Coalescing Random Walks and Universality in Two Dimensions

by

Jamie Peter Lukins

Thesis

Submitted for the degree of

Doctor of Philosophy

in Mathematics and Statistics

Mathematics Institute

The University of Warwick

October 2020

Contents

Acknowledgments

I would like to begin by expressing my sincere gratitude for my supervisors, Doctor Roger Tribe and Professor Oleg Zaboronski, who have both shown an immense level of patience and support during my entire period of study. I am particularly indebted to Roger for his guidance and supervision, and for engaging me in an interesting and satisfying project.

I am grateful for the opportunity afforded to me by the Engineering and Physical Research Council (EPSRC) to undertake this research and produce this work. The funding I have received was an invaluable part of the puzzle. Additionally, I give thanks to the organisers of MASDOC at the Mathematics department for my inclusion to the programme.

To my fellow MASDOC cohort, and those that came before and after, I'd like to thank for your companionship and for stuggling alongside me through all the trials and tribulations of PhD life.

There are of course, too many other people who have supported me in all kinds of ways throughout the course of my studies. I hope that those that are not named here specifically will still know they are appreciated. However, I'd like to acknowledge the impact of my secondary school teachers Melinda Aldred and Sarah Blainey Ansquer in inspiring in me an interest in learning, nurturing my young mind and setting me on the journey that culminates here. Finally, special thanks go to the Lukins family, the Bakewell Stuarts, Endika Arquero Ugarte for their continued encouragement, emotional support and cups of tea.

This thesis is dedicated to the life and memory of my father,

Colin Graham Lukins,

13/04/1956 - 06/02/2018.

♦

"In life, all anyone is looking for is that Second Moment"

-Roxanne Douglas, 15/02/2019.

Declarations

I, Jamie Peter Lukins, declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this dissertation is submitted to the University of Warwick for the degree of Doctor of Philosophy, and has not been submitted to any other university or for any other degree.

Abstract

We study infinite systems of coalescing nearest neighbour random walks on the integer lattice, $\mathbb{Z}^2$. We are interested in the decay of the probability that the origin is occupied as time increases. This is a well known result for the case that the random walks coalesce instantaneuously and was first proved by Bramson and Griffeath in [2]. We rederive this result and strengthen it by providing an error bound by using the methods employed by van den Berg and Kesten in [27], where they worked in dimensions greater than 2. We further study coalescing random walks that do not coalesce immediately on collision, but can occupy the same site for an exponential (rate $\lambda \in (0, \infty)$) random time before coalescing, in this was they have a chance to walk away before coalescing. We derive the analogous asymptotic for the decay of the probability of the occupation of the origin and find that, in two dimensions, this decay is independent of the coalescence rate λ in the leading order and agrees with the decay for the instantaneuously coalescing walks.

Chapter 1

Introduction

1.1 Introduction to Coalescing Random Walks and Relevant Background

We study an infinite system of independent continuous time random walks on the integer lattice that can react on collision. There will be an initial spread of particles on the entire lattice $\mathbb{Z}^2$ and each particle will evolve by independent rate 1 continuous time simple random walks until there is a reaction. Specifically, the reaction we are interested in is coalescence of the colliding particles. When a particle moves to a site that is occupied by another particle there is the potential for the particles to react and coalesce forming a particle that now follows a random walk that is independent of the walks of its parent particles. We will consider two regimes, instantly coalescing particles and slowly coalescing particles. The topic of interest is the decay in the density of the particles in space as time grows large.

Such systems and the decay in density have been studied for many decades.

An infinite system of instantly coalescing nearest neighbour random walks was constructed in [12] in all dimensions by making use of a graphical constuction using the well known methods that can be found in [16] and [6]. Due to the instantaneous nature of their reactions, there can only be at most one particle per site. Both there and in [11], the authors asked the question of the rate of decay in the density of the particles. One sensible measure of the density is the occupation of the origin by a particle at large times. In [11] and [12] it is determined quite naturally that the probability that a particle can be found at the origin at large times t tends to 0 as t tends to infinity. The asymptotic rate at which this quantity decays to 0 was then found for all dimensions in [2]. If we call this quantity $p_d(t)$, with d being the dimension, then the leading order of the rate of decay was found to be

$$p_d(t) \sim \begin{cases} \frac{1}{\sqrt{\pi t}} \text{ when } d = 1 \\ \frac{\log(t)}{\pi t} \text{ when } d = 2 \\ \frac{1}{\gamma_d t} \text{ when } d \geq 3 \end{cases} \tag{1.1}$$

where by $f(t) \sim g(t)$ we mean that $lim_{t\to\infty} f(t)/g(t) = 1$ and γ_d is the probability that a random walk in d dimensions starting at the origin never returns to the origin. These are the so-called Pólya constants. Their methods do not provide a bound on the lower order terms.

Some time later in [27], van den Berg and Kesten studied a variant for dimensions ≥ 3 of the infinite system where the particles do not necessarily coalesce upon meeting. Instead, particles can stack. In their model, if a particle jumps to a site that contains a pile of particles then this particle will coalesce with one of the particles already present with some probability depending on the size of the stack. If the probabilities are chosen so that the probability that particles coalesce given that there is only 1 particle present, then their model exactly recovers the instantaneously coalescing random walk system as Bramson and Griffeath in dimensions ≥ 3.

For their more general process, they were also interested in the decay in the probablility of the occupation of the origin at large times. In particular, they actually study the expected number of particles at the origin at time t since there can be more than 1 particle at a site at any time. Not only does their method recover the original result of Bramson and Griffeath, as this is a specialisation of their model, it also strengthens the result by providing an estimate for the error in the leading order asymptotic that wasn't given originally.

If the probabilities are chosen as to allow for stacks of particles, then it was shown that the expected number of particles at the origin at time t has the same leading order asymptotics as for the probability that there is only 1 particle present at the origin at time t. That this is true, is a result of a property of some cancelative processes such as coalescing systems. These properties are sometimes given the name negative correlation, and in this instance its significance is that in systems of coalescing walks the probability that there are two or more particles at a site should be unlikely. Negative correlation will be a main focus of one of our chapters. Various notions of negative correlation exist and can be reviewed in [20] and [21]. Arratia proved appropriate negative correlation results for the instantaneous coalescing random walks in [1], making use of duality with the voter model. Note, that for van dan Berg and Kesten, while coalescence does not always occur, when it does occur, the reaction is an instantaneous one. They improve the asymptotic for the instantly coalescing random walks in dimensions ≥ 3 and also show that asymptotic for the expected number of particles at the origin is

$$p_d(t) = \frac{1}{C_d t} + O\left(\frac{1}{t^{1+\zeta}}\right)$$

for some $\zeta > 0$ and some absolutely determined constant C_d depending on dimension and the choice of probabilities. In particular, this constant reduces to γ_d when the probability that a particle coalesces given that there is at least 1 particle at the coincident site is chosen to be 1. They are able to achieve the error estimate by using a method that differed greatly from the original result. Their method is based on a heuristic argument which we repeat in the next section for the instantaneous coalescing particle system. To make their arguments rigorous, they were able to derive an approximate differential equation for the quantity in interest, controlling the error in this approximation so that they can prove that the solution to the approximate equation has correct asymptotic behaviour.

We have three aims:

1. The first is to sharpen the result of Bramson and Griffeath for the instantly coalescing random walks in dimension 2 by providing an error bound for the lower order terms like van den Berg and Kesten were able to do in dimensions ≥ 3.

2. The second, is to study a non-instantly coalescing system in dimension 2 for which the coalescing mechanism is governed differently from the mechanism studied by van den Berg and Kesten. We will call these systems slowly coalescing walks. This is a system in which particles may occupy the same site and only coalesce at a certain exponential rate $\lambda > 0$. That is to say, they must occupy the same site for a random exponentially distributed time in order to react so that the particles have a chance to walk away before coalescing.

3. Thirdly, that the decay in the density, as measured by the expected number of particles at the origin at time t, enjoys a certain universality property in dimension 2. Call $p_2^\lambda(t)$ the expected number of particles. The leading order term in the decay in density will be seen to be independent of the rate of coalescence. It will remain unchanged if we replace $p_2(t)$ with $p_2^\lambda(t)$ for any $\lambda > 0$. In fact, it will be shown that

$$p_2^\lambda(t) = \frac{\log(t)}{\pi t}\left(1 + O\left(\frac{1}{\sqrt{\log t}}\right)\right) \tag{1.2}$$

for all $\lambda > 0$. Note, we can consider the instantly coalescing particles as a slowly coalescing system with $\lambda = \infty$. The constant factor, π, in (1.2) is therefore a universal constant amongst this class of coalescing random walks.

This universalilty phenomemon is restricted to processes in dimensions 2 or fewer, and it seems to be entirely determined by the recurrent nature of simple random walks in these dimensions. It can be seen easily by repeating for dimensions ≥ 3 the work done here for dimension 2, that slowly coalescing walks in dimensions ≥ 3 do not have this universality property. Indeed, the asymptotics for $p_d^\lambda(t)$ with

$d \geq 3$ turn out to be

$$p_d^\lambda(t) = \frac{1}{\lambda \beta_d t} + O(t^{-(1+\zeta)}) \tag{1.3}$$

for $\zeta > 0$ and $\beta_d = \gamma_d/(\gamma_d + \lambda)$. (Again the instantaneous system is recovered by letting $\lambda \to \infty$ since then $\lambda\beta_d = \lambda\gamma_d/(\gamma_d + \lambda) \to \gamma_d$.) The constants β_d have the interpretation that they are the probability that in a system of just two particles both starting at the origin, these particles never coalesce. Of course, in the transient dimensions, the Pólya constants γ_d are strictly positive but in the recurrent dimensions $\gamma_d = 0$, it is in this difference that we see the introduction of π and the loss of dependence on λ. There is some intuition to support the claim of this dichotomy of universality at the critical dimension 2. Since recurrent walks are bound to meet each other infinitely often, and also each bound to reach the origin and return infinitely often, it becomes less of a question of if particles will coalesce but when they will. As such, the information of how slowly the particles coalesce is lost in the asymptotic. No matter how quickly or slowly the particles react (compared to the rate at which they walk, in our case rate 1) they are asymptotically bound to have met at a coincident site and remained together for an exponential random time sufficientlly long enough to react. In the transient dimensions, there is positive probability that a pair of particles never meet, let alone spend enough time together to react. As such, in each and every meet that they have, if any, their coalescence depends strongly on the interplay between the coalescence rate and the walk rate. Therefore, the information on the coalescence rate is retained in the asymptotic. The calculations for $d \geq 3$ largely follow the calculations presented in Chapter 4. The proof of these results only differ significantly in two ways. Firstly, the order of the error estimates are different and their calculation is easier since they do not rely on strengthening the bounds from come from negative correlation. Secondly, the random walk estimates are easier due to transience and the appearence of the Pólya constants. The existence and uniqueness of the stochastic differential equation and the negative correlation properties immediately generalise to all dimensions.

The original method of Bramson and Griffeath to prove the decay for instantly coalescing random walks required a deep theorem of Sawyer, given in [24], relating to the voter model which is dual to the the instantly coalescing random walks. This was fundamental for the identification of the constants given in equation (1.1) for the leading order term. For more general coalescing random walks, it is not always clear that a useful dual exists in order to mimic the original method. And even if we could, more work would be required to develop the error bounds that the original result forgoes. However, the heuristic argument of van den Berg and Kesten has been used successfully elsewhere for different processes. See for example [25] and [27]. So not only does it have the benefit that it produces a bound for the error in the large time approximation, but it also proves to be quite robust. Of course, having good control of the error bound has the immediate benefit of also providing tight enough estimation to correctly identify the constant in the leading order.

While we do employ their heuristic, our calculations differ significantly since we construct our processes as the solution to a system of stochastic differential equations (SDEs) driven by Poisson processes rather than by the graphical construction. The constructions are very closely related but in proving the error estimates we can make extensive use of the SDEs through calculus. These calculations are given for the instantly coalescing process in Chapter 4 together with the asymptotic of the decay in density for the finite rate reaction process. Universailty follows immediately from this asymptotic.

While we will be chiefly interested in an initial condition of one particle per site, we can investigate other deterministic and some random initial conditions. Random initial conditions for instantly coalescing random walks have been studied in [2] and [3]. However, as seen in [15], random initial conditions can sometimes disrupt the negative correlation properties that would otherwise have held. So some care will need to be taken about the class of random initial conditions that we can consider.

It is also possible to consider massive coalescing particles. These particles will have a mass associated to them and upon coalescence, they will form a particle that has a mass of the sum of its parents masses. In these models, there is only ever one particle per site, but we can ask the question of the expected mass of a particle at the origin at time t. A system of massive coalescing random walks with spontaneous loss or gain of unit mass (evaporation/deposition respectively) has been studied in [4], for example.

We now give definitions and formulate our problem. In the instantaneous case, we begin with the maximal initial condition of a particle occupying every site of $\mathbb{Z}^2$ and define the family

$$(P_t(x,y) : t \geq 0,\ x, y \in \mathbb{Z}^2,\ x \sim y) \tag{1.4}$$

of independent, identically distributed (IID) rate 1/4 Poisson processes that control the jumps of a particle at x to a neighbouring site y. The total rate of walking away from x is then 1. Let $\xi = \{\xi_t(x) | x \in \mathbb{Z}^2, t \geq 0\}$ be our infinite system of coalescing random walks on $\mathbb{Z}^2$. Then each coordinate of ξ records whether or not the site x is occupied at time t, that is

$$\xi_t(x) = \begin{cases} 1 & \text{if } x \text{ is occupied at time } t, \\ 0 & \text{if } x \text{ is not occupied at time } t. \end{cases}$$

The SDE that governs the occupation of x is

$$d\xi_t(x) = \sum_{y:y\sim x} \Big(\mathbb{1}\{\xi_{t-}(y) = 1, \xi_{t-}(x) = 0\} dP_t(y,x) - \mathbb{1}\{\xi_{t-}(x) = 1\} dP_t(x,y) \Big). \tag{1.5}$$

Indeed the first terms arises from the introduction of a particle to x coming from one of its neighbours y and the second term arises from the departure of a particle from the site x.

In the non-instantaneous case, let $0 < \lambda < \infty$ be the finite reaction rate for coalescence. Spread an initial distribution of particles of one per site. Since particles have a finite rate of reaction, it is entirely possible for mulitple particles to occupy the same site. Therefore, in the slowly coalescing regime $\xi = \{\xi_t(x) | x \in \mathbb{Z}^2, t \geq 0\}$ will be an $\mathbb{N}^{\mathbb{Z}^2}$ valued process with $\xi_t(x) \in \mathbb{N}$ for all $x \in \mathbb{Z}^2$ and $t > 0$ It denote the number of particles present at x at time t. We introduce the following Poisson families

- $(P(i,x,y) : i \in \mathbb{N},\, x, y \in \mathbb{Z}^2,\, x \sim y)$ of IID rate 1/4 Poisson processes that govern the jump of the ith particle at x to the neighbouring site y.
- $(P^c(i,j,x) : i, j \in \mathbb{N},\, x \in \mathbb{Z}^2)$ of IID rate λ Poisson processes that will control the coalescence of the ith particle onto the jth at site x (the total rate of coalesence for each pair will be 2λ).

The SDE that governs the occupation number of x is

$$d\xi_t(x) = \sum_{y: y\sim x} \sum_{i\geq 1} \Big(\mathbb{1}\{i \leq \xi_{t-}(y)\} dP_t(y,x) - \mathbb{1}\{i \leq \xi_{t-}(x)\} dP_t(i,x,y) \Big) - \sum_{i,j\geq 1} \mathbb{1}\Big(\max\{i,j\} \leq \xi_{t-}(x), i \neq j \Big) dP^c(i,j,x). \tag{1.6}$$

Indeed, the only changes that can occur in the particle numbers in the system is accounted for by particles walking to and from a site and the coalescence of particles at a site.

In Chapter 2, we will prove that equations (1.5) and (1.6) have unique solutions amongst a suitable class of processes. Since the particle numbers for (1.6) can be unbounded, the convergence of the infinite sums in (1.6) is called into question. To circumvent this we will construct a solution of (1.6) as a limit of a sequence of solutions to modified SDEs that keep the number at each site capped. In this way, in each approximating system of SDEs there are only finite sums and so no issue of convergence. Some of the proofs of results in this chapter are given in the appendix.

As noted already, the concept of negative correlation will be the focus of Chapter 3. The results here will reflect the intuition that for coalescing random walks the presence of a particle at one site lowers the chance of a particle elsewhere. And, for slowly coalescing random walks, the presence of a particle at a site lowers the chance of another particle sharing the site because they have had the chance to coalesce. They will be needed especially in the estimation of each error in the approximations.

Correlation inequalities are often proved using the van den Berg-Kesten-Reimer (BKR) inequality which was conjectured to hold in full generality in [29] and proved by Reimer [22]. For the application of the inequality we require a finite, discrete time structure. Negative correlation results like the ones we will require were established for van den Berg and Kesten's randomly coalescing random walks in [28] by discretizing the time axis and cutting off sites that are too far from the origin in order to make use of the BKR inequality. We will prove our results in a different way. We will define a discrete time process on a symmetric box about the origin of fixed size that mimics our continuous time process but only allows for a bounded number of particles per site. For this process, negative correlation results can be proved using the BKR inequality. It will then be shown that our full continuous time process can be obtained from the discrete time, finite volume process by means of a chain of limit theorems, first by passing to continuous time, then by allowing for unbounded number of particles and finally by allowing the length of the box tend to infinity. This allows us to prove statements like if $\xi_t(x)$ is the solution to (1.6) then for all $x \neq y$

$$\mathbb{E}[\xi_t(x)\xi_t(y)] \leq \mathbb{E}[\xi_t(x)]\mathbb{E}[\xi_t(y)]$$

and

$$\mathbb{E}[\xi_t(x)(\xi_t(x) - 1)] \leq \mathbb{E}[\xi_t(x)]^2$$

which capture the intuitive properties that slowly coalescing random walks should enjoy, as discussed in the preceding paragraph.

Chapter 5 will collect various random walk estimates that are also necessary for our estimates. One of these estimates has a significant relation to our universality for the class of slowly coalescing random walks. It is an asymptotic estimate for the probability that a random walk starting at the origin never spends more than a rate λ exponential random time there on any of its visits by time t, for now we call it $\gamma_2^\lambda(t)$. It is the analogy to the role played by γ_d in dimensions $d \geq 3$. The other important estimate is the analogy to Lemma 12 of [27] where they prove, for non-interacting random walks (although baring the instantly coalescing random walks in mind as an application) in $d \geq 3$ that the probability that two independent random walks starting at some distinct x and y travel to the origin and one of its neighbours e respectfully by time t without coalescing is well approximated by γ_d times the product of the free random walk probability transition functions for the random walk started at the origin to x and $y - e$ respectfully through time t. We use free here to mean that the trajectories have been decoupled and have no interaction through time t. Since we don't mind if our slowly coalescing particles meet, our result is similar but with γ_d swapped out for $\gamma_2^\lambda(t)$.

1.2 The Heuristic Argument of van den Berg and Kesten

We will repeat here the heuristic argument of van den Berg and Kesten given in either of [27] or [28]. We will give the argument for the instantaneously coalescing random walks. Our starting point is the exact ordinary differential equation given by compensating the poisson processes in (1.5) (by which we mean writing $dP_t = d(P_t - t/4) + dt/4$) and taking expectation to arrive at

$$\frac{d}{dt}\mathbb{E}[\xi_t(x)] = -\mathbb{E}[\xi_t(O)\xi_t(e)]$$

where e is any of the four neighbours of the origin and this is independent of the choice of neighbour by translational and rotational invariance. Write $\hat{\xi}_t = \mathbb{E}[\xi_t(x)]$ which is independent of x by translation invariance. In each approximation below there is an error incurred. We shall not be explicit here as to the order of each error but we will indicate the relevant result in which the correct error is specified. The expectation in the right hand side is equal to the probability that there is a particle present at both the origin and one of its neighbours e at time t since the argument of the expectation is just a product of indicator functions. For there to be a particle at the origin and a neighbour of the origin there must have existed sites x and y such that there had been particles located at each of those sites at some earlier time $t-s$. These particles must then have walked to the origin and a neighbour respectively without meeting along the way, or else the particles will have coalesced. So, using the informal notation $x \to y$ to mean that a particle at x walks to y through time s, we have

$$\begin{aligned}\frac{d\hat{\xi}_t}{dt} = -\sum_{x,y} \mathbb{P}[&x \text{ is occupied at time } t-s, y \text{ is occupied at time} \\ &t-s, x \to O, y \to e, \text{ paths do not meet in the interval } [t-s,t]].\end{aligned}$$

The time s is to be chosen carefully so that it is large enough compared with t so that the sites x and y that contribute to the derivative are sites that are well seperated, but it must be sufficiently small compared to t to avoid, for example, the possibility of the particle at x coinciding with another particle while on its way to the origin since then this other particle would have been located at another site x' at $t-s$. Choosing s small enough will guarantee that the main contribution to the derivative comes from particles that have no coalescing event over the interval $[t-s,t]$ and so the sites x and y are, in a sense, well defined in that we can trace back the location of the particles to unique sites at time $t-s$. The advantage of the sites x and y being "well seperated" is that there is then near independence of the events $\{x \text{ is occupied at time } t-s\}$ and $\{y \text{ is occupied at time } t-s\}$. This near independence is quantified by a variance estimate Lemma 4.1.8. So as a first approximation (by Lemma 4.1.6) we have

$$\begin{aligned}\frac{d\hat{\xi}_t}{dt} &\approx -\sum_{x,y} \mathbb{P}[x \text{ is occupied at time } t-s] \\ &\qquad \times \mathbb{P}[y \text{ is occupied at time } t-s] \\ &\qquad \times \mathbb{P}[x \to O, y \to e, \text{ paths do not meet in the interval } [t-s,t]] \\ &= -\hat{\xi}_{t-s}^2 \sum_{x,y} \mathbb{P}[x \to O, y \to e, \text{ paths do not meet in the interval } [t-s,t]].\end{aligned}$$

By time reversal of the simple random walk we can replace

$$\mathbb{P}[x \to O, y \to e, \text{ paths do not meet in the interval } [t-s,t]]$$

by

$$\mathbb{P}[O \to x, e \to y, \text{ paths do not meet in the interval } [t-s,t]].$$

The random walk paths in the event

$$\{O \to x, e \to y, \text{ paths do not meet in the interval } [t-s,t]\}$$

must not meet through time s. Choosing s carefully will allow us to decouple the random walks and approximate this probability by the product of two free probability transition densities which can meet, together with a correction factor to account for this. This factor has leading order $\pi/\log s$ and is the probability that a random walk in $d=2$ started at the origin does not return to the origin by time s. This is the main diversion from the heuristic of van den Berg and Kesten. In dimensions $d \geq 3$, the transience of the random walks implies the probabilities γ_d are strictly positive. Of course, in two dimensions, simple random walks always return eventually to the origin and so $\gamma_2 = 0$. This random walk estimate is proved in Chapter 5. This leads us to

$$\frac{d\hat{\xi}_t}{dt} \approx -\frac{\pi\hat{\xi}_{t-s}^2}{\log s}. \tag{1.7}$$

With the appropriate choice of s, Lemma 4.1.10 shows that $\hat{\xi}_{t-s}$ is close to $\hat{\xi}_t$ and also $\log s$ will be close to $\log t$ so that we arrive at the approximate ordinary differential equation for $\hat{\xi}_t$

$$\frac{d\hat{\xi}_t}{dt} \approx -\frac{\pi\hat{\xi}_t^2}{\log t}. \tag{1.8}$$

Finally, notice that if we write $p_t = \frac{\log t}{\pi t}$, then taking its derivative we have

$$\begin{aligned}\frac{dp_t}{dt} &= \frac{1}{\pi t^2} - \frac{\log t}{\pi t^2} \\ &= \frac{1}{\pi t^2} - \frac{\pi p_t^2}{\log t}\end{aligned}$$

so that p_t is nearly the solution of the solution to (1.8). The results of Chapter 4 make these approximations rigorous and show that p_t is indeed the leading order of the solution to (1.8) and provides the necessary bounds on the errors to conclude

$$\hat{\xi}_t = \frac{\log t}{\pi t}\left(1 + O\left(\frac{1}{\sqrt{\log t}}\right)\right). \tag{1.9}$$

The same approximations and estimations will allow us to conclude the same asymptotic for the slowly coalescing randoms walks and deduce universality in the second section of Chapter 4.

1.3 Applications and Further Work

We briefly describe some applications of the results of this work.

1.3.1 Application to the N-point Correlation Function and Non-Coalesence Probabilities

Consider the instantaneously coalescing random walks on $\mathbb{Z}^2$ with each site occupied initially. Once the asymptotic decay (1.9) is established for the probability that the origin is occupied at large times, it is natural to ask what is the asymptotic decay of the correlation function

$$\mathbb{P}[x_1, \ldots, x_N \text{ are occupied at time } t] = \mathbb{E}[\xi_t(x_1) \cdots \xi_t(x_N)]$$

for distinct $x_1, \ldots, x_N$. This gives a more general measure of the decay in density of a system of coalescing random walks than $\hat{\xi}_t$. This has been answered by Lukins, Tribe and Zaboronski [17], using results and similar ideas as that of this thesis and confirms the prediction of Munasinghe, Rajesh and Zaboronski [19] where predictions were made in all dimensions using a renormalisation group method. It is given by

$$\mathbb{P}[x_1, \ldots, x_N \text{ are occupied at time } t] \tag{1.10}$$

$$= \frac{c_0(x_1, \ldots, x_N)}{\pi^N} (\log t)^{N - \binom{N}{2}} t^{-N} \left(1 + O\left(\frac{1}{\log^{\frac{1}{2}-\delta} t}\right)\right) \tag{1.11}$$

for some constant $c_0(x_1, \ldots, x_N)$ and any $0 < \delta < 1/2$. The constant $c_0(x_1, \ldots, x_N)$ and the logarithmic correction $(\log t)^{\binom{N}{2}}$ accounts for the discrepancy in the use of the negative correlation that results in a bound of

$$\mathbb{E}[\xi_t(x_1) \cdots \xi_t(x_N)] \leq \hat{\xi}_t^N$$

(see Chapter 3). This result therefore, is of interest not just as a measure of the decay of density of particles but also because it quantifies the error in the use of negative correlation.

The two key tools needed for the application are the sharpened asymptotic

$$\hat{\xi}_t = \frac{\log t}{\pi t} \left(1 + O\left(1 + \frac{1}{\log^{1/2} t}\right)\right)$$

given in Chapter 4 and the non-collision probability $P_{NC}(t)$, that no pair from a finite collection of N two dimensional random walks meet by time t. The asymptotic for $P_{NC}(t)$ is known and was proved by Cox, Merle and Perkins in [5]. There, it is given that $P_{NC}(t) \sim c_0 \log(t)^{-\binom{N}{2}}$, however it is rederived in [17] using the asymptotic given for the probability that only two random walks manage to avoid each other by time t, the full strength of which is given by the asymptotic

$$\frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right)$$

in Lemma 5.1.2. The result of the rederivation is a sharpened asymptotic for $P_{NC}(t)$

given by

$$P_{NC}(t) = c_0 \log(t)^{-\binom{N}{2}} \left(1 + \frac{\log\log t}{\log t}\right).$$

Not only do the asymptotics for both the N-point density

$$\mathbb{P}[x_1, \ldots, x_N \text{ are occupied at time } t]$$

and the non-coalescence probability $P_{NC}(t)$ depend upon and generalise results in this thesis, but their derivations also showcase the strength of van den Berg and Kesten heuristic, in that they also arise as the solution to an approximate SDE. To the best of our knowledge, the instances in this thesis and the paper [17] are the only ones for which this strategy has been employed for these types of problems in 2 dimensions, having only previously been used in dimensions ≥ 3.

There is future work do be done in this area. The dependence on the sites $x_1, \ldots, x_N$ in the constant c_0 is unknown although it has been conjectured in [17] that

$$c_0(x_1, \ldots, x_N) \approx \Pi_{i<j} \log\left(|x_i - x_j|^2\right) \tag{1.12}$$

so long as

$$O(1) << |x_i - x_j| << O(t^{1/2})$$

so that the walks starting at x_i and x_j have a chance to meet.

It is likely that the same method as employed in [17] will work in $d \geq 3$, and similar methods have been used in $d = 1$ in [18] to produce the correct decay in time in the leading order but the estimates were not strong enough to get the correct constant. This constant is known and was found later in [26] where the authors recognised that the system was exactly solvable using the Karlin-McGregor formula for Brownian motions and takes on a similar form as the conjecture (1.12).

This work would also extend easily to the slowly coalescing random walks, where the independence of the coalescing rate is expected to hold firm.

1.3.2 Slowly Coalescing Random Walks in All Dimensions and Related Problems

We have already briefly discussed slowly coalescing random walks in $d \geq 3$. The same methods used in this thesis derive the asymptotics given in equation (1.3). It is not clear if these methods can be extended to $d = 1$. The problem seems to be regarding the strength of the negative correlation bounds. As we will see in this thesis, the pairwise negative correlation bounds as they are, are not strong enough alone to provide an error bound of low enough order for most of our approximations. For us, it is necessary to use recurrence for the random walks to force an a logarithmic correction that is apparent in the formula for the generalised correlation function (1.10). This extra factor of a logarithm in our pairwise correlations proves vital in obtaining an error bound tight enough to solve the SDE. In the transient dimensions, the pairwise correlation bounds are immediately strong enough to be applied and do not need any improvement. In $d = 1$, improvements will certainly need to be made in order to mimic the heuristic presented here. One thing that can be considered in the future, is if the pairwise correlations in $d = 1$ are too strong for this to be done.

At least for $d \geq 2$ we can ask more questions related to slowly coalescing random walks. One potential topic of interest, is a system of multiple species that react with each other, and themselves, at different rates.

We give an example. Suppose in $d \geq 3$ we have a system of particles that we shall call A particles, that in the absence of any other species, behave as a slowly coalescing system with rate λ_A and walk rate 1. Therefore, their rate of decay of density is given by (1.3), with λ_A replacing λ. We introduce another species of particles labelled B that walk with some rate D and have a rate λ_B at which it will coalesce with an A particle and lose its B label if they share a site. We can think of the B particle being deleted or annihilated by the reaction with the A particle. The question now is determine the rate of decay of the B particles. Let $p_d^B(t)$ be the probability that the origin is occupied by a B particle at time t.

We highly suspect that this can be achieved in all $d \geq 2$ through the use of the van den Berg and Kesten heuristic, and making use of the ideas presented in this work, particularly in $d = 2$, where the recurrence will present an issue. However, the existence and uniqueness of the SDEs and the results in the negative correlation chapter for the A particles are applicable in any dimension.

We conjecture that, for $d \geq 3$,

$$p_d^B(t) \sim Ct^{-\lambda_B \beta_B / \lambda_A \beta_d}, \tag{1.13}$$

where $\beta_B = \frac{\gamma_d(1+D)}{\gamma_d(1+D)+\lambda_B}$. This is quite intuitive. When λ_A is small, the A particles will coalesce with themselves more slowly and hence be more abundant. If λ_B is large compared to λ_A, the B particles will be disappearing rapidly as they continue to meet the abundant A particles. These values will give a very negative exponent in (1.13) and so the quantity $P_d^B(t)$ is driven to 0 quickly. On the other hand, λ_A very large could mean that the B particles are able to thrive and decay at an arbitrarily slow rate. Similar rationale follows by considering the interplay between the walk rate D and λ_B, since if the B particles walk at a greater rate than they coalesce with the A particles they may manage to avoid coalescing.

The most difficult step in studying these multiple species processes seems to be in decoupling the pairwise correlations involving A and B type particles. That is because we don't know how to prove negative correlation results for the joint A/B system. Considerable work could be done in this area. It seems possible, in $d \geq 3$ at least, to resort to higher moments in order to perform the necessary decouplings. In $d = 2$, it is likely that these decouplings will be more subtle as they were just in the case of the A particles, even if negative correlation can be established.

It would be interesting to see how the interplay between the walk rate D and the coalesce rate λ_B affect the $d = 2$ asymptotics. We highly suspect that we will once again see some kind of universality in the asymptotic amongst the joint two species system with respect to their coalescence rates. The intuition follows exactly as it would for the A system alone. That the value of λ_A is inconsequential is an immediate corollary of the fact that it is so in the leading asymptotic of the decay of the A particles. The value of λ_B ought not to matter since recurrence will guarantee that B particles always eventually meet A particles and so in the asymptotic the information about how the coalescence rate compares with the walk rate will be lost. Of course, the asymptotic will depend on D.

This work is ongoing.

Chapter 2

The Governing Stochastic Differential Equations

We remark that the results of this chapter hold in all dimensions. In order to highlight this generality, we will prove existence and uniqueness for any dimension d.

2.1 Existence and Uniqueness for the Instantly Coalescing System

We start with an initial condition $\xi_0 \in \{0,1\}^{\mathbb{Z}^d}$ that assigns to each site of $\mathbb{Z}^d$ a 0 or 1 which corresponds to the site initially being empty or occupied by a particle. Integrating (1.5) gives

$$\xi_t(x) = \xi_0(x) + \int_0^t \sum_{y \sim x} \Big(\mathbb{1}\{\xi_{s-}(y) = 1, \xi_{s-}(x) = 0\} dP_s(y,x) \\ - \mathbb{1}\{\xi_{s-}(x) = 1\} dP_s(x,y) \Big). \tag{2.1}$$

Proposition 2.1.1. *For an initial condition satisfying* $\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0^2(x)\right] < \infty$ *for some fixed* $\theta > 0$ *that is independent of the Poisson processes, there exists a unique* *solution to* (2.1) *and the solution satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t(x)|^2\right] < \infty.$$

Proof. Let $\xi_t^{(0)} \equiv \xi_0$ for all t and define sucessive iterates by

$$\begin{aligned}\xi_t^{(n)}(x) = \xi_0(x) &+ \int_0^t \sum_{y\sim x} \mathbb{1}\{\xi_{s-}^{(n-1)}(y) = 1, \xi_{s-}^{(n-1)}(x) = 0\} dP_s(y,x) \\ &- \int_0^t \sum_{y\sim x} \mathbb{1}\{\xi_{s-}^{(n-1)}(x) = 1\} dP_s(x,y). \end{aligned} \tag{2.2}$$

We will use the usual iteration argument to prove existence and a standard Grönwall argument for uniqueness.

While it is clear that if there is a solution to the original equation it has values in $\{0,1\}$, this is not necessarily true for the iterates. Indeed, if $\xi_t^{(0)}(x) = \xi_0(x) = 1$ then

$$\xi_t^{(1)}(x) = 1 - \sum_{y\sim x} P_t(x,y)$$

which agrees with $\xi_t^{(0)}(x)$ until the first jump of one of the $P_t(x,\cdot)$ Poisson processes. Immediately after this jump, $\xi_t^{(1)}(x) = 0$. But then it only takes one more jump from any of the $P_t(x,\cdot)$ for $\xi_t^{(1)}(x)$ to take a negative value. In fact, $\xi_t^{(1)}(x) \in \{\dots,-2,-1,0,1\}$ if $\xi_0(x) = 1$. Similarly, if $\xi_0(x) = 0$, then $\xi_t^{(1)}(x) \in \{0,1,2,\dots\}$. The best that we can say about the iterates is that they lie in $\mathbb{Z}$. Notice however the iterates satisfy the following inequalities,

$$\mathbb{1}\{\xi_t^{(n)}(x) = 1\} \le |\xi_t^{(n)}(x)|,$$

$$\mathbb{1}\{\xi_t^{(n)}(y) = 1, \xi_t^{(n)}(x) = 0\} \le |\xi_t^{(n)}(y)|,$$

$$|\mathbb{1}\{\xi_t^{(n)}(x) = 1\} - \mathbb{1}\{\xi_t^{(n-1)}(x) = 1\}| \le |\xi_t^{(n)}(x) - \xi_t^{(n-1)}(x)|,$$

$$
\begin{aligned}
&|\mathbb{1}\{\xi_t^{(n)}(y) = 1, \xi_t^{(n)}(x) = 0\} - \mathbb{1}\{\xi_t^{(n-1)}(y) = 1, \xi_t^{(n-1)}(x) = 0\}| \\
&\le |\xi_t^{(n)}(y) - \xi_t^{(n-1)}(y)| + |\xi_t^{(n)}(x) - \xi_t^{(n-1)}(x)|. \qquad (2.3)
\end{aligned}
$$

The first of these inequalities is clear. The indicator is only ever 1 if $\xi_t^{(n)}(x)$ also takes the value 1, in which case there is equality. There is also equality if $\xi_t^{(n)}(x) = 0$. If $\xi_t^{(n)}(x)$ takes any other integer value, the indicator is 0 and $\xi_t^{(n)}(x)$ is strictly greater than 0 in modulus. The second inequality is also immediate. For the third, there will be equality if both $\xi_t^{(n)}(x)$ and $\xi_t^{(n-1)}(x)$ take values that agree and both sides of the inequality will vanish. If their values do not coincide the left hand side will be at most 1, being the difference of indicator functions, while the right hand side will be at least 1, being the difference of distinct integer values, hence the third inequality is shown. The final inequality is similar, the left hand side is once again no larger than 1. In the worst case scenario that the left hand side is 1, it must be the case that one indicator achieves the value 1 while the other vanishes. If, without loss of generality, it is the second indicator that vanishes. Suppose that this indicator vanishes because exactly one of the arguments of the indicator fails, then the right hand side of the inequality is at least one since one or other of the summands on the right hand side is the dfference of distinct integers. If the indicator vanishes because both events fail then the right hand side is at least 2, since both summands in the right hand side are the difference of distinct integers. If both indicators vanish, or neither, the worst case for the right hand side is that both summands also vanish, otherwise the right hand side is strictly larger than 0. We will need to make use of these together with the following inequality for positive $f\colon \mathbb{Z}^d \to \mathbb{R}$,

$$
\begin{aligned}
\sum_x e^{-\theta|x|} \sum_{y:y\sim x} f(y) &= \sum_y f(y) \sum_{x\sim y} e^{-\theta|x|} \\
&= \sum_y f(y) e^{-\theta|y|} \sum_{x\sim y} e^{\theta(|y|-|x|)} \\
&\le \sum_y f(y) e^{-\theta|y|} \sum_{x\sim y} e^{\theta(|y-x|)} \\
&= 2de^{\theta} \sum_x f(x) e^{-\theta|x|} \qquad (2.4)
\end{aligned}
$$

where the inequality on the third line is by the reverse triangle inequality since $\theta > 0$, We split the proof into several steps, the structure of which follows from [10].

Step 1 We aim to show that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(n)}(x)^2\right] < \infty.$$

For an induction argument, assume

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(n-1)}(x)^2\right] < \infty,$$

the base case is satisfied by the initial condition. Compensating the Poisson processes in the expression for the n^{th} iterate

$$\begin{aligned}
\xi_t^{(n)}(x) = \xi_0(x) &+ \sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_{s-}^{(n-1)}(y)=1, \xi_{s-}^{(n-1)}(x)=0\}d\left(P_s(y,x)-\frac{s}{2d}\right)\\
&+ \frac{1}{2d}\sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_s^{(n-1)}(y)=1, \xi_s^{(n-1)}(x)=0\}ds\\
&+ \sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_{s-}^{(n-1)}(x)=1\}d\left(P_s(x,y)-\frac{s}{2d}\right)\\
&+ \int_0^t \mathbb{1}\{\xi_s^{(n-1)}(x)=1\}ds\\
&=: \sum_{i=1}^5 I_i(t,x)
\end{aligned}$$

where the i^{th} term in the final line is defined by the i^{th} term in the right hand side of the equality with $\xi_t^{(n)}(x)$. We can expand the square of this by the Cauchy-Schwarz inequality to give

$$\xi_t^{(n)}(x)^2 \le 5\sum_{i=1}^5 I_i^2(t,x). \tag{2.5}$$

We will estimate each of the I_i^2 individually in expectation.

$$\mathbb{E}\left[I_2^2\right] = \mathbb{E}\left[\left(\sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_{s-}^{(n-1)}(y)=1, \xi_{s-}^{(n-1)}(x)=0\} d\left(P_s(y,x)-\frac{s}{2d}\right)\right)^2\right].$$

Expanding the square gives many cross terms that vanish in expectation due to the independence of the Poisson processes, therefore we may extract the sum over the neighbours of x

$$\mathbb{E}\left[I_2^2\right] = \sum_{y\sim x}\mathbb{E}\left[\left(\int_0^t \mathbb{1}\{\xi_{s-}^{(n-1)}(y)=1, \xi_{s-}^{(n-1)}(x)=0\} d\left(P_s(y,x)-\frac{s}{2d}\right)\right)^2\right].$$

The Itô isometry then gives

$$\begin{aligned}
\mathbb{E}\left[I_2^2\right] &= \frac{1}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_s^{(n-1)}(y)=1, \xi_s^{(n-1)}(x)=0\} ds\right] \\
&\le \frac{1}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_s^{(n-1)}(y)=1\} ds\right] \\
&\le \frac{1}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t |\xi_s^{(n-1)}(y)| ds\right] \\
&\le \frac{1}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t \xi_s^{(n-1)}(y)^2 ds\right].
\end{aligned}$$

It can be seen immediately that I_4 satisfies the similar bound of

$$\mathbb{E}\left[I_4^2\right] \le \mathbb{E}\left[\int_0^t \xi_s^{(n-1)}(x)^2 ds\right]. \tag{2.6}$$

Turning our attention to I_3

$$\mathbb{E}\left[I_3^2\right] = \frac{1}{(2d)^2}\mathbb{E}\left[\left(\sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_s^{(n-1)}(y)=1, \xi_s^{(n-1)}(x)=0\} ds\right)^2\right].$$

The Cauchy-Schwarz inequality applied to the sum over y and also to the integral gives the upper bound

$$\begin{aligned}\mathbb{E}\left[I_3^2\right] &\leq \frac{t}{2d}\mathbb{E}\left[\sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_s^{(n-1)}(y)=1,\xi_s^{(n-1)}(x)=0\}ds\right]\\ &\leq \frac{t}{2d}\mathbb{E}\left[\sum_{y\sim x}\int_0^t \mathbb{1}\{\xi_s^{(n-1)}(y)=1\}ds\right]\\ &\leq \frac{t}{2d}\mathbb{E}\left[\sum_{y\sim x}\int_0^t \xi_s^{(n-1)}(y)^2 ds\right].\end{aligned}$$

Once again, it is clear that we have

$$\mathbb{E}\left[I_5^2\right] \leq t\mathbb{E}\left[\int_0^t \xi_s^{(n-1)}(x)^2 ds\right]. \tag{2.7}$$

By multiplying by the $e^{-\theta|x|}$, summing over $x \in \mathbb{Z}^d$ and using (2.4) we find that

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}(I_2^2+I_4^2)\right] \leq (1+t)e^{\theta}\mathbb{E}\left[\int_0^t \xi_s^{(n-1)}(x)^2 ds\right]. \tag{2.8}$$

Returning to (2.5), multiplying by $e^{-\theta|x|}$, summing and substituting (2.6),(2.7) and (2.8) gives for each $t \in [0,T]$

$$\begin{aligned}\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(n)}(x)^2\right] &\leq 5\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0(x)^2\right]\\ &\quad + 5(1+e^{\theta})(1+t)\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \xi_s^{(n-1)}(x)^2 ds\right]\end{aligned}$$

which is finite by assumption. By the principle of mathematical induction the moments are finite for all n. Taking supremum over $t \in [0,T]$ gives

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(n)}(x)^2\right] &\leq C_{\theta,\xi_0}\\ &\quad + 5(1+e^{\theta})(1+T)\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^T \xi_s^{(n-1)}(x)^2 ds\right] < \infty.\end{aligned}$$

Step 2 With finite second moments established, we show that there exists constants C, L such that

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x) - \xi_t^{(n)}(x)|^2\right] \leq \frac{C(Lt)^n}{n!}.$$

That the left hand side is finite follows from the Cauchy-Schwarz inequality and Step 1. For $t \in [0, T]$, Cauchy-Schwarz gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_{|x|\leq N} e^{-\theta|x|}|\xi_t^{(n+1)}(x) - \xi_t^{(n)}(x)|^2\right] \\
&\leq 4\mathbb{E}\Bigg[\sum_{|x|\leq N} e^{-\theta|x|}\Bigg|\int_0^t \sum_{y\sim x} \big(\mathbb{1}\{\xi_{s-}^{(n)}(y) = 1, \xi_{s-}^{(n)}(x) = 0\} \\
&\qquad - \mathbb{1}\{\xi_{s-}^{(n-1)}(y) = 1, \xi_{s-}^{(n-1)}(x) = 0\}\big) d\left(P_s(y,x) - \frac{s}{2d}\right)\Bigg|^2\Bigg] \\
&+ \frac{4}{(2d)^2}\mathbb{E}\Bigg[\sum_{|x|\leq N} e^{-\theta|x|}\Bigg|\int_0^t \sum_{y\sim x} \big(\mathbb{1}\{\xi_s^{(n)}(y) = 1, \xi_s^{(n)}(x) = 0\} \\
&\qquad - \mathbb{1}\{\xi_s^{(n-1)}(y) = 1, \xi_s^{(n-1)}(x) = 0\}\big) ds\Bigg|^2\Bigg] \\
&+ 4\mathbb{E}\Bigg[\sum_{|x|\leq N} e^{-\theta|x|}\Bigg|\int_0^t \sum_{y\sim x} \Big(\mathbb{1}\{\xi_{s-}^{(n)}(x) = 1\} \\
&\qquad - \mathbb{1}\{\xi_{s-}^{(n-1)}(x) = 1\}\Big) d\left(P_s(x,y) - \frac{s}{2d}\right)\Bigg|^2\Bigg] \\
&+ 4\mathbb{E}\left[\sum_{|x|\leq N} e^{-\theta|x|}\left|\int_0^t \left(\mathbb{1}\{\xi_s^{(n)}(x) = 1\} - \mathbb{1}\{\xi_s^{(n-1)}(x) = 1\}\right) ds\right|^2\right] \\
&=: I + II + III + IV
\end{aligned} \tag{2.9}$$

when the Roman numerals represent each of the terms coming before them in order.

