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Moduli space of *J*-holomorphic subvarieties

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Abstract

We study the moduli space of J-holomorphic subvarieties in a 4-dimensional symplectic manifold. For an arbitrary tamed almost complex structure, we show that the moduli space of a sphere class is formed by a family of linear system structures as in algebraic geometry. Among the applications, we show various uniqueness results of J-holomorphic subvarieties, *e.g.* for the fiber and exceptional classes in irrational ruled surfaces. On the other hand, non-uniqueness and other exotic phenomena of subvarieties in complex rational surfaces are explored. In particular, connected subvarieties in an exceptional class with higher genus components are constructed. The moduli space of tori is also discussed, and leads to an extension of the elliptic curve theory.

Mathematics Subject Classification 32Q65 · 32Q60 · 53D35 · 57R57 · 14H15

Contents

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1 Introduction

In this paper, we study the moduli space of J-holomorphic subvarieties where the almost complex structure J is tamed by a symplectic form. Recall J is said to be tamed by a symplectic form ω if the bilinear form $\omega(\cdot, J(\cdot))$ is positive definite. When we say J is tamed, we mean it is tamed by an arbitrary symplectic form unless it is said otherwise. J-holomorphic subvarieties are the analogues of one dimensional subvarieties in algebraic geometry. In our paper, the ambient space M is of dimension four, where subvarieties are just divisors. In [31], Taubes provided systematic local analysis of its moduli space $\mathcal{M}_e$ of J-holomorphic subvarieties in a class $e \in H^2(M, \mathbb{Z})$ with the Gromov-Hausdorff topology, in particular when the almost complex structure J is chosen generically. For precise definitions and basic properties, see Sect. 2.1.

For an almost complex structure J, and a class $e \in H^2(M, \mathbb{Z})$, we introduce the J-genus of e,

$$g_J(e) = \frac{1}{2}(e \cdot e + K_J \cdot e) + 1, \tag{1}$$

where K_J is the canonical class of J. A *K_J-spherical class* (sometimes called sphere class if there is no confusion of choosing a canonical class) is a class e which could be represented by a smoothly embedded sphere and $g_J(e) = 0$. An exceptional curve class E is a K_J-spherical class such that $E^2 = K \cdot E = -1$.[1] For a generic tamed J, any exceptional curve class is represented by a unique embedded J-holomorphic sphere with self-intersection -1.

For an arbitrary J, even it is tamed, the behaviour of reducible J-holomorphic subvarieties could be very wild. There are even some unexpected phenomenon for a K_J-spherical class. For instance, there are classes of exceptional curves, such that the moduli space are of complex dimension 1 and some representatives have an elliptic curve component. One such example is constructed in [32], recalled in section 6.1. It shows that an exceptional curve class in $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ has a $\mathbb{CP}^1$ family of subvarieties and some of them have an elliptic curve as one of irreducible components. Such examples, although very simple, were not generally expected by symplectic geometers. Since the Gromov-Witten invariant is 1, people expected to have uniqueness in some sense. This example is extended to all sphere classes in Proposition 6.3. This sort of examples could be even wilder. The example constructed above Question 4.18 in [32] is disconnected and has a genus 1 component. In Example 6.5, we show the existence of a rational complex surface such that there is a *connected* subvariety with a genus 3 component in an exceptional curve class. Moreover, the graph attached to the subvariety has a loop. This does not contradict to Gromov-Witten theory. In fact,

[1] Since we are in dimension 4, we will identify an element in $H_2(M, \mathbb{Z})$ with its Poincaré dual cohomology class by abusing the notation. Usually, we use e to denote a general class in $H^2(M, \mathbb{Z})$. The letter E is reserved for an exceptional curve class.

none of the subvarieties in a spherical class with higher genus irreducible components contributes to the Gromov-Witten invariant of e, see Remark 6.7.

In [17,18], the notion of J-nefness is introduced. A class is said to be *J-nef* if it pairs non-negatively with all J-holomorphic subvarieties. This condition prevents all the exotic phenomena mentioned in the above. Under this assumption, the topological complexity, *e.g.* the genus of each irreducible component and the intersection theory, is well controlled. The result is particularly nice when $g_J(e) = 0$. In this case, all the irreducible components of subvarieties in class e are rational curves (comparing to Proposition 6.3 and Example 6.5). Moreover, when e is a sphere class with $e \cdot e \geq 0$, we know there is always a smooth J-holomorphic curve in class e. Both results are sensitive to the nefness condition. In particular, they no longer hold when e is an exceptional curve class in a rational surface as we mentioned above. However, there are no such examples in irrational ruled surfaces. Here, irrational ruled surfaces are smooth 4-manifolds diffeomorphic to blowups of sphere bundles over Riemann surfaces with positive genus.

Theorem 1.1 *Let M be an irrational ruled surface, and E an exceptional class. Then for any tamed J and any subvariety in class E, each irreducible component is a rational curve of negative self-intersection. Moreover, the moduli space $\mathcal{M}_E$ is a single point.*

In particular, it confirms Question 4.18 of [32] for irrational ruled surfaces.[2] As other results in this paper, our statement works for an arbitrary tamed almost complex structure, this gives us much more freedom for geometric applications than a generic statement.

The first statement follows from the fact that the positive fiber class of an irrational ruled surface is J-nef for any tamed J (Proposition 3.2). Here the positive fiber class is the unique K_J-spherical class of square 0. Then the J-nefness technique in [18] gives the desired result. The proof of Proposition 3.2 requires a new idea. This is based on a simple observation that the adjunction number of a class e is the Seiberg–Witten dimension of $-e$. When the class is not J-nef and the J-genus of the class is positive, the wall crossing formula of Seiberg–Witten theory would produce non-trivial subvarieties with trivial homology class. To summarize, this observation gives us a strategy to show certain class is J-nef. We expect this observation, along with the nefness technique in [17,18], would lead to more applications. See the discussion in Sect. 3.

The second statement of Theorem 1.1 follows from a uniqueness result of reducible subvarieties, Lemma 2.5. This lemma constraints the reducible subvarieties by intersection theory of subvarieties. This is an important ingredient for almost all the results in this paper.

In fact, it follows directly from the second statement of Theorem 1.1 that the J-holomorphic subvariety in class E is connected and has no cycle in its underlying graph for any tamed J by Gromov compactness, since these properties hold for the Gromov limit of smooth pseudoholomorphic rational curves.

2 Question 4.18 of [32] for other symplectic 4-manifolds is answered affirmatively in the Appendix of [4].

The nefness of the positive fiber class and Lemma 2.5 also lead to the structure of the moduli space of a sphere class in irrational surfaces for an arbitrary tamed almost complex structure.

Theorem 1.2 *Let M be an irrational ruled surface of base genus $h \geq 1$. Then for any tamed J on M,*

(1) *There is a unique subvariety in the positive fiber class T passing through a given point;*
(2) *The moduli space $\mathcal{M}_T$ is homeomorphic to Σ_h, and there are finitely many reducible varieties;*
(3) *Every irreducible rational curve is an irreducible component of a subvariety in class T.*

Theorem 1.1 and Theorem 1.2(1-2) hold for generic tamed J on general ruled surfaces regardless they are rational or not. But they hold for arbitrary tamed J only in irrational case. It is likely the following version of Theorem 1.2(3) is true for general rational surfaces as well: every irreducible negative rational curve is an irreducible component of a subvariety in a sphere class of nonnegative self-intersection.

In algebraic geometry, Theorem 1.2 could be explained by the linear systems. Recall the long exact sequence

$$\cdots \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \longrightarrow \cdots$$

A divisor D gives rise to a line bundle $L_D \in \mathrm{Pic}(M) = H^1(M, \mathcal{O}^*)$. When M is projective, the group of divisor classes modulo linear equivalence is identified with $\mathrm{Pic}(M)$. The Poincaré-Lelong theorem says that $c_1(L_D) = PD[D]$. In our setting, we fix the class $e \in H^2(M, \mathbb{Z})$ (indeed its Poincaré dual, but we will not distinguish them in this paper). Any line bundle L with $c_1(L) = e$ would give a projective space family of effective divisors, *i.e.* the linear system $(\Gamma(M, L) \setminus \{0\})/\mathbb{C}^*$, in the moduli space $\mathcal{M}_e$. The union of such projective spaces with respect to all possible line bundles with $c_1(L) = e$ is exactly $\mathcal{M}_e$. Two fibers of an irrational ruled surface are not linearly equivalent, since they are not connected through a family parametrized by rational curves. Hence each projective space is just a point, and the family of these spaces is parametrized by a section of the ruled surface which is diffeomorphic to Σ_h. In fact, this Σ_h is embedded in its Jacobian which is a complex tori T^{2h}. Theorem 1.1 could also be interpreted by the linear system, where $\mathcal{M}_E = \mathbb{CP}^0$.

When M is simply connected, the long exact sequence implies the uniqueness of the line bundle with given Chern class. Hence the moduli space is always a projective space. It is very interesting to see whether it still holds for a tamed almost complex structure. The following is for rational surfaces.

Theorem 1.3 *Let J be a tamed almost complex structure on a rational surface M. Suppose e is a primitive class and represented by a smooth J-holomorphic sphere. Then $\mathcal{M}_e$ is homeomorphic to $\mathbb{CP}^l$ where $l = \max\{0, e \cdot e + 1\}$.*

In particular, it partially confirms Question 5.25 in [17]. Here, M is called a rational surface if it is diffeomorphic to $S^2 \times S^2$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$. We remark that even the connectedness of the moduli spaces $\mathcal{M}_e$ appearing in Theorems 1.2 and 1.3 was not known.

For the proof of the result, we view $\mathbb{CP}^l$ as $\mathrm{Sym}^l S^2$, the l-th symmetric product of S^2. There are two main steps in the argument. First we need to find a "dual" smooth J-holomorphic rational curve in a class e' whose pairing with e is l. This is achieved by a delicate homological study of J-nef classes and techniques from [17]. Hence the intersection of elements in $\mathcal{M}_e$ with this rational curve would give elements of $\mathrm{Sym}^l S^2$. Then a refined version of Lemma 2.5 would give us the desired identification.

The only possible non-primitive sphere classes are Cremona equivalent to a double line class in $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$. Here Cremona equivalence refers to the equivalence under the group of diffeomorphisms preserving the canonical class K_J. In this case, we can still show the connectedness of the moduli space and its irreducible part (Proposition 4.1). The connectedness is important in the study of symplectic isotopy problem. More interestingly, a potential generalization of our argument for Theorem 1.3 leads us to a larger framework which generalizes certain part of the elliptic curve theory. In particular, a non-associative (because of the failure of Cayley-Bacharach theorem for a non-integrable almost complex structure) addition is introduced to measure the deviation from the integrability.

On the other hand, some arguments and techniques in this paper and that of [17,18] could be extended to study moduli space of subvarieties in higher genus classes, in particular, tori or classes with $g_J(e) = 1$. In this paper, we focus our discussion on the anti-canonical class of $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$. We are able to show the following:

Theorem 1.4 *If there is an irreducible (singular) nodal curve in $\mathcal{M}_{-K}$, then $\mathcal{M}_{smooth,-K}$ and $\mathcal{M}_{-K}$ are both path connected.*

We hope to have a more general discussion of J-holomorphic tori in future work.

Section 6 contains a couple more applications. First, we show that the example mentioned in the beginning is actually a general phenomenon for any non-negative sphere classes. Namely, Proposition 6.3 says that some subvarieties in a sphere class of a complex surface have an elliptic curve component. This immediately implies that no sphere class in $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, $k \geq 8$ is J-nef for every complex structure J. This should be compared with the result mentioned above that the positive fiber class of an irrational ruled surface is J-nef for any tamed J.

The other application is on the symplectic isotopy of spheres to a holomorphic curve. This problem is first studied for plane curves, i.e. symplectic surfaces in $\mathbb{CP}^2$. In this case, the genus of a smooth symplectic surface is totally determined by its degree d. It is now known that any symplectic surface in $\mathbb{CP}^2$ of degree $d \leq 17$ is symplectically isotopic to an algebraic curve. Chronologically, for $d = 1, 2$ (i.e. the sphere case) this result is due to Gromov [9], for $d = 3$ to Sikorav [29], for $d \leq 6$ to Shevchishin [27] and finally $d \leq 17$ to Siebert and Tian [28]. In Theorem 6.9, we give an alternative proof of the fact (see *e.g.* [15]) that any symplectic sphere S with self-intersection $S \cdot S \geq 0$ in a 4-manifold (M, ω) is symplectically isotopic to a holomorphic rational curve.

Besides the techniques of J-holomorphic subvarieties, especially the J-nefness technique, another important ingredient in our arguments is the Seiberg–Witten theory.

In particular, we use SW=Gr and wall-crossing formula frequently. They provide abundant J-holomorphic subvarieties when $b^+(M) = 1$. As an amusing byproduct, we observe in Proposition 2.7 that the corresponding statement of Hodge conjecture for tamed almost complex structure on M with $b^+(M) = 1$ holds. Namely, any element of $H^2(M, \mathbb{Z})$ is the cohomology class of a J-divisor.

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2 *J*-holomorphic subvarieties

In this section, we recall the definition and basic properties of J-holomorphic subvarieties. The first two subsections are essentially from [17,18,31]. Then an useful technical lemma on the intersection of J-holomorphic subvarieties, Lemma 2.5, is proved. Finally, after recalling the basics of Seiberg–Witten theory, we show that the almost Kähler Hodge conjecture holds when $b^+ = 1$.

2.1 *J*-holomorphic subvarieties

A closed set $C \subset M$ with finite, nonzero 2-dimensional Hausdorff measure is said to be an *irreducible J-holomorphic subvariety* if it has no isolated points, and if the complement of a finite set of points in C, called the singular points, is a connected smooth submanifold with J-invariant tangent space. Suppose C is an irreducible subvariety. Then it is the image of a J-holomorphic map $\phi : \Sigma \to M$ from a complex connected curve Σ, where ϕ is an embedding off a finite set. Σ is called the model curve and ϕ is called the tautological map. The map ϕ is uniquely determined up to automorphisms of Σ.

A *J-holomorphic subvariety* Θ is a finite set of pairs $\{(C_i, m_i), 1 \le i \le n\}$, where each C_i is irreducible J-holomorphic subvariety and each m_i is a positive integer. The set of pairs is further constrained so that $C_i \neq C_j$ if $i \neq j$. When J is understood, we will simply call a J-holomorphic subvariety a subvariety. They are the analogues of one dimensional subvarieties in algebraic geometry. Taubes provides a systematic analysis of pseudo-holomorphic subvarieties in [31].

A subvariety $\Theta = \{(C_i, m_i)\}$ is said to be connected if $\cup C_i$ is connected. We call $\Theta > \Theta_0$ if $\Theta - \Theta_0$ is another, possibly empty, subvariety.

The associated homology class e_C (sometimes, we will also write it by $[C]$) is defined to be the push forward of the fundamental class of Σ via ϕ. And for a subvariety Θ, the associated class e_Θ is defined to be $\sum m_i e_{C_i}$.

An irreducible subvariety is said to be *smooth* if it has no singular points. A special feature in dimension 4 is that, by the adjunction formula, the genus of a smooth subvariety C is given by $g_J(e_C)$. For a general class e in $H^2(M; \mathbb{Z})$, recall the J-

genus of e is defined by

$$g_J(e) = \frac{1}{2}(e \cdot e + K_J \cdot e) + 1$$

where K_J is the canonical class of J. In general, $g_J(e)$ could take any integer value. Let $\mathcal{J}^\omega$ be the space of ω-tamed almost complex structures. Notice the J-genus is an invariant for $J \in \mathcal{J}^\omega$ since $\mathcal{J}^\omega$ is path connected and K_J is invariant under deformation. Hence, later we will sometimes write $g_\omega(e) = g_J(e)$ when a symplectic structure ω is fixed.

Moreover, when C is an irreducible subvariety, $g_J(e_C)$ is non-negative. In fact, by the adjunction inequality in [22], $g_J(e_C)$ is bounded from below by the genus of the model curve Σ of C, with equality if and only if C is smooth. Especially, when $g_J(e_C) = 0$, C is a smooth rational curve.

An element Θ, in the moduli space $\mathcal{M}_e$ of subvarieties in the class e, is a subvariety with $e_\Theta = e$. $\mathcal{M}_e$ has a natural topology in the following Gromov-Hausdorff sense. Let $|\Theta| = \cup_{(C,m)\in\Theta} C$ denote the support of Θ. Consider the symmetric, non-negative function, ϱ, on $\mathcal{M}_e \times \mathcal{M}_e$ that is defined by the following rule:

$$\varrho(\Theta, \Theta') = \sup_{z\in|\Theta|} \mathrm{dist}(z, |\Theta'|) + \sup_{z'\in|\Theta'|} \mathrm{dist}(z', |\Theta|). \tag{2}$$

The function ϱ is used to measure distances on $\mathcal{M}_e$, where the distance function $\mathrm{dist}(\cdot, \cdot)$ is defined by an almost Hermitian metric on (M, J).

Given a smooth 2-form ν we introduce the pairing

$$(\nu, \Theta) = \sum_{(C,m)\in\Theta} m \int_C \nu.$$

The topology on $\mathcal{M}_e$ is defined in terms of convergent sequences:

A sequence $\{\Theta_k\}$ in $\mathcal{M}_e$ converges to a given element Θ if the following two conditions are met:

- $\lim_{k\to\infty} \varrho(\Theta, \Theta_k) = 0$
- $\lim_{k\to\infty}(\nu, \Theta_k) = (\nu, \Theta)$ for any given smooth 2-form ν.

That the moduli space $\mathcal{M}_e$ is compact is an application of Gromov compactness, see Proposition 3.1 of [31].

Definition 2.1 A homology class $e \in H_2(M; \mathbb{Z})$ is said to be J-effective if $\mathcal{M}_e$ is nonempty.

We use $\mathcal{M}_{irr,e}$ to denote the moduli space of irreducible subvarieties in class e. Let $\mathcal{M}_{red,e} := \mathcal{M}_e \setminus \mathcal{M}_{irr,e}$.

