# Relaxation of integral functionals depending on the symmetrised gradient

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# Contents

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# Declaration

I hereby declare that this thesis is the work of my own. All external sources used throughout the thesis are acknowledged as references. It has not been submitted for a degree at any other institution. Selected fragments from chapters 3, 4 and 5 of this thesis have been submitted in the form of a journal paper:

Kosiba K. and Rindler F., *On the relaxation of integral functionals depending on the symmetrized gradient*. Preprint available at: `https://arxiv.org/abs/1903.05771`.

# Abstract


In this thesis we aim to advance the variational theory of integral functionals depending on the symmetrised gradient. New contributions to this theory are contained in chapters 3, 4 and 5, where we study relaxations of integral functionals of the form:

$$\mathcal{F} : u \mapsto \int_\Omega f\left(x, \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T\right)\right) \mathrm{d}\, x, \qquad u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$$

under various 'shape' constraints imposed on the integrand $f$. Functionals of this form arise naturally in the mathematical theory of solid mechanics. In Chapter 3 we investigate the linear growth case, that is we additionally assume that $f$ satisfies bounds:

$$m|A| \leq f(A) \leq M(1 + |A|)$$

for all symmetric matrices $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and some constants $0 < m \leq M$. Sometimes this growth is called linear *isotropic*. In Chapter 4 we deal with the case of mixed growth, that is we assume that the inequality

$$m\left((\operatorname{tr} A)^2 + |\operatorname{dev} A|\right) \leq f(A) \leq M\left(1 + (\operatorname{tr} A)^2 + |\operatorname{dev} A|\right)$$

holds for all symmetric matrices $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and some constants $0 < m \leq M$.

In Chapter 5 we look at the special case of mixed-growth functionals, the Hencky's plasticity functional and its inhomogeneous generalisation. The main result of this chapter is the proof of lower semicontinuity of the aforementioned inhomogeneous functional in a sufficiently weak topology. This result relies on the theory of Young measures, which we briefly recall. We also discuss new developments in this theory and state open problems.



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# Chapter 1

# Introduction

The prime objective of this work is to advance the variational theory of integral functionals depending on the symmetrised gradient. By the symmetrised gradient we understand the symmetric part of the gradient of a vector-valued function. In applications, this function represents a displacement of a body that occupies certain region in space and the symmetric gradient is the (linearised) strain tensor, which expresses the relative change in the position of points within a body that has undergone the infinitesimal deformation. Such functionals arise naturally in the mathematical theory of solid mechanics, where they represent the total energy of the material deformation.

A physically relevant problem is to minimise the deformation energy in a suitable class of deformations, subject to some boundary datum, which represents the density of external forces acting on the body. Denoting by $\mathcal{F}$ the energy functional, we can express the minimisation problem more precisely as follows:

$$\min\Big\{\mathcal{F}[u] :\ u \in X,\ u = g \text{ on } \partial\Omega\Big\}, \tag{1.1}$$

where $X$ is a space of functions $u : \Omega \to \mathbb{R}^d$, $g : \partial\Omega \to \mathbb{R}^d$ is a boundary constraint and $\Omega \subset \mathbb{R}^d$ is a domain occupied by the continuum. For practical problems, it suffices to investigate dimensions $d \leq 3$. As we will soon see, the underlying class of functions $X$ is not only determined by the 'shape' of the functional $\mathcal{F}$, but also by the mathematical 'machinery' it offers. A general approach to the problem (1.1) is via the so-called Direct Method of the Calculus of Variations, which essentially means that the minimiser of (1.1) is constructed by taking a *minimising sequence* $(u_j) \subset X$ such that

$$\mathcal{F}[u_j] \to \inf_X \mathcal{F} \quad \text{as} \quad j \to \infty.$$

Then, by the interplay between the *continuity* of $\mathcal{F}$ and the *compactness* of $X$ we can, in principle, obtain a limit $u_\infty$ of the minimising sequence (with respect to the convergence determined by the compactness), which is the desired minimiser. In reality, however, the compactness of the space $X$ often forces us to consider a very weak notion of convergence for the minimising sequence, which in turn may result in the minimiser to be outside of $X$. This leads to the concept of *relaxation*, which is a natural procedure, when one is interested in finding a minimiser, but the poor compactness of the underlying function space $X$ undermines the use of the Direct Method. In this case one may extend (*relax*) the functional to a larger space with a better compactness and seek for a minimiser of the extension, with a property that the minimum of the extension agrees with the minimum (infimum) of the original problem.

In the subsequent chapters we study relaxations of integral functionals of the form:

$$\mathcal{F} : u \mapsto \int_\Omega f\left(x, \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T\right)\right) \mathrm{d}\, x, \qquad u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d \tag{1.2}$$

under various 'shape' constraints imposed on the integrand $f$. Henceforth, we write $\mathcal{E}u(x)$ for the symmetrised derivative $(\nabla u(x) + \nabla u(x)^T)/2$.

In Chapter 2 we assume that the integrand $f$ is a homogeneous function, i.e. $f$ does not explicitly depend on $x$, with linear growth bounds:

$$m|A| \leq f(A) \leq M\,(1 + |A|)$$

for some constants $0 < m \leq M$ and all $A \in \mathbb{R}^{d \times d}_{\mathrm{sym}}$. Here $|A|$ denotes the Frobenius norm of a matrix $A$.

In this case it is natural to study (1.2) over the space of integrable functions $u$ with integrable symmetrised distributional derivative $\mathcal{E}u$, i.e.

$$\mathrm{LD}(\Omega) := \left\{ u \in \mathrm{L}^1(\Omega;\mathbb{R}^d) : \ \mathcal{E}u \in \mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \right\}.$$

Unfortunately, in this space the direct method of the calculus of variations does not provide any solution to the minimization problem. The culprit is the lack of reflexivity and consequently, the inability to infer the (weak) relative compactness from the norm-boundedness of a minimising sequence. In fact, one can see that the sequence

$$u_j(x) := jx\mathbb{1}_{(0,1/j)}(x) + \mathbb{1}_{(1/j,1)}(x), \quad x \in (-1,1)$$

is bounded in $\mathrm{LD}((-1,1))$ with respect to the natural norm $\|u\|_{\mathrm{LD}} := \|u\|_1 + \|\mathcal{E}u\|_1$, but the $\mathrm{L}^1$-limit $u_\infty = \mathbb{1}_{(0,1)} \notin \mathrm{LD}((-1,1))$. The key feature of the sequence $(u_j)$ is that the sequence of derivatives $(u_j')$ develops a singular behaviour – it concentrates at 0. In other words, the sequence of measures $u_j'\mathscr{L}^1 \llcorner (-1,1)$ converges weakly* to the Dirac measure $\delta_0$, which is singular with respect to the Lebesgue measure.

Therefore, the functional (1.2) needs to be extended to account for displacement fields $u$ whose linear strains $Eu$ are measures, since in the space of measures norm-boundedness of a minimising sequence implies weak* relative compactness. Then the usual Direct Method applies. For this, one introduces the space $\mathrm{BD}(\Omega)$ of *functions of bounded deformation* as the space of all functions $u \in \mathrm{L}^1(\Omega;\mathbb{R}^d)$ such that the distributional symmetrised derivative $Eu := \frac{1}{2}(Du + Du^T)$ is representable as a finite Radon measure $Eu \in \mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

The relaxation of $\mathcal{F}$, commonly denoted by $\mathcal{F}_*$, is then defined in an abstract way as the smallest of the lower limits of $\mathcal{F}[u_h]$ over all sequences $(u_h) \subset \mathrm{BD}(\Omega)$, converging to some $u \in \mathrm{BD}(\Omega)$ from the larger space, i.e.

$$\mathcal{F}_*[u] := \inf\left\{ \liminf_{h\to\infty} \mathcal{F}[u_h] : \ (u_h) \subset \mathrm{BD}(\Omega), \ u_h \xrightarrow{*} u \text{ in } \mathrm{BD}(\Omega) \right\}.$$

In this definition it is implicitly assumed that $\mathcal{F}$ is extended by $+\infty$ outside of $\mathrm{LD}(\Omega)$. The choice of convergence in the above definition is effectively determined by the available compactness of the larger space.

As it stands, the relaxation in its abstract form is not particularly appealing. Fortunately, under certain convexity assumption on $f$, one can prove that the relaxation of $\mathcal{F}$ is also an integral functional. Here, we prove a refined relaxation theorem in BD, improving the results of [6, 10, 33] to an essentially optimal (under the following growth conditions) result:

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, \infty)$ be a continuous function such that*

1. *there exist constants $0 < m \leq M$, for which the inequality*
$$m|A| \leq f(A) \leq M(1 + |A|), \quad A \in \mathbb{R}^{d\times d}_{\mathrm{sym}},$$
*holds;*
2. *$f$ is symmetric-quasiconvex, that is for any bounded Lipschitz domain $\omega \subset \mathbb{R}^d$, any symmetric matrix $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and any $\psi \in \mathrm{W}^{1,\infty}_0(\omega; \mathbb{R}^d)$ the inequality*
$$|\omega| f(A) \leq \int_\omega f(A + \mathcal{E}\psi(y)) \,\mathrm{d}\, y$$
*holds.*

*Then, the functional*
$$\overline{\mathcal{F}}[u] := \int_\Omega f(\mathcal{E}u) \,\mathrm{d}\, x + \int_\Omega f^{\#}\left(\frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|}\right) \mathrm{d}\, |E^s u|, \quad u \in \mathrm{BD}(\Omega)$$
*is the relaxation of the functional*
$$\mathcal{F}[u] := \int_\Omega f(\mathcal{E}u) \,\mathrm{d}\, x$$
*with respect to the weak* topology in* $\mathrm{BD}(\Omega)$.

Here, the strain $Eu$ is decomposed into $Eu = E^a u + E^s u = \mathcal{E}u \mathscr{L}^d \llcorner \Omega + E^s u$ according to the Lebesgue decomposition theorem, $\frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|}$ is the polar density of the singular part $E^s u$ with respect to $|E^s u|$, and $f^{\#}$ is the upper recession function of $f$, i.e.
$$f^{\#}(A) := \limsup_{\substack{A' \to A \\ s \to \infty}} \frac{f(sA')}{s}, \quad A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

In Theorem 1.1 in [33], only the weak* lower semicontinuity result, and not a full relaxation result, was established under the assumption that the strong recession function $f^\infty$ (with a limit instead of upper limit) exists. Our result extends [10] and also Corollary 1.10 in [6] to a relaxation theorem without any assumption on the recession function. It was possible due to the recent developments in the theory of functions of bounded deformation (see next chapter for details), namely the Alberti's rank-one analogue by [16]. We note that in view of Theorem 2 in [32], one can construct a function satisfying (1), for which $f^\infty$ does not exist.

In Chapter 4 we investigate the case where the integrand $f$ in (1.2) is a homogeneous function satisfying mixed-growth bounds:

$$m\left((\operatorname{tr} A)^2 + |\operatorname{dev} A|\right) \leq f(A) \leq M\left(1 + (\operatorname{tr} A)^2 + |\operatorname{dev} A|\right) \tag{1.3}$$

for some constants $0 < m \leq M$ and all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. The motivation for such study comes from the classical convex functional of Hencky's plasticity:

$$\int_\Omega \varphi(\operatorname{dev} \mathcal{E}u) + \frac{\kappa}{2}(\operatorname{div} u)^2 \,\mathrm{d}\,x, \tag{1.4}$$

where $\varphi : \mathrm{SD}(d) \to [0, +\infty)$ is a function which grows quadratically on some compact set and linearly outside of this set, and $\kappa = \lambda + 2\mu/3$ is the bulk modulus of the material with the Lamé constants $\lambda$ and $\mu$. Here, $\mathrm{SD}(d)$ denotes the space of symmetric and trace-free matrices in $\mathbb{R}^{d\times d}$ and $\operatorname{dev} A := A - d^{-1}(\operatorname{tr} A)\,\mathrm{id}$ is the deviatoric (trace-free) part of a matrix $A \in \mathbb{R}^{d\times d}$. Our aim is to generalize (1.4) to include possibly non-convex integrands.

A first choice for a function space on which to define the functional (1.2) with the growth constraints (1.3) is the space of integrable functions $u$ with integrable symmetrised distributional derivative $\mathcal{E}u$ and square-integrable distributional divergence, i.e.

$$\mathrm{LU}(\Omega) := \left\{u \in \mathrm{L}^1(\Omega; \mathbb{R}^d) : \ \mathcal{E}u \in \mathrm{L}^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}),\ \operatorname{div} u \in \mathrm{L}^2(\Omega)\right\}.$$

This space of functions, however, shares the same flaws as the space $\mathrm{LD}(\Omega)$. To address them, one is naturally led to considering the *Temam–Strang space*

$\mathrm{U}(\Omega)$ of functions of bounded deformation with square-integrable divergence, i.e.

$$\mathrm{U}(\Omega) := \left\{ u \in \mathrm{BD}(\Omega) : \ \operatorname{div} u \in \mathrm{L}^2(\Omega) \right\}.$$

For more information on BD, U and their applications in the theory of plasticity we refer to [2, 21, 25, 30, 36–39]. In the next chapter we briefly recall results relevant for this thesis.

The appropriate relaxation $\mathcal{F}_*$ of $\mathcal{F}$ for $u \in \mathrm{U}(\Omega)$ is defined as follows:

$$\mathcal{F}_*[u] := \inf \left\{ \liminf_{h\to\infty} \mathcal{F}[u_h] : \ (u_h) \subset \mathrm{U}(\Omega), \ u_h \overset{*}{\rightharpoonup} u \text{ in } \ \mathrm{U}(\Omega) \right\}.$$

Again, we implicitly extend $\mathcal{F}$ by $+\infty$ outside of $\mathrm{LU}(\Omega)$.

It turns out that it also has an integral form under suitable constraints on the integrand $f$:

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, \infty)$ be a continuous function satisfying the following conditions:*

1. *there exist constants $0 < m \le M$ such that for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the growth*
$$m \left( (\operatorname{tr} A)^2 + |\operatorname{dev} A| \right) \le f(A) \le M \left( 1 + (\operatorname{tr} A)^2 + |\operatorname{dev} A| \right)$$
*holds;*
2. *$f$ is symmetric-quasiconvex;*
3. *there exist constants $\gamma \in [0,2)$ and $\delta \in [0,1)$ such that for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the inequality*
$$f(A) \ge f^{\#}_{\mathrm{dev}}(\operatorname{dev} A) - M \left( |\operatorname{tr} A|^\gamma + |\operatorname{dev} A|^\delta + 1 \right) \tag{1.5}$$
*holds.*

*Then, the functional*

$$\overline{\mathcal{F}}[u,\Omega] := \int_\Omega f(\mathcal{E}u) \,\mathrm{d}\, x + \int_\Omega f^{\#}_{\mathrm{dev}} \left( \frac{\mathrm{d}\, E^s u}{\mathrm{d}\,|E^s u|} \right) \mathrm{d}\,|E^s u|, \quad u \in \mathrm{U}(\Omega)$$

*is the relaxation of the functional*

$$\mathcal{F}[u] := \int_\Omega f(\mathcal{E}u) \,\mathrm{d}\, x$$

*with respect to the weak* topology in $\mathrm{U}(\Omega)$, i.e. $\mathcal{F}_* = \overline{\mathcal{F}}$.*

Here $f_{\mathrm{dev}}^{\#}$ is the upper recession function of the restriction $f_{\mathrm{dev}}$ of $f$ to the symmetric deviatoric $(d \times d)$-matrices, i.e.

$$f_{\mathrm{dev}}^{\#}(D) := \limsup_{\substack{D' \to D \\ s \to \infty}} \frac{f_{\mathrm{dev}}(sD')}{s}, \quad D \in \mathrm{SD}(d)\,.$$

The integral representation is substantially harder to obtain in the mixed-growth case than in the linear growth case. The issues arise due to the incompatibility of the standard blow-up argument with mixed-growth integrands, see Chapter 4 for details.

In Chapter 5, by using Young measure methods, we establish the following weak* lower semicontinuity theorem for inhomogeneous (i.e., $x$-dependent) energy functionals:

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and*

1. *the function $g : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ is Carathéodory with linear growth:*

$$m|A| \le g(x, A) \le M(1 + |A|), \quad (x, A) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}},$$

   *for some constants $0 < m \le M$;*

2. *for every $x \in \Omega$ the map $A \mapsto g(x, \operatorname{dev} A)$ is symmetric-quasiconvex;*
3. *the strong recession function $(g \circ \operatorname{dev})^\infty$, defined as the limit*

$$(g \circ \operatorname{dev})^\infty(x, A) := \lim_{\substack{(x', A') \to (x, A) \\ s \to \infty}} \frac{g(x', s \operatorname{dev} A')}{s}, \quad A \in \mathrm{SD}(d),$$

   *exists and is jointly continuous;*

4. *the function $h : \Omega \times \mathbb{R} \to [0, +\infty)$ is Carathéodory, convex and has quadratic growth*

$$0 \le h(x, z) \le M(1 + |z|^2), \quad (x, z) \in \Omega \times \mathbb{R}.$$

*Then, the functional*

$$\overline{\mathcal{G}}[u] := \int_\Omega g(x, \operatorname{dev} \mathcal{E}u) + h(x, \operatorname{div} u) \,\mathrm{d}\,x + \int_\Omega (g \circ \operatorname{dev})^\infty \left( x, \frac{\mathrm{d}\, E^s u}{\mathrm{d}\,|E^s u|} \right) \mathrm{d}\,|E^s u|$$

*is weakly* lower semicontinuous on* $\mathrm{U}(\Omega)$.

Note that here, we need to require the existence of the strong recession function $(g \circ \mathrm{dev})^\infty$, as the argument is based on the theory of Young measures, for which the existence of $(g \circ \mathrm{dev})^\infty$ is crucial.

# Chapter 2

# Prerequisites

## 2.1 General notation

By $\mathbb{R}^d$ we denote the $d$-dimensional Euclidean space with $d \geq 1$. We write $B(x,r)$ for an open ball, $\overline{B(x,r)}$ for a closed ball and $\partial B(x,r)$ for a sphere centred at $x \in \mathbb{R}^d$ with the radius $r > 0$. For any matrix $A \in \mathbb{R}^{d\times d}$ its *deviatoric projection* is defined as $\operatorname{dev} A := A - d^{-1}(\operatorname{tr} A)\operatorname{id}$, where $\operatorname{id} \in \mathbb{R}^{d\times d}$ is the identity matrix. The set of all symmetric and deviatoric matrices in $\mathbb{R}^{d\times d}$ is denoted by

$$\mathrm{SD}(d) := \{M \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \ \operatorname{tr} M = 0\}.$$

We also write $a \odot b := (a \otimes b + b \otimes a)/2$ for the symmetrised tensor product.

In this thesis we always assume that $\Omega \subset \mathbb{R}^d$ is an open bounded Lipschitz domain, unless stated otherwise.

We write $\mathrm{L}^p(\Omega)$, $\mathrm{L}^p(\Omega; X)$, $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, etc. for the Lebesgue spaces and $\mathrm{W}^{p,q}(\Omega)$, $\mathrm{W}^{p,q}(\Omega; X)$, $\mathrm{W}^{p,q}_g(\Omega)$, etc. for the Sobolev spaces with suitable exponents.

## 2.2 Measure theory

We write $\mathcal{B}(X)$ for the Borel $\sigma$-algebra on a topological space $X$. The $d$-dimensional Lebesgue measure is denoted by $\mathscr{L}^d$ and for the $\mathscr{L}^d$-measurable set $A \subseteq \mathbb{R}^d$ we occasionally write $|A|$ instead of $\mathscr{L}^d(A)$.

The cone of (finite) Radon measures is denoted by $\mathrm{M}^+(\mathbb{R}^d)$ and its subspace of probability measures is denoted by $\mathrm{M}^1(\mathbb{R}^d)$. The following theorem provides a simple criterion for a set function to be a Radon measure (for the proof see [4, Theorem 1.53]).

**Theorem 2.1 (De Giorgi-Letta).** *Let $X$ be a metric space and let $\mathcal{U}(X)$ denote the set of open subsets of $X$. Let $\mu : \mathcal{U}(X) \to [0, \infty]$ be a set function such that*

1. *$\mu(\emptyset) = 0$;*
2. *(monotonicity) for $A, B \in \mathcal{U}(X)$ if $A \subset B$ then $\mu(A) \leq \mu(B)$;*
3. *(subadditivity) for $A, B \in \mathcal{U}(X)$ it holds that $\mu(A \cup B) \leq \mu(A) + \mu(B)$;*
4. *(superadditivity) for $A, B \in \mathcal{U}(X)$ with $A \cap B = \emptyset$ it holds that $\mu(A \cup B) \geq \mu(A) + \mu(B)$;*
5. *(inner regularity) $\mu(A) = \sup\{\mu(B) :\ B \in \mathcal{U}(X),\ B \Subset A\}$.*

*Then, the extension of $\mu$ to every $B \subset X$ defined by*

$$\mu(B) := \inf\{\mu(A) :\ A \in \mathcal{U}(X),\ A \supset B\}$$

*is an outer measure. In particular, the restriction of $\mu$ to Borel $\sigma$-algebra is a positive measure.*

Let $\mu$ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^d$ and let $k \geq 0$. We define the *upper $k$-density* of $\mu$ at $x \in \Omega$ as

$$\Theta_k^*(\mu, x) := \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{\omega_k r^k},$$

where $\omega_k := \pi^{k/2}\Gamma(1 + k/2)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^k$. Similarly, one defines the lower $k$-density, by replacing the upper limit with the lower limit.

The following result (see [4, Theorem 2.56] for the proof) asserts that the upper $k$-density can be used to estimate the measure $\mu$ from below by the $k$-dimensional Hausdorff measure $\mathscr{H}^k$.

**Proposition 2.2.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mu$ be a positive Radon measure in $\Omega$. Then, for any $0 < t < \infty$ and any Borel set $B \subset \Omega$ the implication*

$$\Theta^*_k(\mu, x) \geq t \quad \forall\, x \in B \implies \mu \geq t\mathscr{H}^k \llcorner B$$

*holds.*

We also use vector-valued Borel measures $\mu : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}^N$, which are $\sigma$-additive set functions with $\mu(\emptyset) = 0$. The space of all such vector measures is denoted by $\mathrm{M}(\mathbb{R}^d; \mathbb{R}^N)$. The space of local vector measures is denoted by $\mathrm{M}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^N)$. For a vector measure $\mu \in \mathrm{M}(\mathbb{R}^d; \mathbb{R}^N)$ we define its *total variation measure* $|\mu| \in \mathrm{M}^+(\mathbb{R}^d)$ by

$$|\mu|(S) := \sup \left\{ \sum_{k\in\mathbb{N}} |\mu(S_k)| : \ S = \bigcup_{k\in\mathbb{N}} S_k, \ \{S_k\} \text{ is a Borel partition of } S \right\}.$$

The *restriction* of a measure $\mu \in \mathrm{M}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^N)$ to a Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ is defined as $\mu \llcorner B(S) := \mu(B \cap S)$ for all relatively compact Borel sets $S \in \mathcal{B}(\mathbb{R}^d)$.

For a positive measure $\mu$ on a locally compact separable metric space $X$, the *support* of $\mu$, in symbols $\operatorname{supp} \mu$, is the closed set of all points $x \in X$ such that $\mu(U) > 0$ for every neighbourhood $U$ of $x$. For a vector measure $\nu$ we define its support to be the support of its total variation measure $|\nu|$.

**Theorem 2.3 (Besicovitch differentiation theorem).** *Let $\mu \in \mathrm{M}(\mathbb{R}^d; \mathbb{R}^N)$ be a vector-valued Radon measure and let $\nu \in \mathrm{M}^+(\mathbb{R}^d)$ be a positive Radon measure. Then for $\nu$-a.e. $x_0 \in \mathbb{R}^d$ in the support of $\nu$, the limit*

$$\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}(x_0) := \lim_{r\downarrow 0} \frac{\mu(B(x_0, r))}{\nu(B(x_0, r))}$$

*exists and is called the* Radon-Nikodym derivative *of $\mu$ with respect to $\nu$.*

*Moreover, we have the Lebesgue decomposition of $\mu = \frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}\nu + \mu^s$, where $\mu^s = \mu \llcorner E$ is singular with respect to $\nu$ and*

$$E = (\mathbb{R}^d \setminus \operatorname{supp} \nu) \cup \left\{ x \in \operatorname{supp} \nu : \ \lim_{r\downarrow 0} \frac{|\mu|(B(x, r))}{\nu(B(x, r))} = \infty \right\}.$$

For the proof, see Theorem 2.22 in [4]. See also Theorem 5.52 in [4] for a more general version, where a ball $B(x_0, r)$ can be replaced with a set $x_0 + rC$ for any open convex set $C \subset \mathbb{R}^d$ containing the origin.

## 2.3 Convexity

Various notions of convexity play a central role in the calculus of variations, as they affect the statement of necessary and sufficient conditions for many minimisation problems. We briefly recall definitions and basic properties of the two weaker notions of convexity. These convexity concepts are effectively symmetric counterparts of the usual quasiconvexity in the sense of [31] and rank-one convexity.

**Definition 2.4.** *Let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ be a locally bounded Borel function. We call $f$* symmetric-quasiconvex*, provided that for all bounded Lipschitz domains $\omega \subset \mathbb{R}^d$, all test functions $\psi \in \mathrm{W}^{1,\infty}_0(\omega; \mathbb{R}^d)$ and all matrices $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the inequality*

$$f(A) \leq \fint_\omega f(A + \mathcal{E}\psi(y)) \,\mathrm{d}\, y \tag{2.1}$$

*holds.*

If the function $f$ additionally satisfies an asymptotic growth condition of the form $|f(A)| \leq C(1 + |A|^p)$, then (2.1) holds for $\psi \in \mathrm{W}^{1,p}_0(\omega; \mathbb{R}^d)$ (cf. [35, Lemma 5.2(ii)]).

**Definition 2.5.** *Let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ be a Borel function. Then, the* symmetric-quasiconvex envelope *$SQf : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R} \cup \{-\infty\}$ is a function defined as*

$$SQf(A) := \inf \left\{ \fint_\omega f(A + \mathcal{E}\psi(y)) \,\mathrm{d}\, y : \ \psi \in \mathrm{W}^{1,\infty}_0(\omega; \mathbb{R}^d) \right\}. \tag{2.2}$$

**Remark 2.6.**

1. By the Vitali covering argument one can show that the inequality (2.1) and the formula (2.2) are independent of the choice of the domain $\omega$ (cf. [35, Lemma 5.2(i)]). See also Proposition 5.11 in [15] for a different proof.