We will bound each of the terms. The first can be bounded as follows

$$
\begin{aligned}
I &= 4\mathbb{E}\Bigg[\sum_{|x|\le N} e^{-\theta|x|}\bigg|\int_0^t \sum_{y\sim x}\big(\mathbb{1}\{\xi^{(n)}_{s-}(y)=1,\xi^{(n)}_{s-}(x)=0\} \\
&\qquad - \mathbb{1}\{\xi^{(n-1)}_{s-}(y)=1,\xi^{(n-1)}_{s-}(x)=0\}\big)d\left(P_s(y,x)-\frac{s}{2d}\right)\bigg|^2\Bigg] \\
&= 4\mathbb{E}\Bigg[\sum_{|x|\le N} e^{-\theta|x|}\sum_{y\sim x}\bigg|\int_0^t \big(\mathbb{1}\{\xi^{(n)}_{s-}(y)=1,\xi^{(n)}_{s-}(x)=0\} \\
&\qquad - \mathbb{1}\{\xi^{(n-1)}_{s-}(y)=1,\xi^{(n-1)}_{s-}(x)=0\}\big)d\left(P_s(y,x)-\frac{s}{2d}\right)\bigg|^2\Bigg] \\
&\le \frac{8}{2d}\mathbb{E}\Bigg[\sum_{|x|\le N} e^{-\theta|x|}\sum_{y\sim x}\int_0^t\bigg(\left|\xi^{(n)}_s(y)-\xi^{(n-1)}_s(y)\right|^2 \\
&\qquad\qquad + \left|\xi^{(n)}_s(x)-\xi^{(n-1)}_s(x)\right|^2\bigg)ds\Bigg] \\
&\le 8(1+e^{\theta})\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \left|\xi^{(n-1)}_s(x)-\xi^{(n)}_s(x)\right|^2 ds\right] \qquad (2.10)
\end{aligned}
$$

where the first equality is definition, the second equality is by expanding the quadratic and losing the cross terms in expectation, the first inequality is a use of the Itô isometry and then the inequality for the indicators given in (2.3), and the final inequality is from (2.4) and also by including in the sum $|x| > N$. III is similar but easier

$$
III \le 4\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \left|\xi^{(n-1)}_s(x)-\xi^{(n)}_s(x)\right|^2 ds\right]. \qquad (2.11)
$$

Estimating II,

$$
\begin{aligned}
II &\le \frac{4t}{2d}\mathbb{E}\Bigg[\sum_x e^{-\theta|x|}\sum_{y\sim x}\int_0^t \big|\mathbb{1}\{\xi_{s-}^{(n)}(y)=1,\xi_{s-}^{(n)}(x)=0\} \\
&\qquad - \mathbb{1}\{\xi_{s-}^{(n-1)}(y)=1,\xi_{s-}^{(n-1)}(x)=0\}\big|^2 ds\Bigg] \\
&\le \frac{8t}{2d}\mathbb{E}\Bigg[\sum_x e^{-\theta|x|}\sum_{y\sim x}\int_0^t \Big(\big|\xi_s^{(n)}(y)-\xi_s^{(n-1)}(y)\big|^2 \\
&\qquad\qquad + \big|\xi_s^{(n)}(x)-\xi_s^{(n-1)}(x)\big|^2\Big) ds\Bigg] \\
&\le 8(1+e^{\theta})t\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \big|\xi_s^{(n-1)}(x)-\xi_s^{(n)}(x)\big|^2 ds\right] \qquad (2.12)
\end{aligned}
$$

where we have used two applications of Cauchy-Schwarz, firstly on the integral, secondly on the sum and then we have included the x with $|x|>N$. IV is similar but easier.

$$
IV \le 4t\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \big|\xi_s^{(n-1)}(x)-\xi_s^{(n)}(x)\big|^2 ds\right]. \qquad (2.13)
$$

Gathering all of the bounds for $I-IV$ in equations (2.10) through (2.13) and substituting them into (2.9) for $t\in[0,T]$

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\{|x|\le N\}} e^{-\theta|x|}\left|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)\right|^2\right] \\
&\quad \le 4(1+T)(3+2e^{\theta})\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \big|\xi_s^{(n-1)}(x)-\xi_s^{(n)}(x)\big|^2 ds\right].
\end{aligned}
$$

Let $N\to\infty$ and set $L=4(1+T)(3+2e^{\theta})$ and $C=\sup_{t\in[0,T]}\mathbb{E}[\sum_x e^{-\theta|x|}|\xi_t^{(1)}(x)-\xi_t^{(0)}(x)|^2]$ which is finite by Step 1. Then

$$
\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|^2\right] \le \frac{C(Lt)^n}{n!}.
$$

This follows by induction:

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(2)}(x)-\xi_t^{(1)}(x)|^2\right] \\
&\quad \le L\int_0^t \mathbb{E}\left[\sum_x e^{-\theta|x|}\left|\xi_s^{(1)}(x)-\xi_s^{(0)}(x)\right|^2\right] ds \\
&\quad \le CLt
\end{aligned}$$

and if

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(n)}(x)-\xi_t^{(n-1)}(x)|^2\right] \le \frac{C(Lt)^{n-1}}{(n-1)!}$$

then

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|^2\right] \\
&\quad \le L\int_0^t \mathbb{E}\left[\sum_x e^{-\theta|x|}\left|\xi_s^{(n)}(x)-\xi_s^{(n-1)}(x)\right|^2\right] ds \\
&\quad \le L\int_0^t \frac{C(Ls)^{n-1}}{(n-1)!} ds \\
&\quad = \frac{C(Lt)^n}{n!}.
\end{aligned}$$

Step 3 Next we show that

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]}\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|\right)^2\right] \le \frac{C_\theta C(LT)^n}{n!}$$

where C, L are as before and $C_\theta = \sum_x e^{-\theta|x|}$. We take the supremum over $t \in [0,T]$ of the difference in absolute value of successive iterates, and we exploit the increasing sample paths of Poisson processes.

$$\begin{aligned}
&\sup_{t\in[0,T]} |\xi_t^{(n+1)}(x) - \xi_t^{(n)}(x)| \\
&\le \sup_{t\in[0,T]} \sum_{y\sim x} \int_0^t \big|\mathbb{1}\{\xi_{s-}^{(n)}(y)=1, \xi_{s-}^{(n)}(x)=0\} \\
&\quad - \mathbb{1}\{\xi_{s-}^{(n-1)}(y)=1, \xi_{s-}^{(n-1)}(x)=0\}\big| dP_s(y,x) \\
&\quad + \sup_{t\in[0,T]} \sum_{y\sim x} \int_0^t \big|\mathbb{1}\{\xi_{s-}^{(n)}(x)=1\} - \mathbb{1}\{\xi_{s-}^{(n-1)}(x)=1\}\big| dP_s(x,y) \\
&= \sum_{y\sim x} \int_0^T |\mathbb{1}\{\xi_{s-}^{(n)}(y)=1, \xi_{s-}^{(n)}(x)=0\} \\
&\quad - \mathbb{1}\{\xi_{s-}^{(n-1)}(y)=1, \xi_{s-}^{(n-1)}(x)=0\}| dP_s(y,x) \\
&\quad + \sum_{y\sim x} \int_0^T |\mathbb{1}\{\xi_{s-}^{(n)}(x)=1\} - \mathbb{1}\{\xi_{s-}^{(n-1)}(x)=1\}| dP_s(x,y) \\
&\le \sum_{y\sim x} \int_0^T |\xi_{s-}^{(n)}(y) - \xi_{s-}^{(n-1)}(y)| + |\xi_{s-}^{(n)}(x) - \xi_{s-}^{(n-1)}(x)| dP_s(y,x) \\
&\quad + \sum_{y\sim x} \int_0^T |\xi_{s-}^{(n)}(x) - \xi_{s-}^{(n-1)}(x)| dP_s(x,y) \\
&= \sum_{y\sim x} \int_0^T \Big(|\xi_{s-}^{(n)}(y) - \xi_{s-}^{(n-1)}(y)| \\
&\quad + |\xi_{s-}^{(n)}(x) - \xi_{s-}^{(n-1)}(x)| \Big) d\left(P_s(y,x) - \frac{s}{2d}\right) \\
&\quad + \frac{1}{2d} \sum_{y\sim x} \int_0^T \left(|\xi_s^{(n)}(y) - \xi_s^{(n-1)}(y)| + |\xi_s^{(n)}(x) - \xi_s^{(n-1)}(x)| \right) ds \\
&\quad + \sum_{y\sim x} \int_0^T |\xi_{s-}^{(n)}(x) - \xi_{s-}^{(n-1)}(x)| d\left(P_s(x,y) - \frac{s}{2d}\right) \\
&\quad + \frac{1}{2d} \sum_{y\sim x} \int_0^T |\xi_s^{(n)}(x) - \xi_s^{(n-1)}(x)| ds.
\end{aligned}$$

Then, by Cauchy-Schwarz twice, (writing $e^{-\theta|x|} = e^{-\theta|x|/2} e^{-\theta|x|/2}$ in the first inequality)

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]}\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|\right)^2\right]$$
$$\leq C_\theta\mathbb{E}\left[\sum_x e^{-\theta|x|}\left(\sup_{t\in[0,T]}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|\right)^2\right]$$
$$\leq 4C_\theta\mathbb{E}\Bigg[\sum_x e^{-\theta|x|}\Bigg(\sum_{y\sim x}\int_0^T \big(|\xi_s^{(n)}(y)-\xi_s^{(n-1)}(y)|$$
$$+|\xi_s^{(n)}(x)-\xi_s^{(n-1)}(x)|\big)d\left(P_s(y,x)-\frac{s}{2d}\right)\Bigg)^2\Bigg]$$
$$+\frac{4C_\theta}{(2d)^2}\mathbb{E}\Bigg[\sum_x e^{-\theta|x|}\Bigg(\sum_{y\sim x}\int_0^T\Big(|\xi_s^{(n)}(y)-\xi_s^{(n-1)}(y)|$$
$$+|\xi_s^{(n)}(x)-\xi_s^{(n-1)}(x)|\Big)ds\Bigg)^2\Bigg]$$
$$+4C_\theta\mathbb{E}\left[\sum_x e^{-\theta|x|}\left(\sum_{y\sim x}\int_0^T|\xi_{s-}^{(n)}(x)-\xi_{s-}^{(n-1)}(x)|d\left(P_s(x,y)-\frac{s}{2d}\right)\right)^2\right]$$
$$+\frac{4C_\theta}{(2d)^2}\mathbb{E}\left[\sum_x e^{-\theta|x|}\left(\sum_{y\sim x}\int_0^T|\xi_s^{(n)}(x)-\xi_s^{(n-1)}(x)|ds\right)^2\right].$$

A repeat of calculations we have already seen gives

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]}\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|\right)^2\right]$$
$$\leq C_\theta L\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^T\left|\xi_s^{(n-1)}(x)-\xi_s^{(n)}(x)\right|^2ds\right]$$
$$\leq C_\theta L\int_0^T\frac{C(Ls)^{n-1}}{(n-1)!}ds$$
$$\leq C_\theta\frac{C(LT)^n}{n!}$$

which concludes Step 3.

Step 4 Markov's inequality implies that

$$\sup_{t\in[0,T]}\sum_x e^{-\theta|x|}|\xi_t^{(n+1)}(x)-\xi_t^{(n)}(x)|\leq\frac{1}{n^2}$$

almost surely for large n. This in turn implies that

$$\begin{aligned}\sum_x e^{-\theta|x|}\xi_t^{(n)}(x) &= \sum_x e^{-\theta|x|}\left(\xi_0(x)+\sum_{k=1}^n(\xi_t^{(k)}(x)-\xi_t^{(k-1)}(x))\right)\\ &= \sum_x e^{-\theta|x|}\xi_0(x)+\sum_x e^{-\theta|x|}\sum_{k=1}^n(\xi_t^{(k)}(x)-\xi_t^{(k-1)}(x))\\ &= \sum_x e^{-\theta|x|}\xi_0(x)+\sum_{k=1}^n\sum_x e^{-\theta|x|}(\xi_t^{(k)}(x)-\xi_t^{(k-1)}(x))\end{aligned}$$

converges almost surely uniformly in $t\in[0,T]$. Let $\xi_t(x)=\lim_{n\to\infty}\xi_t^{(n)}(x)$. Now,

$$\begin{aligned}&\bigg|\int_0^t\sum_{y\sim x}\mathbb{1}\{\xi_{s-}^{(n)}(y)=1,\xi_{s-}^{(n)}(x)=0\}dP_s(y,x)\\ &\quad-\int_0^t\sum_{y\sim x}\mathbb{1}\{\xi_{s-}(y)=1,\xi_{s-}(x)=0\}dP_s(y,x)\bigg|\\ &\le\int_0^t\sum_{y\sim x}\bigg|\mathbb{1}\{\xi_{s-}^{(n)}(y)=1,\xi_{s-}^{(n)}(x)=0\}\\ &\quad-\mathbb{1}\{\xi_{s-}(y)=1,\xi_{s-}(x)=0\}\bigg|dP_s(y,x)\\ &\le\int_0^t\sum_{y\sim x}\left|\xi_{s-}^{(n)}(y)-\xi_{s-}(y)\right|+\left|\xi_{s-}^{(n)}(x)-\xi_{s-}(x)\right|dP_s(y,x)\to 0\end{aligned}$$

as $n\to\infty$ and similarly

$$\begin{aligned}&\left|\int_0^t\sum_{y\sim x}\mathbb{1}\{\xi_{s-}^{(n)}(x)=1\}dP_s(x,y)-\int_0^t\sum_{y\sim x}\mathbb{1}\{\xi_{s-}(x)=1\}dP_s(x,y)\right|\\ &\le\int_0^t\sum_{y\sim x}\left|\xi_{s-}^{(n)}(x)-\xi_{s-}(x)\right|dP_s(x,y)\to 0\end{aligned}$$

so that taking limits in (2.2) implies that the limit $\xi_t(x)$ is a solution to (2.1).

Step 5 Uniqueness amongst the class of processes satisfying

$$\sup_{t\in[0,T]}\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t(x)^2\right]<\infty$$

is implied by Grönwall's inequality. Let (ξ_t) and (η_t) be two solutions satisfying the finite second moment condition. Indeed, the previous calculations can be repeated and give

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t(x)-\eta_t(x)|^2\right] \le 4(1+t)(3+2e^\theta)\mathbb{E}\left[\sum_x e^{-\theta|x|}\int_0^t \left|\xi_s(x)-\eta_s(x)\right|^2 ds\right].$$

This implies $\xi_t \equiv \eta_t$ for each t almost surely. Since the solutions have càdlàg paths, it follows that they are identically equal for all times simultaneously. □

2.2 The Non-Instantaneous Regime

Integrating (1.6) gives

$$\begin{aligned}\xi_t(x) = \xi_0(x) + \int_0^t \sum_{y:y\sim x}\sum_{i\ge 1}\Big(&\mathbb{1}\{\xi_{s-}(y)\ge i\}dP_s(i,y,x)\\ &- \mathbb{1}\{\xi_{s-}(x)\ge i\}dP_s(i,x,y)\Big)\\ - \int_0^t \sum_{i,j\ge 1}\mathbb{1}\Big(&\xi_{s-}(x)\ge i\vee j, i\ne j\Big)dP^c_s(i,j,x).\end{aligned} \tag{2.14}$$

We are interested in proving existence and uniqueness for this equation. We need to know that the Poisson sums converge since in each of them there infinitely many summands. To establish that the Poisson sums converge, we will instead consider a "finitely reactive" model satisfying the modified SDE

$$\begin{aligned}\xi^{(m)}_t(x) = \xi_0(x) + \int_0^t \sum_{y:y\sim x}\sum_{i=1}^m\Big(&\mathbb{1}\{\xi^{(m)}_{s-}(y)\ge i, \xi^{(m)}_{s-}(x)<m\}dP_s(i,y,x)\\ &- \mathbb{1}\{\xi^{(m)}_{s-}(x)\ge i\}dP_s(i,x,y)\Big)\\ - \int_0^t \sum_{i,j=1}^m\mathbb{1}\Big(&\xi^{(m)}_{s-}(x)\ge i\vee j, i\ne j\Big)dP^c_s(i,j,x).\end{aligned} \tag{2.15}$$

Here, we start we the same initial spread of particles (having enough finite moments, which will be quantified later) but only the first m particles at x or one of its neighbours has the chance to react. The process $\xi^{(m)}_t(x)$ records the total number of particles at x at time t (even if this is larger than m). However, notice that if the particle number at x ever drops beneath m then it will never exceed m since whenever $\xi^{(m)}_{\cdot}(m) = m$ the only Poisson processes that can increase the particle number vanish. In particular, if $\xi_0(x) \leq m$, then $\xi_t^{(m)}(x)$ will, almost surely, never exceed m. Also, note that, while in the infinite system $(\xi_t(x))_{x\in\mathbb{Z}^d}$ governed by (2.14) there are no incidents of instantaneous coalescence. Any particle that walks to x at time t almost surely increases the number of particles by 1. The situation is slightly different for the system $(\xi_t^{(m)}(x))_{x\in\mathbb{Z}^d}$. Consider x with $\xi_t^{(m)}(x) = m$ and a neighbour $w \sim x$ such that $\xi_t^{(m)}(w) > 0$. Then if any of the Poisson processes $P_t(i, w, x)$ for $i \in \{1, \ldots, \xi_t^{(m)}(w)\}$ reacts, then there is a loss of a particle at w, while there is no increase of the number at x because the indicator attached to $P_t(i, w, x)$ vanishes. In a sense, the particle arriving at x from w has instantaneously coalesced with one of the m at x. We aim to show that there exists a unique solution to (2.15) and then that this "finitely reactive" model converges to our infinite model so that a solution to the infinite model arises as the limit of the solutions to successive finitely modified models. The effect of the instantaneously coalescing particles will be lost in this limit. It will then remain to show that the solution that arises in this fashion is the only solution in amongst a certain class of processes.

Proposition 2.2.1. *Let $\xi_0 \in \mathbb{N}^{\mathbb{Z}^d}$ be an initial distribution of particles independent of the driving Poisson processes satisfying $\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0^2(x)\right] < \infty$ for any $\theta \geq 0$, then for each m, there exists a unique solution to* (2.15) *and the solution satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(m)}(x)|^2\right] < \infty.$$

Lemma 2.2.2. *Fix $p \geq 1$ and let $\xi_0 \in \mathbb{N}^{\mathbb{Z}^d}$ be an initial condition satisfying $\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_0(x)|^p\right] < \infty$ for all $\theta > 0$. Let $\xi_t^{(m)}$ be the solution to* (2.15) *with initial condition ξ_0 then*

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} \sum_x e^{-\theta|x|}|\xi_t^{(m)}(x)|\right)^p\right]$$

is finite and bounded uniformly in m.

Refer to Section A of the Appendix for the proofs of the two previous results.

Theorem 2.2.3. *Let $\xi_0 \in \mathbb{N}^{\mathbb{Z}^d}$ satisfy $\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_0(x)|^2\right] < \infty$ for all $\theta > 0$. The sequence of solutions $(\xi_t^{(m)}(x))_m$ to (2.15) converge to an adapted càdlàg process $\xi_t(x)$ that is a solution to (2.14). Furthermore, amongst the class of processes that satisfy $\{\eta \colon \sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}|\eta_t(x)|^2\right] < \infty$ for all $\theta > 0\}$, the solution to (2.14) is unique.*

Proof. We will show that $\xi_t^{(m)}(x)$ forms a Cauchy sequence. Let $m > k$. Our strategy will be to derive a Grönwall inequality that will control the sum $\sum_x e^{-\theta|x|}|\xi_t^{(m)}(x) - \xi_t^{(k)}(x)|^2$ for large k. There are a large number of terms that arise in the difference. We cannot proceed immediately since one of the terms that contributes to the integrand of the Grönwall inequality is a non-linear function of $\sum_x e^{-\theta|x|}|\xi_s^{(m)}(x) - \xi_s^{(k)}(x)|^2$ and the Grönwall Lemma does not produce a bound that vanishes in any sensible way. Instead, we split the sum at $x \in \{z \colon |z| \leq N\} = B_N$. The benefit of this is that for such x, we can take $k > \max_{|x|\leq N} \xi_0(x)$ (this max is almost surely finite by the moment conditions the initial condition satisfies) whereupon it is necessarily true that $\xi_t^{(m)}(x) \geq \xi_t^{(k)}(x)$ for all t almost surely.

To see this, looking at (2.15) it is clear that, almost surely, the particle number at x can only increase if there are fewer than m particles already occupying x by the presence of the indicator $\mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(x) < m\}$. The only way that there can be more than m particles at x is if there were more than m particles at x at time 0. If this is the case, the particle number can only drop until it first dips beneath m, in which case it could then potentially take on more particles, but only up to m. In this way, so long as there are fewer than m particles at x at time 0, the particle number will, almost surely never exceed m. Similarly, if there are fewer particles than k at x at time 0, $\xi_t^{(k)}(x)$ will almost surely not exceed k. Since $\xi_t^{(m)}(x)$ and $\xi_t^{(k)}(x)$ are run from the same initial condition, taking $k > \max_{|x|\leq N} \xi_0(x)$ means that their dynamics are the same (and so $\xi_t^{(m)}(x) = \xi_t^{(k)}(x)$) up until the first time that they take on the value k. If at this point there is a neighbour $w \sim x$

such that $\xi_t^{(m)}(w), \xi_t^{(k)}(w) > 0$, and any of the Poisson processes $P_t(i,w,x)$ for $i \in \{1, \ldots, \xi_t^{(m)}(w) \wedge \xi_t^{(k)}(w)\}$ reacts, then there is a loss of a particle at w in both systems $\xi_t^{(m)}, \xi_t^{(k)}$ but only a gain of a particle at x in the $\xi_t^{(m)}$ system. Beyond this time it is the case that $\xi_t^{(m)}(x) \geq \xi_t^{(k)}(x)$.

This will allow us to throwaway the term that causes us to have a non-linear Grönwall inequality (as well as many other terms). The sum over $x \notin B_N$ will not contribute to the integrand in the Grönwall inequality and will make up the error the results from the Grönwall Lemma and will vanish due to the uniform moment bounds given by Lemma 2.2.2.

As such fix N and let $k > \max_{|x| \leq N} \xi_0(x)$. Then for $x \in B_N$

$$\begin{aligned}
0 \leq & \xi_t^{(m)}(x) - \xi_t^{(k)}(x) \\
= & \sum_{y \sim x} \sum_{i=1}^{k} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(y) < m\} - \mathbb{1}\{\xi_{s-}^{(k)}(y) \geq i, \xi_{s-}^{(k)}(x) < k\} \\
& \quad \times dP_s(i,y,x) \\
& + \sum_{y \sim x} \sum_{i=k+1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(x) < m\} dP_s(i,y,x) \\
& - \sum_{y \sim x} \sum_{i=1}^{k} \int_0^t \underbrace{\mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i\} - \mathbb{1}\{\xi_{s-}^{(k)}(x) \geq i\}}_{\geq 0} dP_s(i,x,y) \\
& - \sum_{y \sim x} \sum_{i=k+1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i\} dP_s(i,x,y) \\
& - \sum_{i,j=1}^{k} \int_0^t \underbrace{\mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_{s-}^{(k)}(x) \geq i \vee j, i \neq j\}}_{\geq 0} \\
& \qquad \times dP_s^c(i,j,x) \\
& - \sum_{i=k+1}^{m} \sum_{j=1}^{k} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
& - \sum_{i=1}^{k} \sum_{j=k+1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
& - \sum_{i,j=k+1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x).
\end{aligned}$$

So for $x \in B_N$,

$$
\begin{aligned}
0 \leq & \xi_t^{(m)}(x) - \xi_t^{(k)}(x) \\
& \leq \sum_{y \sim x} \sum_{i=1}^{k} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(y) < m\} - \mathbb{1}\{\xi_{s-}^{(k)}(y) \geq i, \xi_{s-}^{(k)}(x) < k\} \\
& \qquad \times dP_s(i,y,x) \\
& + \sum_{y \sim x} \sum_{i=k+1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(x) < m\} dP_s(i,y,x) \\
& \leq \sum_{y \sim x} \sum_{i=1}^{k} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} - \mathbb{1}\{\xi_{s-}^{(k)}(y) \geq i\} dP_s(i,y,x) \\
& + \sum_{y \sim x} \sum_{i=1}^{k} \int_0^t \mathbb{1}\{\xi_{s-}^{(k)}(y) \geq i, \xi_{s-}^{(k)}(x) = k\} dP_s(i,y,x) \\
& + \sum_{y \sim x} \sum_{i=k+1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} dP_s(i,y,x) \\
& = \sum_{i=1}^{3} I_i.
\end{aligned}
$$

The I_i each represent (respectfully) the double sums before the last equality.

Since k is random, instead of working with the expectation proper, we need to condition on the initial condition ξ_0. The difference can be estimated by

$$
\begin{aligned}
& \mathbb{E}\left[\sum_x e^{-\theta|x|} |\xi_t^{(m)}(x) - \xi_t^{(k)}(x)|^2 \middle| \xi_0\right] \\
& = \mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|} |\xi_t^{(m)}(x) - \xi_t^{(k)}(x)|^2 \middle| \xi_0\right] + \mathbb{E}\left[\sum_{x \in B_N} e^{-\theta|x|} |\xi_t^{(m)}(x) - \xi_t^{(k)}(x)|^2 \middle| \xi_0\right] \\
& \leq \mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|} |\xi_t^{(m)}(x) - \xi_t^{(k)}(x)|^2 \middle| \xi_0\right] + \mathbb{E}\left[\sum_{x \in B_N} e^{-\theta|x|} \left(\sum_{i=1}^{3} I_i\right)^2 \middle| \xi_0\right] \\
& \leq \mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|} |\xi_t^{(m)}(x) - \xi_t^{(k)}(x)|^2 \middle| \xi_0\right] + 3 \sum_{i=1}^{3} \mathbb{E}\left[\sum_{x \in B_N} e^{-\theta|x|} I_i^2 \middle| \xi_0\right]. \qquad (2.16)
\end{aligned}
$$

Beginning with I_1 carrying out all the familiar calculations (squaring, compensating the Poisson processes, Cauchy-Schwarz etc.)

$$
\begin{aligned}
I_1^2 \le 2 \sum_{y \sim x} \sum_{i=1}^{k} & \left(\int_0^t \left| \mathbb{1}\{\xi_{s-}^{(m)}(y) \ge i\} - \mathbb{1}\{\xi_{s-}^{(k)}(y) \ge i\} \right| d\left(P_s(i,y,x) - \frac{s}{2d} \right) \right)^2 \\
& + \frac{2t}{2d} \sum_{y \sim x} \int_0^t \left| \xi_s^{(m)}(y) \wedge k - \xi_s^{(k)}(y) \right|^2 ds + c.t.
\end{aligned}
$$

where in $c.t.$ we have collected the cross terms that result from expanding the square. Taking conditional expectation gives

$$
\begin{aligned}
\mathbb{E}\left[I_1^2 \middle| \xi_0 \right] \le 2 \sum_{y \sim x} \mathbb{E} & \left[\sum_{i=1}^{k} \left(\int_0^t \left| \mathbb{1}\{\xi_{s-}^{(m)}(y) \ge i\} - \mathbb{1}\{\xi_{s-}^{(k)}(y) \ge i\} \right| d\left(P_s(i,y,x) - \frac{s}{2d} \right) \right)^2 \middle| \xi_0 \right] \\
& + \frac{2t}{2d} \sum_{y \sim x} \mathbb{E}\left[\int_0^t \left| \xi_s^{(m)}(y) - \xi_s^{(k)}(y) \right|^2 ds \middle| \xi_0 \right] + \mathbb{E}[c.t.|\xi_0].
\end{aligned}
$$

Since the Poisson processes are independent of the initial condition and $\xi_t^{(m)}$ and $\xi_t^{(k)}$ are adapted,

$$
\mathbb{E}[c.t.|\xi_0] = \mathbb{E}[c.t.]
$$

and they vanish in the usual way since the Poisson processes are also independent of each other for different choices of i and y, and the compensated processes are martingales. Conditioning on ξ_0 gives us information on k by its definition, which allows us to take out the sum that ranges over $i \in \{1, \dots, k\}$ as known from the expectation. This gives

$$
\begin{aligned}
\mathbb{E}\left[I_1^2 \middle| \xi_0 \right] \le 2 \sum_{y \sim x} \sum_{i=1}^{k} \mathbb{E} \Bigg[\Bigg(\int_0^t & \left| \mathbb{1}\{\xi_{s-}^{(m)}(y) \ge i\} - \mathbb{1}\{\xi_{s-}^{(k)}(y) \ge i\} \right| \\
& \times d\left(P_s(i,y,x) - \frac{s}{2d} \right) \Bigg)^2 \Bigg| \xi_0 \Bigg] \\
+ \frac{2t}{2d} \sum_{y \sim x} \mathbb{E} & \left[\int_0^t \left| \xi_s^{(m)}(y) - \xi_s^{(k)}(y) \right|^2 ds \middle| \xi_0 \right].
\end{aligned}
$$

Now, we again use that $\xi_t^{(m)}$ and $\xi_t^{(k)}$ are adapted and the independence of the Poisson processes from the initial condition, use the stability of conditional expectation and apply the usual isometry to get

$$\mathbb{E}\left[I_1^2\middle|\xi_0\right] \le \frac{2(1+t)}{2d}\sum_{y\sim x}\int_0^t \left|\xi_s^{(m)}(y)-\xi_s^{(k)}(y)\right|^2 ds.$$

Multiplying by $e^{-\theta|x|}$, summing over $x \in B_N$, taking expectation and using the total law of probability gives

$$\begin{aligned}\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|}I_1^2\right] &\le \frac{2(1+t)}{2d}\mathbb{E}\left[\int_0^t \sum_{x\in B_N} e^{-\theta|x|}\sum_{y\sim x}\left|\xi_s^{(m)}(y)-\xi_s^{(k)}(y)\right|^2 ds\right]\\ &\le \frac{2(1+t)}{2d}\mathbb{E}\left[\int_0^t \sum_{x} e^{-\theta|x|}\sum_{y\sim x}\left|\xi_s^{(m)}(y)-\xi_s^{(k)}(y)\right|^2 ds\right]\\ &\le 2(1+t)e^{\theta}\mathbb{E}\left[\int_0^t \sum_{x} e^{-\theta|x|}\left|\xi_s^{(m)}(x)-\xi_s^{(k)}(x)\right|^2 ds\right]. \end{aligned} \tag{2.17}$$

I_3 should not contribute to the main term of $|\xi_t^{(m)}(x)-\xi_t^{(k)}(x)|$ and should be small as $k\to\infty$. We will use the following equality for the square of the sum of indicators. For integer valued X

$$\left(\sum_{i=1}^{k}\mathbb{1}\{X\ge i\}\right)^2 = \sum_{i=1}^{k}(2i-1)\mathbb{1}\{X\ge i\}$$

which follows since for integer valued X and $i<j$, $\mathbb{1}\{X\ge i\}\mathbb{1}\{X\ge j\} = \mathbb{1}\{X\ge j\}$ and can be proved by induction as follows. The base case with one summand is

trivial. Suppose it holds for k. Then

$$\begin{aligned}
&\left(\sum_{i=1}^{k+1} \mathbb{1}\{X \geq i\}\right)^2 \\
&= \left(\mathbb{1}\{X \geq k+1\} + \sum_{i=1}^{k} \mathbb{1}\{X \geq i\}\right)^2 \\
&= \mathbb{1}\{X \geq k+1\} + 2\mathbb{1}\{X \geq k+1\} \sum_{i=1}^{k} \mathbb{1}\{X \geq i\} + \left(\sum_{i=1}^{k} \mathbb{1}\{X \geq i\}\right)^2 \\
&= \mathbb{1}\{X \geq k+1\} + 2\sum_{i=1}^{k} \mathbb{1}\{X \geq k+1\}\mathbb{1}\{X \geq i\} + \sum_{i=1}^{k}(2i-1)\mathbb{1}\{X \geq i\} \\
&= (2k+1)\mathbb{1}\{X \geq k+1\} + \sum_{i=1}^{k}(2i-1)\mathbb{1}\{X \geq i\} \\
&= \sum_{i=1}^{k+1}(2i-1)\mathbb{1}\{X \geq i\}.
\end{aligned}$$

Instead of trying to sum the indicators we manipulate I_3 so that we can use the Markov inequality on $\xi_t^{(m)}(x)$.

$$\begin{aligned}
I_3^2 \leq\ & 2\sum_{y\sim x} \sum_{i=k+1}^{m} \left(\int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} d\left(P_s(i,y,x) - \frac{s}{2d}\right)\right)^2 \\
&+ \frac{2t}{2d} \sum_{y\sim x} \int_0^t \left(\sum_{i=k+1}^{m} \mathbb{1}\{\xi_s^{(m)}(y) \geq i\}\right)^2 ds + c.t. \\
\leq\ & 2\sum_{y\sim x} \sum_{i=k+1}^{m} \left(\int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} d\left(P_s(i,y,x) - \frac{s}{2d}\right)\right)^2 \\
&+ \frac{4t}{2d} \sum_{y\sim x} \int_0^t \sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(y) \geq i\} ds + c.t.. \qquad (2.18)
\end{aligned}$$

Taking conditional expectation with respect to the initial condition

$$
\begin{aligned}
\mathbb{E}\left[I_3^2\middle|\xi_0\right] \leq 2\sum_{y\sim x}\sum_{i=k+1}^{m}\mathbb{E}\left[\left(\int_0^t \mathbb{1}\{\xi_{s-}^{(m)}(y)\geq i\}d\left(P_s(i,y,x)-\frac{s}{2d}\right)\right)^2\middle|\xi_0\right] \\
+\frac{4t}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(y)\geq i\}ds\middle|\xi_0\right] \\
\leq \frac{2}{2d}\sum_{y\sim x}\sum_{i=k+1}^{m}\int_0^t \mathbb{1}\{\xi_s^{(m)}(y)\geq i\}ds \\
+\frac{4t}{2d}\sum_{y\sim x}\int_0^t\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(y)\geq i\}ds \\
\leq \frac{8t}{2d}\sum_{y\sim x}\int_0^t\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(y)\geq i\}ds.
\end{aligned}
$$

Multiplying by $e^{-\theta|x|}$, summing over $x\in B_N$, taking conditional expectation once again with respect to the initial condition

$$
\begin{aligned}
\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|}I_3^2\middle|\xi_0\right] &\leq \frac{8t}{2d}\sum_{x\in B_N} e^{-\theta|x|}\sum_{y\sim x}\mathbb{E}\left[\int_0^t\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(y)\geq i\}ds\middle|\xi_0\right] \\
&\leq \frac{8t}{2d}\sum_{x} e^{-\theta|x|}\sum_{y\sim x}\mathbb{E}\left[\int_0^t\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(y)\geq i\}ds\middle|\xi_0\right] \\
&\leq 8te^{\theta}\mathbb{E}\left[\int_0^t\sum_{x} e^{-\theta|x|}\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(x)\geq i\}ds\middle|\xi_0\right]. \quad (2.19)
\end{aligned}
$$

The sum of the indicators can be bounded by as follows

$$
\begin{aligned}
&\sum_{i=k+1}^{m} i\mathbb{1}\{\xi_s^{(m)}(x) \geq i\} \\
&\leq \sum_{i=k+1}^{m} \xi_s^{(m)}(x)\mathbb{1}\{\xi_s^{(m)}(x) \geq i\} \\
&= \xi_s^{(m)}(x) \sum_{i=1}^{m} \mathbb{1}\{\xi_s^{(m)}(x) \geq i\}\mathbb{1}\{i \geq k+1\} \\
&\leq \xi_s^{(m)}(x)\mathbb{1}\{\xi_s^{(m)}(x) \geq k+1\} \sum_{i=1}^{m} \mathbb{1}\{\xi_s^{(m)}(x) \geq i\} \\
&\leq \xi_s^{(m)}(x)^2\mathbb{1}\{\xi_s^{(m)}(x) \geq k+1\}. \qquad (2.20)
\end{aligned}
$$

Substituting (2.20) into (2.19) gives

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_3^2 \middle| \xi_0\right] \\
&\leq 8te^{\theta}\mathbb{E}\left[\int_0^t \sum_x e^{-\theta|x|}\xi_s^{(m)}(x)^2\mathbb{1}\{\xi_s^{(m)}(x) \geq k+1\}ds \middle| \xi_0\right] \\
&\leq 8te^{\theta}\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\theta|x|}\xi_s^{(m)}(x)^4\right)^{\frac{1}{2}} \left(\sum_x e^{-\theta|x|}\mathbb{1}\{\xi_s^{(m)}(x) \geq k+1\}\right)^{\frac{1}{2}} ds \middle| \xi_0\right] \\
&\leq 8te^{\theta}\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\frac{\theta|x|}{2}}\xi_s^{(m)}(x)^2\right) \left(\sum_x e^{-\theta|x|}\mathbb{1}\{\xi_s^{(m)}(x) \geq k+1\}\right)^{\frac{1}{2}} ds \middle| \xi_0\right].
\end{aligned}
$$

Let $\tau_{R,\theta/2} = \inf\left\{s > 0 \colon \sum_x e^{-\frac{\theta|x|}{2}} \xi_s^{(m)}(x)^2 > R\right\}$. Then

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_3^2(t\wedge\tau_{R,\theta/2})\bigg|\xi_0\right] \\
&\le 8te^{\theta}\mathbb{E}\left[\int_0^{t\wedge\tau_{R,\theta/2}} \left(\sum_x e^{-\frac{\theta|x|}{2}}\xi_s^{(m)}(x)^2\right)\left(\sum_x e^{-\theta|x|}\mathbb{1}\{\xi_s^{(m)}(x)\ge k+1\}\right)^{\frac{1}{2}} ds\bigg|\xi_0\right] \\
&\le 8te^{\theta}R\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\theta|x|}\mathbb{1}\{\xi_s^{(m)}(x)\ge k+1\}\right)^{\frac{1}{2}} ds\bigg|\xi_0\right] \\
&\le 8te^{\theta}R\int_0^t \left(\sum_x e^{-\theta|x|}\mathbb{P}[\xi_s^{(m)}(x)\ge k+1|\xi_0]\right)^{\frac{1}{2}} ds.
\end{aligned}
$$

The final inequality is the Jensen inequality. The Markov inequality and stability of conditional expectation gives

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_3^2(t\wedge\tau_{R,\theta/2})\bigg|\xi_0\right] \\
&\quad\le 8te^{\theta}R\int_0^t \left(\sum_x e^{-\theta|x|}\frac{\xi_s^{(m)}(x)^2}{k+1}\right)^{\frac{1}{2}} ds.
\end{aligned}
$$

Taking expectation, using the total law of probability and Jensen's inequality once more gives

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_3^2(t\wedge\tau_{R,\theta/2})\right] \\
&\quad\le 8te^{\theta}R\int_0^t \left(\sum_x e^{-\theta|x|}\mathbb{E}\left[\frac{\xi_s^{(m)}(x)^2}{k+1}\right]\right)^{\frac{1}{2}} ds. \qquad (2.21)
\end{aligned}
$$

Notice, by bounding $k+1>1$ we can see that the expression given by (2.21) is bounded above by a constant only depending on t, θ, R, due to the uniform moment conditions satisfied by $\xi_s^{(m)}(x)$. Hence, the bounded convergence theorem gives that

$$\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_3^2(t\wedge\tau_{R,\theta/2})\right]\to 0 \tag{2.22}$$

as $k\to\infty$. Finally,

$$\begin{aligned}
\mathbb{E}[I_2^2|\xi_0] &\le \frac{2}{2d}\sum_{y\sim x}\sum_{i=1}^{k}\mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_{s-}^{(k)}(y)\ge i,\xi_{s-}^{(k)}(x)=k\}ds\middle|\xi_0\right]\\
&\quad+\frac{2t}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t\left(\sum_{i=1}^{k}\mathbb{1}\{\xi_s^{(k)}(y)\ge i,\xi_s^{(k)}(x)=k\}\right)^2 ds\middle|\xi_0\right]\\
&\le\frac{2(1+2t)}{2d}\sum_{y\sim x}\mathbb{E}\left[\int_0^t\sum_{i=1}^{k} i\mathbb{1}\{\xi_s^{(k)}(y)\ge i,\xi_s^{(k)}(x)=k\}ds\middle|\xi_0\right].
\end{aligned}\tag{2.23}$$

Bounding the sum of the indicators by $\xi_s^{(k)}(y)^2\mathbb{1}\{\xi_s^{(k)}(x)=k\}$ and summing over $x\in B_N$ gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x\in B_N}e^{-\theta|x|}I_2^2\middle|\xi_0\right]\\
&\le\frac{2(1+2t)}{2d}\mathbb{E}\left[\int_0^t\sum_{x\in B_N}e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\sum_{y\sim x}\xi_s^{(k)}(y)^2ds\middle|\xi_0\right]\\
&\le\frac{2(1+2t)}{2d}\mathbb{E}\left[\int_0^t\sum_{x}e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\sum_{y\sim x}\xi_s^{(k)}(y)^2ds\middle|\xi_0\right]\\
&\le\frac{2(1+2t)}{2d}\mathbb{E}\left[\int_0^t\left(\sum_{x}e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\right)^{\frac12}\left(\sum_{x}e^{-\theta|x|}\left(\sum_{y\sim x}\xi_s^{(k)}(y)^2\right)^2\right)^{\frac12}ds\middle|\xi_0\right]\\
&\le\frac{2(1+2t)}{2d}\mathbb{E}\left[\int_0^t\left(\sum_{x}e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\right)^{\frac12}\left(2d\sum_{x}e^{-\theta|x|}\sum_{y\sim x}\xi_s^{(k)}(y)^4\right)^{\frac12}ds\middle|\xi_0\right]\\
&\le\frac{2(1+2t)}{2d}\mathbb{E}\left[\int_0^t\left(\sum_{x}e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\right)^{\frac12}\left((2d)^2e^{\theta}\sum_{x}e^{-\theta|x|}\xi_s^{(k)}(x)^4\right)^{\frac12}ds\middle|\xi_0\right]\\
&\le 2(1+2t)e^{\frac{\theta}{2}}\mathbb{E}\left[\int_0^t\left(\sum_{x}e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\right)^{\frac12}\left(\sum_{x}e^{-\frac{\theta|x|}{2}}\xi_s^{(k)}(x)^2\right)ds\middle|\xi_0\right].
\end{aligned}$$

Reintroducing $\tau_{R,\theta}$ we find

$$\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_2^2(t\wedge\tau_{R,\theta/2})\middle|\xi_0\right]$$
$$\leq 2(1+2t)e^{\frac{\theta}{2}}\mathbb{E}\left[\int_0^{t\wedge\tau_{R,\theta/2}}\left(\sum_x e^{-\theta|x|}\mathbb{1}\{\xi_s^{(k)}(x)=k\}\right)^{\frac{1}{2}}\left(\sum_x e^{-\frac{\theta|x|}{2}}\xi_s^{(k)}(x)^2\right)ds\middle|\xi_0\right]$$
$$\leq 2(1+2t)e^{\frac{\theta}{2}}R\int_0^t\left(\sum_x e^{-\theta|x|}\mathbb{P}[\xi_s^{(k)}(x)=k|\xi_0]\right)^{\frac{1}{2}}ds.$$

Again, the final inequality is the Jensen inequality and by the Markov inequality once again

$$\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_2^2(t\wedge\tau_{R,\theta/2})\middle|\xi_0\right]$$
$$\leq 2(1+2t)e^{\frac{\theta}{2}}R\int_0^t\left(\sum_x e^{-\theta|x|}\frac{\xi_s^{(k)}(x)^2}{k}\right)^{\frac{1}{2}}ds.$$

Taking expectation and using Jensen's inequality gives

$$\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_2^2(t\wedge\tau_{R,\theta/2})\right]$$
$$\leq 2(1+2t)e^{\frac{\theta}{2}}R\int_0^t\left(\sum_x e^{-\theta|x|}\mathbb{E}\left[\frac{\xi_s^{(k)}(x)^2}{k}\right]\right)^{\frac{1}{2}}ds \tag{2.24}$$

and similarly bounding $k\geq 1$ allows us to justify that

$$\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|} I_2^2(t\wedge\tau_{R,\theta/2})\right]\to 0 \tag{2.25}$$

as $k\to\infty$. Returning to (2.16) evaluated at $t\wedge\tau_{R,\theta/2}$, taking expectation, using the total law of probability and substituting (2.17), (2.21) and (2.24), but collecting (2.21) and (2.24) into $\mathscr{E}_k(t,\theta,R)$ a term that almost surely tend to 0 as $k\to\infty$ by

(2.22) and (2.25), gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)-\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right]\\
&\le \mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|}|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)-\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right]+3\sum_{i=1}^{3}\mathbb{E}\left[\sum_{x\in B_N} e^{-\theta|x|}I_i^2(t\wedge\tau_{R,\theta/2})\right]\\
&\le \mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|}|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)-\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right]\\
&\qquad +3\mathscr{E}_k(t,\theta,R)+6(1+t)e^{\theta}\mathbb{E}\left[\int_0^{t\wedge\tau_{R,\theta/2}}\sum_x e^{-\theta|x|}\left|\xi^{(m)}_s(x)-\xi^{(k)}_s(x)\right|^2 ds\right].
\end{aligned}$$

Grönwall's inequality now implies that for each $R>0$

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)-\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right]\\
&\le\left(\mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|}|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)-\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right]+3\mathscr{E}_k(t,\theta,R)\right)e^{6(1+t)e^{\theta}t}.
\end{aligned}$$

Since $t\wedge\tau_{R,\theta/2}\le t$ for any R, Lemma 2.2.2 gives bounds on

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\left(|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)|^2+|\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right)\right]$$

uniform in R,k and m. Hence, for any $\varepsilon>0$ we can choose N independent of R such that

$$\mathbb{E}\left[\sum_{|x|>N} e^{-\theta|x|}|\xi^{(m)}_{t\wedge\tau_{R,\theta/2}}(x)-\xi^{(k)}_{t\wedge\tau_{R,\theta/2}}(x)|^2\right]<\frac{\varepsilon}{2e^{6(1+t)e^{\theta}t}}.$$

Lemma 2.2.2 also implies that $\tau_{\theta/2,R}\to\infty$ as $R\to\infty$. Therefore, for any t there is a ρ such that $R>\rho$ gives $\tau_{\theta/2,R}>t$ so that $\xi_{t\wedge\tau_{\theta/2,R}}(x)=\xi_t(x)$. In particular, for $R=\rho+1$

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi^{(m)}_t(x)-\xi^{(k)}_t(x)|^2\right]\le\left(\frac{\varepsilon}{2e^{6(1+t)e^{\theta}t}}+\mathscr{E}_k(t,\theta,\rho+1)\right)e^{6(1+t)e^{\theta}t}.$$

Finally, choosing $k > \max_{|x|\leq N}\xi_0(x)$ (such k almost surely exist) large enough implies

$$\mathscr{E}_k(t,\theta,\rho+1) < \frac{\varepsilon}{2e^{6(1+t)e^{\theta}t}}$$

in turn implying that

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(m)}(x)-\xi_t^{(k)}(x)|^2\right] < \varepsilon$$

and so $(\xi_t^{(m)}(x))_m$ is a Cauchy sequence. Let $\xi_t(x) = \lim_{m\to\infty}\xi_t^{(m)}(x)$. Then, since the bound (A.13) (with $p=2$) is independent of m, taking limits shows

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t(x)^2\right] \leq \sum_x e^{-\theta|x|}\mathbb{E}\left[\xi_0(x)^2\right]e^{4t}. \tag{2.26}$$

We will now show that this limit is indeed a solution to (2.14). In order to do this, take $n > m$ and $\xi_t^{(m)}(x)$ to be the solution to (2.15). Consider the difference of the corresponding first integrals of (2.14) and (2.15), with the infinite sum truncated at n

$$\begin{aligned}
\Big|\int_0^t &\sum_{y\sim x}\sum_{i=1}^{n}\mathbb{1}\{\xi_{s-}(y)\geq i\}dP_s(i,y,x) \\
&- \int_0^t\sum_{y\sim x}\sum_{i=1}^{m}\mathbb{1}\{\xi_{s-}^{(m)}(y)\geq i, \xi_{s-}^{(m)}(x) < m\}dP_s(i,y,x)\Big| \\
\leq &\int_0^t\sum_{y\sim x}\sum_{i=1}^{m}\Big|\mathbb{1}\{\xi_{s-}(y)\geq i\} - \mathbb{1}\{\xi_{s-}^{(m)}(y)\geq i\}\Big|dP_s(i,y,x) \\
&+ \int_0^t\sum_{y\sim x}\sum_{i=1}^{m}\Big|\mathbb{1}\{\xi_{s-}^{(m)}(y)\geq i, \xi_{s-}^{(m)}(x) = m\}\Big|dP_s(i,y,x) \\
&+ \int_0^t\sum_{y\sim x}\sum_{i=m+1}^{n}\mathbb{1}\{\xi_{s-}(y)\geq i\}dP_s(i,y,x).
\end{aligned}$$

By squaring and using Cauchy-Schwarz, we may treat each term individually. The second term will vanish in expectation in calculations analogous to equations (2.23) through (2.24). And the last term will vanish because of the second moment bound (2.26) and a calculation analogous to (2.18) through (2.21) that is uniform in n. As for the first term

$$\mathbb{E}\left[\left(\int_0^t \sum_{y\sim x}\sum_{i=1}^m \left|\mathbb{1}\{\xi_{s-}(y)\geq i\} - \mathbb{1}\{\xi_{s-}^{(m)}(y)\geq i\}\right| dP_s(i,y,x)\right)^2\right]$$
$$\leq \frac{2(1+t)}{2d}\sum_{y\sim x}\int_0^t \mathbb{E}\left[\left|\xi_s(y)-\xi_s^{(m)}(y)\right|^2\right] ds \to 0$$

as $m\to\infty$. Since the errors are uniform in n and n is arbitrary, this establishes the almost sure convergence of the integrals

$$\int_0^t \sum_{y\sim x}\sum_{i=1}^m \mathbb{1}\{\xi_{s-}^{(m)}(y)\geq i, \xi_{s-}^{(m)}(x) < m\} dP_s(i,y,x)$$
$$\to \int_0^t \sum_{y\sim x}\sum_{i=1}^\infty \mathbb{1}\{\xi_{s-}(y)\geq i\} dP_s(i,y,x).$$

A similarly, yet easier argument will show the almost sure convergence of the integrals

$$\int_0^t \sum_{y\sim x}\sum_{i=1}^m \mathbb{1}\{\xi_{s-}^{(m)}(x)\geq i\} dP_s(i,x,y) \to \int_0^t \sum_{y\sim x}\sum_{i=1}^\infty \mathbb{1}\{\xi_{s-}(x)\geq i\} dP_s(i,x,y).$$

All that is left to do is demonstrate that the final integral in (2.15) converges to the final integral of (2.14). Once again, taking $n > m$, we have

$$\begin{aligned}
&\Big| \int_0^t \sum_{i,j=1}^{n} \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
&- \int_0^t \sum_{i,j=1}^{m} \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \Big| \qquad (2.27) \\
&\leq \sum_{i,j=1}^{m} \int_0^t \Big| \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} \Big| \\
&\qquad \times dP_s^c(i,j,x) \\
&\quad + \sum_{i=m+1}^{n} \sum_{j=1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
&\quad + \sum_{i=1}^{m} \sum_{j=m+1}^{n} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
&\quad + \sum_{i,j=m+1}^{n} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
&\leq \sum_{i,j=1}^{m} \int_0^t \Big| \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} \Big| \\
&\qquad \times dP_s^c(i,j,x) \\
&\quad + 2 \sum_{i=m+1}^{n} \sum_{j=1}^{m} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i\} dP_s^c(i,j,x) \\
&\quad + 2 \sum_{i=m+2}^{n} \sum_{j=m+1}^{i} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i\} dP_s^c(i,j,x) \\
&\leq \sum_{i,j=1}^{m} \int_0^t \Big| \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} \Big| \\
&\qquad \times dP_s^c(i,j,x) \\
&\quad + 4 \sum_{i=m+1}^{n} \sum_{j=1}^{i} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i\} dP_s^c(i,j,x).
\end{aligned}$$

Squaring and using Cauchy-Schwarz gives

$$
\begin{aligned}
&\Big| \int_0^t \sum_{i,j=1}^{n} \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
&\qquad - \int_0^t \sum_{i,j=1}^{m} \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \Big|^2 \\
&\leq 2 \Big(\sum_{i,j=1}^{m} \int_0^t \Big| \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} \Big| \\
&\qquad\qquad \times dP_s^c(i,j,x) \Big)^2 \\
&\quad + 32 \Big(\sum_{i=m+1}^{n} \sum_{j=1}^{i} \int_0^t \mathbb{1}\{\xi_{s-}(x) \geq i\} dP_s^c(i,j,x) \Big)^2. \qquad (2.28)
\end{aligned}
$$

Ignoring constant factors above, for the first term, taking expectation and summing over all x we have the upper bound

$$
\begin{aligned}
&2\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \sum_x e^{-\theta|x|} |\xi_s(x) + \xi_s^{(m)}(x)|^2 |\xi_s(x) - \xi_s^{(m)}(x)|^2 \right] ds \\
&\leq 2\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \sum_x e^{-\theta|x|} |\xi_s(x) + \xi_s^{(m)}(x)|^3 |\xi_s(x) - \xi_s^{(m)}(x)| \right] ds \\
&\leq 2\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\theta|x|} \left| \xi_s(x) + \xi_s^{(m)}(x) \right|^6 \right)^{\frac{1}{2}} \left(\sum_x e^{-\theta|x|} |\xi_s(x) - \xi_s^{(m)}(x)|^2 \right)^{\frac{1}{2}} \right] ds \\
&\leq 2\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\frac{\theta|x|}{3}} \left| \xi_s(x) + \xi_s^{(m)}(x) \right|^2 \right)^{\frac{3}{2}} \left(\sum_x e^{-\theta|x|} |\xi_s(x) - \xi_s^{(m)}(x)|^2 \right)^{\frac{1}{2}} \right] ds \\
&\leq 2^{\frac{5}{2}} \lambda(1+\lambda t)\mathbb{E}\Big[\int_0^t \left(\sum_x e^{-\frac{\theta|x|}{3}} \left(\xi_s(x)^2 + \xi_s^{(m)}(x)^2 \right) \right)^{\frac{3}{2}} \\
&\qquad \times \left(\sum_x e^{-\theta|x|} |\xi_s(x) - \xi_s^{(m)}(x)|^2 \right)^{\frac{1}{2}} \Big] ds. \qquad (2.29)
\end{aligned}
$$

Evaluating (2.27) at $t \wedge \tau_{R,\theta/3}^{m,\infty}$ and repeating the argument up to equation (2.29) bounding $t \wedge \tau_{R,\theta/3} \leq t$ where necessary gives

$$
\begin{aligned}
&2^{\frac{5}{2}}\lambda(1+\lambda t)\mathbb{E}\Big[\int_0^{t\wedge\tau_{R,\theta/3}} \left(\sum_x e^{-\frac{\theta|x|}{3}}\left(\xi_s(x)^2+\xi_s^{(m)}(x)^2\right)\right)^{\frac{3}{2}} \\
&\quad\times\left(\sum_x e^{-\theta|x|}|\xi_s(x)-\xi_s^{(m)}(x)|^2\right)^{\frac{1}{2}}\Big]ds \\
&\leq 2^4\lambda(1+\lambda t)R^{\frac{3}{2}}\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\theta|x|}|\xi_s(x)-\xi_s^{(m)}(x)|^2\right)^{\frac{1}{2}}\right]ds \\
&\leq 2^4\lambda(1+\lambda t)R^{\frac{3}{2}}\int_0^t \left(\sum_x e^{-\theta|x|}\mathbb{E}\left[|\xi_s(x)-\xi_s^{(m)}(x)|^2\right]\right)^{\frac{1}{2}} ds \to 0
\end{aligned}
$$

as $m \to 0$. For the second term of (2.28)

$$
\begin{aligned}
&2\sum_{i=m+1}^{n}\sum_{j=1}^{i}\mathbb{E}\left[\left(\int_0^t \mathbb{1}\{\xi_{s-}(x)\geq i\}d(P_s^c(i,j,x)-\lambda s)\right)^2\right] \\
&\quad+2\lambda^2 t\mathbb{E}\left[\int_0^t\left(\sum_{i=m+1}^{n} i\mathbb{1}\{\xi_s(x)\geq i\}\right)^2 ds\right] \\
&\leq 2\lambda\mathbb{E}\left[\int_0^t \sum_{i=m+1}^{n} i\mathbb{1}\{\xi_{s-}(x)\geq i\}ds\right] \\
&\quad+4\lambda^2 t\mathbb{E}\left[\int_0^t \xi_s(x)^2\sum_{i=m+1}^{n} i\mathbb{1}\{\xi_s(x)\geq i\}ds\right] \\
&\leq 2\lambda(1+2\lambda t)\mathbb{E}\left[\int_0^t \xi_s(x)^3\sum_{i=m+1}^{n}\mathbb{1}\{\xi_s(x)\geq i\}ds\right] \\
&\leq 2\lambda(1+2\lambda t)\mathbb{E}\left[\int_0^t \xi_s(x)^4\mathbb{1}\{\xi_s(x)\geq m+1\}ds\right].
\end{aligned}
$$

Summing over all x gives the upper bound

$$
\begin{aligned}
&2\lambda(1+2\lambda t)\mathbb{E}\left[\int_0^t\sum_x e^{-\theta|x|}\xi_s(x)^4\mathbb{1}\{\xi_s(x)\geq m+1\}ds\right] \\
&\leq 2\lambda(1+2\lambda t)\mathbb{E}\left[\int_0^t\left(\sum_x e^{-\frac{\theta|x|}{4}}\xi_s(x)^2\right)^2\left(\sum_x e^{-\theta|x|}\mathbb{1}\{\xi_s(x)\geq m+1\}\right)^{\frac{1}{2}}ds\right].
\end{aligned}
$$

Evaluating (2.27) at $t\wedge\tau^{m,\infty}_{R,\theta/4}$, repeating the calculations up to this point and noting

$\tau_{R,\theta/4} \leq \tau_{R,\theta/3}$ almost surely so that $t \leq \tau_{R,\theta/4}$ implies

$$\sum_x e^{-\frac{\theta|x|}{3}} \xi_s(x)^2 \leq R,$$

the calculations for the first term of (2.28) remain valid, while we have the upper bound for the second term of

$$\begin{aligned}
&2\lambda(1+2\lambda t)R^2 \mathbb{E}\left[\int_0^t \left(\sum_x e^{-\theta|x|} \mathbb{1}\{\xi_s(x) \geq m+1\}\right)^{\frac{1}{2}} ds\right] \\
&\leq 2\lambda(1+2\lambda t)R^2 \int_0^t \left(\mathbb{E}\left[\sum_x e^{-\theta|x|} \mathbb{1}\{\xi_s(x) \geq m+1\}\right]\right)^{\frac{1}{2}} ds \\
&\leq \frac{2\lambda(1+2\lambda t)R^2}{m+1} \int_0^t \left(\mathbb{E}\left[\sum_x e^{-\theta|x|} \xi_s(x)^2\right]\right)^{\frac{1}{2}} ds \to 0
\end{aligned}$$

as $m \to \infty$. Therefore, at least up until time $t \wedge \tau_{R,\theta/4}$, all the integrals in (2.15) converge to their counterparts in (2.14). However, since $\tau_{R,\theta/4} \to \infty$ almost surely as $R \to \infty$ we may choose R large enough that $t \wedge \tau_{R,\theta/4} = t$ and that $\xi_t(x)$ solves (2.14) can be seen by taking limits in (2.15).