Given a class e, its J-dimension is

$$\iota_e = \frac{1}{2}(e \cdot e - K_J \cdot e). \tag{3}$$

The integer ι_e is the expected (complex) dimension of the moduli space $\mathcal{M}_e$. When $g_J(e)=0$, we have $\iota_e = e\cdot e+1$. When e is a class represented by a smooth rational curve (*i.e.* J-holomorphic sphere), we introduce

$$l_e = \max\{\iota_e, 0\}.$$

Given a $k(\leq l_e)$-tuple of distinct points Ω, recall that $\mathcal{M}_e^{\Omega}$ is the space of subvarieties in $\mathcal{M}_e$ that contains all entries of Ω. Introduce similarly $\mathcal{M}_{irr,e}^{\Omega}$ and $\mathcal{M}_{red,e}^{\Omega}$. We will often drop the subscript e when there is no confusion.

2.2 *J*-nef classes

In general, all these moduli spaces could behave wildly. The notion of J-nefness provides good control as shown in [17,18].

A class e is said to be *J-nef* if it pairs non-negatively with any J-holomorphic subvariety. When there is a J-holomorphic subvariety in a J-nef class e, *i.e.* e is also effective, we have $e\cdot e\geq 0$. A J-nef class e is said to be *big* if $e\cdot e>0$. The vanishing locus $Z(e)$ of a big J-nef class e is the union of irreducible subvarieties D_i such that $e\cdot e_{D_i}=0$. Denote the complement of the vanishing locus of e by $M(e)$. From the definition and the positivity of intersections of distinct irreducible subvarieties [9,25], it is clear that there does not exist an irreducible subvariety in class e passing through $x\in Z(e)$ when e is big and J-nef.

If the support $|C|=\cup C_i$ of subvariety $\Theta=\{(C_i,m_i)\}$ is connected, then Theorem 1.4 of [18] says that

$$g_J(e)\geq \sum_i g_J(e_{C_i}) \tag{4}$$

for a J-nef class e with $g_J(e)\geq 0$. In this paper, we use the following result which follows from the above genus bound and is read from Theorem 1.5 of [18].

Theorem 2.2 *Suppose J is tamed by some symplectic structure, e is a J-nef class with $g_J(e)=0$ and $\Theta\in\mathcal{M}_e$. Then Θ is connected and each irreducible component of Θ is a smooth rational curve.*

Moreover, when e is J-nef and J-effective with $g_J(e)=0$, we have the following strong bound for the expected dimension of curve configuration for $\Theta\in\mathcal{M}_{red,e}$ (Lemma 4.10 in [18])

$$\sum_{(C_i,m_i)\in\Theta} l_{e_i} \leq \sum_{(C_i,m_i)\in\Theta} m_i l_{e_i} \leq l_e-1. \tag{5}$$

Along with automatic transversality, we have the following which is extracted from Proposition 4.5 and Proposition 4.10 of [17].

Theorem 2.3 *Suppose e is a J-nef spherical class with $e\cdot e\geq 0$. Then $\mathcal{M}_{irr,e}$ is a non-empty smooth manifold of dimension $2l_e$ and $\mathcal{M}_{red,e}$ is a finite union of compact manifolds, each with dimension at most $2(l_e-1)$.*

This is an unobstructedness result for the deformation of symplectic surfaces. In [29], an unobstructed result is obtained. In our circumstance, it implies that when $\mathcal{M}_{irr,e} \neq \emptyset$, it is a smooth manifold. Hence, our main contribution is to show $\mathcal{M}_{irr,e} \neq \emptyset$ when e is J-nef. It is important for our applications, since we will deform J in $\mathcal{J}^{\omega}$ and the irreducible part of moduli space need not to be nonempty *a priori*. Our result for $\mathcal{M}_{red,e}$ is more general since [29] need each component of Θ has multiplicity one and has self-intersection no less than -1.

2.3 Intersection of subvarieties

We first analyze how an intersection point contributes to the intersection number of two subvarieties. Since every component of a subvariety is an irreducible curve, the intersection number will always contribute positively.

There are two typical types of intersections. The first is when two multiple components (C, n) and (C', m) have an intersection point p. If the two irreducible curves C and C' intersect at p transversally, then the point p contributes mn to the intersection numbers. The second type is when two curves C and C' have high contact order at p. If they are tangent to each other at order n, which means the local Taylor expansion coincides up to order $n-1$, then p would contribute n to the intersection number. Notice only the local behavior of the two curves matters for the intersection near p. Hence the two types could interact simultaneously.

Example 2.4 Suppose Θ is a subvariety with two irreducible components (C_1, m_1) and (C_2, m_2), which intersect transversally at point p, and Θ' is another subvariety with a component (C', m), passing through p and tangent to C_1 of order n at p. The point p would contribute $nmm_1 + mm_2$ to the intersection of two subvarieties Θ and Θ'.

Later in this paper, we will see in several occasions to prescribe a subvariety passing through given "points with weight", which will be explained immediately. Corresponding to the above two types of intersections of subvarieties, there are two types of points with weight. The first type, denote by (x, d) with $x \in M$ and $d \in \mathbb{Z}$, means the subvariety Θ passes through point x with multiplicity d. Since no direction or higher order contact is given, the multiplicity here is the sum of weights of all irreducible components of Θ passing through x, say $(C_1, m_1), \ldots, (C_k, m_k)$, *i.e.* $d = m_1 + \cdots + m_k$.

The second type, denote by (x, C, d) with $x \in M$, $d \in \mathbb{Z}$ and C a (local) J-holomorphic curve passing through x, means subvariety Θ passes through point x with multiplicity d counted with contact orders with C. Precisely, if locally there are local components of Θ, say $(C_1, m_1), \ldots, (C_k, m_k)$ passing through point x and tangent to the curve C with order $d_1, \ldots, d_k$ respectively, then $d = d_1 m_1 + \cdots + d_k m_k$. Here we implicitly assume C is of multiplicity one. In the most general case, we consider (C, n), and the corresponding relation is $d = n(d_1 m_1 + \cdots + d_k m_k)$. Sometimes, we call C the "matching" curve at point x.

The following strengthens Lemma 4.18 in [17], considering the first type intersection.

Lemma 2.5 *Let J be an almost complex structure on M^4. Suppose e is J-nef with $l = \max\{e \cdot e + 1, 0\}$ and $\{(x_1, d_1), \ldots, (x_k, d_k)\}$ are points with weight.*

(1) *Suppose two subvarieties $\Theta, \Theta' \in \mathcal{M}_e$ do not share irreducible components. If they both pass through these points with weight, then $d_1 + \cdots + d_k < l$.*
(2) *Let $\Theta = \{(C_i, m_i)\} \in \mathcal{M}_e$ be a connected subvariety passing through these points with weight such that there are at least $m_i e \cdot e_{C_i}$ points (counted with multiplicities) on C_i for each i and all x_i are smooth points. Then there is no other such subvariety in class e that shares an irreducible component with Θ.*

Proof The first statement simply follows from positivity of intersection of two distinct irreducible J-holomorphic curves. This is because the points (x_i, d_i) are in the intersection of Θ and Θ' and each d_i is no greater than the local intersection index of them at x_i. These local intersection indices are positive integers which add up to $e \cdot e$, although there might be intersection points of Θ and Θ' other than x_i. Thus, the inequality follows.

For the second statement, suppose there is another such subvariety Θ', such that Θ and Θ' share at least one common irreducible components.

We rewrite two subvarieties $\Theta, \Theta' \in \mathcal{M}_e$, allowing $m_i = 0$ in the notation, such that they share the same set of irreducible components formally, i.e. $\Theta = \{(C_i, m_i)\}$ and $\Theta' = \{(C_i, m_i')\}$. Then for each C_i, if $m_i \leq m_i'$, we change the components to $(C_i, 0)$ and $(C_i, m_i' - m_i)$. At the same time, if a point x, as one of $x_1, \ldots, x_k$, is on C_i, then the weight is reduced by m_i as well. Similar procedure applies to the case when $m_i > m_i'$. Apply this process to all i and discard finally all components with multiplicity 0 and denote them by Θ_0, Θ_0' and still use (C_i, m_i) and (C_i, m_i') to denote their components. Notice they are homologous, formally having homology class

$$e - \sum_{m_{k_i} < m'_{k_i}} m_{k_i} e_{C_{k_i}} - \sum_{m'_{l_j} < m_{l_j}} m'_{l_j} e_{C_{l_j}} - \sum_{m'_{q_p} = m_{q_p}} m'_{q_p} e_{C_{q_p}}.$$

There are two ways to express the class, by taking $e = e_\Theta$ or $e = e_{\Theta'}$ in the above formula. Namely, it is

$$\sum_{m_{k_i} < m'_{k_i}} (m'_{k_i} - m_{k_i}) e_{C_{k_i}} + \text{others} = e_{\Theta'_0} = e_{\Theta_0} = \sum_{m'_{l_j} < m_{l_j}} (m_{l_j} - m'_{l_j}) e_{C_{l_j}} + \text{others}.$$

Here the term "others" means the terms $m_i e_{C_i}$ or $m_i' e_{C_i}$ where i is not taken from k_i, l_j or q_p.

Now Θ_0 and Θ_0' have no common components. By the process we just applied, counted with weight, there are at least $e \cdot e_{\Theta_0}$ points on Θ_0. These points are also contained in Θ_0' with right weights. Hence Θ_0 and Θ_0' would intersect at least $e \cdot e_{\Theta_0}$ points with weight.

We notice that $e \cdot e_{\Theta_0} \geq e_{\Theta_0} \cdot e_{\Theta_0'}$. In fact, the difference $e - e_{\Theta_0} = e - e_{\Theta_0'}$ has 3 types of terms, any of them pairing non-negatively with the class e_{Θ_0}. For the terms with index k_i, *i.e.* the terms with $m_{k_i} < m'_{k_i}$, we use the expression of $e_{\Theta_0} = \sum_{m'_{l_j} < m_{l_j}} (m_{l_j} - m'_{l_j}) e_{C_{l_j}} + \text{others}$ to pair with. Since the irreducible curves involved in the expression are all different from C_{k_i}, we have $e_{C_{k_i}} \cdot e_{\Theta_0} \geq 0$. Similarly, for C_{l_j}, we use the expression of $e_{\Theta_0'} = \sum_{m_{k_i} < m'_{k_i}} (m'_{k_i} - m_{k_i}) e_{C_{k_i}} +$ others. We have $e_{C_{l_j}} \cdot e_{\Theta_0'} \geq 0$. For C_{q_p}, we could use either e_{Θ_0} or $e_{\Theta_0'}$. Since $e_{\Theta_0} = e_{\Theta_0'}$, we have $(e - e_{\Theta_0}) \cdot e_{\Theta_0} \geq 0$.

Moreover, we have the strict inequality $e \cdot e_{\Theta_0} > e^2_{\Theta_0}$. This is because we assume the original Θ, Θ' are connected and have at least one common component. The first fact implies there is at least one index in k_i, l_j or q_p. The second fact implies at least one of the intersection of C_{k_i}, C_{l_j} or C_{q_p} with e_{Θ_0} as in the last paragraph would take positive value.

As we have shown that Θ_0 and Θ_0' would intersect at least $e \cdot e_{\Theta_0}$ points with weight, the inequality $e \cdot e_{\Theta_0} > e^2_{\Theta_0}$ implies the sum of local intersection indices of Θ_0 and Θ_0' is greater than the homology intersection number $e^2_{\Theta_0}$ of our new subvarieties Θ_0 and Θ_0'. This contradicts to the local positivity of intersection and the fact that Θ_0, Θ_0' have no common component. The contradiction implies that Θ is the unique such subvariety as described in the statement. □

The lemma and its argument will be used later, in particular, Theorem 3.4, Theorem 3.7 and Proposition 4.1. A similar statement for the more general second type intersection will be proved by a similar argument and used in Theorem 4.4.

2.4 Seiberg–Witten invariants and subvarieties

Other than techniques in [17,18], another important ingredient of our method is the Seiberg–Witten invariant. We follow the notation in [32]. However we need a more general setting.

Let M be an oriented 4-manifold with a given Riemannian metric g and a spin^c structure $\mathcal{L}$. Hence there are a pair of rank 2 complex vector bundles $S^{\pm}$ with isomorphisms $\det(S^+) = \det(S^-) = \mathcal{L}$. The Seiberg–Witten equations are for a pair (A, ϕ) where A is a connection of $\mathcal{L}$ and $\phi \in \Gamma(S^+)$ is a section of S^+. These equations are

$$\begin{aligned} D_A\phi &= 0 \\ F_A^+ &= iq(\phi) + i\eta \end{aligned}$$

where q is a canonical map $q : \Gamma(S^+) \to \Omega^2_+(M)$ and η is a self-dual 2-form on M.

The group $C^\infty(M; S^1)$ naturally acts on the space of solutions. Under this action, the map $f \in C^\infty(M; S^1)$ sends a pair (A, ϕ) to $(A + 2fdf^{-1}, f\phi)$. It acts freely at irreducible solutions. Recall a reducible solution has $\phi = 0$, and hence $F_A^+ = i\eta$. The quotient is the moduli space, denoted by $\mathcal{M}_M(\mathcal{L}, g, \eta)$. For generic pairs (g, η), the Seiberg–Witten moduli space $\mathcal{M}_M(\mathcal{L}, g, \eta)$ is a compact manifold of dimension

$$2d(\mathcal{L}) = \frac{1}{4}(c_1(\mathcal{L})^2 - (3\sigma(M) + 2\chi(M)))$$

where $\sigma(M)$ is the signature and $\chi(M)$ is the Euler number. Furthermore, an orientation is given to $\mathcal{M}_M(\mathcal{L}, g, \eta)$ by fixing a homology orientation for M, *i.e.* an orientation of $H^1(M) \oplus H^2_+(M)$. When $b^+(M) = 1$, the space of g-self-dual forms

$\mathcal{H}_g^+(M)$ is spanned by a single harmonic 2-form ω_g of norm 1 agreeing with the homology orientation.

Quotient out the space of triple $(p, (A, \phi))$ where $p \in M$ and (A, ϕ) is a solution of Seiberg–Witten equation by based actions $f \in C^\infty(M; S^1)$ with $f(p) = 1$, we obtain a smooth manifold $\mathcal{E}$. It is a principal S^1 bundle over $M \times \mathcal{M}_M(\mathcal{L}, g, \eta)$. The slant product with $c_1(\mathcal{E})$ defines a natural map ψ from $H_*(M, \mathbb{Z})$ to $H^{2-*}(\mathcal{M}_M(\mathcal{L}, g, \eta), \mathbb{Z})$.

We now assume (M, J) is an almost complex 4-manifold with canonical class K. We denote $e := \frac{c_1(\mathcal{L})+K}{2} \in H^2(M; \mathbb{Z})/(2\text{-torsion})$. For a generic choice of (g, η), the Seiberg–Witten invariant $SW^*_{M,g,\eta}(e)$ takes value in $\Lambda^* H^1(M, \mathbb{Z})$. If $d(\mathcal{L}) < 0$, then the SW invariant is defined to be zero. Otherwise, let $\gamma_1 \wedge \cdots \wedge \gamma_p \in \Lambda^p(H_1(M, \mathbb{Z})/\text{Torsion})$, we define

$$SW^*_{M,g,\eta}(e; \gamma_1 \wedge \cdots \wedge \gamma_p) := \int_{\mathcal{M}_M(\mathcal{L},g,\eta)} \psi(\gamma_1) \wedge \cdots \wedge \psi(\gamma_p) \wedge \psi(pt)^{d-\frac{p}{2}}. \quad (6)$$

If $b^+ > 1$, a generic path of (g, η) contains no reducible solutions. Hence, the Seiberg–Witten invariant is an oriented diffeomorphism invariant in this case. Hence we can use the notation $SW^*(e)$ for the (full) Seiberg–Witten invariant. We will also write

$$\dim_{SW}(e) = 2d(\mathcal{L}) = e^2 - K \cdot e$$

for the Seiberg–Witten dimension. In the case $b^+ = 1$, there might be reducible solutions on a 1-dimensional family. Recall that the curvature F_A represents the cohomology class $-2\pi i c_1(\mathcal{L})$. Hence $F_A^+ = i\eta$ holds only if $-2\pi c_1(\mathcal{L})^+ = \eta$. This happens if and only if the discriminant $\Delta_{\mathcal{L}}(g, \eta) := \int (2\pi c_1(\mathcal{L}) + \eta)\omega_g = 0$. With this in mind, the set of pairs (g, η) with positive (resp. negative) discriminant is called the positive (resp. negative) $\mathcal{L}$ chamber. We use the notation $SW^*_\pm(e)$ for the Seiberg–Witten invariants in these two chambers. The part of $SW^*(e)$ (resp. $SW^*_\pm(e)$) in $\Lambda^i H^1(M, \mathbb{Z})$ will be denoted by $SW^i(e)$ (resp. $SW^i_\pm(e)$). Moreover, in the this paper, we will use $SW^*(e)$ instead of $SW^*_-(e)$ when $b^+ = 1$. For simplicity, the notation $SW(e)$ is reserved for $SW^0(e)$.

We now assume (M, ω) is a symplectic 4-manifold, and J is a ω-tamed almost complex structure. Then the results in [13,30] equate Seiberg–Witten invariants with Gromov-Taubes invariants that are defined by making a suitable counting of J-holomorphic subvarieties. In fact, our $SW^*(e)$ used in this paper is essentially the Gromov-Taubes invariant in the literature. In particular, our $SW^*(e)$ is the original Seiberg–Witten invariant of the class $2e - K$. The key conclusion we will take from this equivalence is that when $SW^*(e) \neq 0$, there is a J-holomorphic subvariety in class e. Moreover, if $SW(e) \neq 0$, there is a J-holomorphic subvariety in class e passing through $\dim_{SW}(e)$ given points.

