredholm maps with positive index as developed with Tromba, is combined with Ramer's unpublished thesis work on finite co-dimensional differential forms. As an illustrative example the approach of Nicolaescu & Savale to the Gauss-Bonnet-Chern theorem for vector bundles is reworked in this framework. Other examples mentioned are Kokarev & Kuksin's approach to periodic differential equations, and to forced harmonic maps. A discussion about how such forms and their constructions and cohomology relate to constructions for diffusion measures on path and loop spaces is also included.
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1 Introduction

Roald Ramer's 1974 thesis [58] was designed to lay the foundations of a finite co-dimensional deRham-Hodge-Kodaira theory on a class of infinite dimensional manifolds. He built it on Gross's existing calculus [37] using Gaussian measures. To do this, he proved several deep and revolutionary results in stochastic analysis, in particular his now well known transformation of integral formula. Here we will avoid most of this analysis by considering situations with more regularity than he assumed.

The principal aims here are to define his forms and volumes, show how they fit into infinite dimensional degree theory of proper Fredholm maps of index $k \geq 0$, and look at possible generalisations of them to path spaces with diffusion measures. We will also observe how integration along fibres for Ramer's forms relates to conditioning, or "filtering out redundant noise", for diffusions; this suggests applications to other geometrically arising measures.

The $(\infty - k)$-volumes are initially defined on manifolds with a special structure for which there is a prescribed measure class of Borel measures which are locally Gaussian in a smooth sense. They are basically sections of the tensor product of the k-order exterior product of a Hilbert bundle of tangent vectors, "admissible directions", with the line bundle of signed measures absolutely continuous with respect to the given measure class. On a finite dimensional manifold such objects would be a special class of k-dimensional currents, as in [35, section 4.1.7]. On a

Banach space with Gaussian measure, the space of all L^2 $(\infty - *)$-forms can be identified with a tensor product of Bosonic and Fermionic Fock spaces.

As such their host manifold needs little structure. We initially present them in their natural home of abstract Wiener manifolds. These are Fredholm manifolds, the natural home of degree theory for proper Φ_k maps, especially if $k > 0$. Ramer's theory leads to a connection between the degree and an element in the $(\infty - *)$-cohomology, see (30) below. As a toy example, in section 4 we use these theories to interpret Nicolaescu & Savale's approach to the Gauss-Bonnet-Chern theorem in this context, and to show how the Thom form appears there naturally. Earlier, section 2.1.3, described natural abstract Wiener manifold structures on infinite dimensional Grassmanians. This introduces the terminology for the GBC discussion, and also raises the challenge of constructing universal Euler or Thom forms by an extension of Ramer's theory.

Finally, section 5 describes the existing geometric analysis on path spaces and give an example of a generalised version of Ramer's forms which has appeared there, pointing out that operations similar to operations on such forms are fairly standard when using Malliavin's approach to hypoellipticity, or in looking at the structure of path spaces. This suggests their applicability to more general measures, especially those arising from Kokarev & Kuksin's examples in sections 2.1 and 2.2.

This paper is dedicated to the memory of Sir Michael Atiyah. It was about 56 years ago that his suggestion that I might be interested in non-linear Fredholm maps sent me on an exciting path. I have always treasured the general philosophy towards mathematics that I had been privileged to learn from him. Writing this article brought back memories of excitement and enjoyment, especially of those lectures at the, very old, Maths Institute building by him and Raoul Bott on currents and distributional sections of vector bundles as a tool for Lefschetz fixed point theorems.

2 Fredholm maps, Abstract Wiener manifolds, Gaussian measures

2.1 Abstract Wiener spaces and manifolds

The basic local object in the theory is an open subset of an *abstract Wiener space*, AWS, as defined by L. Gross in the late 1960's, [37]. This will be a triple $\{i, H, E\}$ consisting of a continuous linear map $i : H \to E$ of a separable Hilbert space $H, \langle -, - \rangle_H$ into a separable Banachable space E; both real. Recall that a complete topological vector space is said to be *Banachable* or *Hilbertable* if its topology can be determined by a norm, or an inner product, respectively. We will take i to be injective with dense range, so the adjoint of i gives a similar map $j : E^* \to H$. To qualify as an AWS there must exist a Borel probability measure, γ say, on E whose Fourier transform

$$\hat{\gamma} : E^* \to \mathbf{R}, \qquad \hat{\gamma}(\ell) = \int_E e^{i\ell(x)}\, d\gamma(x) \qquad \ell \in E^*$$

is given by

$$\hat{\gamma}(\ell) = e^{-\frac{1}{2}|j(\ell)|_H^2}.$$

This is a strong condition on i. If E is also a Hilbert space it holds iff i is Hilbert-Schmidt. The measure γ is a *Gaussian* measure. Every (non-degenerate, centred) Gaussian measure on E arises from some abstract Wiener space structure on E, unique up to the natural equivalence. From Gross, for an AWS $\{i, H, E\}$

1. Translation by an element x of E preserves sets of γ measure 0 iff $x \in i(H)$;
2. The map i is compact and $i \circ j : E^* \to E$ is nuclear;
3. The image $i[H]$ of H is measurable and if E is infinite dimensional
$$\gamma(i[H]) = 0;$$
4. There are factorisations $i = \alpha \circ i_0$ where $\alpha : E_0 \to E$ is a compact injective linear map of a Banachable space E_0 and $i_0 : H \to E_0$ is an AWS.

The standard example is classical Wiener space with $i : L_0^{2,1}([0,1]; R^m) \to C_0([0,1]; \mathbf{R}^m)$, the inclusion of the finite energy paths into the continuous paths. Here the subscript 0 refers to paths in R^m emanating from the origin. The measure γ is then *classical Wiener measure*, or *Brownian motion measure*. In this case for E_0 in item 4. above we can take the closure of smooth paths in the Holder $C^{0+\alpha}$ norm for $\alpha < \frac{1}{2}$ with i_0 the inclusion.

2.1.1 Change of variable formulae

For an AWS $\{i, H, E\}$ let $F : U \to V$ be a diffeomorphism between open subsets of E of the form

$$F(x) = x + i \circ j \circ \alpha(x) \tag{1}$$

where $\alpha : U \to E^*$ is C^1 with range in some finite dimensional subspace of E^*. Then for bounded measurable $f : V \to \mathbf{R}$ we have

$$\int_V f \, d\gamma = \int_U f \circ F(x) |\Lambda(F)(x)| d\gamma(x) \tag{2}$$

for $\Lambda(F) : U \to \mathbf{R}$ given by

$$\Lambda(F)(x) = \det DF(x) \exp\left\{\alpha(x)(x) - \frac{1}{2}|j \circ \alpha(x)|_H^2\right\}. \tag{3}$$

Here the determinant of the derivative of F is evaluated by restricting $DF(x)$ to the image of $j \circ D\alpha(x)$. This is a primitive form of a result by Gross. The proof is an easy application of the finite dimensional result. Note that only the derivative in H-directions is needed; a philosophy expounded by Gross which has guided most subsequent work. The conditions on F were progressively relaxed by H.H Kuo, R. Ramer [59], S.Kusuoka, and Ustunel & Zakai, see [65]. The relaxations were towards allowing α to be H-differentiable on U into H, but with Hilbert-Schmidt derivatives. This required interpretation of $\alpha(x)(x)$ as a stochastic integral, extending Itô's classical definition, and replacing $\det DF(x)$ by a Fredholm -Carleman determinant, with a renormalisation term which got absorbed by Ramer in his stochastic integral. For our purposes the simplest version will generally suffice. Call a map of the form (1) a *(strong) AW map*.

2.1.2 Wiener manifolds

Suppose P is a separable metrisable Banach manifold modelled on E. For us a C^r (strong) Wiener structure on P modelled on an AWS $\{i, H, E\}$ will be given by a C^r atlas, $\{U_i, \phi_i\}_i$ for P whose interchanges of charts $\phi_i \circ \phi_j^{-1}$ are strong AW maps. These are called W^r-structures in [19]. When E has a Schauder base, admits C^∞ partitions of unity, and has contractible general linear group such structures exist. Indeed by [18], such a manifold is diffeomorphic to an open subset of E, so admits a trivial strong Wiener structure.

A codimension $k < \infty$ closed subspace $E^{\infty-k}$ of E has a natural AWS structure inherited from $\{i, H, E\}$:

Set $H^{\infty-k} = i^{-1}[E^{\infty-k}]$ with its induced Hilbert space inner product. Let H_k be the orthogonal complement of $H^{\infty-k}$ in H with its Euclidean structure. It is the image under j of the annihilator E_k of $E^{\infty-k}$ in E^* which is usually identified with H_k and $i[H_k]$. Then $E = E_k \oplus E^{\infty-k}$. Also $\{i^{\infty-k}, H^{\infty-k}, E^{\infty-k}\}$ with $i^{\infty-k}$ the restriction of i is an AWS. Let $\gamma^{\infty-k}$ denote its measure. Then $\gamma = \gamma^k \otimes \gamma^{\infty-k}$ where γ^k is the standard Gaussian measure on E_k as a copy of $\mathbf{R}^k$. We often write $(E^{\infty-k})^\perp$ for E_k, or $E_k^\perp$ for $E^{\infty-k}$.

Using this a codimension k submanifold S of a Wiener manifold modelled on $\{i, H, E\}$ has an induced Wiener structure modelled on $\{i^{\infty-k}, H^{\infty-k}, E^{\infty-k}\}$ for which the inclusion $S \to M$ can be given locally in charts by perturbations of the inclusion $a \mapsto (0, a)$ of $E^{\infty-k} \to E_k \oplus E^{\infty-k}$, the perturbations being maps into $j^{\infty-k}[(E^{\infty-k})^*]$. In particular a finite co-dimensional submanifold of E has a natural Wiener structure.

A Wiener structure as above on P determines a line bundle $\mathcal{W}(P)$ over P with transition functions $\Lambda_{k,\ell} : U_k \cap U_\ell \to \mathbf{R}$ given by

$$\Lambda_{k,\ell}(x) = \Lambda(\phi_k \circ \phi_\ell^{-1})(\phi_\ell(x)) \qquad x \in U_k \cap U_\ell. \tag{4}$$

The Wiener structure is *orientable* if $\mathcal{W}(P)$ is trivializable. Sections of $\mathcal{W}(P)$ are *Wiener densities*. Just as in finite dimensions, by (2) the "modulus" $|\zeta|$ of such a measurable density ζ determines a Borel measure $\mu_{|\zeta|}$ on P. Given orientability an oriented atlas can be chosen, i.e. one with each $\det(\phi_k \circ \phi_\ell^{-1}) > 0$. In this case, we can integrate any measurable Wiener density ζ over a Borel $A \in P$ by

$$\int_{A\cap U_\ell} \zeta = \int_{\phi_\ell[A\cap U_\ell]} \zeta_\ell \circ \phi_\ell^{-1}\, d\gamma \qquad \text{any oriented AW chart } (U_\ell, \phi_\ell) \tag{5}$$

provided $\mu_{|\zeta|}(A) < \infty$.

2.1.3 Example: Infinite dimensional Grassmanians

For an abstract Wiener space $\{i, H, E\}$ we can form the Grassman manifolds $G(k; E^*)$, $G(k; H)$, and $G(k; E)$ of k-dimensional subspaces in E^*, H, E respectively. For a fixed decomposition $E = E_k \times E^{\infty-k}$ as above, these are C^∞, even analytic, manifolds modelled on $L(E_k; (ij)^{-1}[E^{\infty-k}]), L(H_k; H^{\infty-k})$, and $L(E_k; E^{\infty-k})$ respectively, remembering $H_k = j[E_k]$. The chart for $G(k; E)$ centred at a subspace $\alpha \subset E^*$ is given by identifying E_k with α and mapping

nearby subspaces to the linear maps in $L(E_k; E^{\infty-k})$ which have them as their graphs, as in A. Douady's thesis, eg see [41]. Such charts cover $G(k; E)$ and can be seen to give an AWM structure modelled on the AWS
$\{\circ i^{\infty-k}, L(\mathbf{R}^k; H^{\infty-k}), L(\mathbf{R}^k; E^{\infty-k})\}$.