Uniqueness Suppose we have two solutions to (2.14), ξ_t and η_t started with the same inital condition satisfying the moment bounds in the statement. Then looking at their difference

$$\begin{aligned}
&|\xi_t(x) - \eta_t(x)| \\
&\leq \sum_{y \sim x} \sum_{i=1}^{\infty} \int_0^t |\mathbb{1}\{\xi_{s-}(y) \geq i\} - \mathbb{1}\{\eta_{s-}(y) \geq i\}| \, dP_s(i,y,x) \\
&+ \sum_{y \sim x} \sum_{i=1}^{\infty} \int_0^t |\mathbb{1}\{\xi_{s-}(x) \geq i\} - \mathbb{1}\{\eta_{s-}(x) \geq i\}| \, dP_s(i,x,y) \\
&+ \sum_{i,j=1}^{\infty} \int_0^t |\mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\eta_{s-}(x) \geq i \vee j, i \neq j\}| \, dP_s^c(i,j,x) \quad (2.30)
\end{aligned}$$

and squaring

$$
\begin{aligned}
&|\xi_t(x) - \eta_t(x)|^2 \\
&\leq 3 \left(\sum_{y \sim x} \sum_{i=1}^{\infty} \int_0^t |\mathbb{1}\{\xi_{s-}(y) \geq i\} - \mathbb{1}\{\eta_{s-}(y) \geq i\}| \, dP_s(i, y, x) \right)^2 \\
&+ 3 \left(\sum_{y \sim x} \sum_{i=1}^{\infty} \int_0^t |\mathbb{1}\{\xi_{s-}(x) \geq i\} - \mathbb{1}\{\eta_{s-}(x) \geq i\}| \, dP_s(i, x, y) \right)^2 \\
&+ 3 \left(\sum_{i,j=1}^{\infty} \int_0^t |\mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\eta_{s-}(x) \geq i \vee j, i \neq j\}| \, dP_s^c(i, j, x) \right)^2 \\
&= 3 \sum_{i=1}^{3} I_i. \qquad (2.31)
\end{aligned}
$$

We are once again using I_i for the i^{th} term of the finite sum, we will estimate them individually. Beginning with I_1 and taking expectation we find

$$
\mathbb{E}[I_1] \leq \frac{6(1+t)}{2d} \sum_{y \sim x} \mathbb{E} \left[\int_0^t |\xi_s(y) - \eta_s(y)|^2 \, ds \right].
$$

Summing over all x gives

$$
\mathbb{E} \left[\sum_x e^{-\theta|x|} I_1 \right] \leq 6(1+t) e^{\theta} \mathbb{E} \left[\int_0^t \sum_x e^{-\theta|x|} |\xi_s(x) - \eta_s(x)|^2 \, ds \right]. \qquad (2.32)
$$

Similarly,

$$
\mathbb{E} \left[\sum_x e^{-\theta|x|} I_2 \right] \leq 6(1+t) \mathbb{E} \left[\int_0^t \sum_x e^{-\theta|x|} |\xi_s(x) - \eta_s(x)|^2 \, ds \right]. \qquad (2.33)
$$

Repeating familiar calculations we arrive at the following expression involving I_3

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|} I_3\right] \qquad (2.34)\\
&\le 6\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \sum_x e^{-\theta|x|}\left|\xi_s(x)+\eta_s(x)-1\right|^2\left|\xi_s(x)-\eta_s(x)\right|^2 ds\right]\\
&\le 6\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \sum_x e^{-\theta|x|}\left|\xi_s(x)+\eta_s(x)\right|^2\left|\xi_s(x)-\eta_s(x)\right|^2 ds\right]\\
&\le 6\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\frac{\theta|x|}{2}}\left|\xi_s(x)+\eta_s(x)\right|\left|\xi_s(x)-\eta_s(x)\right|\right)^2 ds\right].
\end{aligned}$$

Fix $0 < \varepsilon < \theta$, and write $e^{-\theta|x|} = e^{-(\theta-\varepsilon)|x|}e^{-\varepsilon|x|}$, then Cauchy-Schwarz gives the upper bound of

$$6\lambda(1+\lambda t)\mathbb{E}\left[\int_0^t \left(\sum_x e^{-\varepsilon|x|}\left|\xi_s(x)+\eta_s(x)\right|^2\right)\left(\sum_x e^{-(\theta-\varepsilon)|x|}\left|\xi_s(x)-\eta_s(x)\right|^2\right) ds\right].$$

Let $\tau_{R,\varepsilon} = \inf\{t > 0\colon \sum_x e^{-\varepsilon|x|}\left(|\xi_t(x)|^2 + |\eta_t(x)|^2\right) > R\}$ then evaluating (2.30) at $t \wedge \tau_{R,\varepsilon}$ bounding $t \wedge \tau_{R,\varepsilon} \le t$ wherever necessary will allow us to bound

$$\mathbb{E}\left[\sum_x e^{-\theta|x|} I_3\right] \le 6\lambda(1+\lambda t)R\mathbb{E}\left[\int_0^t \left(\sum_x e^{-(\theta-\varepsilon)|x|}\left|\xi_s(x)-\eta_s(x)\right|^2\right) ds\right]. \qquad (2.35)$$

Now, multiplying (2.31) though by $e^{-\theta|x|}$ and summing over all x gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_{t\wedge\tau_{R,\varepsilon}}(x) - \eta_{t\wedge\tau_{R,\varepsilon}}(x)|^2\right]\\
&\le 3\sum_{i=1}^{3}\mathbb{E}\left[\sum_x e^{-\theta|x|} I_i(t\wedge\tau_{R,\varepsilon})\right]
\end{aligned}$$

and substituting (2.32), (2.33) and (2.35) gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_{t\wedge\tau_{R,\varepsilon}}(x) - \eta_{t\wedge\tau_{R,\varepsilon}}(x)|^2\right] \\
&\leq 18(1+t)(1+e^\theta)\mathbb{E}\left[\int_0^t \sum_x e^{-\theta|x|}\,|\xi_s(x)-\eta_s(x)|^2\,ds\right] \\
&\quad + 18\lambda(1+\lambda t)R\mathbb{E}\left[\int_0^t \left(\sum_x e^{-(\theta-\varepsilon)|x|}\,|\xi_s(x)-\eta_s(x)|^2\right)ds\right].
\end{aligned}$$

Since $\tau_{R,\varepsilon} \to \infty$ as $R \to \infty$, we may choose R large enough that $t\wedge\tau_{R,\varepsilon} = t$. Writing $C_{t,\theta,\lambda,R} = 18\max\{(1+t)(1+e^\theta), \lambda(1+\lambda t)R\}$ we have

$$\begin{aligned}
0 &\leq \sum_x e^{-\theta|x|}\mathbb{E}\left[|\xi_t(x)-\eta_t(x)|^2\right] \\
&\leq \sum_x e^{-\theta|x|}\int_0^t C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\mathbb{E}\left[|\xi_s(x)-\eta_s(x)|^2\right]ds.
\end{aligned}$$

This implies that

$$\sum_x e^{-\theta|x|}\left(C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\int_0^t \mathbb{E}\left[|\xi_s(x)-\eta_s(x)|^2\right]ds - \mathbb{E}\left[|\xi_t(x)-\eta_t(x)|^2\right]\right) \geq 0. \tag{2.36}$$

Since the sum is positive, there exist sites in $\mathbb{Z}^d$ such that the summand is positive. Let $\mathcal{A}$ be the set of sites for which the summand is positive. Then for $x \in \mathcal{A}$

$$\mathbb{E}\left[|\xi_t(x)-\eta_t(x)|^2\right] \leq C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\int_0^t \mathbb{E}\left[|\xi_s(x)-\eta_s(x)|^2\right]ds$$

and Grönwall's Lemma implies that

$$\mathbb{E}\left[|\xi_s(x)-\eta_s(x)|^2\right] = 0 \text{ for all } x \in \mathcal{A} \tag{2.37}$$

and all $s \leq t$. Returning to (2.36) we may bound by ignoring the negative contributions that come from all $x \in \mathcal{A}^c$ (which may be empty)

$$\begin{aligned}
0 &\leq \sum_x e^{-\theta|x|} \left(C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|}) \int_0^t \mathbb{E}\left[|\xi_s(x) - \eta_s(x)|^2 \right] ds - \mathbb{E}\left[|\xi_t(x) - \eta_t(x)|^2 \right] \right) \\
&\leq \sum_{x \in \mathcal{A}} e^{-\theta|x|} \left(C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|}) \int_0^t \underbrace{\mathbb{E}\left[|\xi_s(x) - \eta_s(x)|^2 \right]}_{=0} ds - \underbrace{\mathbb{E}\left[|\xi_t(x) - \eta_t(x)|^2 \right]}_{=0} \right) \\
&= 0.
\end{aligned}$$

In particular, this chain of inequalities is in fact a chain of equality which implies that the sum over $x \in \mathcal{A}^c$ is 0 and since all the summands in the sum over $x \in \mathcal{A}^c$ are less than or equal to 0, we must that for any $x \in \mathcal{A}^c$

$$\mathbb{E}\left[|\xi_t(x) - \eta_t(x)|^2 \right] = C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|}) \int_0^t \mathbb{E}\left[|\xi_s(x) - \eta_s(x)|^2 \right] ds$$

and Grönwall's Lemma once again implies

$$\mathbb{E}\left[|\xi_t(x) - \eta_t(x)|^2 \right] = 0 \text{ for all } x \in \mathcal{A}^c$$

which together with (2.37) implies that $\xi \equiv \eta$ for each t. □

2.3 Properties of Solutions

Let $\xi_t^{(\infty)}$ be the solution to (2.1) and let $\xi^{(\lambda)}$ be the solution to (2.14) with appropriate initial conditions. We will need a lower bound for $\hat{\xi}_t^{(\lambda)} = \mathbb{E}[\xi_t^{(\lambda)}(0)]$ of order $O(\log t/t)$. Such a lower bound is available for $\hat{\xi}_t^{\infty} = \mathbb{E}[\xi_t^{\infty}(0)]$ due to results about the dual process of instantly coalescing random walks given in Kelly [13]. We can exploit the lower bound for the instantaneous process by coupling it with the finite rate process. The following Lemma is intuitively clear. Start the two processes with a particle at each site (this is the maximal initial condition for the instantaneous process). Then since colliding particles instantly coalesce in the former process but coalesce slowly in the latter process (so that the particles have the chance to walk away before they coalesce) there will be more distinct particles at any site in the latter process. In this way the number of particles at any site in the instantaneous process is a lower bound for the number of particles at x in the finite rate process so that the lower bound from [13] is also a lower bound for $\hat{\xi}_t^\lambda$.

Lemma 2.3.1. *Let $\xi_t^{(\lambda)}$ be the solution to (2.14) and $\xi_t^{(\infty)}$ be the solution to (2.1) both having initial condition $\xi_0 \equiv 1$. The processes $\xi_t^{(\lambda)}, \xi_t^{(\infty)}$ can be coupled so that*

$$\xi_t^{(\lambda)}(x) \geq \xi_t^{(\infty)}(x) \qquad \text{for each } x \in \mathbb{Z}^d \text{ and for all } t > 0 \text{ a.s..}$$

Proof. Begin with a collection of independent Poisson processes that drive the equation (2.14) $P_t(i, x, y)$ and $P_t^c(i, j, x)$. Use the $P_t(1, x, y)$ to define $P_t(x, y) = P_t(1, x, y)$ to drive the equation (2.1). Proposition 2.1.1 and Theorem 2.2.3 guarantee the existence and uniqueness of the processes.

Now we demonstrate how this guarantees the inequality is preserved. We colour every particle white initially. We will paint a white particle red the first time it jumps to a site already occupied by a white particle. If there is a white particle present at a site it will occupy the first position (as viewed as a particle in the finite rate system) and in this way obeys the ring of a $P_t(x, y) = P_t(1, x, y)$ clock so that with that ring it will move in either system. The instantaneous process will be made up only of the white particles while the finite rate process will consist of both the white and red particles.

Suppose at time s there is a white particle and i red particles at x totalling $i + 1$ particles in the finite rate process.

- If there is a white particle at a neighbouring y and if the $P_t(y, x)$ clock rings after time s before any $P_t(x, \cdot)$ then the white particle arriving at x from y is instantly repainted red and joins as the $(i + 2)^{th}$ particle of the finite process.
- If there is a red particle at a neighbouring y in some position j and the process $P_t(j, y, x)$ rings first after time s then it simply joins the stack of reds without a repaint as the $(i + 2)^{th}$ particle at x in the fintie rate process.

- Regardless of the occupation of neighbouring sites if any $P_t(x,\cdot) = P_t(1,x,\cdot)$ rings first after s then the white particle at x leaves to one of its neighbouring sites leaving behind the i red particles.

- If any $P_t^c(1,j,x)$ or $P_t^c(j,1,x)$ for $2 \le j \le i+1$ clock rings first then the red particle in position j will be deleted (or equivalently painted white and will coalesce with the white particle in position 1).

- If any $P_t^c(j,k,x)$ rings first for $2 \le j,k \le i+1$ then the red particle present at x in position k will be deleted (or coalesce with the red particle in postion j).

Suppose that there are no white particles at x and i red particles at time s

- If there is a white particle at a neighbour y and $P_t(1,y,x)$ rings first then it'll arrive at x and assume the first position.

According to the above rules the white particles will evolve according to an instantly coalescing system, while the white and red particles together form a slowly coalescing system. □

We will need the corresponding upper bound for the finite rate process and in proving this we will need two more coupling results. The content of the first is that the process is subadditive in that the process started from the union of disjoint sets is bounded above sum of independent copies of the process started from each of the members of the disjoint union. The second results shows that the particle numbers satisfy a monotonicity property in that the particle numbers are increasing in the initial condition.

Lemma 2.3.2. *For any subset $A \subset \mathbb{Z}^d$, let $\hat{\xi}_t^{\lambda,A}$ be the solution to* (2.14) *with initial condition ξ_0 satisfying $\sum_{x\in A} e^{-\theta|x|}\xi_0(x) < \infty$ and $\xi_0 \equiv 0$ on $\mathbb{Z}^d \setminus A$. Then for disjoint $A, B \subset \mathbb{Z}^d$ with $\sum_{x\in A\cup B} e^{-\theta|x|}\xi_0(x) < \infty$,*

$$\hat{\xi}_t^{\lambda,A\cup B}(x) \le \hat{\xi}_t^{\lambda,A}(x) + \hat{\xi}_t^{\lambda,B}(x) \qquad \textit{for all } x \in \mathbb{Z}^d \textit{ and } t > 0 \textit{ a.s.}.$$

Proof. Theorem 2.2.3 guarantees the existence of each of the processes $\hat{\xi}_t^{\lambda,A\cup B}, \hat{\xi}_t^{\lambda,A}$, $\hat{\xi}_t^{\lambda,B}$. The following is very close to the argument given in [25] where the author proves the analogous result for a variant on the instantly coalescing system. Since A and B are disjoint we can colour all the particles that begin in set A blue and colour all particles in set B green. Run all processes according to the same collection of Poisson processes but in such a way that if the blue and green particles start to appear at coincident sites, say there are i blue particles at x and j green particles at x at time s, then we order them so that the blue particles occupy the lower valued positions (1 through i) and the green particles occupy the higher valued positions ($i+1$ through $i+j$). The coalescence of a blue particle with a green particle can only occur in the system started from $A \cup B$. In the A system, no particle can react with a green particle. As in [25] we introduce striped particles of blue or green colour to represent the particles that have not coalesced in the A or B systems respectively that would otherwise have been lost in the $A \cup B$ process. If in addition to the i blue balls and j green balls there are i' striped blue balls and j' striped green balls at x at time s, the stiped blue balls will sit in positions $i + j + 1$ through $i + j + i'$ and the striped green balls will sit in positions $i + j + i' + 1$ through $i + j + i' + j'$. The process started at $A \cup B$ will then be represented by the sum of the solid blue and green balls, while the process started at A (resp. B) will be represented by the sum of solid and striped blue (resp. green) balls. The colouring of the particles adhere to the following rules:

- If any of the $P_t(k, x, y)$ ring first for $1 \leq k \leq i$ then the blue particle at x in position k will leave x and walk to a neighbour reducing the number of particles in the A system and $A \cup B$. Similarly for $i + 1 \leq k \leq i + j$ and the B system.

- If any of the $P^c(k, l, x)$ clocks ring first for $1 \leq k, l \leq i$ then the blue particle at position l is deleted (or coalesces with the particle in position k) and a particle is lost from A and $A \cup B$. Similarly for $i + 1 \leq k, l \leq i + j$ and the B system.

- If any of the $P^c(k, l, x)$ ring for $1 \leq k \leq i$, $i + 1 \leq l \leq i + j$ then the blue particle in position k will become a striped blue ball while the green particle in position l will remain unchanged. Similarly if $1 \leq l \leq i$, $i+1 \leq k \leq i+j$, the green ball in positon k will become a striped green ball, while the blue ball in position l remains unchanged.

- If any of the $P_t(k,x,y)$ ring first for $i+j+1 \leq k \leq i+j+i'$ then the striped blue particle at x in position k will leave x and walk to a neighbour reducing the number of particles in the A system only. Similarly for $i+j+i'+1 \leq k \leq i+j+i'+j'$ and the striped green particle in the B system.

- If any of the $P^c(k,l,x)$ clocks ring first for $1 \leq k \leq i$ and $i+j+1 \leq l \leq i+j+i'$ or for $i+j+1 \leq k, l \leq i+j+i'$ then the striped blue particle at position l is deleted and a particle is lost from the A process only. Similarly for $i+1 \leq k \leq i+j$ and $i+j+i'+1 \leq l \leq i+j+i'+j'$ or $i+j+i'+1 \leq k, l \leq i+j+i'+j'$ and the B system. Also the same will hold with the roles of k, l interchanged except that in this case the striped ball in position l is deleted (rather than the solid ball in position k).

- If any of the $P^c(k,l,x)$ clocks ring first for $1 \leq k \leq i$ and $i+j+i'+1 \leq l \leq i+j+i'+j'$, or $i+1 \leq k \leq i+j$ and $i+j+1 \leq l \leq i+j+i'$, or $i+j+1 \leq k, l \leq i+j+i'+j'$ then there will be no change made to the colour of condition (solidness vs stripedness) of any of the balls. The same will hold with k, l interchanged.

- If a particle of any colour, solid or striped, jumps to x no changes occurs to the colour or condition of the ball that arrives and it just joins the stack of balls of its colour and condition.

As described, the solid blue and green balls together will represent the evolution of the process beginning at $A \cup B$, while the solid and striped blue balls will represent the process starting at A and the solid and striped green balls will represent the process started at B and under the coupling the inequality will hold. $\square$

The next result is perhaps the most obvious. If we start two processes, one with an initial condition starting from a subset A and another started from some restriction of that subset, the particle numbers of the former should always be at least the particle numbers of the latter since there are possibly extant particles that have not reacted with particles that began in the restriction.

Lemma 2.3.3. *Let $A \subset B$. With initial condition satisfying $\sum_{x \in B} e^{-\theta|x|}\xi_0(x)^2 < \infty$ the processes $\hat{\xi}_t^{\lambda,A}, \hat{\xi}_t^{\lambda,B}$ defined in Lemma 2.3.2 satisfy*

$$\hat{\xi}_t^{\lambda,A}(x) \leq \hat{\xi}_t^{\lambda,B}(x)$$

for all $x \in \mathbb{Z}^d$ and all $t > 0$.

Proof. The proof is similar, but easier than, the proof of Lemma 2.3.2. Colour every initial particle of A blue and every initial particle of B green. Driving the processes by the same Poisson processes, with the blue particles occupying the lower value positions at each site and the green particles occupying the higher level positions at each site, we will have the A process represented solely by the number of blue particles while the B process will be represented by the sum of the number of blue and green particles. Suppose there are i blue particles and j green particles at x. The rules are simple:

- If any of the $P_t(l, x, y)$ or $P^c(k, l, x)$ ring first for $1 \leq k, l \leq i$ then the blue particle at position l is lost from both processes either through random walk step to a neighbour or through coalescence with another blue particle.
- If any of the $P_t(k, x, y)$ ring first for $i + 1 \leq k \leq j$ then the green particle at postion k leaves x for one of its neighbours and a particle is lost from the B process at x.
- If any of the $P^c(k, l, x)$ ring first for $1 \leq k \leq i$ and $i + 1 \leq l \leq i + j$ or $i + 1 \leq k, l \leq i + j$ then the green particle at position l is deleted, any blue particle is left alone and a particle is lost from the B system. For $1 \leq l \leq i$ and $i + 1 \leq k \leq i + j$ then the green particle at position k is deleted (or

changes blue and immediately coalesces with the blue particle in postion l) and a particle is lost in the B system.

- Any particle that walks to x simply joins the stack of the particles of the same colour. If the particle is blue, it increases the particle number of both A and B processes, if it is green, it only increases the particle number in the B process.

As described, the worst case is that the B process "catches up" with the lower particle numbers of the A process, but the B particle numbers can never fall beneath the particle numbers of the A process. $\square$

Chapter 3

Negative Correlation Results for Systems of Coalescing Random Walks

As in the previous chapter, the results here do not depend on the dimension and so we prove them in their full generality.

3.1 Introduction to Negative Correlation

We study continuous time random walks that coalesce at some finite rate. We are primarily interested in the decay in the probability that a particular site is occupied and in the study of this object, natural approximations occur. The task in understanding the decay transforms into careful estimation of the error terms produced in each approximation. We find that present in most of the error terms is the expected value of particular functions of our variables and proving various notions of negative dependence enable us to control these errors well enough to understand the decay in the density of particles.

The key tool in proving negative correlation results is the van den Berg-Kesten-Reimer (BKR) inequality. The nature of this inequality requires a finite, discrete time structure and so is not immediately applicable to our continuous time process that is the solution to the system of equations (4) of Chapter 1. As such, we will begin by proving various negative dependence results for a sequence of finite, discrete time systems that approximate our full process and then prove the convergence of this approximating sequence thereby carrying over the results that BKR provides us for the finite, discrete time structures to our full continuous process that solves (4) in Chapter 1.

Let V be a finite set and for each $i \in V$, let S_i be a finite set. Define $\Omega = \prod_{i \in V} S_i$. For $K \subset V$ and $\omega \in \Omega$, let $[\omega]_K$ be a cylinder, by which we mean all $\omega' \in \Omega$ that agree with ω on K. We denote by $A \square B$ the set of all $\omega \in \Omega$ for which there exist disjoint $K, L \subset V$ with $[\omega]_K \subseteq A$ and $[\omega]_L \subseteq B$. The BKR inequality states that for a product measure μ on Ω (that is $\mu = \Pi_{i \in V} \mu_i$), for all events $A, B \in \Omega$

$$\mu(A \square B) \leq \mu(A)\mu(B).$$

For systems of instantly coalescing random walks, certain negative correlation porperties are known due to van den Berg and Kesten [27]. They achieve their results by cutting their process off outside a box and discretising time thereby producing a discrete time approximation from their continuous time process. Since the method we employ to prove correlation inequalities for the finite rate process is significantly different, we will first walk through the discrete time characterisation of the instantly coalescing system to understand how build a structure for which BKR is applicable before turning to the non-instant regime. We will not prove that the discrete time process that approximates the instantly coalescing system converges since the negative correlation results are already known for the full system. The final section of this chapter be dedicated to the convergence for the non-instantly coalescing system only.

3.2 Pairwise Negative Correlation for an Instantly Coalescing Particle System

For our instantly coalescing random walks, the only meaningful result regarding correlation is pairwise negative correlation for disjoint neighbouring sites, this is really due to the nearest neighbour walks. The significance of the result is that if there is a particle at a site y, it should reduce the chance of there simultaneously being a particle at a neighbour x. In order to formulate a problem for which BKR is applicable, we shall consider a discrete time process on a box of $\mathbb{Z}^d$ of finite volume, on which we can define a notion of paths that will describe a discrete time coalescing random walk.

3.2.1 Description of Paths

Let $V = B \times T$, with $B = B_L = \{-L, -L+1, ..., 0, ..., L-1, L\}^d$, for $L \in \mathbb{N}$, and $T = T_n = \{0, ..., n\}$, that is attach a discrete time interval at each $x \in B_L$. Let $(e_i)_{i=1}^d$ be the unit vectors in each of the positive directions, that is

$$e_i = (0, 0 \ldots, \underbrace{1}_{i^{th} \text{ place}}, \ldots, 0).$$

For each $(x, i) \in V$ let the state at (x, i) be an element of the set

$$S_{(x,i)} = \{\underbrace{{}^{-e_1}\nwarrow}_{p_{(x,i,1)}}, \ldots, \underbrace{{}^{-e_d}\nwarrow}_{p_{(x,i,d)}}, \underbrace{\nearrow^{e_1}}_{q_{(x,i,1)}}, \ldots, \underbrace{\nearrow^{e_d}}_{q_{(x,i,d)}}, \underbrace{\uparrow}_{r_{(x,i)}}\}$$

with probabilities given by the underbraces. In particular, these probabilities satisfy that for all $(x, i) \in V$, $r_{(x,i)} + \sum_{j=1}^d (p_{(x,i,j)} + q_{(x,i,j)}) = 1$. In addition, if $x \in B$ has L in its j^{th} coordinate then $q_{(x,i,j)} = 0$ and similarly if x has a $-L$ in its j^{th} coordinate $p_{(x,i,j)} = 0$ for all $i \in \{0, 1, \ldots, n\}$ so that particles cannot leave the box B. The arrows indicate the site that the particle at (x, i) will move to. The northwest arrow, ${}^{-e_1}\nwarrow$ for instance, represents a move from (x, i) to $(x - e_1, i + 1)$, while an up arrow means the particle remains at the same site, $x_{i+1} = x_i$.

Let $\Omega = \prod_{(x,i)\in V} S_{(x,i)}$. Then for $\omega \in \Omega$ we have marginals given by

$$\mathbb{P}[\omega_{(x,i)} = \alpha] = \begin{cases} p_{(x,i,-j)} & \text{if } \alpha = {}^{-e_j}\nwarrow \\ q_{(x,i,j)} & \text{if } \alpha = \nearrow^{e_j} \\ r_{(x,i)} & \text{if } \alpha = \uparrow . \end{cases}$$

We give the following definitions.

Definition 3.2.1. *A path from $(x, 0)$ to (y, n) in V is a sequence $\{x = x_0, x_1, ..., x_n = y\} \subset B$ such that*

$$|x_j - x_{j-1}| \in \{0, 1\} \text{ for all } j \in \{1, ..., n\}.$$

And we will call the sequence, $\{(x_i, i)\}_{i=0}^n \subset V$, a path.

Fix $\omega \in \Omega$, given the definition of a path in our space-time V we will now define a path that our particles may take.

Definition 3.2.2. *An ω-successful path from $(x, 0)$ to (y, n) is a path $\{(x_i, i)\}_{i=0}^n$ from $(x, 0)$ to (y, n) such that for each $i \in \{0, ..., n-1\}$, the direction of the vector $x_{i+1} - x_i$ corresponds to the arrow given by $\omega_{(x_i,i)}$, the identification of the up arrow with the 0 vector. More generally, for a subset $D \subseteq B$ there exists an ω-sucessful path up from $(D, 0)$ to (y, n) if there exists $x \in D$ and an ω-successful path up from $(x, 0)$ to (y, n).*

Now we can define our coalescing walks. Let $x \in B$. Define the random variable $\xi_n(\cdot; x) \colon \Omega \to \{0, 1\}$ by

$$\xi_n(\omega; x) = \mathbb{1}[\text{there exists an } \omega\text{-successful path from } B \times \{0\} \text{ to } (x, n)]. \tag{3.1}$$

We will often supress the dependence on ω in events. For example, we will write

$$\{\xi_n(x) = 1\} = \{\omega \colon \text{there exists an } \omega\text{-successful path from } B \times \{0\} \text{ to } (x, n)\}.$$

3.2.2 A First Example of Negative Correlation

Take $x, y \in \Lambda$ such that $x \neq y$. Let

$$\mathcal{A} = \{\xi_n(x) = 1\} \text{ and } \mathcal{B} = \{\xi_n(y) = 1\}. \tag{3.2}$$

Our aim is now to prove that these events are pairwise negatively correlated in the sense that

$$\mathbb{P}[\mathcal{A} \cap \mathcal{B}] \leq \mathbb{P}[\mathcal{A}]\mathbb{P}[\mathcal{B}] \tag{3.3}$$

and we do so by applying the BKR inequality.

Theorem 3.2.3. *With the events $\mathcal{A}, \mathcal{B}$ defined in* (3.2)*, $\mathcal{A}$ and $\mathcal{B}$ are negatively correlated in the sense of* (3.3)*.*

Remark 3.2.4. *Trivially, $\mathcal{A} \Box \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$.*

Proof. It is sufficient to show that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \Box \mathcal{B}$. For then,

$$\mathbb{P}[\mathcal{A} \cap \mathcal{B}] \leq \mathbb{P}[\mathcal{A} \Box \mathcal{B}]$$

and the result follows by BKR.

As such, let $\omega \in \mathcal{A} \cap \mathcal{B}$. Then there exists $u, v \in B$ such that there is an ω-successful path from $(u, 0)$ to (x, n) and an ω-successful path from $(v, 0)$ to (y, n). Call these paths $\{(u_i, i)\}_{i=0}^n$ and $\{(v_i, i)\}_{i=0}^n$ respectively.

Let $J = \{(u_0, 0), ..., (u_n, n)\}$ and $K = \{(v_0, 0), ..., (v_n, n)\}$. We will build our cylinders from these two sets.

So, the set $[\omega]_J$ is all the $\omega' \in \Omega$ that agree with ω on J. Since J is taken to be the vertices of an ω-successful path from $(u, 0)$ to (x, n), it follows that for each $\omega' \in [\omega]_J$ there is an ω'-successful path from $(u, 0)$ to (x, n), namely $\{(u_i, i)\}_{i=0}^n$. Therefore, we have the inclusion $[\omega]_J \subseteq \mathcal{A}$. Similarly, $[\omega]_K \subseteq \mathcal{B}$.

To conclude that $\omega \in \mathcal{A} \Box \mathcal{B}$ and complete the proof, we must show that $J \cap K = \emptyset$. So suppose $(z, r) \in J \cap K$. Then (z, r) was a vertex in the ω-successful path $\{(u_i, i)\}_{i=0}^n$ and a vertex in the ω-successful path $\{(v_i, i)\}_{i=0}^n$, namely the vertices

(u_r, r) and (v_r, r) respectively. In particular, $u_r = v_r$. Now since (u_r, r) is in the ω-successful path $\{(u_i, i)\}_{i=0}^{n}$, the direction of the vector $u_{r+1} - u_r = u_{r+1} - z$ corresponds to the arrow $\omega_{(z,r)}$. Similarly, the direction of the vector $v_{r+1} - v_r = v_{r+1} - z$ also corresponds to the arrow given by $\omega_{(z,r)}$. Hence, $u_{r+1} = v_{r+1}$. Call this point in Λ, z_1 and define $z_0 = z$. We can repeat the previous line of argument and continue in that way to define a sequence $\{z_i\}_{i=0}^{n-r}$ where for each $m \in \{0, ..., n-r\}$, $z_m = u_{r+m} = v_{r+m}$. The sequence $\{(z_i, r+i)\}_{i=0}^{n-r}$ is an ω-successful path from (z, r) to $(z_{n-r}, n) = (u_n, n) = (v_n, n)$ and in particular $(x, n) = (y, n)$ which contradicts the fact that x and y are distinct. Therefore $J \cap K$ must be disjoint. $\square$

Remark 3.2.5. *Note that the above proof presupposes that the initial condition is that each site of $B \times \{0\}$ is occupied by a particle. The proof can easily be adapted to account for any deterministic initial conditions by defining the for $D \subseteq B$, $\xi_n^D(\omega; x) = \mathbb{1}[$there exists an ω-successful path from $D \times \{0\}$ to $(x, n)]$.*

Viewing $\xi_t(x)$ as a random variable that takes values in $\{0, 1\}$, with the event $\{\xi_t(x) = 1\}$ as interpreted as before and $\{\xi_t(x) = 0\}$ interpreted as the collection of ω for which there is not an ω-successful path, then we can write $\mathbb{E}[\xi_t(x)] = \mathbb{P}[\xi_t(x) = 1]$ and we have an immediate corollary of Theorem 3.2.3

$$\mathbb{E}[\xi_t(x)\xi_t(y)] = \mathbb{P}[\xi_t(x) = 1, \xi_t(y) = 1] \leq \mathbb{P}[\xi_t(x) = 1]\mathbb{P}[\xi_t(y) = 1] = \mathbb{E}[\xi_t(x)]\mathbb{E}[\xi_t(y)].$$

3.2.3 Site Independent Random Initial Conditions

We have dealt, in the previous, with any deterministic initial distribution of particles along $B \times \{0\}$. We now turn to an examination of which random initial conditions can be implemented that negative correlation can be proved for. We can prove the result for any site independent initial conditions such as Bernoulli initial conditions where each site is occupied with probability p (possibly depending on the site) independent of all other sites in $B \times \{0\}$ and unoccupied with probabliity $1 - p$.

Indeed, let us consider an independent Bernoulli initial condition at each site in $B \times \{0\}$ as an example. Now let $V' = B \cup V$ and define $\Omega = \{0, 1\}^B \times \prod_{(x,i) \in V} S_{(x,i)}$.

This is the collection of all possible initial occupancies and all configurations on arrows, the latter as has been described in the last section. In this way, for $\omega \in \Omega$, $x \in B$ and $(x,i) \in V$, we will write $\omega_x = 1$ if x contains a particle initially and denote by $\omega_{(x,i)}$ the element of $S_{(x,i)}$ present in the configuration ω. This induces a product measure on Ω with marginals given by

$$\mathbb{P}[\omega_x = 1] = p_x$$

for all $x \in B$ and

$$\mathbb{P}[\omega_{(x,i)} = \alpha] = \begin{cases} p_{(x,i)} & \text{if } \alpha = \nwarrow_{(x,i)} \\ q_{(x,i)} & \text{if } \alpha = \nearrow_{(x,i)} \\ r_{(x,i)} & \text{if } \alpha = \uparrow_{(x,i)} . \end{cases}$$

Now, for a fixed $\omega \in \Omega$ and remembering definition 3.2.1 we give a slightly different definition of an ω-successful path.

Definition 3.2.6. *A path from $(x,0)$ to (y,n) is ω-successful if $\omega_x = 1$ and for each $i \in \{0,...,n-1\}$, the direction of the vector $x_{i+1} - x_i$ corresponds to the arrow given by $\omega_{(x_i,i)}$.*

For distinct, x and y in B, define the events $\mathcal{A}$ and $\mathcal{B}$ as before. We can prove $\mathcal{A}$ and $\mathcal{B}$ are negatively correlated in much the same as before.

Theorem 3.2.7. *With independent Bernoulli initial conditions, the events $\mathcal{A}$ and $\mathcal{B}$ are negatively correlated.*

Proof. As before, we endeavour to show that the inclusion $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \Box \mathcal{B}$. Let $\omega \in \mathcal{A} \cap \mathcal{B}$. Then there exists $u, v \in V$ such that there is an ω-successful path from $(u,0)$ to (x,n) and an ω-successful path from $(v,0)$ to (y,n). Call these paths $\{(u_i,i)\}_{i=0}^n$ and $\{(v_i,i)\}_{i=0}^n$ respectively.

Let $J = \{u_0\} \cup \{(u_0,0),...,(u_n,n))\}$ and $K = \{v_0\} \cup \{(v_0,0),...,(v_n,n)\}$ be our subsets of V' on which we will build our cylinders. The rest follows in a similar way as for the proof of Theorem 3.2.3

□

3.3 Coloured Paths and Various Notions of Negative Dependence

Here, we introduce a different description of paths, that of coloured paths, that allows for particles to share a site but not necessarily coalesce. In this way we will be able to define the paths that make up a discrete time system of coalescing walks that can have multiple particles at a site. Due to the constuction, the particles will not coalesce instantly unless they have the same colour and occupy the same site. The rules that determine the paths will ensure that in a continuous time and large volume limit we recover the dynamics of the slowly coalescing random walks that solve equation (4) in Chapter 1.

We follow the same structure as in the previous. Build a probability space and describe the marginals of a suitable product measure that will be used in an application of the BKR inequality. The first application we will consider is a natural generalisation of Theorem 3.2.3, while the second application is of a very different flavour. Usually, our statements regarding negative correlation will be statements that refer to behaviours at distinct sites. However, the second application of coloured paths and the BKR inequality refers to occupation of a single site by multiple particles. This then leads to a bound on the factorial moments of the random variable that counts the particles at a given site.

3.3.1 Description of Coloured Paths

Begin as before with the grid of spacetime vertices $B \times T$. Introduce m colours, call the collection of colours C and list the colours in some order $c_1, \ldots, c_m$. Let $V' = B \times T \times C$. We now build our probability space Ω. Initally, each site in $B \times \{0\}$ should be painted a colour uniformly at random from C independent of all other sites and at each spacetime point $(x, i) \in B \times T$ there should be an instruction for a particle of any colour. Accordingly, for each $x \in B$, $i \in T$ and $c \in C$ let

$$S_{(x,i,c)} =$$

$$\left\{ \begin{array}{cccccccc} {}^{-e_1,c_1}\nwarrow_c, & {}^{-e_2,c_1}\nwarrow_c, \dots, & {}^{-e_d,c_1}\nwarrow_c, & \uparrow_c^{c_1}, & {}_c\nearrow^{e_1,c_1}, \dots, & {}_c\nearrow^{e_d,c_1}, \\ {}^{-e_1,c_2}\nwarrow_c, & {}^{-e_2,c_2}\nwarrow_c, \dots, & {}^{-e_d,c_2}\nwarrow_c, & \uparrow_c^{c_2}, & {}_c\nearrow^{e_1,c_2}, \dots, & {}_c\nearrow^{e_d,c_2}, \\ & & & \vdots & & \\ {}^{-e_1,c_m}\nwarrow_c, & {}^{-e_2,c_m}\nwarrow_c, \dots, & {}^{-e_d,c_m}\nwarrow_c, & \uparrow_c^{c_m}, & {}_c\nearrow^{e_1,c_m}, \dots, & {}_c\nearrow^{e_d,c_m} \end{array} \right\}$$

This is the collection of all possible instructions that a particle at (x,i) of colour c can be given. For example, a particle of colour c at (x,i) given the instruction ${}^{e_j,c_k}\nearrow_c$, at time $i+1$ it will arrive at $x+e_j$ and take on the colour c_k. Now let $\Omega = C^B \times \prod_{(x,i,c)\in V'} S_{(x,i,c)}$. This is the collection of all initial colours and all configurations of instructions for coloured particles. That is, take $\omega \in \Omega$ and $x \in B$, then ω_x is the colour of the particle that begins in $(x,0)$. Correspondingly, for $x \in B$, $i \in T$ and $c \in C$, $\omega_{(x,i,c)} = \alpha \in S_{(x,i,c)}$ is the instuction given to a particle of colour c at (x,i). This induces our product measure on Ω once we specify probabilities for our coloured particle instructions. The marginals are given as follows.

$$\mathbb{P}[\omega_x = c] = \frac{1}{m}$$

and

$$\mathbb{P}[\omega_{(x,i,c)} = \alpha] = \begin{cases} p_{x,i,c_j,k} & \text{if } \alpha = {}^{-e_k,c_j}\nwarrow_c \\ q_{x,i,c_j,k} & \text{if } \alpha = {}_c\nearrow^{e_k,c_j} \\ r_{x,i,c_j} & \text{if } \alpha = \uparrow_c^{c_j} \end{cases}$$

where the probabilities satisfy

$$\sum_{j=1}^{m} \left(r_{x,i,c_j} + \sum_{k=1}^{d} (p_{x,i,c_j,k} + q_{x,i,c_j,k}) \right) = 1. \tag{3.4}$$

In this way, a particle at (x,i) of any colour is guaranteed an instruction. And similarly as before, if $x \in B$ has L in the k^{th} then $q_{x,i,c_j,k} = 0$ for any $i \in T, j \in$

$\{1, \ldots, m\}$ and if x has $-L$ in the k^{th} coordinate $p_{x,i,c_j,k} = 0$ for any $i \in T, j \in \{1, \ldots, m\}$ so that no particle that reaches the boundary of B can leave B.

For a coloured arrow α, write $h(\alpha)$ for the head colour of the arrow. Now we can define successful paths in the context of coloured paths.

Definition 3.3.1. *An ω-coloured path of terminal colour c from $(x, 0)$ to (y, n) is a path $\{(x_i, i)\}_{i=0}^n$ from $(x, 0)$ to (y, n) and a sequence of colours $\{c_i\}_{i=0}^n$ such that $\omega_{x_0=x} = c_0$, $c_n = c$ and for all $i \in \{0, \ldots, n-1\}$, $\omega_{(x_i,i,c_i)} = \alpha_i \in S_{(x_i,i,c_i)}$ such that $h(\alpha_i) = c_{i+1}$ and the direction of the vector $x_{i+1} - x_i$ corresponds to the direction of α_i, where again we identify $x_{i+1} - x_i = 0$ with the direction of the "up" arrow.*

Under this definition, if distinct particles have instructions to arrive at a coincident site (x, i) and to change to the same colour c', then their paths will remain together thereafter. This is how coalescence of particles occurs in this system. With the definition of ω-coloured paths of a specified terminal colour, we can define random variables of interest. For each $j \in \{1, \ldots, m\}$, let

$$\xi_n^j(\omega; x)$$
$$= \mathbb{1}[\text{there exists an } \omega\text{-coloured path with terminal colour } c_j \text{ from } B \times \{0\} \text{ to } (x, n)].$$

and
$$\xi_n(\omega; x) = \sum_{j=1}^m \xi_n^j(\omega; x).$$

Now we can make sense of events such as $\{\xi_n(x) \geq k\}$ for $k \in \{1, \ldots, m\}$.

3.3.2 Pairwise Negative Correlation for a System of Slowly Coalescing Particles

Theorem 3.3.2. *Let $x \neq y \in B$. Then*

$$\mathbb{P}[\xi_t(x) \geq 1, \xi_t(y) \geq 1] \leq \mathbb{P}[\xi_t(x) \geq 1]\mathbb{P}[\xi_t(y) \geq 1]$$

Proof. Let $\mathcal{A} = \{\xi_n(x) \geq 1\}$ and $\mathcal{B} = \{\xi_n(y) \geq 1\}$. As before, it is enough to show the inclusion $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \square \mathcal{B}$ then by virtue of our finite construction and the BKR inequality we are done.

Let $\omega \in \mathcal{A} \cap \mathcal{B}$ so that $\xi_n(x) \geq 1$ and $\xi_n(y) \geq 1$. There must exist $k_1, k_2 \in C$, not necessarily distinct, such that $\xi_n^{k_1}(x) = 1$ and $\xi_n^{k_2}(x) = 1$. That is, there are $u, v \in B$ and ω-coloured paths of terminal colours k_1 respectively k_2 from $(u, 0)$ to (x, n) resp. $(v, 0)$ to (y, n). Name these paths $\{(u_i, i) \cup \{c_i\}\}_{i=0}^n$ and $\{(v_i, i) \cup \{c_i'\}\}_{i=0}^n$ respectively.

Now let

$$J = \{u_0\} \cup \{(u_0, 0, c_0), ..., (u_n, n, c_n)\}$$

and

$$K = \{v_0\} \cup \{(v_0, 0, c_0'), ..., (v_n, n, c_n')\}$$

be our subsets of V on which we will build our cylinders. Then, showing the inclusion of $[\omega]_J \subseteq \mathcal{A}$ and $[\omega]_K \subseteq \mathcal{B}$ is the same as in Theorem 3.2.3 and showing that J and K are disjoint is similar. The only comment needed is that if $u_0 = v_0$ then the initial colour of the paths would also be equal, hence the paths would be forced to move together thereafter and contradict the paths terminating at distinct x and y. This follows again from a similar reasoning: suppose $u_0 = v_0$, then $c_0 = \omega_{u_0} = \omega_{v_0} = c_0'$. By the definition of ω-coloured paths, the vector $u_1 - u_0$ has the direction of the arrow $\omega_{(u_0,0,c_0)} = \omega_{(v_0,0,c_0')}$ and so does the vector $v_1 - v_0$. Additionally $c_1 = h(\omega_{(u_0,0,c_0)}) = h(\omega_{(v_0,0,c_0')}) = c_1'$, so that particles that began at that same site move to the same site and take on the same colour. Clearly this is true for each of the finitely many steps, n, and therefore the particles cannot reach distinct sites. Therefore $\omega \in \mathcal{A} \square \mathcal{B}$ as required. □

More generally we can prove the following:

Theorem 3.3.3. *Let $x \neq y \in B$ and $, \sigma, \tau \in \mathbb{N}$. Then*

$$\mathbb{P}[\xi_n(x) \geq \sigma, \xi_n(y) \geq \tau] \leq \mathbb{P}[\xi_n(x) \geq \sigma]\mathbb{P}[\xi_n(y) \geq \tau].$$

Proof. Let $\mathcal{A}_\sigma = \{\xi_n(x) \geq \sigma\}$ and $\mathcal{B}_\sigma$ be the corresponding event for the site point

y. We must show $\mathcal{A}_\sigma \cap \mathcal{B}_\tau \subseteq \mathcal{A}_\sigma \Box \mathcal{B}_\tau$ and appeal to the BKR inequality. Take $\omega \in \mathcal{A}_\sigma \cap \mathcal{B}_\tau$. Firstly, $\omega \in \mathcal{A}_\sigma$ means $\xi_n(x) \geq \sigma$ and so there exist a collection of distinct colours $k_1, k_2, \ldots, k_\sigma \in C$ such that $\xi_n^{k_i}(x) = 1$ for $i \in \{1 \ldots, \sigma\}$. For each $i \in \{1, \ldots \sigma\}$ there is a $u^i \in B$ and an ω-coloured path from $(u^i, 0)$ to (x, n) of terminal colour k_i. The collection $\{u^i\}_{i=1}^\sigma$ must also be pairwise distinct, since the definition of an ω-coloured path of specified terminal colour will not permit two paths beginning in the same initial site to reach the terminus with differing terminal colours. This claim follows from the same proof as for the claim in the previous theorem. Call the path corresponding to terminal colour k_i, $\{(u_j^i, j) \cup \{c_j^i\}\}_{j=0}^n$. Note, $c_n^i = k_i$. Similarly, $\omega \in \mathcal{B}_\tau$ gives rise to the existence of distinct colours $l_1, \ldots, l_\tau$ with corresponding $\{v^i\}_{i=1}^\tau \subset B$ and ω-coloured paths from $(v^i, 0)$ to (y, n) of terminal colour l_i, for each $i \in \{1, \ldots, \tau\}$. Again, the $\{v^i\}_{i=1}^\tau$ are pairwise distinct. Call the path corresponding to terminal colour l_i, $\{(v_j^i, j) \cup \{\gamma_j^i\}\}_{j=0}^n$, so that $\gamma_n^i = l_i$.