Hence, to produce subvarieties in a given class, we will prove nonvanishing results for $SW^*(e)$, usually for $SW(e)$. When $b^+(M) > 1$, an important result of Taubes says that $SW(K) = 1$. When $b^+(M) = 1$, the key tool is the wall-crossing formula, which relates the Seiberg–Witten invariants of classes $K - e$ and e when $\dim_{SW}(e) \geq 0$. The general wall-crossing formula is proved in [12]. In particular, when M is rational or ruled, we have

$$|SW(K-[C])-SW([C])| = \begin{cases} 1 & \text{if } (M,\omega)\text{rational}, \\ |1+[C]\cdot T|^h & \text{if } (M,\omega) \text{ irrationally ruled}, \end{cases}$$

where T is the unique positive fiber class and h is the genus of base surface of irrationally ruled manifolds. For a general symplectic 4-manifold with $b^+(M)=1$, usually the wall-crossing number for $SW(e)$ is hard to determine and sometimes vanishes [12]. However, we still have a simple formula for top degree part of Seiberg–Witten invariant (see Lemma 3.3 (1) of [14]).

Proposition 2.6 *Let M be a symplectic 4-manifold with $b^+=1$ and canonical class K. Suppose $\dim_{SW}(e)\geq b_1$. Let $\gamma_1,\dots,\gamma_{b_1}$ be a basis of $H_1(M,\mathbb{Z})/Torsion$ such that $\gamma_1\wedge\cdots\wedge\gamma_{b_1}$ is the dual orientation of that on $\Lambda^{b_1}(H^1(M,\mathbb{Z}))$. Then*

$$|SW^{b_1}(K-e;\gamma_1\wedge\cdots\wedge\gamma_{b_1})-SW^{b_1}(e;\gamma_1\wedge\cdots\wedge\gamma_{b_1})|=1.$$

Here, b_1 stands for the first Betti number. In particular, it implies a nonvanishing result: let $e\in H^2(M,\mathbb{Z})$ be a class with $e^2\geq 0$, $K\cdot e\leq 0$, and at least one of the inequalities being strict, then $SW^*(ke)\neq 0$ for sufficiently large k.

2.5 Almost Kähler Hodge conjecture

Let X be a non-singular complex projective manifold. The (integral) Hodge conjecture asks whether every class in $H^{2k}(X,\mathbb{Q})\cap H^{k,k}(X)$ (resp. $H^{2k}(X,\mathbb{Z})\cap H^{k,k}(X)$) is a linear combination with rational (resp. integral) coefficients of the cohomology classes of complex subvarieties of X. When $\dim_{\mathbb{C}} X\leq 3$, Hodge conjecture is known to be true and follows from Lefschetz theorem on $(1,1)$ classes. The integral Hodge conjecture, which was Hodge's original conjecture, is known to be false for some projective 3-folds.

In this subsection, we will show an amusing result, which basically says that the integral Hodge conjecture, or Lefschetz theorem on $(1,1)$ classes, is true for almost Kähler 4-manifolds of $b^+=1$.

It is well known that in general the almost Kähler Hodge conjecture statement is not true if $b^+>1$, even when our manifold is Kähler. The most well known counterexample is a generic CM complex tori. It has no subvarieties in general, but the group of integral Hodge classes has dim $H^{1,1}(M,\mathbb{Z})=2$. See the appendix of [33].

In our situation, $H^+_J(M)\cap H^2(M,\mathbb{K})$ plays the role of $H^{1,1}(M,\mathbb{K})$ for $\mathbb{K}=\mathbb{Z}$ or $\mathbb{Q}$. Here $H^+_J(M)$ is called the J-invariant cohomology which is introduced in [7,16] along with the J-anti-invariant $H^-_J(M)$. Recall that an almost complex structure acts on the bundle of real 2-forms Λ^2 as an involution, by $\alpha(\cdot,\cdot)\to\alpha(J\cdot,J\cdot)$. This involution induces the splitting into J-invariant, respectively, J-anti-invariant 2-forms $\Lambda^2=\Lambda^+_J\oplus\Lambda^-_J$. Then we define $H^{\pm}_J(M)=\{\mathfrak{a}\in H^2(M;\mathbb{R})|\exists\ \alpha\in\Lambda^{\pm}_J,\ d\alpha=0 \text{ such that } [\alpha]=\mathfrak{a}\}$.

A *divisor* (resp. $\mathbb{Q}$-*divisor*) with respect to an almost complex structure J is a finite formal sum $\sum a_i C_i$ where C_i are J-holomorphic irreducible curves and $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{Q}$).

Proposition 2.7 *Let M be a symplectic 4-manifold with $b^+(M) = 1$, and J a tamed almost complex structure on it. Any element of $H^2(M, \mathbb{Z})$ is the cohomology class of a divisor (with respect to J).*

Proof When $b^+(M) = 1$, by Corollary 3.4 of [7], we have $h_J^- = \dim H_J^-(M) = 0$ and $H_J^+(M) = H^2(M, \mathbb{R})$. Let $e_1, \ldots, e_{b_2}$ be a $\mathbb{Z}$-basis of $H^2(M, \mathbb{Z})$, and $\alpha_1, \ldots, \alpha_{b_2}$ 2-forms representing them. Since being a J-tamed symplectic form is an open condition, if J is tamed by a symplectic form ω, we can choose ω such that $[\omega] \in H^2(M, \mathbb{Q})$. Then we can find a large integer N and b_2+1 J-tamed symplectic forms $\omega_i = N\omega+\alpha_i$ with $[\omega_i] = N[\omega] + e_i \in H^2(M, \mathbb{Z})$ when $1 \leq i \leq b_2$ and $\omega_0 = N\omega$. Their cohomology classes generate the vector space $H^2(M, \mathbb{Z})$.

If we choose $L > k := \max_i\{0, \frac{K\cdot[\omega_i]}{[\omega_i]\cdot[\omega_i]}\} + b_1$, we have

$$\dim_{SW}(L[\omega_i]) = L(L[\omega_i]^2 - K \cdot [\omega_i]) > L((K \cdot [\omega_i] + b_1) - K \cdot [\omega_i]) \geq b_1.$$

Apply Proposition 2.6, we have $SW^{b_1}(L[\omega_i]) \neq SW^{b_1}(K - L[\omega_i])$. We claim that when $L > k$, $SW^{b_1}(L[\omega_i]) \neq 0$ for any i. By wall-crossing, we only need to show that $SW^{b_1}(K - L[\omega_i]) = 0$. We prove it by contradiction. If $SW^*(K - L[\omega_i]) \neq 0$, then $K - L[\omega_i]$ will be the class of a J-holomorphic subvariety and hence an ω_i-symplectic submanifold. However, when $L > k$, we have $(K - L[\omega_i]) \cdot [\omega_i] < 0$, which is a contradiction. Hence, we have $SW(L[\omega_i]) \neq 0$ for $L > k$ and there are subvarieties in class $L[\omega_i]$ for any i.

Let $a \in H^2(M, \mathbb{Z})$ be an arbitrary class. Because of the way we choose our ω_i, we have $a = \sum_{i=0}^{b_2} a_i[\omega_i]$ with $a_i \in \mathbb{Z}$. Now we further write it as $a = \sum_{i=0}^{b_2} a_i(L + 1)[\omega_i] - \sum_{i=0}^{b_2} a_i L[\omega_i]$, which implies a is the cohomology class of a divisor. □

Remark 2.8 There is another argument to prove $SW(K - L[\omega_i]) = 0$ for large L. This is because $K - L[\omega_i]$ pairs negatively with $2K$ for non-rational or non-ruled manifolds, with H for $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, with a positive fiber class A for $S^2 \times S^2$, and with the positive fiber class T for irrational ruled manifolds. All of the classes mentioned above are SW non-trivial classes with a representative of irreducible J-holomorphic non-negative self-intersections. Hence the contradiction follows from Lemma 3.1 by taking $e = K - L[\omega_i]$.

We remark that the symplectic version of Hodge conjecture holds for any compact symplectic manifolds (M^{2n}, ω). More precisely, in [10], it shows that any element of $H_{2k}(M^{2n}, \mathbb{Z})$ is a symplectic $\mathbb{Q}$-cycle in the form $\frac{1}{N}[S_1^{2k}] - \frac{1}{N}[S_2^{2k}]$ where N is a positive integer and S_i^{2k} are symplectic submanifolds of dimension $2k$.

3 Irrational ruled surfaces

In this section, we use the techniques of [17,18] along with Seiberg–Witten theory to identify the moduli space of J-holomorphic subvarieties in the fiber class of irrational ruled surfaces for any tamed almost complex structure J. When the irrational ruled surface is minimal, it was handled by McDuff in a series of papers, in particular [23]. For non-minimal irrational ruled surfaces, the structure of reducible subvarieties was not clear for a non-generic tamed almost complex structure. The work of [17,18] developed a toolbox to study this kind of problems.

To apply the results and techniques from [17,18], one has to check the J-nefness of the classes we are dealing with. For previous applications, like Nakai-Moishezon type duality and the tamed to compatible question, we could always start with a J-nef class. However, for most other applications like our problem in this section, we do not know J-nefness *a priori*. In the following, we will develop a strategy to verify this technical condition. Then along with the techniques in [17,18], we cook up a general scheme to investigate the moduli space of subvarieties (and its irreducible and reducible parts) in a given class.

The following lemma is Lemma 2.2 in [32]. Since the statement is very useful and the proof is extremely simple, we include in the following.

Lemma 3.1 *If C is an irreducible J-holomorphic curve with $C^2 \geq 0$ and $SW(e) \neq 0$, then $e \cdot [C] \geq 0$.*

Proof Since $SW(e) \neq 0$, we can represent e by a possible reducible J-holomorphic subvariety. Since each irreducible curve C' has $[C'] \cdot [C] \geq 0$, we have $e \cdot [C] \geq 0$. □

Let us now fix the notation. Since the blowups of $S^2 \times \Sigma_h$ and nontrivial S^2 bundle over Σ_h are diffeomorphic, we will write $M = S^2 \times \Sigma_h \# k\overline{\mathbb{CP}^2}$ if it is not minimal. Let U be the class of $\{pt\} \times \Sigma_h$ which has $U^2 = 0$ and T be the class of the fiber $S^2 \times \{pt\}$. Then the canonical class $K = -2U + (2h-2)T + \sum_i E_i$.

On the other hand, if M is a nontrivial S^2 bundle over Σ_h, U represents the class of a section with $U^2 = 1$ and T is the class of the fiber. Then $K = -2U + (2h-1)T$. In this section, we assume $h \geq 1$, *i.e.* M is an irrational ruled surface.

We will first show that there is an embedded curve in the fiber class.

Proposition 3.2 *Let J be a tamed almost complex structure on irrational ruled surface M, then the fiber class T is J-nef. Hence there is an embedded curve in class T.*

Proof The first statement is equivalent to the following: let C be an irreducible curve with $[C] = aU + bT - \sum_i c_i E_i$, then $a \geq 0$. We prove it by contradiction. Assume there is an irreducible curve with $a < 0$. Then we know that $2g_J([C]) - 2 = C^2 + K \cdot [C]$. We take the projection $f : C \to \Sigma_h$ to the base. Its mapping degree is $a = [C] \cdot T$. Since Σ_h has genus at least one, by Kneser's theorem, we have

$$C^2 + K \cdot [C] = 2g_J([C]) - 2 \geq 2g(\Sigma_C) - 2 \geq |a|(2h-2) \geq 0.$$

Here Σ_C is the model curve of the irreducible subvariety C.

Now we look at the class $-[C]$. By the above calculation, we have the Seiberg–Witten dimension $\dim_{SW}(-[C]) = C^2 - K \cdot (-[C]) \geq 0$. Hence, we could apply the wall-crossing formula

$$|SW(K + [C]) - SW(-[C])| = |1 + (-[C]) \cdot T|^h = (1 - a)^h \neq 0. \tag{7}$$

For classes T and $e = K + [C] = (a - 2)U + (2h - 2 + b)T + \sum_i (1 - c_i)E_i$ when M is not a nontrivial S^2 bundle (or $e = K + [C] = (a-2)U + (2h-1+b)T$ when M is a nontrivial S^2 bundle), we have $e \cdot T = a - 2 < 0$. We choose an almost complex structure J' such that there is an embedded J'-holomorphic curve in class T. Then apply Lemma 3.1 for this J' to conclude that $SW(K + [C]) = 0$. Apply (7), we have $SW(-[C]) \neq 0$. Hence the class $0 = [C] + (-[C])$ is a class of subvariety. This contradicts to the fact that J is tamed which implies that any positive combinations of curve classes have positive paring with a symplectic form taming J. This finishes the proof that T is J-nef.

Note $g_J(T) = 0$, any irreducible curve in class T would be smooth. Hence, we only need to show the existence of an irreducible curve in class T. By Theorem 1.5 of [18], all components of reducible curves in class T are rational curves since T is J-nef. Furthermore, all the subvarieties are connected since J is tamed. Then by the dimension counting formula Equation (5) for reducible subvarieties, we know $\sum l_{e_i} \leq l_T - 1 = 0$. Here e_i is the homology class of each irreducible component and $l_{e_i} = \max\{0, e_i \cdot e_i + 1\}$. Hence $l_{e_i} = 0$ and all these irreducible components are rational curves of negative self-intersections. It is direct to see from the adjunction formula that there are finitely many negative J-holomorphic spheres on an irrational ruled surface. For a complete classification of symplectic spheres on irrational ruled surfaces, see [6] section 6.

Since $SW(T) \neq 0$ and $\dim_{SW}(T) = 2$, any point of M lies on a subvariety in class T. Since the part covered by reducible curves is a union of finitely many rational curves, as we have shown above, we conclude that there has to be an irreducible, thus embedded, rational curve in class T. □

Corollary 3.3 *On irrational ruled surfaces, the only irreducible rational curves with nonnegative square are in the fiber class T.*

Proof Let $[C] = aU + bT - \sum_i c_i E_i$ be the class of an irreducible rational curve. By Proposition 3.2, we have $a \geq 0$. Since $g_J([C]) = 0$, as argued in Proposition 3.2 by Kneser's theorem, we will have contradiction if $a > 0$. Hence we must have $a = 0$. Then $C^2 = -\sum_i c_i^2 \geq 0$. Hence $c_i = 0$ for all i and $[C] = bT$. Since C is a rational curve, $-2 = C^2 + K \cdot [C] = -2b$. Hence $b = 1$ and $[C] = T$. □

We can now confirm Question 4.18 of [32] for irrational ruled surfaces, and further show there is a unique subvariety in each exceptional class. We rephrase Theorem 1.1.

Theorem 3.4 *Let M be an irrational ruled surface, and let E be an exceptional class. Then for any subvariety $\Theta = \{(C_i, m_i)\}$ in class E, each irreducible component C_i is a rational curve of negative self-intersection. Moreover, the moduli space $\mathcal{M}_E$ is a single point.*

Notice the statement is not true for a rational surface. See [32] for a disconnected example and Sect. 6.1 for a connected example and related discussion.

Proof As explained in Corollary 3.3, any rational curve class must be like $[C] = bT - \sum_i c_i E_i$. If it is the class of an exceptional curve, then

$$K \cdot [C] = -2b + \sum c_i = -1, \ C^2 = -\sum c_i^2 = -1.$$

Hence the only such classes are E_i and $T - E_i$. Both types have non-trivial Seiberg–Witten invariants. Hence, there are J-holomorphic subvarieties in both types of classes for arbitrary tamed J.

Let $\Theta_i \in \mathcal{M}_{E_i}$ and $\tilde{\Theta}_i \in \mathcal{M}_{T-E_i}$. Since $T = E_i + (T - E_i)$, we have $\{\Theta_i, \tilde{\Theta}_i\} \in \mathcal{M}_T$. Since T is J-nef by Proposition 3.2, we know all irreducible components in Θ_i and $\tilde{\Theta}_i$ are rational curves by Theorem 1.5 of [18]. Moreover, by Equation (5), we have $\sum l_{e_{C_i}} \le l_T - 1 = 0$. Hence $e_{C_i}^2 < 0$. This proves the first statement.

For the second statement, we apply the same trick. If there is another subvariety $\Theta_i' \in \mathcal{M}_{E_i}$. Consider $\Theta = \{\Theta_i, \tilde{\Theta}_i\} \in \mathcal{M}_T$ and $\Theta' = \{\Theta_i', \tilde{\Theta}_i\} \in \mathcal{M}_T$. They have common components including $\tilde{\Theta}_i$. We then follow the argument of Lemma 2.5. After discarding all common components, we have cohomologous subvarieties Θ_0 and Θ_0'. Moreover, we have

$$0 = T^2 \ge T \cdot e_{\Theta_0} > e_{\Theta_0}^2 = e_{\Theta_0} \cdot e_{\Theta_0'}. \tag{8}$$

The first inequality follows from nefness of T. Actually, $T^2 = T \cdot e_{\Theta_0}$ by nefness of T applying to the common components we have discarded. The second inequality is because original Θ, Θ' have common components at least from $\tilde{\Theta}_i$, and because they are connected by Theorem 1.5 of [18].

The inequality (8) implies $\Theta_0 = \Theta_0'$ by local positivity of intersections and in turn $\Theta = \Theta'$. Hence there is a unique subvariety Θ_i in each exceptional class E_i. Similarly, there is a unique subvariety $\tilde{\Theta}_i$ in $T - E_i$. □

By the uniqueness result that $\mathcal{M}_E$ is a single point, we know the J-holomorphic subvariety in class E is connected and has no cycle in its underlying graph for any tamed J by Gromov compactness. This is because E is represented by a smooth rational curve for a generic tamed almost complex structure, and the above properties hold for the Gromov limit of these smooth pseudoholomorphic rational curves.

Corollary 3.5 *Let M be an irrational ruled surface, and E an exceptional class. If an irreducible J-holomorphic curve C satisfies $E \cdot [C] < 0$, then C is a rational curve of negative square.*

Proof Since $SW(E) \neq 0$, we always have a subvariety in class E. By Proposition 3.4, all irreducible components are negative rational curves. Thus, if C has positive genus, then C cannot be an irreducible component of the J-holomorphic subvariety in class E. Hence $E \cdot C \ge 0$ by local positivity of intersections. □

We would like to remark that the technique we use to prove Proposition 3.2 could also be applied to other situations. Let us summarize it in the following. We will focus on the case when $b^+ = 1$. To show certain class A with $A^2 \geq 0$ is J-nef when J is tamed, we would have to show classes B with $A \cdot B < 0$ are not curve classes. If such a curve class exists with $B^2 \geq 0$ and at the same time A is realized by a symplectic surface, then there is a contradiction due to the light cone lemma.