We will not use or analyse this structure here, but it is related to some of the constructions below. We note that there are the tautological vector bundles:

$$p^{E^*} : \mathcal{V}^{\text{univ},E^*} \to G(k; E^*); \qquad p^H : \mathcal{V}^{\text{univ},H} \to G(k; H);$$

and $\qquad p^E : \mathcal{V}^{\text{univ},E} \to G(k; E)$.

We can identify E, H respectively as sections of the first two of these by maps

$$\mathbf{X}^{E^*} : G(k; E^*) \times E \to \mathcal{V}^{\text{univ},E^*} \qquad (\alpha, U) \mapsto P_\alpha(U) \tag{6}$$

where $P_\alpha : E \to \alpha$ is the projection (using H), and similarly for p^H. However E^* is naturally identified as sections of the dual bundle

$$p^{E,*} : (\mathcal{V}^{\text{univ},E})^* \to G(k; E)$$

via $\mathbf{X}^E(\alpha, \ell) = \ell|_\alpha$. These $\mathbf{X}$ project the flat connection on the product vector bundle to a connection on the tautological bundles or their dual. The Hilbert bundle p^H has the inner product induced from that of H and the connection is metric for this. It is the universal connection of Narasimhan & Ramanan, [53]. The same holds for p^{E^*} using $j : E^* \to H$. Suppose H is infinite dimensional. Then $GL(H)$ is contractible by Kuiper's theorem, and so $G(k; H)$ is a classifying space for k-plane bundles, eg see [13] or [41]. Since all three Grassmanians are homotopy equivalent by standard techniques, the others are also classifying spaces.

Now suppose $p : \mathcal{V} \to M$ is a Riemannian k-plane bundle over an n- dimensional manifold M with a metric connection. By Narasimhan & Ramanan [53], see also [41], we can obtain a smooth classifying map $\mathcal{Y} : M \to G(k; E^*)$ with an isometric connection preserving $\tilde{\mathcal{Y}} : \mathcal{V} \to \mathcal{V}^{\text{univ},E^*}$. In particular this pulls back X^{E^*} to give $X : M \times E \to \mathcal{V}$ such that the restrictions $X(x) \circ i : H \to \mathcal{V}_x$ induce the inner products on each $\mathcal{V}_x$, and such that X projects the flat connection to the given one on $\mathcal{V}$. It can then be convenient to replace E by the space of sections $\tilde{E}$ given by $\{X(-, e) : e \in E\}$ and X by the evaluation map $\rho : M \times \tilde{E} \to \mathcal{V}$. Narasimhan & Ramanan's construction leads to a finite dimensional space $\tilde{E}$. It would be satisfying to know whether every metric connection on $\mathcal{V}$ can be obtained with $\tilde{E}$ containing all C^∞ sections, or at least dense in the continuous sections of $\mathcal{V}$.

Conversely, any such $X : M \times E \to \mathcal{V}$ is induced by $\mathcal{Y} : M \to G(k; E^*)$ with $\mathcal{Y}(x) = X(x, -)^*[\mathcal{V}_x]$.

2.1.4 Riemannian metrics and Wiener data

A smooth Wiener structure on P determines a Hilbertable vector bundle $\mathcal{H} := \mathcal{H}P$ over P, with an inclusion $i_P : \mathcal{H} \to TP$. In charts $\mathcal{H}_x$ is tangent to the translate of $i[H]$ through x. It gives each tangent space a pre-AWS structure. Also it is integrable. The relevant differential calculus on P is H-differentiation, differentiation in $\mathcal{H}$ directions, following Gross's philosophy.

Since at each point the derivative of a strong AW map is a finite rank perturbation of the identity, a Wiener structure on P reduces the structure group of TP to the group $GL_N(E)$ of linear isomorphisms of E which are perturbations of the identity by nuclear operators. More precisely it reduces it to the smaller group $GL_W(E)$ where the perturbations are linear maps into $i \circ jE^*$. Similarly $\mathcal{H}$ has structure group $GL_N(H)$. These groups have the homotopy type of $GL(\infty)$, and so TP and $\mathcal{H}$ determine an element of $\widetilde{KO}(P)$. In particular by an *orientation* we will mean a further reduction to the identity component of these groups.

A Wiener density on P can be obtained from a *Wiener-Riemannian* metric $\mathcal{G}$ on $\mathcal{H}$ together with a *position field* Z. This works best if we require $\mathcal{G}$ to give a Hilbert inner product $\langle -, - \rangle_x$ to each $\mathcal{H}_x$, $x \in P$, which in a chart $\{U_i, \phi_i\}$, is given by $\langle G^i_x u, G^i_x v\rangle_H$ for smooth $G^i_- : U_i \to GL_W(E)$. For us the position field can be required to be locally given by $Z^i : U_i \to E$ with $Z^i(x) = \phi_i(x) + i \circ jz^i(x)$ for smooth $z^i : U^i \to E^*$. If P is oriented we use this *Wiener data* $\{\mathcal{G}, Z\}$ to obtain a Wiener density, $\zeta(\mathcal{G}, Z)$, given in an oriented chart by

$$\zeta(\mathcal{G}, Z)(x) = |\det G^i_x|\lambda^i(x) \qquad x \in U_i \tag{7}$$

for

$$\lambda^i(x) = \exp\big\{ -\langle G^i_x Z^i(x) - \phi_i(x), \phi_i(x)\rangle_H - \frac{1}{2}|\phi_i(x) - G^i_x Z^i(x)|^2_H \big\}. \tag{8}$$

If P is not oriented we get a measure $\mu_{(\mathcal{G},Z)}$ using $\zeta(\mathcal{G}, Z)$ as defined above in any chart. Strictly speaking $\zeta(\mathcal{G}, Z)$ should be considered an *absolute scalar density* given in any chart by (7). Given an orientation we will write the oriented version as $\zeta^+(\mathcal{G}|_S, \lambda_Z|_S)$.

For $P = E$ the Gaussian measure γ corresponds to $(\mathcal{G}, Z)$ with G_x the identity and $Z(x) = x$ for all $x \in E$. As for general P this induces Wiener data $(\mathcal{G}|_S, Proj^S Z)$ on finite co-dimensional submanifolds S of P, where $Proj^S_x$: $T_xP \to T_xS$ is the projection. Alternatively there is a measure $\mu_{\zeta(\mathcal{G}|_S, \lambda_Z|_S)}$ on S using the restriction of λ defined in (8). When S is oriented we have the density $\zeta^+(\mathcal{G}|_S, \lambda_Z|_S)$.

This theory was developed particularly by H.-H Kuo, V.Goodman, and M.A. Piech in the 1970's. It gives a direct extension of finite dimensional geometric analysis. However in many infinite dimensional manifolds of interest there appears to be no natural choice of an AW manifold structure. For example this is true for P the space of continuous paths on a compact Riemannian manifold M. The much studied Brownian motion measure on this space in general does not come from a smooth Wiener structure. For differential calculus on such path spaces there is an almost surely defined Hilbert bundle, the "Bismut tangent bundle", and differentiation is usually taken in these directions. However for non-flat M this bundle is not integrable. See section 5.2 below. Nevertheless from Kokarev & Kuksin, [39], we see interesting examples of Wiener manifolds arise from certain geometrically defined Fredholm maps, as described next.

2.2 Fredholm maps and degree theory

Following Smale [63], a C^r map $F : P \to \mathcal{E}$ of connected Banach manifolds, for $r \geq 1$, is *Fredholm* of *index k*, or Φ_k, if its derivative map $T_xF : T_xP \to T_{F(x)}\mathcal{E}$ at any point x is a Fredholm operator of index k. That means

$$\dim\ker T_xF - \dim\left(T_{F(x)}P/T_xF[T_xP]\right) = k < \infty \qquad\qquad x \in P.$$

A Fredholm operator of index zero differs from a linear isomorphism by a finite rank operator. From this and the inverse function theorem, as in [63] we have: *Suppose* $F : P \to \mathcal{E}$ *is a* Φ_0 *map and* $\mathcal{E}$ $\{$*has* a strong Wiener structure modelled on $\{i, H, E\}$. Then there is a Wiener structure on P with respect to which, in the charts of the two structures, F is represented by a strong AW map. It is unique up to equivalence. The proof is essentially a corollary of the result for strong layer structures [32].

In the situation above, a Φ_0-map $F : P \to \mathcal{E}$ determines a linear map $\mathcal{W}(F) : F^*(\mathcal{W}(\mathcal{E})) \to \mathcal{W}(P)$ using the induced Wiener structure on P. Over suitable Wiener charts in which F is represented by an AW map $F^{\alpha,j}$ say, it is given by pointwise multiplication by $\Lambda(F^{\alpha,j})$. A density ξ on $\mathcal{E}$ pulls back this way to a density, $F^*(\xi)$, on P.

2.2.1 Degree theory and the Sard measures

In [63] Smale showed that the set of critical values of a Φ_k-map $F : P \to \mathcal{E}$ which is C^r with $r > \max\{k, 0\}$, has first category in $\mathcal{E}$. When F is proper and Φ_0 this enabled him to define a mod 2 degree, as the number, mod 2, of points in the inverse image of a regular value of F. When there is an oriented Wiener structure on $\mathcal{E}$ an orientation for the induced AW manifold structure on P determines a sign $\operatorname{sgn} T_xF \in \{-1, +1\}$ for the derivative of F at a regular point $x \in P$, using the determinant of its derivative in charts. We can say F is *oriented*. See [4], and its references, and [39] for other definitions. This gives an oriented degree, [32], $\operatorname{Deg}(F) \in \mathbb{Z}$ defined by

$$\operatorname{Deg}(F) = \sum_{x \in F^{-1}(y)} \operatorname{sgn} T_xF \qquad \text{any regular value } \ y \ \text{ of } \ F. \tag{9}$$

We will say that a Borel measure μ on $\mathcal{E}$ is a *Sard measure* if the set of critical values of such F have measure zero. As observed in [19] essentially the same proof as used by Smale shows that this holds for measures on Wiener manifolds determined by a Wiener density. A slight generalisation is given in [4] with some discussion, see also the monograph [65], and there are deep results by Kusuoka, [42], but extending this to a wider class of measures remains a difficult problem.

Given such Sard measure μ on $\mathcal{E}$ and a Φ_0 map F there is a unique pull back measure $F^*(\mu)$ with the property that, with it, the critical points of F have measure zero and F is measure preserving on any open set of P on which it is a diffeomorphism. See [66], [4]. If $\mu = \mu_{|\xi|}$ for a Wiener density ξ on $\mathcal{E}$ then $F^*(\mu_{|\xi|}) = \mu_{|F^*(\xi)|}$. From [4] we have:

Suppose $F : P \to \mathcal{E}$ is a proper Φ_0 map with an orientation and μ a locally finite Sard measure on $\mathcal{E}$ then for any measurable $g : \mathcal{E} \to \mathbf{R}$ such that $g \circ F$ is $F^*(\mu)$-integrable

$$\int_P g(F(x)) \operatorname{sgn} T_x F \, d(F^*\mu)(x) = \operatorname{Deg}(F) \int_{\mathcal{E}} g \, d\mu. \tag{10}$$

2.3 Examples of Φ_0 maps

The first two examples are selected from Kokarev & Kuksin, [39]. Precise details are given there, together with conditions for orientability and calculations of the degrees.

Example 2.1 Periodic orbits: Take $\mathcal{E}$ to be a space of non-autonomous periodic vector fields

$$g : S^1 \times M \to TM$$

on a smooth compact manifold M, and $\mathcal{F}(S^1; M)$ a suitable connected space of differentiable loops, $u : S^1 \to M$. Define P by

$$P = \{(u, g) \in \mathcal{F}(S^1; M) \times \mathcal{E} \text{ such that } \frac{du}{dt} = g(t, u(t)) \quad t \in S^1\}. \tag{11}$$

Take $F : P \to \mathcal{E}$ to be the projection. The spaces can be chosen so that P is a smooth Banach manifold and F is Φ_0 and proper. When defined the degree gives the algebraic number of solutions in a given homotopy class to $\frac{du}{dt} = g(t, u(t))$ for a generic g. For contractible loops the degree is the Euler characteristic χ_M, on other components it vanishes.