Now let

$$J = \bigcup_{i=1}^{\sigma} \{u_0^i\} \cup \{(u_0^i, 0, c_0^i), ..., (u_n^i, n, c_n^i)\}$$

and

$$K = \bigcup_{i=1}^{\tau} \{v_0^i\} \cup \{(v_0^i, 0, \gamma_0^i), ..., (v_n^i, n, \gamma_n^i)\}.$$

The same ideas as before make it clear that $[\omega]_J \subset \mathcal{A}$ and $[\omega]_K \subset \mathcal{B}$ and $J \cap K = \emptyset$. $\Box$

This leads to a simple corollary which captures the idea that the occupation of x by particles reduces the chance of particles also occupying a distinct site y.

Corollary 3.3.4. *For distinct $x, y \in B$*

$$\mathbb{E}[\xi_n(x)\xi_n(y)] \leq \mathbb{E}[\xi_n(x)]\mathbb{E}[\xi_n(y)].$$

Proof.

$$\begin{aligned}\mathbb{E}[\xi_n(x)\xi_n(y)] &= \sum_{i,j\in\mathbb{N}} ij\mathbb{P}[\xi_n(x)=i,\xi_n(y)=j] \\ &= \sum_{i,j}\sum_{n=1}^{i}\sum_{m=1}^{j}\mathbb{P}[\xi_n(x)=i,\xi_n(y)=j] \\ &= \sum_{n,m}\sum_{i=n}^{\infty}\sum_{j=m}^{\infty}\mathbb{P}[\xi_n(x)=i,\xi_n(y)=j] \\ &= \sum_{n,m}\mathbb{P}[\xi_n(x)\geq n,\xi_n(y)\geq m] \\ &\leq \sum_{n}\sum_{m}\mathbb{P}[\xi_n(x)\geq n]\mathbb{P}[\xi_n(y)\geq m] \\ &= \mathbb{E}[\xi_n(x)]\mathbb{E}[\xi_n(y)].\end{aligned}$$

□

In fact, we will need to know about the behaviour of ξ_n at more that just two distinct sites. We can do so in the following, the only difficulty being notation.

Theorem 3.3.5. *Let $\{x_j\}_{j=1}^{\sigma} \subset B$ be a collection of distinct sites in B and $\{i_j\}_{j=1}^{\sigma}$ be natural numbers then*

$$\mathbb{P}[\xi_n(x_1)\geq i_1,\ldots,\xi_n(x_\sigma)\geq i_\sigma] \leq \prod_{j=1}^{\sigma}\mathbb{P}[\xi_n(x_j)\geq i_j]$$

Proof. Let $\mathcal{A}_j = \{\xi_n(x_j)\geq i_j\}$ for $j\in\{1,\ldots,\sigma\}$ and $\mathcal{A}_{1,\ldots,\sigma} = \{\xi_n(x_1)\geq i_1,\ldots\xi_n(x_\sigma)\geq i_\sigma\} = \cap_{j=1}^{\sigma}\mathcal{A}_j$. If we can show $\mathcal{A}_{1,\ldots,\sigma} = \mathcal{A}_{1,\ldots,\sigma-1}\cap\mathcal{A}_\sigma \subset \mathcal{A}_{\sigma-1}\Box\mathcal{A}_\sigma$ we can apply BKR to deduce $\mathbb{P}[\mathcal{A}_\sigma] \leq \mathbb{P}[\mathcal{A}_{\sigma-1}]\mathbb{P}[\mathcal{A}_\sigma]$. The same argument can be repeated by peeling off a factor at a time from each of the remaining $\sigma-1$ the intersections $A_{1,\ldots,j}$, $j\in\{1,\ldots,\sigma-1\}$.

Things have become very notationally heavy so we describe the construction of the cylinder bases J and K necessary to prove the inclusion for the application of BKR without explicitly writing them. Take $\omega\in\mathcal{A}_{1,\ldots,\sigma}$. Then for each $j\in\{1,\ldots,\sigma\}$ there are i_j distinct colours and there are ω-coloured paths with all of the i_j colours represented as terminal colours up to (x_j, n). Label each of these paths, for each j, in the same fashion as we have multiple times before now. From these paths, we can build elements of V as we also have done before now.

First, for each $j \in \{1, \dots, \sigma\}$, take a union over these elements in V. Each of these unions is now a base for a cylinder on which $\mathcal{A}_j$ is bound to occur. Secondly, take the union of the first $\sigma - 1$ of these bases and call this new union J. Now this is a base for a cylinder that guarantees $A_{1,\dots,\sigma-1}$ occurs, i.e $[\omega]_J \subset A_{1,\dots,\sigma-1}$. Now let K be the remaining element of V that guarantees the occurance $\mathcal{A}_\sigma$. The fact that J and K are disjoint can be seen by no more reasoning than in the previous, but writing the argument is convoluted. □

Corollary 3.3.6. *Let $\{x_j\}_{j=1}^{\sigma} \subset B$ be a collection of distinct sites in B then*

$$\mathbb{E}[\xi_n(x_1) \cdot \dots \cdot \xi_n(x_\sigma)] \leq \prod_{i=1}^{\sigma} \mathbb{E}[\xi_n(x_i)]$$

Proof. Same calculation as for Corollary 3.3.4, repeated use of Fubini and of the fact that $\xi_n(x)$ is non-negative integer valued to collect the events. □

3.3.3 A Negative Dependence Result for the Factorial Moments of the Occupancy of a Site in a Slowly Coalescing System

In this section, we take on a negative dependence result of a different flavour than we have studied so far. It states, rather intuitively, that in a system where particles are allowed to coincide at a site but have the possibility to take on the same colour, that stacks of particles are somehow unlikely. We first prove a special case of the main result in this section since the proof of the general case is no more difficult but more notationally heavy.

Theorem 3.3.7. *Our coalescing system ξ_n is reluctant to stack in the sense that for any $x \in B$*

$$\mathbb{P}[\xi_n(x) \geq 2] \leq \mathbb{P}[\xi_n(x) \geq 1]^2.$$

Proof. The idea is let $\mathcal{A} = \{\xi_n(x) \geq 1\}$ and show the inclusion $\{\xi_n(x) \geq 2\} \subseteq$

$\mathcal{A}\Box\mathcal{A}$. Let $\omega \in \{\xi_n(x) \geq 2\}$, then there exist distinct colours $k_1, k_2 \in C$ such that there are sites $u, v \in B$ and ω-coloured paths of terminal colours k_1 respectively k_2 from $(u, 0)$ respectively $(v, 0)$ to (x, n). Call these paths $\{(u_i, i) \cup \{c_i\}\}_{i=0}^{n}$ and $\{(v_i, i) \cup \{c'_i\}\}_{i=0}^{n}$. Necessarily $u_0 \neq v_0$, and this claim follows as it did in Theorem 3.3.3 by the argument given for the claim in Theorem 3.3.2.

Let

$$J = \{u_0\} \cup \{(u_0, 0, c_0), ..., (u_n, n, c_n)\}$$

and

$$K = \{v_0\} \cup \{(v_0, 0, c'_0), ..., (v_n, n, c'_n)\}.$$

Then, for $\omega' \in [\omega]_J$, $\{(u_i, i) \cup \{c_i\}\}_{i=0}^{n}$ is an ω'-coloured path from $(u, 0)$ to (x, n) of terminal colour k_1, hence $\omega' \in \{\xi_n^{k_1}(x) = 1\} \subset \mathcal{A}$, and we have the inclusion $[\omega]_J \subset \mathcal{A}$. We similarly have that for $\omega' \in [\omega]_K$, $\{(v_i, i) \cup \{c'_i\}\}_{i=0}^{n}$ is an ω'-coloured path from $(v, 0)$ to (x, n) with terminal colour k_2 and the inclusion $[\omega]_K \subset \mathcal{A}$ also follows.

Suppose there are indices i, j such that the triples (u_i, i, c_i) and (v_j, j, c'_j) are equal, then it is immediately true that $i = j$. By cutting off the first i steps of the paths $\{(u_i, i) \cup \{c_i\}\}_{i=0}^{t}$ and $\{(v_i, i) \cup \{c'_i\}\}_{i=0}^{n}$, we define paths $\{(u_{i+j}, i + j) \cup \{c_{i+j}\}\}_{j=0}^{n-i}$ and $\{(v_{i+j}, i + j) \cup \{c'_{i+j}\}\}_{j=0}^{n-i}$. These are ω-coloured paths from (u_i, i) to (x, n) of terminal colour k_1 respectively from $(v_i, i) = (u_i, i)$ to (x, n) of terminal colour k_2 beginning with the same initial colour $c_i = c'_i$. But then u_i cannot equal v_i by the same reasoning as for the claim that $u_0 \neq v_0$. Therefore, $J \cap K = \emptyset$ and the proof is complete. $\Box$

Theorem 3.3.8. *Our coalescing system ξ_t is reluctant to stack in the sense that for any $x \in \mathbb{Z}^d$*

$$\mathbb{P}[\xi_t(x) \geq i + j] \leq \mathbb{P}[\xi_t(x) \geq i]\mathbb{P}[\xi_t(x) \geq j].$$

Proof. We must show $\mathcal{A}_{i+j} \subseteq \mathcal{A}_i \Box \mathcal{A}_j$. For $\omega \in \mathcal{A}_{i+j}$, $\xi_n(x) \geq i + j$, so there must exist $i + j$ distinct colours $k_1, \ldots, k_i, l_1, \ldots, l_j \in C$ such that for each $i^* \in \{1, \ldots, i\}$, $\xi_n^{k_{i^*}}(x) = 1$ and each $j^* \in \{1, \ldots, j\}$, $\xi_n^{l_{j^*}}(x) = 1$. That is, there exist

$u^1, \ldots, u^i, v^1, \ldots, v^j \in \Lambda$ that are pairwise distinct (by the same argument as in the previous theorem) and for each $i^* \in \{1, \ldots, i\}$ and $j^* \in \{1, \ldots, j\}$ there is an ω-coloured path from $(u^{i^*}, 0)$ respectively $(v^{j^*}, 0)$ to (x, t) with terminal colours k_{i^*} respectively l_{j^*}. Call these paths $\{(u_s^{i^*}, s) \cup \{c_s^{i^*}\}\}_{s=0}^n$ and $\{(v_s^{j^*}, s) \cup \{\gamma_s^{j^*}\}\}_{s=0}^n$ for $i^* \in \{1, \ldots, i\}$ and $j^* \in \{1, \ldots, j\}$.

Now let

$$J = \bigcup_{i^* \in \{1,\ldots,i\}} \{u_0^{i^*}\} \cup \{(u_0^{i^*}, 0, c_0^{i^*}), \ldots, (u_n^{i^*}, n, c_n^{i^*})\}$$

and

$$L = \bigcup_{j^* \in \{1,\ldots,j\}} \{v_0^{j^*}\} \cup \{(v_0^{j^*}, 0, \gamma_0^{j^*}), \ldots, (v_n^{j^*}, n, \gamma_n^{j^*})\}.$$

It is clear that the same ideas as in the previous theorem, except far more notationally heavy, prove the inclusion and rest follows by BKR. $\square$

The following corollaries are immediate.

Corollary 3.3.9. *For any $x \in B$*

$$\mathbb{P}[\xi_n(x) \geq \sigma] \leq \mathbb{P}[\xi_n(x) \geq 1]^\sigma$$

Proof. For a positive integer σ, write $\{\xi_n(x) \geq \sigma\} = \{\xi_n(x) \geq (\sigma - 1) + 1\}$ and apply Theorem 3.3.8 with $i = \sigma - 1$ and $j = 1$ repeatedly. $\square$

Corollary 3.3.10.

$$\mathbb{P}[\xi_n(x_1) \geq i_1, \ldots \xi_n(x_\sigma) \geq i_\sigma] \leq \prod_{j=1}^{\sigma} \mathbb{P}[\xi_n(x_j) \geq 1]^{i_j}$$

Proof. By Theorem 3.3.5

$$\begin{aligned}\mathbb{P}[\xi_t(x_1) \geq i_1, \ldots \xi_t(x_n) \geq i_n] &\leq \prod_{j=1}^{n} \mathbb{P}[\xi_t(x_j) \geq i_j] \\ &\leq \prod_{j=1}^{n} \mathbb{P}[\xi_t(x_j) \geq 1]^{i_j}\end{aligned}$$

by the previous corollary. □

Remark 3.3.11. *This last corollary might seem superfluous and trivial, but it will be more useful later when we reach our general continuous time coalescing with translation invariant IC which will also be translation invariant as a consequence and therefore by an extra line we'll have an upper bound of* $\mathbb{E}[\xi_t(0)]^{i_1+\cdots+i_n}$.

Remark 3.3.12. *Again, the results of this section in its current form only apply to the deterministic initial condition* $\xi_0 \equiv 1$. *But again, it is easy to change for deterministic IC with at most 1 per site but possible to extend to any deterministic IC and Bernoulli. It is an open question as to how general the initial conditions can be. Unlike positive correlations, negative correlation properties can be destroyed by random initial conditions. Examples of this are covered in [15]. Perhaps a good conjecture is that these properties can be proved for initial conditions with strong decay of correlations.*

Deduction of 'Negative Dependence' for the Falling Factorials

We begin this section with some general results that are useful for random variables that satisfy the type of negative dependence that we have seen in this section.

Lemma 3.3.13. *Let* X *be a non-negative integer random variable. Then we have the following expression for the factorial moments of* X

$$\mathbb{E}[X(X-1)\dots(X-m+1)] = m! \sum_{i_1,\dots,i_m=1}^{\infty} \mathbb{P}\left[X \geq \sum_{j=1}^{m} i_j\right]$$

Proof. Starting with the right hand side without the factor $m!$, we have

$$\sum_{i_1,\ldots,i_m=1}^{\infty} \mathbb{P}\left[X \geq \sum_{j=1}^{m} i_j\right] = \sum_{i_1,\ldots,i_m=1}^{\infty} \sum_{n=\sum_{j=1}^{m} i_j}^{\infty} \mathbb{P}[X=n]$$

$$= \sum_{i_1,\ldots,i_m=1}^{\infty} \sum_{n=i_m+\sum_{j=1}^{m-1} i_j}^{\infty} \mathbb{P}[X=n]$$

$$= \sum_{i_1\ldots,i_{m-1}=1}^{\infty} \sum_{n=1+\sum_{j=1}^{m-1} i_j}^{\infty} \sum_{i_m=1}^{n-\sum_{j=1}^{m-1} i_j} \mathbb{P}[X=n].$$

In the last equality, an application of Fubini's Theorem was used to exchange the order of the two inner most sums. Notice, the summand does not depend on the index i_m so the innermost sum can be evaluated to arrive at

$$\sum_{i_1\ldots,i_{m-1}=1}^{\infty} \sum_{n=1+\sum_{j=1}^{m-1} i_j}^{\infty} \left(n - \sum_{j=1}^{m-1} i_j\right) \mathbb{P}[X=n]. \tag{3.5}$$

We pause briefly to give some notation. Let $(n)_m$ denote the falling factorial of n of length m, i.e. $(n)_m = n(n-1)\ldots(n-m+1)$. Then in particular, $(n)_1 = n$. Let D^+ denote the discrete derivative, for a function f that takes integer values $D^+f(n) = f(n+1) - f(n)$. Then $D^+(n)_m = m(n)_{m-1}$. Indeed,

$$\begin{aligned}
D^+(n)_m &= (n+1)_m - (n)_m \\
&= (n+1)n(n-1)(n-2)\ldots(n-m+2) - n(n-1)(n-2)\ldots(n-m+1) \\
&= \Big((n+1) - (n-m+1)\Big)n(n-1)(n-2)\ldots(n-m+2) \\
&= mn(n-1)\ldots(n-(m-1)+1) \\
&= m(n)_{m-1}.
\end{aligned}$$

So we can write

$$\left(n - \sum_{j=1}^{m-1} i_j\right) = \left(n - \sum_{j=1}^{m-1} i_j\right)_1 = \frac{1}{2}D^+\left(n - \sum_{j=1}^{m-1} i_j\right)_2.$$

Making that substitution and applying Fubini again to swap the two innermost sums in (3.5)

$$\frac{1}{2}\sum_{i_1\ldots,i_{m-2}=1}^{\infty}\ \sum_{n=2+\sum_{j=1}^{m-2}i_j}^{\infty}\ \sum_{i_{m-1}=1}^{n-\sum_{j=1}^{m-2}i_j} D^+\left(n-\sum_{j=1}^{m-2}i_j-i_{m-1}\right)_2 \mathbb{P}\left[X=n\right]. \tag{3.6}$$

Using the trivial discrete version of the fundamental theorem of calculus

$$\sum_{n=a}^{b} D^+ f(n) = f(b+1) - f(a)$$

we can evaluate the innermost sum in (3.6)

$$=\frac{1}{2}\sum_{i_1\ldots,i_{m-2}=1}^{\infty}\ \sum_{n=2+\sum_{j=1}^{m-2}i_j}^{\infty}\left(n-\sum_{j=1}^{m-2}i_j\right)_2 \mathbb{P}\left[X=n\right].$$

Now, the falling factorial of length 2 in the summand can be written as the discrete derivative of the falling factorial of length 3 together with a factor of 1/3. Using Fubini and the discrete fundamental theorem of calculus allows us to evaluate the sum over the index i_{m-2}. Continuing in the same way for all but the last exchange of sums we arrive at

$$\begin{aligned}
&\frac{1}{(m-1)!}\sum_{i_1=1}^{\infty}\ \sum_{n=m+i_1-1}^{\infty}(n-i_1)_{m-1}\,\mathbb{P}\left[X=n\right]\\
&=\frac{1}{m!}\sum_{i_1=1}^{\infty}\ \sum_{n=m+i_1-1}^{\infty} D^+\left(n-i_1\right)_m \mathbb{P}\left[X=n\right]\\
&\underbrace{=}_{Fubini}\frac{1}{m!}\sum_{n=m}^{\infty}\ \sum_{i_1=1}^{n-m+1} D^+\left(n-i_1\right)_m \mathbb{P}\left[X=n\right]\\
&=\frac{1}{m!}\sum_{n=m}^{\infty}(n)_m\,\mathbb{P}\left[X=n\right]\\
&=\frac{1}{m!}\sum_{n=m}^{\infty} n(n-1)\ldots(n-m+1)\mathbb{P}\left[X=n\right].
\end{aligned}$$

□

Corollary 3.3.14. *Let X be a non-negative integer random variable with finite expected value that satisfies*

$$\mathbb{P}[X \geq i + j] \leq \mathbb{P}[X \geq i]\mathbb{P}[X \geq j].$$

Then

$$\mathbb{E}[X(X-1)(X-2)\dots(X-m+1)] \leq m!\mathbb{E}[X]^m$$

Proof. Since X is a non-negative integer random variable, we can apply lemma 3.3.13

$$\begin{aligned}
\mathbb{E}[X(X-1)(X-2)\dots(X-m+1)] \underbrace{=}_{3.3.13} m! \sum_{i_1,\dots,i_m=1}^{\infty} \mathbb{P}\left[X \geq \sum_{j=1}^{m} i_j\right] \\
\underbrace{\leq}_{assumption} m! \sum_{i_1,\dots,i_m=1}^{\infty} \prod_{j=1}^{m} \mathbb{P}[X \geq i_j] \\
= m! \prod_{j=1}^{m} \sum_{i_j=1}^{\infty} \mathbb{P}[X \geq i_j] \\
= m! \prod_{j=1}^{m} \mathbb{E}[X] = m!\mathbb{E}[X]^m
\end{aligned}$$

.

In the inequality, the assumption on the probabilities is used repeatedly by peeling one of the summands off at a time. □

Corollary 3.3.15. *For each $x \in B$*

$$\mathbb{E}[(\xi_n(x)-1)\dots(\xi_n(x)-m+1)] \leq m!\mathbb{E}[\xi_n(x)]^m.$$

Proof. By Theorem 3.3.8, $\xi_n(x)$ is a non-negative integer random variable that satisfies the assumptions of Corollary 3.3.14. □

3.4 Convergence in Distibution of the Approximating Discrete Time Models

In this section, we will prove that our general continuous time coalescing random walks $(\xi_t^{(\lambda)}(x) : x \in \mathbb{Z}^d)$ can be obtained by a sequence of suitable limits that will allow us to carry over the results that have been determined for our discrete time, finite-space model in the previous section. We are required to make an appropriate choice for the probabilities defined in our product measures to lead to the correct rate λ for coalescence. We will then, first keeping the box size $|B_L|$ (equivalently L) and number of colours m fixed, build a sequence of discrete time processes that will have progressively shorter times between jumps and appropriately scaled probabilities to obtain a continuous time process. Then, let $m, L \to \infty$.

3.4.1 Convergence of the Discrete Time Markov Chain to the Continuous Time Markov Process

In discrete time, our state space is $\{0,1\}^{B\times\{1,\ldots,m\}}$. That is, the state at any time is a collection of 1's and 0's at each site of B indicating whether or not there is a particle of each colour of $\{1,\ldots,m\}$ present. Then $(\xi_n^m(i,x)\colon i \in \{1,\ldots,m\}, x \in B)_{n\in\mathbb{N}}$ is a discrete time Markov chain. Suppose initially each site has one particle and that particle is painted with one of the colours taken uniformly from $\{1,\ldots,m\}$. Suppose now we are at time n and we know the state of the configuration. This is all the information we need to predict the state at time $n+1$.

There are a large number of transition probabilities to specify but only very few of them will be needed in what follows. Since we ultimately want a simple symmetric walk we will have to choose $p_{x,i,c_j,k} = q_{x',i',c_{j'},k'} = p/(2dm)$ for all $i, i' \in \{0,1,\ldots,n\}$, $c_j, c_{j'} \in C$, $k, k' \in \{1,\ldots,d\}$ and all x, x' that do not lie on the boundary of B_L (that is, does not contain an L or $-L$ in any of its coordinates. We similarly do not want to favour any colour over another. In the limit, coalescence at a site will happen according to independent Poisson processes between pairs, each pair coalescing at rate 2λ. In this discrete framework, two particles coalesce at a site if, after a jump, they have taken on the same colour. We want the most likely transition from state to state to be one in which most particles are instructed to remain at the same site and with the same colour so that the first change in state has an almost geometric distribution. In the limit from discrete time to continuous time, these approximate geometric distributions will manifest as independent exponential random variables that build up the Poisson processes. As such, let $r_{x,c,c_j} = r_c$ for all $c_j \neq c$, and let $r_{x,c,c} = r$ for all $c \in C$ again for x not on the boundary. For x in the boundary, we will reweight the probabilities so that they still sum to 1. We can do this easily. Suppose x is in the boundary of B_L, then wherever there appears an L of $-L$ in its coordinate representaion, the probabilities associated to the arrows that point in the direction of leaving the box are 0. Sum all the remaining probabilities in the sense of the left hand side of (3.4) and divide each of the probabilities by this weight. Summing these weighted probabilities in the sense of the left hand side of (3.4) will result in a sum of 1 and in such a way as to preserve their pairwise ratios.

Provided the probability r is chosen close to 1, the most likely transistion will be $\xi^m_{n+1} = \xi^m_n$ where all the particles were given the instruction to stay at the same site with the same colour. However, the probability $\mathbb{P}[\xi^m_{n+1} = \xi^m_n | \xi^m_n]$ is not simply $\prod_{x \in B_L} \prod_{i=1}^m r^{\xi^m_n(i,x)}$ since if at any site there is more than one particle, the same configuration can be achieved if any two of these particles swap colours. Also, if neighbouring sites are both occupied and a particle at each of the neighbours swap sites and the arriving particle takes on the colour of the departing particle then the same state is reached once more. Any combination of these swaps will arrive at the same state as at the previous time, so the probability $\mathbb{P}[\xi^m_{n+1} = \xi^m_n | \xi^m_n]$ has a large number of terms. However, choosing r close to 1 (and then correspondingly all other probabilities small due to the restriction (3.4)) ensures that the leading term is $\prod_{x \in B_L} \prod_{i=1}^m r^{\xi^m_n(i,x)} = r^{\sum_{x \in B_L} \xi^m_n(x)}$ and so that the main way in which the configuration remains the same is for there to be no swaps and all the particles obey the instruction to stay at the same site with the same colour.

Similarly, we can discuss transitions where there is only "one" change in the system. For example, a coalescing event at one site but no change in the configuration otherwise, or a departure of a colour at one site and the (possibly) consequential arrival of a colour at a neighbouring site. The meaning of the emphasis on "one" is that, as in the previous discussion, the same state can be achieved by various swaps so the probability of such a transition given the state at the previous time is contributed to by a large number of possibilities. However, the largest contribution will come from a single genuine change to the configuration with most of the particles remaining where they are and with the same colour.

We use some classical results of Ethier and Kurtz [9] regarding the convergence of appropriately timescaled discrete Markov chains to continuous time Markov chains, characterised by the convergence of the transition probabilities to the rate parameters in some appropriate sense.

Proposition 3.4.1. *Let* $(\xi_t^{(m,L)}(i,x)\colon i \in \{1,\dots,m\},\ x \in B_L,\ t \geq 0)$ *be the Markov process of rate* 1 *random walks coalescing at rate* $\lambda > 0$ *with initial condition of one particle at each site whose colour is chosen uniformly at random from* $\{1,\dots,m\}$, *and let* $(\xi_n^{(m,L),k}(i,x))_{k\in\mathbb{N}}$ *be the sequence of discrete time Markov chains each with the same initial condition and whose transition matrices are built from the sequence of triples of probabilities* $(r^{(k)}, r_c^{(k)}, p^{(k)})_{k\in\mathbb{N}}$. *Let* $Y_t^{(k)} = \xi_{\lfloor kt \rfloor}^{(m,L),k}$. *Then the sequence of probabilities can be chosen so that* $Y_t^{(k)}$ *converges in distribution to* $\xi_t^{(m,L)}$ *as* $k \to \infty$.

Proof. Without loss of generality $\lambda \in (0,1)$, since otherwise $\lambda/(1+\lambda)$ is and we can scale time in continuous time.

We must identify the non-zero transition rates for the continuous time process $(\xi_t^m(i,x)\colon i \in \{1,\dots,m\}, x \in B_L, t \geq 0)$ that records the colours of the particles present at a site and construct a sequence of discrete time processes in the style of the by theorems 6.5 of Chapter 1 and 2.6 of Chapter 4 of Ethier and Kurtz [9] whose transition probabilities are chosen so that the off-diagonal terms of the transition matrix converge to corresponding off-diagonal terms of the rate matrix in the appropriate sense.

In continuous time only one Poisson clock can ring at any time, so we can identify the only possible non-zero transitions by considering changes that can occur with a single ring of a Poisson clock. Write ξ and ξ' for two different configurations. We have the following three cases:

- $\xi'(i,x) = 1 - \xi(i,x)$,
  $\xi'(k,z) = \xi(k,z)$ for all $(k,z) \neq (i,x)$ at rate $\xi(i,x)\Big(\lambda\big(\sum_{j=1}^{m}\xi(j,x) - 1\big) + \frac{1}{2dm}\sum_{y\colon y\sim x}\sum_{j=1}^{m}\xi(j,y)\Big)$. This describes the case where the only change in the configuration is the loss of one colour i at one site $x \in B_L$. This can only occur if there was a particle at x that either changed colour to one of the other colours present at x or if that particle walked to any of x's neighbours and took the colour of any of the particles there.

- $\xi'(i,x) = 1 - \xi(i,x)$,
  $\xi'(i',x) = 1 - \xi(i',x)$,
  $\xi'(k,z) = \xi(k,z)$ for all $(k,z) \neq (i,x),(i',x)$ at rate $\lambda\big(\xi(i,x)(1-\xi(i',x)) + \xi(i',x)(1-\xi(i,x))\big)$. This describes the case where there is a loss of a particle of one colour at x as a result of a gain in a different colour at x while there is no other change elsewhere.

- $\xi'(i,x) = 1 - \xi(i,x)$,
  $\xi'(i',y) = 1 - \xi'(i',y)$ for $y \sim x$,
  $\xi'(k,z) = \xi(k,z)$ for all $(k,z) \neq (i,x),(i',y)$ at rate $\frac{1}{2dm}\big(\xi(i,x)(1-\xi(i',y)) + \xi(i',y)(1-\xi(i,x))\big)$. This describes the case where there is a loss of a particle of one colour at site as a result the particle walking to a neighbouring site.

These are the only ways in which the configuration can change as a result of the ring of one Poisson clock.

Now let $(\xi_n^m(i,x)\colon i \in \{1,\dots,M\}, x \in B_L, n \in \mathbb{N})$ be the discrete time Markov chain discussed at the beginning of this chapter with the with the probabilities $r_c, r, p/2dm$ that govern the motion and colours of each particle. Let $(\xi_n^{m,(k)}(i,x)\colon i \in \{1,\dots,m\}, x \in B_L, n \in \mathbb{N})_{k\in\mathbb{N}}$ be a sequence of Markov chains that evolve in the same way but according to the sequence of probabilities given by the triple $(r_c^{(k)}, r^{(k)}, p^{(k)}/2dm)_{k\in\mathbb{N}}$. Now let $Y_t^{m,(k)} = \xi_{\lfloor kt\rfloor}^{m,(k)}$. We want to prove that

$Y_t^{m,(k)}$ converges in distribution to ξ_t^m the continuous time process with m colours as $k \to \infty$.

The idea is that by comparing the probabilities of the transitions that correspond to the non-zero rate transtions for the continuous time model, we can choose the probabilities $(r_c^{(k)}, r^{(k)}, p^{(k)}/2dm)_{k\in\mathbb{N}}$ so that the transtion matrix converges in the right sense and in particular, transitions with non-zero probabilities in discrete time that correspond to zero rate transition in continuous time should vanish. $\xi_n^{m,(k)}$ is the state at time n and assume that it is known.

Let us take a look at the first bullet point, the event that the site x has lost a particular particle of colour i whilst the configuration is otherwise untouched. Firstly, consider the case that all of the particles except for the particle of colour i at x receives the instruction at time n to remain at the same site and remain the same colour. Then the probability that at time $n+1$ we are in the state that we've lost the particle of colour i at x given $\xi_n^{m,(k)}$ is

$$
\begin{aligned}
&(r^{(k)})^{\sum_{z\in B\setminus\{x\}}\sum_{j=1}^m \xi_n^{m,(k)}(j,z)}(r^{(k)})^{\sum_{j=1}^m \xi_n^{m,(k)}(j,x)-1} \\
&\times\left(r_c^{(k)}\xi_n^{m,(k)}(i,x)\left(\sum_{j=1}^m \xi_n^{m,(k)}(j,x)-1\right)+\frac{p^{(k)}}{2dm}\xi_n^{m,(k)}(i,x)\sum_{y\sim x}\sum_{j=1}^m \xi_n^{m,(k)}(j,y)\right) \\
&=(r^{(k)})^{\sum_{z\in B}\xi^{m,(k)}(z)-1}\Bigg((r_c^{(k)}\xi_n^{m,(k)}(i,x)\left(\sum_{j=1}^m \xi_n^{m,(k)}(j,x)-1\right) \\
&+\frac{p^{(k)}}{2dm}\xi_n^{m,(k)}(i,x)\sum_{y\sim x}\sum_{j=1}^m \xi_n^{m,(k)}(j,y)\Bigg).
\end{aligned}
$$

All other contributions would either have to contain at least one swap of colours of two particles at the same site or at least one swap of neighbouring sites of particles and the arriving particle taking on the colour of the departing particle. Otherwise it would not respect the configuration outside of the particle of colour i at x. Supposing there is a swap of colours, then there would have to be at least three copies of $r_c^{(k)}$, the first that changes the colour of the particle of colour i at x and the following two to facilitate the swap. Similarly, if there is a swap of sites, there must be at least one copy of $r_c^{(k)}$ and two copies of of $p^{(k)}/2dm$. Returning to our main term, we want to compare it to the rate $\xi(i,x)\Big(\lambda\big(\sum_{j=1}^m \xi(j,x)-1\big)+\frac{1}{2dm}\sum_{y:\,y\sim x}\sum_{j=1}^m \xi(j,y)\Big)$. A little bit of thought reveals that a sensible choice of probabilities is $r_c^{(k)} = \lambda/k$, $p^{(k)} = 1/k$ and $r^{(k)} = 1-p^{(k)}-(m-1)r_c^{(k)} = 1-(1+(m-1)\lambda)/k$, where the final value is forced upon us by the restriction (3.4). Since then

$$k(r^{(k)})^{\sum_{z\in B_L}\xi^{m,(k)}(z)-1}r_c^{(k)} = k\left(1-\frac{1+(m-1)\lambda}{k}\right)^{\sum_{z\in B_L}\xi^{m,(k)}(z)-1}\frac{\lambda}{k}$$

and for $k>m$, $(1+\lambda(m-1))/k \le 1$ since we have taken $\lambda\in(0,1)$, so

$$\lambda\left(1-\frac{1+(m-1)\lambda}{k}\right)^{m|B_L|} \le \lambda\left(1-\frac{1+(m-1)\lambda}{k}\right)^{\sum_{z\in B_L}\xi^{m,(k)}(z)-1} \le \lambda.$$

Similarly,

$$\frac{1}{2dm}\left(1-\frac{1+(m-1)\lambda}{k}\right)^{|B_L|} \le k(r^{(k)})^{\sum_{z\in B_L}\xi^{m,(k)}(z)-1}\frac{p^{(k)}}{2dm} \le \frac{1}{2dm},$$

and so the probabilities converge in the correct sense to the correponding rates. All other contributions then are of order $O\left((r_c^{(k)})^3+(r_c^{(k)})(p^{(k)}/2dm)^2\right) = O(1/k^3)$ since the number of particles per site is bounded, so that their contribution to the limit is $O(k/k^3) = O(1/k^2)$, that is they vanish in the limit.

The next case to consider is that there is a loss of a particle of colour i as a result of a gain of a colour i' not present at x at time n with no change otherwise. Once again the main contribution will come from all other particles receiving the instruction to remain at the same site with the same colour with probability

$$(r^{(k)})^{\sum_{z\in B_L}\xi_n^{m,(k)}(z)-1}r_c^{(k)}\xi_n^{m,(k)}(i,x)(1-\xi_n^{m,(k)}(i',x))$$

and

$$k(r^{(k)})^{\sum_{z\in B_L}\xi_n^{m,(k)}(z)-1}r_c^{(k)} \to \lambda$$

as $k\to\infty$ as before, with all other contributions vanishing since they of order $1/k^3$.

The final non-zero rate transition is a loss of a particle of colour i at x as a result of the arrival of a colour i' at a neighbour y of x that was not already present at y at time n. The probability associated to the main contribution is

$$(r^{(k)})^{\sum_{z \in B_L} \xi_n^{m,(k)}(z)-1} \frac{p^{(k)}}{2dm} \xi_n^{m,(k)}(i,x)(1-\xi_n^{m,(k)}(i',y))$$

which satisfies

$$k(r^{(k)})^{\sum_{z \in B_L} \xi_n^{m,(k)}(z)-1} \frac{p^{(k)}}{2dm} \to \frac{1}{2dm}$$

as $k \to \infty$ as a result of our choice of probabilities. Again contributions arising from swaps will vanish.

Finally, all that is left to be checked is that the probabilites corresponding to zero rate transitions vanish in the limit in the sense of Ethier and Kurtz. This is easy since any such transition in discrete time must contain at least two copies of $r_c^{(k)}$, a copy each of $r_c^{(k)}$ and $p^{(k)}/2dm$, or two copies of $p^{(k)}/2dm$, these are the best case scenarios where two particles are given non-trivial instructions and all other particles remain at the same site and with the same colour (i.e., no swaps) but these probabilities are already of order $1/k^2$ and hence disappear in the limit.

From the discussion thus far, we can build a rate matrix Q for the continuous time process $(\xi_t^m(i,x))$ and transition matrices $P^{(k)}$ for the sequence of Markov chains $(\xi_n^{m,(k)}(i,x))_k$ such that $kP^{(k)} \to Q$ as $k \to \infty$. Hence by theorems 6.5 of Chapter 1 and 2.6 of Chapter 4 of Ethier and Kurtz [9], the continuous time process given by $(\xi_{\lfloor kt \rfloor}^{m,(k)}(i,x))$ converges in distribution to $(\xi_t^m(i,x))$ as $k \to \infty$. □

3.4.2 Convergence to the Solution of (2.14)

The previous section gives rise to a collection of Poisson processes on the box $B_L = [-L,L]^d$ that should describe the continuous time evolution of the presence of a particular colour at a particular site. Namely, these are the family of processes given by

$$(P_t(i,x,y,j), i,j \in \mathbb{N}, x,y \in B_L)$$

of rate $1/(2dm)$ controlling the jumps of a particle of colour i at x to y arriving with a new colour j, and

$$(P^c(i,j,x), i,j \in \mathbb{N}, x \in B_L)$$

which are the familiar rate λ Poisson processes controlling the coalescence of particle but restricted to the B_L. In the context of coloured particles, these processes control the change of colour i of a particle at x to j.

Equipped with these processes, we can write the equations that govern the occupation of a site by a particle of a specific colour. Define the random variable $\xi_t^{(m,L)}(i,x)$ to be 1 is there is a particle at time t at x with colour i and 0 otherwise. Then, if $\xi_0(i,x)$ is the initial condition corresponding to that for the discrete time model,

$$\begin{aligned}
&\xi_t^{(m,L)}(i,x) \\
&= \xi_0(i,x) + \int_0^t \sum_{j=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x)=0, \xi_{s-}^{(m,L)}(j,y)=1\} dP_s(j,y,x,i) \\
&- \int_0^t \sum_{j=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x)=1\} dP_s(i,x,y,j) \\
&- \int_0^t \sum_{j=1}^m \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x)=1, i\neq j\} dP_s^c(i,j,x) \\
&+ \int_0^t \sum_{j=1}^m \mathbb{1}\{\xi_{s-}^{(m,L)}(j,x)=1, \xi_{s-}^{(m,L)}(i,x)=0, i\neq j\} dP_s^c(j,i,x) \qquad (3.7)
\end{aligned}$$

for $x \in B_L$. The first two integrals account for a gain or loss of a particle of colour i as a result of migration while the final two terms account for the gain or loss as a result of a spontaneous colour change. We will not concern ourselves with proving the existence and uniqueness of the solution to this equation but we note that when a solution exists, it belongs to $\{0,1\}$ for all times. Instead, define $\xi_t^*(x) = \sum_{i=1}^m \xi_t^{(m,L)}(i,x)$ and $\xi_0(x) = \sum_{i=1}^m \xi_0(i,x)$ and also $P_t(i,y,x) = \sum_{j=1}^m P_t(i,y,x,j)$.

Then, summing up (3.7) gives

$$\begin{aligned}
&\xi_t^*(x) \\
&= \xi_0(x) + \int_0^t \sum_{i=1}^m \sum_{j=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x) = 0, \xi_{s-}^{m,L}(j,y) = 1\} dP_s(j,y,x,i) \\
&- \int_0^t \sum_{i=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x) = 1\} dP_s(i,x,y) \\
&- \int_0^t \sum_{i=1}^m \sum_{j=1}^m \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x) = 1, i \neq j\} dP_s^c(i,j,x) \\
&+ \int_0^t \sum_{i=1}^m \sum_{j=1}^m \mathbb{1}\{\xi_{s-}^{(m,L)}(j,x) = 1, \xi_{s-}^{m,L}(i,x) = 0, i \neq j\} dP_s^c(j,i,x).
\end{aligned}$$

Rewriting the indicators in the first and last integrals by decomposing the events $\{\xi_{s-}^{m,L}(j,y) = 1\}$ and $\{\xi_{s-}^{(m,L)}(i,x) = 1, i \neq j\}$ appropriately gives

$$\begin{aligned}
\xi_t^*(x) = \xi_0(x) &+ \int_0^t \sum_{j=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{m,L}(j,y) = 1\} dP_s(j,y,x) \\
&- \int_0^t \sum_{i=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x) = 1\} dP_s(i,x,y) \\
&- \int_0^t \sum_{i=1}^m \sum_{j=1}^m \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x) = 1, \xi_{s-}^{(m,L)}(j,x) = 1, i \neq j\} dP_s^c(i,j,x) \\
&- \int_0^t \sum_{i=1}^m \sum_{j=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^{(m,L)}(i,x) = 1, \xi_{s-}^{m,L}(j,y) = 1\} dP_s(j,y,x,i).
\end{aligned}$$

Since all the sums involved here are finite, we may rearrange and relabel at our leisure so that the indices that give a genuine non-zero contribution to the sums appear first, more explicitly we have equalities such as

$$\sum_{j=1}^m \mathbb{1}\{\xi_{s-}^{m,L}(j,y) = 1\} dP_s(j,y,x) = \sum_{j=1}^m \mathbb{1}\{\xi^*(y) \geq j\} dP_s(j,y,x). \qquad (3.8)$$

This is just essentially a repaint of particles according to their position at the site

x. Exploiting this allows us to write

$$\begin{aligned}
\xi_t^*(x) = \xi_0(x) &+ \int_0^t \sum_{i=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^*(y) \geq i\} dP_s(i,y,x) \\
&- \int_0^t \sum_{i=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^*(x) \geq i\} dP_s(i,x,y) \\
&- \int_0^t \sum_{i=1}^m \sum_{j=1}^m \mathbb{1}\{\xi_{s-}^*(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x) \\
&- \int_0^t \sum_{i=1}^m \sum_{j=1}^m \sum_{y\sim x, y\in B_L} \mathbb{1}\{\xi_{s-}^*(x) \geq j, \xi_{s-}^*(y) \geq i\} dP_s(i,y,x,j) \qquad (3.9)
\end{aligned}$$

for $x \in B_L$. We can extend this to a process on the entire d-dimensional lattice by adding initial conditions for $x \in \mathbb{Z}^d \setminus B_L$ and insisting that $\xi_t^*(x) \equiv \xi_0(x)$ for all t. We will prove the existence, uniqueness and finiteness of moments for (3.9) and following that, the remainder of this section will be to prove that this solution converges in the correct sense to the unique solution $(\xi_t(x) \colon x \in \mathbb{Z}^d)$ of the system of equations (4) in the chapter on existence.

Proposition 3.4.2. *For an initial condition satisfying $\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}\mathbb{E}[\xi_0(x)^2] < \infty$, where $\xi_0(x) = \sum_{i=1}^m \xi_0(i,x)$ for $x \in B_L$, there exists a unique solution to* (3.9) *and for every p*

$$\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}\mathbb{E}[\xi_t^*(x)^p]$$

is finite if it is finite at time 0 *and independent of L and m.*

Proof. The first four terms of (3.9) appear in equation (2.15) albeit with a more simple argument for the indicator in the first integral and also, only summing over neighbours y that belong to B_L. As such it is clear how the usual iteration method that is used to prove existence and uniqueness for equation (2.15) in Section A of the Appendix will work once we check that the additional term in (3.9) does not upset the argument. Since, in establishing existence and uniqueness, we need not worry about the dependence on m, a bound such as

$$|\mathbb{1}\{\xi_{s-}^{*,n}(x) \geq j, \xi_{s-}^{*,n}(y) \geq i\} - \mathbb{1}\{\xi_{s-}^{*,n-1}(x) \geq j, \xi_{s-}^{*,n-1}(y) \geq i\}$$
$$\leq |\xi_{s-}^{*,n}(x) - \xi_{s-}^{*,n-1}(x)| + |\xi_{s-}^{*,n-1}(y) - \xi_{s-}^{*,n-1}(y)|$$

will suffice where similar bounds were needed elsewhere in Section A of the Appendix. It is clear that the argument for existence and uniqueness will carry through. Similarly, with existence established, it is clear that the solution takes natural numbers as values for all times. Therefore, we can bound

$$0 \leq \xi_t^*(x) \leq \xi_0(x) + \int_0^t \sum_{i=1}^m \sum_{y \sim x, y \in B_L} \mathbb{1}\{\xi_{s-}^*(y) \geq i\} dP_s(i, y, x)$$

and hence the finite moments follow in exactly the same way as in Section 1.2 of the Appendix. □

Let $\xi_t(x)$ be the unique solution to (2.14) guaranteed by Theorem 2.2.3. We now prove that the negative correlation results carry over to this solution by proving that $\xi_t^*(x)$ converges to this solution as $m, L \to \infty$.

Proposition 3.4.3. *Given $\tilde{\xi}_0$ satisfying $\mathbb{E}[\sum_x e^{-\theta|x|}\tilde{\xi}_0(x)^2] < \infty$ let $\xi_t^*(x)$ be the unique solution to (3.9) with initial condition given by*

$$\xi_0(x) = \begin{cases} \tilde{\xi}_0(x) \wedge m, & x \in B_L \\ \tilde{\xi}_0(x), & x \notin B_L \end{cases}$$

Let $\xi_t(x)$ be the unique solution (2.14) with initial condition ξ_0, driven by the same Poisson drivers as for (3.9) inside B_L and by independent Poisson drivers outside of B_L. Then

$$\xi_t^*(x) \to \xi_t(x) \qquad \text{as } L, m \to \infty, \text{ for all } x \text{ and } t > 0 \text{ a.s..}$$

Proof. See Appendix Section B. □

Finally, we conclude that the solution to (2.14) with initial condition $\xi_0 \equiv 1$ enjoys all the same negative correlation properties as for the discrete time model. All convergence results in this chapter have been convergence in distribution or stronger, hence we only need the following lemma.

Lemma 3.4.4. *Let X_n, Y_n be positive, jointly distributed, integer valued random variables for each n such that*

$$\mathbb{P}[X_n \geq i, Y_n \geq j] \leq \mathbb{P}[X_n \geq i]\mathbb{P}[Y_n \geq j]$$

then if $(X_n, Y_n) \to (X, Y)$ in distribution, then

$$\mathbb{E}[XY] \leq \mathbb{E}[X]\mathbb{E}[Y].$$

Proof.

$$\begin{aligned}\mathbb{P}[X \geq i, Y \geq j] &= \lim_{n\to\infty} \mathbb{P}[X_n \geq i, Y_n \geq j] \\ &\leq \lim_{n\to\infty} (\mathbb{P}[X_n \geq i]\mathbb{P}[Y_n \geq j]) = \mathbb{P}[X \geq i]\mathbb{P}[Y \geq j]\end{aligned}$$

since X_n, Y_n are negatively correlated for all n. So

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{i,j} \mathbb{P}[X \geq i, Y \geq j] \\ &\leq \sum_{i} \mathbb{P}[X \geq i] \sum_{j} \mathbb{P}[Y \geq j] \\ &= \mathbb{E}[X]\mathbb{E}[Y].\end{aligned}$$

□

Chapter 4

Rate Equations

4.1 Instantaneously Coalescing Particles in $d = 2$

4.1.1 The Main Result

Following van den Berg and Kesten [27], our strategy is to build an approximate ordinary differential equation for the probability of interest, the solution of which will give the leading order asymptotic. For now we take $\xi_0 \equiv 1$.

By subtracting off the mean of each of the Poisson processes, i.e. writing $dP_t = dP_t - dt/4 + dt/4$ in equation (1) in Chapter 1 we gain a martingale term. Since $\xi_t(x)$ only takes values in $\{0, 1\}$ we have that $\mathbb{1}\{\xi_t(x) = 1\} = \xi_t(x)$. Also, we can write $\mathbb{1}\{\xi_t(y) = 1, \xi_t(x) = 0\} = \xi_t(y)(1 - \xi_t(x))$. Together, all of this gives

$$d\xi_t(x) = \Delta\xi_t(x)dt - \frac{1}{4}\sum_{y:y\sim x} \xi_t(x)\xi_t(y)dt + d\text{m.t.} \tag{4.1}$$

where we have collected the martingale terms in (m.t.). After taking expectation, translation invariance implies that the martingale terms and the discrete Laplacian vanish. Letting $\hat{\xi}_t = \mathbb{E}[\xi_t(x)]$ gives us the exact differential equation

$$\frac{d\hat{\xi}_t}{dt} = -\mathbb{E}[\xi_t(0)\xi_t(e)] \tag{4.2}$$

where e is one of the origin's 4 nearest neighbours (the choice is not important due to rotational invariance). It is now the expectation on the right hand side that we wish to approximate in order to find the rate of decay of $\hat{\xi}_t$ given in the following theorem.

Theorem 4.1.1. *Let ξ_t be the solution to* (1.6) *in $d = 2$ with initial condition $\xi_0 \equiv 1$. Then we have*

$$\hat{\xi}_t = \frac{\log t}{\pi t} + O\left(\frac{\log^{1/2} t}{t}\right).$$

4.1.2 A Priori Bounds

Before beginning the calculus, we state an *a priori* estimate for the first moment of ξ_t and a result regarding negative correlation.