Hence we could assume $B^2 < 0$. For this case, the first obvious obstruction is from the adjunction formula. Second type of obstruction is what we have applied above. To show B is not in the curve cone, we look for classes C_i with nontrivial Seiberg–Witten invariants, and $a_0 B + \sum_i a_i C_i = 0$ with each $a_i > 0$. In Proposition 3.2, we choose $a_0 = a_1 = 1$ which are the only nonzero a_i's. For another such application, see Lemma 3.10. The key observation in this case is $2g_J(B) - 2 = \dim_{SW}(-B)$. Hence, if $g_J(B) > 0$ and $(K + B) \cdot A < 0$ we could efficiently apply the general wall crossing formula in [12,14] to get nontriviality of Seiberg–Witten invariant for B. The above argument could have some obvious twists such as taking $C_1 = -kB$.

For the case of $g_J(B) = 0$, we will use a different strategy. We might apply the classifications of negative rational curves, *e.g.* [6,32], and calculate the intersection numbers with A directly.

Now, we will investigate the moduli space of the subvarieties in class T. First, we need a curve to model the moduli space as we did in [17].

Proposition 3.6 *There is a smooth section of the irrational ruled surface, i.e. there is an embedded J-holomorphic curve C of genus h such that $[C] \cdot T = 1$.*

Proof We do our calculation for $M = S^2 \times \Sigma_h \# k\overline{\mathbb{CP}^2}$. When M is a nontrivial S^2 bundle over Σ_h, the calculation is similar.

In Proposition 3.2, we have shown that all curves having the homology class $aU + bT - \sum_i c_i E_i$ must have $a \geq 0$. Especially, for a possibly reducible section which is in the class $U + bT - \sum_i c_i E_i$, there is exactly one irreducible component of it has $a = 1$ (with multiplicity one), all the others have $a = 0$.

Furthermore, let $A = U + hT$, we have $\dim_{SW}(A) = A^2 - K \cdot A = 2h - (-2h + 2h - 2) > 0$. Since $K - A = -3U + (h-2)T + \sum_i E_i$ pairs negatively with T, by Lemma 3.1, $SW(K - A) = 0$. Apply the wall crossing formula, we have $SW(A) = \pm 2^h \neq 0$. Hence there is a subvariety in class $U + hT$. Choose an irreducible component with $a = 1$, call it C.

We show that C has to be smooth. Since $[C] \cdot T = 1$, for any point $x \in C$, there is a subvariety Θ_x in class T passing through it. Since any curve class $aU + bT - \sum_i c_i E_i$ has $a \geq 0$, we know C cannot be an irreducible component of this subvariety Θ_x in class T. If x is a singular point, the contribution to the intersection of C and Θ_x would be greater than 1. Hence by the local positivity of the intersection, we know C is an embedded curve.

Since C is a section, we have $g(C) > 0$ by Kneser's theorem. By Corollary 3.5, for any exceptional rational curve class E, we have $[C] \cdot E \geq 0$. Since $T - E$ is another exceptional rational curve class and $[C] \cdot (T - E) + [C] \cdot E = [C] \cdot T = 1$, we have $0 \leq [C] \cdot E \leq 1$. Because of this,

$$K \cdot [C] + [C]^2 = (2h - 2 - 2b + \sum c_i) + (2b - \sum c_i^2) = 2h - 2.$$

Hence C has genus h. □

We are ready to show the structure of the moduli space $\mathcal{M}_T$.

Theorem 3.7 *Let M be an irrational ruled surface of base genus h. Then for any tamed J on M,*

(1) *There is a unique subvariety in class T passing through a given point;*
(2) *The moduli space $\mathcal{M}_T$ of the subvarieties in class T is homeomorphic to Σ_h;*
(3) *$\mathcal{M}_{red,T}$ is a set of finitely many points.*

Proof Let $C \cong \Sigma_h$ be the smooth J-holomorphic section constructed in Proposition 3.6. First, by Lemma 2.5, for any given point $x \in M$, there is a unique element in $\mathcal{M}_T$ passing through it. We denote this element by Θ_x.

Now, we construct a natural map $h : x \mapsto \Theta_x$ from C to $\mathcal{M}_T$. The map h is surjective because $T \cdot [C] \neq 0$. The map is injective since $T \cdot [C] = 1$ and the positivity of intersection. To show h^{-1} is continuous, consider a sequence $\Theta_i \in \mathcal{M}_T$ approaching to its Gromov-Hausdorff limit Θ. Let the intersection points of Θ_i, Θ with C be p_i, p. Then p_i has to approach p by the first item of the definition of topology on $\mathcal{M}_T$. Now since $C \cong \Sigma_h$ is Hausdorff and $\mathcal{M}_T$ is compact, the fact we just proved that $h^{-1} : \mathcal{M}_T \to C$ is continuous would imply h is also continuous. Hence h is a homeomorphism. This completes the proof of the second statement.

The third bullet, that $\mathcal{M}_{red,T}$ is a set of finitely many points, follows from the following two facts. First, each irreducible component of an element in $\mathcal{M}_{red,T}$ would have negative self-intersection since $\sum l_{e_i} \leq 0$ by Equation (5). Second, there are finitely many negative rational curves as we have seen in Proposition 3.2. □

Corollary 3.8 *Every irreducible rational curve belongs to a fiber, i.e. it is an irreducible component of an element of $\mathcal{M}_T$.*

Proof First, by Corollary 3.3, all irreducible rational curves with nonnegative square have class T. Hence, we could only talk about negative curves. By Kneser theorem, for such a curve C, we have $[C] \cdot T = 0$ as argued in Corollary 3.3. By Theorem 3.7 (1), for any point $x \in C$, there is a unique element $\Theta_x \in \mathcal{M}_T$ passing through it. If C is not an irreducible component of Θ_x, then $[C] \cdot T > 0$ by the positivity of intersection, which contradicts to $[C] \cdot T = 0$. □

Theorem 3.7 and Corollary 3.8 constitute Theorem 1.2 in the introduction.

Along with Corollary 3.3, we have described $\mathcal{M}_e$ for any rational curve class e and an arbitrary tamed almost complex structure on an irrational ruled surface.

Some finer local structures of the moduli space are described in the following.

Corollary 3.9 *The natural map $f : M \to \mathcal{M}_T$, where $f(x)$ is the unique subvariety Θ_x in class T passing through x, is a continuous map.*

Proof We only need to show that for any sequence $\{x_n\}_{n=1}^{\infty}$ converging to x, the subvarieties Θ_{x_n} converge to Θ_x in $\mathcal{M}_T$. We notice that if a sequence satisfies $\lim_{n\to\infty} \rho(\Theta_x, \Theta_{x_n}) = 0$ (the first defining property of the topology of $\mathcal{M}_e$), then a subsequence must converge to Θ_x because of Theorem 3.7(1).

Hence, we can assume on the contrary that there is a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x such that $\rho(\Theta_x, \Theta_{x_n}) > c > 0$ for a constant c. However, since $\mathcal{M}_T$ is compact, we know there is a subsequence of $\{x_n\}$ such that it converges to a subvariety $\Theta' \in \mathcal{M}_T$. Since $\{x_n\}$ converging to x, we know $x \in |\Theta'| \cap |\Theta_x|$, which implies $\Theta' = \Theta_x$ by Theorem 3.7 (1). This contradicts to our assumption $\rho(\Theta_x, \Theta_{x_n}) > c > 0$. Thus we know $f : M \to \mathcal{M}_T$ is a continuous map. $\square$

It is worth pointing out that near a smooth curve $C \subset \mathcal{M}_T$ (or more generally any moduli space $\mathcal{M}_e$), the convergence is very explicit, as described in [31] (see also [17]). Recall that any curve in a neighborhood of C in $\mathcal{M}_T$ can be written as $\exp_C(\eta)$ with η being a section of normal bundle N_C satisfying

$$D_C\eta + \tau_1\partial\eta + \tau_0 = 0. \tag{9}$$

Here, τ_1 and τ_0 are smooth, fiber preserving maps from a small radius disk in N_C to $\mathrm{Hom}(N_C \otimes T^{1,0}C, N_C \otimes T^{0,1}C)$ and to $N_C \otimes T^{0,1}C$ that obey $|\tau_1(b)| \leq c_0|b|$ and $|\tau_0(b)| \leq c_0|b|^2$. Meanwhile, D_C is the $\mathbb{R}$-linear operator that appears in (2.12) of [31], which is used to describe the first order deformations of C as a J-holomorphic submanifold. The L^2-orthogonal projection map from $C^\infty(C; N_C)$ to the kernel of D_C maps an open set of solutions of (9) diffeomorphically to an open ball centered at 0 in $\ker(D_C)$. Notice in our situation, $\ker(D_C)$ has complex dimension one. This description identifies an open neighborhood $\mathcal{N}(C)$ of C in $\mathcal{M}_T$ with a small radius ball about the origin in $\ker(D_C)$. From this description, we know the tangent bundle of each element in $\mathcal{N}(C)$ varies as a smooth family.

We will finish this section by a digression on another example of using the technique of Proposition 3.2, now for rational surfaces $M = \mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$.

Lemma 3.10 *Let M be a rational surface and J be tamed. Let $A \in H^2(M, \mathbb{Z})$ be a class with $A^2 \geq 0$. Moreover, assume there is an embedded J'-holomorphic curve in class A for a tamed J'. Then if a J-holomorphic curve C such that $[C] \cdot A < \min\{0, -K \cdot A\}$, it has to be a rational curve with negative square.*

For example, A could be chosen as H, $H - E$, $3H - E$, *etc.*

Proof We first show C is a rational curve by contradiction. If $g_J([C]) > 0$, we have $C^2 + K \cdot [C] \geq 0$. We look at the class $-[C]$, which has $\dim_{SW}(-[C]) = C^2 + K \cdot [C] \geq 0$. The wall-crossing formula for rational surfaces implies $|SW(K + [C]) - SW(-[C])| = 1$. For classes A and $e = K + [C]$, we have $A \cdot e < 0$. Apply Lemma 3.1, using the conditions $A^2 \geq 0$ and A has an embedded J'-holomorphic representative, we conclude $SW(K + [C]) = 0$. Hence $SW(-[C]) \neq 0$ by wall-crossing. It follows that the class $0 = [C] + (-[C])$ is a class of subvariety, which contradicts to the fact that J is tamed. Hence C has to be a rational curve.

Now we choose an integral symplectic form ω taming J. Hence, for large N we have $\dim_{SW}(N[\omega]) > 0$. Moreover, the class $K - N[\omega]$ pairs negatively with the symplectic form ω for large N. Therefore, we must have $SW(K - N[\omega]) = 0$. By wall-crossing, we have $SW(N[\omega]) \neq 0$ for large N. Then by Lemma 3.1, we have $[\omega] \cdot A \geq 0$. Since C is a J-holomorphic curve, $[\omega] \cdot [C] > 0$. If $C^2 \geq 0$, and because $A^2 \geq 0$, we apply the light cone lemma to conclude that $[C] \cdot A \geq 0$, which contradicts to our assumption. Hence C is a rational curve with negative square. $\square$

With this lemma in hand, we could apply the classification of negative rational curves in [32] to find J-nef classes for rational surfaces with $K^2 > 0$.

The only feature of rational surfaces used in the proof is that they have nonzero wall-crossing number for all the classes with non-negative Seiberg–Witten dimension. Hence, the argument could be extended to a general symplectic 4-manifold with $b^+ = 1$ under this assumption.

4 Rational surfaces

In this section, we will concentrate on rational surfaces, *i.e.* 4-manifolds diffeomorphic to $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ or $S^2 \times S^2$. We study the moduli space of J-holomorphic subvarieties in a sphere class. Our main results, Theorem 4.4 and Proposition 4.1, show that our moduli space behaves like a linear system in algebraic geometry.

4.1 Connectedness of moduli spaces of subvarieties

For the applications, in particular the symplectic isotopy problem, it is important to show that the reducible part $\mathcal{M}_{red}$ would not disconnect the whole moduli space. This is the technical heart of [28] in which it is called Isotopy Lemma. In our setting, we would show the following stronger result.

Proposition 4.1 *Suppose e is a J-nef class with $g_J(e) = 0$. Then $\mathcal{M}_e$ and $\mathcal{M}_{irr,e}$ are path connected. In particular, any two smooth rational curves representing class e are connected by a path of smooth rational curves.*

Proof We divide our argument into five parts.

Part 1: Reduce to rational surfaces

If $\mathcal{M}_e \neq \emptyset$, since e is J-nef, we know the self-intersection $e^2 \geq 0$. By a classical result of McDuff, if furthermore $g_J(e) = 0$ and $\mathcal{M}_{irr,e} \neq \emptyset$, then M has to be a rational or ruled surface. When M is not rational, the results follow from Theorem 3.7 and Corollary 3.8. Hence, in the following, we assume M is a rational surface.

Part 2: Definition of pretty generic tuples

We first assume e is a big J-nef class, *i.e.* a J-nef class with $e \cdot e > 0$. For the proof we need the following definition of [17]. We denote $l = l_e \geq 2$. Let $M^{[k]}$ be the set of k tuples of pairwise distinct points in M.

Definition 4.2 Fix a point $x \in M(e)$ (see Section 2.2 for the definition). An element $\Omega \in M^{[l-2]}$ is called *pretty generic* with respect to e and x if

- x is distinct from any entry of Ω;

For each $\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}^x_{red,e}$ with $x \in C_1$,

- x is not in C_i for any $i \geq 2$;
- $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, where $\Omega_i = \Omega \cap C_i$;
- $1 + w_1 = m_1 e \cdot e_1 (\geq l_{e_1})$, and $w_i = m_i e \cdot e_i (\geq l_{e_i})$ for $i \geq 2$. Here w_i is the cardinality of Ω_i.

Let G_e^x be the set of pretty generic $l-2$ tuples with respect to e and x.

It is indeed a generic set in the sense that the complement of G_e^x has complex codimension at least one in $M^{[l-2]}$ by Proposition 4.8 of [17]. In particular, the set G_e^x is path connected.

Part 3: $\mathcal{M}_{irr,e}$ is path connected when e is a big J-nef class

Now, if C and C' are smooth rational curves in $\mathcal{M}_{irr,e}$, they intersect at $l-1$ points (counted with multiplicities). If one of the intersection points $\tilde{x} \in D$ where D is an irreducible curve in $Z(e)$, we have $e_C \cdot e_D = 0$ by definition. On the other hand, since C is irreducible, we know the irreducible curve D is not identical to C since the class $e = e_C$ is big and thus $e \cdot e_C > 0 = e \cdot e_D$. Then $\tilde{x} \in C \cap D$ implies $e \cdot e_D > 0$ which is a contradiction.

Hence, all of these intersection points are in $M(e)$. First, we will show that we can deform the curve C within $\mathcal{M}_{irr,e}$ to a smooth curve $\tilde{C}$ such that all the intersection points with C' are of multiplicity one, if there are intersection points of C and C' having multiplicity greater than one. We know $\mathcal{M}_{irr,e}$ is a smooth manifold of dimension $2l$. Hence we can choose an open neighborhood U of $C \in \mathcal{M}_{irr,e}$. We look at the intersection points between elements in U and the curve C'. There are $l-1$ intersection points counted with multiplicities. Let $U' \subset U$ be a subset of U such that an element in U' is tangent to the curve C', *i.e.* intersecting at least one point with multiplicity at least two. In particular, $C \in U'$.

The following is a general fact of automatic transversality, see *e.g.* Remark 3.6 of [17]. If we have $k \leq l$ distinct points $x_1, \ldots, x_k$ in C' and $k' < k$ with $k + k' \leq l$, then the set of smooth rational curves in class e passing through $x_1, \ldots, x_k$ and having the same tangent space at the k' points $x_1, \ldots, x_{k'}$ as C' is still a smooth manifold, whose dimension is $2(l-k-k')$. Since we can vary $x_1, \cdots, x_k$ in the curve C', and $k' \geq 1$, we know U' is a submanifold of U with dimension $2(l-k-1)+2k = 2l-2$. In particular, $U \setminus U'$, which is the set of curves in U intersecting C' at points with multiplicity one, is non-empty and path connected. Moreover, elements in $U \setminus U'$ could be connected by paths to the element $C \in U'$ within $U \setminus U'$, in the sense that for any $\tilde{C} \in U \setminus U'$ there is a path $P(t) \subset U$ such that $P(1) = C$, $P(0) = \tilde{C}$ and $P([0,1)) \subset U \setminus U'$. Hence, any curve $\tilde{C} \in \mathcal{M}_{irr,e}$ could be obtained by deforming the curve C within $\mathcal{M}_{irr,e}$, such that all the intersection points of $\tilde{C}$ and C' are of multiplicity one. For simplicity of notation, we will still write the deformed curve $\tilde{C}$ by C.

We can now choose one of the intersection points of C and C', and call it x. For the remaining $l-2$ points $x_3, \ldots, x_l$, they might not be in G_e^x. Choose two more points $y \in C$ and $y' \in C'$ other than all these intersection points. We are able to choose $l-2$ disjoint open neighborhoods N_i of x_i in M with $i = 3, \ldots, l$, such that all curves representing e, passing through x, and y or y', and intersecting all N_i are smooth rational curves. This is because $\mathcal{M}_{irr,e}$ is a smooth manifold by Theorem 2.3 and there is a unique subvariety, smooth or not, passing through l given points on an irreducible curve in class e.

Since the complement of G_e^x has complex codimension at least one in $M^{[l-2]}$, we are able to choose a pretty generic $l-2$-tuple from $N_3 \times \cdots \times N_l$. With these understood, we are able to deform C and C' within $\mathcal{M}_{irr,e}$ to two smooth rational curves intersecting at x and an $l-2$-tuple in G_e^x. We still denote these two curves by C and C'.