Example 2.2 Forced harmonic maps: Let $\mathcal{F}(M; N)$ be a connected space of maps $f : M \to N$ of finite dimensional Riemannian manifolds with M compact and oriented, and let $\mathcal{E}$ be a suitable space of "non-autonomous" vector fields $v : M \to TN$. Define P by

$$P = \{(f, v) \in \mathcal{F}(M; N) \times \mathcal{E} \text{ such that } \triangle f(x) = v(x, f(x)) \quad x \in M\} \tag{12}$$

where $\triangle$ is the non-linear Laplacian of Eells & Sampson. Take $F : P \to \mathcal{E}$ to be the projection. With suitable choices F will be proper and Φ_0.

When N has negative sectional curvature and $\mathcal{F}(M; N)$ is the component of P consisting of null-homotopic maps, the degree of F is again χ_M, with the degree vanishing for other components.

The third example comes from [4] but is based on Kusuoka's approach in [42].

Example 2.3 McKean-Singer formula: Let M be a compact oriented Riemannian manfold. For $T > 0$ and large $s > 0$ take $\mathcal{E}$ to be the manifold $C_{id} \operatorname{Diff}^s M$ of continuous paths $\xi : [0, T] \to \operatorname{Diff}^s M$, starting at the identity where $\operatorname{Diff}^s M$ is the Hilbert manifold of H^s diffeomorphisms of M. Define

$$P = \{(x, \xi) \in M \times \mathcal{E} : \xi_T(x) = x\}, \tag{13}$$

with $F : P \to \mathcal{E}$ the projection.
Again F is Φ_0, proper, and with a natural orientation. We see $\operatorname{Deg} F$ is the fixed

point index of ξ_T for a generic ξ, and so by the Lefschetz theorem it is the Euler characteristic $\chi(M)$ of M.
There is a choice of measures on $\mathcal{E}$. A natural one is the law of the stochastic flow of Brownian motions on M determined by a gradient stochastic differential equation on M, see section 5.1 below. Let μ be one of these. It is only a conjecture that μ is a Sard measure. Assuming it is we can use the integral formula (10) to obtain a formula for $\chi(M)$. Analysis of this shows, via the Berezin formula, that the integral is just the supertrace, Str$(e^{T\triangle})$, of the Hodge-Kodaira heat semigroup acting on forms on M, as in McKean & Singer [50].

To get round the question of the Sard property of μ it is possible to use the more technically sophisticated approach of Kusuoka, or, as in [4], to approximate μ by the laws of more regular processes than Brownian motions. The latter gives a formula for χ_M in terms of a supertrace related to generalised Ornstein-Uhlenbeck operators, as in the hypoelliptic Laplacian theory of Bismut et al. One then has to use their techniques, [10], to obtain convergence of the supertraces. Rough path theory might give a more direct approach, and in greater generality.

For other types of examples see Getzler [36] and the final chapter of Tromba et al [16].

2.4 Φ_k maps for $k > 0$

When $F : P \to \mathcal{E}$ is Fredholm of index $k > 0$ we obtain a Φ_0-map as the composition with the inclusion

$$P \to \mathcal{E} \to \mathbf{R}^k \times \mathcal{E}$$

and so a Wiener structure for $\mathcal{E}$ modelled on $\{i, H, E\}$ induces one on P modelled on $\{id_{\mathbf{R}^k} \times i, \mathbf{R}^k \times H, \mathbf{R}^k \times E\}$, with respect to which F differs locally from the projection $\mathbf{R}^k \times E \to E$ by a finite rank map into E^*. The model abstract Wiener measure, $\tilde{\gamma}$ on $\mathbf{R}^k \times E$ is $\gamma^k \otimes \gamma$ for the standard Gaussian γ^k on $\mathbf{R}^k$.

A version of Pontryagin-Thom theory, [51], now extends the degree theory described for $k = 0$, see [32]. Assuming F is proper the inverse image X say of a regular value y is now a compact k-dimensional framed submanifold of P whose framed co-bordism class in P is independent of the choice of y. If the restriction to X of the tangent bundle of TP is trivial as an element of $\widetilde{KO(X)}$ then X is a framed π-manifold, and by Pontryagin-Thom theory corresponds to an element in the k-th stable homotopy group of spheres. This will hold if P is k-connected.

Examples of non-linear Φ_k-maps, with $k > 0$, have not been studied so much. However Nirenberg in [56] applied these ideas to an oblique boundary value problem. He showed existence of solutions given the non-vanishing of a certain stable homotopy class of spheres. See also Zvagin & Ratiner [67] and their bibliography.

Such maps also arise from modifying Example 2.3 above by starting with a foliated manifold M, and replacing $\mathcal{E}$ by paths on the group of H^s diffeomorphisms of M which preserve the foliation. A McKean-Singer formula in this context is given in [52], as is a Gauss-Bonnet-Chern theorem. To obtain the McKean-Singer formula for foliated manifolds by Kusuoka's approach or its modification described above we need a substitute for pull-back measures when $k > 0$. One approach would be to investigate "transversal measures" , transversality being with respect to the fibres of F. Another, as more common in finite dimensions, is to use differential forms of co-dimension k. These we describe next.

3 Ramer's finite co-dimensional forms and volumes

3.1 H-vector fields and forms

Suppose P is an oriented Wiener manifold modelled on $\{i, H, E\}$. Following Gross it is natural to consider H-vector fields, namely sections of $\mathcal{H} \to P$, and H-forms ie sections of $(\bigwedge^p \mathcal{H})^* \to P$, $p = 0, 1, ...$ Here and in what follows we use the Hilbert space completion of the algebraic tensor product in order to define exterior powers, or more general tensor products, of a Hilbert space, [58], [25]. Write the space of smooth sections as $\Gamma\mathcal{H}$ etc. We can define an exterior derivative

$$d : \Gamma(\wedge^p \mathcal{H})^* \to \Gamma(\wedge^{p+1}\mathcal{H})^*$$

by the usual formula involving Lie derivatives $\mathcal{L}_U$: for an H 1-form ϕ and H-vector fields U, V

$$d\phi(U(x) \wedge V(x)) = \mathcal{L}_U \phi(V(-))(x) - \mathcal{L}_V \phi(U(-))(x) - \phi([U, V])(x). \quad (14)$$

This makes sense because $\Gamma\mathcal{H}$ is closed under Lie brackets. (That this does not hold in general for path spaces on Riemannian manifolds with Wiener measure leads to difficulties there, eg see [29], and the discussion below in section 5.1.)

This gives a de-Rham complex as usual. Its cohomology is the usual real cohomology of P provided E admits smooth partitions of unity, or more generally if we only require some differentiability of our forms in H-directions, see [58].

3.1.1 Finite codimensional forms

For an n-dimensional Hilbert space, $H, \langle -, - \rangle$ an orientation determines an element ω of $\bigwedge^n H^*$ given by $\ell^1 \wedge ... \wedge \ell^n$ for any oriented orthonormal base $\ell^1, .., \ell^n$ of H^*. From this arises a linear isomorphism between $\bigwedge^{n-k} H^*$, which can and will be identified with $(\bigwedge^{n-k} H)^*$, and $\bigwedge^k H$ for each $0 \leq k \leq n$. Namely to any $\phi \in (\bigwedge^{n-k} H)^*$ we assign the interior product $\iota_\phi(e_1 \wedge ... \wedge e_n)$ for $e_1, ..., e_n$ the dual base for H, as in eg [35]. Ramer uses this to define $\bigwedge^{\infty-k} H^*$, for $k = 0, 1, ...$ and then finite co-dimensional forms on Wiener manifolds. For this he constructs a determinantal structure, following ideas of Ruston, Grothendieck, Leżański & Sikorski. We will avoid introducing this, but at the cost of not proving smoothness of some of our constructions over sets of measure zero determined by critical points. This is important when we want to use versions of the divergence theorem, however it enables us to give simplified versions of Ramer's constructions.

Kusuoka in [44] introduced more general types of forms, which he called $(\infty+n)$-forms. See also Stacey's thesis [64] for semi-infinite forms in this context.

For our Wiener manifold P we can take the frame bundle of $\mathcal{H}$ with group $GL_N(H)$ and define $\bigwedge^\infty \mathcal{H}$ to be the associated line bundle with fibre $\mathbf{R}$ using the action of $A \in GL_N(H)$ by multiplication by $\det A$, and similarly $\bigwedge^\infty \mathcal{H}^*$ using the action $A \mapsto \det A^{-1}$. Then $\bigwedge^\infty \mathcal{H}^*$ is canonically isomorphic to the dual of $\bigwedge^\infty \mathcal{H}$. For a more concrete representation we could take the fibre of $\bigwedge^\infty \mathcal{H}$ to be $GL_N(H)/SL_N(H)$, so sections of $\bigwedge^\infty \mathcal{H}$ will give reductions of the group of $\mathcal{H}$ to the special linear group $SL_N(H)$. To represent elements of $\bigwedge^\infty H$ we can fix an orthonormal base $\{e_i\}_{i=1}^\infty$ for $H, \langle -, - \rangle_H$, related to an orientation if that is given, and consider elements as equivalence classes of bases of H of the form $\{Ae_i\}_{i=1}^\infty$ for $A \in GL_N(H)$, with $\det A$ determining the class.

The action of $A \in GL_N(H)$ on $\bigwedge^p H$ and $\bigwedge^p H^*$, (for finite p), will be by the usual $\bigwedge^p(A)$ and $\bigwedge^p(A^{-1})^*$. The determinental structure in [58] led to the action $(\det A)^{-1} \bigwedge^p(A)$ on $\bigwedge^p H$. Consequently we will make the definition:

Definition 3.1 For $k = 0, 1, ...$ an $(\infty - k)$-form is a section of the bundle $\bigwedge^{\infty-k} \mathcal{H}^*$ where

$$\bigwedge^{\infty-k} \mathcal{H}^* := \bigwedge^{k} \mathcal{H} \otimes \bigwedge^{\infty} \mathcal{H}^*. \tag{15}$$

When P is oriented if we choose an *orientation form* ω^P for P, ie a strictly positive section of $\bigwedge^\infty \mathcal{H}$ then we can write any $(\infty - k)$-form Ψ as

$$\Psi \ = V \otimes \omega^P \tag{16}$$

for some $V \in \bigwedge^k \mathcal{H}$.

Note that from the standard interior product pairing for $r \leq k$

$$\bigwedge^{r} H^* \otimes \bigwedge^{k} H \to \bigwedge^{k-r} H$$

we get an *exterior product*

$$\bigwedge^{r} H^* \otimes \bigwedge^{\infty-k} H^* \to \bigwedge^{\infty-k+r} H^* \tag{17}$$

giving an $(\infty - k + r)$-form $\phi \wedge \Psi$ from an r-form ϕ and an $(\infty - k)$-form Ψ.

There is also an *interior product*. For $V \in \bigwedge^r H$:

$$\iota_V : \bigwedge^{\infty-k} H^* \to \bigwedge^{\infty-k-r} H^* \tag{18}$$

defined by $U \otimes \zeta \mapsto (V \wedge U) \otimes \zeta$ for $U \in \bigwedge^k H$ and $\zeta \in \bigwedge^\infty H^*$.

Ramer defined exterior derivative operators taking a class of $(\infty - k)$-forms to $(\infty - k + 1)$-forms and so extended the deRham complex. As in [35] the exterior derivative corresponds to a version of the divergence. See section 3.2.1 below .

3.2 Wiener volumes.

From the change of variable formula (2) and (3) to have an analogue of the integration theory for forms in finite dimensions the use of Gaussian measures requires more than just differential forms. One way to do this is, as done by Ramer, is to take a bundle $\mathcal{G} \to M$, the *Gauss bundle*, fibre $\mathbf{R} > 0$, with transition densities involving only the exponential term in (3) namely

$$\exp\left\{\alpha(x)(x) - \frac{1}{2}|i \circ j\alpha_{k,\ell}(x)|_H^2\right\}$$

when the interchange of chart map $\phi_k \circ \phi_\ell^{-1}(x) = x + i \circ j\alpha_{k.\ell}(x)$. Then define, for k=0,1,...

$$\mathcal{W}^{\infty-k}(P) = \bigwedge^{\infty-k} \mathcal{H}^* \otimes \mathcal{G} \tag{19}$$

Thus our Wiener density bundle $\mathcal{W}(P)$ of Section 2.1.2 is just $\mathcal{W}^{\infty}(P) = \bigwedge^{\infty} \mathcal{H}^* \otimes \mathcal{G}$ and $\mathcal{W}^{\infty-k}(P) = \wedge^k(\mathcal{H}) \otimes \mathcal{W}(P)$.