2. For a non-negative continuous function $f$ with $p$-growth, $1 \leq p < \infty$, the symmetric-quasiconvex envelope $SQf$ is symmetric-quasiconvex and also has $p$-growth (see [35, Lemma 7.1]).
3. For a function $f$ as in (2), the symmetric-quasiconvex envelope of $f$ can be equivalently expressed as the greatest symmetric-quasiconvex function, no larger than $f$, i.e.

$$SQf(A) = \sup \{g(A) : \ g \text{ is symmetric-quasiconvex and } g \leq f\}\,.$$

**Example 2.7.** Let $\operatorname{dist}(e, S) := \inf \{|x - e| : \ x \in S\}$. A non-trivial example of a symmetric-quasiconvex function which is not convex is the map

$$\mathbb{R}^{d\times d}_{\mathrm{sym}} \ni A \mapsto SQ\left(\operatorname{dist}(A, \{F, -F\})^p\right), \quad 1 \leq p < 2,$$

where $F \neq a \odot b$ for any $a, b \in \mathbb{R}^d$. Indeed, it is clear that the distance function $\operatorname{dist}(\cdot, \{F, -F\})^p$ is non-negative, continuous and with $p$-growth. Hence, by Remark 2.6(2), the function $SQ\left(\operatorname{dist}(\cdot, \{F, -F\})^p\right)$ is symmetric-quasiconvex. It can be shown that this function is not convex at the zero matrix (see [35, Lemma 7.3]).

**Definition 2.8.** *Let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ be a locally bounded Borel function. We call $f$* symmetric rank-one convex*, if*

$$\mathbb{R} \ni t \mapsto f(A + ta \odot b),$$

*is convex for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $a, b \in \mathbb{R}^d$.*

**Remark 2.9.**

1. As for the quasiconvexity and the rank-one convexity, it can be shown, that symmetric-quasiconvexity implies symmetric rank-one convexity. More precisely, by the one-directional oscillations argument, similar to the one in the proof of Proposition 5.3 in [35], one can prove that for a symmetric-quasiconvex function $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ the inequality

$$f(\theta A + (1 - \theta)B) \leq \theta f(A) + (1 - \theta)f(B)$$

holds for $A, B \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ with $B - A = a \odot b$ for some $a, b \in \mathbb{R}^d$ and $\theta \in [0,1]$. This is equivalent to $f$ being symmetric rank-one convex.

2. Every symmetric rank-one convex function $f$ is (locally) Lipschitz continuous. If $f$ additionally satisfies $p$-growth condition (with a constant $M > 0$), $1 \le p < \infty$, then the inequality

$$|f(A) - f(B)| \le C(1 + |A|^{p-1} + |B|^{p-1})|A - B|, \quad A, B \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

holds with a constant $C = C(d, M) > 0$. In particular for $p = 1$ we have a global Lipschitz continuity of such $f$. The proof of these assertions is contained in [35, Lemma 5.6].

Recall that a function $f : \mathbb{R}^N \to \mathbb{R}$ is called *positively 1-homogeneous*, if for all $A \in \mathbb{R}^N$ and all $t \ge 0$ the equality

$$f(tA) = tf(A)$$

holds.

The following convexity result for positively 1-homogeneous functions in conjunction with the BD-analogue of Alberti's rank-one theorem (Theorem 2.26) plays a vital role in the study of relaxations in chapter 4.

**Theorem 2.10 (Kirchheim-Kristensen).** *Let $\mathcal{C}$ be an open convex cone in a normed finite dimensional real vector space $\mathcal{V}$, and let $\mathcal{D}$ be a cone of directions in $\mathcal{V}$ such that $\mathcal{D}$ spans $\mathcal{V}$.*

*If $f : \mathcal{C} \to \mathbb{R}$ is $\mathcal{D}$-convex (i.e. its restrictions to line segments in $\mathcal{C}$ in directions of $\mathcal{D}$ are convex) and positively 1-homogeneous, then $f$ is convex at each point of $\mathcal{C} \cap \mathcal{D}$.*

*More precisely, and in view of homogeneity, for each $x_0 \in \mathcal{C} \cap \mathcal{D}$ there exists a linear function $\ell : \mathcal{V} \to \mathbb{R}$ satisfying $\ell(x_0) = f(x_0)$ and $f \ge \ell$ on $\mathcal{C}$.*

For the proof we refer to [24]. We also record the following simple fact.

**Proposition 2.11.** *The set of symmetric and deviatoric matrices* $\mathrm{SD}(d)$ *is spanned by the subset*

$$\mathcal{S} := \left\{ a \odot b : \ a, b \in \mathbb{R}^d, \ a \cdot b = 0 \right\}.$$

*Proof.* It is elementary to see that basis for $\mathrm{SD}(d)$ consists of matrices $(e_i + e_{i+1}) \odot (e_i - e_{i+1})$ for $i = 1, \ldots, d-1$ and $e_i \odot e_j$ for $i \neq j$. $\square$

We draw the following important conclusion from Theorem 2.10 and Proposition 2.11.

**Corollary 2.12.** *A symmetric rank-one convex and positively 1-homogeneous function* $f : \mathrm{SD}(d) \to \mathbb{R}$ *is convex at each point of the symmetric rank-one cone* $\mathcal{S}$.

## 2.4 Abstract relaxation

In this section we introduce the concept of the *relaxation* of functionals in an abstract topological space equipped with a metrizable topology.

As a motivating example, suppose that we want to solve a minimisation problem

$$\min \left\{ \int_\Omega f(x, u(x), \nabla u(x)) \, \mathrm{d}\, x : \ u \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^m) \right\}, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and the continuous integrand $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$ satisfies linear growth bounds

$$m|A| \leq f(x, z, A) \leq M(1 + |A|)$$

for all $(x, z, A) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ and some positive constants $0 < m \leq M$. It turns out that the direct method of the calculus of variations does not provide a solution to the problem (2.3). This is due to the fact that the Sobolev space $\mathrm{W}^{1,1}$ is not reflexive, and thus one cannot infer (weak) relative compactness of minimising sequences from boundedness.

However in this case, due to the bounds on the integrand $f$, one can see that a minimising sequence converges weakly* to a function of a bounded variation (cf. [4, Proposition 3.13]). This suggests that one should seek some form of *relaxation* of the functional $\mathcal{F}$, extended by $+\infty$ beyond $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^m)$, i.e.

$$\mathcal{F}_\infty[u] := \begin{cases} \displaystyle\int_\Omega f(x,u(x),\nabla u(x))\,\mathrm{d}\,x & \text{for } u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^m) \\ +\infty & \text{for } u \in (\mathrm{BV}\setminus \mathrm{W}^{1,1})(\Omega;\mathbb{R}^m) \end{cases}$$

to a functional $\mathcal{F}_*$ which is lower semicontinuous with respect to the weak* topology of $\mathrm{BV}(\Omega;\mathbb{R}^m)$ and such that the equality of infima holds, i.e.

$$\inf_{\mathrm{W}^{1,1}} \mathcal{F} = \min_{\mathrm{BV}} \mathcal{F}_*.$$

Below we present the basic theory of an abstract relaxation, which provides these features. The remainder of this section is based on [8, Chapter 11].

Let $(X,d)$ be a metrizable topological space and let $\mathcal{F} : X \to \mathbb{R}\cup\{+\infty\}$ be an extended real-valued functional. We define the *relaxation* $\mathcal{F}_* : X \to \mathbb{R}\cup\{+\infty\}$ as

$$\mathcal{F}_*[x] := \inf\left\{\liminf_{j\to\infty} \mathcal{F}[x_j] :\ (x_j) \subset X,\ x_j \rightsquigarrow x \text{ as } j \to \infty\right\},$$

where the convergence $x_j \rightsquigarrow x$ is understood with respect to the metric $d$.

We shall prove in Proposition 2.15 that the relaxation $\mathcal{F}_*$ is lower semicontinuous with respect to the convergence '$\rightsquigarrow$', that is, the inequality

$$\mathcal{F}_*[x] \leq \liminf_{j\to\infty} \mathcal{F}_*[x_j] \tag{2.4}$$

holds for any sequence $x_j \rightsquigarrow x$. Note, that this is not immediately clear from the definition of $\mathcal{F}_*$, since we only have the inequality $\mathcal{F}_*[x] \leq \liminf_{j\to\infty} \mathcal{F}[x_j]$ for $x_j \rightsquigarrow x$ and to obtain (2.4) one needs to use a suitable diagonal argument, see Lemma 2.13 below. Moreover, we prove that the relaxation $\mathcal{F}_*$ is the greatest lower semicontinuous functional no larger than $\mathcal{F}$. Such a functional is often called a *lower semicontinuous envelope* of $\mathcal{F}$.

**Lemma 2.13 (Diagonalisation lemma).** *Let $(a_{k,l})_{k,l} \subset X$ be a doubly-indexed sequence in a first countable topological space $X$ such that*

1. $\lim\limits_{l\to\infty} a_{k,l} = a_k$,
2. $\lim\limits_{k\to\infty} a_k = a$.

*Then, there exists a non-decreasing map $l \mapsto k(l)$, such that*

$$\lim_{l \to \infty} a_{k(l),l} = a.$$

The proof of Lemma 2.13 can be found in [7]. We begin with the following technical lemma.

**Lemma 2.14 (Recovery sequence).** *Let $x \in X$. Then, there exists a sequence $(x_j) \subset X$ such that $x_j \rightsquigarrow x$ and $\mathcal{F}_*[x] = \lim_{j \to \infty} \mathcal{F}[x_j]$.*

*Proof.* Fix arbitrary $x \in X$ and $k \in \mathbb{N}$. By the definition of $\mathcal{F}_*$, there exists a sequence $(x_j^{(k)})_j$ such that $x_j^{(k)} \rightsquigarrow x$ as $j \to \infty$ and

$$\mathcal{F}_*[x] \leq \liminf_{j \to \infty} \mathcal{F}[x_j^{(k)}] < \mathcal{F}_*[x] + \frac{1}{k}.$$

Let $\sigma_k : \mathbb{N} \to \mathbb{N}$ be an increasing map, which may depend on $k$, such that

$$\liminf_{j \to \infty} \mathcal{F}[x_j^{(k)}] = \lim_{j \to \infty} \mathcal{F}[x_{\sigma_k(j)}^{(k)}].$$

We thus have

$$\lim_{k \to \infty} \lim_{j \to \infty} \mathcal{F}[x_{\sigma_k(j)}^{(k)}] = \mathcal{F}_*[x].$$

By Lemma 2.13, applied to the sequence $(x_{\sigma_k(j)}^{(k)}, \mathcal{F}[x_{\sigma_k(j)}^{(k)}])_{k,j}$ and the first countable space $X \times (\mathbb{R} \cup \{+\infty\})$, we can choose a non-decreasing map $j \mapsto k(j)$ such that $x_{\sigma_{k(j)}(j)}^{(k(j))} \rightsquigarrow x$ and

$$\lim_{j \to \infty} \mathcal{F}[x_{\sigma_{k(j)}(j)}^{(k(j))}] = \mathcal{F}_*[x].$$

Therefore, the desired recovery sequence is given by $x_j := x_{\sigma_{k(j)}(j)}^{(k(j))}$. $\square$

**Proposition 2.15.** *The relaxation $\mathcal{F}_*$ is the greatest lower semicontinuous functional less than $\mathcal{F}$.*

*Proof.* Note that for an arbitrary $x \in X$, taking a constant sequence $x_j = x$ for all $j \in \mathbb{N}$ in the definition of $\mathcal{F}_*$ yields the inequality $\mathcal{F}_*[x] \leq \mathcal{F}[x]$.

We now prove that $\mathcal{F}_*$ is lower semicontinuous. Let $(x_j)_j \subset X$ be a sequence such that $x_j \rightsquigarrow x$ for some $x \in X$. Let $(x_k)_k := (x_{j_k})_k$ be a subsequence of

$(x_j)_j$ such that

$$\lim_{k\to\infty} \mathcal{F}_*[x_k] = \liminf_{j\to\infty} \mathcal{F}_*[x_j].$$

By Lemma 2.14 there exists a recovery sequence $(y_k^{(l)})_l \subset X$ such that $y_k^{(l)} \rightsquigarrow x_k$ for each $k \in \mathbb{N}$ and such that

$$\mathcal{F}_*[x_k] = \lim_{l\to\infty} \mathcal{F}[y_k^{(l)}],$$

hence

$$\liminf_{j\to\infty} \mathcal{F}_*[x_j] = \lim_{k\to\infty} \mathcal{F}_*[x_k] = \lim_{k\to\infty} \lim_{l\to\infty} \mathcal{F}[y_k^{(l)}].$$

Since

$$\lim_{k\to\infty} \lim_{l\to\infty} y_k^{(l)} = \lim_{k\to\infty} x_k = x,$$

by Lemma 2.13 applied to the doubly-indexed sequence $(y_k^{(l)}, \mathcal{F}[y_k^{(l)}])_{k,l}$, there exists a sequence $(k_l)_l$ such that $y_{k_l}^{(l)} \rightsquigarrow x$ as $l \to \infty$ and

$$\lim_{l\to\infty} \mathcal{F}[y_{k_l}^{(l)}] = \liminf_{j\to\infty} \mathcal{F}_*[x_j].$$

We have

$$\begin{aligned}\liminf_{j\to\infty} \mathcal{F}_*[x_j] &= \lim_{l\to\infty} \mathcal{F}[y_{k_l}^{(l)}] \\ &\geq \inf\left\{\liminf_{j\to\infty} \mathcal{F}[x_j] :\ (x_j)_j \subset X,\ x_j \rightsquigarrow x\right\} = \mathcal{F}_*[x],\end{aligned}$$

which proves the lower semicontinuity of the relaxation $\mathcal{F}_*$.

It remains to prove that if $\mathcal{G} : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary lower semicontinuous functional such that $\mathcal{G} \leq \mathcal{F}$, then the inequality $\mathcal{G} \leq \mathcal{F}_*$ holds. This proves that $\mathcal{F}_*$ is the greatest such functional.

Let $\mathcal{G}$ be a functional as above and let $x_j \rightsquigarrow x$ in $X$. Since $\mathcal{G}$ is lower semicontinuous and $\mathcal{G} \leq \mathcal{F}$ it holds that

$$\mathcal{G}[x] \leq \liminf_{j\to\infty} \mathcal{G}[x_j] \leq \liminf_{j\to\infty} \mathcal{F}[x_j].$$

Taking infimum over all sequences $x_j \rightsquigarrow x$ yields the inequality $\mathcal{G}[x] \leq \mathcal{F}_*[x]$, which ends the proof. $\square$

We end this section with the proof of the following relaxation principle.

**Proposition 2.16.** *Let $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, i.e. there exists $x \in X$ such that $\mathcal{F}[x] < \infty$, extended real-valued function. Suppose that there exists a minimising sequence $(x_j) \subset X$, i.e. $\mathcal{F}[x_j] \to \inf_X \mathcal{F}$ and $(x_j)$ is relatively compact in $X$. Then*

1. $\inf_X \mathcal{F} = \min_X \mathcal{F}_*$,
2. *every cluster point $x \in X$ of the sequence $(x_j)$ is a solution to the minimisation problem $\min_X \mathcal{F}_*$, that is $x \in \arg\min_X \mathcal{F}_*$.*

*Proof.* Let $x \in X$ be a cluster point of $(x_j)$ and let $(x_{j_k})$ be a subsequence of $(x_j)$ such that $x_{j_k} \rightsquigarrow x$ as $k \to \infty$. By the definition of $\mathcal{F}_*$ we have

$$\mathcal{F}_*[x] \leq \liminf_{j\to\infty} \mathcal{F}[x_j] = \inf_X \mathcal{F}.$$

On the other hand, by Proposition 2.14, for any $y \in X$ we can find a recovery sequence $(y_j)$ such that $y_j \rightsquigarrow y$ and

$$\mathcal{F}_*[y] = \lim_{j\to\infty} \mathcal{F}[y_j].$$

Therefore we obtain

$$\mathcal{F}_*[x] \leq \inf_X \mathcal{F} \leq \mathcal{F}_*[y]$$

for any $y \in X$, hence $x \in \arg\min_X \mathcal{F}_*$. Choosing $y = x$ we conclude that $\mathcal{F}_*[x] = \min_X \mathcal{F}_* = \inf_X \mathcal{F}$. □

## 2.5 Function spaces

In this section we recall definitions and basic properties of function spaces used throughout this thesis.

### Functions of bounded deformation

In the applications coming from plasticity theory, see for instance [36, 37, 39], one is often concerned with the class of functions

$$\mathrm{LD}(\Omega) := \left\{u \in \mathrm{L}^1(\Omega;\mathbb{R}^d) : \ \mathcal{E}u \in \mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\right\},$$

where $\mathcal{E}u := (\nabla u + \nabla u^T)/2$ is the distributional symmetrised gradient of a displacement $u : \Omega \to \mathbb{R}^d$. The space $\mathrm{LD}(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathrm{LD}} := \|u\|_1 + \|\mathcal{E}u\|_1.$$

However, in general we cannot infer weak relative compactness from boundedness, since $\mathrm{LD}(\Omega)$ is not reflexive. If a bounded sequence in $\mathrm{LD}(\Omega)$ has equiintegrable symmetric gradients, then in virtue of the Dunford-Pettis theorem, we could infer the weak relative compactness. The equiintegrability, however, is rare in applications, so we need to consider a larger space instead.

Therefore, we define the space $\mathrm{BD}(\Omega)$ of *functions of bounded deformation* [2, 36, 37, 39] as the space of all functions $u \in \mathrm{L}^1(\Omega; \mathbb{R}^d)$ such that the distributional symmetrised derivative $Eu := (Du + Du^T)/2$ is representable as a finite Radon measure $Eu \in \mathrm{M}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, i.e.

$$\mathrm{BD}(\Omega) := \left\{ u \in \mathrm{L}^1(\Omega; \mathbb{R}^d) : \ Eu \in \mathrm{M}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \right\}.$$

The space $\mathrm{BD}(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathrm{BD}} := \|u\|_1 + |Eu|(\Omega),$$

but the norm topology is too strong for applications in the theory of elasto-plasticity, hence we usually work in weaker topologies. We distinguish three such topologies.

**Definition 2.17 (Weak* convergence).** *We say that $(u_h) \subset \mathrm{BD}(\Omega)$ converges* weakly* *to $u$ in $\mathrm{BD}(\Omega)$ if $u_h \to u$ strongly in $\mathrm{L}^1(\Omega; \mathbb{R}^d)$ and $Eu_h \xrightarrow{*} Eu$ weakly* in $\mathrm{M}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$.*

The topology of the weak* convergence is useful, due to the following compactness property (cf. [38]).

**Theorem 2.18 (Compactness).** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $(u_h) \subset \mathrm{BD}(\Omega)$ be a uniformly norm-bounded sequence. Then, there exists a subsequence converging weakly* to some $u \in \mathrm{BD}(\Omega)$.*

We have the following simple fact.

**Lemma 2.19.** *Let* $(u_h) \subset \mathrm{BD}(\Omega)$ *be a sequence such that* $u_h \to u$ *strongly in* $\mathrm{L}^1(\Omega;\mathbb{R}^d)$ *and* $(u_h)$ *is uniformly norm-bounded in* $\mathrm{BD}(\Omega)$. *Then,* $(u_h)$ *converges weakly* to* $u$ *in* $\mathrm{BD}(\Omega)$.

*Proof.* Let $(u_h)$ be bounded in $\mathrm{BD}(\Omega)$ and $u_h \to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^d)$. We need to establish the convergence $Eu_h \overset{*}{\rightharpoonup} Eu$ weakly* in $\mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

From the boundedness of $(u_h)$ in $\mathrm{BD}(\Omega)$ we have in particular that $(Eu_h)$ is bounded in $\mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Therefore, up to a (not relabelled) subsequence we have $Eu_h \overset{*}{\rightharpoonup} \mu$ for some measure $\mu \in \mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. For $\Phi \in \mathrm{C}^1_c(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ we have

$$\begin{aligned}\sum_{i,j=1}^{d}\int_\Omega \Phi_i^j \,\mathrm{d}\,\mu_i^j &= \lim_{h\to\infty}\sum_{i,j=1}^{d}\int_\Omega \Phi_i^j \,\mathrm{d}(Eu_h)_i^j \\ &= -\lim_{h\to\infty}\sum_{j=1}^{d}\int_\Omega \operatorname{div}\Phi^j\, u_h^j \,\mathrm{d}\,x \\ &= -\sum_{j=1}^{d}\int_\Omega \operatorname{div}\Phi^j\, u^j \,\mathrm{d}\,x.\end{aligned}$$

The proof is finished. □

**Remark 2.20.** The weak* topology is metrisable on bounded sets of $\mathrm{BD}(\Omega)$ (see [13] for details).

**Definition 2.21 (Strict convergence).** *A sequence* $(u_h) \subset \mathrm{BD}(\Omega)$ *converges* strictly *to* $u$ *in* $\mathrm{BD}(\Omega)$ *if* $u_h \to u$ *strongly in* $\mathrm{L}^1(\Omega;\mathbb{R}^d)$, $Eu_h \overset{*}{\rightharpoonup} Eu$ *weakly* in* $\mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *and* $|Eu_h|(\Omega) \to |Eu|(\Omega)$.

For a measure $\mu \in \mathrm{M}(\mathbb{R}^d;\mathbb{R}^d)$ with the Lebesgue decomposition

$$\mu = \frac{\mathrm{d}\,\mu}{\mathrm{d}\,\mathscr{L}^d}\mathscr{L}^d + \mu^s$$

we define a Borel measure $\langle\mu\rangle : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ by

$$\langle\mu\rangle(A) := \int_A \sqrt{1 + \left|\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\mathscr{L}^d}\right|^2}\,\mathrm{d}\,x + |\mu^s|(A).$$

**Definition 2.22 (Area-strict convergence).** *A sequence* $(u_h) \subset \mathrm{BD}(\Omega)$ *converges* area-strictly *to* $u$ *in* $\mathrm{BD}(\Omega)$ *if* $u_h \to u$ *strictly and* $\langle Eu_h\rangle(\Omega) \to \langle Eu\rangle(\Omega)$.

The last type of convergence is particularly important, as it allows approximation of functions in $\mathrm{BD}(\Omega)$ by smooth functions (which is not possible in the norm topology). The proof of this density result follows along the same lines as the proof of Lemma 11.1 in [35].

**Remark 2.23.** Clearly, the weak* convergence is weaker than the strict convergence, which in turn is weaker than the area-strict convergence.

In order to see that the opposite implications do not hold we consider sequences $(u_j), (v_j) \subset \mathrm{BD}((0, 2\pi))$ defined by

$$u_j(x) := \frac{1}{j}\sin(jx) \quad \text{and} \quad v_j(x) := x + u_j(x), \quad x \in (0, 2\pi).$$

Then, we can see that $u_j \overset{*}{\rightharpoonup} 0$ weakly* in $\mathrm{BD}((0, 2\pi))$, but not strictly, as $|Eu_j|((0, 2\pi)) = 4$ for each $j$. We can also see that $v_j$ converges to $x$ weakly* and strictly, but not area-strictly, since the integrand $\sqrt{1 + |A|^2}$ is strictly convex away from 0.

According to the Lebesgue decomposition theorem, we split the measure $Eu$ into

$$Eu = \mathcal{E}u\mathscr{L}^d + E^s u,$$

where $\mathcal{E}u := \frac{\mathrm{d}\,Eu}{\mathrm{d}\,\mathscr{L}^d} \in \mathrm{L}^1(\Omega, \mathscr{L}^d; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is the Radon-Nikodym derivative of $Eu$ with respect to the Lebesgue measure $\mathscr{L}^d$ (called the *approximate symmetrised gradient*) and $E^s u \perp \mathscr{L}^d$ is the *singular part* of $Eu$.

We have the following trace theorem in $\mathrm{BD}(\Omega)$ (cf. [9, 39]).

**Theorem 2.24.** *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. Then, there exists a unique linear continuous map* $\gamma : \mathrm{BD}(\Omega) \to \mathrm{L}^1(\partial\Omega, \mathscr{H}^{d-1} \llcorner \partial\Omega; \mathbb{R}^d)$, *called the* trace *such that*

1. *for* $u \in \mathrm{BD}(\Omega) \cap \mathrm{C}(\overline{\Omega}; \mathbb{R}^d)$ *it holds that* $\gamma(u) = u|_{\partial\Omega}$,

*2. for $u \in \mathrm{BD}(\Omega)$ and $\varphi \in \mathrm{C}^1(\mathbb{R}^d)$ the integration-by-parts formula*

$$\int_\Omega u \odot \nabla\varphi \,\mathrm{d}\,x = -\int_\Omega \varphi \,\mathrm{d}\,Eu + \int_{\partial\Omega} \varphi\gamma(u) \odot n \,\mathrm{d}\,\mathscr{H}^{d-1}$$

*holds. Here $n : \partial\Omega \to \mathbb{S}^{d-1}$ denotes an outward pointing unit normal to the boundary $\partial\Omega$, and $\mathscr{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure.*

*Moreover, the trace $\gamma$ is continuous with respect to the topology of strict convergence in* $\mathrm{BD}(\Omega)$.

We usually write $u|_{\partial\Omega}$ instead of $\gamma(u)$ for the trace of $u \in \mathrm{BD}(\Omega)$.

**Theorem 2.25 (Poincaré inequality).** *For every $u \in \mathrm{BD}(\Omega)$ there exists a rigid deformation $r$, i.e. a skew-symmetric affine map $r : \mathbb{R}^d \to \mathbb{R}^d$ of the form $r(x) = Sx + b$, where $S \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$, such that*

$$\|u + r\|_{\mathrm{L}^{d/(d-1)}} \leq C|Eu|(\Omega), \tag{2.5}$$

*where a constant $C = C(\Omega) > 0$ depends only on the domain $\Omega$.*

For the proof see [39, Proposition 2.4] or [38, Remark II.2.5]. Moreover, if $u|_{\partial\Omega} = 0$, then (2.5) simplifies to

$$\|u\|_{\mathrm{L}^{d/(d-1)}} \leq C|Eu|(\Omega). \tag{2.6}$$

The following BD-analogue of Alberti's rank-one theorem in BV (cf. [1, 29]) is proved in [16].

**Theorem 2.26 (DePhilippis-Rindler).** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathrm{BD}(\Omega)$. Then, for $|E^s u|$-a.e. $x \in \Omega$, there exist $a(x), b(x) \in \mathbb{R}^d \setminus \{0\}$ such that*

$$\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}(x) = a(x) \odot b(x).$$

For every $u \in \mathrm{L}^1(\Omega; \mathbb{R}^d)$ there exists an $\mathscr{L}^d$-negligible set $S_u \subset \Omega$, called the *Lebesgue discontinuity set* of $u$, such that for every $x_0 \in \Omega \setminus S_u$ there exists

$\tilde{u}(x_0) \in \mathbb{R}^d$ for which

$$\lim_{r\downarrow 0} \frac{1}{r^d} \int_{B(x_0,r)} |u(x) - \tilde{u}(x_0)| \,\mathrm{d}\, x = 0.$$

The function $\tilde{u} : \Omega \setminus S_u \to \mathbb{R}^d$ is called the *precise representative* of $u$.