By Proposition 4.9 of [17], the subset $\mathcal{M}_e^{x,x_3,\dots,x_l} \subset \mathcal{M}_e$ is homeomorphic to $\mathbb{CP}^1 = S^2$ when $(x_3,\dots,x_l) \in G_e^x$. Moreover, $\mathcal{M}_e^{x,x_3,\dots,x_l} \cap \mathcal{M}_{red,e}$ is a finite set of points. Since $C, C' \in \mathcal{M}_e^{x,x_3,\dots,x_l}$, they are connected by a family of smooth rational curve in $\mathcal{M}_e^{x,x_3,\cdots,x_l} \cap \mathcal{M}_{irr,e}$. This finishes the proof that $\mathcal{M}_{irr,e}$ is path connected when e is big J-nef.

Part 4: $\mathcal{M}_e$ is path connected when e is a big J-nef class

To show $\mathcal{M}_e$ is path connected, we only need to prove that any element in $\mathcal{M}_{red,e}$ is connected by a path to an element in $\mathcal{M}_{irr,e}$. This would imply $\mathcal{M}_e$ is path connected since we have shown $\mathcal{M}_{irr,e}$ is path connected. Let $\Theta \in \mathcal{M}_{red,e}$, we choose $l' = \sum e\cdot e_{C_i}$ distinct points $x_1,\dots,x_{l'}$ from the smooth part of $\Theta = \{(C_i, m_i)\}$. We choose the l' points such that there are exactly $e\cdot e_{C_i}$ points on C_i for each i, each with type (x_i, m_i) in the sense of Sect. 2.3. Counted with weights, there are $\sum m_i e\cdot e_{C_i} = l-1$ points. We then choose another point, labeled by x_l, from the smooth part of Θ and different from $x_1,\dots,x_{l'}$. By Lemma 2.5, there is a unique element in $\mathcal{M}_e$ passing through points $x_1,\dots,x_{l'}$ with multiplicities and another point x_l.

We take l disjoint open sets $N_1,\dots,N_l \subset M$ as following. For each $x_i, i \le l'$, assume it is on the irreducible component (C_j, m_j). We choose m_j disjoint open sets, say $N'_1,\dots,N'_{m_j}$, such that $\overline{N'_1}\cup\cdots\cup\overline{N'_{m_j}}$ is a neighborhood of x_i in M and $x_i \in \overline{N'_k}$ for each $1\le k\le m_i$. Considering all the points $x_1,\dots,x_{l'}$, there are $l-1=\sum m_i e\cdot e_{C_i}$ such open sets. We relabel them by $N_1,\dots,N_{l-1}$. Finally, we take a neighborhood N_l of x_l in M. Apparently, we can choose these open sets such that they are disjoint from each others.

We denote by $\mathcal{M}_{irr,e,k}$ (resp. $\mathcal{M}_{red,e,k}$) the subset of $\mathcal{M}_{irr,e}\times M^{[k]}$ (resp. $\mathcal{M}_{red,e}\times M^{[k]}$) that consists of elements of the form $(C, x_1,\dots,x_k)$ with $x_i \in C$ and distinct. There are natural projections $\pi_{irr,l}: \mathcal{M}_{irr,e,l} \to M^{[l]}$ and $\pi_{red,l}: \mathcal{M}_{red,e,l} \to M^{[l]}$. First, we notice that the diagonal elements $Z_{diag} = M^l \setminus M^{[l]}$ is a finite union of submanifolds of dimension at least two. Proposition 4.5 in [17] shows that the image of $\pi_{red,l}$, say $Z_{red} \subset M^{[l]}$, is a finite union of submanifolds of codimension at least two, and $\pi_{irr,e}$ maps onto its complement. Moreover, the map $\pi_{irr,l}$ is one-to-one. Hence, $M^l \setminus (Z_{diag} \cup Z_{red})$ is path connected. In particular, we can choose a path $P(t)$ in M^l such that $P(0) \in M^l$ is the l points with weight $(x_1, m_{k_1}),\dots,(x_{l'}, m_{k_{l'}})$ and x_l that determine Θ uniquely and $P((0,1]) \subset N_1\times\cdots\times N_l \setminus Z_{red}$. Since all the l tuples $P(t)$ determines the subvariety uniquely, the path $P(t) \subset M^l$ gives rise to a path connecting Θ to $\mathcal{M}_{irr,e}$.

Part 5: $\mathcal{M}_e$ is homeomorphic to S^2 when $e\cdot e = 0$.

When $e\cdot e = 0$, we no longer need the technicalities of pretty generic tuples. In fact, the argument here is similar to that of Theorem 3.7. Instead of finding a smooth section as in Proposition 3.6, we will use a general construction in [17] of a “dual” J-nef class. This will be used as our model for moduli space.

By Theorem 2.3, $\mathcal{M}_{irr,e}$ is a manifold of complex dimension 1 and $\mathcal{M}_{red,e}$ is a union of finitely many points. We will show that $\mathcal{M}_e = \mathcal{M}_{irr,e} \cup \mathcal{M}_{red,e}$ is actually homeomorphic to S^2. By Proposition 4.6 of [17], there is another J-nef class H_e with $g_J(H_e) = 0$ such that $H_e \cdot e = 1$. We choose a smooth rational curve S representing

H_e. Given any $z \in S$, there is a unique (although possibly reducible) rational curve C_z in class e passing through z. Thus we obtain a map $h : z \mapsto C_z$ from S to $\mathcal{M}_e$.

The map h is surjective since $H_e \cdot e \neq 0$. Since S is also J-holomorphic and $H_e \cdot e = 1$ any curve in $\mathcal{M}_e$ intersects with S at a unique point by the positivity of intersection. Therefore h is also one-to-one.

Now let us show that h is a homeomorphism, namely both h and h^{-1} are continuous. Since $S = S^2$ is Hausdorff and $\mathcal{M}_e$ is compact, if we can show that $h^{-1} : \mathcal{M}_e \to S$ is continuous, it follows that h is also continuous. To show h^{-1} is continuous, consider a sequence $C_i \in \mathcal{M}_e$ approaching to its Gromov-Hausdorff limit C. Let the intersection of C_i (resp. C) with S be p_i (resp. p). Then p_i has to approach p by the first item of the definition of topology on $\mathcal{M}_e$. Therefore h is a homeomorphism. □

4.2 $\mathcal{M}_e = \mathbb{CP}^l$ when e is primitive

In the argument of Proposition 4.1, we have shown $\mathcal{M}_e = \mathbb{CP}^1$ when $e \cdot e = 0$. We will next generalize it to show that $\mathcal{M}_e$ is homeomorphic to $\mathbb{CP}^l$ when e is primitive, hence confirm Question 5.25 of [17] in the topological sense in this circumstance. This is Theorem 1.3 and we state it again below as Theorem 4.4.

We first need a lemma to adapt the discussion of section 4.3 in [17]. This lemma is crucial in our construction of the model for the moduli space.

Lemma 4.3 *Let $M = S^2 \times S^2$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ with $k \geq 1$. Suppose $e \in H^2(M, \mathbb{Z})$ is a primitive (i.e. e is not divisible by an integer $k > 1$) J-nef class with $g_J(e) = 0$. Then there is a J-nef class H_e such that $g_J(H_e) = 0$ and $H_e \cdot e = 1$. Moreover, H_e can be assumed to be not proportional to e.*

Proof We take the class H_e to be the same ones chosen in the proof of Lemma 4.13 of [17] except if e is Cremona equivalent to H or $2H - E_1 - E_2$ when $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$.

When e is equivalent to H, without loss of generality, we assume $e = H$. We will show that at least one of $H - E_1, \ldots, H - E_k$ is J-nef. Let us first take $H'_e = H - E_1$, and assume there is a curve pairing negatively with it. By Lemma 4.15 of [17], we know an irreducible curve class $e_C = aH - b_1 E_1 - \cdots - b_k E_k$, pairing negatively with $H - E_1$, must have $a \leq 0$. Hence $a = 0$, otherwise it contradicts to the assumption that $e = H$ is J-nef. But when $a = 0$, we have $b_1 = -H'_e \cdot e_C > 0$. Moreover, since $SW(E_i) \neq 0$, we know there are J-holomorphic subvarieties in classes E_i. At least one $b_i < 0$, otherwise 0 is a linear combination of e_C and e_{E_i} which contradicts to the fact that J is tamed. Then we look at the adjunction number

$$e_C \cdot e_C + K_J \cdot e_C = -b_1^2 - \cdots - b_k^2 + b_1 + \cdots + b_k \leq 0.$$

To make sure the adjunction number is no less than -2, we will exactly have one negative b_i, say $b_2 = -1$. Other b_i's are 0 or 1. In particular, $b_1 = 1$.

Then we take the class $H - E_2$. If it is not J-nef, we can argue as in the last paragraph for $H - E_1$ to show that there is a curve class $e_{C_2} = -b_1^{(2)} E_1 - \cdots - b_k^{(2)} E_k$ with only one negative coefficient which is -1, and others are 0 or 1. If the negative coefficient is some $b_i^{(2)} = -1$ such that $b_i = 1$, then $e_C + e_{C_2}$ is a linear combination of $E_1, \ldots, E_k$

with non-positive coefficients. This contradicts to the fact that J is tamed. If the negative coefficient is some $b_i^{(2)}$ ($i \neq 2$ since $b_2^{(2)} = 1$) with $b_i = 0$, say $b_3^{(2)} = -1$, we could continue our argument with the class $H - E_3$. Since k is a finite number, this process will stop at some finite number j, such that when we argue it with a non J-nef class $H - E_j$, we will get a curve class e_{C_j} with one negative $b_i^{(j)}$ and $i < j$. Then the sum $e_{C_i} + \cdots + e_{C_j}$ is a linear combination of E_i with non-positive coefficients, which contradicts to the tameness of J again. Hence, we have shown that at least one of $H - E_1, \dots, H - E_k$ is J-nef. We choose it as H_e, which is a class satisfying the requirements of our lemma.

When e is equivalent to $2H - E_1 - E_2$, say $e = 2H - E_1 - E_2$, we claim that one of the classes, $H - E_1$ or $H - E_2$, is J-nef. We assume both $H - E_1$ and $H - E_2$ are not J-nef. By Lemma 4.15 of [17], we know an irreducible curve class $e_C = aH - b_1E_1 - \cdots - b_kE_k$ pairing negatively with $H - E_1$ or $H - E_2$ must have $a \le 0$. We are able to determine all the possible classes that pair negatively with $H - E_1$. In this case, $(H - E_1) \cdot e_C \le -1$ implies $a \le b_1 - 1$. Since $2H - E_1 - E_2$ is J-nef, we know $H - E_2$ pairs positively with e_C, which implies $b_2 \le a - 1 \le -1$. We calculate the K_J-adjunction number

$$e_C \cdot e_C + K_J \cdot e_C \le a^2 - b_2^2 - 3a + b_2 \le 2a - 1 - 3a + b_2 \le -2.$$

The equality holds only when $b_2 = a - 1$ (the second inequality) and all other b_i are 0 or 1 (the first inequality). Furthermore $b_2 = a - 1$ would imply $a = b_1 - 1$ also holds. Hence the only possible classes are $E_2 - E_1 - b_3E_3 - \cdots - b_kE_k$ and $-H + 2E_2 - b_3E_3 - \cdots - b_kE_k$ with all $b_3, \dots b_k$ are 0 or 1.

Similarly, if the class $H - E_2$ is not J-nef, then there is a curve class $E_1 - E_2 - b_3E_3 - \cdots - b_kE_k$ or $-H + 2E_1 - b_3E_3 - \cdots - b_kE_k$ with all $b_3, \dots b_k$ are 0 or 1.

We notice the classes $E_2 - E_1 - b_3E_3 - \cdots - b_kE_k$ and $E_1 - E_2 - b_3'E_3 - \cdots - b_k'E_k$ cannot coexist. We assume there is no curve class of type $E_2 - E_1 - b_3E_3 - \cdots - b_kE_k$. Then there is a curve in class $-H + 2E_2 - b_3E_3 - \cdots - b_kE_k$. At the same time, there is a curve in class $E_1 - E_2 - b_3'E_3 - \cdots - b_k'E_k$ or $-H + 2E_1 - b_3'E_3 - \cdots - b_k'E_k$. In particular, it implies that there are J-holomorphic subvarieties in classes $-H + 2E_2 - b_3E_3 - \cdots - b_kE_k$ and $-H + 2E_1 - b_3'E_3 - \cdots - b_k'E_k$. Again, $SW(E_i) \neq 0$ implies that there are J-holomorphic subvarieties in classes E_i. In turn, it would imply that there are subvarieties in classes $-H + 2E_1$ and $-H + 2E_2$. Finally $SW(H - E_1 - E_2) \neq 0$, hence $H - E_1 - E_2$ is the class of a subvariety. However, then we know $0 = (-H + 2E_1) + (-H + 2E_2) + 2(H - E_1 - E_2)$ is the class of a subvariety, which contradicts to the tameness of J.

Hence, $H - E_1$ or $H - E_2$ has to be J-nef. It is our H_e when $e = 2H - E_1 - E_2$. It satisfies all the requirements. This finishes the proof of the lemma. □

Theorem 4.4 *Suppose J is a tamed almost complex structure on a rational surface M, and e is a primitive class which is represented by a smooth J-holomorphic sphere. Then $\mathcal{M}_e$ is homeomorphic to $\mathbb{CP}^l$ where $l = \max\{0, e \cdot e + 1\}$.*

Proof When $e \cdot e < 0$, we have $l = 0$. It follows from positivity of local intersections that the smooth J-holomorphic sphere representing e is the unique element in $\mathcal{M}_e$.

When $e \cdot e \geq 0$, we know e is J-nef. We could assume $e \cdot e > 0$ otherwise it is verified in Proposition 4.1. For $M = \mathbb{CP}^2$, it is well known $\mathcal{M}_H = \mathbb{CP}^2$, see *e.g.* [9]. Actually, $\mathcal{M}_{2H} = \mathbb{CP}^5$ by the result of [29]. Hence, let us discuss $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ or $S^2 \times S^2$. Our first goal is to find a class e' such that it is J-nef, $g_J(e') = 0$ and $e' \cdot e = l$.

In Lemma 4.3, we have found J-nef class H_e such that $g_J(H_e) = 0$ and $H_e \cdot e = 1$ when e is primitive. Moreover, we could choose H_e not proportional to e.

Let $e' = e + H_e$. By adjunction formula

$$(e + H_e)^2 + K \cdot (e + H_e) = (-2) + (-2) + 2 = -2,$$

$g_J(e') = 0$ and e' is J-nef since both e and H_e are so. Moreover, the intersection number $e' \cdot e = e^2 + 1 = l$. Since $e'^2 > 0$, we have $\dim_{SW}(e') > 0$. If $SW(K - e') \neq 0$, it will contradict to the nefness of e' by $e \cdot (K - e') = -\dim_{SW}(e') < 0$. Hence by Seiberg–Witten wall-crossing, we have $SW(e') = 1$. By Proposition 4.5 of [17], we choose a smooth rational curve S in class e'. Notice by our choice of class e', the smooth rational curve S cannot be an irreducible component of any element in $\mathcal{M}_e$. Since J is tamed, any subvariety in $\mathcal{M}_e$ is connected. Take points $x_1, \ldots, x_l \in S$. Some of the points x_i might be identical. Since the curve S is given *a priori*, when we talk about the intersection of subvarieties as in Sect. 2.3, we could also include the second type where the "matching" curve at x_i is given by S.

We will show that there is a unique (possibly reducible) rational curve in class e passing through all x_i. The argument is similar to that of Lemma 2.5, with slight modifications with regard to the existence of the curve S and the corresponding second type intersections. We assume there are two such subvarieties, say $\Theta = \{(C_i, m_i)\}$, $\Theta' = \{(C_i', m_i')\}$. If Θ, Θ' have no common components, then the result follows from positivity of local intersection since $e_\Theta \cdot e_{\Theta'} < l$.

Hence we assume they have at least one common components. In particular, none of Θ and Θ' is a smooth variety.

We rewrite two subvarieties $\Theta, \Theta' \in \mathcal{M}_e$, allowing $m_i = 0$ in the notation, such that they have the same set of irreducible components formally, i.e. $\Theta = \{(C_i, m_i)\}$ and $\Theta' = \{(C_i, m_i')\}$. Then for each C_i, if $m_i \leq m_i'$, we change the components to $(C_i, 0)$ and $(C_i, m_i' - m_i)$. Similar procedure applies to the case when $m_i > m_i'$. Apply this process to all i and discard finally all components with multiplicity 0 and denote them by Θ_0, Θ_0' and still use (C_i, m_i) and (C_i, m_i') to denote their components. Notice they are homologous, formally have homology class

$$e - \sum_{m_{k_i} < m'_{k_i}} m_{k_i} e_{C_{k_i}} - \sum_{m'_{l_j} < m_{l_j}} m'_{l_j} e_{C_{l_j}} - \sum_{m'_{q_p} = m_{q_p}} m'_{q_p} e_{C_{q_p}}.$$

There are two ways to express the class, by taking $e = e_\Theta$ or $e = e_{\Theta'}$ in the above formula. Namely, it is

$$\sum_{m_{k_i} < m'_{k_i}} (m'_{k_i} - m_{k_i}) e_{C_{k_i}} + \text{others} = e_{\Theta_0'} = e_{\Theta_0} = \sum_{m'_{l_j} < m_{l_j}} (m_{l_j} - m'_{l_j}) e_{C_{l_j}} + \text{others}.$$

Here the term “others” means the terms $m_ie_{C_i}$ or $m'_ie_{C_i}$ where i is not taken from k_i, l_j or q_p.

Now Θ_0 and Θ'_0 have no common components. They intersect the rational curve S at least $e'\cdot e_{\Theta_0}\geq e\cdot e_{\Theta_0}$ points (as a subset of $\{x_1,\dots,x_l\}$) with multiplicities. In the inequality above, we make use of the fact that H_e is J-nef. Hence Θ_0 and Θ'_0 would intersect at least $e\cdot e_{\Theta_0}$ points with multiplicities.