Lemma 4.1.2. *Suppose that ξ_t is a solution to* (1.6) *with $\xi_0 \equiv 1$ in $d = 2$. Then there exist constants $t_0 > e$ and $0 < c_1 < c_2 < \infty$, so that*

$$c_1 \frac{\log t}{t} < \hat{\xi}_t < c_2 \frac{\log t}{t} \qquad \textit{for all } t \geq t_0.$$

Proof. This follows from Bramson and Griffeath [2] where they study the same process but in all dimensions $d \geq 1$. Rather than constructing their process as the solution to a system of stochastic differential equations as we do, they employ a graphical construction which is closely related to our construction. Therefore we require a little explanation. Van den Berg and Kesten proved a corresponding result for dimensions $d \geq 3$ in Lemma 8 of their paper [27] on a slight generalisation of the instantly coalescing random walks. In their system, particles coalesce on contact immediately but only with a probability that depends on the number of particles at the site of contact. They also allow for a more general random walk than nearest neighbour. The construction of their process is an appropriate variation of the graphical construction of Bramson and Griffeath which is given in detail in [12]. According to this construction, they are able to show that the expected value of the number of particles at a site satisfies a differential equation, namely equation (3.9) of Lemma 9. All of this is independent of dimension and in particular can be carried out for $d = 2$. Our instantly coalescing random walks arise as a special case of their randomly instantly coalescing walks as soon as we insist that the probability that a particle coalesces is 1 if there is a single particle already present (this corresponds in their notation as letting $p_1 = 1$) and as long as we take their jump function to be $q(y-x) = 1/4$ if and only if $y \sim x$ and 0 otherwise. And in this case the differential equation (3.9) in Lemma 9 of that paper reduces to the very same equation that we have in equation (4.2). By the uniqueness of the solution given by Proposition 2.1.1 whatever holds for one construction will hold for the other. From this point, the proof as it is in Bramson and Griffeath applies, as a special case of the proof of Lemma 8 in [27]. □

Lemma 4.1.3. *Suppose that ξ_t is a solution to (1.6) with $\xi_0 \equiv 1$ in $d = 2$. Then, for all $k \in \mathbb{N}$,*

$$\mathbb{E}\left[\xi_t(x_1) \cdot \ldots \cdot \xi_t(x_k)\right] \leq \left(c_2 \frac{\log t}{t}\right)^k \quad \text{for all disjoint } x_1, \ldots, x_k \text{ and all } t > t_0$$

where t_0, c_2 are the same constants as in Lemma 4.1.2.

Proof. This follows from van den Berg and Kesten [28], where they prove more general negative correlation results then we did for the instantly coalescing system in Chapter 3 Section 3.2. As in the proof of Lemma 4.1.2 our instantly coalescing random walks are a special case of their randomly coalescing random walks as soon as the coalescing probability p_1 is taken to be 1 and the walks are restricted to nearest neighbour. This allows us to bound the expectation above by k copies of $\hat{\xi}_t$ and the upper bound comes from Lemma 4.1.2. □

4.1.3 A Two Point Estimate

Our first step in the approximation of the expectation in (4.2) comes from a two point estimate. For $f, g : \mathbb{Z}^2 \to \mathbb{R}$ we will write $\langle f, g \rangle = \sum_x f(x)g(x)$, and similarly for $f, g : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$, $\langle f, g \rangle = \sum_{x,y} f(x,y)g(x,y)$. We will use $*$ to view the product of functions $f, g : \mathbb{Z}^2 \to \mathbb{R}$ as a function $f * g : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$. Notice that with the test function $\varphi_s = \psi_{t-s}$, where

$$\psi_t(x,y) = \mathbb{P}[x + S_t^1 = 0, y + S_t^2 = e, \tau > t] \tag{4.3}$$

and $\tau = \inf\{t : x + S_t^1 = y + S_t^2\}$ that

$$\begin{aligned} \mathbb{E}[\langle \xi_t * \xi_t, \varphi_t \rangle] &= \mathbb{E}\left[\sum_{x,y} \xi_t(x)\xi_t(y)\varphi_t(x,y)\right] \\ &= \mathbb{E}\left[\sum_{x,y} \xi_t(x)\xi_t(y)\mathbb{1}\{x = 0, y = e\}\right] = \mathbb{E}[\xi_t(0)\xi_t(e)]. \end{aligned} \tag{4.4}$$

The right hand side of (4.4) is the expression that appears in the right hand side of (4.2). With this in mind, we will use some calculus to pull back the left hand side of (4.4) to time $t - s$ and control the error. This corresponds exactly to the idea that at the earlier time $t - s$, there are particles located at some sites x, y that walk without coalescing to sites $0, e$ respectively by time t. The error in this approximation accounts for the possibility that the particles might have coalesced over the interval $[t - s, t]$. In order to do the calculus, we find the compensator for the quadratic variation process and use the integration by parts formula. The quadratic variation, defined by

$$[\xi(x), \xi(y)]_t = \sum_{s \le t} (\xi_s(x) - \xi_{s-}(x))(\xi_s(y) - \xi_{s-}(y)), \tag{4.5}$$

is constant except at each of the jumps of $\xi_t(x)$ or $\xi_t(y)$. If $x = y$,

$$[\xi(x), \xi(x)]_t = \sum_{s \le t} (\xi_s(x) - \xi_{s-}(x))^2$$

and we see that

$$\begin{aligned} d[\xi(x), \xi(x)]_t = \sum_{z: z \sim x} & (-1)^2 \mathbb{1}\{\xi_{t-}(x) = 1\} dP_t(x,z) \\ & + (1)^2 \mathbb{1}\{\xi_{t-}(z) = 1, \xi_{t-}(x) = 0\} dP_t(z,x) \end{aligned}$$

since the only change that can occur are the evacuation of a particle if x was occupied or the occupation of x by a particle if x was empty.

If $x \sim y$, we get

$$
\begin{aligned}
d[\xi(x), \xi(y)]_t = & -\mathbb{1}\{\xi_{t-}(x) = 1, \xi_{t-}(y) = 0\} dP_t(x, y) \\
& - \mathbb{1}\{\xi_{t-}(x) = 0, \xi_{t-}(y) = 1\} dP_t(y, x)
\end{aligned}
$$

since the only change that can occur that contributes is a particle occupying x while y is empty that evacuates x to occupy y or vice versa. There is no change if the sites are either both occupied or both empty since there are no simultaneous events for the collection of independent Poisson processes. Similarly, if $|x - y| > 1$ then there can be no change that contributes to $d[\xi(x), \xi(y)]_t$, since the walks are simple and there are no simultaneous events. Compensating the Poisson processes and bringing together all the cases, the compensator of the quadratic variation process should satisfy

$$
d\langle\langle \xi(x), \xi(y) \rangle\rangle_t = \begin{cases} \frac{1}{4} \sum_{z: z \sim x} \Big(\xi_t(x) + \xi_t(z) - \xi_t(x)\xi_t(z) \Big) dt & x = y \\ -\frac{1}{4} \Big(\xi_t(x) + \xi_t(y) - 2\xi_t(x)\xi_t(y) \Big) dt & x \sim y \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}
$$

The integration by parts formula reads as

$$
d(\xi_t(x)\xi_t(y)) = \xi_{t-}(x) d\xi_t(y) + \xi_{t-}(y) d\xi_t(x) + d[\xi(x), \xi(y)]_t. \tag{4.7}
$$

Using the compensated equation (4.1) and the compensator for the quadratic covariance process we arrive at

$$\begin{aligned}
d\Big(\xi_t(x)\xi_t(y)\Big) &= \xi_{t-}(x)\left[\Delta\xi_t(y)dt - \frac{1}{4}\sum_{z:z\sim y}\xi_t(z)\xi_t(y)dt\right] \\
&\quad + \xi_{t-}(y)\left[\Delta\xi_t(x)dt - \frac{1}{4}\sum_{w:w\sim x}\xi_t(x)\xi_t(w)dt\right] \\
&\quad + d\langle\langle\xi(x),\xi(y)\rangle\rangle_t + \text{m.t.}
\end{aligned}$$

Rearranging

$$\begin{aligned}
d\Big(\xi_t(x)\xi_t(y)\Big) &= \xi_t(x)\Delta\xi_t(y)dt + \xi_t(y)\Delta\xi_t(x)dt \\
&\quad - \frac{1}{4}\sum_{z:z\sim y}\xi_t(x)\xi_t(y)\xi_t(z)dt - \frac{1}{4}\sum_{w:w\sim x}\xi_t(x)\xi_t(y)\xi_t(w)dt \\
&\quad + d\langle\langle\xi(x),\xi(y)\rangle\rangle_t + \text{m.t.} \\
&= \Delta\xi_t(x)\xi_t(y)dt - \frac{1}{4}\xi_t(x)\xi_t(y)\left(\sum_{w:w\sim x}\xi_t(w) + \sum_{z:z\sim y}\xi_t(z)\right)dt \\
&\quad + d\langle\langle\xi(x),\xi(y)\rangle\rangle_t + \text{m.t.}
\end{aligned}$$

where Δ in the last equality is the Laplacian acting in two variables with the product $\xi_t(x)\xi_t(y)$ being treated as a single function of two variables. Now multiply by an arbitrary test function $\varphi : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ that satisfies

$$\sum_{x,y}|\varphi(x,y)| + |\dot{\varphi}(x,y)| < \infty$$

and sum over all x and y,

$$\begin{aligned}
d\langle \xi_u * \xi_u, \varphi_u \rangle &= \sum_x \sum_y d\Big(\xi_u(x)\xi_u(y)\Big)\varphi_u(x,y) + \langle \xi_u * \xi_u, \dot{\varphi_u}\rangle du \\
&= \sum_x \sum_y \Delta\xi_u(x)\xi_u(y)\varphi_u(x,y)du \\
&\quad - \frac{1}{4}\xi_u(x)\xi_u(y)\left(\sum_{w:w\sim x}\xi_u(w) + \sum_{z:z\sim y}\xi_u(z)\right)\varphi_u(x,y)du \\
&\quad + \sum_x \sum_y d\langle\langle \xi(x), \xi(y)\rangle\rangle_u \varphi_u(x,y) \\
&\quad + \langle \xi_u * \xi_u, \dot{\varphi_u}\rangle du + \text{m.t.}.
\end{aligned}$$

The sum over all x, y converges since if we first truncate (writing $\langle \xi_u * \xi_u, \varphi_u\rangle_N$ to be the sum truncated at $|x|, |y| \leq N$) then

$$\begin{aligned}
&\mathbb{E}[\langle \xi_u * \xi_u, |\varphi_u|\rangle_N] = \sum_{|x|,|y|\leq N} \mathbb{E}[\xi_u(x)\xi_u(y)]|\varphi_u(x,y)| \\
&\leq \sum_{|x|\leq N} \mathbb{E}[\xi_u(x)]|\varphi_u(x,x)| + \sum_{|x|,|y|\leq N, y\neq x} \mathbb{E}[\xi_u(x)]\mathbb{E}[\xi_u(y)]|\varphi_u(x,y)| \\
&\leq \hat{\xi}_t \sum_{|x|\leq N} |\varphi_u(x,x)| + \hat{\xi}_t^2 \sum_{|x|,|y|\leq N, y\neq x} |\varphi_u(x,y)| \\
&\leq \hat{\xi}_t \sum_{|x|,|y|\leq N} |\varphi_u(x,y)| \\
&\leq \hat{\xi}_t \sum_{x,y} |\varphi_u(x,y)| < \infty
\end{aligned}$$

So we may take $N \to \infty$ and the sum converges almost surely. Discrete integration by parts for the Laplacian in one variable is achieved by

$$\begin{aligned}\sum_x g(x)\Delta f(x) &= \frac{1}{4}\sum_x \sum_{y:y\sim x} g(x)f(y) - \frac{1}{4}\sum_x \sum_{y:y\sim x} g(x)f(x)\\ &= \frac{1}{4}\sum_y \sum_{x:x\sim y} g(x)f(y) - \frac{1}{4}\sum_x \sum_{y:y\sim x} g(x)f(x)\\ &= \frac{1}{4}\sum_x \sum_{y:y\sim x} g(y)f(x) - \frac{1}{4}\sum_x \sum_{y:y\sim x} g(x)f(x)\\ &= \sum_x f(x)\Delta g(x). \end{aligned} \tag{4.8}$$

Define Δ_i as the Laplacian in one variable acting on variable i of a multivariable function, for example

$$\Delta_1 f(x,y) = \frac{1}{4}\sum_{e:e\sim x} f(e,y) - f(x,y).$$

For the Laplacian in two variables we have for $f, g\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$

$$\begin{aligned}\langle \Delta f, g\rangle &= \sum_{x,y} g(x,y)\Delta f(x,y)\\ &= \sum_{x,y} g(x,y)\left(\Delta_1 f(x,y) + \Delta_2 f(x,y)\right)\\ &= \sum_y \sum_x g(x,y)\Delta_1 f(x,y) + \sum_x \sum_y g(x,y)\Delta_2 f(x,y)\\ &= \sum_y \sum_x f(x,y)\Delta_1 g(x,y) + \sum_x \sum_y f(x,y)\Delta_2 g(x,y)\\ &= \sum_{x,y} f(x,y)\Delta g(x,y)\\ &= \langle f, \Delta g\rangle. \end{aligned} \tag{4.9}$$

Using discrete integration by parts on the product of the first summand with φ and the relation in (4.6) for the compensator of the quadratic covariance process we get

$$\begin{aligned} d\langle \xi_u * \xi_u, \varphi_u \rangle = {} & \langle \xi_u * \xi_u, \dot{\varphi}_u + \Delta\varphi_u \rangle \\ & - \frac{1}{4} \sum_x \sum_y \xi_u(x)\xi_u(y) \left(\sum_{w:w\sim x} \xi_u(w) + \sum_{z:z\sim y} \xi_u(z) \right) \varphi_u(x,y)du \\ & + \frac{1}{4} \sum_x \sum_{w:w\sim x} \Big(\xi_u(x) + \xi_u(w) - \xi_u(x)\xi_u(w) \Big) \varphi_u(x,x)du \\ & - \frac{1}{4} \sum_x \sum_{y:y\sim x} \Big(\xi_u(x) + \xi_u(y) - 2\xi_u(x)\xi_u(y) \Big) \varphi_u(x,y)du \\ & + \text{m.t.}. \end{aligned} \tag{4.10}$$

Now, we make a specific choice of a test function. Let $\varphi_u = \psi_{t-u}$ as in (4.3). Then since $\psi_t(x,y) \le p_t(x)p_t(y)$ we have $\sum_{x,y} \psi_t(x,y) \le 1$. Now on the diagonal and for any $u \in [0,t]$, $\varphi_u \equiv 0$, since if the two sites are coincident the two particles will coalesce immediately so that $\tau = 0$. The probability is then the indicator of the event that the site x is simultaneously the origin and its neighbour e. This is an impossible event so the indicator is identically 0. Therefore, all the terms in (4.10) on the diagonal vanish leaving

$$\begin{aligned} d\langle \xi_u * \xi_u, \varphi_u \rangle = {} & \langle \xi_u * \xi_u, \dot{\varphi}_u + \Delta\varphi_u \rangle du \\ & - \frac{1}{4} \sum_x \sum_{y\neq x} \xi_u(x)\xi_u(y) \left(\sum_{w\sim x} \xi_s(w) + \sum_{z\sim y} \xi_u(z) \right) \varphi_u(x,y)du \\ & - \frac{1}{4} \sum_x \sum_{y\sim x} \Big(\xi_u(x) + \xi_u(y) - 2\xi_u(x)\xi_u(y) \Big) \varphi_u(x,y)du + \text{m.t.} \end{aligned} \tag{4.11}$$

Also, we have that the test function φ satisfies

$$\dot{\varphi}_u(x,y) + \Delta\varphi_u(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ \Delta\varphi_u(x,y) & \text{if } x = y \end{cases}$$

since the only contribution to the derivative of φ when x and y are not coincident comes from walking and there is no contribution when the sites are coincident since the walks have met instantaneously so that

$$
\begin{aligned}
\langle \xi_u * \xi_u, \dot{\varphi_u} + \Delta\varphi_u \rangle &= \sum_x \sum_y \xi_u(x)\xi_u(y)\Big(\dot{\varphi}_u(x,y) + \Delta\varphi_u(x,y)\Big) \\
&= \sum_x \sum_{y=x} \xi_u(x)\xi_u(y)\Delta\varphi_u(x,y) \\
&= \sum_x \xi_u^2(x)\Delta\varphi_u(x,x) \\
&= \sum_x \xi_u(x)\Delta\varphi_u(x,x).
\end{aligned}
$$

Here the last equality follows since $\xi_u(x) = 0$ or 1. Now,

$$
\begin{aligned}
\Delta\varphi_u(x,x) &= \Delta_1\varphi_u(x,x) + \Delta_2\varphi_u(x,x) \\
&= \frac{1}{4}\sum_{y\sim x}\Big(\varphi_u(y,x) - \varphi_u(x,x)\Big) + \frac{1}{4}\sum_{y\sim x}\Big(\varphi_u(x,y) - \varphi_u(x,x)\Big) \\
&= \frac{1}{4}\sum_{y\sim x}\Big(\varphi_u(y,x) + \varphi_u(x,y)\Big).
\end{aligned}
$$

This gives, by substitution and a change of variables

$$
\begin{aligned}
\langle \xi_u * \xi_u, \dot{\varphi_u} + \Delta\varphi_u \rangle &= \frac{1}{4}\sum_x \xi_u(x) \sum_{y\sim x}\Big(\varphi_u(y,x) + \varphi_u(x,y)\Big) \\
&= \frac{1}{4}\sum_x \sum_{y\sim x}\Big(\xi_u(x) + \xi_u(y)\Big)\varphi_u(x,y).
\end{aligned}
$$

This term now cancels with the first two summands of the first factor in the last sum of (4.11), giving us

$$
\begin{aligned}
d\langle \xi_u * \xi_u, \varphi_u \rangle = &- \frac{1}{4}\sum_x \sum_{y\neq x} \xi_u(x)\xi_u(y)\left(\sum_{w\sim x}\xi_u(w) + \sum_{z\sim y}\xi_u(z)\right)\varphi_u(x,y)du \\
&+ \frac{1}{4}\sum_x \sum_{y\sim x} 2\xi_u(x)\xi_u(y)\varphi_u(x,y)du + \text{m.t.}.
\end{aligned}
$$

Taking expectation gives

$$d\mathbb{E}\left[\langle \xi_u * \xi_u, \varphi_u \rangle\right] = -\frac{1}{4}\sum_x \sum_{y \neq x} \Bigg(\sum_{w \sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{z \sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)] \Bigg) \varphi_u(x,y)du + \frac{1}{4}\sum_x \sum_{y \sim x} 2\mathbb{E}\left[\xi_u(x)\xi_u(y)\right]\varphi_u(x,y)du.$$

Now, since the inner sum of the first term is taken over all $y \neq x$, there will be terms appearing in the case that y neighbours x. In this instance, the value of $\xi_u(y)$ will of course coincide exactly with the value of ξ_u at one of x's 4 neighbours w and we'll lose a term in the expectation since in this instance $\xi_u(y)\xi_u(w) = \xi_u(y)\xi_u(y) = \xi_u(y)^2 = \xi_u(y)$. Similarly, the value of $\xi_u(x)$ will coincide with the value of ξ_u at one of y's 4 neighbours z. We have to be careful in estimating the expectation because Lemma 4.1.3 only gives a result for disjoint sites. This motivates us to split the sum over $y \neq x$ into a sum over y such that $|y - x| > 1$ and a sum over $y \sim x$. Here $|\cdot|$ is the standard Euclidean norm.

$$\begin{aligned}
&d\mathbb{E}\left[\langle \xi_u * \xi_u, \varphi_u \rangle\right] \\
&= -\frac{1}{4}\sum_x \sum_{y:\,|y-x|>1} \left(\sum_{w \sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{z \sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)] \right) \varphi_u(x,y)du \\
&- \frac{1}{4}\sum_x \sum_{y \sim x} \left(\sum_{w \sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{z \sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)] \right) \varphi_u(x,y)du \\
&+ \frac{1}{4}\sum_x \sum_{y \sim x} 2\mathbb{E}\left[\xi_u(x)\xi_u(y)\right]\varphi_u(x,y)du.
\end{aligned}$$

For the case that y neighbours x, we have

$$
\begin{aligned}
&d\mathbb{E}\left[\langle \xi_u * \xi_u, \varphi_u\rangle\right] \\
&= -\frac{1}{4}\sum_x \sum_{y\colon |y-x|>1} \left(\sum_{w\sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{z\sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)]\right) \\
&\quad \times \varphi_u(x,y)du \\
&\quad - \frac{1}{4}\sum_x \sum_{y\sim x} \left(\sum_{\{w:w\sim x\}\setminus y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{\{z:z\sim y\}\setminus x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)]\right) \\
&\quad \times \varphi_u(x,y)du \\
&\quad - \frac{1}{4}\sum_x \sum_{y\sim x} \mathbb{E}\left[\xi_u(x)\xi_u(y)\Big(\xi_u(y)+\xi_u(x)\Big)\right]\varphi_u(x,y)du \\
&\quad + \frac{1}{4}\sum_x \sum_{y\sim x} 2\mathbb{E}\left[\xi_u(x)\xi_u(y)\right]\varphi_u(x,y)du \\
&= -\frac{1}{4}\sum_x \sum_{y\colon |y-x|>1} \left(\sum_{w\sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{z\sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)]\right) \\
&\quad \times \varphi_u(x,y)du \\
&\quad - \frac{1}{4}\sum_x \sum_{y\sim x} \left(\sum_{\{w:w\sim x\}\setminus y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{\{z:z\sim y\}\setminus x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)]\right) \\
&\quad \times \varphi_u(x,y)du
\end{aligned}
\tag{4.12}
$$

where in the last equality the last two sums cancelled since $\xi^2 \equiv \xi$. We interupt the consideration of the two point estimate briefly, just to state a result about non-interacting random walks in two dimensions which we will need moving on. We will defer the proof of this result until the end of Section 5.2. Recall that $p_t(x)$ is the probability that a simple, rate 1, continuous time random walk started at the origin is at x at time t and e is any neighbour of the origin.

Lemma 4.1.4. *Let* $\Lambda_t(x,y) = \psi_t(x,y) - \frac{\pi}{\log t} p_t(x)p_t(y-e)$*. Then there exists* $c_5 > 0$

and $\zeta \in (0,1)$ *such that*

$$\langle 1, |\Lambda_t(\cdot,\cdot)|\rangle \leq \frac{c_5}{\log^{1+\zeta} t}$$

for large enough t.

This next lemma is a small result but it is important. In many places we need to estimate the expected value of the occupation of neighbouring sites, $\mathbb{E}[\xi_t(0)\xi_t(e)]$. If we used only negative correlation here we would find $\mathbb{E}[\xi_t(0)\xi_t(e)] \leq \left(c_3\hat{\xi}_t\right)^2$. However, we can improve this by a logarithm at the cost of rewinding time a little bit using the random walk estimate Lemma 4.1.4 and indeed it is necessary to do so, without this extra logarithm our errors will not be small enough.

Lemma 4.1.5. *Suppose that* ξ *is a solution to (1.6) with* $\xi_0 \equiv 1$ *in* $d = 2$*. Let* $s = t/\log^\alpha t$*, for some* $\alpha > 0$*, and let* $r \in [t-s, t]$*. Then there exists* $c_6(\alpha) < \infty$ *such that*

$$\mathbb{E}[\xi_r(0)\xi_r(e)] \leq c_6 \frac{\hat{\xi}^2_{t-2s}}{\log t}.$$

Proof. Note that (4.12) is negative so that the quantity decreases, the same is true with the test function $\varphi'_u = \psi_{r-u}$, for $u \leq r$, that is

$$\begin{aligned}
&d\mathbb{E}\left[\langle \xi_u * \xi_u, \varphi'_u\rangle\right] \\
&= -\frac{1}{4}\sum_x \sum_{y:\,|y-x|>1} \left(\sum_{w\sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_s(w)] + \sum_{z\sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)]\right)\varphi'_u(x,y)du \\
&- \frac{1}{4}\sum_x \sum_{y\sim x} \left(\sum_{\{w:w\sim x\}\setminus y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{\{z:z\sim y\}\setminus x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)]\right)\varphi'_u(x,y)du
\end{aligned}$$

so, by Lemma 4.1.4

$$
\begin{aligned}
&\mathbb{E}[\xi_r(0)\xi_r(e)] \\
&= \mathbb{E}[\langle \xi_r * \xi_r, \varphi_r' \rangle] \le \mathbb{E}[\langle \xi_{t-2s} * \xi_{t-2s}, \psi_{r-t+2s} \rangle] \\
&\le \sup_{x,y} \mathbb{E}[\xi_{t-2s}(x)\xi_{t-2s}(y)] \Bigg(\frac{\pi}{\log(r-t+2s)} \sum_{x,y} p_{r-t+2s}(x) p_{r-t+2s}(y-e) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{C}{\log^{1+\zeta}(r-t+2s)} \Bigg) \\
&\le C \frac{\hat{\xi}^2_{t-2s}}{\log(r-t+2s)} \\
&\le C \frac{\hat{\xi}^2_{t-2s}}{\log(s)} \\
&\le c_6 \frac{\hat{\xi}^2_{t-2s}}{\log(t)}
\end{aligned} \tag{4.13}
$$

since $s = t/\log^\alpha t$ and $r \ge t - s$. $\square$

Returning to (4.12), both of the terms in the derivative are small in expectation. Consider the first term of (4.12). We estimate for $u \in [t-s, t]$, with $s = t/\log^\alpha t$

$$
\begin{aligned}
&\sum_x \sum_{y:|y-x|>1} \left(\sum_{w \sim x} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(w)] + \sum_{z \sim y} \mathbb{E}[\xi_u(x)\xi_u(y)\xi_u(z)] \right) \varphi_u(x,y) \\
&\le \sum_x \sum_{y:|y-x|>1} \left(\sum_{w \sim x} \mathbb{E}[\xi_u(x)\xi_u(w)]\mathbb{E}[\xi_u(y)] + \sum_{z \sim y} \mathbb{E}[\xi_u(z)\xi_u(y)]\mathbb{E}[\xi_u(x)] \right) \varphi_u(x,y) \\
&\le C\mathbb{E}[\xi_u(0)\xi_u(e)]\mathbb{E}[\xi_u(0)] \sum_x \sum_{y:|y-x|>1} \varphi_u(x,y) \\
&\le c_6 \frac{\hat{\xi}^2_{t-2s}}{\log t} \cdot \frac{\log u}{u} \sum_x \sum_{y:|y-x|>1} \varphi_u(x,y) \qquad \text{by Lemma 4.1.3 and (4.13)} \\
&\le C \frac{\log^2(t-2s)}{(t-2s)^2 \log t} \cdot \frac{\log(t-s)}{t-s} \sum_x \sum_y \varphi_u(x,y) \\
&\le C \frac{\log^2 t}{t^3} \sum_x \sum_y \varphi_u(x,y)
\end{aligned}
$$

where the constant is changing line by line. The second term is similar since the arguments of the expectation are all disjoint and we can peal off one of the terms in each of the expectations by negative correlation, leaving a pair of neighbouring sites in each for which (4.13) can be applied. So (4.12) can be bounded by

$$\frac{\log^2 t}{t^3} \sum_x \sum_y \varphi_u(x, y)$$

up to some constant.

Lemma 4.1.6. *With φ as defined in* (4.3)*, there exists $c_7 < \infty$ so that for $s = t/\log^\alpha t$ and $t \geq 2$*

$$\left| \mathbb{E}[\xi_t(0)\xi_t(e)] - \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s} \rangle] \right| \leq c_7 \frac{\log^{1-\alpha} t}{t^2}.$$

Proof. When r is small φ_r is well approximated as in Lemma 4.1.4

$$\begin{aligned}
&\left| \mathbb{E}[\xi_t(0)\xi_t(e)] - \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s} \rangle] \right| \\
&= \left| \mathbb{E}[\langle \xi_t * \xi_t, \varphi_t \rangle] - \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s} \rangle] \right| \\
&= \left| \int_{t-s}^{t} d\mathbb{E}[\langle \xi_r * \xi_r, \varphi_r \rangle] \right| \\
&\leq C \frac{\log^2 t}{t^3} \int_{t-s}^{t} \sum_x \sum_y \varphi_r(x, y) ds \\
&\leq C \frac{\log^2 t}{t^3} \left(\int_{t-s}^{t-\frac{s}{\log s}} \sum_x \sum_y \varphi_r(x, y) ds + \int_{t-\frac{s}{\log s}}^{t} \sum_x \sum_y \varphi_r(x, y) ds \right)
\end{aligned}$$

in the first integral we use Lemma 4.1.4 and in the second we simply bound the entire double sum by 1.

$$\begin{aligned}
&\left| \mathbb{E}[\xi_t(0)\xi_t(e)] - \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s} \rangle] \right| \\
&\leq C \frac{\log^2 t}{t^3} \left(\left(s - \frac{s}{\log s} \right) \frac{C'}{\log\left(\frac{s}{\log s}\right)} + \frac{s}{\log s} \right) \leq C \frac{\log^2 t}{t^3} \frac{s}{\log s} \leq C \frac{\log^{1-\alpha} t}{t^2}
\end{aligned}$$

where C' is some positive constant and the value of C is changing in each inequality.

□

Recall the definition of Λ given in 4.1.4.

Lemma 4.1.7. *With* φ *as defined in* (4.3)*, there exists* $c_8 < \infty$ *such that for* $s = t/\log^\alpha t$ *and* $t \geq 2$

$$\left| \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s} \rangle] - \frac{\pi}{\log s} \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s(\cdot - e) \rangle] \right| \leq c_8 \frac{\log^{(1-\zeta)\vee\alpha} t}{t^2}.$$

Proof.

$$\begin{aligned}
&\left| \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s} \rangle] - \frac{\pi}{\log s} \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s(\cdot - e) \rangle] \right| \\
&= \left| \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \psi_s - \frac{\pi}{\log s} p_s * p_s(\cdot - e) \rangle] \right| \\
&= \left| \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \Lambda_s \rangle] \right| \\
&\leq \sum_x \sum_y \left| \mathbb{E}[\xi_{t-s}(x) \xi_{t-s}(y)] \Lambda_s(x,y) \right| \\
&= \sum_x \sum_{y \neq x} \left| \mathbb{E}[\xi_{t-s}(x) \xi_{t-s}(y)] \Lambda_s(x,y) \right| + \sum_x \left| \mathbb{E}[\xi_{t-s}(x)] \Lambda_s(x,x) \right|.
\end{aligned}$$

We have split the sum so that the expectation in the first sum can be estimated by the moment bounds in Lemma 4.1.3. The bound on the second sum will follow from the bound on the transition density in Lemma 5.1.1.

$$\begin{aligned}
&\sum_x \sum_{y \neq x} \left| \mathbb{E}[\xi_{t-s}(x) \xi_{t-s}(y)] \Lambda_s(x,y) \right| + \sum_x \left| \mathbb{E}[\xi_{t-s}(x)] \Lambda_s(x,x) \right| \\
&\leq C \left(\frac{\log(t-s)}{t-s} \right)^2 \sum_x \sum_{y \neq x} \left| \Lambda_s(x,y) \right| + \frac{C}{\log s} \left(\frac{\log(t-s)}{t-s} \right) \sum_x p_s(x) p_s(x-e) \\
&\leq C \left(\frac{\log(t-s)}{t-s} \right)^2 \langle 1, |\Lambda_s| \rangle_{x,y} + \frac{C}{s \log s} \left(\frac{\log(t-s)}{t-s} \right) \sum_x p_s(x) \\
&\leq c \left(\frac{1}{\log^{1+\zeta} s} \left(\frac{\log t}{t} \right)^2 + \frac{1}{s \log s} \left(\frac{\log t}{t} \right) \right) \qquad \text{by (4.1.4).}
\end{aligned}$$

$\square$

4.1.4 A Variance Estimate

In the last approximation we were left at

$$\begin{aligned}
&\frac{\pi}{\log s}\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s(\cdot - e)\rangle] \\
&= \frac{\pi}{\log s}\mathbb{E}\left[\sum_{x,y} \xi_{t-s}(x)\xi_{t-s}(y)p_s(x)p_s(y-e)\right] \\
&= \frac{\pi}{\log s}\mathbb{E}\left[\sum_{x} \xi_{t-s}(x)p_s(x)\sum_{y} \xi_{t-s}(y)p_s(y-e)\right] \\
&= \frac{\pi}{\log s}\mathbb{E}[\langle \xi_{t-s}, p_s\rangle\langle \xi_{t-s}, p_s(\cdot - e)\rangle]
\end{aligned}$$

and wanting to estimate this expectation. There will be need for an application of a variance estimate to decouple the terms in the final equality of the above. This is easy using negative correlation.

Lemma 4.1.8. *Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be such that $\langle f^2, 1\rangle < \infty$. Then there exists $c_9 < \infty$ such that for $t \geq e$*

$$Var(\langle \xi_t, f\rangle) \leq c\langle f^2, 1\rangle \frac{\log t}{t}.$$

Proof.

$$\begin{aligned}
Var(\langle \xi_t, f\rangle) &= \mathbb{E}\left[\left(\langle \xi_t, f\rangle - \mathbb{E}[\langle \xi_t, f\rangle]\right)^2\right] \\
&= \mathbb{E}\left[\sum_x (\xi_t(x) - \hat{\xi}_t)f(x)\sum_y (\xi_t(y) - \hat{\xi}_t)f(y)\right] \\
&= \sum_x\sum_y \mathbb{E}\left[(\xi_t(x) - \hat{\xi}_t)(\xi_t(y) - \hat{\xi}_t)\right] f(x)f(y). \qquad (4.14)
\end{aligned}$$

Ignore for now the diagonal terms $y = x$, then expanding the expectation and using Lemma 4.1.3 we find

$$\begin{aligned}
&\sum_x \sum_{y:y\neq x} \mathbb{E}\left[(\xi_t(x) - \hat{\xi}_t)(\xi_t(y) - \hat{\xi}_t)\right] f(x)f(y) \\
&= \sum_x \sum_{y:y\neq x} \mathbb{E}\left[(\xi_t(x)\xi_t(y) - \hat{\xi}_t\xi_t(y) - \xi_t(x)\hat{\xi}_t + \hat{\xi}_t^2)\right] f(x)f(y) \\
&= \sum_x \sum_{y:y\neq x} \left(\mathbb{E}[\xi_t(x)\xi_t(y)] - \hat{\xi}_t^2\right) f(x)f(y) \\
&\leq C\sum_x \sum_{y:y\neq x} \left(\mathbb{E}[\xi_t(x)]\mathbb{E}[\xi_t(y)] - \hat{\xi}_t^2\right) f(x)f(y) \\
&= C\sum_x \sum_{y:y\neq x} \left(\hat{\xi}_t^2 - \hat{\xi}_t^2\right) f(x)f(y) = 0.
\end{aligned}$$

We can use this to bound (4.14) above by just the sum over the diagonal terms so that

$$\begin{aligned}
Var(\langle \xi_t, f\rangle) &\leq \sum_x \sum_{y=x} \mathbb{E}\left[(\xi_t(x) - \hat{\xi}_t)(\xi_t(y) - \hat{\xi}_t)\right] f(x)f(y) \\
&= \sum_x \mathbb{E}\left[(\xi_t(x) - \hat{\xi}_t)^2\right] f(x)^2 \\
&= \sum_x \left(\mathbb{E}[\xi_t(x)^2] - \hat{\xi}_t^2\right) f(x)^2 \\
&= \sum_x \left(\mathbb{E}[\xi_t(x)] - \hat{\xi}_t^2\right) f(x)^2 \\
&\leq c\left(\frac{\log t}{t}\right) \sum_x f(x)^2 = c\left(\frac{\log t}{t}\right) \langle f^2, 1\rangle
\end{aligned}$$

□

If we choose f to be either p_s or $p_s(\cdot - e)$ then since $\sum_x p_s^2(x-e) = \sum_x p_s^2(x)$ we have $\langle f^2, 1\rangle = \sum_x p_s^2(x) \leq \frac{c}{s}\sum_x p_s(x) = \frac{c}{s}$. For $s \leq t/2$, using Cauchy-Schwarz we get the estimate

$$\begin{aligned}
&\Big|\mathbb{E}\left[\langle \xi_{t-s}, p_s\rangle \langle \xi_{t-s}, p_s(\cdot - e)\rangle\right] - \mathbb{E}[\langle \xi_{t-s}, p_s\rangle]\mathbb{E}[\langle \xi_{t-s}, p_s(\cdot - e)\rangle]\Big| \\
&= |Cov(\langle \xi_{t-s}, p_s\rangle, \langle \xi_{t-s}, p_s(\cdot - e)\rangle)| \\
&\le \sqrt{Var(\langle \xi_{t-s}, p_s\rangle) Var(\langle \xi_{t-s}, p_s(\cdot - e)\rangle)} \\
&\le c\left(\frac{\log(t-s)}{t-s}\right)\langle (p_s)^2, 1\rangle \\
&\le c\left(\frac{\log t}{t}\right)\sum_x p_s(x)^2 \\
&\le \frac{c}{s}\left(\frac{\log t}{t}\right)\sum_x p_s(x) = \frac{c}{s}\left(\frac{\log t}{t}\right) \qquad (4.15)
\end{aligned}$$

where the constant c is changing in the inequalities.

Remark 4.1.9. $\mathbb{E}[\langle \xi_{t-s}, p_s(\cdot - e)\rangle] = \mathbb{E}\left[\sum_x \xi_{t-s}(x) p_s(x-e)\right] = \hat{\xi}_{t-s}\sum_x p_s(x) = \hat{\xi}_{t-s} = \mathbb{E}[\langle \xi_{t-s}, p_s\rangle]$ *so* $\mathbb{E}[\langle \xi_{t-s}, p_s\rangle]\mathbb{E}[\langle \xi_{t-s}, p_s(\cdot - e)\rangle] = \hat{\xi}_{t-s}^2$.

4.1.5 A One Point Estimate

Our next task is approximating $\mathbb{E}[\langle \xi_{t-s}, p_s\rangle] = \hat{\xi}_{t-s}$ so that we can replace it by the corresponding value at time t. For the sake of brevity, let

$$\Gamma\xi_t(x) = \sum_{y:y\sim x} \xi_t(x)\xi_t(y).$$

Now let $\varphi_t(x)$ be a suitably smooth and integrable test function, then we calculate

$$\begin{aligned}
d\langle \xi_t, \varphi_t\rangle &= \langle d\xi_t, \varphi_t\rangle + \langle \xi_t, \dot{\varphi}_t\rangle dt \\
&= \left\langle \Delta\xi_t - \frac{1}{4}\Gamma\xi_t, \varphi_t \right\rangle dt + \langle \xi_t, \dot{\varphi}_t\rangle dt + \text{m.t.} \\
&= \langle \Delta\xi_t, \varphi_t\rangle dt - \frac{1}{4}\langle \Gamma\xi_t, \varphi_t\rangle dt + \langle \xi_t, \dot{\varphi}_t\rangle dt + \text{m.t.}.
\end{aligned}$$

Discrete integration by parts for the Laplacian in one variable (4.8) gives

$$d\langle \xi_t, \varphi_t\rangle = \langle \xi_t, \dot{\varphi}_t + \Delta\varphi_t\rangle dt - \frac{1}{4}\langle \Gamma\xi_t, \varphi_t\rangle dt + \text{m.t.}$$

For $f\colon \mathbb{Z}^2 \to \mathbb{R}$ such that $\langle |f|, 1\rangle < \infty$, choosing $\varphi_s(x) = P_{t-s}f(x) := \sum_y p_{t-s} f(x-y)$ we have for all $r \in [0,t]$, $\dot{\varphi}_r + \Delta\varphi_r = 0$. So

$$d\langle \xi_r, \varphi_r\rangle = -\frac{1}{4}\langle \Gamma\xi_r, \varphi_r\rangle\, dr + \text{m.t.}$$

and taking expectation gives

$$d\mathbb{E}[\langle \xi_r, \varphi_r\rangle] = -\frac{1}{4}\mathbb{E}\left[\langle \Gamma\xi_r, \varphi_r\rangle\right] dr$$

this leads to the following one point estimate.

Lemma 4.1.10. *There exists $c_{10} < \infty$ so that for $s = t/\log^\alpha t$, $t \geq e$ and $f\colon \mathbb{Z}^2 \to \mathbb{R}$ satisfying $\langle |f|, 1\rangle < \infty$*

$$\left|\mathbb{E}[\langle \xi_t, f\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, P_s f\rangle\right]\right| \leq c_{10}\langle |f|, 1\rangle \frac{\log^{1-\alpha} t}{t}.$$

In particular, $|\hat{\xi}_t^2 - \hat{\xi}_{t-s}^2| \leq c_{11}\frac{\log^{2-\alpha} t}{t^2}$.

Proof.

$$\begin{aligned}
\left|\mathbb{E}[\langle \xi_t, f\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, P_s f\rangle\right]\right| &= \left|\mathbb{E}\left[\langle \xi_t, P_0 f\rangle\right] - \mathbb{E}\left[\langle \xi_{t-s}, P_s f\rangle\right]\right| \\
&= \left|\int_{t-s}^t d\mathbb{E}\left[\langle \xi_r, P_{t-r} f\rangle\right]\right| \\
&= \frac{1}{4}\left|\int_{t-s}^t \mathbb{E}\left[\langle \Gamma\xi_r, P_{t-r} f\rangle\right] dr\right| \\
&= \frac{1}{4}\left|\int_{t-s}^t \mathbb{E}\left[\sum_x \sum_{y:y\sim x} \xi_r(x)\xi_r(y) P_{t-r}f(x)\right] dr\right| \\
&= \frac{1}{4}\left|\int_{t-s}^t \mathbb{E}\left[\sum_x \sum_{y:y\sim x} \xi_r(x)\xi_r(y) \sum_z p_{t-r}(z) f(x-z)\right] dr\right| \\
&= \left|\int_{t-s}^t \mathbb{E}[\xi_r(0)\xi_r(e)] \sum_z p_{t-r}(z) \sum_x f(x-z) dr\right| \\
&\leq \int_{t-s}^t \left|\mathbb{E}[\xi_r(0)\xi_r(e)] \sum_z p_{t-r}(z) \sum_x f(x-z)\right| dr
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t-s}^{t} \left| \mathbb{E}[\xi_r(0)\xi_r(e)]\langle f, 1\rangle \sum_z p_{t-r}(z) \right| dr \\
&= \int_{t-s}^{t} \left| \mathbb{E}[\xi_r(0)\xi_r(e)]\langle f, 1\rangle \right| dr \\
&\leq c\langle |f|, 1\rangle \frac{\hat{\xi}_{t-2s}}{\log t} \int_{t-s}^{t} dr \qquad \text{by (4.13)} \\
&\leq c\langle |f|, 1\rangle \frac{\log^{1-\alpha} t}{t}.
\end{aligned}
$$

Note that the constant c changes from line to line. Choose $f = p_0$. Then, $\langle p_0, 1\rangle = \sum_x p_0(x) = \sum_x \mathbb{1}\{x = 0\} = 1$ and

$$
\begin{aligned}
|\hat{\xi}_t - \hat{\xi}_{t-s}| &= \left| \mathbb{E}[\langle \xi_t, p_0\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, p_s\rangle\right] \right| \\
&= \left| \mathbb{E}[\langle \xi_t, p_0\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, P_s p_0\rangle\right] \right| \\
&\leq c_{10} \frac{\log^{1-\alpha} t}{t}.
\end{aligned}
$$

This gives (using Lemma 4.1.2) the estimate,

$$
\begin{aligned}
|\hat{\xi}_t^{\,2} - \hat{\xi}_{t-s}^2| &= |\hat{\xi}_t - \hat{\xi}_{t-s}||\hat{\xi}_t + \hat{\xi}_{t-s}| \\
&\leq c \frac{\log^{1-\alpha} t}{t} \left(\frac{\log t}{t} + \frac{\log(t-s)}{t-s} \right) \\
&\leq c_{11} \frac{\log^{2-\alpha} t}{t^2}
\end{aligned}
\tag{4.16}
$$

for $s \leq t/2$, where the constant changes again from line to line. □

4.1.6 Proof of the Theorem

Bringing all of the estimates together we have the following rehashing of the heuristic albeit now all true equalities keeping track of the errors

$$\begin{aligned}
\frac{d\hat{\xi}_t}{dt} &= -\mathbb{E}[\xi_t(0)\xi_t(e)] \\
&= -\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \varphi_{t-s}\rangle] + \mathscr{E}_t^{(1)} \qquad \text{by Lemma 4.1.6} \\
&= -\frac{\pi}{\log s}\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s(\cdot - e)\rangle] + \mathscr{E}_t^{(2)} \qquad \text{by Lemma 4.1.7} \\
&= -\frac{\pi}{\log s}\mathbb{E}\left[\langle \xi_{t-s}, p_s\rangle\langle \xi_{t-s}, p_s(\cdot - e)\rangle\right] + \mathscr{E}_t^{(2)} \\
&= -\frac{\pi}{\log s}\mathbb{E}[\langle \xi_{t-s}, p_s\rangle]^2 + \mathscr{E}_t^{(3)} \qquad \text{by Lemma 4.1.8, (4.15) and the following remark} \\
&= -\frac{\pi}{\log s}\hat{\xi}_{t-s}^2 + \mathscr{E}_t^{(3)} \qquad \text{again by the remark} \\
&= -\frac{\pi}{\log s}\hat{\xi}_t^2 + \mathscr{E}_t^{(4)} \qquad \text{by (4.16)}
\end{aligned}$$

we have noted carefully in each step where there was a genuine contribution to the error by updating the index. The final error satisfies

$$|\mathscr{E}_t^{(4)}| \le \frac{C}{t^2}\left(\log^{1-\alpha} t + \log^{1-\zeta} t + \log^{\alpha} t\right).$$

Since the *a priori* estimates tell us to expect the leading order of our approximation for $d\hat{\xi}_t/dt$ to be

$$\frac{1}{\log s}\left(\frac{\log t}{t}\right)^2 = \frac{1}{\log t - \alpha \log\log t}\left(\frac{\log t}{t}\right)^2 \tag{4.17}$$

the second of the error terms is immediately of small enough order so as not to contribute to the derivative at large times. The first error is of small enough order for any $\alpha > 0$, while the final error term requires us to take $\alpha < 1$.

We now solve the approximate differential equation.

$$\begin{aligned}
\frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log s}\hat{\xi}_t^2 + \mathscr{E}_t^{(4)} \\
\hat{\xi}_t^{-2}\frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log s} + \hat{\xi}_t^{-2}\mathscr{E}_t^{(4)} \\
\hat{\xi}_t^{-2}\frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log t}\frac{\log t}{\log t - \alpha \log\log t} + \hat{\xi}_t^{-2}\mathscr{E}_t^{(4)} \\
\hat{\xi}_t^{-2}\frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log t} - \frac{\pi\alpha \log\log t}{(\log t)(\log t - \alpha\log\log t)} + \mathscr{E}_t^{(5)}
\end{aligned}$$

where now

$$|\mathscr{E}_t^{(5)}| \leq C\left(\log^{-(1+\alpha)} t + \log^{-(1+\zeta)} t + \log^{\alpha-2} t\right)$$

In Lemma 4.1.4, we can choose $\zeta = 1 - \delta \geq 1/2$, also choosing $\alpha = 1/2$ gives the further bound of

$$|\mathscr{E}_t^{(5)}| \leq C(\log t)^{-3/2}.$$

Returning to our equation and integrating over t larger than a suitable t_0

$$\begin{aligned}
-\hat{\xi}_t^{-1} + \hat{\xi}_{t_0}^{-1} = -\pi & \left(\underbrace{\int_{t_0}^{t} \frac{1}{logs} ds}_{I} + \frac{1}{2} \underbrace{\int_{t_0}^{t} \frac{\log\log s}{(\log s)(\log s - \frac{1}{2}\log\log s)} ds}_{II} \right) \\
& + O\left(\underbrace{\int_{t_0}^{t} \frac{1}{(\log s)^{3/2}} ds}_{III} \right)
\end{aligned}$$

We will apply the following trivial lemma a number of times.

Lemma 4.1.11. *Let f be an increasing, continuous function on some interval $[t_0, \infty]$ satisfying $f(t) \geq 1$. Suppose $f(t) = O(\log\log t)$. Then, for all $\beta > 1$,* *there exists* $t_0 > e^{e^{1/\beta}}$ *such that*

$$\int_{t_0}^{t} \frac{f(s)}{(\log s)^\beta} ds = O\left(\frac{tf(t)}{(\log t)^\beta}\right).$$

Proof. Fix $\beta > 1$. Since f is positive and of order $\log\log t$, there is a t_0 such that $f(t)/(\log t)^\beta$ is decreasing for $t > t_0$. Also (taking t_0 larger if necessary), for all $t > t_0$, $\sqrt{t} \geq (\log t)^\beta$ since $\sqrt{t}$ eventually grows quicker than any power of a logarithm. Take such t_0, then for $t > t_0^2$

$$\begin{aligned}
\int_{t_0}^{t} \frac{f(s)}{(\log s)^\beta} ds &= \int_{\sqrt{t}}^{t} \frac{f(s)}{(\log s)^\beta} ds + \int_{t_0}^{\sqrt{t}} \frac{f(s)}{(\log s)^\beta} ds \\
&\leq \frac{(t-\sqrt{t})f(\sqrt{t})}{(\log \sqrt{t})^\beta} + \frac{(\sqrt{t}-t_0)f(t_0)}{(\log t_0)^\beta} \qquad \text{since } f(t) = O(\log\log t) \\
&\leq \frac{2^\beta t f(t)}{(\log t)^\beta} + \frac{\sqrt{t} f(t_0)}{(\log t_0)^\beta} \qquad \text{since } f(t) \text{ is increasing} \\
&\leq C\left(\frac{tf(t)}{(\log t)^\beta} + \sqrt{t}\right) \\
&= C\left(\frac{tf(t)}{(\log t)^\beta} + \frac{t}{\sqrt{t}}\right) \\
&\leq C\left(\frac{tf(t)}{(\log t)^\beta} + \frac{t}{(\log t)^\beta}\right) \\
&\leq 2C\left(\frac{tf(t)}{(\log t)^\beta}\right) \qquad \text{since } f(t) \geq 1,
\end{aligned}$$

that is

$$\int_{t_0}^{t} \frac{f(s)}{(\log s)^\beta} ds = O\left(\frac{tf(t)}{(\log t)^\beta}\right).$$

□

Let's take I,

$$\begin{aligned}
I = \int_{t_0}^{t} \frac{1}{\log s} ds &= \frac{t}{\log t} - \frac{t_0}{\log t_0} + \int_{t_0}^{t} \frac{1}{(\log s)^2} ds \\
&= \frac{t}{\log t} - \frac{t_0}{\log t_0} + O\left(\frac{t}{(\log t)^2}\right)
\end{aligned}$$

where we have used Lemma 4.1.11 and taken $f(t) = 1$. For II

$$\begin{aligned} II &= \int_{t_0}^{t} \frac{\log\log s}{(\log s)(\log s - \frac{1}{2}\log\log s)} ds \\ &\leq 2\int_{t_0}^{t} \frac{\log\log s}{(\log s)^2} ds \\ &= O\left(\frac{t\log\log t}{(\log t)^2}\right) \end{aligned}$$

again by Lemma 4.1.11, with $f(t) = \log\log t$. Finally, for III

$$III = \int_{t_0}^{t} \frac{1}{(\log s)^{3/2}} ds = O\left(\frac{t}{(\log t)^{3/2}}\right)$$

similarly with $f(t) = 1$ once again. Plugging all this into our equation we get

$$\begin{aligned} -\hat{\xi}_t^{-1} + \hat{\xi}_{t_0}^{-1} &= -\frac{\pi t}{\log t} + \frac{\pi t_0}{\log t_0} + O\left(\frac{t}{(\log t)^{3/2}}\right) \\ \frac{\pi t}{\log t} &= \hat{\xi}_t^{-1} + O\left(\frac{t}{(\log t)^{3/2}}\right) \\ \hat{\xi}_t &= \frac{\log t}{\pi t} + O\left(\frac{\hat{\xi}_t}{\log^{1/2} t}\right) \\ \hat{\xi}_t &= \frac{\log t}{\pi t} + O\left(\frac{\log^{1/2} t}{t}\right) \end{aligned}$$

and this proves Theorem 4.1.1.