Let $\Omega^{\infty-k}$ or $\Omega^{\infty-k}(P)$ denote the space of sections (in default smooth) of $\mathcal{W}^{\infty-k}(P)$. These will be called $(\infty - k)$-volumes. When $P = E$ with E an AWS with measure γ we write ζ_γ for the corresponding element of Ω^∞ when an orientation is chosen for E.

Note that given an orientation and a positive $\zeta \in \Omega^\infty$ any $\Psi \in \Omega^{\infty-k}$ is determined by some $V \in \Gamma \bigwedge^k \mathcal{H}$. We write

$$\Psi = V \otimes \zeta. \tag{20}$$

Given a Wiener Riemannian metric on $\mathcal{H}$ we will say that Ψ as in (20) is in L^p if $|V| \in L^p(P, \mu_\zeta)$. For $p = 1$ this is independent of the representation (20); for other p we should specify the density ζ being used.

There is no canonical section of $\mathcal{G}$, but Ramer showed it has a canonical connection. This determines an extension of exterior differentiation to an operator $\tilde{d}$ acting from a domain of $(\infty - k)$-volumes to $(\infty - k + 1)$-volumes which is determined by a generalised divergence. Using (27) below it satisfies

$$\tilde{d}(\phi \wedge \Psi) = d\phi \wedge \Psi + (-1)^k \phi \wedge \tilde{d}\Psi \tag{21}$$

for smooth k-forms ϕ and $(\infty - p)$-form Ψ when $p \geq k$.

3.2.1 Generalised divergences and exterior differentiation

Given a measure μ on a smooth Banach manifold P and an adequately large class $\mathcal{C}$ of smooth differential forms on P a k-vector field V on M is said to *have a divergence*, with respect to μ and $\mathcal{C}$ if there exists a $(k-1)$-vector field $Div_\mu V$ on P such that for all $(k-1)$-forms $\phi \in \mathcal{C}$ there is an integration by parts formula

$$\int_P d\phi(V)\, d\mu = -\int_P \phi(Div_\mu V)\, d\mu \tag{22}$$

with both integrals existing. Here μ could be a signed measure. We will take $\mathcal{C}$ to consist of C^1 forms ϕ which together with $d\phi$ are in $\bigcap_{1\leq p\leq\infty} L^p$ for some given metric.

As usual for a C^1 function f we have

$$Div_{f\mu}(V) = f Div_\mu V + df(V). \tag{23}$$

When such a divergence exists and Riemannian metrics or Finsler metrics are available it is often possible to consider such V which lie in L^p, some $1 \leq p < \infty$, and restrict and then take closure of Div_μ to obtain a closed linear operator $\mathrm{div}_\mu^{(p)}$ between L^p spaces. For P an AWM with Wiener-Riemannian metric and μ given by a smooth density this gives a densely defined closed operator on $L^p\Gamma \bigwedge^k \mathcal{H}$, the space of L^p k-H-vector fields on P,

$$\mathrm{div}_\mu^{(p)} : \mathrm{Dom}(\mathrm{div}_\mu^{(p)}) \subset L^p\Gamma \bigwedge^k \mathcal{H} \to L^p\Gamma \bigwedge^{k-1} \mathcal{H}.$$

This involves non-anticipating stochastic integrals, as developed by Ramer [58], and independently for vector fields, at essentially the same time, by Skorohod. For discussions in generality for vector fields see Bogachev [12] and [30]. For diffusion measures see section 5.1 below.

An important result for AWM from [31], [59] is that if there is a complete Wiener-Riemannian metric for which an H-vector field V is in $\mathrm{Dom}(\mathrm{div}_\mu^{(1)})$ then V has a divergence, namely $\mathrm{div}^{(1)} V$ and in particular

$$\int_P \mathrm{div}^{(1)} V d\mu = 0. \tag{24}$$

Suppose P has a complete Wiener Riemannian metric. If we require the exterior derivative $\tilde{d}$ on finite co-dimensional volumes to satisfy (21) and to be such that for an $(\infty - 1)$-form Ψ with $\tilde{d}\Psi$ integrable we have

$$\int_P \tilde{d}\Psi = 0 \tag{25}$$

then the exterior derivative of an $(\infty - k)$-form Ψ is given by

$$\tilde{d}\Psi = (-1)^k Div_{\mu_{|\zeta|}}(V) \otimes \zeta \tag{26}$$

for $\Psi = V \otimes \zeta$ when V has a divergence which is a section of $\bigwedge^{k+1} \mathcal{H}$.

When the metric on P is complete and $\mu_{|\zeta|}$ is finite, (25), shows that the infinity L^2 cohomology is not trivial. However if we work for L^2 volumes the adjoint d^* acting on k-H-forms corresponds via the duality with k-H-vector fields to $\mathrm{div}_\mu^{(p)}$. Thus Shigekawa's version of Hodge theory for an abstract Wiener space E together with his vanishing results, [62], imply that for E the L^2 finite co-dimensional cohomology vanishes except for the top dimension and the top dimensional cohomology is one-dimensional.

Note that the L^1 $(\infty - *)$ -cohomology is independent of the choice of measure in the given measure class. That's not surprising since so is the usual L^∞-cohomology.

3.2.2 Main properties of $(\infty - k)$-volumes

Exterior products: For $0 \leq m \leq k$ exterior multiplication determines a pairing

$$\Gamma \wedge^m \mathcal{H}^* \otimes \Omega^{\infty-k} \to \Omega^{\infty-k+m} \quad , \quad \alpha \otimes \Psi \mapsto \alpha \wedge \Psi \tag{27}$$

with

$$\tilde{d}(\alpha \wedge \Psi) = d\alpha \wedge \Psi + (-1)^m \alpha \wedge \tilde{d}\Psi.$$

In particular, if $m = k$, we can consider the integral $\int_P \alpha \wedge \Psi$. It will exist at least for α of sufficiently small support

Interior products: Given $V \in \Gamma \bigwedge^r(\mathcal{H})$, from (18) we have the interior product

$$\iota_V : \Omega^{\infty-k} \to \Omega^{\infty-k-r}.$$

For $k = 0$ we have $\iota_V \zeta = V \otimes \zeta$.

Pull backs: For an abstract Wiener space $\{i, H, E\}$ and a Banach manifold $\mathcal{M}$, a smooth Φ_k-map $F : \mathcal{M} \to E$, $k \geq 0$, induces a Wiener structure on $\mathcal{M}$ as in Section 2.4, with model $\{\tilde{i}, R^k \times H, R^k \times E\}$ for $\tilde{i}(e, h) = (e, i(h))$. If this structure is orientable a choice of orientation determines an $(\infty - k)$-volume $F^*(\zeta_\gamma)$ on $\mathcal{M}$ such that for k-H-forms ϕ on $\mathcal{M}$ we have

$$\int_{\mathcal{M}} \phi \wedge F^*(\zeta_\gamma) = (2\pi)^{-\frac{k}{2}} \int_E \{\int_{F^{-1}(y)} \phi\} \, d\gamma(y). \tag{28}$$

Moreover $F^*(\zeta_\gamma)$ is closed.

Here we use the orientation induced on the k-dimensional submanifolds $F^{-1}(y)$ when y is a regular value of F. It is induced by the normal framing together with the orientation of $\mathcal{M}$. We have also used the Sard property of γ.

To construct $F^*(\zeta_\gamma)$ consider a regular point x of F. Then F is a submersion near x. Let S_x be the fibre through x. Choose an oriented basis $\tilde{e}_1, , ..., \tilde{e}_k$ for $T_x S_x = \ker T_x F$ in $T_x\mathcal{M}$. Set $V = \tilde{e}_1 \wedge ... \wedge \tilde{e}_k \in \bigwedge^k \mathcal{H}_x\mathcal{M}$. Let $\tilde{\ell}^1, ..., \tilde{\ell}^k$ be the dual base for $T_x^* S$ extended to vanish on a chosen transversal to $\ker T_x F$. Let $\tilde{\ell}^{k+1}, \tilde{\ell}^{k+2}, ...$ be the pull backs $T_x F^* \ell^1, T_x F^* \ell^2, ...$ of an oriented orthonormal base for H^*. Then $\{\tilde{\ell}^a\}_{a=1}^{\infty}$ restricts to a base for $\mathcal{H}_x^*\mathcal{M}$ and together with V determines an element of $\bigwedge^{\infty-k} {\mathcal{H}_x}^*$.

For example we can find a chart about x in which F is represented as the projection $\pi_2 : U^1 \times U^2 \to U^2$ with U^1 open in $\mathbf{R}^k$ and U^2 open in E. Our pull back $(\infty - k)$-form is then represented by $\frac{\partial}{\partial x_1} \wedge ... \wedge \frac{\partial}{\partial x_k}$ for the standard basis of $\mathbf{R}^k$ tensored with the canonical element of $\bigwedge^\infty (\mathbf{R}^k \times H)^*$.

In formula (28) the integral of ϕ over each fibre of F is taken in the usual Lebesgue measure sense, not Gaussian. Since we took a Gaussian measure $\gamma^k \otimes \gamma$ on $\mathbf{R}^k \times H$ rather than a hybrid one we need to get rid of the finite dimensional Gaussian density by taking our section of $G(\mathcal{M})$ in the above chart to be $(e, z) \mapsto \exp\left\{\frac{1}{2}||e||^2\right\}$. Thus at $x = (e, z)$ in our chart the pull back volume is given by

$$F^*(\zeta_\gamma) = \pm \frac{\partial}{\partial x_1} \wedge ... \wedge \frac{\partial}{\partial x_k} \otimes \exp\left\{\frac{1}{2}||e||^2\right\} \omega_{\gamma \otimes \gamma^k} \tag{29}$$

with the sign depending on the orientation of the chart.

At critical points we will set $F^*(\zeta_\gamma) = 0$ neglecting to discuss regularity questions. Formula (28) is immediate from Fubini's theorem applied in the local chart.

To see that $\tilde{d}F^*(\zeta_\gamma) = 0$ we can either use (28) with $\phi = d\psi$ some ψ together with (21), and (25), or argue directly using the local expression above.

This is a special case of a more general result on pulling back $(\infty - m)$ - volumes on Wiener manifolds by Φ_k-maps.

Note that if also F is proper and $\phi \in \Gamma \wedge^k \mathcal{H}^*$, is closed then because of the framed, and therefore oriented, cobordism property of inverse images of regular points we have $\int_{F^{-1}(y)} \phi$ is independent of the regular value y. Denote it by $\mathrm{Deg}(F)(\phi)$, so as for the $k = 0$ case, formula (10), assuming integrability we have

$$\int_{\mathcal{M}} \phi \wedge F^*(\zeta_\gamma) = (2\pi)^{-\frac{k}{2}} \,\mathrm{Deg}(F)(\phi) \tag{30}$$

showing that $\mathrm{Deg}(F)$ induces a map

$$H^p(\mathcal{M}; \mathbf{R}) \to \mathbf{R}$$

" represented" by $(2\pi)^{\frac{k}{2}} F^*(\zeta_\gamma)$. Formally at least this vanishes if $F^*(\zeta_\gamma)$ is exact under $\tilde{d}$.

Restriction to finite co-dimensional submanifolds: Suppose S is a submanifold of an oriented Wiener manifold P, with codimension $r < \infty$. Assume that with its induced Wiener structure it is orientable. An orientation of S then determines "restriction" maps:

$$|_S : \Omega^{\infty-k}(P) \to \Omega^{\infty-k+r}(S) \qquad r \le k = 0, 1, ...$$

.