We notice that $e\cdot e_{\Theta_0}\geq e_{\Theta_0}\cdot e_{\Theta_0}$. In fact, the difference $e-e_{\Theta_0}=e-e_{\Theta'_0}$ has 3 types of terms, any of them pairing non-negatively with $e_{\Theta_0}=e_{\Theta'_0}$. For the terms with index k_i, *i.e.* the terms with $m_{k_i}<m'_{k_i}$, we use the expression of $e_{\Theta_0}=\sum_{m'_{l_j}<m_{l_j}}(m_{l_j}-m'_{l_j})e_{C_{l_j}}+\text{others}$ to pair with. Since the irreducible curves involved in the expression are all different from C_{k_i}, we have $e_{C_{k_i}}\cdot e_{\Theta_0}\geq 0$. Similarly, for C_{l_j}, we use the expression of $e_{\Theta'_0}=\sum_{m_{k_i}<m'_{k_i}}(m'_{k_i}-m_{k_i})e_{C_{k_i}}+\text{others}$. We have $e_{C_{l_j}}\cdot e_{\Theta'_0}\geq 0$. For C_{q_p}, we could use either e_{Θ_0} or $e_{\Theta'_0}$. Since $e_{\Theta_0}=e_{\Theta'_0}$, we have $(e-e_{\Theta_0})\cdot e_{\Theta_0}\geq 0$.

Moreover, we have the strict inequality $e\cdot e_{\Theta_0}>e^2_{\Theta_0}$. This is because we assume the original Θ, Θ' have at least one common component and because they are connected by Theorem 1.5 of [18]. The first fact implies there is at least one index in k_i, l_j or q_p. The second fact implies at least one of the intersection of C_{k_i}, C_{l_j} or C_{q_p} with e_{Θ_0} as in the last paragraph would take positive value.

The inequality $e\cdot e_{\Theta_0}>e^2_{\Theta_0}$ implies there are more intersections than the homology intersection number $e^2_{\Theta_0}$ of our new subvariety Θ_0 and Θ'_0. This contradicts to the positivity of local intersection and the fact that Θ_0, Θ'_0 have no common component. Hence $\Theta=\Theta'$.

We will use $C_{x_1,\dots,x_l}$ to denote the unique subvariety passing through the l points $\{x_1,\dots,x_l\}$. Apparently, changing the order of x_i gives the same curve. Thus we obtain a well-defined map $h:(x_1,\dots,x_l)\mapsto C_{x_1,\dots,x_l}$ from $\mathrm{Sym}^l S^2\cong\mathbb{CP}^l$ to $\mathcal{M}_e$.

Since S is J-holomorphic and $e'\cdot e=e^2+1=l>0$, any curve in $\mathcal{M}_e$ intersects with S at exactly l points by the positivity of local intersection of distinct irreducible J-holomorphic subvarieties. Therefore h is one-to-one and surjective.

Now let us show that h is a homeomorphism, namely both h and h^{-1} are continuous. Since $\mathrm{Sym}^l S^2$ is Hausdorff and $\mathcal{M}_e$ is compact, if we can show that $h^{-1}:\mathcal{M}_e\to\mathrm{Sym}^l S^2$ is continuous, it follows that h is also continuous. To show h^{-1} is continuous, consider a sequence $C_i\in\mathcal{M}_e$ approaching to its Gromov-Hausdorff limit C. Let the intersection of C_i (resp. C) with S be $(x^i_1,\cdots,x^i_l)$ (resp. $(x_1,\dots,x_l)$). Then $(x^i_1,\dots,x^i_l)$ has to approach $(x_1,\dots,x_l)$ by the first item of the definition of topology on $\mathcal{M}_e$. Therefore h is a homeomorphism. □

We remark that the subvarieties determined by $(x^i_1,\dots,x^i_l)$ with $x^i_j\in M$ will not converge in general, especially when two points in the tuple converge to same point by a simple dimension counting. However when $x_i\in S$, they indeed converge in the Gromov-Hausdorff sense since the tangent plane is fixed as that of S.

Notice that since the configuration of a subvariety of a sphere class is a tree [18], a non-primitive class will never be an irreducible component in a reducible subvariety. This is because there will be another irreducible component intersecting the non-primitive class more than once to form cycles, since the reducible subvariety is connected.

Usually, nefness is a numerical way to guarantee the existence of smooth J-holomorphic curves.

Lemma 4.5 *If e is a J-nef class with $g_J(e) = 0$ and $\mathcal{M}_e \neq \emptyset$, then it is represented by a smooth J-holomorphic sphere of non-negative self-intersection.*

Proof Since e is J-nef and J-effective, we have $e \cdot e \geq 0$. Then by Proposition 4.5 of [17], we know e is represented by a smooth rational curve. □

Corollary 4.6 *Suppose J is a tamed almost complex structure on a rational surface M. If e is a J-nef class with $g_J(e) = 0$ and $\mathcal{M}_e \neq \emptyset$, then $\mathcal{M}_e$ is homeomorphic to $\mathbb{CP}^l$ with $l = e \cdot e + 1$.*

Proof It is a combination of Lemma 4.5 and Theorem 4.4. □

Theorem 4.4 also gives a vanishing result for sheaf cohomology of complex rational surfaces.

Proposition 4.7 *Let M be a complex rational surface and D a smooth rational curve with $D^2 \geq 0$. Then $H^p(M, \mathcal{O}(D)) = 0$ for $p > 0$.*

Proof First, $K - D$ is not an effective divisor since $(K - D) \cdot D < 0$ and D is a smooth divisor with $D^2 \geq 0$. Hence, $H^2(M, \mathcal{O}(D)) = H^0(M, \mathcal{O}(K - D)) = 0$. Moreover, for $p > 2$, $H^p(M, \mathcal{O}(D)) = 0$ by dimension reason.

To show $H^1(M, \mathcal{O}(D)) = 0$, we first compute the Euler characteristic $\chi(D)$. By Riemann-Roch theorem for non-singular projective surfaces,

$$\begin{aligned}
\chi(D) &= \chi(0) + \tfrac{1}{2}([D] \cdot [D] - K \cdot [D]) \\
&= \chi(0) + D^2 + 1 \\
&= \tfrac{1}{12}(c_1^2 + c_2) + D^2 + 1 \\
&= D^2 + 2
\end{aligned} \tag{10}$$

when M is a rational surface. On the other hand, it follows from Theorem 4.4 that

$$\dim H^0(M, \mathcal{O}(D)) = l + 1 = D^2 + 2.$$

Since $\chi(D) = \dim H^0(M, \mathcal{O}(D)) - \dim H^1(M, \mathcal{O}(D)) + \dim H^2(M, \mathcal{O}(D))$, we have $H^1(M, \mathcal{O}(D)) = 0$. □

5 *J*-holomorphic tori

This section is on the J-holomorphic tori, *i.e.* a subvariety in a class e with $g_J(e) = 1$. The first part is on a non-associative addition on elliptic curve induced from almost complex structures of the rational surface. In the second part, we will explore the method in last two sections to study the moduli space $\mathcal{M}_e$. We will explain our method through an example.

5.1 Non-associative addition on elliptic curve

In this subsection, we will show that the primitive case of the statement of Theorem 4.4 follows from a more general framework on the generalization of the addition in the elliptic curve theory. Hence, we start with the classical algebraic curve theory.

For an algebraic curve C of genus g, there is a natural map from the symmetric product to the Jacobian of the curve $u : \mathrm{Sym}^n C \to J(C)$ by

$$u(p_1, p_2, \dots, p_n) \to \left(\sum_i \int_{p_0}^{p_i} \omega_1, \dots, \sum_i \int_{p_0}^{p_i} \omega_g \right),$$

where $\omega_1, \dots, \omega_g$ form a basis of $H^0(C, K)$. When $n \geq 2g - 2$, this map is very useful in determining the topology of $\mathrm{Sym}^n \Sigma_g$. This matches with the philosophy of our discussion in previous two sections. The elements of symmetric product $\mathrm{Sym}^n C$ are just the divisors of degree n on C. The subset $\mathrm{Pic}^n(C)$ of the Picard group is the isomorphism classes of degree n line bundles. It is just the quotient of $\mathrm{Div}^n(C)$ modulo linear equivalence. Abel's theorem says that the map u factors through $\mathrm{Pic}^n(C)$ and the induced $\phi : \mathrm{Pic}^n(C) \to J(C)$ is a bijection.

When we take the curve C to be an elliptic curve, then the Jacobian is identified with the elliptic curve with its addition structure. In particular, the map u is now a map

$$u : \mathrm{Sym}^n(C) \to C, \ (p_1, \dots, p_n) \mapsto p_1 + \cdots + p_n,$$

where the addition is the one of the elliptic curve C. We first show that, when $n > 0$, the map is surjective. The canonical divisor K is trivial and $h^0(K - D) = 0$ for any effective divisor D. By Riemann-Roch theorem $h^0(D) = n - g + 1 + h^0(K - D) = n > 0$. Hence u is surjective. The preimage $u^{-1}(c)$ for a point $c \in C$ is the set of all possible $(p_1, \dots, p_n) \in \mathrm{Sym}^n(C)$ with $p_1 + \cdots + p_n = c$. By above calculation $h^0(D) = n$, hence the map u is a holomorphic fibration over the elliptic curve C with fiber $\mathbb{CP}^{n-1}$. In particular, we have shown the following, which we want to generalize it to our setting later.

Lemma 5.1 *The space of unordered n-tuples $(y_1, \dots, y_n)$ on an elliptic curve with $y_1 + \cdots + y_n = 0$ is homeomorphic to the space of unordered $n - 1$-tuples $(x_1, \dots, x_{n-1})$ on a rational curve, i.e. to $\mathrm{Sym}^{n-1} S^2 = \mathbb{CP}^{n-1}$.*

Topologically, we have shown the $\mathrm{Sym}^n(T^2)$ is a $\mathbb{CP}^{n-1}$ bundle over T^2. Moreover, it can be shown that the fibration is non-trivial. For simplicity of notation, we argue it for $n = 2$. A section of u is given by $c \mapsto (x, c - x)$ where c varies in the base C. We take two such sections corresponding to x and x' with $x \neq x'$. Then they intersect at only one point corresponding to the value $c = x + x'$. Hence $\mathrm{Sym}^2(T^2)$ is the nontrivial S^2 bundle over T^2.

Let it digress a bit more on the symmetric product to put the results of previous sections into our current discussion. In fact, when n is small there are less rational curves embedded in $\mathrm{Sym}^n \Sigma_g$. For example, when $n = 1$, it is clear that there is no embedded sphere unless $g = 0$. When $n = 2$ and $g > 1$, $\mathrm{Sym}^2 \Sigma_g$ always has an embedded symplectic sphere. It could be seen by choosing a hyperelliptic structure τ on Σ_g. Then $h^0(p+\tau(p)) = 2$ which provides a $\mathbb{CP}^1$. Hence, when C is hyperelliptic, the map u has a fiber $\mathbb{CP}^1$ and each other fiber a single point. This rational curve has self-intersection $1 - g(C)$. Since, a -1 curve has non-trivial Gromov-Witten invariant and T^4 does not admit any such curve, this is one way to see any genus 2 curve is hyperelliptic. When C with $g(C) \geq 3$ is not hyperelliptic, this $\mathbb{CP}^1$ is not holomorphic and is mapped to the image in its Jacobian. When n gets larger, we have more non-trivial linear systems which give us embedded projective spaces in the symmetric product.

Now we end the digression and generalize the above discussion to our setting.

As mentioned in the proof of Theorem 4.4, all non-primitive sphere classes are Cremona equivalent to $2H$ in $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, hence of self-intersection 4 and $l = 5$. Hence, for the convenience of notation, we will write $e = 2H$ in the following. When $k < 9$, we can apply the classification of negative rational curves in [32] to identify all possible subvarieties in class $2H$ and show the moduli space is $\mathbb{CP}^5$.

When $k \geq 9$, one might hope to use a method similar to what was done for primitive classes. Since both $2H$ and H are J-nef classes with $g_J(e) = 0$, we could find smooth rational curves representing $2H$ and H. Hence $\{(2H, 1), (H, 1)\}$ is a nodal curve in the case of Corollary 2 of [29]. Hence, one could find a smooth (elliptic curve) representative of $3H$. Let it be S.

For any 5 points tuples $(y_1, \dots, y_5) \in \mathrm{Sym}^5 S$, there is a rational curve in class e passing through it. However $e \cdot [S] = 6$ and each intersection is positive, we have to show that the sixth intersection with S (possibly coinciding with one of y_i) is fixed by the first five. In other words, we want to show there is a unique (possibly reducible) rational curve $C_{y_1,\cdots,y_5}$ in class e passing through $(y_1, \dots, y_5)$. Although S is an elliptic curve, the class $2H$ is spherical. Hence, the corresponding argument in Theorem 4.4 confirms the uniqueness of the rational curve $C_{y_1,\dots,y_5}$.

When J is an integrable complex structure, it is the topological interpretation of the abelian group structure of elliptic curves: for any divisor $(y_1, \dots, y_5)$ of an elliptic curve, we can canonically choose the sixth point as $-(y_1 + \cdots + y_5)$. Hence, the intersection points correspond to an unordered 6-tuple $(y_1, \cdots, y_6)$ on an elliptic curve with $y_1 + \cdots + y_6 = 0$. A conical curve determines such a 6-tuple on S by its intersection with the elliptic curve S as we will show that it follows from Cayley-Bacharach theorem. On the other hand, any such 6-tuple determines the cubic curve uniquely as we show in the last paragraph. Since the conical curves, *i.e.* the curves in class $2H$, form a linear system of dimension 5, we have shown such $(y_1, \dots, y_6)$ form a $\mathbb{CP}^5$. This is a special case and another interpretation of Lemma 5.1.

In particular, we obtain a map $h : (x_1, \dots, x_5) \mapsto (y_1, \cdots, y_6) \mapsto C_{y_1,\dots,y_5}$ from $\mathrm{Sym}^5 S^2 \cong \mathbb{CP}^5$ to $\mathcal{M}_e$. It is clear h is surjective, injective and it is continuous.

When J is a non-integrable almost complex structure, we could define a commutative addition by the exactly same way. Recall that $[S] = 3H$. Then any rational curves in class H would intersect S at three points. Let them be z_1, z_2, z_3. Since two points are enough to determine the rational curve in class H, we could define a symmetric function $z_3 = f_1^-(z_1.z_2)$, which could be viewed as the negative of an "addition". If we further take a point $O \in S$ to be the "zero", we are able to determine the addition $f_1(z_1, z_2) = f_1^-(f_1^-(z_1, z_2), O)$. Notice, in the integrable case the zero point has to be an inflection point. But we do not require this to be true, since we do not require $f_1^-(O, O) = O$. Similarly, we could associate another symmetric function f_2 to five points on S: $z_6 = f_2^-(z_1, \dots, z_5)$ where $z_1, \dots, z_6$ are intersection points of a conic with S, and $f_2(z_1, \dots, z_5) = f_1^-(f_2^-(z_1, \dots, z_5), O)$. The functions f_1, f_2 could be viewed as deformations of addition structure on the elliptic curve S.

However, in general, this new "addition" f_1 is not associative albeit commutative, thus gives rise only a loop structure instead of an abelian group. Even if we have shown that the moduli space $\mathcal{M}_H = \mathbb{CP}^2$ for any tamed almost complex structure, we do not have an authentic linear system structure in general. While our desired result $\mathcal{M}_{2H} = \mathbb{CP}^5$ follows from the generalization of Lemma 5.1 for $n = 6$ by replacing the addition by the function f_2. Notice that f_2 is not determined by f_1 in general, although it is true for the integrable case as we will explain in a moment. More generally, we expect a generalization of the fibration from $\mathrm{Sym}^6(T^2)$ to T^2 where the projection is given by $f_1^-(f_2(x_1, \dots, x_5), x_6)$.

Return to integrable complex structure, the associativity of the addition is equivalent to

Cayley-Bacharach theorem: *If two (possibly degenerate) curves in* $3H$ *intersect at* 9 *points, then any other curve passing through* 8 *of them also passes through the ninth point, where intersection points are counted with multiplicities.*

In fact, we first choose points x, y, z and the zero. Then by drawing lines, the intersections with the elliptic curve would give other 4 points $x + y, -(x + y), y + z, -(y + z)$. The ninth point has two expressions, $(x + y) + z$ or $x + (y + z)$, from two different sets of 3 lines. Then the associativity follows from the Cayley-Bacharach theorem.

This result has many other implications. In particular, it could make our previous discussion more accurate. Precisely, the addition on the elliptic curve S is determined by its intersection with lines. Namely, if S intersects a line in three points x_1, x_2, x_3, then $x_1 + x_2 = -x_3$. A conical curve C intersect S in six points $x_1, \dots, x_6$, and we can show $x_1 + \cdots + x_6 = 0$. Namely, the line L_1 passing through x_1, x_2 intersect S at a third point, say x_7. Similarly, the line passing through x_3, x_4 would intersect S at another point x_8. The line L_3 passing through x_7, x_8 intersect S at another point x_9. Finally, there is a line L_4 passing through x_5, x_9 and intersects S with another point. Cayley-Bacharach theorem implies that the point is x_6: the two set of curves $\{L_1, L_2, L_4\}$ and $\{C, L_3\}$ are both in class $3H$ and pass through 8 points $x_1, \dots, x_5, x_7, x_8, x_9$, hence $\{L_1, L_2, L_4\}$ would also intersect S at the ninth intersection point, x_6, of $\{C, L_3\}$ and

S. This implies

$$\begin{aligned} x_6 &= -(x_5 + x_9) = -x_5 + x_7 + x_8 = -x_5 - (x_1 + x_2) - (x_3 + x_4) \\ &= -(x_1 + \cdots + x_5). \end{aligned}$$

The Cayley-Bacharach theorem is not true for general tamed almost complex structure as we see above. By dimension counting, the expected dimension of $\mathcal{M}_{3H}$ is 9, hence there will be only finite many curves passing through 9 points for a generic almost complex structure.

It will be interesting to know whether this actually gives another criterion of integrability of almost complex structures.