Every function $u \in \mathrm{BD}(\Omega)$ is *approximately differentiable* at $\mathscr{L}^d$-almost every $x_0 \in \Omega \setminus S_u$, i.e.

$$\lim_{r\downarrow 0} \frac{1}{r^d} \int_{B(x_0,r)} \frac{|u(x) - \tilde{u}(x_0) - \nabla u(x_0)(x - x_0)|}{r} \,\mathrm{d}\, x = 0. \tag{2.7}$$

For the proof, see Theorem 7.4 in [2].

## Temam-Strang space

For the theory of elasto-plasticity in the geometrically linear setting the class of functions defined as

$$\mathrm{LU}(\Omega) := \left\{ u \in \mathrm{LD}(\Omega) : \ \operatorname{div} u \in \mathrm{L}^2(\Omega) \right\}$$

becomes a natural choice [14, 21, 23, 38]. Unfortunately, the space $\mathrm{LU}(\Omega)$ inherits the poor compactness property of $\mathrm{LD}(\Omega)$ and again, it is reasonable to look for a larger space which could be used instead of $\mathrm{LU}(\Omega)$ to overcome this issue. Therefore, we define the *Temam-Strang space* $\mathrm{U}(\Omega)$ as a subspace of functions of bounded deformation $\mathrm{BD}(\Omega)$:

$$\mathrm{U}(\Omega) := \left\{ u \in \mathrm{BD}(\Omega) : \ \operatorname{div} u \in \mathrm{L}^2(\Omega) \right\}.$$

The space $\mathrm{U}(\Omega)$ is usually endowed with the norm

$$\|u\|_{\mathrm{U}} := \|u\|_{\mathrm{BD}} + \|\operatorname{div} u\|_2,$$

which turns it into a Banach space.

**Remark 2.27.** For $u \in \mathrm{U}(\Omega)$ we have that $\operatorname{dev} E^s u = E^s u$, since the trace part of $Eu$ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^d$. In conjunction with Theorem 2.26, this implies that for $u \in \mathrm{U}(\Omega)$ the polar of the measure $E^s u$ is a symmetric tensor product of two non-zero *orthogonal* vectors.

Similarly to the space BD, one usually works in weaker topologies than the norm topology. We distinguish three such topologies.

**Definition 2.28 (Weak* convergence).** *We say that* $(u_h) \subset \mathrm{U}(\Omega)$ *converges* weakly* *to* $u$ *in* $\mathrm{U}(\Omega)$ *if* $u_h \to u$ *strongly in* $\mathrm{L}^1(\Omega;\mathbb{R}^d)$, $Eu_h \overset{*}{\rightharpoonup} Eu$ *weakly* in* $\mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *and* $\operatorname{div} u_h \rightharpoonup \operatorname{div} u$ *weakly in* $\mathrm{L}^2(\Omega)$.

We have an analogue of Lemma 2.19 for the Temam-Strang space.

**Lemma 2.29.** *Let* $(u_h) \subset \mathrm{U}(\Omega)$ *be a sequence such that* $u_h \to u$ *strongly in* $\mathrm{L}^1(\Omega;\mathbb{R}^d)$ *and* $(u_h)$ *is uniformly norm-bounded in* $\mathrm{U}(\Omega)$. *Then,* $(u_h)$ *converges weakly* to* $u$ *in* $\mathrm{U}(\Omega)$.

The proof follows along the same lines, so we omit it here.

**Definition 2.30 (Strict convergence).** *We say that a sequence* $(u_h) \subset \mathrm{U}(\Omega)$ *converges* strictly *to* $u$ *in* $\mathrm{U}(\Omega)$ *if* $u_h \to u$ *strongly in* $\mathrm{L}^1(\Omega;\mathbb{R}^d)$, $Eu_h \overset{*}{\rightharpoonup} Eu$ *weakly* in* $\mathrm{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $|Eu_h|(\Omega) \to |Eu|(\Omega)$ *and* $\operatorname{div} u_h \to \operatorname{div} u$ *strongly in* $\mathrm{L}^2(\Omega)$.

**Definition 2.31 (Area-strict convergence).** *We say that* $(u_h) \subset \mathrm{U}(\Omega)$ *converges* area-strictly *to* $u$ *in* $\mathrm{U}(\Omega)$ *if* $u_h \to u$ *strictly,* $\langle Eu_h\rangle(\Omega) \to \langle Eu\rangle(\Omega)$ *and* $\langle \operatorname{dev} Eu_h\rangle(\Omega) \to \langle \operatorname{dev} Eu\rangle(\Omega)$.

The following theorem was proved by Jesenko and Schmidt [23]:

**Theorem 2.32.** *Let* $f : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty)$ *be a continuous function satisfying the following conditions:*

1. *there exist constants* $0 < m \le M$ *such that for all* $(x,A) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *the growth estimates*

$$m((\operatorname{tr} A)^2 + |\operatorname{dev} A|) \le f(x,A) \le M(1 + (\operatorname{tr} A)^2 + |\operatorname{dev} A|) \tag{2.8}$$

*hold;*

2. $f(x,\cdot)$ *is symmetric rank-one convex;*

*3. for every fixed $D \in \mathrm{SD}(d)$ the map $x \mapsto f^{\#}_{\mathrm{dev}}(x, D)$ is continuous; here $f^{\#}_{\mathrm{dev}}$ is the recession function of the restriction $f_{\mathrm{dev}} := f|_{\Omega \times \mathrm{SD}(d)}$ defined by*

$$f^{\#}_{\mathrm{dev}}(x, D) := \limsup_{\substack{D' \to D \\ s \to \infty}} \frac{f_{\mathrm{dev}}(x, sD')}{s}. \tag{2.9}$$

*Then, the functional*

$$\mathcal{F}[u, \Omega] := \int_{\Omega} f(x, \mathcal{E}u(x)) \,\mathrm{d}\, x, \quad u \in \mathrm{LU}(\Omega)$$

*extends continuously, with respect to the area-strict convergence in* $\mathrm{U}(\Omega)$*, to the functional*

$$\overline{\mathcal{F}}[u, \Omega] := \int_{\Omega} f(x, \mathcal{E}u(x)) \,\mathrm{d}\, x + \int_{\Omega} f^{\#}_{\mathrm{dev}} \left( x, \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u|, \quad u \in \mathrm{U}(\Omega).$$

**Remark 2.33.** For $u \in \mathrm{U}(\Omega)$ there exists a sequence $(v_h) \subset \mathrm{LU}(\Omega) \cap \mathrm{C}^{\infty}(\Omega; \mathbb{R}^d)$ such that $v_h \to u$ area-strictly in $\mathrm{U}(\Omega)$, see [8, Theorem 14.1.4] (the proof is similar to the proof of Lemma 11.1 in [35], with the strong $\mathrm{L}^2$-convergence of $(\operatorname{div} v_h)$ being a consequence of the mollification). In virtue of Theorem 2.32 we have that

$$\int_{\Omega} f(x, \mathcal{E}v_h) \,\mathrm{d}\, x \to \int_{\Omega} f(x, \mathcal{E}u) \,\mathrm{d}\, x + \int_{\Omega} f^{\#}_{\mathrm{dev}} \left( x, \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u|.$$

# Chapter 3

# Linear growth functionals

In this chapter we study homogeneous integral functionals of the form:

$$\mathcal{F}[u] := \int_\Omega f(\mathcal{E}u(x)) \,\mathrm{d}\, x$$

with the linear *isotropic* growth bounds:

$$m|A| \leq f(A) \leq M(1 + |A|)$$

for some constants $0 < m \leq M$.

The function $u : \Omega \to \mathbb{R}^d$ describes the displacement of a body that occupies the region $\Omega \subset \mathbb{R}^d$ and the functional $\mathcal{F}$ represents the total energy of the deformation.

A physically relevant problem is to minimise the energy $\mathcal{F}$ in a suitable class of deformations, subject to some boundary datum which represents the density of external forces acting on the continuum. Mathematically, this can be formulated as the following *minimisation problem*:

$$\min \{\mathcal{F}[u] : \; u \in X, \; u = g \text{ on } \partial\Omega\}, \tag{3.1}$$

where $X$ is some space of functions $u : \Omega \to \mathbb{R}^d$ and $g : \partial\Omega \to \mathbb{R}^d$ is a boundary datum.

Ideally, we would like to study the minimisation problem (3.1), modelled over the space $X$ which consists of differentiable functions or at least is a subspace of $\mathrm{LD}(\Omega)$ defined in the previous chapter. This, however, is not realistic, since in problems coming from the elastoplasticity, we often encounter discontinuities across the so-called *slippage surfaces.* No differentiable (or even Sobolev) function can account for such behaviour.

For the most part of this chapter, we investigate the minimisation problem, for which the displacement field is not constrained by the boundary datum. It is, however, possible to account for the Dirichlet boundary condition through the so-called *penalisation term* (see Remark 3.14 and Theorem 3.15).

## 3.1 Recession function

In the sequel we often need to 'encode' the information about the asymptotic behaviour of an integrand $f$. To this end, we define the *(strong) recession function* $f^\infty : \mathbb{R}^N \to \mathbb{R}$ as:

$$f^\infty(A) := \lim_{\substack{A' \to A \\ s \to \infty}} \frac{f(sA')}{s}, \quad A \in \mathbb{R}^N \tag{3.2}$$

if the limit exists and is finite. It is straightforward to see that $f^\infty$ is *positively 1-homogeneous*, i.e.

$$f^\infty(tA) = tf^\infty(A), \quad t > 0,\ A \in \mathbb{R}^N.$$

Moreover, for a Lipschitz function $f$ the definition of $f^\infty$ reduces to the following:

$$f^\infty(A) = \lim_{s \to \infty} \frac{f(sA)}{s}, \quad A \in \mathbb{R}^N. \tag{3.3}$$

Indeed, by the Lipschitz continuity of $f$ we obtain:

$$f^\infty(A) \le \lim_{\substack{A' \to A \\ s \to \infty}} \frac{f(sA) + sL|A' - A|}{s} = \lim_{s \to \infty} \frac{f(sA)}{s}$$

and similarly we estimate from below to conclude.

The existence of the strong recession function $f^\infty$ is a subtle matter. One can easily construct a continuous function $f$ with linear growth at infinity, for which

$$\liminf_{\substack{A' \to A \\ s \to \infty}} \frac{f(sA')}{s} < \limsup_{\substack{A' \to A \\ s \to \infty}} \frac{f(sA')}{s}.$$

Indeed, define $f(A) := |A| \sin A$ for $A \in \mathbb{R}$. Then, for $A = \pm 1$ we have

$$\liminf_{s \to \infty} \frac{f(s)}{s} = -1 < 1 = \limsup_{s \to \infty} \frac{f(s)}{s}.$$

It turns out that even in the presence of quasiconvexity the existence of $f^\infty$ is not guaranteed (cf. [32, Theorem 2]).

Nevertheless, we can always define weaker recession functions $f^\#$ and $f_\#$, where the limit in (3.2) is replaced by the upper limit and lower limit respectively and we call $f^\#$ and $f_\#$ the *upper* and *lower recession function* respectively. It is also clear that the positive 1-homogeneity property and the simplification for Lipschitz functions carry over to $f^\#$ and $f_\#$.

For a (symmetric) rank-one convex function $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ with linear growth at infinity, the upper recession function $f^\#$ agrees with the lower recession function $f_\#$ on matrices from (symmetric) rank-one cone. Indeed, let $A = a \odot b$ for some vectors $a, b \in \mathbb{R}^d$. Then we have

$$\frac{f(sA)}{s} = \frac{f(sA) - f(0)}{s} + \frac{f(0)}{s} =: g_s(A) + \frac{f(0)}{s}.$$

Clearly, the second term disappears as $s \to \infty$. For the first term, we note that the (symmetric) rank-one convexity of $f$ implies that $g_s(A) \geq g_{\theta s}(A)$ for $\theta \in (0,1)$, so the map $s \mapsto g_s(A)$ is non-decreasing and we have a pointwise limit

$$g_s(A) \nearrow \sup_{t>0} g_t(A) \quad \text{as} \quad s \to \infty.$$

Hence,

$$\limsup_{s \to \infty} \frac{f(sA)}{s} = \liminf_{s \to \infty} \frac{f(sA)}{s} = \sup_{t>0} g_t(A).$$

Moreover, for a (symmetric-)quasiconvex function $f$, by Fatou's lemma, the upper recession function $f^\#$ is also (symmetric-)quasiconvex. Unfortunately, we cannot infer this property for the lower recession function.

## 3.2 Lower semicontinuity

In this section we prove the following lower semicontinuity result.

**Theorem 3.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f : \mathbb{R}^{d\times d}_{\text{sym}} \to [0, +\infty)$ be a continuous function satisfying the following conditions:*

1. *there exist constants $0 < m \le M$ such that for all $A \in \mathbb{R}^{d\times d}_{\text{sym}}$ the inequality*
$$m|A| \le f(A) \le M(1 + |A|)$$
*holds;*
2. *$f$ is symmetric-quasiconvex.*

*Then, the functional*

$$\overline{\mathcal{F}}[u] := \int_\Omega f(\mathcal{E}u(x))\, \mathrm{d}\, x + \int_\Omega f^{\#}\left(\frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|}(x)\right) \mathrm{d}\, |E^s u|(x), \quad u \in \mathrm{BD}(\Omega)$$

*is weakly* lower semicontinuous in* $\mathrm{BD}(\Omega)$.

**Remark 3.2.**

1. Theorem 3.1 is an immediate corollary of Theorem 3.3 below, thanks to the properties of the relaxation outlined in Chapter 2.
2. It is possible to relax the coercivity assumption on the integrand $f$, and assume only that $f \ge 0$. Then, thanks to Theorem 2.26, the proof of Theorem 3.1 follows along the same lines as the proof of Theorem 11.7 in [35]. Nevertheless, the coercivity assumption is important when one is interested in minimisation problems, so we keep it here.
3. We do not assume the existence of the strong recession function $f^\infty$ in Theorem 3.1. This is the main novelty and a significant improvement over the previously available weak* lower semicontinuity result in BD for homogeneous functionals [33]. See also [6].
4. We also remark that a more general non-homogeneous version of Theorem 3.1 is available in [33], however the result presented there is proved using the theory of generalised Young measures, for which the existence of $f^\infty$ is fundamental.

**Theorem 3.3.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ be a continuous function satisfying the following conditions:*

1. *there exist constants $0 < m \leq M$ such that for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the inequality*
$$m|A| \leq f(A) \leq M(1 + |A|) \tag{3.4}$$
*holds;*
2. *$f$ is symmetric-quasiconvex.*

*Then, the relaxation of the extended real-valued functional*

$$\mathcal{F}_\infty[u,\Omega] := \begin{cases} \displaystyle\int_\Omega f(\mathcal{E}u(x))\,\mathrm{d}\,x & \text{for } u \in \mathrm{LD}(\Omega) \\ +\infty & \text{for } u \in \mathrm{BD}(\Omega)\setminus \mathrm{LD}(\Omega), \end{cases}$$

*with respect to the weak* topology in $\mathrm{BD}(\Omega)$ is given by*

$$\mathcal{F}_*[u,\Omega] = \int_\Omega f(\mathcal{E}u(x))\,\mathrm{d}\,x + \int_\Omega f^{\#}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}(x)\right)\mathrm{d}\,|E^s u|(x), \quad u \in \mathrm{BD}(\Omega).$$

In order to prove Theorem 3.3 we begin with a series of lemmas.

**Lemma 3.4.** *Let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ satisfy conditions (1) and (2) of Theorem 3.3, let $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and let $(u_h) \subset \mathrm{BD}(\Omega)$ be a sequence such that $u_h \xrightarrow{*} Ax$ weakly* in $\mathrm{BD}(\Omega)$. Then,*

$$|\Omega| f(A) \leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E}u_h)\,\mathrm{d}\,x. \tag{3.5}$$

*Proof.* Without loss of generality assume that $(u_h) \subset \mathrm{LD}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d)$. The proof is divided into two steps. In the first step we prove (3.5) for a sequence $(u_h)$ which has linear boundary values. Then, in the second step we prove, using a cut-off argument, that the assumption of the linear boundary values can be dropped.

*Step 1.* Suppose that $u_h(x) - Ax$ is compactly supported inside $\Omega$ for all $h \in \mathbb{N}$ and take $\psi_h(x) := u_h(x) - Ax$. Clearly $\psi_h \in \mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^d)$. Then, by the symmetric-quasiconvexity of $f$, we obtain

$$|\Omega| f(A) \leq \int_\Omega f(A + \mathcal{E}\psi_h(y))\,\mathrm{d}\,y = \int_\Omega f(\mathcal{E}u_h(y))\,\mathrm{d}\,y$$

for all $h \in \mathbb{N}$. Therefore

$$|\Omega| f(A) \leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E}u_h)\, \mathrm{d}\, x.$$

*Step 2.* Let $u_h \xrightarrow{*} Ax$ weakly* in $\mathrm{BD}(\Omega)$. The argument below is due to De Giorgi. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$ and choose a Lipschitz subdomain $\Omega_0 \Subset \Omega$ such that $|\Omega \setminus \Omega_0| \leq \varepsilon$. Let $R := \mathrm{dist}(\Omega_0, \partial\Omega)$ and for $i = 1, \ldots, n$ define sets

$$\Omega_i := \left\{ x \in \Omega : \ \mathrm{dist}(x, \Omega_0) < \frac{iR}{n} \right\}.$$

Now, choose cut-off functions $\varphi_i \in \mathrm{C}^1_c(\Omega; [0,1])$ such that

$$\mathbb{1}_{\Omega_{i-1}} \leq \varphi_i \leq \mathbb{1}_{\Omega_i} \quad \text{and} \quad \|\nabla\varphi_i\|_\infty \leq \frac{2n}{R}$$

and for $x \in \Omega$ define

$$u_h^{(i)}(x) := Ax + \varphi_i(x)(u_h(x) - Ax).$$

We have

$$\mathcal{E}u_h^{(i)}(x) = A + \varphi_i(x)(\mathcal{E}u_h(x) - A) + \nabla\varphi_i(x) \odot (u_h(x) - Ax). \tag{3.6}$$

Note that $u_h^{(i)} \xrightarrow{*} Ax$ weakly* in $\mathrm{BD}(\Omega)$ as $h \to \infty$ and $u_h^{(i)}|_{\partial\Omega} = Ax$ for every $i = 1, \ldots, n$. We obtain

$$\begin{aligned}
\int_\Omega f(\mathcal{E}u_h^{(i)})\, \mathrm{d}\, x &= \int_{\Omega_{i-1}} f(\mathcal{E}u_h)\, \mathrm{d}\, x + \int_{\Omega_i \setminus \Omega_{i-1}} f(\mathcal{E}u_h^{(i)})\, \mathrm{d}\, x + \int_{\Omega\setminus\Omega_i} f(A)\, \mathrm{d}\, x \\
&\leq \int_\Omega f(\mathcal{E}u_h)\, \mathrm{d}\, x + \int_{\Omega_i \setminus \Omega_{i-1}} f(\mathcal{E}u_h^{(i)})\, \mathrm{d}\, x + |\Omega \setminus \Omega_0| f(A) \\
&\leq \int_\Omega f(\mathcal{E}u_h)\, \mathrm{d}\, x + M \int_{\Omega_i \setminus \Omega_{i-1}} |\mathcal{E}u_h^{(i)}|\, \mathrm{d}\, x + |\Omega \setminus \Omega_0|(M + f(A)).
\end{aligned}$$

We now estimate the middle integral on the right-hand side using (3.6):

$$\begin{aligned}
&\int_{\Omega_i \setminus \Omega_{i-1}} |\mathcal{E}u_h^{(i)}|\, \mathrm{d}\, x \\
&\quad \leq |A|\, |\Omega_i \setminus \Omega_{i-1}| + \int_{\Omega_i \setminus \Omega_{i-1}} |\varphi_i(x)|\, |\mathcal{E}u_h(x) - A|\, \mathrm{d}\, x \\
&\qquad + \int_{\Omega_i \setminus \Omega_{i-1}} |\nabla\varphi_i(x) \odot (u_h(x) - Ax)|\, \mathrm{d}\, x \\
&\quad \leq |A|\, |\Omega \setminus \Omega_0| + \int_{\Omega_i \setminus \Omega_{i-1}} |\mathcal{E}u_h - A|\, \mathrm{d}\, x \\
&\qquad + \int_{\Omega_i \setminus \Omega_{i-1}} |\nabla\varphi_i(x) \odot (u_h(x) - Ax)|\, \mathrm{d}\, x.
\end{aligned}$$

Since the embedding $\mathrm{BD}(\Omega) \Subset \mathrm{L}^q(\Omega;\mathbb{R}^d)$ is compact for $1 \le q < d/(d-1)$, in particular we have that $u_h \to Ax$ strongly in $\mathrm{L}^q(\Omega;\mathbb{R}^d)$ for $1 < q < d/(d-1)$. Hence

$$
\begin{aligned}
&\int_{\Omega_i\setminus\Omega_{i-1}} |\nabla\varphi_i(x) \odot (u_h(x) - Ax)| \,\mathrm{d}\,x \\
&\quad\le \int_{\Omega_i\setminus\Omega_{i-1}} |\nabla\varphi_i(x)|\,|u_h(x) - Ax| \,\mathrm{d}\,x \\
&\quad\le \frac{2n}{R}\int_{\Omega} |u_h(x) - Ax| \mathbb{1}_{\Omega_i\setminus\Omega_{i-1}}(x) \,\mathrm{d}\,x \\
&\quad\le \frac{2n}{R}\left(\int_{\Omega} |u_h(x) - Ax|^q \,\mathrm{d}\,x\right)^{1/q} |\Omega_i \setminus \Omega_{i-1}|^{1/q'} \\
&\quad\le c_1|\Omega_i \setminus \Omega_{i-1}|^{1/q'} \\
&\quad\le c_1|\Omega \setminus \Omega_0|^{1/q'},
\end{aligned}
$$

where $1/q + 1/q' = 1$ and $c_1 = c_1(n, R, q) > 0$ is a $h$-independent constant. Combining the above estimates yields

$$
\begin{aligned}
\int_{\Omega} f(\mathcal{E}u_h^{(i)}) \,\mathrm{d}\,x \le \int_{\Omega} f(\mathcal{E}u_h) \,\mathrm{d}\,x + M\int_{\Omega_i\setminus\Omega_{i-1}} |\mathcal{E}u_h - A| \,\mathrm{d}\,x& \\
+ (M|A| + M + f(A))|\Omega \setminus \Omega_0| + c_1 M|\Omega \setminus \Omega_0|^{1/q'}&.
\end{aligned}
$$

By Step 1 we have

$$
\begin{aligned}
|\Omega| f(A) &\le \liminf_{h\to\infty} \int_{\Omega} f(\mathcal{E}u_h^{(i)}) \,\mathrm{d}\,x \\
&\le \liminf_{h\to\infty}\left[\int_{\Omega} f(\mathcal{E}u_h) \,\mathrm{d}\,x + M\int_{\Omega_i\setminus\Omega_{i-1}} |\mathcal{E}u_h - A| \,\mathrm{d}\,x\right] \\
&\quad + (M|A| + M + f(A))|\Omega \setminus \Omega_0| + c_1 M|\Omega \setminus \Omega_0|^{1/q'}.
\end{aligned}
$$

Summing up over $i = 1, \dots, n$, dividing by $n$, and using the superadditivity of a lower limit as well as $|\Omega \setminus \Omega_0| \le \varepsilon$ yields

$$
\begin{aligned}
|\Omega| f(A) \le \liminf_{h\to\infty} \int_{\Omega} f(\mathcal{E}u_h) \,\mathrm{d}\,x + \frac{M}{n}\sup_h \int_{\Omega} |\mathcal{E}u_h - A| \,\mathrm{d}\,x& \\
+ (M|A| + M + f(A))\varepsilon + c_1 M\varepsilon^{1/q'}&.
\end{aligned}
$$

Letting $n \to \infty$ and $\varepsilon \downarrow 0$ yields

$$
|\Omega| f(A) \le \liminf_{h\to\infty} \int_{\Omega} f(\mathcal{E}u_h) \,\mathrm{d}\,x. \qquad \square
$$

**Remark 3.5.** Clearly, Lemma 3.4 also holds for affine limits.

We now investigate the *relaxation* $\mathcal{F}_*$ of the functional $\mathcal{F}_\infty$, i.e. the functional defined as

$$\mathcal{F}_*[u,\Omega] := \inf\left\{\liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] : \ (u_h) \subset \mathrm{BD}(\Omega),\ u_h \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BD}(\Omega)\right\}.$$

Since the topology of weak* convergence in $\mathrm{BD}(\Omega)$ is metrizable on bounded sets, it follows that the relaxation $\mathcal{F}_*$ is lower semicontinuous with respect to this topology (see Section 2.4 for details).

**Proposition 3.6.** *The relaxation $\mathcal{F}_*$ can be equivalently written as*

$$\mathcal{G}[u,\Omega] := \inf\Big\{\liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] : \ (u_h) \subset \mathrm{LD}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d), \\ u_h \overset{*}{\rightharpoonup} u \textit{ in } \mathrm{BD}(\Omega)\Big\}.$$

*Proof.* Clearly, it suffices to prove the inequality $\mathcal{G} \leq \mathcal{F}_*$. Take arbitrary sequence $(u_h) \subset \mathrm{BD}(\Omega)$ such that $u_h \overset{*}{\rightharpoonup} u$ in $\mathrm{BD}(\Omega)$. By a similar argument to the one contained in the proof of Lemma 11.1 in [35], for each $h \in \mathbb{N}$ we can find a sequence $(v_h^{(k)})_k \subset \mathrm{LD}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d)$ such that $v_h^{(k)} \to u_h$ area-strictly as $k \to \infty$. We choose a subsequence $(v_h^{(k_h)})_h$ such that $v_h^{(k_h)} \overset{*}{\rightharpoonup} u$ in $\mathrm{BD}(\Omega)$ and

$$\mathcal{F}_\infty[v_h^{(k_h)},\Omega] \leq \mathcal{F}_\infty[u_h,\Omega] + \frac{1}{h}.$$

Indeed, if $u_h \in \mathrm{BD}(\Omega)$ for all $h \in \mathbb{N}$ we have $\mathcal{F}_\infty[v_h^{(k_h)},\Omega] \leq +\infty$, whereas if there exists $h \in \mathbb{N}$ such that $u_h \in \mathrm{LD}(\Omega)$, then the above inequality is a consequence of the area-strict continuity of $\mathcal{F}_\infty$.