4.2 Non-Instantaneously Coalescing Particles in $d = 2$

Suppose that we have independent and identically distributed continuous time rate 1 random walks describing the trajectories of particles in $\mathbb{Z}^2$, and further that these particles coalesce at rate λ. Let $\xi_t(x)$ be the number of particles occupying x at time t with initial state $\xi_0 \equiv 1$. The equation that governs the occupation number of x is

$$d\xi_t(x) = \sum_{y:y\sim x}\sum_{i\geq 1}\Big(\mathbb{1}\{\xi_{t-}(y)\geq i\}dP_t(i,y,x) - \mathbb{1}\{\xi_{t-}(x)\geq i\}dP_t(i,x,y)\Big) \qquad (4.18)$$
$$- \sum_{i,j\geq 1}\mathbb{1}\Big(\xi_{t-}(x)\geq i\vee j, i\neq j\Big)dP^c(i,j,x).$$

The existence and uniqueness of a solution to the equation (4.18) is guaranteed by Theorem 2.2.3. Compensating gives

$$d\xi_t(x) = \Delta\xi_t(x)dt - \lambda\xi_t(x)(\xi_t(x)-1)dt + \text{m.t.} \qquad (4.19)$$

and taking expectation and exploiting translation invariance gives us the following equality for the derivative of $\hat{\xi}_t = \mathbb{E}[\xi_t(0)]$,

$$\frac{d\hat{\xi}_t}{dt} = -\lambda\mathbb{E}[\xi_t(0)(\xi_t(0)-1)]. \qquad (4.20)$$

4.2.1 A Priori Bounds

As with the instantaneously coalescing random walks, we need an *a priori* estimate for the first moment.

Lemma 4.2.1. *Suppose that ξ is a solution to* (1.6) *in $d = 2$ with initial condition $\xi_0 \equiv 1$. Then there exist constants $0 < c_1 < c_2 < \infty$ and $t_0 \geq e$, depending on λ so that for any $x \in \mathbb{Z}^2$*

$$c_1\frac{\log t}{t} \leq E[\xi_t(x)] < c_2\frac{\log t}{t} \qquad \textit{for all } t > t_0.$$

Proof. The lower bound follows immediately from Lemma 2.3.1 and [13]. Fix box $\Lambda = \Lambda_R$ of side R. Let V be the unique collection of centres that enable us to tile $\mathbb{Z}^2$ by translates of Λ. That is $\mathbb{Z}^2 = \bigcup_{v\in V}\{\Lambda + v\}$. Let $e_t(\Lambda) = \sum_{x\in\Lambda}\mathbb{E}[\xi_t(x)]$. By translation invariance $e_t(\Lambda) = \widehat{\xi}_t|\Lambda|$. Also, for $s < t$,

$$|\Lambda| = \frac{e_t(\Lambda)}{\widehat{\xi}_t} = \frac{e_s(\Lambda)}{\widehat{\xi}_s}$$

or equivalently

$$\widehat{\xi}_t = \widehat{\xi}_s \left(1 - \frac{e_s(\Lambda) - e_t(\Lambda)}{e_s(\Lambda)}\right).$$

Note that (in the notation given in Proposition 2.3.2) $\xi_t = \xi_t^{\mathbb{Z}^2}$ so that by the tiling of $\mathbb{Z}^2$ by copies of Λ centred at the sites of V and Proposition 2.3.2 and Markov property, writing ξ_s for the state of the entire system at time s and conditioning on that so as to treat s as the origin of the timeline

$$\begin{aligned}
e_t(\Lambda) &= \sum_{x\in\Lambda} \mathbb{E}[\xi_t(x)] \\
&= \sum_{x\in\Lambda} \mathbb{E}[\xi_{t-s}^{\xi_s}(x)] \\
&\leq \sum_{x\in\Lambda} \sum_{v\in V} \mathbb{E}[\xi_{t-s}^{\xi_s\cap\{\Lambda+v\}}(x)] \\
&= \sum_{x\in\Lambda} \sum_{v\in V} \mathbb{E}[\xi_{t-s}^{\xi_s\cap\Lambda}(x-v)] \\
&= \sum_{x\in\mathbb{Z}^2} \mathbb{E}[\xi_{t-s}^{\xi_s\cap\Lambda}(x)].
\end{aligned}$$

The inequality is due to the coupling result in Lemma 2.3.2 since we have decomposed $\mathbb{Z}^d$ into translates of Λ. Let

$$\Delta_{s,t}(\Lambda) = e_s(\Lambda) - \sum_{x\in\mathbb{Z}^2} \mathbb{E}[\xi_{t-s}^{\xi_s\cap\Lambda}(x)]$$

We interupt the proof briefly to provide a result to help us bound $\Delta_{s,t}$.

Lemma 4.2.2. *For any* $\Lambda \subset \mathbb{Z}^2$

$$\sum_{x\in\Lambda} \xi_0(x) - \sum_{x\in\mathbb{Z}^2} \mathbb{E}[\xi_s^{\Lambda}(x)] \geq \frac{\lambda}{\lambda+1}(|\Lambda| - 1) \min_{x,y\in\Lambda} H_{2(s-1)}(x-y)$$

where $H_s(x)$ *is the probability that the first hitting time of the origin of a random* *walk started at x is before time s.*

Proof. We choose to employ a different construction for our process than have used thus far. We will construct the process inductively by adding in particles one by one. Firstly, run a random walk that will describe the trajectory of particle that will be numbered one. This particle will never be lost to the system. Run a second, independently identically distributed, random walk that will give the trajectory of particle 2 up until it hits particle one and reacts with it. After it reacts with particle 1, the trajectory that particle 2 would have followed is deleted and we'll interpret it that particle 2 has coalesced with particle 1 and follows its path thereafter. Now run a third IID random walk until the particle on its trajectory meets and reacts with either of the paths of the first or second particle. And we continue this inductively. We will not prove that this construction is equivalent to the one that is described by the differential equations but we note that the equivalence is clear for any configuration starting from finitely many particles. Now,

$$\begin{aligned}
&\sum_{x\in\Lambda}\xi_0(x) - \sum_{x\in\mathbb{Z}^2}\mathbb{E}[\xi_s^\Lambda(x)] \\
&= \mathbb{E}\left[\sum_{i=1}^{|\Lambda|}\mathbb{1}\{\text{particle } i \text{ is killed by time } s\}\right] \\
&= \sum_{i=2}^{|\Lambda|}\mathbb{P}\left[\text{particle } i \text{ is killed by time } s\right] \\
&= \sum_{i=2}^{|\Lambda|}\mathbb{P}\left[\text{particle } i \text{ reacts with any particle in } \{1,2,\dots,i-1\} \text{ by time } s\right] \\
&\geq \sum_{i=2}^{|\Lambda|}\mathbb{P}\left[\text{particle } i \text{ reacts with particle 1 by time } s\right].
\end{aligned}$$

We now decompose this event according to the time that particle i meets particle 1.

$$\begin{aligned}
&\sum_{x\in\Lambda}\xi_0(x) - \sum_{x\in\mathbb{Z}^2}\mathbb{E}[\xi_s^\Lambda(x)] \\
&\geq \sum_{i=2}^{|\Lambda|}\int_0^s \mathbb{P}\left[\text{particle } i \text{ meets particle 1 by time } r\right]\mathbb{P}\left[\text{particles react by time } s-r\right]dr \\
&\geq \sum_{i=2}^{|\Lambda|}\int_0^{s-1} \mathbb{P}\left[\text{particle } i \text{ meets particle 1 by time } r\right]\mathbb{P}\left[\text{particles react by time } s-r\right]dr \\
&\geq \mathbb{P}\left[\text{particles react by time } 1\right]\sum_{i=2}^{|\Lambda|}\int_0^{s-1} \mathbb{P}\left[\text{particle } i \text{ meets particle 1 by time } r\right]dr \\
&= \frac{\lambda}{\lambda+1}\sum_{i=2}^{|\Lambda|}\int_0^{s-1} \mathbb{P}\left[\text{particle } i \text{ meets particle 1 by time } r\right]dr \\
&\geq \frac{\lambda}{\lambda+1}\sum_{i=2}^{|\Lambda|}\min_{x,y\in\Lambda} H_{2(s-1)}(x-y) \\
&= \frac{\lambda}{\lambda+1}(|\Lambda|-1)\min_{x,y\in\Lambda} H_{2(s-1)}(x-y).
\end{aligned}$$

□

Conclusion of the proof of Lemma 4.2.1 The rest of the proof of Lemma 4.1.2 follows in exactly the same manner as the proof of Theorem 1 in [2]. □

4.2.2 A One Point Estimate

We turn first to a one point estimate that allows us to control the error in winding an instance of $\xi_t(x)$ back to time $t-s$. As was the case for the instantaneously coalescing particles, it will be necessary for us to develop a two point function using calculus and rely on our random walk estimates from Chapter 5 to introduce an extra logarithm, without which the estimate is not strong enough. This is an improvement on the bound that would be achieved by negative correlation alone.

To begin the one point estimates, we must develop using calculus $\langle \xi_r, \varphi_r \rangle$ for a test function $\varphi\colon [0,\infty]\times\mathbb{Z}^2 \to \mathbb{R}$ satisfying $\sum_x |\varphi_r(x)| + |\dot{\varphi}_r(x)| < \infty$. In particular, this implies $\sum_x \Delta\varphi_r(x) < 0$. By first truncating the sum $\langle \xi_r, \varphi_r \rangle$ at

$|x| \le N$ for fixed N, and writing this as $\langle \xi_r, \varphi_r \rangle_N$ we have

$$
\begin{aligned}
\mathbb{E}\left[\langle \xi_r, |\varphi_r| + |\dot{\varphi}_r| \rangle_N\right] &= \sum_{|x| \le N} \mathbb{E}[\xi_r(x)] \left(|\varphi_r(x)| + |\dot{\varphi}_r(x)|\right) \\
&= \hat{\xi}_t \sum_{|x| \le N} \left(|\varphi_r(x)| + |\dot{\varphi}_r(x)|\right) \\
&\le \hat{\xi}_t \sum_{x} \left(|\varphi_r(x)| + |\dot{\varphi}_r(x)|\right) < \infty
\end{aligned} \tag{4.21}
$$

so we may take $N \to \infty$ and in particular $\langle \xi_r, \varphi_r \rangle$ converges almost surely. Differentiating this we see that

$$
\begin{aligned}
d\langle \xi_r, \varphi_r \rangle &= d \sum_x \xi_r(x) \varphi_r(x) = \langle d\xi_r, \varphi_r \rangle + \langle \xi_r, \dot{\varphi}_r \rangle \\
&= \langle \Delta \xi_r dr - \lambda \xi_r(\xi_r - 1) dr + dM_r, \varphi \rangle + \langle \xi_r, \dot{\varphi}_r \rangle
\end{aligned}
$$

where M_r are the m.t. from (4.19) that are true martingales due to the moment conditions for our solution to (2.1). The second sum in the right hand side is well defined and this follows from (4.21). Using discrete integration by parts (see (4.8)) on the term $\langle \Delta \xi_r(\cdot) dr, \varphi_r \rangle$ gives

$$
d\langle \xi_r, \varphi_r \rangle = \langle \xi_r, \dot{\varphi}_r + \Delta \varphi_r \rangle dr - \lambda \langle \xi_r(\xi_r - 1), \varphi_r \rangle dr + \langle dM_r, \varphi_r \rangle.
$$

For $f\colon \mathbb{Z}^2 \to \mathbb{R}$ satisfying $\sum_x |f(x)|$, choosing $\varphi_s(x) = P_{t-s} f(x) := \sum_y p_{t-s}(y) f(x - y)$ we have for all $r \in [0, t]$ and $x \in \mathbb{Z}^2$

$$
\sum_x |\varphi_r(x)| \le \sum_x |f(x)| < \infty
$$

and, $\dot{\varphi}_r + \Delta \varphi_r = 0$. Furthermore $\langle dM_r, \varphi \rangle$ is a martingale by the integrability of φ and the moment conditions imposed on ξ_r, so

$$
d\langle \xi_r, \varphi_r \rangle = -\lambda \left\langle \xi_r(\xi_r - 1), \varphi_r \right\rangle dr + \langle dM_r, \varphi \rangle
$$

and taking expectation gives

$$d\mathbb{E}[\langle \xi_r, \varphi_r \rangle] = -\lambda \mathbb{E}\left[\langle \xi_r(\xi_r - 1), \varphi_r \rangle\right] dr.$$

The idea is that by estimating the right hand side, we will be able to estimate the errors in approximations such as

$$|\mathbb{E}[\langle \xi_t, \varphi_t \rangle] - \mathbb{E}[\langle \xi_{t-s}, \varphi_{t-s} \rangle]| = \lambda \int_{t-s}^{t} \mathbb{E}\left[\langle \xi_r(\xi_r - 1), \varphi_r \rangle\right] dr. \tag{4.22}$$

Negative correlation results give us an immediate bound of

$$\lambda \int_{t-s}^{t} \hat{\xi}_r^2 \sum_x |f(x)| dr \leq \lambda \langle |f|, 1 \rangle s \hat{\xi}_{t-s}^2,$$

since by (4.20), $\hat{\xi}_r$ is decreasing. This will not be a strong enough bound for our purposes. In order to strengthen it, we will have to look ahead at a two point estimate to improve the bound on $\mathbb{E}\left[\xi_r(0)(\xi_r(0) - 1)\right]$ by a logarithmic correction.

Let

$$\psi_t(x, y) = \mathbb{P}[x + S_t^1 = 0, y + S_t^2 = 0, NC_{2\lambda}[0, t]] \tag{4.23}$$

where $NC_\lambda[0, t]$ is the probability that a random walk start at the origin doesn't spend more that an exponential time rate λ at the origin on each of its visits there up until time t. In explanation as for why this is the right test function to study, notice the event that a particle walks from x to be occupying the origin at time t and another from y to occupying the origin at time t without two particles coalescing is the same event as for two particles starting at the origin, one occupying x at time t while the other occupies y at time t without the particles coalescing. Now, since each particle coalesces with another at rate λ, each pair of particles coalesce at rate 2λ. In order not to coalesce, their paths must not coincide at a site for longer than an exponential time rate 2λ. This is equivalent to the statement that the difference of their random walk paths (each of rate 1), which is itself a random walk (of rate 2), must not rest for more than an exponential time rate 2λ at the origin. Now for $s \leq t/4$ and $r \in [t - s, t]$ define for $u \in [0, r]$ $\kappa_u = \psi_{r-u}$ and similarly let $\bar{\kappa}_u(x) = \psi_{r-u}(x,x)$. Observe, formally for now, that

$$\mathbb{E}\left[\xi_r(0)(\xi_r(0)-1)\right] = \mathbb{E}[\langle \xi_r * \xi_r, \kappa_r\rangle - \langle \xi_r, \bar{\kappa}_r\rangle] \tag{4.24}$$

since $\kappa_r(x,y) = \mathbb{1}\{x=y=0\}$ and $\bar{\kappa}_r(x) = \mathbb{1}\{x=0\}$. We will demonstrate that the right hand side is decreasing and hence

$$\begin{aligned}
|\mathbb{E}\left[\xi_r(0)(\xi_r(0)-1)\right]| &\leq |\mathbb{E}[\langle \xi_{r-(r-t+2s)} * \xi_{r-(r-t+2s)}, \kappa_{r-(r-t+2s)}\rangle \\
&\qquad - \langle \xi_{r-(r-t+2s)}, \bar{\kappa}_{r-(r-t+2s)}\rangle]| \\
&= |\mathbb{E}[\langle \xi_{t-2s} * \xi_{t-2s}, \psi_{r-t+2s}\rangle - \langle \xi_{t-2s}, \bar{\psi}_{r-t+2s}\rangle]| \\
&= \Bigg| \sum_{x,y} \mathbb{E}[\xi_{t-2s}(x)\xi_{t-2s}(y)]\psi_{r-t+2s}(x,y) \\
&\qquad - \sum_{x} \mathbb{E}[\xi_{t-2s}(x)]\bar{\psi}_{r-t+2s}(x) \Bigg| \\
&= \sum_{x} \sum_{y: y\neq x} \mathbb{E}[\xi_{t-2s}(x)\xi_{t-2s}(y)]\psi_{r-t+2s}(x,y) \\
&\qquad + \sum_{x} \mathbb{E}[\xi_{t-2s}(x)(\xi_{t-2s}(x)-1)]\bar{\psi}_{r-t+2s}(x).
\end{aligned}$$

Now we use negative correlation on each of the above expectations to bound this by

$$\hat{\xi}^2_{t-2s} \sum_{x,y} \psi_{r-t+2s}(x,y)$$

We can bound this further by the following lemma which is proved in Section 5.3 of Chapter 5.

Lemma 4.2.3. *Let ψ be given by (4.23). Then there exists $C > 0$ and $\zeta > 0$ depending on λ such that*

$$\sum_x \sum_y \left| \psi_t(x,y) - \frac{\pi}{\lambda \log t} p_t(x) p_t(y) \right| \leq \frac{C}{\log^{1+\zeta} t}.$$

Now we may bound

$$|\mathbb{E}\left[\xi_r(0)(\xi_r(0)-1)\right]| \leq C\frac{\hat{\xi}_{t-2s}^2}{\log\left(r-t+2s\right)}.$$

Using this bound to estimate the right hand side of (4.22) give the improved bound of

$$C\langle |f|, 1\rangle \frac{s\hat{\xi}_{t-2s}^2}{\log s}. \tag{4.25}$$

Later, we will make this more explicit by choosing an appropriate value for s. The extra logarithm gained in (4.25) is crucial for the estimation of the errors.

To show that (4.24) is a decreasing quantity, we will have to develop

$$\mathbb{E}\left[\xi_t(x)(\xi_t(x)-1)\right]$$

by calculus. This calculation will be of further use to us later. It will enable us to bound the error in the first approximation in the heuristic argument given in the introduction.

The integration by parts formula (4.7) and the substitution of equation (4.19) gives

$$\begin{aligned} d(\xi_t(x)\xi_t(y)) &= \Delta(\xi_t(x)\xi_t(y))dt - \lambda\xi_t(x)\xi_t(y)(\xi_t(x)+\xi_t(y)-2)dt \\ &\quad + \xi_{t-}(x)dM_t(y) + \xi_{t-}(y)dM_t(x) + d[\xi(x),\xi(y)]_t \end{aligned} \tag{4.26}$$

where the Laplacian Δ now acts in both variables. The final term represents the jumps and is defined in (4.5). There are unique continuous adapted compensators $\langle\!\langle \xi(x), \xi(y)\rangle\!\rangle_t$ satisfying

$$[\xi(x),\xi(y)]_t = \langle\!\langle \xi(x),\xi(y)\rangle\!\rangle_t + \text{m.t.}.$$

We can find these compensators by studying $d[\xi(x),\xi(y)]_t$ with some first order bookkeeping. Firstly, notice that if $|x-y|>1$ then there can be no contribution to

$d[\xi(x), \xi(y)]_t$. Next, consider the case $x = y$, then $[\xi(x), \xi(y)]_t$ reduces to

$$[\xi_t(x), \xi_t(y)] = \sum_{s \le t} (\xi_t(x) - \xi_{t-}(x))^2.$$

If any of the partcles at x at time $t-$ coalesces (at rate λ) with any of the other particles also present, or walks to a neighbour, $\xi_t(x) - \xi_{t-}(x) = -1$ since there can be no other change to the system. The only other possibility is that any particle at y at time $t-$ can walk to x and increase its particle number 1. So, in the case $x = y$ we have

$$\begin{aligned} d[\xi_t(x), \xi_t(y)] = &\sum_{y \sim x} \sum_{i=1}^{\infty} (-1)^2 \mathbb{1}\{\xi_{t-}(x) \ge i\} dP_t(i, x, y) \\ &+ \sum_{y \sim x} \sum_{i=1}^{\infty} \mathbb{1}\{\xi_{t-}(y) \ge i\} dP_t(i, y, x) \\ &+ \sum_{i,j=1}^{\infty} (-1)^2 \mathbb{1}\{\xi_{t-}(x) \ge i \vee j, i \ne j\} dP_t^c(i, j, x). \end{aligned}$$

If $y \sim x$, then there will be loss of a particle at x and a gain at y, if any particle at x at time $t-$ walks to y and an analgous thing can be said for a gain of a particle at x with a loss of one at y. There can be no other change. This gives, for $y \sim x$

$$d[\xi_t(x), \xi_t(y)] = \sum_{y \sim x} \sum_{i=1}^{\infty} (-1) \mathbb{1}\{\xi_{t-}(x) \ge i\} dP_t(i, x, y) + (-1) \mathbb{1}\{\xi_{t-}(y) \ge i\} dP_t(i, y, x).$$

Compensating the Poisson processes we find the compensator of $d[\xi_t(x), \xi_t(y)]$ is of the form

$$d\langle\langle \xi(x), \xi(y) \rangle\rangle_t = \begin{cases} \frac{1}{4} \sum_{y: y \sim x} (\xi_t(y) + \xi_t(x)) dt + \lambda \xi_t(x)(\xi_t(x) - 1)\, dt & \text{if } x = y, \\ -\frac{1}{4} \left(\xi_t(x) + \xi_t(y)\right)\, dt & \text{if } x \sim y. \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

Take a test function $\varphi_t(x, y)$ that satifies $\sum_{x,y} |\varphi(x, y)| + |\dot{\varphi}(x, y)| < \infty$ and write $\langle \xi_t * \xi_t, \varphi_t \rangle$ for the sum $\sum_x \sum_y \xi_t(x) \xi_t(y) \varphi_t(x, y)$. Truncating the sums at $|x|, |y| \le$

N, and taking expection

$$
\begin{aligned}
\mathbb{E}[\langle \xi_t * \xi_t, |\varphi_t| \rangle_N] &= \sum_{|x|,|y| \le N} \mathbb{E}[\xi_t(x)\xi_t(y)] |\varphi_t(x,y)| \\
&= \sum_{|x| \le N} \sum_{|y| \le N, y \ne x} \mathbb{E}[\xi_t(x)\xi_t(y)] |\varphi_t(x,y)| \\
&\quad + \sum_{|x| \le N} \mathbb{E}[\xi_t^2(x)] |\varphi_t(x,x)| \\
&\le \sum_{|x| \le N} \sum_{|y| \le N, y \ne x} \mathbb{E}[\xi_t(x)] \mathbb{E}[\xi_t(y)] |\varphi_t(x,y)| \\
&\quad + \mathbb{E}[\xi_t(0)(\xi_t(0) - 1)] \sum_{|x| \le N} |\varphi_t(x,x)| \\
&\quad + \hat{\xi}_t \sum_{|x| \le N} |\varphi_t(x,x)| \\
&= \hat{\xi}_t^2 \sum_{|x|,|y| \le N} |\varphi_t(x,y)| \\
&\quad + \hat{\xi}_t \sum_{|x| \le N} |\varphi_t(x,x)| \\
&\le 2\hat{\xi}_t \sum_{x,y} |\varphi_t(x,y)| < \infty. \qquad (4.28)
\end{aligned}
$$

So by the bounded convergence theorem, the left shand side converges almost surely. Differentiating,

$$
\begin{aligned}
d\langle \xi_t * \xi_t, \varphi_t \rangle &= d \sum_{x,y} \xi_r(x)\xi_r(y)\varphi_r(x,y) \\
&= \sum_{x,y} d\left(\xi_r(x)\xi_r(y)\right) \varphi_r(x,y) + \langle \xi_t * \xi_t, \dot{\varphi}_t \rangle
\end{aligned}
$$

where the final term is well defined by the regularity and integrability of φ which can be seen by repeating the calculation (4.28) for $\dot{\varphi}$ in place of φ. Substituting

(4.26), (4.19) and (4.27) gives

$$\begin{aligned}d\langle \xi_t * \xi_t, \varphi_t\rangle = \langle \Delta\xi_r * \xi_r, \varphi\rangle dt - \lambda \sum_{x,y} \xi_t(x)\xi_t(y)(\xi_t(x) + \xi_t(y) - 2)\varphi_t(x,y)dt \\ + \sum_x \left(\frac{1}{4} \sum_{y:y\sim x} (\xi_t(y) + \xi_t(x)) + \lambda\xi_t(\xi_t(x) - 1) \right) \varphi_t(x,x)dt \\ - \frac{1}{4} \sum_x \sum_{y:y\sim x} (\xi_t(x) + \xi_t(y))\varphi_t(x,y)dt \\ + \langle \xi_t * \xi_t, \dot{\varphi}_t\rangle + \text{m.t.}. \end{aligned} \tag{4.29}$$

Applying (4.9) to the first term of the right hand side of (4.29)

$$\begin{aligned}d\langle \xi_t * \xi_t, \varphi_t\rangle = \langle \xi_t * \xi_t, \dot{\varphi}_t + \Delta\varphi_t\rangle dt \\ - \lambda \sum_{x,y} \xi_t(x)\xi_t(y)(\xi_t(x) + \xi_t(y) - 2)\varphi_t(x,y)dt \\ + \frac{1}{4} \sum_x \sum_{y:y\sim x} (\xi_t(y) + \xi_t(x))\varphi_t(x,x) + \lambda \sum_x \xi_t(\xi_t(x) - 1)\varphi_t(x,x)dt \\ - \frac{1}{4} \sum_x \sum_{y:y\sim x} (\xi_t(x) + \xi_t(y))\varphi_t(x,y)dt + \text{m.t.}. \end{aligned} \tag{4.30}$$

We split the sum in the second term down the diagonal and collect the sum along the diagonal with the sum in the fourth term. We also collect the third and fifth terms as follows.

$$\begin{aligned}&\frac{1}{4} \sum_x \sum_{y\sim x} (\xi_t(x) + \xi_t(y))(\varphi_t(x,x) - \varphi_t(x,y)) \\ &= \frac{1}{4} \sum_x \sum_{y\sim x} \xi_t(x)(\varphi_t(x,x) - \varphi_t(x,y)) + \frac{1}{4} \sum_x \sum_{y\sim x} \xi_t(y)(\varphi_t(x,x) - \varphi_t(x,y)).\end{aligned}$$

For the second term here we will use the earlier trick of exchanging the order of the sums which leads us to

$$\begin{aligned}
&\sum_x \sum_{y\sim x} (\xi_t(x)+\xi_t(y))(\varphi_t(x,x)-\varphi_t(x,y)) \\
&\quad = \sum_x \sum_{y\sim x} \xi_t(x)(\varphi_t(x,x)+\varphi_t(y,y)-\varphi_t(x,y)-\varphi_t(y,x)) \\
&\quad = 4\langle \xi_t, \Box\varphi_t\rangle
\end{aligned}$$

where we define

$$\Box\varphi_t(x) = \frac{1}{4}\sum_{y:y\sim x} \left(\varphi_t(x,x)+\varphi_t(y,y)-\varphi_t(x,y)-\varphi_t(y,x)\right).$$

Bring all of this together and returning to equation (4.30)

$$\begin{aligned}
d\langle \xi_t * \xi_t, \varphi_t\rangle &= \langle \xi_t * \xi_t, \dot\varphi_t + \Delta\varphi_t\rangle dt \\
&\quad - \lambda \sum_x \sum_{y\neq x} \xi_t(x)\xi_t(y)(\xi_t(x)+\xi_t(y)-2)\varphi_t(x,y)dt \\
&\quad - \lambda \sum_x \left(2\xi_t^3(x) - 3\xi_t^2(x) + \xi_t(x)\right)\varphi_t(x,x)dt \\
&\quad + \langle \xi_t, \Box\varphi_t\rangle dt + \text{m.t.}. \qquad (4.31)
\end{aligned}$$

Recall the definition of ψ given in (4.23). Then ψ satisfies

$$\dot\psi_t(x,y) = \Delta\psi_t(x,y) - 2\lambda\psi_t(x,x)$$

since if at time $t-$ all the subevents that make up the event in the probability of ψ is satisfied except $\{x+S^1_{t-}=0\}$ which fails because $\{x+S^1_{t-}=e\}$ is true instead for some neighbour of the origin e, then the event in ψ is realised at time t only if there is a random walk step from e to the origin. This happens at rate 1. Translating the walk path S^1 by e and summing over the neighbours, introduces terms of the form $\psi_t(x+e,y)$. If the even has been realised by time $t-$ but the random walk path $x+S^1_{t-}$ jumps to a neighbour, this will introduce a copy of $-\psi_t(x,y)$ for each neighbour of x. Similar considerations for the walk path of S^2_t. This accounts for all of the terms that make up the Laplacian. The only further case not accounted for is is the event is realised at time $t-$ but that there is a coalescing event by time t. Since the coalescence is non-instant and independent exponential clocks cannot ring simultaneously, this can only occur on the event that $x = y$. Then choosing $\varphi_s = \psi_{t-s}$ over $s \in [0, t]$ we have that

$$\dot{\varphi} + \Delta\varphi = 2\lambda\varphi I(x = y), \quad \text{and } \varphi_t(x, y) = I(x = y = 0). \tag{4.32}$$

Using this test function in (4.31) we find, for $s \in [0, t]$,

$$\begin{aligned}
d\langle \xi_t * \xi_t, \varphi_t \rangle &= \lambda \sum_x 2\xi_t^2(x)\varphi_t(x, x)dt \\
&\quad - \lambda \sum_x \sum_{y \neq x} \xi_t(x)\xi_t(y)(\xi_t(x) + \xi_t(y) - 2)\varphi_t(x, y)dt \\
&\quad - \lambda \sum_x \left(2\xi_t^3(x) - 3\xi_t^2(x) + \xi_t(x)\right) \varphi_t(x, x)dt \\
&\quad + \langle \xi_t, \Box\varphi_t \rangle dt + \text{l.m.i.}. \\
&= -\lambda \sum_x \sum_{y:y \neq x} \xi_s(x)\xi_s(y)(\xi_s(x) + \xi_s(y) - 2)\varphi_s(x, y)ds \\
&\quad - \lambda \sum_x \left(2\xi_s^3(x) - 5\xi_s^2(x) + \xi_s(x)\right) \varphi_s(x, x)ds + \langle \xi_s, \Box\varphi_s \rangle ds + \text{m.t.}
\end{aligned} \tag{4.33}$$

We now define $\tilde{\varphi} : [0, t] \times \mathbb{Z}^2 \to \mathbb{R}$ by $\tilde{\varphi}_s(x) = \varphi_s(x, x)$. The idea is to get at

$$E[\langle \xi_t * \xi_t, \varphi_t \rangle - \langle \xi_t, \tilde{\varphi}_t \rangle] = E[\xi_t(0)(\xi_t(0) - 1)].$$

The Laplacian of $\tilde{\varphi}_s$ satisfies

$$\begin{aligned}
\Delta\tilde{\varphi}_s(x) &= \frac{1}{4}\sum_{y\sim x}\varphi_s(y,y) - \varphi_s(x,x) \\
&= \frac{1}{4}\sum_{y\sim x}\varphi_s(y,x) + \varphi_s(x,y) - 2\varphi_s(x,x) \\
&\qquad + \varphi_s(y,y) + \varphi_s(x,x) - \varphi_s(y,x) - \varphi_s(x,y) \\
&= \Delta\varphi_s(x,x) + \Box\varphi_s(x).
\end{aligned}$$

Then

$$\begin{aligned}
\dot{\tilde{\varphi}} + \Delta\tilde{\varphi} &= (\dot{\varphi} + \Delta\varphi)\mathbb{1}\{x=y\} + \Box\varphi \\
&= 2\lambda\tilde{\varphi} + \Box\varphi
\end{aligned}$$

and using (4.19) we have

$$d\langle\xi_s,\tilde{\varphi}_s\rangle = 2\lambda\langle\xi_s,\tilde{\varphi}_s\rangle ds - \lambda\langle\xi_s(\xi_s-1),\tilde{\varphi}_s\rangle ds + \langle\xi_s,\Box\varphi_s\rangle ds + \text{m.t.}. \tag{4.34}$$

Combining (4.33) and (4.34) we find

$$\begin{aligned}
d\langle\xi_s*\xi_s,\varphi_s\rangle - d\langle\xi_s,\tilde{\varphi}_s\rangle &= -\lambda\sum_x\sum_{y:y\neq x}\xi_s(x)\xi_s(y)(\xi_s(x)+\xi_s(y)-2)\varphi_s(x,y)ds \\
&\quad -2\lambda\langle\xi_s(\xi_s-1)(\xi_s-2),\tilde{\varphi}_s\rangle ds + \text{m.t.}. \qquad (4.35)
\end{aligned}$$

Since $\xi_t(x)$ is almost surely integer valued for all x, the factor $\xi_s(x)+\xi_s(y)-2$ is strictly positive unless both $\xi_t(x), \xi_t(y)$ are equal to 1, or either is 1 while the other is 0, but in either case the entire first term of (4.35) vanishes by virtue of the additional factors of $\xi_t(x), \xi_t(y)$ in the sum. So (4.35) is negative in expectation as required. This leads us to an estimate of the error in approximating $\hat{\xi}_t$ by $\hat{\xi}_{t-s}$, which accounts of the possibility of the coalescence of particles over the interval $[t-s,t]$.

Lemma 4.2.4. *With $s = t/\log^\alpha t$ there exists $c = c(\lambda,\alpha) < \infty$ so that for $t \geq e$*

and $f\colon \mathbb{Z}^2 \to \mathbb{R}$ *satisfying* $\langle |f|, 1\rangle < \infty$

$$\Big|\mathbb{E}[\langle \xi_t, f\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, P_s f\rangle\right]\Big| \le c\langle |f|, 1\rangle \frac{\log^{1-\alpha} t}{t}.$$

In particular, $|\hat{\xi}_t - \hat{\xi}_{t-s}| \le c\frac{\log^{1-\alpha} t}{t}$.

Proof.

$$\begin{aligned}
\Big|\mathbb{E}[\langle \xi_t, f\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, P_s f\rangle\right]\Big| &= \Big|\mathbb{E}\left[\langle \xi_t, P_0 f\rangle\right] - \mathbb{E}\left[\langle \xi_{t-s}, P_s f\rangle\right]\Big| \\
&= \left|\int_{t-s}^{t} d\mathbb{E}\left[\langle \xi_r, P_{t-r} f\rangle\right]\right| \\
&= \lambda \left|\int_{t-s}^{t} \mathbb{E}\left[\langle \xi_r(\xi_r - 1), \varphi_r\rangle\right] dr\right| \\
&= \lambda \left|\int_{t-s}^{t} \mathbb{E}[\xi_r(0)(\xi_r(0)-1)] \sum_x \varphi_r(x) dr\right| \\
&= \lambda \left|\int_{t-s}^{t} \mathbb{E}[\xi_r(0)(\xi_r(0)-1)] \sum_x \sum_y p_{t-r}(y) f(x-y) dr\right| \\
&\le \lambda \langle |f|, 1\rangle \int_{t-s}^{t} \left|\mathbb{E}[\xi_r(0)(\xi_r(0)-1)]\right| dr \\
&\le C\langle |f|, 1\rangle \left(\frac{\log t}{t}\right)^2 \int_{t-s}^{t} \frac{1}{\log(r-t+2s)} dr \\
&\le C\langle |f|, 1\rangle \left(\frac{\log t}{t}\right)^2 \frac{s}{\log s} \\
&\le c\langle |f|, 1\rangle \frac{\log^{1-\alpha} t}{t}.
\end{aligned}$$

Choose $f = p_0$. Then, $\langle p_0, 1\rangle = \sum_x p_0(x) = \sum_x \mathbb{1}\{x = 0\} = 1$ and

$$\begin{aligned}
|\hat{\xi}_t - \hat{\xi}_{t-s}| &= \Big|\mathbb{E}[\langle \xi_t, p_0\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, p_s\rangle\right]\Big| \\
&= \Big|\mathbb{E}[\langle \xi_t, p_0\rangle] - \mathbb{E}\left[\langle \xi_{t-s}, P_s p_0\rangle\right]\Big| \\
&\le c_3 \frac{\log^{1-\alpha} t}{t}.
\end{aligned}$$

□

Notice this gives for free (using Lemma 4.1.2) the estimate,

$$\begin{aligned} |\hat{\xi}_t^{\,2} - \hat{\xi}_{t-s}^2| &= |\hat{\xi}_t - \hat{\xi}_{t-s}||\hat{\xi}_t + \hat{\xi}_{t-s}| \\ &\le c\frac{\log^{1-\alpha} t}{t}\left(\frac{\log t}{t} + \frac{\log(t-s)}{t-s}\right) \\ &\le c\frac{\log^{2-\alpha} t}{t^2} \end{aligned} \tag{4.36}$$

for $s \le t/4$, where the constant changes from line to line.

4.2.3 A Two Point Estimate

A two point estimate gives us control of the first approximation step.

Lemma 4.2.5. *With* ψ *as defined in* (4.23) *and* s *as in Lemma 4.2.4, there exists* $c = c(\lambda, \alpha) < \infty$ *so that for large* t

$$\left|\mathbb{E}[\xi_t(0)(\xi_t(0)-1)] - \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \psi_s\rangle]\right| \le c\frac{\log^{(1-\alpha)\vee\alpha} t}{t^2}.$$

Proof.

$$\begin{aligned} &\left|\mathbb{E}[\xi_t(0)(\xi_t(0)-1)] - \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \psi_s\rangle]\right| \\ &\le \left|\mathbb{E}[\langle \xi_t * \xi_t, \psi_0\rangle - \langle \xi_t, \bar{\psi}_0\rangle] - \mathbb{E}\left[\langle \xi_{t-s} * \xi_{t-s}, \psi_s\rangle - \langle \xi_{t-s}, \bar{\psi}_s\rangle\right]\right| \\ &\quad + \left|\mathbb{E}[\langle \xi_{t-s}, \bar{\psi}_s\rangle]\right| \\ &= \left|\int_{t-s}^{t} d\mathbb{E}\left[\langle \xi_r * \xi_r, \psi_{t-r}\rangle - \langle \xi_r, \bar{\psi}_{t-r}\rangle\right]\right| + \left|\mathbb{E}[\langle \xi_{t-s}, \bar{\psi}_s\rangle]\right| \end{aligned}$$

The test function $\varphi_s = \psi_{t-s}$ as before together with equation (4.35) gives

$$
\begin{aligned}
&\left| \int_{t-s}^{t} d\mathbb{E}\left[\langle \xi_r * \xi_r, \psi_{t-r}\rangle - \langle \xi_r, \bar{\psi}_{t-r}\rangle\right] \right| + \left|\mathbb{E}[\langle \xi_{t-s}, \bar{\psi}_s\rangle]\right| \\
&\leq \lambda \left| \int_{t-s}^{t} \sum_x \sum_{y \neq x} \mathbb{E}\Big[\xi_r(x)\xi_r(y)\big(\xi_r(x)-1\big)\Big] \psi_{t-r}(x,y)dr \right| \\
&\quad + \lambda \left| \int_{t-s}^{t} \sum_x \sum_{y \neq x} \mathbb{E}\Big[\xi_r(x)\xi_r(y)\big(\xi_r(y)-1\big)\Big] \psi_{t-r}(x,y)dr \right| \\
&\quad + 2\lambda \left| \int_{t-s}^{t} \sum_x \mathbb{E}[\xi_r(x)(\xi_r(x)-1)(\xi_r(x)-2)] \bar{\psi}_{t-r}(x)dr \right| \\
&\quad + \left|\mathbb{E}[\langle \xi_{t-s}, \bar{\psi}_s\rangle]\right| \\
&\leq \lambda \left| \int_{t-s}^{t} \sum_x \sum_{y \neq x} \mathbb{E}[\xi_r(y)]\mathbb{E}\big[\xi_r(x)\big(\xi_r(x)-1\big)\big] \psi_{t-r}(x,y)dr \right| \\
&\quad + \lambda \left| \int_{t-s}^{t} \sum_x \sum_{y \neq x} \mathbb{E}[\xi_r(x)]\mathbb{E}\big[\xi_r(y)\big(\xi_r(y)-1\big)\big] \psi_{t-r}(x,y)dr \right| \\
&\quad + 2\lambda \left| \int_{t-s}^{t} \sum_x \mathbb{E}[\xi_r(x)]\mathbb{E}[\xi_r(x)(\xi_r(x)-1)] \bar{\psi}_{t-r}(x)dr \right| \\
&\quad + \left|\mathbb{E}[\langle \xi_{t-s}, \bar{\psi}_s\rangle]\right|
\end{aligned}
$$

where in each of the first three terms a single use of negative correlation was used to peel off an expectation. Now the first two terms are immediately comparable, call the first term I. Label the third term II and the final term III. Then I can be approximated as follows.

$$
\begin{aligned}
&\left|\int_{t-s}^{t}\sum_{x}\sum_{y\neq x}\mathbb{E}[\xi_r(y)]\mathbb{E}\big[\xi_r(x)\big(\xi_r(x)-1\big)\big]\psi_{t-r}(x,y)dr\right|\\
&\leq C\left|\int_{t-s}^{t}\frac{\log r}{r}\mathbb{E}\big[\xi_r(0)\big(\xi_r(0)-1\big)\big]\sum_{x}\sum_{y\neq x}\psi_{t-r}(x,y)dr\right|\\
&\leq C\left(\frac{\log t}{t}\right)^2\int_{t-s}^{t}\frac{\log r}{r}\frac{1}{\log\left(r-t+2s\right)}\sum_{x}\sum_{y\neq x}\psi_{t-r}(x,y)dr \qquad \text{by (4.25).}
\end{aligned}
$$

Providing that $t-r$ is large, Lemma 4.1.4 provides a good approximation for ψ_{t-r}, therefore we split the integral at $t-s/\log s$ and over the short interval $[t-s/\log s,t]$ we bound the test function by 1 and over the long interval $[t-s,t-s/\log s]$ we use Lemma 4.1.4 to recover a logarithm giving

$$
\begin{aligned}
I \leq &C\left(\frac{\log t}{t}\right)^2\Bigg(\int_{t-\frac{s}{\log s}}^{t}\frac{\log r}{r}\frac{1}{\log\left(r-t+2s\right)}dr\\
&\qquad\qquad+\int_{t-s}^{t-\frac{s}{\log s}}\frac{\log r}{r}\frac{1}{\log\left(r-t+2s\right)}\frac{1}{\log\left(t-r\right)}dr.\Bigg)\\
&\leq C\left(\frac{\log t}{t}\right)^2\left(\frac{s}{(\log s)\log\left(2s-\frac{s}{\log s}\right)}\frac{log(t-\frac{s}{\log s})}{t-\frac{s}{\log s}}+\frac{s-\frac{s}{\log s}}{(\log s)\log\frac{s}{\log s}}\frac{\log\left(t-s\right)}{t-s}\right)\\
&\leq C\left(\frac{\log t}{t}\right)^3\left(\frac{s}{\log^2 s}\right)\\
&\leq C\frac{\log^{1-\alpha}t}{t^2}
\end{aligned}
$$

with the constant changing as usual. An analogous calculation shows without trouble that II satisfies the same upper bound. Turning our attention to III we have

$$\begin{aligned}
\mathbb{E}[\langle \xi_{t-s}, \bar{\psi}_s \rangle] &= \sum_x \mathbb{E}[\xi_{t-s}(x)]\bar{\psi}_s(x) \\
&\leq C \frac{\log (t-s)}{t-s} \sum_x \bar{\psi}_s(x) \\
&\leq C \frac{\log (t)}{t} \frac{1}{\log s} \sum_x p_s^2(x) \\
&\leq C \frac{1}{s \log s} \frac{\log t}{t} \\
&\leq C \frac{\log^\alpha t}{t^2}.
\end{aligned}$$

which completes the proof. □

4.2.4 A Variance Estimate

We need an analogous variance estimate to Lemma 4.1.8 to control the error in the approximation step that asserts that there is approximate independence of the pairs of paths of the walks that result with the particles occupying the origin.

Lemma 4.2.6. *There exists $c < \infty$ depending on λ such that for $t \geq e$*

$$Var(\langle \xi_t, f \rangle) \leq c \langle f^2, 1 \rangle \frac{\log t}{t}.$$

Proof. As in the proof of Lemma 4.1.8 we can use negative correlation to in order to ignore off-diagonal terms.

$$\begin{aligned}
Var(\langle \xi_t, f\rangle) &\le \sum_x \sum_{y=x} \mathbb{E}\left[(\xi_t(x)-\hat{\xi}_t)(\xi_t(y)-\hat{\xi}_t)\right] f(x)f(y) \\
&= \sum_x \mathbb{E}\left[(\xi_t(x)-\hat{\xi}_t)^2\right] f(x)^2 \\
&= \sum_x \left(\mathbb{E}[\xi_t(x)^2] - \hat{\xi}_t^2\right) f(x)^2 \\
&\le \left(\mathbb{E}[\xi_t(0)(\xi_t(0)-1)] + \mathbb{E}[\xi_t(0)]\right) \sum_x f(x)^2 \\
&\le c\left(\frac{\log t}{t}\right) \sum_x f(x)^2 = c\left(\frac{\log t}{t}\right)\langle f^2, 1\rangle.
\end{aligned}$$

□

Choosing $f = p_s$, for $s \le t/2$ we get the estimate

$$\begin{aligned}
\left|\mathbb{E}\left[\langle \xi_{t-s}, p_s\rangle^2\right] - \mathbb{E}[\langle \xi_{t-s}, p_s\rangle]^2\right| &= Var(\langle \xi_{t-s}, p_s\rangle) \\
&\le c\left(\frac{\log(t-s)}{t-s}\right)\langle (p_s)^2, 1\rangle \\
&\le c\left(\frac{\log t}{t}\right)\sum_x p_s(x)^2 \\
&\le \frac{c}{s}\left(\frac{\log t}{t}\right)\sum_x p_s(x) = \frac{c}{s}\left(\frac{\log t}{t}\right) \qquad (4.37)
\end{aligned}$$

where the constant c is changing in the inequalities.

4.2.5 Proof of Theorem

We begin at the equality from (4.20)

$$\begin{aligned}
\frac{d\hat{\xi}_t}{dt} &= -\lambda \mathbb{E}[\xi_t(0)(\xi_t(0)-1)] \\
&= -\lambda \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \psi_s\rangle] + \mathscr{E}^{(1)}_{s,t} \qquad \text{by Lemma 4.2.5}
\end{aligned}$$

with $\mathscr{E}_{s,t}^{(1)} = O\left(\frac{\log^{(1-\alpha)\vee\alpha} t}{t^2}\right)$. Now we seek to estimate this expectation by use of Lemma 4.1.4.

Lemma 4.2.7. *For ψ and s previously defined, for large t we have*

$$\left|\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \psi_s\rangle] - \frac{\pi}{\lambda \log s}\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s\rangle]\right| \leq C\left(\frac{\log^{1-\zeta} t}{t^2} + \frac{\log^{\alpha-\zeta} t}{t^2}\right)$$

Proof.

$$\begin{aligned}
&\left|\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, \psi_s\rangle] - \frac{\pi}{\lambda \log s}\mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s\rangle]\right| \\
&= \left|\sum_{x,y} \mathbb{E}[\xi_{t-s}(x)\xi_{t-s}(y)]\left[\psi_s(x,y) - \frac{\pi}{\lambda\log s}p_s(x)p_s(y)\right]\right| \\
&\leq \left|\sum_x \sum_{y:y\neq x} \mathbb{E}[\xi_{t-s}(x)\xi_{t-s}(y)]\left[\psi_s(x,y) - \frac{\pi}{\lambda\log s}p_s(x)p_s(y)\right]\right| \\
&\quad + \left|\sum_x \mathbb{E}[\xi_{t-s}^2(x)]\left[\bar{\psi}_s(x) - \frac{\pi}{\lambda\log s}p_s^2(x)\right]\right| \\
&\leq \frac{C}{\log^{1+\zeta} s}\left(\frac{\log(t-s)}{t-s}\right)^2 + \frac{C}{s\log^{1+\zeta} s}\frac{\log(t-s)}{t-s} \\
&\leq C\frac{\log^{1-\zeta} t}{t^2} + C\frac{\log^{\alpha-\zeta} t}{t^2}.
\end{aligned}$$

□

This lemma married with a result of Chapter 5 is seemingly where all the work is done for universality, in that in this step the dependence on λ is canceled out. We can continue estimating the derivative of $\hat{\xi}$ keeping careful track of the error in each approximation with reference to the relevant results.

$$\begin{aligned}
\frac{d\hat{\xi}_t}{dt} &= -\lambda \frac{\pi}{\lambda \log s} \mathbb{E}[\langle \xi_{t-s} * \xi_{t-s}, p_s * p_s \rangle] + \mathscr{E}^{(2)}_{s,t} && \text{by Lemma 4.2.7} \\
&= -\frac{\pi}{\log s} \mathbb{E}\left[\langle \xi_{t-s}, p_s \rangle^2 \right] + \mathscr{E}^{(2)}_{s,t} \\
&= -\frac{\pi}{\log s} \mathbb{E}[\langle \xi_{t-s}, p_s \rangle]^2 + \mathscr{E}^{(3)}_{s,t} && \text{by Lemma 4.2.6 and (4.37)} \\
&= -\frac{\pi}{\log s} \hat{\xi}^2_{t-s} + \mathscr{E}^{(3)}_{s,t} \\
&= -\frac{\pi}{\log s} \hat{\xi}^2_t + \mathscr{E}^{(4)}_{s,t} && \text{by the Lemma 4.2.4 and (4.36)}
\end{aligned}$$

where the size of the error can be estimated by

$$|\mathscr{E}^{(4)}_{s,t}| \le C \frac{\log^{(1-\zeta)\vee(1-\alpha)\vee\alpha} t}{t^2}.$$

As before, we can choose $\alpha = 1/2$ and $\zeta \ge 1/2$.