Question 5.2 *Assume the two-variable symmetric function* f_1 *is associative, i.e.* $f_1(x, f_1(y, z)) = f_1(f_1(x, y), z)$ *for* $x, y, z \in S$*, is it true that the almost complex structure is integrable?*

In fact, we expect to express f_1 as a perturbation of addition with the extra terms determined by the Nijenhuis tensor. Apparently, Question 5.2 holds only when the zero O is chosen as an inflection point, *i.e.* a point in S which has a third order contact with a rational curve in class H.

5.2 Moduli space of tori: a case study

In this subsection, we would like to show that the method applied in last two sections could be used to study the moduli space $\mathcal{M}_e$ when $g_J(e) = 1$ and e is J-nef. Here we only analyze a single example, *i.e.* $M = \mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ and $e = -K = 3H - E_1 - \cdots - E_8$. The method could be pushed to more general cases. Some of the discussion in [20] might be useful for our discussion. However, all the subvarieties in [20] are reduced, while we are also working with general non-reduced subvarieties, *i.e.* we allow multiplicities.

We assume the class $-K$ is J-nef. We discuss the possible singular subvarieties in class $-K$ of $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$. First, we will show that the combinatorial behaviour of a reducible variety will not be too bad.

Lemma 5.3 *Suppose* e *is a* J*-nef class with* $g_J(e) = 1$*. If* $\Theta = \{(C_i, m_i)\}_{i=1}^n$ *is a connected subvariety in class* e *with* $g_J(e_1) = 1$ *and* $e_1^2 \geq 0$*, then* C_i *are rational curves for* $2 \leq i \leq n$*, and*

$$\sum_{i=1}^{n} m_i l_{e_i} \leq l_e - 1.$$

Proof By Theorem 1.4 of [18], $\sum g_J(e) \geq \sum g_J(e_i)$. Hence there is at most one C_i has $g_J(e_i) = 1$, others have $g_J = 0$. This component is just our C_1.

Recall that $l_e = \max\{\frac{1}{2}(e \cdot e - K_J \cdot e), 0\}$. It is $\max\{e^2, 0\}$ when $g_J(e) = 1$, and $\max\{e^2 + 1, 0\}$ when $g_J(e) = 0$.

Use $1, \ldots, k$ to label the irreducible components whose classes have self-intersection at least 0. In particular, e_1 is just the one in our statement satisfying $g_J(e_1) = 1$ and $e_1^2 \geq 0$. Notice $l_{e_i} = 0$ for $i = k+1, \ldots, n$.

Since Θ is connected, $e_j \cdot (e - m_j e_j) \geq 1$ for each j. Therefore l_e can be estimated as follows:

$$\begin{aligned} l_e &= e \cdot e \\ &= \textstyle\sum_{j=1}^{k} (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + \sum_{i=k+1}^{n} m_i e_i \cdot e \\ &\geq 1 + \textstyle\sum_{j=1}^{k} m_j l_{e_j} + \sum_{i=k+1}^{n} m_i e_i \cdot e \\ &= 1 + \textstyle\sum_{j=1}^{n} m_j l_{e_j} + \sum_{i=k+1}^{n} m_i e_i \cdot e. \end{aligned} \tag{11}$$

The number 1 appears in the inequality since $l_{e_1} = e_1^2$. Since e is J-nef, we have $l_e \geq 1 + \sum m_j l_{e_j}$. □

The following proposition describes reducible varieties in the J-nef class $-K$.

Proposition 5.4 *Any subvariety in $\mathcal{M}_{-K}$ is connected. All irreducible components of a subvariety $\Theta = \{(C_i, m_i)\} \in \mathcal{M}_{red,-K}$ are smooth rational curves. Moreover, they are of negative self-intersection.*

Proof Suppose there is a disconnected variety $\Theta = \cup \Theta_i$, where Θ_i are connected components. Hence $-K = \sum e_{\Theta_i}$. Since $-K$ is J-nef, we have $-K \cdot e_{\Theta_i} \geq 0$. Since $(-K)^2 = 1$, we know $-K \cdot e_{\Theta_i} = e_{\Theta_i}^2 = 0$ or 1. Since $-K = 3H - E_1 - \cdots - E_8 = \sum e_{\Theta_i}$, we know there is a Θ_i such that $e_{\Theta_i} \cdot H > 0$. Then the argument of Lemma 4.7(2) of [32] implies $e_{\Theta_i}^2 = 1$ and actually $e_{\Theta_i} = -K$. Hence Θ has only one connected component. Thus any element in $\mathcal{M}_{-K}$ is connected.

Hence, by Theorem 1.4 of [18], we have at most one C_i, say C_1, has $g_J(e_{C_1}) = 1$. Moreover, by Lemma 5.3, if $g_J(e_{C_1}) = 1$ and $e_{C_1}^2 \geq 0$, then we have $0 = l_{-K} - 1 \geq \sum_{i=1}^{n} m_i l_{e_{C_i}}$. Hence, all $l_{e_i} = 0$. In particular, $e_{C_1}^2 \leq l_{e_{C_1}} = 0$. Hence we have $e_{C_1}^2 \leq 0$ in any case. However, this contradicts to Lemma 4.7(2) of [32]. Therefore, all irreducible components are rational curves.

Now, we can similarly argue as (11) for $e = -K$

$$\begin{aligned} 1 &= e \cdot e \\ &= \textstyle\sum_{j=1}^{k} (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + (\sum_{i=k+1}^{n} m_i e_i \cdot e) \\ &\geq \textstyle\sum_{j=1}^{k} m_j l_{e_j} + (\sum_{i=k+1}^{n} m_i e_i \cdot e) \\ &= \textstyle\sum_{j=1}^{n} m_j l_{e_j} + (\sum_{i=k+1}^{n} m_i e_i \cdot e). \end{aligned} \tag{12}$$

Hence at most one index, say 1, has $e_1^2 \geq 0$. That is, $k = 1$. This happens only when the equality of (12) holds. Moreover, $e_1^2 = 0$ and $m_1 = 1$. Hence we know $K \cdot e_1 = -2$

by adjunction formula. In particular, this implies $e_1 \cdot (e - e_1) = 2$. This inequality would prevent the equality in (12) holding. Thus all the irreducible components have negative self-intersection. $\square$

In fact, since $-K$ is assumed to be J-nef, there are only a few possibilities for the negative curve classes. If a curve C has $e_C^2 < 0$, then $K \cdot e_C + e_C^2 \geq -2$ and $K \cdot e_C \leq 0$ imply C is a rational curve with $e_C^2 = -1$ or $e_C^2 = -2$.

The main obstacle to generalize the rational curve argument is that l_e points no longer determine a unique curve in class with $g_J(e) \geq 1$. However, the following lemma gives us a bit room to extend our strategy in last two sections.

Lemma 5.5 *Let $\Theta \in \mathcal{M}_{-K}$. If $x \in |\Theta|$ is a singular point. Then Θ is the unique element of $\mathcal{M}_{-K}$ passing through x.*

Proof Assume there is another $\Theta' \in \mathcal{M}_{-K}$ intersecting Θ at x. If they do not share irreducible components passing through x, then x would contribute at least 2 to the intersection number of Θ and Θ'. However, it contradicts to the local positivity of irreducible J-holomorphic curves, since $e_\Theta \cdot e_{\Theta'} = (-K)^2 = 1$.

If Θ and Θ' share irreducible components passing through x, then they are both in $\mathcal{M}_{-K}^{red}$. By Proposition 5.4, each component is a rational curve. Since x is a singular point, it must be the intersection point of at least two irreducible components of Θ. Then we can apply the same argument of Lemma 2.5 to get two cohomologous subvarieties Θ_0 and Θ_0' with no common components. If $x \in \Theta_0$, we have $1 \geq (-K) \cdot e_{\Theta_0} > e_{\Theta_0}^2$. If $(-K) \cdot e_{\Theta_0} = 0$, we have $e_{\Theta_0} \cdot e_{\Theta_0'} = e_{\Theta_0}^2 < 0$, contradicting to the local positivity of intersection. Hence $-K \cdot e_{\Theta_0} = 1$, or equivalently $-K \cdot (-K - e_{\Theta_0}) = 0$. In other words, all the removed components are -2 rational curves. However, in this case $(-K - e_{\Theta_0}) \cdot e_{\Theta_0} = \sum m_i e_{C_i}(-K - \sum m_i e_{C_i}) > 0$. Since all C_i are -2 rational curves, we have

$$\sum m_i e_{C_i} \cdot (-K - \sum m_i e_{C_i}) = -(\sum m_i e_{C_i})^2 = 2\sum m_i^2 - 2\sum m_i m_j e_{C_i} \cdot e_{C_j}$$

is an even number. Hence $1 = (-K) \cdot e_{\Theta_0} \geq e_{\Theta_0}^2 + 2$, which implies the impossible relation $e_{\Theta_0}^2 < 0$ again.

Hence Θ is the unique subvariety in class $-K$ passing through the point x. $\square$

From now on, we will further assume that there is no irreducible J-holomorphic curves with cusp singularity in $\mathcal{M}_{-K}$. Notice there is a confusion of irreducible cuspidal curves: From our subvariety viewpoint, they are tori, since $g_J = 1$; While from the J-holomorphic map viewpoint, they are rational curve, since the model curve is a sphere. By [2], all such almost complex structures form an open dense set. Under our assumption, an irreducible curve in $\mathcal{M}_{-K}$ has at most nodal singularities.

The main point to prevent the cusps is the unobstructedness result.

Proposition 5.6 *Let C be an irreducible nodal curve in class $-K$. There is a local homeomorphism $(\mathcal{M}_{-K}, C) \to (\mathbb{C}, 0)$.*

Proof Essentially, it is due to [29], which proves that the moduli space of J-holomorphic maps is smooth at an immersed J-curve with $K \cdot e_C < 0$, and locally the moduli space is $(\mathbb{C}, 0)$. In our situation, an irreducible nodal curve is the image of an immersion ϕ. As we recalled in the beginning of Sect. 2.1, if C is irreducible, it determines the map ϕ up to automorphisms. Moreover, $e_C = -K$ thus $K \cdot e_C = -1 < 0$. Hence, the result applies to our case. In particular, the neighborhood of ϕ, which is identified with our $\mathcal{M}_{-K}$ locally at C, is homeomorphic to $\mathbb{C}$ with all elements in $\mathbb{C} \setminus \{0\}$ smooth J-holomorphic maps. Hence the conclusion holds. □

Besides irreducible nodal curves, the remaining elements in $\mathcal{M}_{-K}$ are reducible varieties whose irreducible components are all smooth rational curves. These rational curves are of self-intersection -1 or -2. There are only finitely many such curves, see *e.g.* Proposition 4.4 of [32]. Hence $\mathcal{M}_{red,-K}$ is a set of isolated points in $\mathcal{M}_{-K}$. However, we do not know if it is locally Euclidean. Nonetheless, we have the path connectedness of $\mathcal{M}_{-K}$.

Proposition 5.7 *Suppose there is a singular subvariety in class $-K$. Then the moduli space $\mathcal{M}_{-K}$ is path connected when there is no irreducible J-holomorphic cuspidal curve in class $-K$.*

Proof The Gromov compactness implies the moduli space $\mathcal{M}_{-K}$ is compact. In particular, there are finitely many irreducible nodal curves by Proposition 5.6, and finitely many reducible subvarieties by the discussion in the last paragraph. Since $l_{-K} = 1$, by Seiberg–Witten theory, there is an J-holomorphic subvariety passing through any given point. However, such a subvariety might not be unique.

We will show that a smooth subvariety in $\mathcal{M}_{-K}$ is connected to any singular subvarieties in $\mathcal{M}_{-K}$ by paths. Since there are finitely many singular subvarieties, the complement M' of their support in M is path-connected. For a smooth subvariety C, choose a point $x \in C$ which is also in M'. Then for any singular subvariety Θ, choose a singular point $y \in |\Theta|$. Take a path $P(t)$ such that $P(0) = x$, $P(1) = y$ and $P[0, 1) \subset M'$. We look at the set

$$T = \{t \in [0, 1] | C(t) \text{ passes through } P(t), \text{ and is connected to } C \text{ by path}\},$$

where $C(t) \in \mathcal{M}_{-K}$ and $C(0) = C$. Because of our choice of $P(t)$, each $C(t)$, $t \in [0, 1)$, is a smooth curve. By Lemma 5.5, $C(1) = \Theta$ since y is a singular point of Θ.

The set T is non-empty because $0 \in T$, open because Proposition 5.6, and closed because of Gromov compactness. Hence $T = [0, 1]$ and we have constructed a path $C(t) \in \mathcal{M}_{-K}$ from a smooth subvariety C to a singular subvariety Θ. Hence $\mathcal{M}_{-K}$ is path connected. □

The following is Theorem 1.4.

Theorem 5.8 *If there is an irreducible (singular) nodal curve in $\mathcal{M}_{-K}$, then $\mathcal{M}_{smooth,-K}$ and $\mathcal{M}_{-K}$ are both path connected.*

Proof The path connectedness of the moduli space $\mathcal{M}_{-K}$ follows directly from Proposition 5.7.

The space $\mathcal{M}_{smooth,-K}$ is the subset of $\mathcal{M}_{-K}$ where all the elements are smooth curves. By Proposition 5.7, for any smooth curves $C, C' \in \mathcal{M}_{smooth,-K}$, we have paths $C(t), C'(t) \subset \mathcal{M}_{-K}$ with $C(0) = C, C(0) = C', C(1) = C'(1) = C_{nd}$ and $C([0, 1)), C'([0, 1)) \subset \mathcal{M}_{smooth,-K}$. Here, C_{nd} is an irreducible singular nodal curve. By Proposition 5.6, there is a locally Euclidean neighborhood U of $C(1) = C'(1)$ in $\mathcal{M}_{-K}$. Hence for t close to 1, $C(t), C'(t) \in U$. Since $(U, \{C(1)\})$ is homeomorphic to $(\mathbb{R}^2, 0)$, $U \setminus \{C(1)\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$ and $C(t)$ and $C'(t)$ are connected by a path in it. Hence C and C' are connected by a path in $\mathcal{M}_{smooth,-K}$. □

Our method does not apply to the case when $\mathcal{M}_{smooth,-K} = \mathcal{M}_{-K}$, *i.e.* when all subvarieties in class $-K$ are smooth. However, we believe it cannot happen.

Question 5.9 *For any tamed J, do we always have a singular J-holomorphic subvariety in class $-K$?*

In the integrable case, this is apparently correct. Moreover, the total number of singular points for the curves in class $-K$ is 12 by a general Euler characteristic argument. Namely, all the curves in class $-K$ form a pencil. After blowing up at the common intersection, we have the universal curve $\mathcal{C} \to \mathcal{M}_{-K} = S^2$ which is diffeomorphic to $M\#\overline{\mathbb{CP}^2} = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. The Euler number

$$e(\mathcal{C}) = e(S^2) \cdot e(T^2) + \#\text{ singular points} = \#\text{ singular points}$$

and $e(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}) = 12$. Hence we have 12 singular points in total for the curves in class $-K$.

6 More applications

In this section, we give several applications on spaces of tamed almost complex structures and symplectic isotopy. In particular, we exhibit the exotic behaviour of subvarieties of rational surfaces in a sphere class. In particular, Example 6.5 gives a connected subvariety with a genus 3 component in an exceptional curve class. Moreover, the graph corresponding to the subvariety has a loop.

6.1 Spaces of tamed almost complex structures

It is known that the space $\mathcal{J}^\omega$ of ω-tamed almost complex structures is contractible, thus path connected. Next we define $\mathcal{J}_{e-nef} \subset \mathcal{J}^\omega$.

Definition 6.1 Suppose $e \in H^2(M, \mathbb{Z})$. An ω-tamed J is in $\mathcal{J}_{e-nef}$ if e is J-nef.

This subspace is also path connected.

Lemma 6.2 *If e is represented by a smooth ω-symplectic sphere with non-negative self-intersection, then $\mathcal{J}_{e-nef}$ is path connected.*

Proof The assumption tells us $\mathcal{J}_{e-nef} \neq \emptyset$ and the ambient manifold is rational or ruled. Then the conclusion basically follows from a well-known argument [24,27]. It is known that $\mathcal{J}_{reg}$ is path connected by the argument of Theorem 3.1.5 of [24]. Recall that the subset $\mathcal{J}_{reg} \subset \mathcal{J}^{\omega}$ is defined as the set of almost complex structures such that all J-holomorphic maps $\phi : \Sigma \to M$ are regular (or the Fredholm operator D_{ϕ} is onto). When S is a smooth J-holomorphic sphere, J is regular with respect to the class $e = [S]$ if and only if $e \cdot e \geq -1$. And we know that for $J \in \mathcal{J}_{reg}$, $\mathcal{M}_{irr,e,J} \neq \emptyset$ (see e.g. [24,31]). Hence e is J-nef. This implies $\mathcal{J}_{reg} \subset \mathcal{J}_{e-nef}$. This observation ensures us to apply the same argument of Theorem 3.1.5 in [24]. Namely, the projection of $\cup\mathcal{M}_{e-non-nef,J} \to \mathcal{J}^{\omega}$ is Fredholm with index at most -2. □

When J is tamed, there are some occasions that $\mathcal{J}_{e-nef} = \mathcal{J}^{\omega}$. For example, as we have shown in Proposition 3.2, $e = T$ in an irrational ruled surface is such a class. Another such example is $e = H - E$ in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. In general, $\mathcal{J}_{e-nef} \neq \mathcal{J}^{\omega}$ however. For example, when $e = 11H - 4E_1 - \cdots - 4E_7 - 3E_8$, any J on $M = \mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ such that there are smooth curves in classes $3H - 2E_1 - E_2 - \cdots - E_8$ and $3H - E_1 - \cdots - E_8$ will do. For such a J, the sphere classes (11)-(15) in the list of Proposition 4.7 of [32], *e.g.* $e = 11H - 4E_1 - \cdots - 4E_7 - 3E_8$, are not J-nef. In fact, this example works more generally.

Proposition 6.3 *For any K-spherical class e of $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, $k \geq 8$ and $e \cdot e \geq -1$, there is an integrable complex structure J such that there are subvarieties in $\mathcal{M}_e$ with an elliptic curve irreducible component.*

Proof We start with $k = 8$.