To construct $|_S$ suppose $\Psi \in \Omega^{\infty-k}(P)$. Write $\Psi = V \otimes \omega^P \otimes \lambda$ for ω^P an orientation form as in (16) , λ a section of the Gauss bundle $\mathcal{G}$, and V a section of $\bigwedge^k \mathcal{H}$. The choice of an orientation form ω^S for S together with ω^P determines an orientation of the normal bundle of S and orientation form $\omega^N \in \Gamma \bigwedge^r (\frac{\mathcal{H}(P)|_S}{\mathcal{H}(S)})^*$. This pulls back to some $\tilde{\omega}^N \in \Gamma \bigwedge^r \mathcal{H}(P)^*_{|S}$
Define the restriction by

$$|_S \Psi = \iota_{\tilde{\omega}^N}(V) \otimes \omega^S \otimes \lambda|_S.$$

For example take $P = E = E_k \times E^{\infty-k}$ and $S = \{a\} \times E^{\infty-k}$. Suppose Ψ is represented by $e_1 \wedge ... \wedge e_k \otimes \zeta_\gamma$, for the standard top form of E_k as H_k.Then the restriction $|_S\Psi$ of Ψ to S is $e^{-\frac{1}{2}|a|^2_H} \zeta_{\gamma^{\infty-k}}$, corresponding to the measure $e^{-\frac{1}{2}|a|^2_H} \gamma^{\infty-k}$.

Also suppose P has Wiener data $\{\mathcal{G}, Z\}$. Our orientation for S determines one for the normal bundle $N(S)$ of S, considered as a sub-bundle of $\mathcal{H}|_S$ and so a positive unit section ω^N of $\bigwedge^r N(S)$. This can be extended over P to some $\overline{\omega^N}$

and gives an $(\infty - r)$-volume $\overline{\omega^N} \otimes \zeta^+(\mathcal{G}, Z)$ on P, and so an ∞ -volume on S by restriction. We see

$$|_S \overline{\omega^N} \otimes \zeta^+(\mathcal{G}, Z) = \zeta^+(\mathcal{G}|_S, \lambda_Z|_S). \tag{31}$$

Suppose further that we have a submersion $\pi : P \to M$ where M is an oriented n-manifold. For $u \in P$ and $x = \pi(u)$, form the Gram matrix

$$G_u^\pi := T_u\pi \circ T_u\pi^* : T_x^* M \to T_x M$$

in this context sometimes called the Malliavin covariance matrix. It is invertible and gives a lift map,

$$h_u : T_x M \to \mathcal{H}_u \quad \text{by} \;\; h_u = T_u\pi^* (G_u^\pi)^{-1}. \tag{32}$$

By general principals, see for example [11], Vol. II 10.4.2 , there is a disintegration of the measure μ corresponding to $\zeta(\mathcal{G}, Z)$ with a normalised regular conditional measure $\{\mu_x\}_{x\in M}$ and the image measure $\pi_*(\mu)$ on M. We are now assuming μ is finite. Suppose M is Riemannian. As in Bismut's introduction in [9], we see that $\pi_*(\mu)$ has a density $p : M \to \mathbf{R}$ with respect to the Riemannian measure on M given by

$$p(x) = (2\pi)^{-\frac{n}{2}} \int_{\pi^{-1}(x)} \frac{\zeta(\mathcal{G}|_S, \lambda_Z|_S)(u)}{\sqrt{\det G_u^\pi}}$$

with $s = \pi^{-1}(x)$.

Also for $x \in M$ and $\pi(u) = x$, slightly abusing notation:

$$d\mu_x(u) = \frac{(2\pi)^{-\frac{n}{2}}}{p(x)} \frac{\zeta(\mathcal{G}|_S, \lambda_Z|_S)(u)}{\sqrt{\det G_u^\pi}}. \tag{33}$$

Integration along fibres: Ramer does not discuss integration along fibres of his finite co-dimensional volumes; it is analogous to useful constructions for diffusion measures, as we see later. We do not discuss the regularity of the push forwards here. It will work as in finite dimensions for the AWM set up, as exposed in [47] following Bismut. Other claims about the properties of the construction given below are fairly straightforward to prove.

For an oriented n-dimensional M and a smooth submersion $\pi : P \to M$. If P has an oriented Wiener structure there is a mapping

$$\pi_* : \Omega^{\infty-k}(P) \to \Omega^{n-k}(M)$$

for $k \le n$ such that for all k-forms ϕ on M

$$\int_P \pi^*(\phi) \wedge \Psi = (2\pi)^{-\frac{n}{2}} \int_M \phi \wedge \pi_*(\Psi) \tag{34}$$

whenever the integrals exist.

It is defined, as usual, by

$$\pi_*(\Psi)(v^1 \wedge \ldots \wedge v^{n-k}) = \int_{P_z} |_{P_z} \iota_{\tilde{v}^1 \wedge \ldots \wedge \tilde{v}^{n-k}}(\Psi). \tag{35}$$

at each $z \in M$, where $v^j \in T_z M$ for $j = 1, ..., n-k$ and $\tilde{v}^j, \quad j = 1, ..., k$ are their lifts to give vector fields on P defined on the fibre P_z over z. Moreover each submanifold P_x is oriented so that the orientation of P orients the normal bundle of P_x compatibly with $T_x M$ via $T_x \pi$. Note that the restriction to the fibre, $|_{P_z} \iota_{\tilde{v}^1 \wedge ... \wedge \tilde{v}^{n-k}}(\Psi)$, is independent of the choice of lift.

For example by (34) with ϕ a function on M we see that if $\Psi \in \Omega^\infty$ then the measure on M given by the top form $\pi_* \Psi$ is $(2\pi)^{\frac{n}{2}} \pi_*(\mu_\Psi)$ for the push forward measure $\pi_*(\mu_\Psi)$.

Properties of integration along fibres

1. From formulae (34), (21), and (25), we see that π_* commutes with exterior differentiation ie, for Ψ in the common domain:
$$\pi_* \tilde{d} \Psi = d \pi_* \Psi. \tag{36}$$
2. If $\Psi = V \otimes \zeta$ and $\bigwedge^k (T_u \pi) V_u = 0$ for all $u \in P$ then $\pi_*(\Psi) = 0$.
3. We can write $\pi_*(\Psi)$ in terms of the disintegration, (33), of measures determined by π. For this first suppose we have Wiener data $\{\mathcal{G}, Z\}$ on P and $\Psi = V \otimes \zeta^+(\mathcal{G}, Z)$. Take $x \in M$. For $u \in P_x$ let $h_u : T_x M \to (N)_u(P_x) \subset \mathcal{H}_u(P)$ be the horizontal lift to the normal space using the Gram matrix as in (32). Take an oriented orthonormal base $e_1, ..., e_n$ for $T_x M$. Write
$$\bigwedge^k (T_u \pi)(V_u) = \sum_J a_J(u) e_J \qquad \text{for} \quad e_J := e_{j_1} \wedge ... \wedge e_{j_k}$$
where the sum runs over the set $\mathcal{C}(k, n)$ of choices of increasing sequences $J := 1 \le j_1 < ... < j_k \le n$. Set $\tilde{e}_i = h_u e_i$. By item 2 above we can replace V_u by $\sum_{J \in \mathcal{C}(k,n)} a_J(u) \tilde{e}_J$.
For $I \in \mathcal{C}(n-k, n)$ by definition and (31)
$$F_*(\Psi)(e_I) = \int_{P_x} \omega_u^N \left(\tilde{e_I} \wedge \sum_{J \in \mathcal{C}(k,n)} a_J(u) \tilde{e}_J \right) \zeta(\mathcal{G}|_{P_x}, \lambda_Z|_{P_x})(u)$$
for ω^N the orientation form for the normal bundle, using $\mathcal{G}$.

Let $I' \in \mathcal{C}(k, n)$ be the complimentary sequence to $I \in \mathcal{C}(n-k, n)$ and $\operatorname{sgn}(I)$ the sign of the permutation of $1, ..., n$ determined by I followed by I'. We see
$$F_*(\Psi)(e_I) = \int_{P_x} \frac{1}{\sqrt{\det G_u^\pi}} \operatorname{sgn} I a_{I'}(u) \zeta(\mathcal{G}|_{P_x}, \lambda_Z|_{P_x})(u).$$
In terms of the normalized disintegration (33) of $\mu_{\zeta(\mathcal{G},Z)}$
$$F_*(\Psi)(e_I) = \operatorname{sgn}(I)(2\pi)^{\frac{n}{2}} p(x) \int_{P_x} a_{I'}(u)\, d\mu_x(u) \tag{37}$$

where $p : M \to \mathbf{R}$ is the density of $\pi_*(\mu_{\zeta(\mathcal{G},Z)})$ with respect to the Riemannian volume measure. Note that for the right hand side of (37) to make sense the only structures needed are a measure on P with a density on M, a section V of $\bigwedge^k \frac{TP}{\ker T\pi}$ and an orientation and Riemannian metric on M. It can be used for a local co-ordinate formula. The basic property (34) follows easily from (37).
Alternatively for, $\underline{\alpha} \in \bigwedge^{n-k} T_x M$, from (37) we get

$$F_*(\Psi)(\underline{\alpha}) = (2\pi)^{\frac{n}{2}} p(x) \int_{P_x} \omega_x^M(\underline{\alpha} \wedge \bigwedge^k (T_u\pi)V_u)\, d\mu_x(u) \tag{38}$$

for ω^M the volume form for M.

Conditioning A related construction to that of π_* above works in more generality and has proved very useful for diffusion measures, see section 5.1 below. Given a submersion, as above, $\pi : P \to M$, with $L^1\Omega$ refering to L^1 volumes

$$\pi_*^\infty : L^1\Omega^{\infty-k}(P) \to \Omega^{\infty-k}(M),$$

where for M n-dimensional and oriented $\Omega^{\infty-k}$ consists of sections of $\bigwedge^k TM \otimes \bigwedge^n T^*M$.

For this, keeping with our notation $\Psi = V \otimes \zeta$, define

$$\begin{aligned}(\pi_*^\infty \Psi)_x &:= (2\pi)^{-\frac{n}{2}} \mathbf{E}^\zeta\{\bigwedge^k T\pi(V)|\pi = x\} \otimes \pi_*(\zeta)\} &(39)\\ &= (2\pi)^{-\frac{n}{2}} \int_{P_x} \bigwedge^k T_u\pi(V_u)\, d\mu_x(u) \otimes \pi_*(\zeta) &(40)\end{aligned}$$

where $\mathbf{E}^\zeta\{-|\pi = x\}$ refers to the conditional expectation given $\pi = x$ with respect to μ_ζ. In general this is defined for $\pi_*(\mu_\zeta)$ almost all $x \in M$, but in our case there is a smooth version if ζ is positive and smooth. To take conditional expectations of such vector valued functions we can take local, or measurable, global, trivialisations. The result is independent of the choice since they are measurable with respect to x. This is discussed in detail in [22] together with weakening of the integrability condition. For measure theoretic aspects see [11], Volume II.

A direct argument shows that formula (39) is independent of the decomposition $\Psi = V \otimes \zeta$ with μ_ζ a finite positive measure. In fact, since for any integrable $\lambda : P \to \mathbf{R}(>0)$ we have $\pi_*(\lambda\zeta)_x = \mathbf{E}^\zeta\{\lambda|\pi = x\}\pi_*(\zeta)_x$, this claim is essentially the Kallianpur-Striebel formula, a version of Bayes' formula, eg [28] Lemma 5.1.1.

Write $\bar{V}_x$ or $\bar{V}_x^{\zeta,\pi}$ for $\mathbf{E}^\zeta\{\bigwedge^k T\pi(V)|\pi = x\} \in \bigwedge^k T_x M$. From (34) we have, for any horizontal lift mapping $h : \pi^*TM \to \mathcal{H}$,

$$\pi_*(V \otimes \zeta) = \pi_*\left(\bigwedge^k (h)(\bar{V}) \otimes \zeta\right) \tag{41}$$

and then for any k-form ϕ on M, writing $\wedge^\infty$ for the exterior product in the $(\infty -$

k)-sense:

$$\begin{aligned}
\phi \wedge \pi_*(V \otimes \zeta) &= \phi \wedge^\infty \left(\bar{V} \otimes \pi_*(\zeta)\right) && (42)\\
&= \phi(\bar{V}) \otimes \pi_* \zeta && (43)\\
&= (2\pi)^{\frac{n}{2}} \phi \wedge^\infty \pi_*^\infty (V \otimes \zeta). && (44)
\end{aligned}$$

This shows that $(2\pi)^{\frac{n}{2}} \pi_*^\infty (V \otimes \zeta)$ is the $(\infty - k)$- form on M corresponding to the $(n-k)$-form $\pi_*(V \otimes \zeta)$.