We now solve the approximate differential equation.

$$\begin{aligned}
\frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log s} \hat{\xi}^2_t + \mathscr{E}^{(4)}_{s,t} \\
\hat{\xi}^{-2}_t \frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log s} + \hat{\xi}^{-2}_t \mathscr{E}^{(4)}_{s,t} \\
\hat{\xi}^{-2}_t \frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log t} \frac{\log t}{\log t - \alpha \log\log t} + \hat{\xi}^{-2}_t \mathscr{E}^{(4)}_{s,t} \\
\hat{\xi}^{-2}_t \frac{d\hat{\xi}_t}{dt} &= -\frac{\pi}{\log t} - \frac{\pi\alpha \log\log t}{(\log t)(\log t - \alpha \log\log t)} + \mathscr{E}^{(5)}_{s,t}.
\end{aligned}$$

In dividing through by $\hat{\xi}^2_t$ we have used the lower bound in Lemma 4.2.1 which bounds $\hat{\xi}_t$ away from 0 and we can bound

$$|\mathscr{E}^{(5)}_t| \le C(\log t)^{-3/2}.$$

Returning to our equation and integrating over t larger than suitable t_0

$$
\begin{aligned}
-\hat{\xi}_t^{-1} + \hat{\xi}_{t_0}^{-1} = -\pi \left(\int_{t_0}^{t} \frac{1}{\log s} ds + \frac{1}{2} \int_{t_0}^{t} \frac{\log \log s}{(\log s)(\log s - \frac{1}{2} \log \log s)} ds \right) \\
+ O\left(\int_{t_0}^{t} \frac{1}{(\log s)^{\frac{3}{2}}} ds \right)
\end{aligned}
$$

From this point, the proof can be concluded as in the previous section with the same applications of Lemma 4.1.11.

Chapter 5

Random Walk Estimates

In this chapter we compile all the random walk estimates that are necessary throughout our approximations.

5.1 General Estimates for Continuous Time Random Walks

We will make frequent use of an estimate on the transition probabilities for continuous time random walks.

Lemma 5.1.1. *Let S_t be a continuous time rate 1 random walk on the integer lattice $\mathbb{Z}^2$ and $p_t(x) = \mathbb{P}[S_t = x | S_0 = 0]$. Then there exists $c_3 < \infty$, such that for all $t > 0$ and $x \in \mathbb{Z}^2$*

$$p_t(x) = \frac{2}{\pi t} \exp\left\{-\frac{|x|^2}{t}\right\} + O\left(\frac{1}{t^{3/2}}\right).$$

Proof. Let $\tilde{S}_n$ be a discrete time random walk and $\tilde{p}_n$ be its corresponding transition probability. Then Theorem 2.3.5 of Lawler [14] says there exists $C < \infty$ such that

$$\tilde{p}_n(x) = \frac{2}{\pi n} \exp\left\{-\frac{|x|^2}{n}\right\} + O\left(\frac{1}{n^{3/2}}\right) \qquad \text{for all } x \in \mathbb{Z}^2, \text{ for all } n.$$

This can be Poissonised, by which we mean writing

$$p_t(x) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \tilde{p}_n(x)$$

by a calculation similar to that carried out in the proof of Theorem 2.5.6 of [14]. $\square$

This, together with the estimate

$$\begin{aligned} \left| \exp\left\{ -\frac{|x|^2}{2t} \right\} - \exp\left\{ -\frac{|x+y|^2}{2t} \right\} \right| &\leq 2 \left| \int_{|x|/\sqrt{(2t)}}^{|x+y|/\sqrt{(2t)}} u e^{-u^2} du \right| \\ &\leq 2 \left| \int_{|x|/\sqrt{(2t)}}^{|x+y|/\sqrt{(2t)}} du \right| \\ &\leq \frac{C}{\sqrt{t}} \left| |x+y| - |x| \right| \\ &\leq \frac{C|y|}{\sqrt{t}} \end{aligned}$$

gives the following estimate on the spatial difference of the transition density

$$|p_t(x) - p_t(x+y)| \leq \frac{C}{t^{\frac{1}{2}}} + \frac{C|y|}{t^{3/2}}. \tag{5.1}$$

5.2 A Random Walk Estimate in $d = 2$

A crucial random walk estimate that we make use of for the rate equation for the instantaneously coalescing random walk model allowed us approximate the trajectory of two indendent random walks that are not allowed to react with each other by the trajectories of two non-interacting independent random walks with a multiplicative correction that accounts for the interaction. Van den Berg and Kesten give an estimate in $d \geq 3$ for $\psi_t(x,y) = \mathbb{P}\big[x+S^1_t = 0, y+S^2_t = e, x+S^1_s \neq y+S^2_s \text{ for all } s \in [0,t]\big]$ where S^1, S^2 are independent copies of identically distributed random walks that start at the origin. They are able to show that it is well approximated by the product of the free transition probabilities of a walk starting at the origin to the site x and $y - e$ and γ_d, the probability that a random walk never returns to its starting point after leaving. The estimate is not valid for $d = 2$. Of course, the recurrent nature of a two dimensional random walk ensures it will almost surely return to its starting point infinitely often, and so $\gamma_2 = 0$. However, an we prove an analogous approximation replacing γ_d with a function that decays in time.

Let $NC(0,t]$ be the event that up until time t, the continuous time rate 1 random walk started at e does not hit the origin. That is

$$NC[0,t] = \{e + S_s \neq 0, \text{ for all } s \in [0,t]\}.$$

To see that this is the correct object to study, look at the definition of ψ given above or equivalently in equation (4.3). Asking that two independent random walks travel from a site x to the origin and a site y to a neighbour without the trajectories meeting up till time t is the same as requiring two independent random walks to travel from the origin to a site x and from a neighbour to a site y without their trajectories meeting up till time t by translation invariance. This, in turn is the same as asking that a random walk starting at e does not hit the origin before time t and reach $y-x$ since the difference of two independent random walks is again a random walk with twice the rate.

Remark 5.2.1. *Notice, if we have S^1, S^2 two independent copies of the same random walk both starting at the origin, then the difference between S^1 and $e + S^2$ is a random walk starting at e run twice as fast. However, the following calculation shows that the asymptotic is unchanged.*

$$\begin{aligned}\frac{\pi}{\log 2t} &= \frac{\pi}{\log t}\left(\frac{\log t}{\log 2 + \log t}\right)\\ &= \frac{\pi}{\log t}\left(1 - \frac{\log 2}{\log 2 + \log t}\right) = \frac{\pi}{\log t}\left(1 + O\left(\frac{1}{\log t}\right)\right).\end{aligned}$$

If this difference reaches the origin then the two walks have met (and hence coalesced in the instant regime), by the following lemma and the above calculation the probability of this not occuring is $\pi/\log t$ for large t which is the reason for the notation NC (no coalescence).

Lemma 5.2.2. *The asymptotic behaviour of $NC[0,t]$ as defined above is*

$$\mathbb{P}[NC[0,t]] = \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right) \qquad \textit{as } t \to \infty.$$

Proof. Dvoretzky and Erdős proved in their paper [7] that the probability that a discrete time simple random walk starting at the origin avoids the origin by time n is of order

$$\frac{1}{\log(n)}$$

and Erdős and Taylor improved this in [8] to give the leading order explicitly and quantify the error the asymptotic

$$\frac{\pi}{\log(n)} + O\left(\frac{1}{\log^2(n)}\right).$$

Let $(\tilde{S}_n)_n$ be discrete time random walk on the planar lattice started at the origin. Fix $x \in \mathbb{Z}^2$ and let $\gamma(x,n) = \#\{\text{number of visits of } \tilde{S} \text{ to } x \text{ up to time } n\}$ so that the above asymptotic is for the probability of the event $\{\gamma(0,n) = 0\}$. Then Révész [23] gives the asymptotic probability of there being no visits to x by time n, that is, $\gamma(x,n) = 0$ as

$$\mathbb{P}[\gamma(x,n) = 0] = \frac{\pi}{\log n} + O_{||x||^2}\left(\frac{1}{\log^2 n}\right) \qquad \text{as } n \to \infty \tag{5.2}$$

where the subscript on the O is meant to make clear the error is not uniform in x. Let $\widetilde{NC}(0,n]$ be the discrete analogy of $NC(0,t]$, then we have equality of the probabilities $\mathbb{P}\big[\widetilde{NC}(0,n]\big] = \mathbb{P}[\gamma(e,n) = 0]$, since event that a random walk beginning at e avoids the origin by time n is the same as the event that a random walk started at the origin makes no visits to e by time n by translation invariance of the random walk $\tilde{S}$. We only need to prove that this estimate is still valid in continuous time.

Let N_t be a rate 1 Poisson process so that $\mathbb{P}[N_t = n] = e^{-t}\frac{t^n}{n!}$. Then $S_t = e + \sum_{i=1}^{N_t} X_i$ is a rate 1 continuous time random walk starting from e. We decompose the event $NC(0,t]$ according to the number of steps in the walk by time t. We expect

t events of N_t by time t hence we write

$$\{NC(0,t]\} = \left\{NC(0,t], N_t \in \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right\}$$
$$\cup \left\{NC(0,t], N_t \notin \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right\}$$

for some $\varepsilon > 0$ so that

$$\mathbb{P}[NC(0,t]] = \mathbb{P}\left[NC(0,t], N_t \in \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right]$$
$$+\mathbb{P}\left[NC(0,t], N_t \notin \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right].$$

Consider the second term. This can be bounded above by

$$\mathbb{P}\left[N_t \notin \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right]$$
$$= \mathbb{P}[N_t < (1-\varepsilon)t] + \mathbb{P}[N_t > (1+\varepsilon)t] \leq 2e^{-t\varepsilon^2/2}$$

which is exponentially small by the usual Chernoff bound. Now treating the main term

$$\mathbb{P}\left[NC(0,t], N_t \in \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right].$$

On this event, by time t the walk has made no fewer than $\lfloor (1-\varepsilon)t \rfloor$ steps of its walk, and not reaching the origin by time t guarantees the walk has not reached the origin in its first $\lfloor (1-\varepsilon)t \rfloor$ steps hence we have

$$\left\{NC(0,t], N_t \in \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right\}$$
$$\subset \left\{\widetilde{NC}(0,\lfloor (1-\varepsilon)t \rfloor], N_t \in \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right\}$$
$$\subset \{\widetilde{NC}(0,\lfloor (1-\varepsilon)t \rfloor]\}$$

and so

$$\mathbb{P}\left[NC(0,t], N_t \in \Big(\lfloor (1-\varepsilon)t \rfloor \ , \ \lceil (1+\varepsilon)t \rceil \Big)\right] \leq \mathbb{P}[\widetilde{NC}(0,\lfloor (1-\varepsilon)t \rfloor]]. \tag{5.3}$$

Using (5.2) we can continue estimating with

$$
\begin{aligned}
&= \frac{\pi}{\log\lfloor(1-\varepsilon)t\rfloor} + O\left(\frac{1}{\log^2\lfloor(1-\varepsilon)t\rfloor}\right) \\
&\leq \frac{\pi}{\log t}\frac{\log t}{\log\frac{(1-\varepsilon)}{2} + \log t} + O\left(\frac{1}{\log^2 t}\right) \\
&= \frac{\pi}{\log t}\left(1 - \frac{\log(1-\varepsilon)/2}{\log(1-\varepsilon)/2 + \log t}\right) + O\left(\frac{1}{\log^2 t}\right) \\
&= \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right)
\end{aligned}
$$

for significantly large t. Similarly,

$$
\begin{aligned}
&\mathbb{P}\left[NC(0,t], N_t \in \Big(\lfloor(1-\varepsilon)t\rfloor\ ,\ \lceil(1+\varepsilon)t\rceil\Big)\right] \\
&\geq \mathbb{P}\left[\widetilde{NC}(0,\lceil(1+\varepsilon)t\rceil], N_t \in \Big(\lfloor(1-\varepsilon)t\rfloor\ ,\ \lceil(1+\varepsilon)t\rceil\Big)\right] \\
&= \mathbb{P}[NC(0,\lceil(1+\varepsilon)t\rceil]] \\
&\quad - \mathbb{P}\left[NC(0,\lceil(1+\varepsilon)t\rceil], N_t \notin \Big(\lfloor(1-\varepsilon)t\rfloor\ ,\ \lceil(1+\varepsilon)t\rceil\Big)\right] \\
&\geq \mathbb{P}[NC(0,\lceil(1+\varepsilon)t\rceil]] \\
&\quad - \mathbb{P}\left[N_t \notin \Big(\lfloor(1-\varepsilon)Dt\rfloor\ ,\ \lceil(1+\varepsilon)Dt\rceil\Big)\right] \\
&\geq \mathbb{P}[NC(0,\lceil(1+\varepsilon)Dt\rceil] - 2e^{-t\varepsilon^2/2} \\
&= \frac{\pi}{\log\lceil(1+\varepsilon)t\rceil} + O\left(\frac{1}{\log^2\lceil(1+\varepsilon)t\rceil}\right) \\
&\geq \frac{\pi}{\log t}\frac{\log t}{\log\big(2(1+\varepsilon)\big) + \log t} + O\left(\frac{1}{\log^2 t}\right) \\
&= \frac{\pi}{\log t}\left(1 - \frac{\log 2(1+\varepsilon)}{\log 2(1+\varepsilon) + \log t}\right) + O\left(\frac{1}{\log^2 t}\right) \\
&= \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right).
\end{aligned}
$$

□

We now prove an approximation for the test function $\psi_t(x,y) = \mathbb{P}[S_t^1 = x, e + S_t^2 = y, NC[(0,t]]]$. By the remark 5.2.1 we interpret NC not as a failure to return to the origin as in Lemma 5.2.2 but as the event that the two walks do not meet by time t so that ψ agrees with the expression we gave at the start of this section when discussing $d \geq 3$. The argument follows very strongly the method of van den Berg and Kesten Lemma 12 from [27] and only differs in the places they require $d \geq 3$. For $f\colon \mathbb{Z}^2 \to \mathbb{R}$ and $g\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ we define $\langle 1, f(\cdot)\rangle = \sum_{x\in\mathbb{Z}^2} f(x)$ and $\langle 1, g(\cdot,\cdot)\rangle = \sum_{x,y\in\mathbb{Z}^2} g(x,y)$.

Proof of Lemma 4.1.4 We first use Lemma 5.2.2 to replace $\pi/\log t$ by $\mathbb{P}[NC[(0,t]]]$ since the lemma gives that the leading order is independent of the walk rate. By the triangle inequality, We make the following approximations and let $s = s(t) = t/\log^\alpha t$ for some $\alpha > 0$,

$$
\begin{aligned}
&\sum_{x,y}\left|\mathbb{P}[S_t^1 = x, e + S_t^2 = y, NC(0,t]] - \frac{\pi}{\log t}p_t(x)p_t(y-e)\right| \\
&= \sum_{x,y}\left|\mathbb{P}[S_t^1 = x, e + S_t^2 = y, NC(0,s]] - \frac{\pi}{\log s}p_t(x)p_t(y-e)\right| + \mathscr{E}_1(t) \\
&= \sum_{x,y}\left|\sum_{u,v}\Bigg(\mathbb{P}[S_s^1 = u, e + S_s^2 = v, NC(0,s]]\right. \\
&\qquad \left. - \frac{\pi}{\log s}p_s(u)p_s(v-e)\Bigg)p_{t-s}(x-u)p_{t-s}(y-v)\right| + \mathscr{E}_1(t) \\
&= \sum_{x,y}\left|\sum_{u,v}\Bigg(\mathbb{P}[S_s^1 = u, e + S_s^2 = v, NC(0,s]]\right. \\
&\qquad \left. - \frac{\pi}{\log s}p_s(u)p_s(v-e)\Bigg)p_{t-s}(x)p_{t-s}(y)\right| + \mathscr{E}_2(t) \\
&= \sum_{x,y}p_{t-s}(x)p_{t-s}(y-e)\left|\sum_{u,v}\Bigg(\mathbb{P}[S_s^1 = u, e + S_s^2 = v, NC(0,s]]\right. \\
&\qquad \left. - \frac{\pi}{\log s}p_s(u)p_s(v-e)\Bigg)\right| + \mathscr{E}_2(t) \\
&= \left|\sum_{u,v}\mathbb{P}[S_s^1 = u, e + S_s^2 = v, NC(0,s]] - \frac{\pi}{\log s}\sum_{u,v}p_s(u)p_s(v-e)\right| + \mathscr{E}_2(t) \\
&= \left|\mathbb{P}[NC(0,s]] - \frac{\pi}{\log s}\right| + \mathscr{E}_2(t) \\
&= O\left(\frac{1}{\log^2 t}\right) + \mathscr{E}_2(t).
\end{aligned}
$$

What remains to be shown is that $\mathscr{E}_2(t)$ is of order $O\left(\frac{\log\log t}{\log^2 t}\right)$. Now, if particles haven't met by time t than they certainly haven't met by time s, we have the inclusion $NC(0,t] \subset NC(0,s]$. So we can bound the first error term by

$$\begin{aligned}
\mathscr{E}_1(t) \le \sum_{x,y} & \mathbb{P}[S_t^1 = x, e + S_t^2 = y, NC[(0,s]] \\
& - \mathbb{P}[S_t^1 = x, e + S_t^2 = y, NC(0,t]] \\
& + \left(\frac{\pi}{\log s} - \frac{\pi}{\log t} \right) \sum_{x,y} p_t(x) p_t(y-e) \\
\le \mathbb{P}&[NC(0,s]] - \mathbb{P}[NC(0,t]] + \left(\frac{\pi}{\log s} - \frac{\pi}{\log t} \right).
\end{aligned}$$

Lemma 5.2.2 then gives

$$\begin{aligned}
\mathscr{E}_1(t) &\le \pi \frac{(\log t - \log s)}{\log t \log s} + O\left(\frac{1}{\log^2 t} \right) \\
&= O\left(\frac{\log\log t}{\log^2 t} \right)
\end{aligned}$$

where the last equalilty follows from the choice of s. The difference $\mathscr{E}_2(t) - \mathscr{E}_1(t)$ equal to

$$\begin{aligned}
\sum_{x,y} \Bigg| \sum_{u,v} & \left(\mathbb{P}[S_s^1 = u, e + S_s^2 = v, NC(0,s]] - \frac{\pi}{\log s} p_s(u) p_s(v-e) \right) \\
& \times \left(p_{t-s}(x-u) p_{t-s}(y-v) - p_t(x) p_t(y-e) \right) \Bigg|.
\end{aligned}$$

Let $\sigma(t) = t^{1/2} \log^{\delta} t$ for some $\delta > 0$. Consider for now, only the terms contributed to the sum by u, v with at least one of $|u|, |v-e|$ exceeding $\sigma(s)$. By the triangle inequality,

$$\begin{aligned}
&\sum_{x,y}\Bigg|\sum_{|u| \text{ or } |v-e|>\sigma(s)}\left(\mathbb{P}[S_s^1=u, e+S_s^2=v, NC(0,s]]-\frac{\pi}{\log s}p_s(u)p_s(v-e)\right) \\
&\qquad \times (p_{t-s}(x-u)p_{t-s}(y-v)-p_{t-s}(x)p_{t-s}(y-e))\Bigg| \\
&\leq \sum_{|u| \text{ or } |v-e|>\sigma(s)} \mathbb{P}[S_s^1=u, e+S_s^2=v, NC(0,s]]+\frac{\pi}{\log s}p_s(u)p_s(v-e) \\
&\leq 4\mathbb{P}[|S_s^1|>\sigma(s)] \\
&= 4\mathbb{P}[|S_s^1|^{2+\varepsilon}>\sigma(s)^{2+\varepsilon}] \leq C\frac{s^{1+\frac{\varepsilon}{2}}}{\sigma(s)^{2+\varepsilon}} \leq \frac{C}{\log^{(2+\varepsilon)\delta} s}
\end{aligned}$$

for some $\varepsilon > 0$. For the remaining contribution

$$\begin{aligned}
&\sum_{x,y}\Bigg|\sum_{|u|,|v-e|\leq\sigma(s)}\left(\mathbb{P}[S_s^1=u, e+S_s^2=v, NC(0,s]]-\frac{\pi}{\log s}p_s(u)p_s(v-e)\right) \\
&\qquad \times (p_{t-s}(x-u)p_{t-s}(y-v)-p_{t-s}(x)p_{t-s}(y-e))\Bigg|
\end{aligned}$$

we split the outside sum similarly and deal first with the case that at least one of

$|x|, |y-e|$ exceeds $\sigma(t-s)$.

$$\begin{aligned}
&\sum_{|x| \text{ or } |y-e|>\sigma(t-s)} \Bigg| \sum_{|u|,|v-e|\leq\sigma(s)} \Bigg(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s)] \\
&\quad - \frac{\pi}{\log s} p_s(u)p_s(v-e)\Bigg)(p_{t-s}(x-u)p_{t-s}(y-v) - p_{t-s}(x)p_{t-s}(y-e))\Bigg| \\
&\leq \sum_{|x| \text{ or } |y-e|>\sigma(t-s)} \sum_{|u|,|v-e|\leq\sigma(s)} \Bigg(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s)] \\
&\quad + \frac{\pi}{\log s} p_s(u)p_s(v-e)\Bigg)(p_{t-s}(x-u)p_{t-s}(y-v) + p_{t-s}(x)p_{t-s}(y-e)) \\
&\leq \sum_{|x| \text{ or } |y-e|>\sigma(t-s)} \sum_{|u|,|v-e|\leq\sigma(s)} \Bigg(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s)] \\
&\quad + \frac{\pi}{\log s} p_s(u)p_s(v-e)\Bigg)p_{t-s}(x-u)p_{t-s}(y-v) \\
&\quad + \frac{C}{\log s} \sum_{|x| \text{ or } |y-e|>\sigma(t-s)} p_{t-s}(x)p_{t-s}(y-e) \\
&\leq \sum_{|x| \text{ or } |y-e|>\sigma(t-s)} \sum_{|u|,|v-e|\leq\sigma(s)} \Bigg(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s)] \\
&\quad + \frac{\pi}{\log s} p_s(u)p_s(v-e)\Bigg)p_{t-s}(x-u)p_{t-s}(y-v) \\
&\quad + \frac{2C}{\log s}\mathbb{P}[|S_{t-s}^1| > \sigma(t-s)]
\end{aligned}$$

where in the second inequality we have summed over all u, v in the final term which only introduces positive terms. On the final term, Markov's inequality gives

$$\frac{2C}{\log s}\mathbb{P}[|S_{t-s}^1| > \sigma(t-s)] \leq \frac{2C}{\log s}\frac{(t-s)^{1+\varepsilon/2}}{\sigma(t-s)^{2+\varepsilon}} \leq \frac{4C}{\log^{1+\delta(2+\varepsilon)} t}$$

and then for this contribution we are left with

$$\begin{aligned}
&\sum_{|x| \text{ or } |y-e|>\sigma(t-s)} \sum_{|u|,|v-e|\leq\sigma(s)} \Big(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s]] \\
&\quad + \frac{\pi}{\log s} p_s(u)p_s(v-e)\Big) p_{t-s}(x-u)p_{t-s}(y-v) \\
&= \sum_x \sum_{|y-e|>\sigma(t-s)} \sum_{|u|,|v-e|\leq\sigma(s)} \Big(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s]] \\
&\quad + \frac{\pi}{\log s} p_s(u)p_s(v-e)\Big) p_{t-s}(x-u)p_{t-s}(y-v) \\
&+ \sum_{|x|>\sigma(t-s)} \sum_y \sum_{|u|,|v-e|\leq\sigma(s)} \Big(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s]] \\
&\quad + \frac{\pi}{\log s} p_s(u)p_s(v-e)\Big) p_{t-s}(x-u)p_{t-s}(y-v).
\end{aligned}$$

In the region of summation, for large enough t,

$$|x-u| \geq |x| - |u| > \sigma(t-s) - \sigma(s) > \frac{\sigma(t)}{2},$$

by the reverse triangle inequality and similarly for $|y-v| = |y-e-(v-e)|$. So by bounding some instances of $p_s(u), p_s(v-e)$ by 1 and throwing away some instances the events $\{S_s^1 = u\}, \{e+S_s^2 = v\}$ we can bound

$$\begin{aligned}
&\sum_{|y-e|>\sigma(t-s)} \sum_u \sum_{|v-e|\leq\sigma(s)} \left(\mathbb{P}[S_s^1 = u, NC[(0,s]] + \frac{\pi}{\log s} p_s(u)\right) p_{t-s}(y-v) \\
&+ \sum_{|x|>\sigma(t-s)} \sum_{|u|\leq\sigma(s)} \sum_v \left(\mathbb{P}[e+S_s^2 = v, NC[(0,s]] + \frac{\pi}{\log s} p_s(v-e)\right) p_{t-s}(x-u) \\
&= \frac{C}{\log t} \sum_{|y-e|>\sigma(t-s)} \sum_{|v-e|\leq\sigma(s)} p_{t-s}(y-v) \\
&+ \frac{C}{\log t} \sum_{|x|>\sigma(t-s)} \sum_{|u|\leq\sigma(s)} p_{t-s}(x-u) \\
&\leq \frac{2C}{\log t} \mathbb{P}\left[|S_{t-s}^1| > \frac{\sigma(t)}{2}\right] \\
&\leq \frac{C}{\log^{1+\delta(2+\varepsilon)} t}
\end{aligned}$$

constant changing in the last line. All that is left to estimate is

$$\sum_{|x|,|y-e|\le\sigma(t-s)} \Bigg| \sum_{|u|,|v-e|\le\sigma(s)} \Bigg(\mathbb{P}[S_s^1 = u, e + S_s^2 = v, NC(0,s)] - \frac{\pi}{\log s} p_s(u) p_s(v-e) \Bigg) (p_{t-s}(x-u)p_{t-s}(y-v) - p_{t-s}(x)p_{t-s}(y-e)) \Bigg|.$$

The difference

$$p_{t-s}(x-u)p_{t-s}(y-v) - p_{t-s}(x)p_{t-s}(y-e)$$

is equal to

$$p_{t-s}(x-u)\Big(p_{t-s}(y-v) - p_{t-s}(y-e)\Big) + \Big(p_{t-s}(x-u) - p_{t-s}(x)\Big)p_{t-s}(y-e)$$

which can be bounded by the local central limit theorem and corollaries thereof (see (5.1)) by

$$\frac{c}{t}\left(\frac{|v-e|+|u|}{t^{3/2}} + O\left(\frac{1}{t^{1+\frac{\varepsilon}{2}}}\right)\right).$$

This bound gives

$$\begin{aligned}
&\frac{c}{t}\left(\frac{2\max_{w:\,|w|\le\sigma(s)}|w|}{t^{3/2}} + O\left(\frac{1}{t^{1+\frac{\varepsilon}{2}}}\right)\right) \\
&\times \sum_{|x|,|y-e|\le\sigma(t-s)} \sum_{|u|,|v-e|\le\sigma(s)} \left(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s)] \frac{\pi}{\log s} p_s(u)p_s(v-e)\right) \\
&\le \left(\frac{Cs^{1/2}\log^{\delta} s}{t^{5/2}} + O\left(\frac{1}{t^{2+\frac{\varepsilon}{2}}}\right)\right) \\
&\times \sum_{|x|,|y-e|\le\sigma(t-s)} \sum_{u,v} \left(\mathbb{P}[S_s^1 = u, e+S_s^2 = v, NC(0,s)] + \frac{\pi}{\log s} p_s(u)p_s(v-e)\right) \\
&\le \frac{C}{\log s}\left(\frac{Cs^{1/2}\log^{\delta} s}{t^{5/2}} + O\left(\frac{1}{t^{2+\frac{\varepsilon}{2}}}\right)\right) C\sigma(t-s)^4 \\
&\le \frac{C}{\log t}\left(\frac{C}{\log^{\frac{\alpha}{2}-5\delta} t} + O\left(\frac{\log^{4\delta} t}{t^{\frac{\varepsilon}{2}}}\right)\right).
\end{aligned}$$

Collecting all the errors that contribute to $\mathscr{E}_2(t) - \mathscr{E}_1(t)$, we see that it is of order

$$O\left(\frac{1}{\log^{(2+\varepsilon)\delta} t} + \frac{1}{\log^{1+\frac{\alpha}{2}-5\delta} t}\right)$$

which can be made of order

$$O\left(\frac{1}{\log^2 t}\right)$$

by choosing $\delta = 1$ and any $\alpha \geq 12$, which completes the proof of Lemma 4.1.4. $\square$

5.3 λ Reluctant Random Walk

Let S_t be a rate 1 simple random walk on $\mathbb{Z}^2$ starting at the origin and let $NC_\lambda(0,t]$ be the event that on each of the visits of S_t to the origin it rests no longer than an exponential time with mean λ^{-1}. If S^1, S^2 are two independent copies of a continuous time random walk starting at the origin, then their difference is a continuous time random walk starting at the origin and run at twice speed, we will be interested in quantifying the probability of $NC_{2\lambda}(0,t]$ for this difference and that will describe the probability that two independent random walks have not coalesced by time t (2λ since if we have two particles resting at the same site, the first is trying to coalesce onto the second at rate λ as is the second onto the first so that the total rate is 2λ so that we want the difference in the random walks to remain at the origin for no more than $exp(2\lambda)$ amount of time.)

Lemma 5.3.1.

$$\mathbb{P}[NC_\lambda(0,t]] = \frac{\pi}{\lambda \log t} + O\left(\frac{\log\log t}{\log^2 t}\right).$$

Proof. First, it will be convenient to give some notation to aid the manipulations. Let

$$p_e^{(\infty)}(t) = \mathbb{P}[NC(0,t]],$$

the probability that a random walk starting at a neighbour, e of the origin does not reach the origin by time t (the infinity meant to signifiy that the walk is infinitely reluctant to remain at the origin and hence not even visit it at all). This we already know has asymptotics given by Lemma 5.2.2. Let

$$p_0^{(\lambda)}(t) = \mathbb{P}[NC_\lambda(0,t]],$$

which is the object of study and let $p_e^{(\lambda)}(t)$ be the probability that starting from e the random walk spends no more than exponential rate λ time at the origin.

We first sketch the argument. The walk alarm must go off before the λ alarm since the walk starts at the origin. Therefore there must be a jump from the origin to one of its four neighbours. Decomposing the event according to what happens after the first jump we see that either the particle can wander away and not return to the origin by time t or it can return but then again mustn't remain for more than rate λ. Ignoring the fact that we have used up time to make jumps, this gives the approximate relation

$$p_0^{(\lambda)}(t) \approx \frac{1}{1+\lambda} p_e^{(\infty)}(t) + \frac{1}{1+\lambda}(1 - p_e^{(\infty)}(t))p_0^{(\lambda)}(t).$$

Rearranging this expression for $p_0^{(\lambda)}(t)$ and substituting for $p_e^{(\infty)}(t)$ gives

$$p_0^{(\lambda)}(t) \approx \frac{\pi}{\lambda \log t} + O\left(\frac{1}{\log^2 t}\right).$$

We now make this argument rigorous.

We do this in two steps. The first of which (although we will prove this later) is that

$$p_e^{\lambda}(t) = \frac{1+\lambda}{\lambda} \frac{\pi}{\log t} + O\left(\frac{\log \log t}{\log^2 t}\right) \tag{5.4}$$

using what we know about $p_e^{(\infty)}(t)$. We will delay the proof of this until after we have shown how the above equality produces the result. Clearly $p_0^{(\lambda)}(t)$ and $p_e^{\lambda}(t)$ have the following relationship, a particle at the origin must make a jump to a neighbouring site before spending rate λ time there, and now at this neighbour it must not visit the origin for more than rate λ time which is given by $p_e^{(\lambda)}$ evaluated at time t less the amount of time it took to jump. Firstly, we use the fact that $p_e^{(\lambda)}$ is decreasing in t to get the lower bound.

$$
\begin{aligned}
p_0^{(\lambda)}(t) &= \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)s}p_e^{(\lambda)}(t-s)ds \\
&\geq \frac{1}{1+\lambda}p_e^{(\lambda)}(t)(1-e^{-(1+\lambda)t}) \\
&= \frac{1}{1+\lambda}p_e^{(\lambda)}(t) - \frac{1}{1+\lambda}p_e^{(\lambda)}(t)e^{-(1+\lambda)t} \\
&= \frac{1}{1+\lambda}\frac{1+\lambda}{\lambda}\frac{\pi}{\log(t)} + O\left(\frac{\log\log t}{\log^2(t)}\right) \\
&= \frac{\pi}{\lambda\log(t)} + O\left(\frac{\log\log t}{\log^2(t)}\right).
\end{aligned}
$$

This is necessary lower bound for $p_0^{(\lambda)}(t)$. And for the upper bound

$$
\begin{aligned}
p_0^{(\lambda)}(t) &= \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)s}p_e^{(\lambda)}(t-s)ds \\
&= \frac{1}{1+\lambda}\bigg(\int_0^{t/2}(1+\lambda)e^{-(1+\lambda)s}p_e^{(\lambda)}(t-s)ds \\
&\qquad + \int_{t/2}^t (1+\lambda)e^{-(1+\lambda)s}p_e^{(\lambda)}(t-s)ds\bigg) \\
&\leq \frac{1}{1+\lambda}\bigg(p_e^{(\lambda)}(t/2)\int_0^{t/2}(1+\lambda)e^{-(1+\lambda)s}ds \\
&\qquad + \int_{t/2}^t (1+\lambda)e^{-(1+\lambda)s}ds\bigg) \\
&\leq \frac{1}{1+\lambda}\bigg(p_e^{(\lambda)}(t/2)\int_0^{\infty}(1+\lambda)e^{-(1+\lambda)s}ds \\
&\qquad + \int_{t/2}^{\infty} (1+\lambda)e^{-(1+\lambda)s}ds\bigg) \\
&= \frac{1}{1+\lambda}p_e^{(\lambda)}(t/2) + O(e^{-(1+\lambda)t/2}) \\
&= \frac{1}{1+\lambda}\frac{1+\lambda}{\lambda}\frac{\pi}{\log(t/2)} + O\left(\frac{\log\log t}{\log^2(t)}\right) \\
&= \frac{\pi}{\lambda\log t}\frac{\log t}{\log t - \log 2} + O\left(\frac{\log\log t}{\log^2(t)}\right) \\
&= \frac{\pi}{\lambda\log t} + O\left(\frac{\log\log t}{\log^2(t)}\right).
\end{aligned}
$$

Now to prove (5.4), we decompose $p_e^{(\lambda)}(t)$ according to whether the random walk hits the origin before time t. That is

$$p_e^{(\lambda)}(t) = p_e^{(\infty)}(t) + \int_0^t \frac{\Psi(s)}{1+\lambda} \int_0^{t-s} (1+\lambda)e^{-(1+\lambda)r} p_e^{(\lambda)}(t-s-r) dr ds.$$

For a lower bound, since $p_e^{(\lambda)}(t)$ is decreasing in t (the event that the random walk hasn't spent the appropriate amount of time at the origin by time t is contained in the correponding event by time $s < t$) we can bound the copy in the inner integral by its value at 0 and the same again for the outer integral giving

$$p_e^{(\lambda)}(t) \geq p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \int_0^t \Psi(s) \int_0^{t-s} (1+\lambda)e^{-(1+\lambda)r} dr ds.$$

Now evaluating the inner integral and rearranging gives

$$\begin{aligned}
p_e^{(\lambda)}(t) &\geq p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \int_0^t \Psi(s) \left(1 - e^{-(1+\lambda)(t-s)}\right) ds \\
&= p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \left(\int_0^t \Psi(s) ds - \int_0^t \Psi(s) e^{-(1+\lambda)(t-s)} ds\right) \\
&= p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \left(1 - p_e^{(\infty)}(t) - \int_0^t \Psi(s) e^{-(1+\lambda)(t-s)} ds\right) \\
&= p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \Bigg(1 - p_e^{(\infty)}(t) - \int_0^{t/2} \Psi(s) e^{-(1+\lambda)(t-s)} ds \\
&\qquad - \int_{t/2}^t \Psi(s) e^{-(1+\lambda)(t-s)} ds\Bigg) \\
&\geq p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \left(1 - p_e^{(\infty)}(t) - e^{-(1+\lambda)t/2} \int_0^{t/2} \Psi(s) ds - \int_{t/2}^t \Psi(s) ds\right) \\
&\geq p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \left(1 - p_e^{(\infty)}(t) - e^{-(1+\lambda)t/2} \left(1 - p_e^{(\infty)}(t/2)\right) - p_e^{(\infty)}(t/2)\right).
\end{aligned}$$

Now, we know the asymptotic behaviour of $p_e^{(\infty)}(t)$, and by the properties of the logarithm, $p_e^{(\infty)}(t/2)$ has the same limiting behaviour (as is done explicitly in the upper bound for $p_0^{(\lambda)}(t)$ above.)

$$p_e^{(\lambda)}(t) \geq p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda} \left(1 - 2p_e^{(\infty)}(t) + O\left(\frac{1}{\log^2(t)}\right)\right).$$

Notice, we can throw away the second term on the right hand side since it is positive for large t and achieve a trivial lower bound $p_e^{(\lambda)}(t) \geq p_e^{(\infty)}(t)$ which allows us to rewrite

$$p_e^{(\lambda)}(t) \geq p_e^{(\infty)}(t) + \frac{p_e^{(\lambda)}(t)}{1+\lambda}\left(1 - 2p_e^{(\infty)}(t)\right) + O\left(\frac{1}{\log^3(t)}\right)$$

and now we can rearrange as in the sketch.

$$\begin{aligned} p_e^{(\lambda)}(t) &\geq \frac{(1+\lambda)p_e^{(\infty)}(t)}{\lambda + 2p_e^{(\infty)}(t)} + O\left(\frac{1}{\log^2(t)}\right) \\ &= \frac{1+\lambda}{\lambda}\frac{\pi}{\log(t)} + O\left(\frac{1}{\log^2(t)}\right). \end{aligned}$$

To complete the proof of this step we need the corresponding upper bound. Let $\varepsilon \in (0,1)$. Beginning with Fubini and then splitting the inner integral,

$$\begin{aligned} p_e^{(\lambda)}(t) &= p_e^{(\infty)}(t) + \int_0^t \frac{\Psi(s)}{1+\lambda}\int_0^{t-s}(1+\lambda)e^{-(1+\lambda)r}p_e^{(\lambda)}(t-s-r)drds \\ &= p_e^{(\infty)}(t) + \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)r}\int_0^{t-r}\Psi(s)p_e^{(\lambda)}(t-s-r)dsdr \qquad (5.5) \\ &= \frac{\pi}{\log t} + \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)r}\left(\int_0^{\varepsilon(t-r)}\Psi(s)p_e^{(\lambda)}(t-s-r)ds\right. \\ &\qquad \left. + \int_{\varepsilon(t-r)}^{t-r}\Psi(s)p_e^{(\lambda)}(t-s-r)ds\right)dr + O\left(\frac{1}{\log^2 t}\right). \end{aligned}$$

Now over the interval $[0, \varepsilon(t-r)]$ we can bound the copy of $p_e^{(\lambda)}$ by its value at the earlier time $\varepsilon(t-r)$ and over the interval $[\varepsilon(t-r), t-r]$ we simply bound the probability by 1 to give

$$\begin{aligned} p_e^{(\lambda)}(t) &\leq \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right) \\ &+ \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)r}p_e^{(\lambda)}\left((1-\varepsilon)(t-r)\right)\int_0^{\varepsilon(t-r)}\Psi(s)dsdr \\ &\quad + \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)r}\int_{\varepsilon(t-r)}^{t-r}\Psi(s)dsdr. \end{aligned}$$

The inner integral of the second term is the tail of the distribution of the visit to the origin. That is, $1 - p_e^{(\infty)}(\varepsilon(t-r)) \leq 1$. Noting that, $p_e^{(\infty)}(t-r) \geq p_e^{(\infty)}(t)$ we evaluate the inner integral of the third term and so bound

$$\begin{aligned}
&p_e^{(\lambda)}(t) \leq \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right) \\
&+ \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)r} p_e^{(\lambda)}\left((1-\varepsilon)(t-r)\right) dr \\
&\quad + \frac{1}{1+\lambda}\int_0^t (1+\lambda)e^{-(1+\lambda)r}(p_e^{(\infty)}(\varepsilon(t-r) - p_e^{(\infty)}(t))dr.
\end{aligned}$$

We split the first integral at εt and the second integral at $t/2$ since we do not have to be as careful.

$$\begin{aligned}
&p_e^{(\lambda)}(t) \leq \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right) \\
&+ \frac{1}{1+\lambda}\int_0^{\varepsilon t} (1+\lambda)e^{-(1+\lambda)r} p_e^{(\lambda)}\left((1-\varepsilon)(t-r)\right) dr \\
&\quad + \frac{1}{1+\lambda}\int_{\varepsilon t}^t (1+\lambda)e^{-(1+\lambda)r} p_e^{(\lambda)}\left((1-\varepsilon)(t-r)\right) dr \\
&\quad + \frac{1}{1+\lambda}\int_0^{t/2} (1+\lambda)e^{-(1+\lambda)r}(p_e^{(\infty)}(\varepsilon(t-r) - p_e^{(\infty)}(t))dr \\
&\quad + \frac{1}{1+\lambda}\int_{t/2}^t (1+\lambda)e^{-(1+\lambda)r}(p_e^{(\infty)}(\varepsilon(t-r) - p_e^{(\infty)}(t))dr \\
&: = \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right) + \frac{1}{1+\lambda}(I + II + III + IV).
\end{aligned}$$

II and IV are easy to deal with,

$$\begin{aligned}
II &= \int_{\varepsilon t}^t (1+\lambda)e^{-(1+\lambda)r} p_e^{(\lambda)}\left((1-\varepsilon)(t-r)\right) dr \\
&\leq \int_{\varepsilon t}^t (1+\lambda)e^{-(1+\lambda)r} dr \\
&\leq \int_{\varepsilon t}^\infty (1+\lambda)e^{-(1+\lambda)r} dr \\
&= e^{-\varepsilon(1+\lambda)t},
\end{aligned}$$

$$
\begin{aligned}
IV &= \int_{t/2}^{t} (1+\lambda)e^{-(1+\lambda)r}(p_e^{(\infty)}(\varepsilon(t-r) - p_e^{(\infty)}(t))dr \\
&\leq \int_{t/2}^{t} (1+\lambda)e^{-(1+\lambda)r}dr \\
&\leq \int_{t/2}^{\infty} (1+\lambda)e^{-(1+\lambda)r}dr \\
&= e^{-(1+\lambda)t/2}
\end{aligned}
$$

which are both tiny. Now for I which constitutes our main term. Bounding by values taken at later times

$$
\begin{aligned}
I &= \int_{0}^{\varepsilon t} (1+\lambda)e^{-(1+\lambda)r}p_e^{(\lambda)}\left((1-\varepsilon)(t-r)\right)dr \\
&\leq p_e^{(\lambda)}\left((1-\varepsilon)^2 t\right) \int_{0}^{\varepsilon t} (1+\lambda)e^{-(1+\lambda)r}dr \\
&\leq p_e^{(\lambda)}\left((1-\varepsilon)^2 t\right) \int_{0}^{\infty} (1+\lambda)e^{-(1+\lambda)r}dr \\
&= p_e^{(\lambda)}\left((1-\varepsilon)^2 t\right).
\end{aligned}
$$

III is the trickiest to handle. Once again, bounding by values at later times

$$
\begin{aligned}
III &= \int_{0}^{t/2} (1+\lambda)e^{-(1+\lambda)r}(p_e^{(\infty)}(\varepsilon(t-r) - p_e^{(\infty)}(t)))dr \\
&\leq (p_e^{(\infty)}(\varepsilon t/2) - p_e^{(\infty)}(t)) \int_{0}^{t/2} (1+\lambda)e^{-(1+\lambda)r}dr \\
&\leq (p_e^{(\infty)}(\varepsilon t/2) - p_e^{(\infty)}(t)) \int_{0}^{\infty} (1+\lambda)e^{-(1+\lambda)r}dr \\
&\leq (p_e^{(\infty)}(\varepsilon t/2) - p_e^{(\infty)}(t)) \\
&= \pi \left(\frac{\log t - \log(\varepsilon t/2)}{\log(t)\log(\varepsilon t/2)} \right) + O\left(\frac{1}{\log^2 t}\right) \\
&\leq O\left(\frac{1}{\log^2 t}\right).
\end{aligned}
$$

Combining everything we have

$$
p_e^{(\lambda)}(t) \leq \frac{\pi}{\log t} + \frac{1}{1+\lambda} p_e^{(\lambda)}\left((1-\varepsilon)^2 t\right) + O\left(\frac{1}{\log^2 t}\right).
$$

Using (5.5) evaluated $(1-\varepsilon)^2 t$ and substituting

$$\begin{aligned}
p_e^{(\lambda)}(t) \le & \frac{\pi}{\log t} + O\left(\frac{1}{\log^2 t}\right) + \frac{1}{1+\lambda} p_e^{(\infty)}((1-\varepsilon)^2 t) \\
& + \left(\frac{1}{1+\lambda}\right)^2 \int_0^t (1+\lambda) e^{-(1+\lambda)r} \\
& \times \int_0^{(1-\varepsilon)^2 t - r} \Psi(s) p_e^{(\lambda)}((1-\varepsilon)^2 t - s - r) ds dr.
\end{aligned}$$

The integrals can be evaluated in exactly the same way as before by splitting the intervals at the appropriate time and this will lead to

$$p_e^{(\lambda)}(t) \le \frac{\pi}{\log t} + \frac{1}{1+\lambda}\frac{\pi}{\log t} + \left(\frac{1}{1+\lambda}\right)^2 p_e^{(\lambda)}\left((1-\varepsilon)^4 t\right) + 2O\left(\frac{1}{\log^2 t}\right)$$

and by induction, collecting a copy of $O(\log^{-2} t)$ on each go

$$\begin{aligned}
p_e^{(\lambda)}(t) \le & \frac{\pi}{\log t} \sum_{k=0}^{n} \left(\frac{1}{1+\lambda}\right)^k + (n+1) O\left(\frac{1}{\log^2 t}\right) \\
& + \left(\frac{1}{1+\lambda}\right)^{n+1} p_e^{(\lambda)}\left((1-\varepsilon)^{2(n+1)} t\right) \\
= & \frac{\pi}{\log t} \left(\frac{1 - \left(\frac{1}{1+\lambda}\right)^{n+1}}{1 - \frac{1}{1+\lambda}}\right) + (n+1) O\left(\frac{1}{\log^2 t}\right) \\
& + \left(\frac{1}{1+\lambda}\right)^{n+1} p_e^{(\lambda)}\left((1-\varepsilon)^{2(n+1)} t\right) \\
\le & \frac{(1+\lambda)\pi}{\lambda \log t} + \left(\frac{1}{1+\lambda}\right)^{n+1} \left| p_e^{(\lambda)}\left((1-\varepsilon)^{2(n+1)} t\right) - \frac{(1+\lambda)\pi}{\lambda \log t} \right| \\
& + (n+1) O\left(\frac{1}{\log^2 t}\right) \\
\le & \frac{(1+\lambda)\pi}{\lambda \log t} + 2\left(\frac{1}{1+\lambda}\right)^{n+1} + (n+1) O\left(\frac{1}{\log^2 t}\right) \qquad (5.6)
\end{aligned}$$

for large enough t. We need to chose the number of iterations n so that $(1/(1+$

$\lambda))^{n+1} \leq \log^{-2} t$. Let $\rho = \log\left((1+\lambda)/\lambda\right)$, then letting $n = \left\lceil \frac{2 \log \log t}{\rho} \right\rceil - 1$ gives

$$\begin{aligned}
\left(\frac{1}{1+\lambda}\right)^{n+1} &= e^{-(n+1)\rho} \\
&= e^{-\left\lceil \frac{2 \log \log t}{\rho} \right\rceil \rho} \\
&\leq e^{-\frac{2 \log \log t}{\rho} \rho} \\
&= \frac{1}{\log^2 t}.
\end{aligned}$$

With this choice of n, (5.6) becomes

$$p_e^{(\lambda)}(t) \leq \frac{(1+\lambda)\pi}{\lambda \log t} + O\left(\frac{\log \log t}{\log^2 t}\right)$$

which completes the proof. □

This allows us to prove the analogy to (4.3) for the non-instantly coalescing random walks. Recall the definition of ψ given in (4.23)

Proof of Lemma 4.2.3

The proof is the same as for Lemma 4.1.4 but with NC_λ playing the role of NC. □

Appendices

Appendix A

Proofs from Chapter 2

A.1 Proof of Proposition 2.2.1

To prove existence we follow the usual iteration procedure. Let $\xi_t^{(m,0)} \equiv \xi_0$ and define

$$\begin{aligned}\xi_t^{(m,n)}(x) = \xi_0(x) + \int_0^t \sum_{y:y\sim x} \sum_{i=1}^m \Big(&\mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \geq i, \xi_{s-}^{(m,n-1)}(x) < m\} dP_s(i,y,x) \\ &- \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i\} dP_s(i,x,y)\Big) \\ - \int_0^t \sum_{i,j=1}^m \mathbb{1}&\Big(\xi_{s-}^{(m,n-1)}(x) \geq i \vee j, i \neq j\Big) dP_s^c(i,j,x). \end{aligned} \tag{A.1}$$

As for for the proof of 2.1.1, we follow [10] and split the proof into steps.

Step 1 Firstly, we will show that for each $n \in \mathbb{N}$ and any $\theta > 0$

$$\sup_{t\in[0,T]} \sum_x e^{-\theta|x|} \mathbb{E}[\xi_t^{(m,n)}(x)^2] < \infty.$$

Suppose, for an induction argument, that

$$\sup_{t\in[0,T]} \sum_x e^{-\theta|x|} \mathbb{E}[\xi_t^{(m,n-1)}(x)^2] < \infty$$

and note that the base case is satisfied by the assumptions on the initial condition.