First, we discuss a special -1 curve class. In the first construction of section 4.2 in [1], we constructed an elliptic $E(1)$ with a I_5 fiber and seven I_1 fibers. The homology classes of the components of I_5 are $H - E_1 - E_6 - E_7$, $H - E_2 - E_8 - E_9$, $H - E_1 - E_2 - E_3$, $E_2 - E_5$ and $E_1 - E_4$. And there are 7 disjoint -1 sections in classes $E_3, \ldots, E_9$. Since $(H - E_2 - E_8 - E_9) \cdot E_9 = 1$, after blowing down E_9, we have a complex structure J_0 on $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ such that there are subvarieties $\{(E_1 - E_4, 1), (F, 1)\} \in \mathcal{M}_E$ with $e_F = -K$ and the -1 class $E = 3H - E_2 - E_3 - 2E_4 - E_5 - \cdots - E_8$, where F are induced from smooth fibers of the above elliptic fibration. Notice our -1 class E is Cremona equivalent to E_1, hence sphere representable.

Then we claim that for any non-negative K-spherical class S with $S \cdot E = 0$, $\mathcal{M}_S$ also contains an element with elliptic curve irreducible component. By adjunction $g_{J_0}(S - E) = 0$. We also have $(S - E)^2 \geq -1$, $\dim_{SW}(S - E) \geq 0$ and $\dim_{SW}(S) > 0$. Hence $SW(S - E) \neq 0$, see *e.g.* Proposition 2.3 of [32], and there is a J_0-holomorphic subvariety Θ_1 in class $S - E$. However,

$$\Theta = \{(E_1 - E_4, 1), (3H - E_1 - \cdots - E_8, 1), \Theta_1\} \in \mathcal{M}_S,$$

and $g_J(3H - E_1 - \cdots - E_8) = 1$.

Now we discuss a general sphere class. It is a classical result that the -2-rational curve classes (with respect to a fixed canonical class K, say $-3H + E_1 + \cdots + E_8$) correspond to the root system of the exceptional Lie algebra E_8 (this should not be confused with the exceptional class E_8). In particular, there are 240 such classes. The Weyl group $W(E_8)$ is the group of permutations of the roots generated by the reflections in the roots. By Corollary 26.7 of [21], $W(E_8)$ acts transitively on the collection of the -1-rational curve classes (with respect to a fixed canonical class K). Since the canonical class is fixed, the Weyl group action could be realized by a Cremona transformation [14], *i.e.* a diffeomorphism preserving the canonical class. Hence, for a -1 rational curve class E' which is related to E by a diffeomorphism f, we know $\mathcal{M}_{E'}$ would also contain an element with an elliptic curve irreducible component with respect to the complex structure $f^{-1}(J_0)$.

Any K-spherical class e of non-negative square is Cremona equivalent to one of the following classes [11]

- $H - E_1$,
- $2H$,
- H,
- $(n+1)H - nE_1, n \geq 1$,
- $(n+1)H - nE_1 - E_2, n \geq 1$.

In particular, the class E_8 is perpendicular to all the above classes. Hence, for any K-spherical class e, there is an exceptional curve class E' such that $e \cdot E' = 0$. By the above discussion, there is a Cremona transformation f which transforms E' to an exceptional curve class $E = 3H - E_2 - E_3 - 2E_4 - E_5 - \cdots - E_8$. Hence $f^*(e) \cdot E = 0$. Hence, for the tamed almost complex structure $J = f^{-1}(J_0)$, we know $\mathcal{M}_e$ also has the desired property by the above discussion for the sphere classes perpendicular to E.

When $k \geq 9$, for any -1 curve class E, we know there is another -1 class E' orthogonal to it. Blow down E', we can use the induction to get the desired result for class E. Similarly, for a non-negative sphere class e with $e \cdot E = 0$, it is at least orthogonal to another -1 curve class E' with $E \cdot E' = 0$. Blow down this E', the induction would give the result for e. This completes the argument. $\square$

This proposition implies $\mathcal{J}_{e-nef} \neq \mathcal{J}^{\omega}$ for sphere classes on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k \geq 8$.

Corollary 6.4 *When $M = \mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, for any K-spherical class e there is an integrable complex structure such that e is not J-nef.*

Proof When $e \cdot e < 0$, the statement follows from gathering information of [32]. If $e \cdot H < 0$, then e is not J-nef for any tamed J since $\mathcal{M}_H \neq \emptyset$ because $SW(H) \neq 0$. If $e \cdot H = 0$, then $e = E_i - \sum_{k_j \neq i} E_{k_j}$ by Lemma 3.5 of [32]. Thus, $e \cdot E_i < 0$. Since $SW(E_i) \neq 0$, we know e is not J-nef. Finally, if $e \cdot H > 0$, we have the list in Proposition 4.6 of [32]. From the list, we know there is always a -1 curve class E_0 such that $e \cdot E_0 = -1 < 0$. Since $SW(E_0) \neq 0$, we know e is not J-nef.

When $e \cdot e \geq 0$, the statement follows from Proposition 6.3. We choose the same J as in the proof of it. If on the contrary, e is J-nef, then by Theorem 1.5 in [18], any irreducible component of a subvariety in $\mathcal{M}_e$ is a rational curve. This contradicts to Proposition 6.3. $\square$

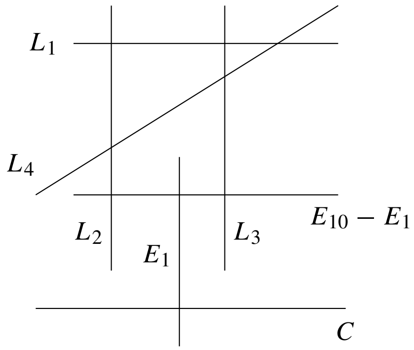
We remark that similar construction could lead to other types of interesting examples. For instance, there are examples in classes of J-genus g with nontrivial Seiberg–Witten invariant and with genus $g' > g$ irreducible components in a subvariety.

In previous examples, the subvarieties are disconnected with an elliptic curve component. Below we construct a complex surface such that there is a connected subvariety with a genus 3 component in an exceptional curve class.

Example 6.5 We start with 4 lines of general position in $\mathbb{CP}^2$. We choose one of them, say L_1, and an intersection point $p = L_2 \cap L_3$. We find a genus 3 J-holomorphic curve C in class $4H$ passing through p such that other intersection points with L_i are transversal and are not the intersection points $L_i \cap L_j$. There are 14 such intersection points.

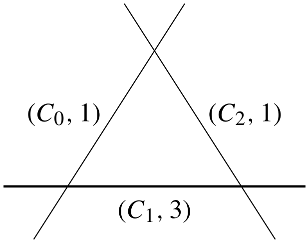
We blow up once at each of these 14 points and twice consecutively at p. The latter means that we first blow up at p to get an exceptional curve in class E_{10}, then blow up again at the intersection of this curve and the proper transform of C to get an exceptional curve, say in class E_1.

This gives us a genus 3 fibration structure on $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$. A general fiber has class $4H - E_1 - \cdots - E_{16}$. The lines L_i become rational curves in classes $H - E_2 - E_3 - E_4 - E_{16}$, $H - E_9 - E_{10} - E_{11} - E_{12}$, $H - E_{10} - E_{13} - E_{14} - E_8$, and $H - E_5 - E_6 - E_7 - E_{15}$. The curve in class E_{10} is transformed to a curve in class $E_{10} - E_1$. Below is the configuration of these curves on $\mathbb{CP}^2\#16\overline{\mathbb{CP}^2}$.



We blow down 5 sections of the fibration in classes $E_{12}, \cdots, E_{16}$. We choose a curve C_0 inherited from the general fiber. Hence $[C_0] = 4H - E_1 - \cdots - E_{11}$. There exist a curve C_1 in class $H - E_2 - E_3 - E_4$ and a curve C_2 in class $H - E_5 - E_6 - E_7$.

It is straightforward to check that $\Theta = \{(C_0, 1), (C_1, 3), (C_2, 1)\} \in \mathcal{M}_E$ is a connected subvariety, where $E = 8H - E_1 - 4E_2 - 4E_3 - 4E_4 - 2E_5 - 2E_6 - 2E_7 - E_8 - \cdots - E_{11}$ is an exceptional curve class in $\mathbb{CP}^2\#11\overline{\mathbb{CP}^2}$. Since any K_J-spherical class of square 1 is Cremona equivalent to H, we know the class $8H - 2E_1 - 4E_2 - 4E_3 - 4E_4 - 2E_5 - 2E_6$ is Cremona equivalent to $2H$. In fact, the diffeomorphism could be realized by the composition of two Dehn twists along the -2-spheres in classes $2H - E_1 - \cdots - E_6$ and $H - E_2 - E_3 - E_4$. Under this diffeomorphism, class E is transformed to $3H - E_5 - E_6 - 2E_7 - E_8 - \cdots - E_{11}$, which is representable by a smooth sphere. Hence, E is an exceptional curve class.



The graph corresponding to the subvariety Θ has a loop. The curve C_0 is a genus 3 curve with $[C_0]^2 = 5$. Moreover, $\dim_{\mathbb{C}} \mathcal{M}_{[C_0]} = 3$ locally since $K \cdot [C_0] = -1 < 0$.

For a J-nef sphere class, it was shown in [18] that any subvariety in it has two features: 1) every irreducible component is a rational curve; 2) the graph of the subvariety is a tree. In the above example, both properties fail.

Example 6.6 We use the same notation as in the previous example. Choose $\Theta' = \{(C_0, 1), (C_1, 2)\} \in \mathcal{M}_T$, where $T = 6H - E_1 - 3E_2 - 3E_3 - 3E_4 - E_5 - \cdots - E_{11}$ is a class with $g_J(T) = 1$, and $SW(T) \neq 0$. Moreover, Θ' is a connected subvariety with $g_J(C_0) = 3 > g_J(T)$.

Remark 6.7 If we look at the space $\overline{\mathcal{M}}_g(M, J, e)$ of stable J-holomorphic curves in the homology class e. There is a natural forgetful map from $\overline{\mathcal{M}}_g(M, J, e)$ to our $\mathcal{M}_e$ by just looking at the image. In general, this map is not surjective since any irreducible component of an element in the image is the image of a rational curve for a sphere class e by Gromov compactness and meanwhile our above examples show the contrary. In particular, our examples above will not contribute to any Gromov-Witten invariant.

Take an exceptional class $e = E$ for example. By Gromov compactness, for any tamed J, there is always a subvariety in class E whose irreducible components are rational curves. However, the J-holomorphic subvarieties in class E might not be unique as in our examples. In fact, for such a J, a general member of J-holomorphic subvarieties in class E will have a higher genus component.

The map from $\overline{\mathcal{M}}_g(M, J, e)$ to $\mathcal{M}_e$ is also not always injective. For example, when $M = \mathbb{CP}^2$ and $e = 2H$, $\mathcal{M}_e = \mathbb{CP}^5$ and $\overline{\mathcal{M}}_g(M, J, e)$ is the blow up of the space of double lines in $\mathbb{CP}^5$.

6.2 Symplectic isotopy of spheres

In this section, we will prove Theorem 6.9, which claims a smooth symplectic sphere S with self-intersection $[S] \cdot [S] \geq 0$ is symplectically isotopic to a holomorphic curve. Here, by symplectic isotopy, we have a two-fold meaning. Let $\mathcal{I}_\omega$ be the space of integrable complex structures tamed by ω. When $\mathcal{I}_\omega \cap \mathcal{J}_{e-nef} \neq \emptyset$, S and the holomorphic curve C are symplectic isotopic if they are connected by a path inside the space of smooth ω-symplectic submanifolds. When $\mathcal{I}_\omega \cap \mathcal{J}_{e-nef} = \emptyset$, (S, ω) and (C, I) are called symplectic isotopic if there is a path of symplectic form ω_t with $\omega_0 = \omega$ and $\omega_1 = \Omega$ such that S is symplectic with respect to all ω_t, and S and C are symplectic isotopic with respect to Ω.

By [22], the ambient manifold M is rational or ruled if $[S]\cdot[S]\geq 0$. Especially, when $[S]\cdot[S]>0$, then M has to be rational, *i.e.* $M=\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ or $S^2\times S^2$.

Lemma 6.8 *Let (M,ω) be a symplectic manifold with $b^+(M)=1$, and S is a smooth symplectic sphere with self-intersection $[S]\cdot[S]\geq 0$. Then we can find a Kähler form Ω and a path of symplectic form ω_t with $\omega_0=\omega$ and $\omega_1=\Omega$ such that S is a symplectic submanifold with respect to any ω_t.*

Moreover, $[S]$ is represented by a smooth rational curve with respect to some integrable I compatible with Ω.

Proof First notice that when $b^+=1$ cohomologous symplectic forms are symplectomorphic (see [14] for example). Hence any symplectic form cohomologous to a Kähler form is actually a Kähler form.

The result of Theorem 2.7 in [5] could be restated to adapt to our situation. It says that a cohomology class e is represented by a symplectic form with canonical class K_ω such that S is a symplectic submanifold if $e^2>0$, $e\cdot E>0$ with all K_ω-exceptional spheres E and $e\cdot[S]>0$. Since S is a symplectic sphere of non-negative self-intersection, $[S]$ is represented by a subvariety for any tamed J. Hence any Kähler form will be in above mentioned set. Moreover, since the construction of [5] is through symplectic inflation, which is a deformation of symplectic structures, then any two such cohomology classes could be connected by a path of symplectic forms. Combining the above two facts, we prove the statement.

For the second statement, we know that there is an integrable complex structure I such that $[S]$ is represented by a smooth rational curve. Then we just choose Ω such that (Ω,I) is a Kähler structure. □

Notice by [3], there are symplectic forms on ruled surface which are not cohomologous to any Kähler form (with respect to any integrable complex structure).

Now, let us prove symplectic isotopy of spheres. The result is essentially known, see *e.g.* [15] Proposition 3.2. Here we provide a proof based on our study of J-holomorphic subvarieties.

Theorem 6.9 *Any symplectic sphere S with self-intersection $S\cdot S\geq 0$ in a 4-manifold (M,ω) is symplectically isotopic to an (algebraic) rational curve. Any two homologous spheres with self-intersection -1 are symplectically isotopic to each other.*

Proof Applying Lemma 6.8, we first deform the symplectic form ω to a Kähler form Ω, and with S invariant. Then we choose an Ω-tamed almost complex structure J on M such that S is a J-holomorphic curve. Since $\mathcal{I}_\Omega\cap\mathcal{J}_{e-nef}\neq\emptyset$ by the second statement of Lemma 6.8, we know the existence of a path $\{J_t\}_{t\in[0,1]}\subset\mathcal{J}_{e-nef}$ with $J_0=J$, and $J_1=I$ by Lemma 6.2.

Consider the set T of $\tau\in[0,1]$ so that for every $t\leq\tau$ a smooth J_t-holomorphic curve $S_t\subset M$ exists that is isotopic to S. By definition, T is an interval, and within this interval S_t is symplectically isotopic to S. It is an open subset of $[0,1]$ by the unobstructedness result Theorem 2.3). It remains to show it is closed. Let $t_n\in T$ and $t_n\to\tau$. By Gromov compactness, S_{t_n} converge to a J_τ-holomorphic subvariety $\Theta=\{(C_1,m_1),\cdots,(C_k,m_k)\}$.

By Proposition 4.1, we could choose the following path consecutively:

- The first path $S_{\tau,t} \subset \mathcal{M}_{e,J_\tau}$ with $t \in [0,1]$ such that $S_{\tau,0} = \Theta$ and all other $S_{\tau,t} \in \mathcal{M}_{irr,e,J_\tau}$. This is possible because of Proposition 4.1.
- The second path $S_{t,1} \subset \mathcal{M}_{irr,e,J_t}$ with $t \in [\tau-\epsilon, \tau]$. Recall that for a given J, the image $\pi_{red,l}(\mathcal{M}_{red,e,l}) \in M^{[l]}$ is a finite union of submanifolds of codimension at least two. Hence, the complement of the union of all these images for J_t, $t \in [\tau-\epsilon, \tau]$, is an open dense set $U \subset M^{[l]}$. Choose an l-tuple in U such that all these points are on $S_{\tau,1}$. If not, we perturb the path in step 1 to achieve it. By our choice of the l-tuple, they determine J_t-holomorphic smooth curves $S_{t,1}$ uniquely for $t \in [\tau-\epsilon, \tau]$.
- The third path $S_{\tau-\epsilon,t} \subset \mathcal{M}_{irr,e,J_{\tau-\epsilon}}$ connects $S_{\tau-\epsilon,0} = S_{\tau-\epsilon}$ and $S_{\tau-\epsilon,1}$ in the second path above. This is guaranteed by Proposition 4.1.
- The last path is the original S_t which connects $S_{t-\epsilon}$ to S through symplectic isotopy.

The four paths together ensure T is closed. Hence $T = [0,1]$. Finally, any Kähler structure I is projective and holomorphic curves are algebraic since $p_g = 0$ for rational and ruled surfaces. This completes the proof of Proposition 6.9 when $[S]\cdot[S] \geq 0$.

When $[S]\cdot[S] = -1$, we have $SW([S]) \neq 0$. Hence there is always a J-holomorphic subvariety Θ_J representing $[S]$ if $J \in \mathcal{J}^\omega$. If we choose J from $\mathcal{J}_{reg}^{[S]}$, which means any J-holomorphic map in class $[S]$ is regular, this representative is a smooth rational curve.

Since $\mathcal{J}_{reg}^{[S]}$ is also path connected, any Θ_J would be symplectically isotopic to $\Theta_{J'}$ when both $J, J' \in \mathcal{J}_{reg}^{[S]}$. This follows the second statement. $\square$

The same result does not hold for higher genus curves in rational surfaces: the hyperelliptic branch loci of the examples of [26] provide symplectic surfaces not homologous to holomorphic ones in rational surfaces. On the other hand, a fairly general construction of homologous nonisotopic tori in nonrational 4-manifolds has been given by Fintushel and Stern [8], and later by many others. However, our discussion in Section 5.2 and the argument we provide above could lead to some positive results on symplectic isotopy of tori.

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