4 The Gauss-Bonnet-Chern theorem and finite co-dimensional forms:

To see how these forms can be used we next re-interpret the approach of Nicolaescu and Savale to the Gauss-Bonnet-Chern theorem for vector bundles over a closed oriented n-dimensional manifold M, see [55], [54]. We also show how Thom forms fit naturally into the same scheme.

Gaussian spaces of sections and connections: Let $p : \mathcal{V} \to M$ be a smooth oriented k-plane bundle over M. Take a Hilbert space $H, \langle -, - \rangle_H$ of sections of p whose evaluations $\rho_x : H \to \mathcal{V}_x$ at each point x of M span the fibre $\mathcal{V}_x$ at x, and such that there is a Banach space E of C^r sections , $r \geq 2$, containing H and for which the inclusion $i : H \to E$ forms an abstract Wiener space. For example we could take H to be a Sobolev space of H^r sections for $r > \frac{n}{2} + 2$ and E the space of C^2 sections, or the closure of H in the space of C^{2+a} Holder class sections for $2 + a < r - \frac{n}{2}$, [7]. The Hilbert space is determined by its reproducing kernel, $k(x,y) := \rho_x \rho_y^* : \mathcal{V}_y \to \mathcal{V}_x^*$ for $x, y \in M$. Choose an orientation for E.

The evaluation map determines a Riemannian metric on $\mathcal{V}$ and a metric connection by projecting the metric and trivial connection on $M \times H$ using $\rho_- : M \times H \to \mathcal{V}$. Extend the evaluation map to $\rho_- : M \times E \to \mathcal{V}$.

Alternatively, as described in section 2.1.3 above, given a smooth metric on $\mathcal{V}$ and a smooth metric connection, from Narasimhan & Ramanan, [53], we can find a Hilbert space H which induces them, and also have H finite dimensional, so as sets $E = H$, see [57],[21],[22], and [55] for a more explicit construction. However for the main result, formula (63), it makes sense to have H dense in the space of continuous sections.

Set $P_x = \ker \rho_x \subset ij[E^*]$ with $K_x^\perp$ its orthogonal complement in E. The basic properties, from [21], [22], relating the connection and H, are in terms of the covariant derivative of sections U of $\mathcal{V}$, written $\breve{\nabla}_- U : TM \to \mathcal{V}$:

$$\breve{\nabla}_v U = 0 \quad \text{if } \mathrm{v} \in \mathrm{T_x M} \ \& \ \mathrm{U} \in \mathrm{K_x^\perp} \tag{45}$$

for any $x \in M$, and

$$\int_E \breve{\nabla}_- U \wedge \breve{\nabla}_- U \, d\gamma(U) = \breve{\mathcal{R}} : \wedge^2 TM \to \wedge^2 \mathcal{V} \tag{46}$$

for $\breve{\mathcal{R}}$ the curvature operator of $\mathcal{V}$. The integral in (46) could also be written as the trace $\sum_j \breve{\nabla}_- E^j \wedge \breve{\nabla}_- E^j$ for $\{E^j\}_j$ an orthonormal base of H.

The integral over zero sets as a generalised degree: Set

$$P = \{(x, U) \in M \times E : U(x) = 0\}.$$

Let $F : P \to E$ and $\pi : P \to M$ be the restrictions of the projections.

Then , as we see from below, P is a C^r submanifold of $M \times E$ and F is a proper C^r Fredholm map of index $n - k$. For ϕ a closed $(n - k)$-form on M we will see

$$\mathrm{Deg}(F)(\pi^*(\phi)) = \int_{Z_U} \phi \tag{47}$$

for γ -almost all sections U of $\mathcal{V}$ where Z_U denote the zero set of U in M.

In equation (47) we need an orientation of Z_U and of P. We are given orientations for M, $\mathcal{V}$, and E. As in [13] we use the surjectivity of $\breve{\nabla}_- U : TM|_{Z_U} \to \mathcal{V}|Z_U$, for any regular point U of F, to give an isomorphism of any normal bundle NZ_U to Z_U in M with $\mathcal{V}|Z_u$. This orients NZ_U and so TZ_U using the ordering convention

$$TM|_{Z_U} = NZ_U \oplus TZ_U \tag{48}$$

as in [13].

Moreover $\rho_- : \underline{E} \to \mathcal{V}$ splits the trivial bundle $\underline{E} = M \times E$ into the Whitney sum $K^\perp \oplus K$ where $K := \ker \rho_-$ and $K^\perp$ its orthogonal complement, fibres modelled on E_k, in the notation of Section 2.1.2. Then $P = K$ and ρ_- gives a vector bundle isomorphism of $K^\perp$ with $\mathcal{V}$. Its inverse is given by its adjoint $\rho_x^t : \mathcal{V}_x \to E^*$. Thus $\pi : P \to M$ can be considered as a Riemannian vector bundle, fibres $E^{\infty-k}$, group $O_N^+(E^{\infty-k})$, the elements of $SL_N^+(E^{\infty-k})$ which act orthogonally on H. Its orientation as a bundle comes from that of $\underline{E}$ and $\mathcal{V}$.

For an oriented vector bundle V and natural number m write $(-1)^m V$ for V with its given orientation if m is even and the reverse orientation otherwise. Below various powers of $(-1)^k$ will appear. However we will see that the terms involving them will vanish unless k is even, so for us their precise description proves unnecessary.

To orient P as a manifold, at $(x, U) \in P$ write

$$T_{(x,U)} P = \ker T_{(x,U)} F \oplus (\ker T_{(x,U)} F)^\perp$$

for a transverse bundle $(\ker TF)^\perp$. Also note

$$T_{(x,U)} P = \big\{(v, W) \in T_x M \times E : W(x) = -\breve{\nabla}_v U\big\}. \tag{49}$$

Thus $\ker T_{(x,U)} F = \{(v, 0) \in T_x M \times E : \breve{\nabla}_v U = 0\}$ which is isomorphic by $T_{(x,U)}\pi$ to $(NZ_U)_x$. Give $\ker T_{(x,U)} F$ the orientation induced by $T_{(x,U)}\pi$.

We have to give $(\ker T_{(x,U)} F)^\perp$ the orientation induced by the projection $T_{(x,U)} F$ onto E. To describe this use the isomorphism:

$$(NZ_U)_x \oplus P_x \to (\ker T_{(x,U)}F)^{\perp}, \quad (v, V) \mapsto (v, -\rho_x^t \check{\nabla}_v U) + (0, V) \quad (50)$$

and observe that for this to be orientation preserving when projected onto E, and using our orientation for P_x, the NZ_U factor has to be taken to be $(-1)^k NZ_U$. Thus as oriented spaces

$$\begin{aligned} T_{(x,U)}P &\simeq T_x Z_U \oplus (-1)^k (NZ_U)_x \oplus P_x & (51)\\ &\simeq (-1)^k(-1)^{k(n-k)} T_x M \oplus P_x. & (52)\end{aligned}$$

Give P the orientation as a manifold induced by this. With this orientation, formula (47) holds.

The geometric Euler class For a general smooth k-form ϕ on M we have by formulae (28), (34), cf [55], [54]:

$$\int_M \phi \wedge \pi_* F^* \zeta_\gamma = (2\pi)^{\frac{n}{2}} \int_P \pi^* \phi \wedge F^* \zeta_\gamma = (2\pi)^{\frac{k}{2}} \int_E \{ \int_{Z_U} \phi \} \, d\gamma(U). \quad (53)$$

We will calculate $\pi_* F^* \zeta_\gamma$.

A. We first consider the $(\infty - (n-k))$-volume $F^*\zeta_\gamma$ on P. For this fix a point $(x, U) \in P \subset M \times E$ which is a regular point for F. For the tangent space to P we can write

$$T_{(x,U)}P = \{(v, W) \in T_x M \times E \ \text{ with } W(x) = -\check{\nabla}_v U\}. \quad (54)$$

The adjoint $\rho_x^t : \mathcal{V}_x \to E^*$ is the identification of $\mathcal{V}_x$ with the subspace $K_x^\perp$ orthogonal to P_x in E. We therefore have a horizontal lift map

$$\mathfrak{H}_{(x,U)} : T_x M \to T_{(x,U)}P \qquad v \mapsto (v, -\rho_x^t \check{\nabla}_v U). \quad (55)$$

We have the decomposition

$$\begin{aligned} \mathcal{H}_{(x,U)}P &= (\{0\} \times (P_x \cap H)) \oplus \mathfrak{H}_{(U,x)}(T_x M) & (56)\\ &= (P_x \cap H) \oplus \mathfrak{H}_{(U,x)}[T_x Z_U] \oplus \mathfrak{H}_{(U,x)}[(NZ_U)_x] & (57)\end{aligned}$$

for NZ_U our normal bundle of Z_U in TM.

Choose an oriented orthonormal basis $\nu_1, ... \nu_k$ for $\mathcal{V}_x$ and an oriented base $\eta_1, ..., \eta_k, \kappa_1, ..., \kappa_{n-k}$ for $T_x M$ with $\kappa_1, ..., \kappa_k$ in $T_x Z_U$ and $\check{\nabla}_{\eta_j} U = \nu_j$ for $j = 1$ to k.
Give H an oriented orthonormal basis $\rho_x^t \nu_1, ..., \rho_x^t \nu_k, e_{k+1}, ...$ with dual basis $\{\ell_j\}_j$.

Then $(\kappa_1, 0), ...(\kappa_{n-k}, 0), -\mathfrak{H}_{(x,U)}(\eta_1), ..., -\mathfrak{H}_{x,U}(\eta_k), e_{k+1}, ...$ is an oriented base for $\mathcal{H}P_{(x,U)}$. Let $Q^P_{(x,U)}$ be the element of $\wedge^\infty(\mathcal{H}P_{(x,U)})^*$ determined by its dual base $\{\tilde{\ell_j}\}_{j=1}^\infty$. Note that $\tilde{\ell_j} = TF^*_{(x,U)}(\ell_{j-(n-k)})$ for $j > n-k$. From this

$$F^*(\zeta_\gamma)_{(x,U)} = (\kappa_1, 0) \wedge ... \wedge (\kappa_{n-k}, 0) \otimes Q^P_{(x,U)} \otimes \lambda_{(x,U)} \quad (58)$$

where λ is the section of $\mathcal{G}$ which "deGaussifies" the fibres of F.

B. Next take $\{v_j\}_{j=1}^k$ in T_xM . Set $V = v_1 \wedge ... \wedge v_k$. To compute $\pi_* F^*(\zeta_\gamma)(V)$ we need the restriction to $(-1)^{nk} P_x$ of the $(\infty - n)$-volume

$$\mathfrak{H}_{(x,U)} v_1 \wedge ... \wedge \mathfrak{H}_{(x,U)} v_k \wedge (\kappa_1, 0) \wedge ... \wedge (\kappa_{n-k}, 0) \otimes Q^P_{(x,U)} \otimes \lambda_{(x,U)}.$$

Writing $\psi_x^{\mathcal{V}} = \ell_1 \wedge ... \wedge \ell_k$ considered as in $(\wedge^k K_x^\perp)^*$, this restriction at (x, U) will be

$$(-1)^{nk} \psi_x^{\mathcal{V}} (\rho_x^t \breve{\nabla}_{v_1} \wedge ... \wedge \rho_x^t \breve{\nabla}_{v_k} U) \zeta_{\gamma_x},$$

where ζ_{γ_x} is the ∞-volume of P_x as a co-dimension n subspace of E with its measure γ_x.

Thus

$$\pi_* F^*(\zeta_\gamma) = (-1)^{nk} \int_{P_x} \psi_x^{\mathcal{V}} \bigwedge^k \left(\rho_x^t \breve{\nabla}_{(-)} U \right) d\gamma_x(U) \in \bigwedge^k T_x^* M. \qquad (59)$$

Since our Gaussian measures are symmetric this vanishes if k is odd, so assume $k = 2p$. Now we can combine the basic property $\breve{\nabla}_v U = 0$ if $v \in T_xM$ and $U \in K_x^\perp$ with Fubini's theorem applied to the decomposition $E = P_x \otimes K_x^\perp$ to see

$$\pi_* F^*(\zeta_\gamma) = \int_E \psi_x^{\mathcal{V}} \bigwedge^k \left(\rho_x^t \breve{\nabla}_{(-)} U \right) \, d\gamma(U). \qquad (60)$$

C. To calculate the integral write $\breve{\nabla}_{(v_i)} U = \sum_{j=1}^k U_i^j \nu_j$ for our $v_i \in T_xM$, and set $A_\ell = U_\ell^{j(\ell)}$ for $\ell = 1, ..., 2p$ and for $\underline{j}$ a given permutation of 1,...,2p. Now apply Wick's formula for each permutation $\underline{j}$:

For a real valued , mean zero, Gaussian family, $A_1, ..., A_{2p}$

$$\mathbf{E}\{A_1 A_2 .... A_{2p}\} = \sum_\pi \mathbf{E}\left\{A_{\pi(1)} A_{\pi(2)}\right\} .... \mathbf{E}\left\{A_{\pi(2p-1)} A_{\pi(2p)}\right\}$$

where the sum is over π *such that* $\pi(2r-1) < \pi(2r)$, eg see [1].