Therefore, since $(v_h^{(k_h)})_h$ is admissible in the definition of $\mathcal{G}$, we obtain:

$$\mathcal{G}[u,\Omega] \leq \liminf_{h\to\infty} \mathcal{F}_\infty[v_h^{(k_h)},\Omega] \leq \liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega].$$

Taking infimum over all sequences $(u_h)$ yields the desired inequality $\mathcal{G} \leq \mathcal{F}_*$. $\square$

In the remaining part of this chapter we will establish an integral representation for $\mathcal{F}_*$, that is

$$\mathcal{F}_*[u,\Omega] = \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^{\#}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|. \tag{3.7}$$

More specifically, we will establish the upper and the lower estimate on the relaxation $\mathcal{F}_*$ by the right-hand side of (3.7). We begin with the upper estimate.

Let us denote by $\mathrm{D}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ a class of continuous functions $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ with a linear growth at infinity, i.e. $|f(A)| \le C(1+|A|)$ for some $C > 0$, and for which the *strong* recession function

$$f^\infty(A) := \lim_{\substack{A'\to A\\ s\to\infty}} \frac{f(sA')}{s}$$

exists. For such functions we have the following continuity result.

**Theorem 3.7 (Reshetnyak).** *Let $(\mu_h) \subset \mathrm{M}(\Omega;\mathbb{R}^d)$ be a sequence of measures such that $\mu_h \to \mu$ area-strictly for some $\mu \in \mathrm{M}(\Omega;\mathbb{R}^d)$. Then, for $f \in \mathrm{D}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ it holds that*

$$\int_\Omega f\left(\frac{\mathrm{d}\,\mu_h}{\mathrm{d}\,\mathscr{L}^d}\right)\mathrm{d}\,x + \int_\Omega f^\infty\left(\frac{\mathrm{d}\,\mu_h^s}{\mathrm{d}\,|\mu_h^s|}\right)\mathrm{d}\,|\mu_h^s| \to \int_\Omega f\left(\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\mathscr{L}^d}\right)\mathrm{d}\,x + \int_\Omega f^\infty\left(\frac{\mathrm{d}\,\mu^s}{\mathrm{d}\,|\mu^s|}\right)\mathrm{d}\,|\mu^s|$$

*as $h\to\infty$.*

For the proof we refer to [28]. Furthermore, it turns out that the admissible integrands $f$ in Theorem 3.1 can be approximated by functions in $\mathrm{D}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ (cf. [27, Lemma 2.2]).

**Lemma 3.8 (Pointwise approximation).** *For every continuous function $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ with linear growth at infinity, there exists a decreasing sequence $(f_k) \subset \mathrm{D}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that*

$$\inf_k f_k = \lim_{k\to\infty} f_k = f \quad \textit{and} \quad \inf_k f_k^\infty = \lim_{k\to\infty} f_k^\infty = f^{\#},$$

*and the convergence is pointwise.*

We are now ready to establish the upper bound.

**Lemma 3.9 (Upper estimate).** *For $u \in \mathrm{BD}(\Omega)$ the inequality*

$$\mathcal{F}_*[u,\Omega] \leq \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^\#\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\,\mathrm{d}\,|E^s u|.$$

*holds.*

*Proof.* Fix $u \in \mathrm{BD}(\Omega)$. Then, there exists a sequence $(u_h) \subset \mathrm{LD}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d)$ such that $u_h \to u$ area-strictly. Let $(f_k) \subset \mathrm{D}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be a sequence as in Lemma 3.8. By Theorem 3.7 we have for each $k \in \mathbb{N}$:

$$\lim_{h\to\infty} \int_\Omega f_k(\mathcal{E}u_h)\,\mathrm{d}\,x = \int_\Omega f_k(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f_k^\infty\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\,\mathrm{d}\,|E^s u|.$$

By the monotonicity of $(f_k)$ we obtain

$$\limsup_{h\to\infty} \int_\Omega f(\mathcal{E}u_h)\,\mathrm{d}\,x \leq \int_\Omega f_k(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f_k^\infty\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\,\mathrm{d}\,|E^s u|.$$

Since the area-strict convergence is stronger than the weak* convergence, by the definition of $\mathcal{F}_*$, it follows that

$$\mathcal{F}_*[u,\Omega] \leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E}u_h)\,\mathrm{d}\,x \leq \int_\Omega f_k(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f_k^\infty\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\,\mathrm{d}\,|E^s u|.$$

By the monotone convergence theorem, letting $k \to \infty$ ends the proof. □

In order to prove the lower estimate, we first prove that for a given $u \in \mathrm{BD}(\Omega)$ the map $V \mapsto \mathcal{F}_*[u,V]$ is the restriction to the open subsets of $\Omega$ of some Radon measure, which we still denote by $\mathcal{F}_*[u,\cdot]$. Then, we decompose this measure into the absolutely continuous and singular parts with respect to the Lebesgue measure, i.e.

$$\mathcal{F}_*[u,\cdot] = \mathcal{F}_*^a[u,\cdot] + \mathcal{F}_*^s[u,\cdot], \quad \mathcal{F}_*^a[u,\cdot] \ll \mathscr{L}^d \llcorner \Omega, \quad \mathcal{F}_*^s[u,\cdot] \perp \mathscr{L}^d \llcorner \Omega$$

and then prove that

$$\mathcal{F}_*^a[u,B] \geq \int_B f(\mathcal{E}u)\,\mathrm{d}\,x \quad \text{and} \quad \mathcal{F}_*^s[u,B] \geq \int_B f^\#\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|$$

for any Borel set $B \subset \Omega$.

**Lemma 3.10.** *For all $u \in \mathrm{BD}(\Omega)$ the set function $V \mapsto \mathcal{F}_*[u,V]$ is a restriction to the open subsets of $\Omega$ of a finite Radon measure.*

*Proof.* Fix $u \in \mathrm{BD}(\Omega)$.

*Step 1.* Let $A', A'', B$ be open subsets of $\Omega$ such that $A' \Subset A''$. We first prove that

$$\mathcal{F}_*[u, A' \cup B] \le \mathcal{F}_*[u, A''] + \mathcal{F}_*[u, B]. \tag{3.8}$$

Fix $\varepsilon > 0$. By the definition of relaxation we can find sequences $(u_h^\varepsilon) \subset \mathrm{LD}(A'')$ and $(v_h^\varepsilon) \subset \mathrm{LD}(B)$ such that $u_h^\varepsilon \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{BD}(A'')$, $v_h^\varepsilon \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{BD}(B)$,

$$\mathcal{F}[u_h^\varepsilon, A''] \le \mathcal{F}_*[u, A''] + \varepsilon,$$

and

$$\mathcal{F}[v_h^\varepsilon, B] \le \mathcal{F}_*[u, B] + \varepsilon.$$

Henceforth, we omit the dependence of sequences $u_h$ and $v_h$ on $\varepsilon$. For each $h \in \mathbb{N}$ extend the functions $u_h$ and $v_h$ by zero outside $A''$ and $B$, respectively. Let

$$C_\varepsilon := \sup_{h \in \mathbb{N}} \left( \int_{A''} 1 + |\mathcal{E} u_h| \,\mathrm{d}\,x + \int_B 1 + |\mathcal{E} v_h| \,\mathrm{d}\,x \right) < \infty.$$

Fix $k \in \mathbb{N}$ and an increasing family of sets

$$A' = A_0 \Subset A_1 \Subset \ldots \Subset A_k \Subset A''.$$

For each $i = 1, \ldots, k$ define the cut-off function $\varphi_i \in \mathrm{C}_c^\infty(A_i; [0,1])$ such that $\varphi_i \equiv 1$ on $A_{i-1}$. Next, define functions

$$w_{h,i} := \varphi_i u_h + (1 - \varphi_i) v_h, \quad h \in \mathbb{N}, \quad i = 1, \ldots, k.$$

It is clear that $w_{h,i} \in \mathrm{LD}(A' \cup B)$. We have

$$\begin{aligned}
\mathcal{F}[w_{h,i}, A' \cup B] &= \int_{A' \cup B} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\,x \\
&= \int_{(A' \cup B) \cap A_{i-1}} f(\mathcal{E} u_h) \,\mathrm{d}\,x + \int_{B \setminus \overline{A}_i} f(\mathcal{E} v_h) \,\mathrm{d}\,x \\
&\quad + \int_{B \cap S_i} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\,x,
\end{aligned}$$

where $S_i := A_i \setminus \overline{A}_{i-1}$ for $i = 1, \ldots, k$. Hence

$$\mathcal{F}[w_{h,i}, A' \cup B] \le \mathcal{F}[u_h, A''] + \mathcal{F}[v_h, B] + \int_{B \cap S_i} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\,x.$$

The last integral can be estimated as follows:

$$\begin{aligned}\int_{B\cap S_i} f(\mathcal{E}w_{h,i})\,\mathrm{d}\,x &\le M\int_{B\cap S_i} 1+|\mathcal{E}w_{h,i}|\,\mathrm{d}\,x\\ &\le M\int_{B\cap S_i} 1+C_k|u_h-v_h|+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x,\end{aligned}$$

where $C_k := \sup\{\|\nabla\varphi_i\|_\infty :\ 1\le i\le k\}$.

Next, for a fixed $h\in\mathbb{N}$ we can choose $i_h\in\{1,\dots,k\}$ such that

$$\int_{B\cap S_{i_h}} 1+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x = \min_{\ell\in\{1,\dots,k\}}\int_{B\cap S_\ell} 1+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x.$$

Then, we have

$$\begin{aligned}\int_{B\cap S_{i_h}} 1+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x &= \frac{1}{k}\sum_{\ell=1}^{k}\int_{B\cap S_{i_h}} 1+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x\\ &\le \frac{1}{k}\sum_{\ell=1}^{k}\int_{B\cap S_\ell} 1+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x\\ &\le \frac{1}{k}\int_{B\cap(A_k\setminus A_0)} 1+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x\\ &\le \frac{C_\varepsilon}{k}.\end{aligned}$$

Therefore, combining the above estimates yields

$$\int_{B\cap S_{i_h}} f(\mathcal{E}w_{h,i_h})\,\mathrm{d}\,x \le M\left(\frac{C_\varepsilon}{k}+C_k\|u_h-v_h\|_1\right).$$

Hence

$$\begin{aligned}\mathcal{F}[w_{h,i_h},A'\cup B] &\le \mathcal{F}[u_h,A'']+\mathcal{F}[v_h,B]+MC_k\|u_h-v_h\|_1+\frac{MC_\varepsilon}{k}\\ &\le \mathcal{F}_*[u,A'']+\mathcal{F}_*[v,B]+2\varepsilon+MC_k\|u_h-v_h\|_1+\frac{MC_\varepsilon}{k}\end{aligned}$$

Note that $w_{h,i_h}\to u$ strongly in $\mathrm{L}^1(A'\cup B;\mathbb{R}^d)$ and $(w_{h,i_h})_h$ is uniformly norm-bounded in $\mathrm{BD}(A'\cup B)$. Lemma 2.19 thus implies that $(w_{h,i_h})_h$ converges weakly* to $u$ in $\mathrm{BD}(A'\cup B)$. Moreover, $(u_h-v_h)_h$ converges strongly to zero in $\mathrm{L}^1(A'\cup B;\mathbb{R}^d)$. Therefore, we obtain

$$\begin{aligned}\mathcal{F}_*[u,A'\cup B] &\le \liminf_{h\to\infty}\mathcal{F}[w_{h,i_h},A'\cup B]\\ &\le \mathcal{F}_*[u,A'']+\mathcal{F}_*[v,B]+2\varepsilon+\frac{MC_\varepsilon}{k}.\end{aligned}$$

Letting $k\to\infty$ followed by $\varepsilon\downarrow 0$ yields the inequality (3.8).

*Step 2.* We now prove that for any open subset $A \subset \Omega$ it holds that

$$\mathcal{F}_*[u, A] = \sup \{\mathcal{F}_*[u, A'] : \; A' \Subset A, \; A' \text{ open}\}\,. \tag{3.9}$$

Firstly, note that the estimate

$$\mathcal{F}_*[u, A] \leq M \left(\mathscr{L}^d(A) + |Eu|(A)\right), \tag{3.10}$$

holds. Indeed, for $u \in \mathrm{BD}(A)$ there exists a sequence $(u_h) \subset \mathrm{LD}(A) \cap \mathrm{C}^\infty(A; \mathbb{R}^d)$ converging strictly to $u$. Since the strict convergence is stronger than the weak* convergence we obtain using the growth bound (3.4):

$$\begin{aligned}
\mathcal{F}_*[u, A] &\leq \liminf_{h \to \infty} \mathcal{F}[u_h, A] \\
&\leq M \left(\mathscr{L}^d(A) + \lim_{h \to \infty} |Eu_h|(A)\right) \\
&= M \left(\mathscr{L}^d(A) + |Eu|(A)\right).
\end{aligned}$$

Therefore, for a fixed $\varepsilon > 0$ we can choose a compact set $K \subset A$ such that $\mathcal{F}_*[u, A \setminus K] < \varepsilon$. Choose open sets $A'$ and $A''$ such that $K \subset A' \Subset A'' \Subset A$. By Step 1 with $B = A \setminus K$ we have

$$\mathcal{F}_*[u, A] \leq \mathcal{F}_*[u, A''] + \mathcal{F}_*[u, A \setminus K] \leq \mathcal{F}_*[u, A''] + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives (3.9).

*Step 3.* Let $A, B$ be open subsets of $\Omega$. We now prove that

$$\mathcal{F}_*[u, A \cup B] \leq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B]. \tag{3.11}$$

Fix $\varepsilon > 0$. By Step 2 there exists an open set $U \Subset A \cup B$ such that

$$\mathcal{F}_*[u, A \cup B] - \varepsilon \leq \mathcal{F}_*[u, U].$$

Choose $A' \Subset A$ open, such that $U \subset A' \cup B$. By Step 1 we have

$$\mathcal{F}_*[u, A \cup B] - \varepsilon \leq \mathcal{F}_*[u, A' \cup B] \leq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B].$$

Letting $\varepsilon \downarrow 0$ yields (3.11).

*Step 4.* Finally, we prove that for open sets $A, B$ such that $A \cap B = \emptyset$ the inequality

$$\mathcal{F}_*[u, A \cup B] \geq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B] \tag{3.12}$$

holds. We can choose a recovery sequence $(u_h) \subset \mathrm{LD}(A \cup B)$ converging weakly* to $u \in \mathrm{BD}(A \cup B)$ and such that

$$\lim_{h \to \infty} \mathcal{F}[u_h, A \cup B] = \mathcal{F}_*[u, A \cup B].$$

Since sets $A$ and $B$ are disjoint we have

$$\begin{aligned} \mathcal{F}_*[u, A \cup B] &= \lim_{h \to \infty} \mathcal{F}[u_h, A \cup B] \\ &\geq \liminf_{h \to \infty} \mathcal{F}[u_h, A] + \liminf_{h \to \infty} \mathcal{F}[u_h, B] \\ &\geq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B], \end{aligned}$$

hence we proved (3.12). By Theorem 2.1 we infer that the set function $V \mapsto \mathcal{F}_*[u, V]$ is a restriction to open sets of a finite Radon measure. □

**Remark 3.11.** The relaxation $\mathcal{F}_*$ satisfies the following properties.

(1) For a rigid deformation, that is a function $R : \mathbb{R}^d \to \mathbb{R}^d$ of the form $R(x) = Sx + b$, where $S \in \mathbb{R}^{d \times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$, we have the *rigid invariance*

$$\mathcal{F}_*[u + R, \Omega] = \mathcal{F}_*[u, \Omega].$$

(2) For $x_0 \in \mathbb{R}^d$ we have the *translation invariance*

$$\mathcal{F}_*[u(\cdot - x_0), x_0 + \Omega] = \mathcal{F}_*[u, \Omega].$$

(3) Let $(R_r)_{r>0} : \mathbb{R}^d \to \mathbb{R}^d$ be a family of rigid deformations. Then, for a blow-up of the form

$$u_r(y) = \frac{u(x_0 + ry) - u(x_0)}{r} + R_r(y)$$

where $r > 0$ and $y \in (\Omega - x_0)/r$, we have the scaling property

$$\mathcal{F}_*\left[u_r, \frac{\Omega - x_0}{r}\right] = r^{-d} \mathcal{F}_*[u, \Omega].$$

*Proof.* Let $R : \mathbb{R}^d \to \mathbb{R}^d$ be a rigid deformation and let $(u_h) \subset \mathrm{LD}(\Omega)$ be a sequence such that $u_h \xrightarrow{*} u$ weakly* in $\mathrm{BD}(\Omega)$. Then, we clearly have the equality

$$\mathcal{F}[u_h + R, \Omega] = \mathcal{F}[u_h, \Omega].$$

Taking the lower limit on both sides, followed by the infimum over all sequences $(u_h)$ yields

$$\begin{aligned}
&\mathcal{F}_*[u + R, \Omega] \\
&\quad = \inf\left\{\liminf_{h\to\infty} \mathcal{F}[u_h, \Omega] :\ (u_h) \subset \mathrm{LD}(\Omega),\ u_h \xrightarrow{*} u + R \text{ in } \mathrm{BD}(\Omega)\right\} \\
&\quad = \inf\left\{\liminf_{h\to\infty} \mathcal{F}[u_h + R, \Omega] :\ (u_h) \subset \mathrm{LD}(\Omega),\ u_h \xrightarrow{*} u \text{ in } \mathrm{BD}(\Omega)\right\} \\
&\quad = \mathcal{F}_*[u, \Omega].
\end{aligned}$$

The proof of translation invariance is analogous. To see that the scaling property holds, note that

$$u_r(y) = \frac{u(x_0 + ry) - u(x_0)}{r} + R_r(y) = \frac{1}{r} u(x_0 + ry) + \tilde{R}_r(y),$$

where $\tilde{R}_r(y) := R_r(y) + u(x_0)/r$. Hence, by the rigid and translation invariances, we obtain

$$\mathcal{F}_*\left[u_r, \frac{\Omega - x_0}{r}\right] = \mathcal{F}_*\left[y \mapsto \frac{u(ry)}{r}, \frac{\Omega}{r}\right] = r^{-d}\mathcal{F}_*[u, \Omega],$$

where the second equality follows from

$$r^{-d}\mathcal{F}[u, \Omega] = \mathcal{F}\left[y \mapsto \frac{u(ry)}{r}, \frac{\Omega}{r}\right]$$

by the change of variables. □

**Lemma 3.12.** *Let $Q$ be an open $d$-cube with side length 1 and faces either parallel or orthogonal to $a$, let $v \in \mathrm{BD}(Q)$ be representable in $Q$ as*

$$v(y) := g(y \cdot a)b + c(a \otimes b)y + Wy + \bar{v},$$

*where $g : \mathbb{R} \to \mathbb{R}$ is a locally bounded and increasing function, $a, b \in \mathbb{R}^d \setminus \{0\}$, $c > 0$, $W \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $\bar{v} \in \mathbb{R}^d$. Let $u \in \mathrm{BD}(Q)$ be such that $\mathrm{supp}(u - v) \Subset Q$.*

*Then, the inequality*

$$\mathcal{F}_*[u, Q] \geq f(Eu(Q))$$

*holds.*

*Proof.* We only treat the case where $a, b$ are not parallel. The case $a, b$ parallel is in fact easier. In virtue of Remark 3.11, we may without loss of generality assume that $a = e_1$, $b = e_2$ and $Q = (0,1)^d$. Then

$$v(y) = g(y_1)e_2 + cy_2e_1 + Wy + \bar{v}.$$

Let

$$q := |Dg|(0,1) = g(1^-) - g(0^+).$$

Since $u \in \mathrm{BD}(Q)$, the function

$$w(x) := u(x - \lfloor x \rfloor) + qe_2\lfloor x_1 \rfloor + ce_1\lfloor x_2 \rfloor + Wx + \bar{v}, \quad x \in \mathbb{R}^d,$$

is in $\mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^d)$. Let $u_h(y) := w(hy)/h$. For $u_0(y) := qe_2y_1 + ce_1y_2 + Wy + \bar{v}$ it holds that

$$\begin{aligned}
&\int_Q |u_h(y) - u_0(y)| \,\mathrm{d}\, y \\
&\quad = \frac{1}{h}\int_Q |u(hy - \lfloor hy \rfloor) - qe_2(hy_1 - \lfloor hy_1 \rfloor) - ce_1(hy_2 - \lfloor hy_2 \rfloor)| \,\mathrm{d}\, y \\
&\quad = \frac{1}{h^{d+1}}\int_{(0,h)^d} |u(x - \lfloor x \rfloor) - qe_2(x_1 - \lfloor x_1 \rfloor) - ce_1(x_2 - \lfloor x_2 \rfloor)| \,\mathrm{d}\, x \\
&\quad = \frac{1}{h}\int_Q |w(y) - (qe_2y_1 + ce_1y_2 + Wy + \bar{v})| \,\mathrm{d}\, y,
\end{aligned}$$

hence $u_h \to u_0$ as $h \to \infty$ in $\mathrm{L}^1(Q;\mathbb{R}^d)$. The sequence $(u_h)$ is uniformly norm-bounded in $\mathrm{BD}(Q)$, so by Lemma 2.19 we also have that $u_h \overset{*}{\rightharpoonup} u_0$ weakly* in $\mathrm{BD}(Q)$.

Let $Q_1, \ldots, Q_{h^d}$ be the canonical decomposition of $Q$ into open cubes with sides parallel to those of $Q$ and side length $1/h$. Then, by the scaling property of $\mathcal{F}_*$, for all $i = 1, \ldots, h^d$ it holds that

$$\mathcal{F}_*[u_h, Q_i] = \mathcal{F}_*[u_h, (0, 1/h)^d] = h^{-d}\mathcal{F}_*[u, Q].$$

Moreover, since $\mathrm{supp}(u - v) \Subset Q$, the measure $|Ew|$ vanishes on every hyperplane of the form $x_j = k$, with $k \in \mathbb{Z}$, $j = 1, \dots, d$. Thus we have that $|Eu_h|(Q \cap \partial Q_i) = 0$ for all $i = 1, \dots, h^d$. By the estimate (3.10) we also have

$$\mathcal{F}_*[u_h, Q \cap \partial Q_i] = 0.$$

Therefore, for any $h \in \mathbb{N}$ we obtain

$$\mathcal{F}_*[u_h, Q] = \sum_{i=1}^{h^d} \mathcal{F}_*[u_h, Q_i] = \sum_{i=1}^{h^d} h^{-d} \mathcal{F}_*[u, Q] = \mathcal{F}_*[u, Q].$$

By the weak* lower semicontinuity of $\mathcal{F}_*$ we obtain

$$\mathcal{F}_*[u, Q] = \lim_{h \to \infty} \mathcal{F}_*[u_h, Q] \geq \mathcal{F}_*[u_0, Q].$$

Let $S \in \mathbb{R}^{d \times d}_{\text{skew}}$ be a skew-symmetric matrix defined as

$$S := \frac{q - c}{2}(e_1 \otimes e_2 - e_2 \otimes e_1).$$

Then, by Remark 3.11 we obtain

$$\begin{aligned}
\mathcal{F}_*[u_0, Q] &= \mathcal{F}_*[q(e_2 \otimes e_1)y + c(e_1 \otimes e_2)y + Wy + \bar{v}, Q] \\
&= \mathcal{F}_*[q(e_2 \otimes e_1)y + c(e_1 \otimes e_2)y, Q] \\
&= \mathcal{F}_*[q(e_2 \otimes e_1)y + c(e_1 \otimes e_2)y + Sy, Q] \\
&= \mathcal{F}_*[(q + c)(e_1 \odot e_2)y, Q].
\end{aligned}$$

In virtue of Lemma 3.4, for every $(v_h) \subset \mathrm{LD}(Q)$ such that $v_h \xrightarrow{*} (q+c)(e_1 \odot e_2)y$ weakly* in $\mathrm{BD}(Q)$ it holds that

$$\liminf_{h \to \infty} \mathcal{F}[v_h, Q] \geq \mathcal{F}[(q + c)(e_1 \odot e_2)y, Q].$$

Taking the infimum over all such sequences yields

$$\mathcal{F}_*[(q + c)(e_1 \odot e_2)y, Q] \geq \mathcal{F}[(q + c)(e_1 \odot e_2)y, Q].$$

Since $Eu(Q) = Ev(Q) = Eu_0(Q) = (q + c)(e_1 \odot e_2)$, we can write

$$\mathcal{F}_*[u, Q] \geq \mathcal{F}_*[u_0, Q] \geq \mathcal{F}[(q + c)(e_1 \odot e_2)y, Q] = f(Eu(Q)).$$

This proves the lemma. $\square$

**Lemma 3.13 (Lower estimate).** *For $u \in \mathrm{BD}(\Omega)$ the inequality*

$$\mathcal{F}_*[u,\Omega] \geq \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^\#\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|$$

*holds.*

*Proof.* We treat separately $\mathscr{L}^d$-a.e. regular point $x_0 \in \Omega$ and $|E^s u|$-a.e. singular point $x_0 \in \Omega$.

*Regular points.* Fix $x_0 \in \Omega$ such that $u$ is approximately differentiable at $x_0$ and

$$\lim_{r\downarrow 0}\frac{|Eu|(B(x_0,r))}{\omega_d r^d} = \frac{\mathrm{d}\,|Eu|}{\mathrm{d}\,\mathscr{L}^d}(x_0) = |\mathcal{E}u(x_0)|.$$

Since $u \in \mathrm{BD}(\Omega)$, these properties hold for $\mathscr{L}^d$-almost every $x \in \Omega$ (see Sections 2.2 and 2.5 for details). For $y \in B(0,1)$ define maps

$$u_r(y) := \frac{u(x_0+ry)-\tilde{u}(x_0)}{r}, \quad 0 < r < \mathrm{dist}(x_0,\partial\Omega),$$

where $\tilde{u}$ is the precise representative of $u$. For $u_0(y) := \nabla u(x_0)y$ we have the strong convergence $u_r \to u_0$ in $\mathrm{L}^1(B(0,1);\mathbb{R}^d)$. Indeed, by the approximate differentiability we have

$$\begin{aligned}&\int_{B(0,1)} |u_r(y)-u_0(y)|\,\mathrm{d}\,y\\ &\quad= \frac{1}{r^d}\int_{B(x_0,r)}\frac{|u(z)-\tilde{u}(x_0)-\nabla u(x_0)(z-x_0)|}{r}\,\mathrm{d}\,z \to 0\end{aligned}$$

as $r \downarrow 0$. Moreover, we have strict convergence:

$$\lim_{r\downarrow 0}|Eu_r|(B(0,1)) = \omega_d \lim_{r\downarrow 0}\frac{|Eu|(B(x_0,r))}{\omega_d r^d} = \omega_d|\mathcal{E}u(x_0)| = |Eu_0|(B(0,1)),$$

thus $(u_r)$ is bounded in $\mathrm{BD}(B(0,1))$, so we have that $u_r \overset{*}{\rightharpoonup} u_0$ weakly* in $\mathrm{BD}(B(0,1))$ by Lemma 2.19. In virtue of Proposition 2.15, Lemma 3.4 and scaling properties of $\mathcal{F}_*$ we obtain

$$\begin{aligned}\liminf_{r\downarrow 0}\frac{\mathcal{F}_*[u,B(x_0,r)]}{r^d} &= \liminf_{r\downarrow 0}\mathcal{F}_*[u_r,B(0,1)]\\ &\geq \mathcal{F}_*[u_0,B(0,1)]\\ &\geq \int_{B(0,1)} f(\mathcal{E}u_0(y))\,\mathrm{d}\,y = \omega_d f(\mathcal{E}u(x_0)).\end{aligned}$$

Therefore, by Proposition 2.2 we obtain

$$\mathcal{F}_*^a[u,B] \geq \int_B f(\mathcal{E}u)\,\mathrm{d}\,x$$

for any Borel set $B \subset \Omega$.