Application of formula (46) as in [55], [54] then leads to

$$\int_E \psi^{\mathcal{V}} \rho_x^t \bigwedge^k \breve{\nabla}_{(-)} U \, d\gamma(U) = (-1)^p \mathbf{P} f(\breve{\Omega}) \qquad (61)$$

for $\mathbf{P}f(\breve{\Omega})$ the Pfaffian of the curvature $\breve{\Omega}$ of $\mathcal{V}$ with its given connection, for example see [61].

Since the Euler form $e(\mathcal{V})$ is defined by $e(\mathcal{V}) = (\frac{-1}{2\pi})^p \mathbf{P} f(\breve{\Omega})$ formula (53) now gives the result of [55], [54]: *For all* $(n-k)$*-forms* ϕ *on* M*:*

$$\int_M \phi \wedge e(\mathcal{V}) = \int_E \{ \int_{Z_U} \phi \} \, d\gamma(U). \qquad (62)$$

In particular this shows that the current $\phi \mapsto \int_E \{ \int_{Z_U} \phi \} \, d\gamma(U)$ depends only on the curvature of the connection induced by the Gaussian measure γ.

It reduces for closed ϕ and generic $Z \in E$ to the usual Gauss-Bonnet-Chern formula for vector bundles:

$$\int_M \phi \wedge e(\mathcal{V}) = \int_{Z_U} \phi. \tag{63}$$

4.0.1 A Thom-like form.

A. Consider the projections

$$\rho_- : M \times E \to \mathcal{V} \qquad \rho_x(U) = \rho(x, U) = U(x),$$

$$p_1 : M \times E \to M \quad \text{and} \quad p_2 : M \times E \to E$$

and $p : \mathcal{V} \to M$. Set $\tau = (2\pi)^{-\frac{k}{2}} \rho_* p_2^*(\zeta_\gamma)$. It is a k-form on $\mathcal{V}$. By the discussion of pull backs above and (36) it is closed.
We can see $p_*\tau = 1$, either directly, with care, by using $p \circ \rho_- = p_1$, or from (65) proved below taking θ to be the pull back under p of a top form on M. Thus τ is like a Thom class, eg see [13], though not of compact support.

B. Let ζ denote the zero section of $\mathcal{V}$. We claim that, as forms,

$$\zeta^*(\tau) = e(\mathcal{V}). \tag{64}$$

For this, fix $x \in M$ and observe $\rho^{-1}\zeta(x) = P_x$. We will therefore be integrating over P_x again. Take $v_1, ..., v_k$ in T_xM, and let $(v_1, 0), ..., (v_k, 0)$ be their lift to $T_{\zeta(x)}\mathcal{V} \simeq T_xM \oplus \mathcal{V}_x$. For $U \in P_x$ we must choose $\tilde{v_j} \in T_xM \times E$ with $T_{(x,U)}\rho(\tilde{v_j}) = (v_j, 0)$.

For this suppose $\tilde{v}_j = (\alpha_j, u_j) \in T_xM \oplus E$. Then

$$\begin{aligned} T_{(x,U)}\rho(\tilde{v_j}) &= (\alpha_j, u_j(x) + \breve{\nabla}_{\alpha_j} U) \\ &= (v_j, 0) \end{aligned}$$

if and only if $\alpha_j = v_j$ and $u_j(x) = -\breve{\nabla}_{v_j} U$. We can therefore take

$$\tilde{v}_j = (v_j, -\rho^t \breve{\nabla}_{\bar{v}_j} U) = \mathfrak{H}_x(v_j)$$

as in computing $\pi_* F^* \zeta_\gamma$, leading to (64).

C. A basic property of Thom classes is that the Poincaré dual of a closed , oriented submanifold is essentially the Thom class of its normal bundle considered as a tubular neighbourhood, [13]. For our τ it suffices to show that for all closed n-forms θ on $\mathcal{V}$, if S is the image of the zero section of $\mathcal{V}$:

$$\int_S \theta = \int_{\mathcal{V}} \theta \wedge \tau. \tag{65}$$

To see this, using the basic properties of integration along fibres (34) and pullbacks (28)

$$
\begin{aligned}
(2\pi)^{\frac{k}{2}}\int_{\mathcal{V}}\theta\wedge\tau &= (2\pi)^{\frac{n+k}{2}}\int_{M\times E}\rho^*(\theta)\wedge p_2^*\zeta_\gamma && (66)\\
&= (2\pi)^{\frac{k}{2}}\int_E\left(\int_{p_2^{-1}(U)}\rho^*(\theta)\right)\,d\gamma(U) && (67)\\
&= (2\pi)^{\frac{k}{2}}\int_E\left(\int_{M\times\{U\}}\rho^*(\theta)\right)\,d\gamma(U) && (68)\\
&= (2\pi)^{\frac{k}{2}}\int_{M\times\{U\}}\rho^*(\theta) && (69)
\end{aligned}
$$

for any $U\in E$ since $\int_{M\times\{U\}}\rho^*(\theta)$ is independent of U because θ is closed. Take U to be the zero section, ζ, with $i_\zeta : M\to M\times\{\zeta\}\subset M\times E$ the inclusion. We have

$$
\begin{aligned}
\int_{\mathcal{V}}\theta\wedge\tau=\int_{M\times\{\zeta\}}\rho^*(\theta) &= \int_M i_\zeta^*\rho^*(\theta)\\
&= \int_M(\rho\circ i_\zeta)^*(\theta).
\end{aligned}
$$

Since $\rho\circ i_\zeta=\zeta$ we have (65) as claimed.

D. A challenge would be to carry a version of this construction over to the infinite dimensional tautological bundle $p^{E^*} : \mathcal{V}^{\text{univ},E^*}\to G(k;E^*)$ of section 2.1.3. Kusuoka's more general forms, [44], might help. Particularly worthwhile would be to do it in the infinite dimensional situation described by Atiyah & Jeffrey in [6].

5 Extensions to less regular situations

It is clear that many of the constructions concerning $(\infty-k)$-forms do not require the full trappings of Frechet differentiability or abstract Wiener manifolds. The discussion above leads to considering objects defined on a Banach manifold $\mathcal{M}$, say, consisting of tensor products of a space of k-vector fields , i.e. sections of $\bigwedge^k T\mathcal{M}$, and the line bundle of all (possibly signed) measures absolutely continuous with respect to some given measure class. We describe some examples of these which arise in stochastic analysis.

5.1 Itô maps as submersions

For background to this and for many references see [49].

A. For $T>0$ consider classical Wiener space

$$i : L_0^{2,1}([0,T];R^m)\to C_0([0,T];\mathbf{R}^m).$$

Here $H = L_0^{2,1}(R^m) := L_0^{2,1}([0,T];R^m)$ with inner product $\langle \sigma_1, \sigma_2 \rangle_H := \int_0^T \langle \dot{\sigma}_1(t), \dot{\sigma}_2(t) \rangle \, dt$. Wiener measure is the induced Gaussian measure γ, and Brownian motion on $\mathbf{R}^m$ is given by the evaluation maps $B_t : C_0(\mathbf{R}^m) \to \mathbf{R}^m$. Almost surely, with respect to γ, paths in $C_0(\mathbf{R}^m)$ do not have bounded variation.

Consider a compact Riemannian connected n-dimensional manifold M. Let $\triangle$ denote the Hodge-Kodaira Laplacian of M with the sign convention $\triangle = -(dd^* + d^*d)$. As described in Section 2.1.3 above, from [53], or by taking an isometric embedding into $\mathbf{R}^m$, there exists $X : M \times \mathbf{R}^m \to TM$, for some $m \geq n$, which induces the metric and its Levi-Civita connection. The Stratonovich stochastic differential equation

$$dx_t = X(x_t) \circ dB_t \quad \text{with given } x_0 \in M \tag{70}$$

has solution $x : [0,T] \times C_0(\mathbf{R}^m) \to M$. It is Brownian motion on M. The solution is almost surely C^∞ in x_0 giving a stochastic flow of diffeomorphisms: an almost surely defined map $\xi : [0,T] \times C_0(\mathbf{R}^m) \to \mathrm{Diff}(M)$, eg see [20], [38]. For each t the *Itô map* $\mathcal{I}_t : C_0(\mathbf{R}^m) \to M$ is just the solution x_t. It can be defined by taking piecewise linear approximations $P_\pi(\sigma)$ to the paths in $C_0(\mathbf{R}^m)$ solving equation (70) for $B_t \circ P_\pi$ replacing B_t and taking the limit in measure of the solutions as mesh $\pi \to 0$. This approximation procedure also shows that $\mathcal{I}_t$ lies in the domain of an H-derivative operator giving an almost surely defined $T_\sigma \mathcal{I}_t : H \to T_{\mathcal{I}_t(\sigma)} M$. This is given by a formula of Bismut: For $h \in H$ and almost all $\sigma \in C_0(\mathbf{R}^m)$

$$T_\sigma \mathcal{I}_t(h) = T\xi_t(\sigma) \int_0^t T\xi_s(\sigma)^{-1} X(x_s(\sigma))(\dot{h}(s)) \, ds. \tag{71}$$

Because the solutions are Brownian motions, for any bounded measurable $f : M \to \mathbf{R}$ the solution $P_t(f)$ of the heat equation

$$\frac{d}{dt} P_t f = \frac{1}{2} \triangle P_t f \quad \text{with } \ P_0 f = f$$

is given by

$$P_t f = \int_{C_0(\mathbf{R}^m)} f \circ \xi_t \, d\gamma$$

and the push forward measure $(\mathcal{I}_t)_*(\gamma)$ has density the heat kernel $p_t(x_0, -)$ on M.

Because the connection on TM induced by the SDE is Levi-Civita, for any bounded measurable k-form ϕ on M the solution $P_t^{(k)} \phi$ of the heat equation

$$\frac{d}{dt} P_t^{(k)} \phi = \frac{1}{2} \triangle P_t^{(k)} \phi \quad \text{with } \ P_0^{(k)} \phi = \phi$$

is given by

$$P_t^{(k)} \phi = \int_{C_0(\mathbf{R}^m)} \xi_t^*(\phi) \, d\gamma, \tag{72}$$

see [26] or in greater generality, though with a sign misprint in (2.4.5), [22], but it appeared in [42].

B. Recall $Y_x : T_xM \to \mathbf{R}^m$ is the pseudo-inverse, and adjoint, of $X(x)$ for each $x \in M$. For $v^1, ..., v^k \in T_{x_0}M$, some $1 \le k \le n$, and $\lambda \in L^1([0,T];\mathbf{R})$ with $\int_0^T \lambda(s)\,ds = 1$ set

$$h_t^i = \int_0^t \lambda(s) Y_{x_s} T\xi_s(v^i)\,ds \quad 0 \le t \le T. \tag{73}$$

This defines, almost surely, measurable H-vector fields $h^1, ...h^k$ on $C_0(\mathbf{R}^m)$. They commute and are in the domain of the divergence operators $\mathrm{div}_\gamma^{(p)}$ of section 3.2.1 for $1 \le p < \infty$ by [23]. Write $\underline{h}(\sigma) = h^1(\sigma) \wedge ... \wedge h^k(\sigma)$ to get an L^p map into $\bigwedge^k H$, a k-vector field. Thus we have an $(\infty - k)$-volume $\underline{h} \otimes \zeta_\gamma$ on $C_0(\mathbf{R}^m)$.