*Singular points.* We want to prove that for all Borel sets $B \subset \Omega$ the inequality

$$\mathcal{F}_*^s[u,B] \geq \int_B f^\# \left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right) \mathrm{d}\,|E^s u|$$

holds. We fix $x_0 \in \Omega$ such that

1. $\dfrac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}(x_0) = a \odot b$ for some $a, b \in \mathbb{R}^d \setminus \{0\}$,
2. $\alpha_r := r^{-d}|Eu|(Q(x_0,r)) \to \infty$ as $r \downarrow 0$, where $Q(x_0,r) := x_0 + rQ$ and $Q$ is a (fixed) open $d$-cube with a centre 0, side-length 1 and sides either parallel or orthogonal to $a$.

These properties hold for $|E^s u|$-a.e. $x_0 \in \Omega$ in virtue of Theorem 2.26 and Theorem 2.3. It suffices to establish the inequality

$$\lim_{r\downarrow 0} \frac{\mathcal{F}_*[u,Q(x_0,r)]}{|Eu|(Q(x_0,r))} \geq f^\#(a \odot b)$$

at any $|Eu|$-Lebesgue point $x_0 \in \Omega$ for which the limit on the left-hand side exists, which is the case at $|Eu|$-a.e. $x_0 \in \Omega$. Define a blow-up sequence

$$v_r(y) := \frac{u(x_0+ry) - [u]_{Q(x_0,r)}}{r\alpha_r} + R_r(y), \quad y \in Q, \quad 0 < r < \operatorname{dist}(x_0, \partial\Omega),$$

where $R_r : \mathbb{R}^d \to \mathbb{R}^d$ is a family of rigid deformations and $[u]_{Q(x_0,r)} := \fint_{Q(x_0,r)} u\,\mathrm{d}\,x$ is the average of $u$ over $Q(x_0,r)$.

In virtue of Lemma 2.14 in [17], up to a subsequence, the blow-up sequence $(v_r)$ converges weakly* in $\mathrm{BD}(Q)$ to the function

$$v_0(y) := h(y\cdot a)b + c(a\otimes b)y + Wy + \bar{v},$$

with a bounded and increasing function $h : (-1/2, 1/2) \to \mathbb{R}$, $c > 0$, and a rigid deformation $Wy + \bar{v}$, where $W \in \mathbb{R}^{d\times d}_{\text{skew}}$, $\bar{v} \in \mathbb{R}^d$.

Note that for any Borel set $B \subset Q$ we have

$$Ev_r(B) = \frac{r^{1-d}Eu(x_0+rB)}{r\alpha_r} = \frac{Eu(x_0+rB)}{|Eu|(Q(x_0,r))} \tag{3.13}$$

hence $|Ev_r|(Q) = 1$. Consequently, by Proposition 1.62(b) in [4], we also have $|Ev_0|(Q) \leq 1$.

Fix $0 < t < 1$ and let $Q_t := tQ$ be a re-scaled cube. There exists a (not particularly labelled) sequence of radii such that

$$\lim_{r\downarrow 0} \frac{|Eu|(Q(x_0, tr))}{|Eu|(Q(x_0, r))} \geq t^d. \tag{3.14}$$

Indeed, if it was not true, then for some $0 < t_0 < 1$ we could find $0 < r_0 < 1$ such that

$$|Eu|(Q(x_0, t_0 r)) \leq t_0^d |Eu|(Q(x_0, r))$$

for all $r < r_0$. Iterating the above inequality yields:

$$|Eu|(Q(x_0, t_0^k r_0)) \leq t_0^{kd} |Eu|(Q(x_0, r_0))$$

for all $k \in \mathbb{N}$. Since any $0 < r < r_0$ is in the interval $(t^{k+1} r_0, t^k r_0]$ for some $k \in \mathbb{N}$ we obtain

$$|Eu|(Q(x_0, r)) \leq |Eu|(Q(x_0, t_0^k r_0)) \leq t_0^{kd} |Eu|(Q(x_0, r_0)) \leq \frac{|Eu|(Q(x_0, r_0))}{t_0^d r_0^d} r^d.$$

Hence for any $0 < r < r_0$

$$\alpha_r \leq \frac{|Eu|(Q(x_0, r_0))}{t_0^d r_0^d}$$

which is a contradiction, since $\alpha_r \to +\infty$ as $r \downarrow 0$. So, (3.14) holds.

Note that (3.14) yields

$$\lim_{r\downarrow 0} |Ev_r|(\overline{Q}_t) \geq t^d. \tag{3.15}$$

Then, for any weak* limit $\nu$ of $|Ev_r|$ in $Q$ we get (by Example 1.63 in [4]) that $\nu(\overline{Q}_t) \geq t^d$. On the other hand, $Ev_r \overset{*}{\rightharpoonup} Ev_0$ and $Ev_0(Q) = \frac{a\odot b}{|a\odot b|}\nu(Q)$ by Theorem 2.3, (3.13) and (1). Moreover

$$|Ev_0|(Q) \leq \nu(Q) = |Ev_0(Q)| \leq |Ev_0|(Q),$$

hence, together with $\nu \geq |Ev_0|$ we obtain $\nu = |Ev_0|$ on $Q$. Thus $|Ev_0|(\overline{Q}_t) \geq t^d$.

Define $w_r := \varphi v_r + (1-\varphi)v_0$, where $\varphi \in \mathrm{C}^1_c(Q; [0,1])$ with $\varphi \equiv 1$ on the neighbourhood of $\overline{Q}_t$. Clearly, the sequence $(w_r)$ converges to $v_0$ strongly in

$\mathrm{L}^1(Q;\mathbb{R}^d)$ and

$$
\begin{aligned}
|E(w_r - v_r)|(Q) &\le |E(v_r - v_0)|(Q \setminus \overline{Q}_t) + \int_Q |\mathcal{E}\varphi|\,|v_r - v_0|\,\mathrm{d}\,y \\
&\le |Ev_r|(Q \setminus \overline{Q}_t) + |Ev_0|(Q \setminus \overline{Q}_t) + \int_Q |\mathcal{E}\varphi|\,|v_r - v_0|\,\mathrm{d}\,y.
\end{aligned}
$$

Therefore, by (3.15), we have

$$
\limsup_{r\downarrow 0} |E(w_r - v_r)|(Q) \le 2(1 - t^d).
$$

Similarly,

$$
|Ew_r|(Q \setminus \overline{Q}_t) \le |Ev_r|(Q \setminus \overline{Q}_t) + |Ev_0|(Q \setminus \overline{Q}_t) + \int_Q |\mathcal{E}\varphi|\,|v_r - v_0|\,\mathrm{d}\,y
$$

and thus we also have

$$
\limsup_{r\downarrow 0} |Ew_r|(Q \setminus \overline{Q}_t) \le 2(1 - t^d).
$$

Using scaling properties of $\mathcal{F}_*$ and the estimate (3.10), we obtain

$$
\begin{aligned}
\frac{\mathcal{F}_*[u, Q(x_0, r)]}{|Eu|(Q(x_0, r))} &= \frac{\mathcal{F}_*[\alpha_r v_r, Q]}{\alpha_r} \\
&\ge \frac{\mathcal{F}_*[\alpha_r w_r, \overline{Q}_t]}{\alpha_r} \\
&= \frac{\mathcal{F}_*[\alpha_r w_r, Q]}{\alpha_r} - \frac{\mathcal{F}_*[\alpha_r w_r, Q \setminus \overline{Q}_t]}{\alpha_r} \\
&\ge \frac{\mathcal{F}_*[\alpha_r w_r, Q]}{\alpha_r} - M\left(\alpha_r^{-1}|Q \setminus \overline{Q}_t| + |Ew_r|(Q \setminus \overline{Q}_t)\right).
\end{aligned}
$$

Since $\alpha_r \to +\infty$ as $r \downarrow 0$ we obtain

$$
\lim_{r\downarrow 0} \frac{\mathcal{F}_*[u, Q(x_0, r)]}{|Eu|(Q(x_0, r))} \ge \limsup_{r\downarrow 0} \frac{\mathcal{F}_*[\alpha_r w_r, Q]}{\alpha_r} - 2M(1 - t^d).
$$

By Lemma 3.12 in conjunction with the Lipschitz continuity of $f$ (see Remark 2.9(2)), we obtain

$$
\mathcal{F}_*[\alpha_r w_r, Q] \ge f(\alpha_r Ew_r(Q)) \ge f(\alpha_r Ev_r(Q)) - \alpha_r L|E(w_r - v_r)|(Q)
$$

for all $r > 0$. Here $L > 0$ denotes the Lipschitz constant of $f$. Therefore

$$
\lim_{r\downarrow 0} \frac{\mathcal{F}_*[u, Q(x_0, r)]}{|Eu|(Q(x_0, r))} \ge \limsup_{r\downarrow 0} \frac{f(\alpha_r Ev_r(Q))}{\alpha_r} - 2(L + M)(1 - t^d).
$$

Since

$$Ev_r(Q) = \frac{Eu(Q(x_0, r))}{|Eu|(Q(x_0, r))} \to \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|}(x_0) = a \odot b \quad \text{as} \quad r \downarrow 0,$$

we obtain

$$\limsup_{r\downarrow 0} \frac{f(\alpha_r E v_r(Q))}{\alpha_r} = f^{\#}(a \odot b),$$

We thus have

$$\lim_{r\downarrow 0} \frac{\mathcal{F}_*[u, Q(x_0, r)]}{|Eu|(Q(x_0, r))} \geq f^{\#}(a \odot b) - 2(L + M)(1 - t^d).$$

Letting $t \uparrow 1$ concludes the proof. □

**Remark 3.14.** Consider a larger Lipschitz domain $\Omega' \subset \mathbb{R}^d$ such that $\Omega \Subset \Omega'$. Extend the function $u \in \mathrm{BD}(\Omega)$ to $\Omega'$ by some function $v \in \mathrm{BD}(\Omega' \setminus \overline{\Omega})$ and denote this extension by $\overline{u}$. Let $g \in \mathrm{L}^1(\partial\Omega; \mathscr{H}^{d-1} \llcorner \partial\Omega; \mathbb{R}^d)$ be a trace of $v$ on $\partial\Omega$. Applying Theorem 3.1 to $\overline{u}$ and $\Omega'$ yields the weak* lower semicontinuity of the functional:

$$\begin{aligned}\mathcal{F}[u] := \int_\Omega f(\mathcal{E}u)\, \mathrm{d}\, x + \int_\Omega f^{\#}\left(\frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|}\right) \mathrm{d}\, |E^s u| \\ + \int_{\partial\Omega} f^{\#}\left((u - g) \odot n_\Omega\right) \mathrm{d}\, \mathscr{H}^{d-1}, \quad u \in \mathrm{BD}(\Omega),\end{aligned}$$

where $u$ in the surface energy component, called the *penalisation term*, is understood in a sense of trace (see Theorem 2.24) and $n_\Omega : \partial\Omega \to \mathbb{S}^{d-1}$ is an inward pointing unit normal. Just like in Theorem 3.1, we assume that $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ is symmetric-quasiconvex with linear growth at infinity.

As a direct consequence of the weak* lower semicontinuity of $\mathcal{F}$ and the Poincaré inequality (2.5), we have the existence of minimisers of the following minimisation problem:

**Theorem 3.15 (Minimisation).** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ be a continuous function satisfying the following conditions:*

1. *there exist constants* $0 < m \leq M$ *such that for all* $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *the inequality*

$$m|A| \leq f(A) \leq M(1+|A|)$$

*holds;*

2. $f$ *is symmetric-quasiconvex.*

*Let* $g \in \mathrm{L}^1(\partial\Omega; \mathscr{H}^{d-1} \llcorner \partial\Omega; \mathbb{R}^d)$ *be a boundary datum. Then, the functional*

$$\begin{aligned}\mathcal{F}[u] := \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^{\#}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u| \\ + \int_{\partial\Omega} f^{\#}\left((u-g)\odot n_\Omega\right)\mathrm{d}\,\mathscr{H}^{d-1}, \quad u \in \mathrm{BD}(\Omega),\end{aligned}$$

*has a minimiser over the space* $\mathrm{BD}(\Omega)$.

## 3.3 Relaxation

In this section we consider the case, where the integrand $f$ in $\mathcal{F}$ is not symmetric-quasiconvex. We will prove the following relaxation theorem:

**Theorem 3.16.** *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain and let* $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty)$ *be a continuous function such that the inequality*

$$m|A| \leq f(A) \leq M(1+|A|) \tag{3.16}$$

*holds for all* $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *and some constants* $0 < m \leq M$. *Then, the relaxation of the extended real-valued functional*

$$\mathcal{F}_\infty[u] := \begin{cases} \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x & \text{for } u \in \mathrm{LD}(\Omega) \\ +\infty & \text{for } u \in \mathrm{BD}(\Omega)\setminus \mathrm{LD}(\Omega), \end{cases}$$

*with respect to the weak* topology in* $\mathrm{BD}(\Omega)$ *is given by*

$$\mathcal{F}_*[u] = \int_\Omega SQf(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega (SQf)^{\#}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|, \quad u \in \mathrm{BD}(\Omega), \tag{3.17}$$

*where* $SQf$ *denotes the symmetric-quasiconvex envelope of* $f$.

Recall that a function $f : \Omega \to \mathbb{R}^d$ is *countably piecewise affine* if there exists a disjoint open partition $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\Omega$, such that $f|_{\Omega_k}$ is affine for every

$k \in \mathbb{N}$. It is well-known (cf. [18, Proposition 2.8]), that every Sobolev function $u \in \mathrm{W}^{1,p}$, $1 \le p < \infty$, can be approximated by countably piecewise affine functions $(u_h)$ in the corresponding Sobolev norm and such that the boundary trace of each $u_h$ agrees with the boundary trace of $u$.

In order to prove Theorem 3.16 we will need the following density result.

**Lemma 3.17 (Affine density).** *For $u \in \mathrm{BD}(\Omega)$ there exists a sequence of countably piecewise affine functions $v_h : \Omega \to \mathbb{R}^d$ such that $v_h|_{\partial\Omega} = u|_{\partial\Omega}$ and $v_h \to u$ area-strictly.*

*Proof.* Following the same procedure as in the proof of Lemma 11.1 in [35], for every $u \in \mathrm{BD}(\Omega)$ one can find a family of smooth maps $(u_h) \subset \mathrm{C}^\infty(\Omega; \mathbb{R}^d)$ such that $u_h \to u$ area-strictly and $u_h|_{\partial\Omega} = u|_{\partial\Omega}$. In particular, for each $h \in \mathbb{N}$, the function $u_h$ is in some $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$, so we can find a countably piecewise affine function $v_k^{(h)} : \Omega \to \mathbb{R}^d$ such that $\|v_k^{(h)} - u_h\|_{1,p} \to 0$ as $k \to \infty$ and $v_k^{(h)}|_{\partial\Omega} = u_h|_{\partial\Omega}$. Finally, we select a diagonal subsequence such that $v_{k_h}^{(h)} \to u$ area-strictly as $h \to \infty$. Clearly, $v_{k_h}^{(h)}$ has the same trace on $\partial\Omega$ as $u$. $\square$

*Proof of Theorem 3.16.* By Remark 2.6(2), the symmetric-quasiconvex envelope $SQf$ is symmetric-quasiconvex with linear growth. Let us denote the right-hand side of (3.17) by $\mathcal{G}$. Then, by Theorem 3.1, the functional $\mathcal{G}$ is weakly* lower semicontinuous in $\mathrm{BD}(\Omega)$. In virtue of Proposition 2.15 we conclude that $\mathcal{G} \le \mathcal{F}_*$. The proof will be finished once we show the opposite inequality.

*Step 1.* Firstly, note that

$$\mathcal{F}_*[u] = \inf \left\{ \liminf_{h\to\infty} \mathcal{F}_\infty[u_h] : \ (u_h) \subset \mathrm{LD}(\Omega), \ u_h \to u \text{ in } \mathrm{L}^1(\Omega; \mathbb{R}^d) \right\}. \quad (3.18)$$

Suppose (3.18) is not true. Then, we could find $(u_h) \subset \mathrm{LD}(\Omega)$ such that $u_h \to u$ strongly in $\mathrm{L}^1(\Omega; \mathbb{R}^d)$ and

$$\mathcal{F}_*[u] > \lim_{h\to\infty} \mathcal{F}_\infty[u_h] \ge \limsup_{h\to\infty} m\|\mathcal{E}u_h\|_{\mathrm{L}^1}.$$

Therefore, the sequence $(\mathcal{E}u_h)$ is uniformly norm-bounded in $\mathrm{L}^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and, by Lemma 2.19, $u_h \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{BD}(\Omega)$, whereby we get that $\mathcal{F}_*[u] > \mathcal{F}_*[u]$, which is absurd.

*Step 2.* In virtue of Lemma 11.1 in [35], we choose a sequence $(u_h) \subset \mathrm{LD}(\Omega)$ such that $u_h \to u$ area-strictly and $u_h|_{\partial\Omega} = u|_{\partial\Omega}$. Fix $\varepsilon > 0$ and let $\Omega_\varepsilon \Subset \Omega$ be a Lipschitz subdomain such that $\sup_{x\in\Omega_\varepsilon} \operatorname{dist}(x, \partial\Omega) < \varepsilon$. By Lemma 3.17 we may assume that functions $(u_h)$ are countably piecewise affine in $\Omega_\varepsilon$, that is $u_h(x) = A_i^{(h)} x + b_i^{(h)}$ almost everywhere in $\Omega_i^{(h)} \subset \Omega_\varepsilon$ for some symmetric matrices $A_i^{(h)} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and some vectors $b_i^{(h)} \in \mathbb{R}^d$. The sets $\Omega_i^{(h)}$ constitute to Vitali's covering of $\Omega_\varepsilon$. By the formula (2.2) we can choose functions $\psi_i^{(h)} \in \mathrm{W}_0^{1,\infty}(\Omega_i^{(h)}; \mathbb{R}^d)$ such that

$$\int_{\Omega_i^{(h)}} |\psi_i^{(h)}(x)| \,\mathrm{d}\, x \le h^{-1} |\Omega_i^{(h)}|$$

and

$$\int_{\Omega_i^{(h)}} f\left(A_i^{(h)} + \mathcal{E}\psi_i^{(h)}(x)\right) \mathrm{d}\, x \le \left(SQf(A_i^{(h)}) + h^{-1}\right) |\Omega_i^{(h)}|.$$

Let $(v_h) \subset \mathrm{LD}(\Omega)$ be a sequence of functions defined as

$$v_h(x) := \begin{cases} u_h(x) + \psi_i^{(h)}(x), & \text{for } x \in \Omega_i^{(h)}, \\ u_h(x), & \text{for } x \in \Omega \setminus \overline{\Omega}_\varepsilon. \end{cases}$$

Clearly $v_h|_{\partial\Omega} = u|_{\partial\Omega}$ and $v_h \to u$ strongly in $\mathrm{L}^1(\Omega; \mathbb{R}^d)$. Hence, by (3.18), we have

$$\mathcal{F}_*[u] \le \liminf_{h\to\infty} \mathcal{F}_\infty[v_h].$$

We have:

$$\begin{aligned} \int_\Omega f(\mathcal{E}v_h(x)) \,\mathrm{d}\, x &= \int_{\Omega_\varepsilon} f(\mathcal{E}v_h(x)) \,\mathrm{d}\, x + \int_{\Omega\setminus\overline{\Omega}_\varepsilon} f(\mathcal{E}u_h(x)) \,\mathrm{d}\, x \\ &\le \sum_i \int_{\Omega_i^{(h)}} f(A_i^{(h)} + \mathcal{E}\psi_i^{(h)}) \,\mathrm{d}\, x + M \int_{\Omega\setminus\overline{\Omega}_\varepsilon} 1 + |\mathcal{E}u_h(x)| \,\mathrm{d}\, x \\ &\le \sum_i |\Omega_i^{(h)}| SQf(A_i^{(h)}) + h^{-1}|\Omega| + M \int_{\Omega\setminus\overline{\Omega}_\varepsilon} 1 + |\mathcal{E}u_h(x)| \,\mathrm{d}\, x \\ &\le \int_\Omega SQf(\mathcal{E}u_h(x)) \,\mathrm{d}\, x + h^{-1}|\Omega| + M \int_{\Omega\setminus\overline{\Omega}_\varepsilon} 1 + |\mathcal{E}u_h(x)| \,\mathrm{d}\, x. \end{aligned}$$

Passing to the lower limit as $h \to \infty$ yields

$$\mathcal{F}_*[u] \le \liminf_{h\to\infty} \int_\Omega SQf(\mathcal{E}u_h(x))\, \mathrm{d}\, x + M(\mathscr{L}^d + |Eu|)(\Omega \setminus \overline{\Omega}_\varepsilon).$$

By the approximation argument, analogous to the one in the proof of Lemma 3.9, we obtain the inequality

$$\liminf_{h\to\infty} \int_\Omega SQf(\mathcal{E}u_h(x))\, \mathrm{d}\, x \le \int_\Omega SQf(\mathcal{E}u)\, \mathrm{d}\, x + \int_\Omega (SQf)^\# \left( \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u|.$$

Hence

$$\begin{aligned}\mathcal{F}_*[u] \le \int_\Omega SQf(\mathcal{E}u)\, \mathrm{d}\, x + \int_\Omega (SQf)^\# \left( \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u| \\ + M(\mathscr{L}^d + |Eu|)(\Omega \setminus \overline{\Omega}_\varepsilon).\end{aligned}$$

By letting $\varepsilon \downarrow 0$ we obtain

$$\mathcal{F}_*[u] \le \int_\Omega SQf(\mathcal{E}u)\, \mathrm{d}\, x + \int_\Omega (SQf)^\# \left( \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u| = \mathcal{G}[u]. \qquad \square$$

## Conclusion

In this chapter we established the optimal relaxation result for integral functionals with integrands that have linear growth at infinity. We proved an integral representation of the relaxation in the case when an integrand is symmetric-quasiconvex and when this assumption is not satisfied. Our results extend [10] and also Corollary 1.10 in [6] to relaxation theorems without any assumption on the recession function. Due to the recent developments in the theory of functions of bounded deformation, we could utilise the classical blow-up argument in its elementary form to establish these results.

# Chapter 4

# Mixed-growth functionals

In this chapter we study homogeneous integral functionals of the form:

$$\mathcal{F}[u] := \int_\Omega f(\mathcal{E}u(x)) \, \mathrm{d}\, x$$

with the *anisotropic* growth bounds:

$$m\left((\operatorname{tr} A)^2 + |\operatorname{dev} A|\right) \le f(A) \le M\left(1 + (\operatorname{tr} A)^2 + |\operatorname{dev} A|\right)$$

for some constants $0 < m \le M$.

An example of such functional comes from the perfectly plastic elastic model known as Hencky's model [5, 38]:

$$\int_\Omega \varphi(\operatorname{dev} \mathcal{E}u) + \frac{\kappa}{2}(\operatorname{div} u)^2 \, \mathrm{d}\, x, \tag{4.1}$$

where $\varphi : \mathrm{SD}(d) \to [0, +\infty)$ is a convex function which grows quadratically on some compact set and linearly outside of this set, and $\kappa = \lambda + 2\mu/3$ is the bulk modulus of the material, i.e. a measure of how resistant to compression is the material, with the Lamé constants $\lambda$ and $\mu$. We will return to the Hencky functional in the next chapter.

## 4.1 Recession function

As in the previous chapter, we need a suitable notion of recession function. For $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, \infty)$ we write $f_{\mathrm{dev}}$ for the restriction of $f$ to the subspace of deviatoric matrices $\mathrm{SD}(d)$. We can now define the recession function $f^{\#}_{\mathrm{dev}}$ as the upper limit:

$$f^{\#}_{\mathrm{dev}}(A) := \limsup_{\substack{A' \to A \\ s \to \infty}} \frac{f_{\mathrm{dev}}(sA')}{s}$$

The recession function $f^{\#}_{\mathrm{dev}}$ shares the same properties as the recession function from the previous chapter, hence we omit the details here.

## 4.2 Relaxation

In this section we prove the following result.

**Theorem 4.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty)$ be a continuous function satisfying the following conditions:*

1. *there exist constants $0 < m \leq M$ such that for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the growth*
$$m\left((\operatorname{tr} A)^2 + |\operatorname{dev} A|\right) \leq f(A) \leq M\left(1 + (\operatorname{tr} A)^2 + |\operatorname{dev} A|\right) \tag{4.2}$$
*holds;*
2. *$f$ is symmetric-quasiconvex;*
3. *there exist constants $\gamma \in [0,2)$ and $\delta \in [0,1)$ such that for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the inequality*
$$f(A) \geq f^{\#}_{\mathrm{dev}}(\operatorname{dev} A) - M\left(|\operatorname{tr} A|^{\gamma} + |\operatorname{dev} A|^{\delta} + 1\right) \tag{4.3}$$
*holds.*

*Then, the relaxation of the extended real-valued functional*

$$\mathcal{F}[u,\Omega] := \begin{cases} \displaystyle\int_\Omega f(\mathcal{E}u(x))\,\mathrm{d}\,x & \text{for } u \in \mathrm{LU}(\Omega) \\ +\infty & \text{for } u \in \mathrm{U}(\Omega) \setminus \mathrm{LU}(\Omega), \end{cases} \tag{4.4}$$

*with respect to the weak* topology in $\mathrm{U}(\Omega)$ is given by*

$$\overline{\mathcal{F}}[u,\Omega] := \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^{\#}_{\mathrm{dev}}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|, \quad u \in \mathrm{U}(\Omega). \tag{4.5}$$

**Remark 4.2.**

1. Since the set $\mathcal{S}$ from Proposition 2.11 spans $\mathrm{SD}(d)$, the function $f_{\mathrm{dev}}$ is globally Lipschitz. This is a consequence of $f_{\mathrm{dev}}$ being separately convex, i.e. convex in each variable, with linear growth at infinity and Lemma 5.42 in [4];
2. Since $f_{\mathrm{dev}}$ is a symmetric rank-one convex function with linear growth at infinity, the recession function $f^{\#}_{\mathrm{dev}}$ is also symmetric rank-one convex and by (1) we can write:
$$f^{\#}_{\mathrm{dev}}(A) = \limsup_{s\to\infty} \frac{f_{\mathrm{dev}}(sA)}{s};$$
3. By Corollary 2.12 the recession function $f^{\#}_{\mathrm{dev}}$ is convex at each point of $\mathcal{S}$.

**Remark 4.3.** The lower bound with subcritical growth in both trace and deviatoric directions in the condition (3) is essential for the proof. It remains an open question whether it can be deduced from the conditions (1) and (2).

The proof of Theorem 4.1 is structured as follows. First, in Lemma 4.6 we prove that the conclusion of Theorem 4.1 holds for the linear weak* limits. This step is essential for the blow-up argument in the proof of the first part of Proposition 4.13.

Next, we investigate the relaxation $\mathcal{F}_*$ of $\mathcal{F}$ defined by
$$\mathcal{F}_*[u,\Omega] := \inf\left\{ \liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] : \ (u_h) \subset \mathrm{U}(\Omega),\ u_h \overset{*}{\rightharpoonup} u \text{ in } \mathrm{U}(\Omega) \right\}.$$
Note that, by the argument similar to the proof of Proposition 3.6, we can consider the above relaxation along sequences $(u_h) \subset \mathrm{LU}(\Omega)$.

We establish that for all $u \in \mathrm{U}(\Omega)$ the map $V \mapsto \mathcal{F}_*[u,V]$ is a restriction to open sets of a finite Radon measure. We then decompose this measure into the absolutely continuous part $\mathcal{F}^a_*$ and the singular part $\mathcal{F}^s_*$ (with respect to the Lebesgue measure) and prove the lower bounds:
$$\mathcal{F}^a_*[u,B] \geq \int_B f(\mathcal{E}u)\,\mathrm{d}\,x \tag{4.6}$$

and

$$\mathcal{F}^s_*[u,B] \geq \int_B f^\#_{\mathrm{dev}}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u| \tag{4.7}$$

for all Borel sets $B \subset \Omega$. For the proof of the *regular bound* (4.6) we use the blow-up sequence argument like in the previous chapter, whereas the proof of the *singular bound* (4.7) relies on Theorem 2.10.

Finally, together with the upper bound $\mathcal{F}_* \leq \overline{\mathcal{F}}$ from Proposition 4.12 we obtain that $\mathcal{F}_* = \overline{\mathcal{F}}$, thus Theorem 4.1 follows.

**Remark 4.4.** It does not seem possible to prove Theorem 4.1 using the blow-up argument for both regular and singular estimates as in the usual BV or BD lower semicontinuity results [3, 19, 34]. Originally, the blow-up argument was tailored for the functionals with an isotropic linear growth imposed on the integrands. This, however, is not the case here, as the admissible integrands in Theorem 4.1 grow quadratically in the trace direction and the blow-up argument does not work. The problem is that if one attempts to utilise the blow-up argument for the singular estimate (4.7), one eventually faces the problem of controlling the blow-up rate of the divergence terms of the blow-up sequence. A priori it seems not possible to obtain a sufficient decay of the sequence of divergences, and so a different strategy based on asymptotic convexity via the Kirchheim-Kristensen convexity result (Theorem 2.10) needs to be employed.

In order to prove Theorem 4.1 we use cut-off arguments (see Lemmas 4.6 and 4.11). For a given function $u \in \mathrm{U}(\Omega)$ and some smooth cut-off function $\varphi \in \mathrm{C}^1_c(\Omega)$ the product $\varphi u$ is in $\mathrm{BD}(\Omega)$, but not necessarily in $\mathrm{U}(\Omega)$ (this property is called *non-locality*). Indeed, we have

$$\operatorname{div}(\varphi u) = \nabla\varphi \cdot u + \varphi \operatorname{div} u$$

and the first term on the right-hand side does not belong to $\mathrm{L}^2(\Omega)$ in general.

The following result due to Bogovskii (see [11, 12] or section III.3 in [22] for the proof) is essential, since it provides a suitable correction term $v$ such that $\varphi u + v \in \mathrm{U}(\Omega)$.

**Theorem 4.5 (Bogovskii).** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < q < \infty$. There exists a linear operator $\mathcal{B} : \mathrm{L}^q(\Omega) \to \mathrm{W}_0^{1,q}(\Omega;\mathbb{R}^d)$ with the following properties:*

*(i) for every $f \in \mathrm{L}^q(\Omega)$ such that $\int_\Omega f \,\mathrm{d}\,x = 0$ it holds that*

$$\operatorname{div} \mathcal{B}f = f \quad \text{in } \Omega;$$

*(ii) for every $f \in \mathrm{L}^q(\Omega)$ the estimate*

$$\|\nabla(\mathcal{B}f)\|_q \le C\|f\|_q$$

*holds with a translation and scaling invariant constant $C > 0$, depending only on $\Omega$ and $q$;*

*(iii) if $f \in \mathrm{C}_c^\infty(\Omega)$, then $\mathcal{B}f \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^d)$.*

We begin with a series of lemmas.

**Lemma 4.6.** *Let $f : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty)$ satisfy conditions (1) and (2) of Theorem 4.1, $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and let $(u_h) \subset \mathrm{U}(\Omega)$ be a sequence such that $u_h \xrightarrow{*} Ax$ weakly* in $\mathrm{U}(\Omega)$. Then,*

$$|\Omega| f(A) \le \liminf_{h\to\infty} \int_\Omega f(\mathcal{E}u_h)\,\mathrm{d}\,x. \tag{4.8}$$

*Proof.* Without loss of generality assume that $(u_h) \subset \mathrm{LU}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d)$. The proof is divided into two steps. In the first step we prove (4.8) for a sequence $(u_h)$ which has linear boundary values. Then, in the second step we prove, using a cut-off argument, that the assumption of the linear boundary values can be dropped.

*Step 1.* Suppose that $u_h - Ax$ is compactly supported inside $\Omega$ for all $h \in \mathbb{N}$ and take $\psi_h(x) := u_h(x) - Ax$. Clearly, $\psi_h \in \mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^d)$. Then, by the symmetric-quasiconvexity of $f$ we obtain

$$|\Omega| f(A) \le \int_\Omega f(A + \mathcal{E}\psi_h(y))\,\mathrm{d}\,y = \int_\Omega f(\mathcal{E}u_h(y))\,\mathrm{d}\,y$$

for all $h \in \mathbb{N}$. Therefore,

$$|\Omega| f(A) \le \liminf_{h\to\infty} \int_\Omega f(\mathcal{E}u_h)\,\mathrm{d}\,x.$$

*Step 2.* Let $u_h \xrightharpoonup{*} Ax$ weakly* in $\mathrm{U}(\Omega)$. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$ and choose a Lipschitz subdomain $\Omega_0 \Subset \Omega$ such that $|\Omega \setminus \Omega_0| \leq \varepsilon$. Let $R := \operatorname{dist}(\Omega_0, \partial\Omega)$ and for $i = 1, \ldots, n$ define sets

$$\Omega_i := \left\{x \in \Omega : \ \operatorname{dist}(x, \Omega_0) < \frac{iR}{n}\right\}.$$

Now, choose cut-off functions $\varphi_i \in \mathrm{C}^1_c(\Omega; [0,1])$ such that

$$\mathbb{1}_{\Omega_{i-1}} \leq \varphi_i \leq \mathbb{1}_{\Omega_i} \quad \text{and} \quad \|\nabla\varphi_i\|_\infty \leq \frac{2n}{R} \tag{4.9}$$

and for $x \in \Omega$ define

$$u_{h,i}(x) := Ax + \varphi_i(x)(u_h(x) - Ax).$$

We have

$$\mathcal{E}u_{h,i} = A + \varphi_i(\mathcal{E}u_h - A) + \nabla\varphi_i \odot (u_h - Ax) \tag{4.10}$$

and

$$\operatorname{div} u_{h,i} = \operatorname{tr} A + \varphi_i(\operatorname{div} u_h - \operatorname{tr} A) + \nabla\varphi_i \cdot (u_h - Ax). \tag{4.11}$$

Note that the last term in (4.11) belongs only to $\mathrm{L}^{d/(d-1)}(\Omega)$ by the embedding $\mathrm{BD}(\Omega) \subset \mathrm{L}^q(\Omega; \mathbb{R}^d)$ for $1 \leq q \leq d/(d-1)$ (cf. Proposition 1.2 in [39]), thus $u_{h,i} \notin \mathrm{U}(\Omega)$ for $d > 2$. In order to overcome this problem we fix some $1 < q < d/(d-1)$ and define numbers

$$\xi_{h,i} := \frac{1}{|S_i|}\int_{S_i} \nabla\varphi_i(x) \cdot (u_h(x) - Ax) \,\mathrm{d}\,x,$$

where $S_i := \Omega_i \setminus \overline{\Omega}_{i-1}$ is the open strip between $\Omega_{i-1}$ and $\Omega_i$.
Note that $\operatorname{supp} \nabla\varphi_i \subset S_i$. Define

$$f_{h,i} := -\nabla\varphi_i \cdot (u_h - Ax) + \xi_{h,i} \in \mathrm{L}^q(S_i). \tag{4.12}$$

By Theorem 4.5 there exist functions $z_{h,i} \in \mathrm{W}^{1,q}_0(S_i; \mathbb{R}^d)$ such that

$$\operatorname{div} z_{h,i} = f_{h,i} \quad \text{in } S_i$$

and such that the estimate

$$\|\nabla z_{h,i}\|_q \leq C_q \|f_{h,i}\|_q \tag{4.13}$$

holds. We also extend the functions $z_{h,i}$ by zero outside $S_i$. Let $w_{h,i} \in \mathrm{U}(\Omega)$ be defined as

$$w_{h,i} := u_{h,i} + z_{h,i}.$$

The correction term $z_{h,i}$ ensures that $\operatorname{div} w_{h,i} \in \mathrm{L}^2(\Omega)$.

Henceforth, for simplicity we write $C > 0$ for a generic constant that changes from line to line, possibly depending on $\Omega, M, A, R, n, q$, but never on $h, i$. Note that we have the following estimate:

$$\|f_{h,i}\|_q \le C\|u_h - Ax\|_q. \tag{4.14}$$

This estimate, in conjunction with the Poincaré inequality, (4.13), and the compactness of the embedding $\mathrm{BD}(\Omega) \Subset \mathrm{L}^q(\Omega;\mathbb{R}^d)$ (cf. [36]), implies that $z_{h,i} \to 0$ in $\mathrm{W}^{1,q}(\Omega;\mathbb{R}^d)$ as $h \to \infty$. Since $w_{h,i} \to Ax$ in $\mathrm{L}^1(\Omega;\mathbb{R}^d)$ and $(w_{h,i})_h$ is bounded in $\mathrm{U}(\Omega)$ for all $i = 1,\dots,n$, by Lemma 2.29 it follows that $w_{h,i} \overset{*}{\rightharpoonup} Ax$ weakly* in $\mathrm{U}(\Omega)$. Moreover, $w_{h,i}|_{\partial\Omega} = Ax$ for every $i = 1,\dots,n$ and $h \in \mathbb{N}$.

By the upper growth bound (4.2) we obtain

$$\begin{aligned}
\int_\Omega f(\mathcal{E}w_{h,i}) \,\mathrm{d}\,x &= \int_{\Omega_{i-1}} f(\mathcal{E}u_h) \,\mathrm{d}\,x + \int_{S_i} f(\mathcal{E}w_{h,i}) \,\mathrm{d}\,x + \int_{\Omega\setminus\Omega_i} f(A) \,\mathrm{d}\,x \\
&\le \int_\Omega f(\mathcal{E}u_h) \,\mathrm{d}\,x + \int_{S_i} f(\mathcal{E}w_{h,i}) \,\mathrm{d}\,x + |\Omega\setminus\Omega_0| f(A) \\
&\le \int_\Omega f(\mathcal{E}u_h) \,\mathrm{d}\,x + M\int_{S_i} |\operatorname{dev}\mathcal{E}w_{h,i}| + |\operatorname{div} w_{h,i}|^2 \,\mathrm{d}\,x \\
&\quad + C|\Omega\setminus\Omega_0|.
\end{aligned}$$

The estimates (4.13) and (4.14) together with Hölder's inequality yield

$$\int_{S_i} |\operatorname{dev}\mathcal{E}z_{h,i}| \,\mathrm{d}\,x \le C|\Omega\setminus\Omega_0|^{1/q'} \sup_h \|u_h - Ax\|_q,$$

where $1/q + 1/q' = 1$. We have

$$\begin{aligned}
\int_{S_i} |\operatorname{dev}\mathcal{E}w_{h,i}| \,\mathrm{d}\,x &\le |\operatorname{dev} A|\,|S_i| + \int_{S_i} |\varphi_i|\,|\operatorname{dev}\mathcal{E}u_h - \operatorname{dev} A| \,\mathrm{d}\,x \\
&\quad + \int_{S_i} |\operatorname{dev}[\nabla\varphi_i \odot (u_h - Ax)]| \,\mathrm{d}\,x + \int_{S_i} |\operatorname{dev}\mathcal{E}z_{h,i}| \,\mathrm{d}\,x \\
&\le |\operatorname{dev} A|\,|\Omega\setminus\Omega_0| + \int_{S_i} |\operatorname{dev}\mathcal{E}u_h - \operatorname{dev} A| \,\mathrm{d}\,x \\
&\quad + \int_{S_i} |\operatorname{dev}[\nabla\varphi_i \odot (u_h - Ax)]| \,\mathrm{d}\,x + C|\Omega\setminus\Omega_0|^{1/q'}.
\end{aligned}$$

Since $|\Omega \setminus \Omega_0| \le \varepsilon$, we obtain

$$\begin{aligned}
\int_{S_i} |\operatorname{dev} \mathcal{E} w_{h,i}| \, \mathrm{d}\, x &\le C(\varepsilon + \varepsilon^{1/q'}) + \int_{S_i} |\operatorname{dev} \mathcal{E} u_h - \operatorname{dev} A| \, \mathrm{d}\, x \\
&\quad + \frac{4n}{R} \int_\Omega |u_h(x) - Ax| \mathbb{1}_{S_i}(x) \, \mathrm{d}\, x \\
&\le C(\varepsilon + \varepsilon^{1/q'}) + \int_{S_i} |\operatorname{dev} \mathcal{E} u_h - \operatorname{dev} A| \, \mathrm{d}\, x \\
&\quad + \frac{4n}{R} \sup_h \|u_h - Ax\|_q \, |S_i|^{1/q'} \\
&\le C(\varepsilon + \varepsilon^{1/q'}) + \int_{S_i} |\operatorname{dev} \mathcal{E} u_h - \operatorname{dev} A| \, \mathrm{d}\, x.
\end{aligned}$$

Next, we estimate the divergence term:

$$\begin{aligned}
&\int_{S_i} |\operatorname{div} w_{h,i}|^2 \, \mathrm{d}\, x \\
&\quad \le \int_{S_i} |\operatorname{tr} A + \varphi_i(\operatorname{div} u_h - \operatorname{tr} A) + \xi_{h,i}|^2 \, \mathrm{d}\, x \\
&\quad \le 3 \int_{S_i} |\operatorname{tr} A|^2 + |\operatorname{div} u_h - \operatorname{tr} A|^2 + \xi_{h,i}^2 \, \mathrm{d}\, x \\
&\quad \le 3|\operatorname{tr} A|^2 |\Omega \setminus \Omega_0| + 3 \int_{S_i} |\operatorname{div} u_h - \operatorname{tr} A|^2 \, \mathrm{d}\, x + \frac{12n^2}{R^2 |S_i|} \|u_h - Ax\|_1^2 \\
&\quad \le C\varepsilon + 3 \int_{S_i} |\operatorname{div} u_h - \operatorname{tr} A|^2 \, \mathrm{d}\, x + \frac{12n^2}{R^2 |S_i|} \|u_h - Ax\|_1^2,
\end{aligned}$$

where we used the inequality

$$\xi_{h,i}^2 = \frac{1}{|S_i|^2} \left( \int_{S_i} \nabla \varphi_i(x) \cdot (u_h(x) - Ax) \, \mathrm{d}\, x \right)^2 \le \frac{4n^2}{R^2 |S_i|^2} \|u_h - Ax\|_1^2.$$

Combining the above estimates yields

$$\begin{aligned}
\int_\Omega f(\mathcal{E} w_{h,i}) \, \mathrm{d}\, x &\le \int_\Omega f(\mathcal{E} u_h) \, \mathrm{d}\, x + M \int_{S_i} |\operatorname{dev} \mathcal{E} u_h - \operatorname{dev} A| \, \mathrm{d}\, x \\
&\quad + 3M \int_{S_i} |\operatorname{div} u_h - \operatorname{tr} A|^2 \, \mathrm{d}\, x + C(\varepsilon + \varepsilon^{1/q'}) \\
&\quad + \frac{12n^2 M}{R^2 |S_i|} \|u_h - Ax\|_1^2.
\end{aligned}$$

By Step 1 we have

$$\begin{aligned}|\Omega| f(A) &\leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E} w_{h,i}) \,\mathrm{d}\,x \\ &\leq \liminf_{h\to\infty} \left[ \int_\Omega f(\mathcal{E} u_h) \,\mathrm{d}\,x + M \int_{S_i} |\operatorname{dev} \mathcal{E} u_h - \operatorname{dev} A| \,\mathrm{d}\,x \right. \\ &\qquad \left. + 3M \int_{S_i} |\operatorname{div} u_h - \operatorname{tr} A|^2 \,\mathrm{d}\,x + \frac{12n^2 M}{R^2 |S_i|} \|u_h - Ax\|_1^2 \right] \\ &\qquad + C(\varepsilon + \varepsilon^{1/q'}).\end{aligned}$$

Since $u_h \to Ax$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^d)$, the term

$$\frac{12n^2 M}{R^2|S_i|}\|u_h - Ax\|_1^2$$

vanishes as $h \to \infty$. Summing up over $i = 1, \ldots, n$, dividing by $n$, and using the superadditivity of a lower limit yields

$$\begin{aligned}|\Omega| f(A) &\leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E} w_{h,i}) \,\mathrm{d}\,x \\ &\leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E} u_h) \,\mathrm{d}\,x + \frac{M}{n} \sup_h \int_\Omega |\operatorname{dev} \mathcal{E} u_h - \operatorname{dev} A| \,\mathrm{d}\,x \\ &\quad + \frac{3M}{n} \sup_h \int_\Omega |\operatorname{div} u_h - \operatorname{tr} A|^2 \,\mathrm{d}\,x + C(\varepsilon + \varepsilon^{1/q'}).\end{aligned}$$

Letting $\varepsilon \downarrow 0$ and $n \to \infty$ yields

$$|\Omega| f(A) \leq \liminf_{h\to\infty} \int_\Omega f(\mathcal{E} u_h) \,\mathrm{d}\,x. \qquad \square$$

**Remark 4.7.** Clearly, Lemma 4.6 also holds for affine limits.

We are now going to prove that the relaxation

$$\mathcal{F}_*[u,\Omega] := \inf\left\{ \liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] : \ (u_h) \subset \mathrm{U}(\Omega),\ u_h \overset{*}{\rightharpoonup} u \text{ in } \mathrm{U}(\Omega) \right\}$$

satisfies the lower bound

$$\mathcal{F}_*[u,\Omega] \geq \int_\Omega f(\mathcal{E} u) \,\mathrm{d}\,x + \int_\Omega f^\#_{\mathrm{dev}} \left( \frac{\mathrm{d}\, E^s u}{\mathrm{d}\,|E^s u|} \right) \mathrm{d}\,|E^s u|. \tag{4.15}$$

**Remark 4.8.** Note that the relaxation $\mathcal{F}_*$ can be written as

$$\mathcal{F}_*[u,\Omega] = \inf\left\{ \liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] : \ (u_h) \subset \mathrm{LU}(\Omega),\ u_h \to u \text{ in } \mathrm{L}^1(\Omega;\mathbb{R}^d) \right\}.$$

Indeed, if this was false, we could find a sequence $(u_h) \subset \mathrm{LU}(\Omega)$ with $u_h \to u$

strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^d)$ such that

$$\mathcal{F}_*[u,\Omega] > \lim_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] \geq \limsup_{h\to\infty} m\left(\|\operatorname{div} u_h\|_2 + \|\operatorname{dev} \mathcal{E}u_h\|_1\right),$$

where the last inequality follows from the lower bound on the integrand $f$. We see that $(u_h)$ is uniformly norm-bounded in $\mathrm{U}(\Omega)$, hence $u_h \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{U}(\Omega)$ by Lemma 2.29, whereby we get the contradiction $\mathcal{F}_*[u,\Omega] > \mathcal{F}_*[u,\Omega]$.

We also have the analogue of Proposition 3.6:

**Proposition 4.9.** *The relaxation $\mathcal{F}_*$ can be equivalently written as*

$$\mathcal{G}[u,\Omega] = \inf\Big\{ \liminf_{h\to\infty} \mathcal{F}_\infty[u_h,\Omega] : \ (u_h) \subset \mathrm{LU}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d), \\ u_h \overset{*}{\rightharpoonup} u \textit{ in } \mathrm{U}(\Omega)\Big\}.$$

**Remark 4.10.** The functional $\mathcal{F}_*$ satisfies the same invariance properties as its BD counterpart from Chapter 3 (see Remark 3.11).

In order to prove the lower bound, we appeal to Lemma 4.11 below, which asserts that for a given $u \in \mathrm{U}(\Omega)$ the map $V \mapsto \mathcal{F}_*[u,V]$ is the restriction to the open subsets of $\Omega$ of a Radon measure on $\Omega$, which we still denote by $\mathcal{F}_*[u,\cdot]$. Then, we decompose this measure into the absolutely continuous and singular parts with respect to the Lebesgue measure, i.e.

$$\mathcal{F}_*[u,\cdot] = \mathcal{F}_*^a[u,\cdot] + \mathcal{F}_*^s[u,\cdot], \quad \mathcal{F}_*^a[u,\cdot] \ll \mathscr{L}^d \llcorner \Omega, \quad \mathcal{F}_*^s[u,\cdot] \perp \mathscr{L}^d \llcorner \Omega$$

and then prove that

$$\mathcal{F}_*^a[u,B] \geq \int_B f(\mathcal{E}u)\,\mathrm{d}\,x \quad \text{and} \quad \mathcal{F}_*^s[u,B] \geq \int_B f_{\mathrm{dev}}^{\#}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|$$

for any Borel set $B \subset \Omega$.

We begin with the following technical lemma.

**Lemma 4.11.** *For all $u \in \mathrm{U}(\Omega)$ the set function $V \mapsto \mathcal{F}_*[u,V]$ is a restriction to the open subsets of $\Omega$ of a finite Radon measure.*

*Proof.* Fix $u \in \mathrm{U}(\Omega)$.

*Step 1.* Let $A', A'', B$ be open subsets of $\Omega$ such that $A' \Subset A''$. We first prove that

$$\mathcal{F}_*[u, A' \cup B] \leq \mathcal{F}_*[u, A''] + \mathcal{F}_*[u, B]. \tag{4.16}$$

Fix $\varepsilon > 0$. By the definition of relaxation we can find sequences $(u_h^\varepsilon) \subset \mathrm{LU}(A'')$ and $(v_h^\varepsilon) \subset \mathrm{LU}(B)$ such that $u_h^\varepsilon \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{U}(A'')$, $v_h^\varepsilon \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{U}(B)$,

$$\mathcal{F}[u_h^\varepsilon, A''] \leq \mathcal{F}_*[u, A''] + \varepsilon,$$

and

$$\mathcal{F}[v_h^\varepsilon, B] \leq \mathcal{F}_*[u, B] + \varepsilon.$$

Henceforth, we omit the dependence of sequences $u_h$ and $v_h$ on $\varepsilon$. For each $h \in \mathbb{N}$ extend the functions $u_h$ and $v_h$ by zero outside $A''$ and $B$, respectively. Let

$$C_\varepsilon := \sup_{h \in \mathbb{N}} \left( \int_{A''} 1 + |\operatorname{div} u_h|^2 + |\mathcal{E} u_h| \,\mathrm{d}\,x + \int_B 1 + |\operatorname{div} v_h|^2 + |\mathcal{E} v_h| \,\mathrm{d}\,x \right) < \infty. \tag{4.17}$$

Fix $k \in \mathbb{N}$ and an increasing family of open sets

$$A' = A_0 \Subset A_1 \Subset \ldots \Subset A_k \Subset A''.$$

For each $i = 1, \ldots, k$ choose the cut-off function $\varphi_i \in \mathrm{C}_c^1(A_i; [0,1])$ such that $\varphi_i \equiv 1$ on $A_{i-1}$. Next, define maps $\tilde{w}_{h,i} \in \mathrm{L}^1(A' \cup B; \mathbb{R}^d)$ via

$$\tilde{w}_{h,i} := \varphi_i u_h + (1 - \varphi_i) v_h, \quad h \in \mathbb{N}, \quad i = 1, \ldots, k.$$

It is clear that $\tilde{w}_{h,i} \in \mathrm{LU}(A_{i-1})$, but $\tilde{w}_{h,i} \notin \mathrm{LU}(A' \cup B)$, since

$$\operatorname{div} \tilde{w}_{h,i} = \varphi_i \operatorname{div} u_h + (1 - \varphi_i) \operatorname{div} v_h + \nabla \varphi_i \cdot (u_h - v_h)$$

and the last term on the right-hand side belongs only to $\mathrm{L}^{d/(d-1)}(A' \cup B)$. To overcome this problem, as before we fix some $1 < q < d/(d-1)$ and define

$$\xi_{h,i} := \frac{1}{|S_i|} \int_{S_i} \nabla \varphi_i(x) \cdot (u_h(x) - v_h(x)) \,\mathrm{d}\,x,$$

where $S_i := A_i \backslash \overline{A}_{i-1}$ for $i = 1, \dots, k$. Note that $\operatorname{supp} \nabla \varphi_i \Subset S_i$. By Theorem 4.5 applied in $S_i$ and with the right-hand side

$$f_{h,i} := -\nabla \varphi_i \cdot (u_h - v_h) + \xi_{h,i} \in \mathrm{L}^q(S_i),$$

there exist functions $z_{h,i} := \mathcal{B} f_{h,i} \in \mathrm{W}^{1,q}_0(S_i; \mathbb{R}^d)$ such that

$$\operatorname{div} z_{h,i} = f_{h,i} \quad \text{on } S_i$$

and the estimate

$$\|\nabla z_{h,i}\|_q \leq C \|f_{h,i}\|_q \tag{4.18}$$

holds. We also extend $z_{h,i}$ by zero outside $S_i$. Define

$$w_{h,i} := \tilde{w}_{h,i} + z_{h,i}.$$

The correction term $z_{h,i}$ guarantees that $w_{h,i} \in \mathrm{LU}(A' \cup B)$. Indeed,

$$\operatorname{div} w_{h,i} = \varphi_i \operatorname{div} u_h + (1 - \varphi_i) \operatorname{div} v_h + \xi_{h,i} \mathbb{1}_{S_i},$$

which clearly belongs to $\mathrm{L}^2(A' \cup B)$. We have

$$\begin{aligned} \mathcal{F}[w_{h,i}, A' \cup B] &= \int_{A' \cup B} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\, x \\ &= \int_{(A' \cup B) \cap \overline{A}_{i-1}} f(\mathcal{E} u_h) \,\mathrm{d}\, x + \int_{B \backslash A_i} f(\mathcal{E} v_h) \,\mathrm{d}\, x \\ &\quad + \int_{B \cap S_i} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\, x, \end{aligned}$$

where we used the fact that the corrector $z_{h,i}$ vanishes outside of $S_i$. Hence,

$$\mathcal{F}[w_{h,i}, A' \cup B] \leq \mathcal{F}[u_h, A''] + \mathcal{F}[v_h, B] + \int_{B \cap S_i} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\, x.$$