Suppose M is oriented. Using the H-differentiability of $\mathcal{I}_T$ and formula (38) we obtain a measurable $(n-k)$-form $(\mathcal{I}_T)_*(\underline{h} \otimes \zeta_\gamma)$ on M. Given a k-form ϕ on M we know

$$\int_M \phi \wedge (\mathcal{I}_T)_*(\underline{h} \otimes \zeta_\gamma) = (2\pi)^{\frac{n}{2}} \int_{C_0(\mathbf{R}^m)} \mathcal{I}_T^*(\phi) \wedge (\underline{h} \otimes \zeta_\gamma). \tag{74}$$

However by (71) writing $\underline{v} = v^1 \wedge ... \wedge v^k$

$$\mathcal{I}_T^*(\phi)(\underline{h}) = \phi(\bigwedge^k T\mathcal{I}_T(\underline{h})) = \phi\left(\bigwedge^k T\xi_T(\underline{v})\right)$$

giving

$$\begin{aligned}(2\pi)^{-\frac{n}{2}} \int_M \phi \wedge (\mathcal{I}_T)_*(\underline{h} \otimes \zeta_\gamma) &= \int_{C_0(\mathbf{R}^m)} \phi(\bigwedge^k T\xi_T(\underline{v}))\,d\gamma &(75)\\ &= P_T^{(k)}\phi(\underline{v}) &(76)\end{aligned}$$

by (72). Thus $\underline{v} \mapsto (2\pi)^{-\frac{n}{2}}(\mathcal{I}_T)_*(\underline{h} \otimes \zeta_\gamma)$ could be considered as a fundamental solution for the heat equation for k-forms on M, though from (78) below $\underline{v} \mapsto (\mathcal{I}_T)_*^\infty(\underline{h} \otimes \zeta_\gamma)$ would be the more usual representation.

Remark 5.1 If $\phi = d\psi$ for some smooth $(k-1)$-form ψ, since $\underline{h}$ has a divergence in L^1 we can integrate by parts in (74) to get

$$d(P_T^{(k-1)}(\psi))(\underline{v}) = -\int_M \psi \wedge (\mathcal{I}_T)_*(\mathrm{div}(\underline{h}) \otimes \zeta_\gamma). \tag{77}$$

As described in [23] this gives an extension of Bismut's formula, [9], for the case $k = 0$. This method of lifting tangent vectors on M through an Itô map and then applying an infinite dimensional integration by parts formula was basic to Malliavin's approach to hypo-ellipticity eg see [8], [9].

C. If we apply $(\mathcal{I}_T)_*^\infty$, i.e.conditioning, to $\underline{h} \otimes \zeta_\gamma$ rather than fibre integration, by (42) and (75) we see

$$P_T^{(k)}\phi(v^1 \wedge ... \wedge v^k) = (2\pi)^{-\frac{n}{2}} \int_M \phi(\overline{\underline{h}}_y) p_T(x_0, y)\,dy \tag{78}$$

where $\overline{\underline{h}}_y = \mathbf{E}^\zeta\{\bigwedge^k T\mathcal{I}_T(\underline{h}) | \mathcal{I}_T = y\}$. These conditional expectations have been computed in [23], see also [22]. They involve parallel translations damped by Weitzenböck curvature terms.

5.2 $C_{x_0}M$ and its generalised finite co-dimensional forms

Given our SDE as before, the Itô map can be considered as a map $\mathcal{I} : C_0(\mathbf{R}^m) \to C_{x_0}(M)$ into the smooth Banach manifold of continuous paths on M starting at x_0 and defined on $[0,T]$. It is only defined almost surely, but as above we will neglect to emphasise this. It has an H-derivative with continuous linear maps $T_\sigma \mathcal{I} : H \to T_{\mathcal{I}(\sigma)} C_{x_0}(M)$. The image measure $\mu_{x_0} := \mathcal{I}_*(\gamma)$ is Wiener, or Brownian motion, measure on $C_{x_0}(M)$.

In general $C_{x_0}M$ does not have a natural smooth AWM structure. However it does have a measurable almost surely defined Hilbert bundle $\mathcal{H}$, the Bismut bundle, of admissible directions, with Riemannian metric and with inclusion $i_{\mathcal{H}} : \mathcal{H} \to TC_{x_0}M$ giving a pre-AWS at each point. For a generic $\{x_t\}_{0\le t\le T}$ in $C_{x_0}M$ let $/\!\!/_t : T_{x_0}M \to T_{x_t}M$ denote parallel translation along x_- using the Levi-Civita connection of M. This exists almost surely, by approximation, as formulated by Itô in 1962, see [20], [38]. Then $\mathcal{H}_{x_-}$ consists of those vector fields V_t along x_- with $/\!\!/_t^{-1} V_t : 0 \le t \le T$ in $L^{2,1}([0,T]; T_{x_0}M)$. It can be given the obvious inner product or, equivalently and usually more conveniently, one replacing $/\!\!/_t$ by damping it with the Ricci curvature, eg see [22], or in greater generality [24]. We will use the latter.

Driver showed that $\mathcal{H}$ does consist of admissible directions, i.e. suitably regular H-vector fields have divergences, and even flows which preserve sets of measure zero, eg see [17]. This led to a Sobolev calculus based on H-differentiation on such path spaces. Since parallel translation trivialises $\mathcal{H}$ it could be considered orientable, if need be. As well as its metric there is also a candidate for a position field, though its role is unclear:

$$Z(x_-)_t = /\!\!/_t \int_0^t /\!\!/_s^{-1} \circ dx_s \quad 0 \le t \le T \tag{79}$$

or the damped version.

In general Lie brackets of H-vector fields are not sections of H, [15], which shows that our set up does not arise from an AWM structure. Moreover, from formula 14, it gives difficulty in defining exterior differentiation of H-k-forms on $C_{x_0}M$ if we require such a form to be a section of $(\bigwedge^k \mathcal{H})^*$. Leandre got over this problem by using stochastic integration to define the exterior derivative, see [45] or the survey [46], but this did not lead to a Kodaira-Hodge theory. An alternative approach by Elworthy & Li, [25], replaced $\bigwedge^k \mathcal{H}$ by a perturbation of it, $\mathcal{H}^k$, inside $\bigwedge_\pi^k TC_{x_0}M$, the projective exterior power, using curvature terms from M. For $k = 1$ it was possible to take the closure of exterior differentiation of geometric forms to obtain a self-adjoint Kodaira -Hodge operator on L^2 H one-forms with an L^2 Hodge decomposition. In fact L^2 harmonic one forms in this sense were shown to vanish in [27]. For $k \ge 2$ this is still in progress.

The structure we have can be obtained by conditioning with the Itô map, rather as done by Airault to project onto a finite dimensional manifold, [3]. Indeed if $\underline{h}$ is a constant H-vector field on $C_0(\mathbf{R}^m)$ we can define

$$\overline{T\mathcal{I}}_{x_-}(h) := \mathbf{E}\{T_-\mathcal{I}(h) | \mathcal{I} = x_-\}.$$

Thus, extending the previous notation, $\mathcal{I}_*^\infty(\underline{h}\otimes\gamma)=\overline{T\mathcal{I}}_-(h)\otimes\mu_{x_0}$.
It turns out, [22], [24], that this takes values in $\mathcal{H}$, determining an H-vector field on $C_{x_0}(M)$. This gives an isometric $\overline{T\mathcal{I}}_{x_-}: H\to\mathcal{H}_{x_-}$ for almost all x_-.

However if we take the analogous construction of

$$\overline{\bigwedge^k T\mathcal{I}}_{x_-}:\bigwedge^k H\to\bigwedge^k_\pi T_{x_-}C_{x_0}(M)$$

for $k\geq 2$ it no longer has values in $\bigwedge^k\mathcal{H}$ in general, and we define $\mathcal{H}^k$ to be its image with induced inner product. It is suitably regular sections of this which have well defined divergences. For $k=2$ these divergences are sections of $\mathcal{H}$, and probably more generally they are sections of $\mathcal{H}^{k-1}$, [23]. It is natural to define an $(\infty-k)$-volume to be a section of $\mathcal{H}^k\otimes\Xi_{\mu_{x_0}}$ where $\Xi_{\mu_{x_0}}$ is the line bundle of all signed Borel measures which are absolutely continuous with respect to μ_{x_0}. For h^j as in (73) it follows from [23] that $\mathcal{I}_*^\infty((h^1\wedge h^2)\otimes\mu_{x_0})$ is an $(\infty-2)$ volume in this sense. By the vanishing of L^2 harmonic one forms in [27] the L^2 $(\infty-1)$-cohomology of $C_{x_0}M$ also vanishes.

5.2.1 Remarks

1. We can use different metric connections on M. These give different $\mathcal{H}$ on $C_{x_0}M$ and the heat semigroups involved may have a first order term, see [22]. This can be compensated for by adding a vector field as drift to our SDE (70). The measures involved will be equivalent to our Brownian motion measure μ_{x_0}.
2. If M is a compact Lie group G with bi-invariant metric we can use a left-invariant SDE, which corresponds to the right invariant flat connection. This gives the same μ_{x_0}. In this case the Itô map $\mathcal{I}$ is essentially an equivalence between $C_0(\mathbf{R}^m)$ and $C_{x_0}G$, though still only defined almost surely, as shown by Fang & Franchi [34]. They are able to use it to pull over Shigekawa's Hodge theory for $C_0(R^m)$ to give a complete version for $C_{x_0}G$, with L^2 harmonic k-forms vanishing for $k\geq 1$. This will give vanishing of the L^2 codimension k cohomology. They similarly obtained a full Hodge theory for based loops on G, [33]. Aida, [2], employed rough path techniques and Kusuoka's alternative approach to Hodge Theory [43] to obtain vanishing for L^2 harmonic one-forms on the based loop space of G when G is simply connected. See also [48] for equivariant deRham cohomology on free loop groups.
3. For based, or free, loops on a general compact M, a Bismut tangent space $\mathcal{H}$ of admissible directions still exists with a Gross -Sobolev calculus, but no Kodaira -Hodge theory in the sense above (but see [43]) . For surveys see [46],[17], [14].
4. The structure theory for $C_{x_0}(M)$ based on the heat equation on M, goes over to a large extent with the Laplacian replaced by a degenerate elliptic 2nd order operator $\mathcal{A}$ on M, at least provided the principal symbol of $\mathcal{A}$ has constant rank, [24].
5. The problem concerning Sard's theorem mentioned in section 2.2.1 is an obstruction to using degree theory with diffusion, and other, measures.

6. For Hodge-deRham theory on configuration spaces of open manifolds see [5]; and for a geometric analysis on Radon measures see [40]. For generalities and references see [12].

5.3 Structures determined by Kokarev & Kuksin's examples

Returning to the examples 2.1 and 2.2 recall that they show the projection gives a proper Φ_0 map

$$F : P = \{(u, g) \in \mathcal{F}(S^1; M) \times \mathcal{E} \text{ such that } \frac{du}{dt} = g(t, u(t)) \quad t \in S^1\} \to \mathcal{E}$$

for a suitable space of periodic non-autonomous vector fields $\mathcal{E}$. Suppose the space $\mathcal{F}(S^1; M)$ of loops refers to the component containing the constant loops. Then, given orientability, a Gaussian measure γ on $\mathcal{E}$ can be pulled back to a signed measure $F^*(\zeta_\gamma)$ on $\mathcal{F}(S^1; M)$ with total mass the degree of F, which is χ_M. In turn this can be projected down to a signed measure $(p_1)_*(F^*(\zeta_\gamma))$ on $\mathcal{F}(S^1; M)$ which again will have total mass χ_M.

Presumably as with the Itô maps $\mathcal{I}$ in section 5.2 we can use the projection p_1 to push some of the AWM structure induced on P down to $\mathcal{F}(S^1; M)$, with an associated Gross-Sobolev calculus. Of course this and the measure will depend on the choice of γ, and $\mathcal{E}$. Free loop spaces are harder to handle by stochastic analysis than based loops or paths. Whether or not such a choice could be made so that the measure is related to one of those which have been already studied, this procedure might be helpful for analysis on these loop spaces. The same holds for their second example: we may obtain signed measures on a component $\mathcal{F}(M; N)$ of maps from $M \to N$. For $M = S^1$ stochastically forced heat equations have been studied, initially by Funaki, then Brzezniak et al, and M.Hairer, see [60] for a discussion and references, but maybe not the stochastic harmonic map equation.

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