The last integral can be estimated as follows:

$$\begin{aligned} \int_{B \cap S_i} f(\mathcal{E} w_{h,i}) \,\mathrm{d}\, x &\leq M \int_{B \cap S_i} 1 + |\operatorname{div} w_{h,i}|^2 + |\mathcal{E} w_{h,i}| \,\mathrm{d}\, x \\ &\leq 3M \int_{B \cap S_i} 1 + |\operatorname{div} u_h|^2 + |\operatorname{div} v_h|^2 + \xi_{h,i}^2 \\ &\qquad + C_k |u_h - v_h| + |\mathcal{E} u_h| + |\mathcal{E} v_h| + |\mathcal{E} z_{h,i}| \,\mathrm{d}\, x, \end{aligned}$$

where $C_k := \sup\{\|\nabla\varphi_i\|_\infty :\ 1 \le i \le k\}$. We have for $1 \le i \le k$ that

$$
\begin{aligned}
\xi_{h,i}^2 &= \frac{1}{|S_i|^2}\left(\int_{S_i} \nabla\varphi_i(x)\cdot(u_h(x)-v_h(x))\,\mathrm{d}\,x\right)^2 \\
&\le \frac{C_k^2}{|S_i|^2}\|u_h-v_h\|_1^2 \\
&\le C_k^2|S_i|^{-2/q}\|u_h-v_h\|_q^2.
\end{aligned}
$$

Here and in all of the following the norms are with respect to the domain $A'\cup B$. Since $|S_i|>0$ for all $i\in\{1,\dots,k\}$, we get

$$
\xi_{h,i}^2 \le C_k^2\left(\min_{\ell\in\{1,\dots,k\}}|S_\ell|\right)^{-2/q}\|u_h-v_h\|_q^2 \le \tilde{C}_{\Omega,q,k}\|u_h-v_h\|_q^2.
$$

By the estimate (4.18) and Hölder's inequality we obtain similarly

$$
\int_{B\cap S_i}|\mathcal{E}z_{h,i}|\,\mathrm{d}\,x \le \|\nabla z_{h,i}\|_q\,|B\cap S_i|^{1/q'} \le \tilde{C}_{\Omega,q,k}\|u_h-v_h\|_q,
$$

where $1/q+1/q'=1$. Note that for every $h\in\mathbb{N}$ there exists $i_h\in\{1,\dots,k\}$ such that

$$
\begin{aligned}
&\int_{B\cap S_{i_h}} 1+|\operatorname{div}u_h|^2+|\operatorname{div}v_h|^2+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x \\
&\quad\le \frac{1}{k}\int_{B\cap(\overline{A}_k\setminus A_0)} 1+|\operatorname{div}u_h|^2+|\operatorname{div}v_h|^2+|\mathcal{E}u_h|+|\mathcal{E}v_h|\,\mathrm{d}\,x \\
&\quad\le \frac{C_\varepsilon}{k},
\end{aligned}
$$

where $C_\varepsilon$ is defined in (4.17). Therefore, combining the above estimates yields

$$
\int_{B\cap S_{i_h}} f(\mathcal{E}w_{h,i_h})\,\mathrm{d}\,x \le C_{\Omega,M,q,k}\left(\|u_h-v_h\|_q^2+\|u_h-v_h\|_1+\|u_h-v_h\|_q\right)+\frac{3MC_\varepsilon}{k}.
$$

Hence,

$$
\begin{aligned}
&\mathcal{F}[w_{h,i_h},A'\cup B] \\
&\quad\le \mathcal{F}[u_h,A'']+\mathcal{F}[v_h,B] \\
&\qquad + C_{\Omega,M,q,k}\left(\|u_h-v_h\|_q^2+\|u_h-v_h\|_1+\|u_h-v_h\|_q\right)+\frac{3MC_\varepsilon}{k} \\
&\quad\le \mathcal{F}_*[u,A'']+\mathcal{F}_*[u,B]+2\varepsilon \\
&\qquad + C_{\Omega,M,q,k}\left(\|u_h-v_h\|_q^2+\|u_h-v_h\|_1+\|u_h-v_h\|_q\right)+\frac{3MC_\varepsilon}{k}.
\end{aligned}
$$

Note that $w_{h,i_h} \to u$ strongly in $\mathrm{L}^1(A' \cup B; \mathbb{R}^d)$ and $(w_{h,i_h})_h$ is uniformly norm-bounded in $\mathrm{U}(A' \cup B)$. Lemma 2.29 thus implies that $(w_{h,i_h})_h$ converges weakly* to $u$ in $\mathrm{U}(A' \cup B)$. Moreover, $(u_h - v_h)_h$ converges strongly to zero in $\mathrm{L}^q(A' \cup B; \mathbb{R}^d)$. Therefore, we obtain

$$\begin{aligned}\mathcal{F}_*[u, A' \cup B] &\leq \liminf_{h\to\infty} \mathcal{F}[w_{h,i_h}, A' \cup B] \\ &\leq \mathcal{F}_*[u, A''] + \mathcal{F}_*[u, B] + \frac{3MC_\varepsilon}{k} + 2\varepsilon.\end{aligned}$$

Letting $k \to \infty$ followed by $\varepsilon \downarrow 0$ yields the inequality (4.16).

*Step 2.* We now prove that for any open subset $A \subset \Omega$ it holds that

$$\mathcal{F}_*[u, A] = \sup \{\mathcal{F}_*[u, A'] : \ A' \Subset A, \ A' \text{ open}\}. \tag{4.19}$$

It can be easily seen that

$$\mathcal{F}_*[u, A] \leq M \left((1 + |\operatorname{div} u|^2)\mathscr{L}^d(A) + |Eu|(A)\right), \tag{4.20}$$

where $M > 0$ is the constant from the upper growth bound on the integrand of $\mathcal{F}$.

Therefore, for a fixed $\varepsilon > 0$ we can choose a compact set $K \subset A$ such that $\mathcal{F}_*[u, A \setminus K] \leq \varepsilon$. Choose open sets $A'$ and $A''$ such that $K \subset A' \Subset A'' \Subset A$. By Step 1 with $B = A \setminus K$ we have

$$\mathcal{F}_*[u, A] \leq \mathcal{F}_*[u, A''] + \mathcal{F}_*[u, A \setminus K] \leq \mathcal{F}_*[u, A''] + \varepsilon$$

Letting $\varepsilon \downarrow 0$ gives (4.19).

*Step 3.* Let $A, B$ be open subsets of $\Omega$. We now prove that

$$\mathcal{F}_*[u, A \cup B] \leq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B]. \tag{4.21}$$

Fix $\varepsilon > 0$. By Step 2 there exists an open set $U \Subset A \cup B$ such that

$$\mathcal{F}_*[u, A \cup B] - \varepsilon \leq \mathcal{F}_*[u, U].$$

Choose $A' \Subset A$ open, such that $U \subset A' \cup B$. By Step 1 we have

$$\mathcal{F}_*[u, A \cup B] - \varepsilon \leq \mathcal{F}_*[u, A' \cup B] \leq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B].$$

Letting $\varepsilon \downarrow 0$ yields (4.21).

*Step 4.* Finally, we prove that for open sets $A, B$ such that $A \cap B = \emptyset$ the inequality

$$\mathcal{F}_*[u, A \cup B] \geq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B] \tag{4.22}$$

holds. We can choose a sequence $(u_h) \subset \mathrm{LU}(A \cup B)$ converging weakly* to $u \in \mathrm{U}(A \cup B)$ and such that

$$\lim_{h\to\infty} \mathcal{F}[u_h, A \cup B] = \mathcal{F}_*[u, A \cup B].$$

Since sets $A$ and $B$ are disjoint we have

$$\begin{aligned}\mathcal{F}_*[u, A \cup B] &= \lim_{h\to\infty} \mathcal{F}[u_h, A \cup B] \\ &\geq \liminf_{h\to\infty} \mathcal{F}[u_h, A] + \liminf_{h\to\infty} \mathcal{F}[u_h, B] \\ &\geq \mathcal{F}_*[u, A] + \mathcal{F}_*[u, B],\end{aligned}$$

hence we proved (4.22). By Theorem 2.1 we infer that the set function $V \mapsto \mathcal{F}_*[u, V]$ is a restriction to open sets of a finite Radon measure. $\square$

**Proposition 4.12 (Upper estimate).** *The relaxation $\mathcal{F}_*$ satisfies the upper bound*

$$\mathcal{F}_*[u, \Omega] \leq \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^\#_{\mathrm{dev}}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|.$$

*Proof.* By Remark 2.33 we can find a sequence $(u_h) \subset \mathrm{LU}(\Omega) \cap \mathrm{C}^\infty(\Omega; \mathbb{R}^d)$ converging area-strictly to $u \in \mathrm{U}(\Omega)$. Since the area-strict convergence is stronger than the weak* convergence, by the definition of $\mathcal{F}_*$, it follows that

$$\mathcal{F}_*[u, \Omega] \leq \liminf_{h\to\infty} \mathcal{F}[u_h, \Omega] = \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^\#_{\mathrm{dev}}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|,$$

where the equality follows from Remark 2.33. $\square$

The conclusion of Theorem 4.1 will follow once we prove the lower bound.

**Proposition 4.13 (Lower estimate).** *For $u \in \mathrm{U}(\Omega)$ the inequality*

$$\mathcal{F}_*[u, \Omega] \geq \int_\Omega f(\mathcal{E}u)\,\mathrm{d}\,x + \int_\Omega f^\#_{\mathrm{dev}}\left(\frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|}\right)\mathrm{d}\,|E^s u|$$

*holds.*

*Proof.* We treat separately $\mathscr{L}^d$-a.e. regular point $x_0 \in \Omega$ and $|E^s u|$-a.e. singular point $x_0 \in \Omega$.

*Regular points.* The proof is based on a blow-up argument. Fix $x_0 \in \Omega$ such that

1. $u$ is approximately differentiable at $x_0$,
2. $\lim_{r \downarrow 0} \dfrac{|Eu|(B(x_0, r))}{\omega_d r^d} = \dfrac{\mathrm{d}\,|Eu|}{\mathrm{d}\,\mathscr{L}^d}(x_0) = |\mathcal{E}u(x_0)|$,
3. $x_0$ is an $\mathscr{L}^d$-Lebesgue point of $\operatorname{div} u$.

Since $u \in \mathrm{U}(\Omega)$, these properties hold for $\mathscr{L}^d$-almost every $x \in \Omega$. In particular (1) is a consequence of Theorem 7.4 in [2], whereas (2) follows from Theorem 2.3. For $y \in B(0,1)$ define maps

$$u_r(y) := \frac{u(x_0 + ry) - \tilde{u}(x_0)}{r}, \quad 0 < r < \operatorname{dist}(x_0, \partial\Omega),$$

where $\tilde{u}$ is the precise representative of $u$. For $u_0(y) := \nabla u(x_0) y$ we have the strong convergence $u_r \to u_0$ in $\mathrm{L}^1(B(0,1); \mathbb{R}^d)$. Indeed, by the approximate differentiability we have

$$\begin{aligned}
&\int_{B(0,1)} |u_r(y) - u_0(y)| \;\mathrm{d}\, y \\
&\qquad = \frac{1}{r^d} \int_{B(x_0,r)} \frac{|u(z) - \tilde{u}(x_0) - \nabla u(x_0)(z - x_0)|}{r} \;\mathrm{d}\, z \to 0
\end{aligned}$$

as $r \downarrow 0$. Moreover, we have strict convergence:

$$\lim_{r \downarrow 0} |Eu_r|(B(0,1)) = \omega_d \lim_{r \downarrow 0} \frac{|Eu|(B(x_0, r))}{\omega_d r^d} = \omega_d |\mathcal{E}u(x_0)| = |Eu_0|(B(0,1)),$$

thus $(u_r)$ is bounded in $\mathrm{BD}(B(0,1))$. Note that for $\varphi \in \mathrm{L}^2(B(0,1))$ we have

$$\begin{aligned}
&\left| \int_{B(0,1)} \varphi(y)(\operatorname{div} u_r(y) - \operatorname{div} u_0(y)) \;\mathrm{d}\, y \right| \\
&\qquad \le \|\varphi\|_2 \int_{B(0,1)} |\operatorname{div} u(x_0 + ry) - \operatorname{div} u(x_0)|^2 \;\mathrm{d}\, y \\
&\qquad = \omega_d \|\varphi\|_2 \fint_{B(x_0,r)} |\operatorname{div} u(z) - \operatorname{div} u(x_0)|^2 \;\mathrm{d}\, z.
\end{aligned}$$

The right-hand side vanishes as $r \downarrow 0$ by the Lebesgue point property (3). Hence, $u_r \overset{*}{\rightharpoonup} u_0$ weakly* in $\mathrm{U}(B(0,1))$. In virtue of Proposition 2.15, Lemma 4.6 and the scaling properties of $\mathcal{F}_*$ we obtain

$$\begin{aligned}\liminf_{r\downarrow 0} \frac{\mathcal{F}_*[u, B(x_0, r)]}{r^d} &= \liminf_{r\downarrow 0} \mathcal{F}_*[u_r, B(0,1)] \\ &\geq \mathcal{F}_*[u_0, B(0,1)] \\ &\geq \int_{B(0,1)} f(\mathcal{E}u_0(y)) \,\mathrm{d}\, y \\ &= \omega_d f(\mathcal{E}u(x_0)).\end{aligned}$$

Therefore, by Proposition 2.2 we obtain

$$\mathcal{F}^a_*[u, B] \geq \int_B f(\mathcal{E}u) \,\mathrm{d}\, x$$

for any Borel set $B \subset \Omega$.

*Singular points.* We want to prove that for all Borel sets $B \subset \Omega$ the inequality

$$\mathcal{F}^s_*[u, B] \geq \int_B f^{\#}_{\mathrm{dev}} \left( \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u|$$

holds. In order to do that we fix $x_0 \in \Omega$ such that

$$\frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|}(x_0) = a \odot b, \qquad a, b \in \mathbb{R}^d \setminus \{0\},\ a \perp b.$$

This property holds for $|E^s u|$-a.e. $x_0 \in \Omega$ by Theorem 2.26 (see also Remark 2.27). It suffices to establish the inequality

$$\lim_{r\downarrow 0} \frac{\mathcal{F}_*[u, B(x_0, r)]}{|Eu|(B(x_0, r))} \geq f^{\#}_{\mathrm{dev}}(a \odot b)$$

at any $|Eu|$-Lebesgue point $x_0 \in \Omega$ for which the limit on the left-hand side exists. By the coercivity of $\mathcal{F}$ and a diagonal argument similar to the one contained in the proof of Lemma 2.14, we can choose a sequence $(u_h) \subset \mathrm{LU}(B(x_0, r))$ such that $u_h \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{U}(B(x_0, r))$ and

$$\lim_{h\to\infty} \mathcal{F}[u_h, B(x_0, r)] = \mathcal{F}_*[u, B(x_0, r)].$$

We then have

$$
\begin{aligned}
&\mathcal{F}_*[u, B(x_0, r)] \\
&\quad = \lim_{h\to\infty} \mathcal{F}[u_h, B(x_0, r)] \\
&\quad = \lim_{h\to\infty} \int_{B(x_0,r)} f(\mathcal{E}u_h) \,\mathrm{d}\, x \\
&\quad = \lim_{h\to\infty} \int_{B(x_0,r)} f(\mathcal{E}u_h) - f^{\#}_{\mathrm{dev}}(\operatorname{dev}\mathcal{E}u_h) \,\mathrm{d}\, x + \int_{B(x_0,r)} f^{\#}_{\mathrm{dev}}(\operatorname{dev}\mathcal{E}u_h) \,\mathrm{d}\, x \\
&\quad =: \lim_{h\to\infty} \left(I^{(1)}_{h,r} + I^{(2)}_{h,r}\right).
\end{aligned}
$$

In virtue of (4.3) we have

$$
\begin{aligned}
I^{(1)}_{h,r} &= \int_{B(x_0,r)} f(\mathcal{E}u_h) - f^{\#}_{\mathrm{dev}}(\operatorname{dev}\mathcal{E}u_h) \,\mathrm{d}\, x \\
&\geq -M \int_{B(x_0,r)} |\operatorname{div} u_h|^{\gamma} + |\operatorname{dev}\mathcal{E}u_h|^{\delta} + 1 \,\mathrm{d}\, x.
\end{aligned}
$$

We can assume that

$$
|\operatorname{dev}\mathcal{E}u_h|^{\delta} \rightharpoonup \xi \quad \text{weakly in } \mathrm{L}^{1/\delta}(B(x_0, r))
$$

for some $\xi \in \mathrm{L}^{1/\delta}(B(x_0, r))$.

For $0 \leq \gamma < 2$ by Hölder's inequality we obtain

$$
\int_{B(x_0,r)} |\operatorname{div} u_h|^{\gamma} \,\mathrm{d}\, x \leq \sup_h \|\operatorname{div} u_h\|_2^{\gamma} \, |B(x_0, r)|^{1-\gamma/2}.
$$

Thus,

$$
\lim_{h\to\infty} I^{(1)}_{h,r} \geq -C_{M,\gamma} \left( |B(x_0, r)|^{1-\gamma/2} + |B(x_0, r)| + \int_{B(x_0,r)} \xi \,\mathrm{d}\, x \right).
$$

Therefore,

$$
\lim_{r\downarrow 0} \lim_{h\to\infty} I^{(1)}_{h,r} \geq 0.
$$

By Proposition 2.11 the set

$$
\mathcal{S} := \left\{ a \odot b : \; a, b \in \mathbb{R}^d, \; a \cdot b = 0 \right\}
$$

spans the space of symmetric and deviatoric matrices $\mathrm{SD}(d)$. Moreover, the recession function $f^{\#}_{\mathrm{dev}}$ is positively 1-homogeneous and convex at points of $\mathcal{S}$ (see Remark 4.2). In virtue of Theorem 2.10 for each orthogonal $a, b \in \mathbb{R}^d$ there exists a linear function $\ell : \mathrm{SD}(d) \to \mathbb{R}$ such that $f^{\#}_{\mathrm{dev}}(D) \geq \ell(D)$ for all

$D \in \mathrm{SD}(d)$ and $f^{\#}_{\mathrm{dev}}(a \odot b) = \ell(a \odot b)$. For all but finitely many $r > 0$ we can assume that $\Lambda(\partial B(x,r)) = 0$, where $\Lambda \in \mathrm{M}^+(\Omega)$ is the weak* limit of (a subsequence of) the measures $|\ell(\mathrm{dev}(\mathcal{E}u_h))|\mathscr{L}^d$. Therefore, we have

$$\begin{aligned}\lim_{h\to\infty} I^{(2)}_{h,r} &= \lim_{h\to\infty} \int_{B(x_0,r)} f^{\#}_{\mathrm{dev}}(\mathrm{dev}\,\mathcal{E}u_h)\,\mathrm{d}\,x\\ &\geq \limsup_{h\to\infty} \int_{B(x_0,r)} \ell(\mathrm{dev}\,\mathcal{E}u_h)\,\mathrm{d}\,x\\ &= \ell(\mathrm{dev}\,Eu(B(x_0,r))),\end{aligned}$$

where the last equality follows from the linearity of $\ell$. Combining the above estimates yields

$$\begin{aligned}\lim_{r\downarrow 0} \frac{\mathcal{F}_*[u, B(x_0,r)]}{|Eu|(B(x_0,r))} &\geq \limsup_{r\downarrow 0} \frac{\ell(\mathrm{dev}\,Eu(B(x_0,r)))}{|Eu|(B(x_0,r))}\\ &= \limsup_{r\downarrow 0} \ell\left(\mathrm{dev}\left(\frac{Eu(B(x_0,r))}{|Eu|(B(x_0,r))}\right)\right)\\ &= \ell\left(\mathrm{dev}\left(\lim_{r\downarrow 0} \frac{Eu(B(x_0,r))}{|Eu|(B(x_0,r))}\right)\right)\\ &= \ell\left(\mathrm{dev}(a \odot b)\right)\\ &= \ell(a \odot b)\\ &= f^{\#}_{\mathrm{dev}}(a \odot b).\end{aligned}$$

This finishes the proof. □

## Conclusion

In this chapter we established the relaxation result for integral functionals with symmetric-quasiconvex integrands satisfying the mixed-growth condition. We proved an integral representation of the relaxation with respect to the weak* convergence in the Temam-Strang space. This result extends the previous result by Jesenko and Schmidt in [23]. Our proof is a mixture of the standard blow-up argument and the convexity argument based on the Kirchheim-Kristensen result [24]. This argument was possible due to the subcritical assumption (3) in Theorem 4.1. It remains an open question, whether this condition can be deduced from conditions (1) and (2) of the aforementioned theorem.

# Chapter 5

# Inhomogeneous Henky's model

## 5.1 Introduction

Recall that the classical minimisation problem in the theory of Hencky plasticity (cf. [5, 38]) involves the following convex functional:

$$\int_\Omega \varphi(\operatorname{dev}\mathcal{E}u) + \frac{\kappa}{2}(\operatorname{div} u)^2 \,\mathrm{d}\,x, \tag{5.1}$$

where $\varphi : \mathrm{SD}(d) \to [0, +\infty)$ is a convex function which grows quadratically on some compact set and linearly outside of this set, and $\kappa = \lambda + 2\mu/3$ is the bulk modulus of the material with the Lamé constants $\lambda$ and $\mu$.

In this chapter we generalise the energy functional to the following inhomogeneous one:

$$\mathcal{G}[u] := \int_\Omega g(x, \operatorname{dev}\mathcal{E}u) + h(x, \operatorname{div} u) \,\mathrm{d}\,x, \tag{5.2}$$

where the functions $g$ and $h$ satisfy certain continuity, convexity and growth properties (see Theorem 5.1 for the precise formulation).

As in Hencky's plasticity, a natural underlying function space for the functional $\mathcal{G}$ is the space $\mathrm{LU}(\Omega)$. Unfortunately, it is not possible to apply the direct method to $\mathcal{G}$ in this space. By Theorem 2.32, the functional $\mathcal{G}$ extends continuously, with respect to the area-strict convergence, to the functional defined on the Temam-Strang space $\mathrm{U}(\Omega)$:

$$\overline{\mathcal{G}}[u] := \int_\Omega g(x, \operatorname{dev} \mathcal{E}u) + h(x, \operatorname{div} u) \, \mathrm{d}\, x + \int_\Omega (g \circ \operatorname{dev})^{\#} \left( x, \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u|, \tag{5.3}$$

where

$$(g \circ \operatorname{dev})^{\#}(x, A) := \limsup_{\substack{(x', A') \to (x, A) \\ s \to \infty}} \frac{g(x', s \operatorname{dev} A')}{s}, \quad A \in \mathrm{SD}(d) \,.$$

The main result of this chapter is the following weak* lower semicontinuity:

**Theorem 5.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and*

1. *the function $g : \Omega \times \mathbb{R}^{d \times d}_{\mathrm{sym}} \to [0, +\infty)$ is Carathéodory with linear growth:*

$$m|A| \leq g(x, A) \leq M(1 + |A|), \quad (x, A) \in \Omega \times \mathbb{R}^{d \times d}_{\mathrm{sym}},$$

   *for some constants $0 < m \leq M$;*

2. *for every $x \in \Omega$ the map $A \mapsto g(x, \operatorname{dev} A)$ is symmetric-quasiconvex;*
3. *the strong recession function $(g \circ \operatorname{dev})^\infty$, defined as the limit*

$$(g \circ \operatorname{dev})^{\infty}(x, A) := \lim_{\substack{(x', A') \to (x, A) \\ s \to \infty}} \frac{g(x', s \operatorname{dev} A')}{s}, \quad A \in \mathrm{SD}(d),$$

   *exists and is jointly continuous;*

4. *the function $h : \Omega \times \mathbb{R} \to [0, +\infty)$ is Carathéodory, convex and has quadratic growth*

$$0 \leq h(x, z) \leq M(1 + |z|^2), \quad (x, z) \in \Omega \times \mathbb{R}.$$

*Then, the functional*

$$\overline{\mathcal{G}}[u] := \int_\Omega g(x, \operatorname{dev} \mathcal{E}u) + h(x, \operatorname{div} u) \, \mathrm{d}\, x + \int_\Omega (g \circ \operatorname{dev})^{\infty} \left( x, \frac{\mathrm{d}\, E^s u}{\mathrm{d}\, |E^s u|} \right) \mathrm{d}\, |E^s u| \tag{5.4}$$

*is weakly* lower semicontinuous on* $\mathrm{U}(\Omega)$.

**Remark 5.2.** Note that in Theorem 5.1 the upper recession function $g^{\#}$ is replaced with the strong recession function in the functional $\overline{\mathcal{G}}$. This requirement comes from the theory of generalised Young measures, which is used here to prove Theorem 5.1, and cannot be dropped.

## 5.2 Young measures

In this section we briefly recall basics of the theory of generalised Young measures. This exposition is based on [27, 35], where one can find all the proofs of results mentioned here.

In all the following we assume that $\Omega \subset \mathbb{R}^d$ is an open bounded Lipschitz domain, unless stated otherwise.

For a function $f \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and a function $g \in \mathrm{C}(\overline{\Omega} \times \mathbb{B}^{d\times d}_{\mathrm{sym}})$ we define a linear operator $S : \mathrm{C}(\overline{\Omega} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}) \to \mathrm{C}(\overline{\Omega} \times \mathbb{B}^{d\times d}_{\mathrm{sym}})$

$$(Sf)(x, \hat{A}) := (1 - |\hat{A}|) f\left(x, \frac{\hat{A}}{1 - |\hat{A}|}\right)$$

for $(x, \hat{A}) \in \overline{\Omega} \times \mathbb{B}^{d\times d}_{\mathrm{sym}}$, and its inverse $S^{-1}$,

$$(S^{-1}g)(x, A) := (1 + |A|) g\left(x, \frac{A}{1 + |A|}\right)$$

for $(x, A) \in \overline{\Omega} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Clearly $S^{-1}Sf = f$ and $SS^{-1}g = g$.

Now we define a space of admissible integrands,

$$\mathrm{E}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) := \left\{ f \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}) : \ Sf \in \mathrm{C}(\overline{\Omega \times \mathbb{B}^{d\times d}_{\mathrm{sym}}}) \right\}. \tag{5.5}$$

Here $Sf \in \mathrm{C}(\overline{\Omega \times \mathbb{B}^{d\times d}_{\mathrm{sym}}})$ is to be understood as the statement that $Sf$ extends to a bounded and continuous function on $\overline{\Omega \times \mathbb{B}^{d\times d}_{\mathrm{sym}}}$. Note that the recession function $f^\infty$ (if it exists) is a unique extension of the function $Sf$ to $\overline{\Omega \times \mathbb{B}^{d\times d}_{\mathrm{sym}}}$, i.e. for $(x, A) \in \overline{\Omega} \times \partial\mathbb{B}^{d\times d}_{\mathrm{sym}}$,

$$\lim_{\substack{(x', A') \to (x, A) \\ |A'| < 1}} (1 - |A'|) f\left(x', \frac{A'}{1 - |A'|}\right) = f^\infty(x, A).$$

Therefore, one can equivalently express the fact that $f \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$ lies in $\mathrm{E}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ by requiring $f^\infty$ to exist.

**Definition 5.3 (Young measure).** *A* generalised Young measure *on an open set $\Omega \subset \mathbb{R}^d$ with values in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a triple $\boldsymbol{\nu} := (\nu_x, \lambda_\nu, \nu_x^\infty)$ with*

1. *a parametrized family of probability measures $(\nu_x)_{x\in\Omega} \subset \mathrm{M}^1(\mathbb{R}^{d\times d}_{\mathrm{sym}})$, called the* oscillation measure*;*

2. *a positive finite measure* $\lambda_\nu \in \mathrm{M}^+(\overline{\Omega})$, *called the* concentration measure*;*

3. *a parametrized family of probability measures* $(\nu_x^\infty)_{x\in\overline{\Omega}} \subset \mathrm{M}^1(\partial\mathbb{B}^{d\times d}_{\mathrm{sym}})$, *called the* concentration-direction measure,

*such that the following conditions hold:*

1. *the map* $x \mapsto \nu_x$ *is weakly** $\mathscr{L}^d \llcorner \Omega$*-measurable, i.e. the function* $x \mapsto \langle f(x,\cdot), \nu_x\rangle$ *is* $\mathscr{L}^d \llcorner \Omega$*-measurable for all bounded Borel functions* $f : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$*;*
2. *the map* $x \mapsto \nu_x^\infty$ *is weakly** $\lambda_\nu$*-measurable (defined analogously to (1));*
3. *the map* $x \mapsto \langle |\cdot|, \nu_x\rangle$ *is in* $\mathrm{L}^1(\Omega)$*.*

*We denote by* $\mathrm{Y}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *the set of all such generalised Young measures.*

The generalised Young measures $\boldsymbol{\nu} \in \mathrm{Y}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ are dual objects to functions $f \in \mathrm{E}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ via the duality pairing

$$\begin{aligned}\langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle &:= \int_\Omega \langle f(x,\cdot), \nu_x\rangle \,\mathrm{d}\, x + \int_{\overline{\Omega}} \langle f^\infty(x,\cdot), \nu_x^\infty\rangle \,\mathrm{d}\,\lambda_\nu(x) \\ &:= \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f(x,A) \,\mathrm{d}\,\nu_x(A)\,\mathrm{d}\, x \\ &\quad + \int_{\overline{\Omega}} \int_{\partial\mathbb{B}^{d\times d}_{\mathrm{sym}}} f^\infty(x,A)\,\mathrm{d}\,\nu_x^\infty(A)\,\mathrm{d}\,\lambda_\nu(x).\end{aligned}$$

**Definition 5.4 (Generation).** *Let* $(\gamma_j) \subset \mathrm{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *be a sequence of Radon measures. We say that* $\gamma_j$ generates *a generalised Young measure* $\boldsymbol{\nu} \in \mathrm{Y}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *(in symbols* $\gamma_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$*), if for all* $f \in \mathrm{E}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$*,*

$$\begin{aligned}& f\left(x, \frac{\mathrm{d}\,\gamma_j}{\mathrm{d}\,\mathscr{L}^d}\right)\mathscr{L}^d \llcorner \Omega + f^\infty\left(x, \frac{\mathrm{d}\,\gamma_j^s}{\mathrm{d}\,|\gamma_j^s|}\right)|\gamma_j^s| \\ &\quad \overset{*}{\rightharpoonup} \langle f(x,\cdot),\nu_x\rangle \mathscr{L}^d \llcorner \Omega + \langle f^\infty(x,\cdot), \nu_x^\infty\rangle \lambda_\nu \qquad \text{in } \mathrm{M}(\overline{\Omega}).\end{aligned} \tag{5.6}$$

**Definition 5.5 (Elementary Young measure).** *Let* $\gamma \in \mathrm{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *be a Radon measure. Then, the triple* $\boldsymbol{\delta}[\gamma] := (\delta_A, |\gamma^s|, \delta_B) \in \mathrm{Y}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *with*

$$A := \frac{\mathrm{d}\,\gamma}{\mathrm{d}\,\mathscr{L}^d} \in \mathrm{L}^1(\Omega, \mathscr{L}^d \llcorner \Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \quad B := \frac{\mathrm{d}\,\gamma^s}{\mathrm{d}\,|\gamma^s|} \in \mathrm{L}^1(\overline{\Omega}, |\gamma^s|; \mathbb{R}^{d\times d}_{\mathrm{sym}})$$

*is called the* $\gamma$-elementary Young measure.

Note that the convergence (5.6) can be now rephrased as $\langle\!\langle f, \boldsymbol{\delta}[\gamma_j]\rangle\!\rangle \to \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle$ for all $f \in \mathrm{E}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

The following result is a cornerstone of the generalised Young measure theory.

**Theorem 5.6 (Fundamental Theorem).** *Let $(\gamma_j) \subset \mathrm{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be a uniformly bounded (in the total variation norm) sequence of Radon measures. Then, there exist a (not relabelled) subsequence of $(\gamma_j)$ and a Young measure $\boldsymbol{\nu} \in \mathrm{Y}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that $\gamma_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$.*

It turns out that it suffices to test the Young measure convergence $\gamma_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$ with a countable family of functions in $\mathrm{E}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ of a particular form.

**Lemma 5.7 (Density).** *There exists a countable set of functions $\{f_k\} = \{\varphi_k \otimes h_k : \ k \in \mathbb{N}\} \subset \mathrm{E}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, where $\varphi_k \in \mathrm{C}(\overline{\Omega})$ and $h_k \in \mathrm{C}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$, such that the knowledge of $\langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle$ completely determines the Young measure $\boldsymbol{\nu} \in \mathrm{Y}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. Moreover, the functions $h_k$ can be taken Lipschitz continuous.*

We have the following extended Young measure limit representation.

**Proposition 5.8 (Extended representation).** *Let $(\gamma_j) \subset \mathrm{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be a sequence of Radon measures generating a generalised Young measure $\boldsymbol{\nu} \in \mathrm{Y}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. Let $f : \overline{\Omega} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ be a Carathéodory function such that the recession function $f^\infty$ exists and is jointly continuous. Then*

$$\lim_{j\to\infty} \langle\!\langle f, \boldsymbol{\delta}[\gamma_j]\rangle\!\rangle = \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle. \tag{5.7}$$

As a consequence of Theorem 4.1, we can establish the following Jensen-type inequalities.

**Proposition 5.9.** *Let $(u_j) \subset \mathrm{U}(\Omega)$ be a sequence such that $u_j \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{U}(\Omega)$. Let $Eu_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$ for some Young measure $\boldsymbol{\nu} = (\nu_x, \lambda_\nu, \nu_x^\infty) \in \mathrm{Y}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, where*

$$\lambda_\nu = \lambda_\nu^a + \lambda_\nu^s \quad \textit{with} \quad \lambda_\nu^a \ll \mathscr{L}^d \llcorner \Omega, \quad \lambda_\nu^s \perp \mathscr{L}^d \llcorner \Omega$$

*and let $(\operatorname{div} u_j)$ generate a classical Young measure $(\mu_x)_{x\in\Omega}$ (see [35] for details).*

*Then, it holds that*

$$g \circ \operatorname{dev}(\mathcal{E}u) + h(\operatorname{div} u) \leq \langle g \circ \operatorname{dev}, \nu_x \rangle + \langle (g \circ \operatorname{dev})^\infty, \nu_x^\infty \rangle \frac{\mathrm{d}\,\lambda_\nu^a}{\mathrm{d}\,\mathscr{L}^d} + \langle h, \mu_x \rangle \tag{5.8}$$

*for $\mathscr{L}^d$-a.e. $x \in \Omega$ and*

$$(g \circ \operatorname{dev})^\infty \left( \frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|} \right) |E^s u| \leq \langle (g \circ \operatorname{dev})^\infty, \nu_x^\infty \rangle \lambda_\nu^s \quad \textit{as measures,} \tag{5.9}$$

*for all continuous functions $g \in \mathrm{C}(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $h \in \mathrm{C}(\mathbb{R})$ such that*

1. *$g \circ \operatorname{dev}$ is symmetric-quasiconvex and the recession function $(g \circ \operatorname{dev})^\infty$ exists,*
2. *$h$ is convex and bounded from below.*

*Proof.* This proposition follows directly from the Jensen-type inequalities for BD-Young measures, see Theorem 4 in [33] (for $\boldsymbol{\nu}$) together with the classical Jensen inequality (for $\mu$). □

We are now ready to prove Theorem 5.1.

*Proof.* Let $u_j \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{U}(\Omega)$. Selecting a subsequence if necessary, we can assume that $(Eu_j)_j$ generates a generalised Young measure $\boldsymbol{\nu} \in \mathbf{Y}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and that $(\operatorname{div} u_j)_j$ generates a classical Young measure $(\mu_x)_{x\in\Omega}$.

Then,

$$\begin{aligned}
\liminf_{j\to\infty} \overline{\mathcal{G}}[u_j] &\geq \liminf_{j\to\infty} \langle\!\langle g \circ \operatorname{dev}, \boldsymbol{\delta}[Eu_j] \rangle\!\rangle + \int_\Omega \langle h(x,\cdot), \mu_x \rangle \,\mathrm{d}\,x \\
&= \int_\Omega \left( \langle g \circ \operatorname{dev}(x,\cdot), \nu_x \rangle + \langle h(x,\cdot), \mu_x \rangle \right) \mathrm{d}\,x \\
&\quad + \int_\Omega \langle (g \circ \operatorname{dev})^\infty(x,\cdot), \nu_x^\infty \rangle \,\mathrm{d}\,\lambda_\nu(x) \\
&= \int_\Omega \Bigg[ \langle g \circ \operatorname{dev}(x,\cdot), \nu_x \rangle + \langle (g \circ \operatorname{dev})^\infty(x,\cdot), \nu_x^\infty \rangle \frac{\mathrm{d}\,\lambda_\nu}{\mathrm{d}\,\mathscr{L}^d}(x) \\
&\quad + \langle h(x,\cdot), \mu_x \rangle \Bigg] \mathrm{d}\,x + \int_\Omega \langle (g \circ \operatorname{dev})^\infty(x,\cdot), \nu_x^\infty \rangle \,\mathrm{d}\,\lambda_\nu^s(x) \\
&\geq \int_\Omega g \circ \operatorname{dev}(x, \mathcal{E}u) + h(x, \operatorname{div} u) \,\mathrm{d}\,x \\
&\quad + \int_\Omega (g \circ \operatorname{dev})^\infty \left( x, \frac{\mathrm{d}\,E^s u}{\mathrm{d}\,|E^s u|} \right) \mathrm{d}\,|E^s u| \\
&= \overline{\mathcal{G}}[u],
\end{aligned}$$

where, by a slight abuse of notation, we write $g \circ \operatorname{dev}(x, A) := g(x, \operatorname{dev} A)$. The first equality follows from Proposition 5.8, and the second inequality is a consequence of Proposition 5.9. Since the above holds for any subsequence, this ends the proof of the lower semicontinuity of $\overline{\mathcal{G}}$ in $\mathrm{U}(\Omega)$. $\square$

## 5.3 Young measures revisited

In this chapter we applied techniques from the theory of Young measures to the mixed growth functional. This functional, however, has a very concrete form, where deviatoric part and trace part are additive components.

A natural question is whether one can apply the theory of Young measures to more general (inhomogeneous) functionals with mixed growth integrands, similar to the ones investigated in Chapter 4. The development of generalised Young measure theory for mixed-growth integrands is not a trivial matter. In fact problems already arise at the level of the functional analytic framework (cf. [27]). Since we are interested in the case of U-Young measures, we can use the existing framework of BD-Young measures – every U-Young measure (i.e. a generalised Young measure generated by a sequence in U) is necessarily a BD-Young measure.

In this section we present a few results regarding U-Young measures, which may shed some light on the path to further developments.

We begin with the definition.

**Definition 5.10.** *We say that $\boldsymbol{\nu}$ is a* U*-Young measure, in symbols $\boldsymbol{\nu} \in \mathrm{UY}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, if there exists a bounded sequence $(u_j) \subset \mathrm{U}(\Omega)$ such that for all $f \in \mathrm{E}(\Omega; \mathbb{R}^d)$ the convergence $\langle\!\langle f, \boldsymbol{\delta}[Eu_j] \rangle\!\rangle \to \langle\!\langle f, \boldsymbol{\nu} \rangle\!\rangle$ holds.*

Firstly, we want to establish that there are 'good' generating sequences for U-Young measures. The following lemma asserts that Young measures generated by sequences in $\mathrm{LU}(\Omega)$ and sequences in $\mathrm{U}(\Omega)$ coincide.

**Lemma 5.11.** $\mathrm{UY}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) = \mathrm{LUY}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$.

*Proof.* It suffices to prove that $\mathrm{UY}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \subset \mathrm{LUY}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Let $\boldsymbol{\nu} \in \mathrm{UY}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be a U-Young measure such that $Eu_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$ for some sequence $(u_j) \subset \mathrm{U}(\Omega)$. For each $j \in \mathbb{N}$ we can find a smooth function $v_j \in \mathrm{LU}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d)$ such that

$$\left| \int_\Omega f_k(x, \mathcal{E}u_j)\,\mathrm{d}\,x + \int_\Omega f_k^\infty\left(x, \frac{\mathrm{d}\,E^s u_j}{\mathrm{d}\,|E^s u_j|}\right)\mathrm{d}\,|E^s u_j| - \int_\Omega f_k(x, \mathcal{E}v_j)\,\mathrm{d}\,x \right| \leq \frac{1}{j}$$

for $k \leq j$. This is a consequence of Theorem 3.7 and the fact that smooth functions are area-strictly dense in $\mathrm{U}(\Omega)$. Hence

$$\lim_{j\to\infty} \int_\Omega f_k(x, \mathcal{E}v_j)\,\mathrm{d}\,x = \langle\!\langle f_k, \boldsymbol{\nu} \rangle\!\rangle$$

for all $k \in \mathbb{N}$. In virtue of Lemma 5.7, $Ev_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$, so $\boldsymbol{\nu} \in \mathrm{LUY}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. □

The following proposition is due to M. Jesenko and B. Schmidt:

**Proposition 5.12.** *Let $\boldsymbol{\nu} \in \mathrm{UY}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be a* U*-Young measure. Then, there exists a bounded sequence $(u_j) \subset \mathrm{LU}(\Omega)$ such that $Eu_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$ and $(|\operatorname{div} u_j|^2)$ is equiintegrable.*

*Proof.* Let $(u_j) \subset \mathrm{LU}(\Omega) \cap \mathrm{C}^\infty(\Omega;\mathbb{R}^d)$ be a bounded sequence such that $Eu_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$ for some $\boldsymbol{\nu} \in \mathrm{UY}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. By the embedding $\mathrm{BD}(\Omega) \subset \mathrm{L}^{d/(d-1)}(\Omega;\mathbb{R}^d)$, we have that $(u_j) \subset \mathrm{L}^{d/(d-1)}(\Omega;\mathbb{R}^d)$. By the Helmholtz decomposition, we can write

$$u_j = v_j + \nabla\varphi_j,$$

where $v_j \in \mathrm{L}^{d/(d-1)}(\Omega;\mathbb{R}^d)$ with $\operatorname{div} v_j = 0$ and $\varphi_j \in \mathrm{W}^{1,d/(d-1)}_0(\Omega)$. Since $(\operatorname{div} u_j)$ is uniformly $\mathrm{L}^2$-bounded and $\operatorname{div} u_j = \Delta\varphi_j$, we obtain that $\varphi_j \in (\mathrm{W}^{1,d/(d-1)}_0 \cap \mathrm{W}^{2,2})(\Omega)$ and $\sup_j \|\varphi_j\|_{2,2} < \infty$.

Let $w_j := \nabla\varphi_j$. Then, up to a subsequence, $w_j \rightharpoonup w$ weakly in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)$ for some $w \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)$. According to the decomposition lemma (cf. [20, Lemma 1.2], [26]) there exists a further subsequence $(w_{j_k})$ and a sequence $(\tilde{w}_k) \subset w + \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^d)$ such that

1. $\tilde{w}_k \rightharpoonup w$ in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)$,
2. $(|\nabla\tilde{w}_k|^2)$ is equiintegrable,

3. $\lim_{k\to\infty} |\{x : \ w_{j_k} \neq \tilde{w}_k \text{ or } \nabla w_{j_k} \neq \nabla \tilde{w}_k\}| = 0.$

Next, define a sequence

$$\tilde{u}_k := v_{j_k} + \tilde{w}_k.$$

It is clear that $(\tilde{u}_k)$ is a bounded sequence in $\mathrm{LU}(\Omega)$ with $(\operatorname{div}\tilde{u}_k)^2 = (\operatorname{div}\tilde{w}_k)^2$ being equiintegrable. It remains to prove that $E\tilde{u}_k \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$.

By Lemma 5.7, there exists a countable family of functions of the form $\xi \otimes h$ with $\xi \in \mathrm{C}(\overline{\Omega})$ and $h : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ Lipschitz continuous, which determines the generated BD-Young measure. We thus have

$$\lim_{j\to\infty} \langle\!\langle \xi \otimes h, \boldsymbol{\delta}[Eu_j] \rangle\!\rangle = \langle\!\langle \xi \otimes h, \boldsymbol{\nu} \rangle\!\rangle.$$

On the other hand

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega |h(\mathcal{E}u_{j_k}(x)) - h(\mathcal{E}\tilde{u}_k(x))| \,\mathrm{d}\,x \\
&\quad\le \lim_{k\to\infty} \int_\Omega \mathrm{Lip}(h) |\mathcal{E}u_{j_k}(x) - \mathcal{E}\tilde{u}_k(x)| \,\mathrm{d}\,x \\
&\quad\le \mathrm{Lip}(h) \lim_{k\to\infty} \int_{\{\nabla w_{j_k} \neq \nabla \tilde{w}_k\}} |\nabla w_{j_k}(x) - \nabla \tilde{w}_k(x)| \,\mathrm{d}\,x \\
&\quad= 0,
\end{aligned}$$

where $\mathrm{Lip}(h)$ denotes the Lipschitz constant of $h$. The last equation follows from the fact that $(\nabla w_{j_k} - \nabla \tilde{w}_k)$ is bounded in $\mathrm{L}^2(\Omega; \mathbb{R}^{d\times d})$ and is therefore equiintegrable. Consequently, $(E\tilde{u}_k)$ generates the same U-Young measure as the initial sequence. $\square$

**Proposition 5.13.** *Let $\boldsymbol{\nu} \in \mathrm{UY}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. Then,*

$$\int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f(x, A) \,\mathrm{d}\,\nu_x(A) \,\mathrm{d}\,x < \infty, \tag{5.10}$$

*for any Carathéodory function $f : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ satisfying*

$$0 \le f(x, A) \le M(1 + |\operatorname{tr} A|^2), \quad M > 0$$

*for $x \in \Omega$ and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.*

*Proof.* Let $(u_j) \subset \mathrm{LU}(\Omega)$ be a bounded sequence such that $Eu_j \overset{\mathbf{Y}}{\to} \boldsymbol{\nu}$ and $(|\operatorname{div} u_j|^2)$ is equiintegrable. Fix $k \in \mathbb{N}$ and define functions:

$$f_k(x,A) := \begin{cases} f(x,A) & \text{for } |\operatorname{tr} A| \leq k, \\ \frac{1+k^2}{1+|\operatorname{tr} A|^2} f(x,A) & \text{for } k < |\operatorname{tr} A|. \end{cases}$$

Then, for each $k \in \mathbb{N}$, $f_k$ is a Carathéodory function, $f_k \uparrow f$ and $f_k^\infty \equiv 0$, since

$$0 \leq f_k(x,A) \leq M(1+k^2)$$

for any $x \in \Omega$ and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. We also have the estimate:

$$\begin{aligned} |f_k(x,A) - f(x,A)| &\leq \mathbb{1}_{\{\operatorname{tr} A:\ |\operatorname{tr} A|>k\}}(\operatorname{tr} A) \frac{|\operatorname{tr} A|^2 - k^2}{1+|\operatorname{tr} A|^2} f(x,A) \\ &\leq M \mathbb{1}_{\{\operatorname{tr} A:\ |\operatorname{tr} A|>k\}}(\operatorname{tr} A)(|\operatorname{tr} A|^2 - k^2) \end{aligned}$$

and so we obtain

$$\begin{aligned} |\langle\!\langle f_k - f, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle| &= \left| \int_\Omega f_k(x,\mathcal{E}u_j) - f(x,\mathcal{E}u_j)\ \mathrm{d}\,x \right| \\ &\leq M \int_{\{x\in\Omega:\ |\operatorname{div} u_j(x)|>k\}} (|\operatorname{div} u_j|^2 - k^2)\ \mathrm{d}\,x \\ &\leq M \sup_j \int_{\{x\in\Omega:\ |\operatorname{div} u_j(x)|>k\}} (|\operatorname{div} u_j|^2 - k^2)\ \mathrm{d}\,x. \end{aligned}$$

By the equiintegrability of $(|\operatorname{div} u_j|^2)$ we obtain:

$$\sup_j \int_{\{x\in\Omega:\ |\operatorname{div} u_j|>k\}} (|\operatorname{div} u_j|^2 - k^2)\ \mathrm{d}\,x \to 0 \quad \text{as} \quad k \to \infty. \tag{5.11}$$

By the monotone convergence theorem we obtain:

$$\lim_{k\to\infty} \langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle = \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle. \tag{5.12}$$

Next, by Proposition 5.8, we obtain:

$$\lim_{j\to\infty} \langle\!\langle f_k, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle = \langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle. \tag{5.13}$$

We have

$$
\begin{aligned}
|\langle\!\langle f, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle - \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle| &\le |\langle\!\langle f, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle - \langle\!\langle f_k, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle| \\
&\qquad + |\langle\!\langle f_k, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle - \langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle| + |\langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle - \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle| \\
&\le \sup_l |\langle\!\langle f, \boldsymbol{\delta}[Eu_l]\rangle\!\rangle - \langle\!\langle f_k, \boldsymbol{\delta}[Eu_l]\rangle\!\rangle| \\
&\qquad + |\langle\!\langle f_k, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle - \langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle| + |\langle\!\langle f_k, \boldsymbol{\nu}\rangle\!\rangle - \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle|.
\end{aligned}
$$

By (5.13), for a fixed $k \in \mathbb{N}$, the second term on the right-hand side vanishes as $j \to \infty$. By (5.11) and (5.12) respectively the first and the last term vanish as $k \to \infty$. Therefore

$$
\lim_{j\to\infty} \langle\!\langle f, \boldsymbol{\delta}[Eu_j]\rangle\!\rangle = \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle. \tag{5.14}
$$

Since $(u_j)$ is uniformly norm-bounded in $\mathrm{LU}(\Omega)$, expanding double brackets yields:

$$
\int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f(x, A) \,\mathrm{d}\,\nu_x(A) \,\mathrm{d}\,x \le \langle\!\langle f, \boldsymbol{\nu}\rangle\!\rangle \le M \sup_j \int_\Omega 1 + |\operatorname{div} u_j|^2 \,\mathrm{d}\,x < \infty. \quad \square
$$

**Remark 5.14.** It is possible to prove Proposition 5.13 in a different way. Indeed, let $(u_j) \subset \mathrm{LU}(\Omega)$ be such that $Eu_j \xrightarrow{\mathbf{Y}} \boldsymbol{\nu}$ for some $\boldsymbol{\nu} \in \mathrm{UY}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. Since $(u_j)$ is uniformly norm-bounded in $\mathrm{LU}(\Omega)$, we have

$$
\sup_j \int_\Omega f(x, \mathcal{E}u_j(x)) \,\mathrm{d}\,x < \infty.
$$

Fix $h \in \mathbb{N}$ and define $f_h(x, A) := \min\{h, f(x, A)\}$. Then, by the classical Young measures convergence, we have

$$
\lim_{j\to\infty} \int_\Omega f_h(x, \mathcal{E}u_j(x)) \,\mathrm{d}\,x = \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f_h(x, A) \,\mathrm{d}\,\nu_x(A) \,\mathrm{d}\,x.
$$

Since $f \ge f_h$, we have

$$
\liminf_{j\to\infty} \int_\Omega f(x, \mathcal{E}u_j(x)) \,\mathrm{d}\,x \ge \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f_h(x, A) \,\mathrm{d}\,\nu_x(A) \,\mathrm{d}\,x.
$$

By the monotone convergence theorem, letting $h \to \infty$ yields

$$
\liminf_{j\to\infty} \int_\Omega f(x, \mathcal{E}u_j(x)) \,\mathrm{d}\,x \ge \int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} f(x, A) \,\mathrm{d}\,\nu_x(A) \,\mathrm{d}\,x.
$$

The proof is finished.

The following conjecture is the ultimate, yet elusive goal of the theory of U-Young measures.

**Conjecture 5.15 (Characterisation).** *Let* $\boldsymbol{\nu} \in \mathrm{BDY}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *be a* BD-*Young measure with* $\lambda_\nu(\partial\Omega) = 0$. *Then,* $\boldsymbol{\nu}$ *is a* U-*Young measure if and only if*

1. $\displaystyle\int_\Omega \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} |\operatorname{tr} A|^2 \,\mathrm{d}\,\nu_x(A)\,\mathrm{d}\,x < \infty,$
2. $\operatorname{supp} \nu_x^\infty \subset \{m \in \mathrm{SD}(d) : \ |m| = 1\}$ *for* $\lambda_\nu$*-a.e.* $x \in \overline{\Omega}$.

In virtue of Proposition 5.13, we see that the first condition of the necessity part of Conjecture 5.15 follows. It seems natural to expect that the concentration-direction measure $\nu_x^\infty$ is supported in the unit sphere of symmetric and deviatoric matrices, as Proposition 5.12 suggests that the concentration in the trace direction does not occur.

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