Then, by Holder's inequality, we have the simple inequality

$$\left(\sum_{i=1}^{n}|x_i|\right)^p \le n^{p-1}\sum_{i=1}^{n}|x_i|^p$$

which gives

$$\begin{aligned}
\mathbb{E}[\xi_t^{(m,n)}(x)^2] = \mathbb{E}\Bigg[\Bigg(\xi_0(x) + \int_0^t \sum_{y:y\sim x}\sum_{i=1}^{m}\Big(&\mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \ge i\}dP_s(i,y,x) \\
&- \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \ge i\}dP_s(i,x,y)\Big) \\
&- \int_0^t \sum_{i,j=1}^{m}\mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \ge i \vee j, i \ne j\}dP_s^c(i,j,x)\Bigg)^2\Bigg] \\
\le 4\mathbb{E}[\xi_0(x)^2] + 4\mathbb{E}\left[\left(\int_0^t \sum_{y:y\sim x}\sum_{i=1}^{m}\left(\mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \ge i, \xi_{s-}^{(m,n-1)}(x) < m\}dP_s(i,y,x)\right)^2\right]\right. \\
+ 4\mathbb{E}\left[\left(\int_0^t \sum_{y:y\sim x}\sum_{i=1}^{m}\left(\mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \ge i\}dP_s(i,x,y)\right)^2\right]\right. \\
+ 4\mathbb{E}\left[\left(\int_0^t \sum_{i,j=1}^{m}\mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \ge i \vee j, i \ne j\}dP_s^c(i,j,x))\right)^2\right] \\
= 4\sum_{i=1}^{4} I_i. \qquad \text{(A.2)}
\end{aligned}$$

We first consider the second term. Compensating the Poisson processes and using Cauchy-Schwarz we have

$$\begin{aligned}
I_2 &= \mathbb{E}\left[\left(\int_0^t \sum_{y:y\sim x}\sum_{i=1}^{m}\left(\mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \ge i\}dP_s(i,x,y)\right)^2\right]\right. \\
&\le 2\mathbb{E}\left[\left(\sum_{y:y\sim x}\sum_{i=1}^{m}\int_0^t \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \ge i, \xi_s^{(m,n-1)}(x) < m\}d\left(P_s(i,y,x) - \frac{s}{2d}\right)\right)^2\right] \\
&\quad + \frac{2}{(2d)^2}\mathbb{E}\left[\left(\int_0^t \sum_{y:y\sim x}\sum_{i=1}^{m}\mathbb{1}\{\xi_s^{(m,n-1)}(y) \ge i, \xi_s^{(m,n-1)}(x) < m\}ds\right)^2\right].
\end{aligned}$$

Since the covariation between independent compensated Poisson processes is 0, the cross terms in each of the stochastic integrals vanish so that the sums can be removed from the expectations.

$$I_2 \leq 2 \sum_{y:y\sim x} \sum_{i=1}^{m} \mathbb{E}\left[\left(\int_0^t \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \geq i, \xi_s^{(m,n-1)}(x) < m\} d\left(P_s(i,y,x) - \frac{s}{2d}\right)\right)^2\right]$$
$$+ \frac{2}{(2d)^2}\mathbb{E}\left[\left(\int_0^t \sum_{y:y\sim x} \sum_{i=1}^{m} \mathbb{1}\{\xi_s^{(m,n-1)}(y) \geq i, \xi_s^{(m,n-1)}(x) < m\} ds\right)^2\right].$$

Now applying the Itô isometry the stochastic integral and Cauchy-Schwarz to the Lebesgue integral

$$I_2 \leq \frac{2}{2d} \sum_{y:y\sim x} \sum_{i=1}^{m} \mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \geq i, \xi_s^{(m,n-1)}(x) < m\} ds\right]$$
$$+ \frac{2}{(2d)^2}\mathbb{E}\left[\int_0^t \left(\sum_{y:y\sim x} \sum_{i=1}^{m} \mathbb{1}\{\xi_s^{(m,n-1)}(y) \geq i, \xi_s^{(m,n-1)}(x) < m\}\right)^2 ds\right].$$

By bounding the indicators above by $\mathbb{1}\{\xi_s^{(m,n-1)}(y) \geq i\}$ we can sum over i and find

$$\sum_{i=1}^{m} \mathbb{1}\{\xi_s^{(m,n-1)}(y) \geq i\} = |\xi_s^{(m,n-1)}(y)| \wedge m \leq |\xi_s^{(m,n-1)}(y)|.$$

We use this in each term and also use Cauchy-Schwarz to remove the sum over the neighbours of x from the square in the second term. Since $\xi_t^{(m,n)}(x)$ is integer valued for all values of m, n, t, we have that $\xi_t^{(m,n)}(x) \leq \xi_t^{(m,n)}(x)^2$ we find

$$I_2 \leq \frac{2(1+t)}{2d} \sum_{y:y\sim x} \int_0^t \mathbb{E}[\xi_s^{(m,n-1)}(y)^2] ds. \tag{A.3}$$

It is similar, but easier, to show

$$I_3 \leq 2\,(1+t) \int_0^t \mathbb{E}[\xi_s^{(m,n-1)}(x)^2] ds. \tag{A.4}$$

In a similar fashion, we have

$$
\begin{aligned}
I_4 \leq & 2\sum_{i,j=1}^{m}\mathbb{E}\left[\left(\int_0^t \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i \vee j, i \neq j\} d(P_s^c(i,j,x) - \lambda s)\right)^2\right] \\
& + 2\lambda^2 \mathbb{E}\left[\left(\int_0^t \sum_{i,j=1}^{m} \mathbb{1}\left\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i \neq j\right\} ds\right)^2\right] \\
\leq & 2\lambda \sum_{i,j=1}^{m}\mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i \vee j, i \neq j\} ds\right] \\
& + 2\lambda^2 t \mathbb{E}\left[\int_0^t \left(\sum_{i,j=1}^{m} \mathbb{1}\left\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i \neq j\right\}\right)^2 ds\right].
\end{aligned}
$$

In order to use our inductive hypothesis, we need to only have squares of ξ appearing. We must therefore remove the sum from the square in the second term before summing at the expense of some powers of m, since this would otherwise produce a quartic expression for ξ. Doing so, and then summing up the indicators and bounding the integer valued integrands we arrive at

$$
\begin{aligned}
I_4 & \leq 2\lambda \sum_{i,j=1}^{m}\mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i \vee j, i \neq j\} ds\right] \\
& \quad + 2(\lambda m)^2 t \mathbb{E}\left[\int_0^t \sum_{i,j=1}^{m} \mathbb{1}\left\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i \neq j\right\} ds\right] \\
& \leq 2(\lambda + (\lambda m)^2 t)\mathbb{E}\left[\int_0^t |\xi_s^{(m,n-1)}(x)|(|\xi_s^{(m,n-1)}(x)| - 1) ds\right] \\
& \leq 2(\lambda + (\lambda m)^2 t)\mathbb{E}\left[\int_0^t \xi_s^{(m,n-1)}(x)^2 ds\right]. \qquad \text{(A.5)}
\end{aligned}
$$

Substituting (A.3), (A.4) and (A.5) into (A.2) gives

$$
\begin{aligned}
\mathbb{E}[\xi_t^{(m,n)}(x)^2] \leq 4\mathbb{E}[\xi_0^2(x)] + \frac{8(1+t)}{2d} \sum_{y:y\sim x} \int_0^t \mathbb{E}[\xi_s^{(m,n-1)}(y)^2] ds \\
+ 8\big(1 + \lambda + t + (\lambda m)^2 t\big) \int_0^t \mathbb{E}[\xi_s^{(m,n-1)}(x)^2] ds.
\end{aligned}
$$

Then, for any $\theta > 0$

$$
\begin{aligned}
&\sup_{t\le T}\sum_x e^{-\theta|x|}\mathbb{E}[\xi_t^{(m,n)}(x)^2] \\
&\le 4\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0(x)^2\right] \\
&\quad + \frac{8(1+t)}{2d}\sum_x e^{-\theta|x|}\sum_{y:y\sim x}\int_0^T \mathbb{E}[\xi_s^{(m,n-1)}(y)^2]ds \\
&\quad + 8\big(1+\lambda+T+(\lambda m)^2 T\big)\int_0^T\sum_x e^{-\theta|x|}\mathbb{E}[\xi_s^{(m,n-1)}(x)^2]ds.
\end{aligned}
$$

The inequality given in (2.4) gives

$$
\begin{aligned}
&\sup_{t\le T}\sum_x e^{-\theta|x|}\mathbb{E}[\xi_t^{(m,n)}(x)^2] \\
&\le 4\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0(x)^2\right] \\
&\quad + 8\big(2e^{\theta}(1+T)+\lambda+(\lambda m)^2 T\big)\int_0^T\sum_x e^{-\theta|x|}\mathbb{E}[\xi_s^{(m,n-1)}(x)^2]ds
\end{aligned}
$$

which is finite by the induction hypothesis.

Step 2 With the second moments established to be finite, it makes sense to compare successive iterates in mean squared since we now know that

$$\mathbb{E}[|\xi_t^{(m,n+1)}(x)-\xi_t^{(m,n)}(x)|^2] \le 2\mathbb{E}[\xi_t^{(m,n+1)}(x)^2+\xi_t^{(m,n)}(x)^2] < \infty$$

by Cauchy-Schwarz. The aim of this step is to show that for $t\in[0,T]$ there exists C, L such that

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\left|\xi_t^{(m,n+1)}(x)-\xi_t^{(m,n)}(x)\right|^2\right] \le \frac{C(Lt)^n}{n!}.$$

Firstly, we use Cauchy-Schwarz to distribute the square. As before, we we will label each of the terms and estimate them individually.

$$
\begin{aligned}
&\mathbb{E}[|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)|^2] \\
&\leq 3\mathbb{E}\Bigg[\bigg(\int_0^t \sum_{y\sim x}\sum_{i=1}^m \Big|\mathbb{1}\{\xi_{s-}^{(m,n)}(y) \geq i, \xi_{s-}^{(m,n)}(x) < m\} \\
&\qquad\qquad - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \geq i, \xi_{s-}^{(m,n-1)}(x) < m\}\Big| dP_s(i,y,x)\bigg)^2\Bigg] \\
&\quad + 3\mathbb{E}\Bigg[\bigg(\int_0^t \sum_{y\sim x}\sum_{i=1}^m \Big|\mathbb{1}\{\xi_{s-}^{(m,n)}(x) \geq i\} - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i\}\Big| dP_s(i,x,y)\bigg)^2\Bigg] \\
&\quad + 3\mathbb{E}\Bigg[\bigg(\int_0^t \sum_{i,j=1}^m \Big|\mathbb{1}\{\xi_{s-}^{(m,n)}(x) \geq i \vee j, i \neq j\} \\
&\qquad\qquad - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i \vee j, i \neq j\}\Big| dP_s^c(i,j,x)\bigg)^2\Bigg] \\
&= 3\sum_{i=1}^3 I_i. \qquad \text{(A.6)}
\end{aligned}
$$

Compensating the Poisson processes in I_1 and expanding the squares in the integral with respect to the compensated Poisson processes gives

$$
\begin{aligned}
I_1 &\leq 2\sum_{y\sim x}\sum_{i=1}^m \mathbb{E}\Bigg[\bigg(\int_0^t \Big|\mathbb{1}\{\xi_{s-}^{(m,n)}(y) \geq i, \xi_{s-}^{(m,n)}(x) < m\} \\
&\qquad\qquad - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \geq i, \xi_{s-}^{(m,n-1)}(x) < m\}\Big| d\left(P_s(i,y,x) - \frac{2}{2d}\right)\bigg)^2\Bigg] \\
&\quad + \frac{2}{(2d)^2}\mathbb{E}\Bigg[\bigg(\int_0^t \sum_{y\sim x}\sum_{i=1}^m \Big|\mathbb{1}\{\xi_{s-}^{(m,n)}(y) \geq i, \xi_{s-}^{(m,n)}(x) < m\} \\
&\qquad\qquad - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \geq i, \xi_{s-}^{(m,n-1)}(x) < m\}\Big| ds\bigg)^2\Bigg].
\end{aligned}
$$

The Itô isometry and Cauchy-Schwarz lead us to

$$
\begin{aligned}
I_1 \le \frac{2}{2d} \sum_{y \sim x} \sum_{i=1}^{m} \mathbb{E}\Big[\int_0^t \Big| & \mathbb{1}\{\xi_{s-}^{(m,n)}(y) \ge i, \xi_{s-}^{(m,n)}(x) < m\} \\
& - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \ge i, \xi_{s-}^{(m,n-1)}(x) < m\} \Big| ds \Big] \\
+ \frac{2t}{2d} \sum_{y \sim x} \mathbb{E}\Big[\int_0^t \Big(\sum_{i=1}^{m} \Big| & \mathbb{1}\{\xi_{s-}^{(m,n)}(y) \ge i, \xi_{s-}^{(m,n)}(x) < m\} \\
& - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \ge i, \xi_{s-}^{(m,n-1)}(x) < m\} \Big| \Big)^2 ds \Big].
\end{aligned}
$$

The difference of the indicators

$$
\Big| \mathbb{1}\{\xi_{s-}^{(m,n)}(y) \ge i, \xi_{s-}^{(m,n)}(x) < m\} - \mathbb{1}\{\xi_{s-}^{(m,n-1)}(y) \ge i, \xi_{s-}^{(m,n-1)}(x) < m\} \Big|
$$

is at most 1 and achieves the value 1 if the arguments of one of the indicators are realised while one or both of the arguments of the other indicator fails to be realised. In any case, we can bound the difference in the indicators by

$$
|\xi_{s-}^{(m,n)}(y) - \xi_{s-}^{(m,n-1)}(y)| + |\xi_{s-}^{(m,n)}(x) - \xi_{s-}^{(m,n-1)}(x)|
$$

which is at least 1 whenever the difference in the indicators is 1. Bounding the difference in the indicators and then summing over i gives

$$
\begin{aligned}
I_1 \le & \frac{2m}{2d} \sum_{y \sim x} \mathbb{E}\Big[\int_0^t |\xi_s^{(m,n)}(y) - \xi_s^{(m,n-1)}(y)| + |\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)| ds \Big] \\
& + \frac{2tm^2}{2d} \sum_{y \sim x} \mathbb{E}\Big[\int_0^t \Big(\Big|\xi_s^{(m,n)}(y) - \xi_s^{(m,n-1)}(y)| + |\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\Big| \Big)^2 ds \Big].
\end{aligned}
$$

We now raise the integer valued integrand of the first integral to the square, sum the integrals and use Cauchy-Schwarz to distribute the square on the integrand

$$
I_1 \le \frac{4m^2(1+t)}{2d} \sum_{y \sim x} \mathbb{E}\Big[\int_0^t \Big|\xi_s^{(m,n)}(y) - \xi_s^{(m,n-1)}(y)\Big|^2 + \Big|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\Big|^2 ds \Big]. \tag{A.7}
$$

Similarly, we can show

$$I_2 \leq 2(1+t)\mathbb{E}\left[\int_0^t \left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|^2 ds\right]. \tag{A.8}$$

We begin the estimation of I_3 in the same way

$$\begin{aligned} I_3 \leq 2\lambda \sum_{i,j=1}^{m} \mathbb{E}\Big[\int_0^t \big|&\mathbb{1}\{\xi_{s-}^{(m,n)}(x) \geq i \vee j, i \neq j\} \\ &- \mathbb{1}\{\xi_{s-}^{(m,n-1)}(x) \geq i \vee j, i \neq j\}\big| ds\Big] \\ + 2\lambda^2 t\mathbb{E}\Big[\int_0^t \Big(\sum_{i,j=1}^{m} &|\mathbb{1}\{\xi_s^{(m,n)}(x) \geq i \vee j, i \neq j\} \\ &- \mathbb{1}\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i \neq j\}\big|\Big)^2 ds\Big]. \end{aligned}$$

Summing the indicators over i and j by considering the possible values $\xi_s^{(m,\cdot)}(x)$ can take in relation to m we see that

$$\begin{aligned} &\sum_{i,j=1}^{m} \big|\mathbb{1}\{\xi_s^{(m,n)}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i \neq j\}\big| \\ &= 2\sum_{i=2}^{m}\sum_{j=1}^{i-1} \big|\mathbb{1}\{\xi_s^{(m,n)}(x) \geq i\} - \mathbb{1}\{\xi_s^{(m,n-1)}(x) \geq i\}\big| \\ &\leq \Big|(|\xi_s^{(m,n)}(x)| \wedge m)((|\xi_s^{(m,n)}(x)| \wedge m) - 1) \\ &\qquad - (|\xi_s^{(m,n-1)}(x)| \wedge m)((|\xi_s^{(m,n-1)}(x)| \wedge m) - 1)\Big| \end{aligned}$$

Using the following identity

$$X(X-1) - Y(Y-1) = (X-Y)(X+Y) - (X-Y) = (X+Y-1)(X-Y)$$

we can bound

$$\sum_{i,j=1}^{m} \left|\mathbb{1}\{\xi_s^{(m,n)}(x) \geq i \vee j, i \neq j\} - \mathbb{1}\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i \neq j\}\right|$$
$$\leq (2m-1) \left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|$$

So

$$\begin{aligned} I_3 &\leq 2\lambda(2m-1)\mathbb{E}\bigg[\int_0^t \left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right| ds\bigg] \\ &\quad + 2(\lambda(2m-1))^2 t\mathbb{E}\bigg[\int_0^t \left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|^2 ds\bigg] \\ &\leq 2\lambda(2m-1)^2(1 \vee \lambda t)\mathbb{E}\bigg[\int_0^t \left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|^2 ds\bigg]. \end{aligned} \tag{A.9}$$

Substituting (A.7), (A.8) and (A.9) into (A.6), multiplying by $e^{-\theta|x|}$ and summing over all x gives

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\left|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)\right|^2\right]$$
$$\leq 6((1 + 2m^2(1+e^\theta)(1+t)) + \lambda(2m-1)^2(1 \vee \lambda t))$$
$$\times \int_0^t \mathbb{E}\bigg[\sum_x e^{-\theta|x|}\left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|^2\bigg] ds$$

Writing

$$L = L(m,T) = 6(1 + 4m^2(e^\theta(1+T) + \lambda(1 \vee \lambda T))) \tag{A.10}$$

gives

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)|^2\right]$$
$$\leq L \int_0^t \mathbb{E}\bigg[\sum_x e^{-\theta|x|}\left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|^2\bigg] ds.$$

Let

$$\begin{aligned}C &= \sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}\big|\xi_t^{(m,1)}(x) - \xi_t^{(m,0)}(x)\big|^2\right] \\ &\leq 2 \sup_{t\in[0,T]} \mathbb{E}\left[\sum_x e^{-\theta|x|}(\xi_t^{(m,1)}(x)^2 + \xi_t^{(m,0)}(x)^2)\right] < \infty,\end{aligned}$$

we can apply the above bound iteratively to give

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)|^2\right] \leq \frac{C(Lt)^n}{n!}. \tag{A.11}$$

Step 3 Now, for a uniform bound in $t \in [0,T]$ let

$$M_n^{(m)} = \sup_{t\leq T}\sum_x e^{-\theta|x|}\big|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)\big|.$$

Firstly, by the triangle inequality

$$\begin{aligned}\sup_{t\in[0,T]} |\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)\big| \leq \int_0^T \sum_{y\sim x}\sum_{i=1}^m \Big|\mathbb{1}\{\xi_s^{(m,n)}(y) \geq i, \xi_s^{(m,n)}(x) < m\} \\ - \mathbb{1}\{\xi_s^{(m,n-1)}(y) \geq i, \xi_s^{(m,n-1)}(x) < m\}\Big| dP_s(i,y,x) \\ + \int_0^T \sum_{y\sim x}\sum_{i=1}^m |\mathbb{1}\{\xi_s^{(m,n)}(x) \geq i\} - \mathbb{1}\{\xi_s^{(m,n-1)}(x) \geq i\}| dP_s(i,x,y) \\ + \int_0^T \sum_{i,j=1}^m |\mathbb{1}\{\xi_s^{(m,n)}(x) \geq i \vee j, i \neq j\} \\ - \mathbb{1}\{\xi_s^{(m,n-1)}(x) \geq i \vee j, i\neq j\}| dP_s^c(i,j,x)\end{aligned}$$

Secondly, by writing $e^{-\theta|x|} = e^{-\theta|x|/2}e^{-\theta|x|/2}$ and using Cauchy-Schwarz

$$\begin{aligned}\mathbb{E}\left[(M_n^{(m)})^2\right] &\leq \left(\sum_x e^{-\theta|x|}\right)\mathbb{E}\left[\sum_x e^{-\theta|x|}\left(\sup_{t\in[0,T]}|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)|\right)^2\right] \\ &= C_\theta \mathbb{E}\left[\sum_x e^{-\theta|x|}\left(\sup_{t\in[0,T]}|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)|\right)^2\right].\end{aligned}$$

Finally, by performing completely analogous calculations that have already been made gives

$$\mathbb{E}[(M_n^{(m)})^2] \leq C_\theta L \mathbb{E}\left[\int_0^T \sum_x e^{-\theta|x|} \left|\xi_s^{(m,n)}(x) - \xi_s^{(m,n-1)}(x)\right|^2 ds\right]$$

where L is the same as given in (A.10) and so by (A.11)

$$\begin{aligned}\mathbb{E}[(M_n^{(m)})^2] &\leq C_\theta L \int_0^T \frac{C(Ls)^{n-1}}{(n-1)!} ds \\ &= \frac{CC_\theta (LT)^n}{n!}.\end{aligned}$$

Step 4 So by Chebyshev's inequality

$$\begin{aligned}&\mathbb{P}\left[\sup_{t\in[0,T]} \sum_x e^{-\theta|x|} \left|\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)\right| > \frac{1}{(n+1)^2}\right] \\ &\leq (n+1)^4 \mathbb{E}[(M_n^{(m)})^2] \\ &\leq \frac{CC_\theta (n+1)^4 (LT)^n}{n!}\end{aligned}$$

which is summable. The Borel-Cantelli lemma tells us that

$$\sum_{n=0}^{\infty} \mathbb{P}\left[M_n^{(m)} > \frac{1}{(n+1)^2}\right] < \infty$$

which implies that

$$\mathbb{P}\Bigg[\omega \colon \exists N(\omega) \text{ such that} \forall n > N, \\ \sup_{t\in[0,T]} \sum_x e^{-\theta|x|} |\xi_t^{(m,n+1)}(x) - \xi_t^{(m,n)}(x)| \leq \frac{1}{(n+1)^2}\Bigg] = 1$$

and so

$$\begin{aligned}\sum_x e^{-\theta|x|}\xi_t^{(m,n+1)}(x) &= \sum_x e^{-\theta|x|}\left(\xi^{(m,0)}(x) + \sum_{i=0}^{n}[\xi_t^{(m,i+1)}(x) - \xi_t^{(m,i)}(x)]\right) \\ &= \sum_x e^{-\theta|x|}\xi^{(m,0)}(x) \\ &\qquad + \sum_x e^{-\theta|x|}\sum_{i=0}^{n}[\xi_t^{(m,i+1)}(x) - \xi_t^{(m,i)}(x)] \\ &= C_{\theta,\xi_0} + \sum_{i=0}^{n}\sum_x e^{-\theta|x|}[\xi_t^{(m,i+1)}(x) - \xi_t^{(m,i)}(x)]\end{aligned}$$

converges uniformly in t with probability 1. In particular, for each x, $\xi_t^{(m,n)}(x)$ converges uniformly in t almost surely.

Step 5 The process $\xi_t^m(x) = \lim_{n\to\infty}\xi_t^{(m,n)}(x)$ is adapted with càdlàg paths since it is the uniform limit of terms with the same properties and satisfies the SDE (2.15). Indeed, for any x and neighbour $y \sim x$,

$$\begin{aligned}&\Bigg|\int_0^t \sum_{i=1}^{m} \mathbb{1}\{\xi_{s-}^{(m,n)}(y) \geq i, \xi_{s-}^{(m,n)}(x) < m\}dP_s(i,x,y) \\ &\qquad - \int_0^t \sum_{i=1}^{m} \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(x) < m\}dP_s(i,x,y)\Bigg| \\ &\leq \sum_{i=1}^{m}\int_0^t |\mathbb{1}\{\xi_{s-}^{(m,n)}(y) \geq i, \xi_{s-}^{(m,n)}(x) < m\} - \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(x) < m\}|dP_s(i,x,y) \\ &\leq \sum_{i=1}^{m}\int_0^t |\xi_{s-}^{(m,n)}(y) - \xi_{s-}^{(m)}(y)| + |\xi_{s-}^{(m,n)}(x) - \xi_{s-}^{(m)}(x)|dP_s(i,x,y) \to 0\end{aligned}$$

as $n \to \infty$. A similar analysis can be carried out to show

$$\begin{aligned}&\Bigg|\int_0^t \sum_{i,j=1}^{m} \mathbb{1}\{\xi_{s-}^{(m,n)}(x) \geq i \vee j, i \neq j\}dP_s^c(i,j,x) \\ &\qquad - \int_0^t \sum_{i,j=1}^{m} \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\}dP_s^c(i,j,x)\Bigg| \to 0\end{aligned}$$

as $n \to \infty$. Then, that the limit $\xi_t^{(m)}(x) = \lim_{n\to\infty}\xi_t^{(m,n)}(x)$ satisfies (2.15), follows from taking limits in (A.1).

Step 6 Define the norm $||f||_{2,\theta} = \sup_{t\in[0,T]}(\mathbb{E}[\sum_x e^{-\theta|x|}|f(x,t)|^2])^{1/2}$, then

$$||\xi_\cdot^{(m,n+1)}(\cdot) - \xi_\cdot^{(m,n)}(\cdot)||_{2,\theta} \leq \left(\frac{C(LT)^n}{n!}\right)^{1/2}$$

and so for $n > k$, Minkowski's inequality gives

$$\begin{aligned} &||\xi_\cdot^{(m,n+1)}(\cdot) - \xi_\cdot^{(m,k)}(\cdot)||_{2,\theta} \\ &\leq \sum_{i=k}^{n} ||\xi_\cdot^{(m,i+1)}(\cdot) - \xi_\cdot^{(m,i)}(\cdot)||_{2,\theta} \\ &\leq \sum_{i=k}^{n} \left(\frac{C(LT)^i}{i!}\right)^{1/2} \\ &\leq \sum_{i=k}^{\infty} \left(\frac{C(LT)^i}{i!}\right)^{1/2}. \end{aligned}$$

The right hand side converges, hence for all $\varepsilon > 0$ there exists K such that for $n > k \geq K$,

$$||\xi_\cdot^{(m,n+1)}(\cdot) - \xi_\cdot^{(m,k)}(\cdot)|| \leq \sum_{i=k}^{\infty} \left(\frac{C(LT)^i}{i!}\right)^{1/2} < \varepsilon.$$

Hence, for each $t \in [0,T]$, $(\xi_t^{(n)}(x))_n$ converges in a weighted L^2. Since it converges almost surely to $\xi_t^{(m)}(x)$, this is also the limit in weighted space. That the sequence converges in the weighted L^2 norm establishes that

$$||\xi_\cdot^{(m)}(\cdot)||_{2,\theta} = \sup_{t\leq T} \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(m)}(x)^2\right]$$

is finite.

Step 7 We shall now prove that any other solution $\eta^{(m)}$ of (2.15) with finite weighted L^2 norm must satisfy $\eta^{(m)} \equiv \xi^{(m)}$. It is easy to check that the preceeding calculations suffice to show that

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_t^{(m)}(x) - \eta_t^{(m)}(x)|^2\right] \leq L(m,t)\int_0^t \mathbb{E}\left[\sum_x e^{-\theta|x|}|\xi_s^{(m)}(x) - \eta_s^{(m)}(x)|^2\right] ds$$

for the same L given in (A.10). Grönwall's Lemma can then be used to conclude.$\square$

A.2 Proof of Lemma 2.2.2

We will first treat the case $p = 1$. Since the solution to (2.15) is non-negative, integer valued, we can bound

$$\xi_t^{(m)}(x) \leq \xi_0(x) + \int_0^t \sum_{y\sim x} \sum_{i=1}^{m} \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} dP_s(i, y, x).$$

Compensating and taking expectation gives

$$\mathbb{E}\left[\xi_t^{(m)}(x)\right] \leq \mathbb{E}\left[\xi_0(x)\right] + \frac{1}{2d}\mathbb{E}\left[\int_0^t \sum_{y\sim x} \xi_s^{(m)}(y) ds\right].$$

Summing over all x gives

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(m)}(x)\right] \leq \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0(x)\right] + \int_0^t \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_s^{(m)}(x)\right] ds.$$

From which, the Grönwall inequality implies

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(m)}(x)\right] \leq \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_0(x)\right] e^t$$

which is finite and independent of m. By the Itô formula for jump processes, we find that

$$\begin{aligned}
&\xi_t^{(m)}(x)^p \\
&= \xi_0(x)^p + \int_0^t \sum_{y\sim x}\sum_{i=1}^m \left((\xi_{s-}^{(m)}(x)+1)^p - \xi_{s-}^{(m)}(x)^p\right) \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i, \xi_{s-}^{(m)}(x) < m\} \\
&\quad \times dP_s(i,y,x) \\
&\quad + \int_0^t \sum_{y\sim x}\sum_{i=1}^m \left((\xi_{s-}^{(m)}(x)-1)^p - \xi_{s-}^{(m)}(x)^p\right) \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i\} dP_s(i,x,y) \\
&\quad + \int_0^t \sum_{i,j=1}^m \left((\xi_{s-}^{(m)}(x)-1)^p - \xi_{s-}^{(m)}(x)^p\right) \mathbb{1}\{\xi_{s-}^{(m)}(x) \geq i \vee j, i \neq j\} dP_s^c(i,j,x).
\end{aligned}$$

Since $\xi_t^{(m)}(x)$ is integer valued for all $x \in \mathbb{Z}^d, t \geq 0$, $\xi_t^{(m)}(x)^p$ is non-negative for any p while the integrands of the final two integrals are necessarily negative. Hence we can bound

$$\xi_t^{(m)}(x)^p \leq \xi_0(x)^p + \int_0^t \sum_{y\sim x}\sum_{i=1}^m \left((\xi_{s-}^{(m)}(x)+1)^p - \xi_{s-}^{(m)}(x)^p\right) \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} dP_s(i,y,x).$$

Using the simple inequality

$$(n+1)^p - n^p \leq p(2n)^{p-1}$$

which holds for all $p > 1$ and all $n \in \mathbb{N}$ before compensating gives

$$\begin{aligned}
\xi_t^{(m)}(x)^p &\leq \xi_0(x)^p + p2^{p-1} \int_0^t \sum_{y\sim x}\sum_{i=1}^m \xi_{s-}^{(m)}(x)^{p-1} \mathbb{1}\{\xi_{s-}^{(m)}(y) \geq i\} dP_s(i,y,x) \qquad \text{(A.12)} \\
&\leq \xi_0(x)^p + \frac{p2^{p-1}}{2d} \int_0^t \sum_{y\sim x} \xi_s^{(m)}(x)^{p-1} \xi_s^{(m)}(y) ds + m.t..
\end{aligned}$$

Taking expectation leads us to

$$\mathbb{E}\left[\xi_t^{(m)}(x)^p\right] \leq \mathbb{E}\left[\xi_0(x)^p\right] + \frac{p2^{p-1}}{2d} \int_0^t \sum_{y\sim x} \mathbb{E}\left[\xi_s^{(m)}(x)^{p-1} \xi_s^{(m)}(y)\right] ds$$

whereupon Young's inequality gives

$$\mathbb{E}\left[\xi_t^{(m)}(x)^p\right] \le \mathbb{E}\left[\xi_0(x)^p\right] + \frac{p2^{p-1}}{2d}\int_0^t \sum_{y\sim x}\mathbb{E}\left[\frac{(p-1)\xi_s^{(m)}(x)^p}{p} + \frac{\xi_s^{(m)}(y)^p}{p}\right]ds$$
$$= \mathbb{E}\left[\xi_0(x)^p\right] + \frac{2^{p-1}}{2d}\int_0^t 2d(p-1)\mathbb{E}[\xi_s^{(m)}(x)^p] + \sum_{y\sim x}\mathbb{E}[\xi_s^{(m)}(y)^p]ds.$$

For any $\theta > 0$, multiply through by $e^{-\theta|x|}$ and sum over all x gives

$$\mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_t^{(m)}(x)^p\right] \le \sum_x e^{-\theta|x|}\mathbb{E}\left[\xi_0(x)^p\right] + p2^{p-1}\int_0^t \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_s^{(m)}(x)^p\right]ds$$
$$\le \sum_x e^{-\theta|x|}\mathbb{E}\left[\xi_0(x)^p\right]e^{p2^{p-1}t} \tag{A.13}$$

where we have used the Grönwall Lemma in the final inequality. Taking the supremum over $t \in [0,T]$ in (A.12) gives

$$\sup_{t\in[0,T]}\xi_t^{(m)}(x)^p \le \xi_0(x)^p + p2^{p-1}\int_0^T \sum_{y\sim x}\sum_{i=1}^m \xi_{s-}^{(m)}(x)^{p-1}\mathbb{1}\{\xi_{s-}^{(m)}(y) \ge i\}dP_s(i,y,x)$$

and so

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]}\sum_x e^{-\theta|x|}|\xi_t^{(m)}(x)|\right)^p\right]$$
$$\le \mathbb{E}\left[\left(\sum_x e^{-\frac{(p-1)\theta|x|}{p}}e^{-\frac{\theta|x|}{p}}\sup_{t\in[0,T]}|\xi_t^{(m)}(x)|\right)^p\right]$$
$$\le \left(\sum_x e^{-(p-1)\theta|x|}\right)\mathbb{E}\left[\sum_x e^{-\theta|x|}\left(\sup_{t\in[0,T]}|\xi_t^{(m)}(x)|\right)^p\right]$$
$$\le C_{\theta,p}\left(\sum_x e^{-\theta|x|}\mathbb{E}\left[\xi_0(x)^p\right] + p2^{p-1}\int_0^T \mathbb{E}\left[\sum_x e^{-\theta|x|}\xi_s^{(m)}(x)^p\right]ds\right)$$
$$\le C_{\theta,p}\sum_x e^{-\theta|x|}\mathbb{E}\left[\xi_0(x)^p\right]\left(1 + p2^{p-1}\int_0^T e^{p2^{p-1}s}ds\right) \quad \text{by (A.13)}$$
$$\le C_{\theta,p}\sum_x e^{-\theta|x|}\mathbb{E}\left[\xi_0(x)^p\right]\left(1 + e^{p2^{p-1}T}\right)$$

and the right hand side is finite for all T and independent of m.

□

Appendix B

Proof of Proposition 3.4.3

We aim to show here that $\xi_t^*(x) \to \xi_t(x)$ as $m, L \to \infty$. By our choice of initial condition $\xi_0(x) = \tilde{\xi}_0(x) \wedge m$ for $x \in B_L$, there are no more than m particles initially at any site of B_L. Therefore we can assign colours to positions at a site in the sense of (3.8) and make sense of the evolution of the finil term of (3.9)

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}|\xi_t(x) - \xi_t^*(x)|^2\right] \leq \mathbb{E}\left[\sum_{x\in B_L} e^{-\theta|x|}|\xi_t(x) - \xi_t^*(x)|^2\right] + 2\mathbb{E}\left[\sum_{x\notin B_L} e^{-\theta|x|}(\xi_t(x)^2 + \tilde{\xi}_0(x)^2)\right] \tag{B.1}$$

where the left hand side converges and the second term of the right hand side tends to 0 as $L \to \infty$ in a way that is independent of m. Both of these facts are due to the moment conditions satisfied. Therefore, we only need to treat

$$\mathbb{E}\left[\sum_{x\in B_L} e^{-\theta|x|}|\xi_t(x) - \xi_t^*(x)|^2\right].$$

Now for $x \in B_L$, consider the difference $|\xi_t(x) - \xi_t^*(x)|$. We will expand this difference collecting like terms and also collecting terms that are vanishingly small with large

m, L.

$$
\begin{aligned}
|\xi_t(x) - \xi_t^*(x)| = \Bigg| & \int_0^t \sum_{i=1}^{\infty} \sum_{y \sim x} \mathbb{1}\{\xi_{s-}(y) \geq i\} dP_s(i, y, x) \\
& - \int_0^t \sum_{i=1}^{\infty} \sum_{y \sim x} \mathbb{1}\{\xi_{s-}(x) \geq i\} dP_s(i, x, y) \\
& - \int_0^t \sum_{i,j=1}^{\infty} \mathbb{1}\{\xi_{s-}(x) \geq i \vee j, i \neq j\} dP_s^c(i, j, x) \\
& - \Bigg(\int_0^t \sum_{i=1}^{m} \sum_{y \in B_L, y \sim x} \mathbb{1}\{\xi_{s-}^*(y) \geq i\} dP_s(i, y, x) \\
& - \int_0^t \sum_{i=1}^{m} \sum_{y \in B_L, y \sim x} \mathbb{1}\{\xi_{s-}^*(x) \geq i\} dP_s(i, x, y) \\
& - \int_0^t \sum_{i,j=1}^{m} \mathbb{1}\{\xi_{s-}^*(x) \geq i \vee j, i \neq j\} dP_s^c(i, j, x) \\
& - \int_0^t \sum_{i,j=1}^{m} \sum_{y \in B_L, y \sim x} \mathbb{1}\{\xi_{s-}^*(y) \geq i, \xi_{s-}^*(x) \geq j\} dP_s(i, y, x, j) \Bigg) \Bigg|.
\end{aligned}
$$

We split the infinite sums at $i, j = m$. Since the Poisson sums in equation (2.14) converge, the tail sums all vanish as $m \to \infty$. For example,

$$
\mathbb{E}\left[\sum_{x \in B_L} e^{-\theta|x|} \left(\int_0^t \sum_{i=m+1}^{\infty} \sum_{y \sim x} \mathbb{1}\{\xi_{s-}(y) \geq i\} dP_s(i, y, x) \right)^2 \right] \to 0 \text{ as } m \to \infty.
$$

So we write

$$
\begin{aligned}
&|\xi_t(x) - \xi_t^*(x)| \\
&\le \int_0^t \sum_{i=1}^m \sum_{y \in B_L, y \sim x} |\mathbb{1}\{\xi_{s-}(y) \ge i\} - \mathbb{1}\{\xi_{s-}^*(y) \ge i\}| dP_s(i, y, x) \\
&+ \int_0^t \sum_{i=1}^m \sum_{y \in B_L, y \sim x} |\mathbb{1}\{\xi_{s-}(x) \ge i\} - \mathbb{1}\{\xi_{s-}^*(x) \ge i\}| dP_s(i, x, y) \\
&+ \int_0^t \sum_{i,j=1}^m |\mathbb{1}\{\xi_{s-}(x) \ge i \vee j, i \ne j\} - \mathbb{1}\{\xi_{s-}^*(x) \ge i \vee j, i \ne j\}| dP_s^c(i, j, x) \\
&+ \int_0^t \sum_{i=1}^m \sum_{y \notin B_L, y \sim x} \mathbb{1}\{\xi_{s-}(y) \ge i\} dP_s(i, y, x) \qquad \text{(B.2)} \\
&+ \int_0^t \sum_{i=1}^m \sum_{y \notin B_L, y \sim x} \mathbb{1}\{\xi_{s-}(x) \ge i\} dP_s(i, x, y) \qquad \text{(B.3)} \\
&+ \int_0^t \sum_{i,j=1}^m \sum_{y \in B_L, y \sim x} \mathbb{1}\{\xi_{s-}^*(y) \ge i, \xi_{s-}^*(x) \ge j\} dP_s(i, y, x, j) \\
&+ \text{ negligible terms.}
\end{aligned}
$$

Similar arguments that we have seen numerous times before show that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{x \in B_L} e^{-\theta|x|} \left(\int_0^t \sum_{i=1}^m \sum_{y \notin B_L, y \sim x} \mathbb{1}\{\xi_{s-}(y) \ge i\} dP_s(i, y, x)\right)^2\right] \\
&\le \frac{2(1+t)}{2d} \sum_{|x|=L} e^{-\theta|x|} \sum_{y \sim x, |y|=L+1} \int_0^t \mathbb{E}[\xi_s(y)^2] ds \\
&\le 2(1+t)e^{\theta} \int_0^t \mathbb{E}\left[\sum_{|y|=L+1} e^{-\theta|y|} \xi_s(y)^2\right] ds
\end{aligned}
$$

which tends to 0 as $L \to \infty$. Therefore, we absorb (B.2) and (B.3) into the negligible terms.

$$
\begin{aligned}
|\xi_t(x) - \xi_t^*(x)| \le & \int_0^t \sum_{i=1}^m \sum_{y \in B_L, y \sim x} |\mathbb{1}\{\xi_{s-}(y) \ge i\} - \mathbb{1}\{\xi_{s-}^*(y) \ge i\}| dP_s(i, y, x) \\
& + \int_0^t \sum_{i=1}^m \sum_{y \in B_L, y \sim x} |\mathbb{1}\{\xi_{s-}(x) \ge i\} - \mathbb{1}\{\xi_{s-}^*(x) \ge i\}| dP_s(i, x, y) \\
& + \int_0^t \sum_{i,j=1}^m |\mathbb{1}\{\xi_{s-}(x) \ge i \vee j, i \ne j\} - \mathbb{1}\{\xi_{s-}^*(x) \ge i \vee j, i \ne j\}| \\
& \times dP_s^c(i, j, x) \\
& + \int_0^t \sum_{i,j=1}^m \sum_{y \in B_L, y \sim x} \mathbb{1}\{\xi_{s-}^*(y) \ge i, \xi_{s-}^*(x) \ge j\} dP_s(i, y, x, j) \\
& + \text{ negligible terms.} \\
& = I + II + III + IV + \text{ negligible terms.}
\end{aligned} \tag{B.4}
$$

The term labelled IV will form part of the negligible terms. Indeed, consider

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^t \sum_{i,j=1}^m \sum_{y \in B_L, y \sim x} \mathbb{1}\{\xi_{s-}^*(y) \ge i, \xi_{s-}^*(x) \ge j\} dP_s(i, y, x, j)\right)^2\right] \\
& \le 2 \sum_{i,j=1}^m \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\left(\int_0^t \mathbb{1}\{\xi_{s-}^*(y) \ge i, \xi_{s-}^*(x) \ge j\} d(P_s(i, y, x, j) - \frac{s}{2dm})\right)^2\right] \\
& + \frac{2t}{2dm^2} \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t (\xi_s^*(y)\xi_s^*(x))^2 \, ds\right] \\
& \le \frac{2}{2dm} \sum_{i,j=1}^m \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t \mathbb{1}\{\xi_{s-}^*(y) \ge i, \xi_{s-}^*(x) \ge j\} ds\right] \\
& + \frac{2t}{2dm^2} \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t (\xi_s^*(y)\xi_s^*(x))^2 \, ds\right] \\
& \le \frac{2}{2dm} \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t \xi_s^*(y)\xi_s^*(x) ds\right] \\
& + \frac{2t}{2dm^2} \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t (\xi_s^*(y)\xi_s^*(x))^2 \, ds\right]
\end{aligned}
$$

which gives a bound of

$$\frac{4(1 \vee t)}{2dm} \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t (\xi_s^*(y)\xi_s^*(x))^2\, ds\right].$$

By Young's inequality we can bound this by

$$\frac{2(1 \vee t)}{2dm} \sum_{y \in B_L, y \sim x} \mathbb{E}\left[\int_0^t \left(\xi_s^*(y)^4 + \xi_s^*(x)^4\right) ds\right].$$

Evaluating at $\tau_{R,\theta/2} = \inf\left\{t > 0\colon \sum_x e^{-\frac{\theta|x|}{2}} \xi_s^*(x)^2 > R\right\}$ and summing over all $x \in B_L$

$$\begin{aligned}
&\frac{2(1 \vee t)(1+e^\theta)}{m} \mathbb{E}\left[\int_0^{t \wedge \tau_{R,\theta/2}} \sum_{x \in B_L} e^{-\theta|x|} \xi_s^*(x)^4 ds\right] \\
&\leq \frac{2(1 \vee t)(1+e^\theta)}{m} \mathbb{E}\left[\int_0^{t \wedge \tau_{R,\theta/2}} \left(\sum_{x \in B_L} e^{-\theta|x|/2} \xi_s^*(x)^2\right)^2 ds\right] \\
&\leq \frac{2(1 \vee t)(1+e^\theta)tR^2}{m} \to 0 \text{ as } m \to \infty.
\end{aligned}$$

The first two terms in (B.4) will easily give us a bound suitable for a Grönwall type argument. That is, we have that there exists some constant depending only t, θ, independent of m, L such that

$$\mathbb{E}\left[\sum_{x \in B_L} e^{\theta|x|}(I^2 + II^2)\right] \leq C \int_0^t \mathbb{E}\left[\sum_{x \in B_L} e^{\theta|x|} |\xi_s(x) - \xi_s^*(x)|^2\right] ds.$$

Like in the proof of uniqueness for equation (2.14), the quadratic terms that arise from the coalescence cannot be dealt with so straightforwardly. However, we may repeated the calculations that follow after (2.34) with ξ^* in place of η, by introducing an $\varepsilon \in (0, \theta)$ and a stopping time $\tilde{\tau}_{R,\varepsilon} = \inf\{t > 0\colon \sum_x e^{-\varepsilon|x|}\left(|\xi_t(x)|^2 + |\xi_t^*(x)|^2\right) >$

$R\}$ and we will find a bound of

$$\mathbb{E}\left[\sum_{x\in B_L} e^{\theta|x|}(III(t\wedge\tilde{\tau}_{R,\varepsilon})^2)\right] \leq C\mathbb{E}\left[\int_0^t \left(\sum_x e^{-(\theta-\varepsilon)|x|}\left|\xi_s(x)-\xi_s^*(x)\right|^2\right) ds\right]$$

for some C depending only on t, λ, R. Returning to (B.1), evaluating at $t\wedge\tau_{R,\theta/2}\wedge\tilde{\tau}_{R,\varepsilon}$ and collecting into $\epsilon_{m,L}$ all the contributions we have show to be vanishing, we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}|\xi_{t\wedge\tau_{R,\theta/2}\wedge\tilde{\tau}_{R,\varepsilon}}(x)-\xi^*_{t\wedge\tau_{R,\theta/2}\wedge\tilde{\tau}_{R,\varepsilon}}(x)|^2\right]\\ &\quad\leq \epsilon_{m,L}+C_{t,\theta,\lambda,R}\mathbb{E}\left[\int_0^{t\wedge\tau_{R,\theta/2}\wedge\tilde{\tau}_{R,\varepsilon}}\sum_{x\in B_L} e^{-\theta|x|}(1+e^{\varepsilon|x|})|\xi_s(x)-\xi_s^*(x)|^2 ds\right]\end{aligned}$$

Choosing R large enough guarantees $t\wedge\tau_{R,\theta/2}\wedge\tilde{\tau}_{R,\varepsilon}=t$ almost surely due to the moment conditions satisfied by ξ and ξ^* which allows the exchange of espectation and integral. Then adding the positive quantity

$$C_{t,\theta,\lambda,R}\int_0^t \mathbb{E}\left[\sum_{x\notin B_L} e^{-\theta|x|}(1+e^{\varepsilon|x|})|\xi_s(x)-\xi_s^*(x)|^2\right] ds$$

to the right hand side gives

$$\begin{aligned}&\mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}|\xi_t(x)-\xi_t^*(x)|^2\right]\\ &\quad\leq \epsilon_{m,L}+C_{t,\theta,\lambda,R}\int_0^t \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}(1+e^{\varepsilon|x|})|\xi_s(x)-\xi_s^*(x)|^2\right] ds.\end{aligned}$$

The right hand side is finite since

$$\mathbb{E}\left[\sum_x e^{-\alpha|x|}\xi_0(x)^2\right] < \infty$$

for all $\alpha > 0$, and ε is such that $0 < \varepsilon < \theta$. Rearranging the finite sums we have

$$\varepsilon_{m,L} + \sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}\mathbb{E}\left[C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\int_0^t |\xi_s(x)-\xi_s^*(x)|^2 ds - |\xi_t(x)-\xi_t^*(x)|^2\right] \geq 0. \tag{B.5}$$

Now, for fixed m and L, the sum in the left hand side is either weakly positive or strictly negative. If it is that case for this choice of m and L that the sum is positive, then there is a non-empty set $A \subseteq \mathbb{Z}^d$ such that the summands corresponding to $x \in A$ are positive. Equivalently, for $x \in A$ we have

$$\mathbb{E}[|\xi_t(x)-\xi_t^*(x)|^2] \leq C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\int_0^t \mathbb{E}[|\xi_s(x)-\xi_s^*(x)|^2]ds$$

and Grönwall's inequality implies for all $x \in A$ that

$$\mathbb{E}[|\xi_t(x)-\xi_t^*(x)|^2] = 0.$$

Since (B.5) was assumed to be weakly positive and the all contributions from A are 0, then it must also be true that the negative contributions are 0 and so the entire sum must be 0. If the sum in (B.5) is strictly negative for the fixed choice of m, L then we have

$$\begin{aligned} 0 &< -\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}\mathbb{E}\left[C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\int_0^t |\xi_s(x)-\xi_s^*(x)|^2 ds - |\xi_t(x)-\xi_t^*(x)|^2\right] \\ &\leq \varepsilon_{m,L}. \end{aligned}$$

However, as m and L vary, the sign of the sum in (B.5) may change. Or rather, it may jump from taking the value 0 to take a genuinely negative value but in either case, it necessarily holds that

$$0 \leq -\sum_{x\in\mathbb{Z}^d} e^{-\theta|x|}\mathbb{E}\left[C_{t,\theta,\lambda,R}(1+e^{\varepsilon|x|})\int_0^t |\xi_s(x)-\xi_s^*(x)|^2 ds - |\xi_t(x)-\xi_t^*(x)|^2\right] \leq \varepsilon_{m,L}.$$

Then, taking the limit $m, L \to \infty$ shows that

$$0 \leq \mathbb{E}[|\xi_t(x) - \xi_t^*(x)|^2] - C_{t,\theta,\lambda,R}(1 + e^{\varepsilon|x|}) \int_0^t \mathbb{E}[|\xi_s(x) - \xi_s^*(x)|^2] ds \to 0$$

or equivalently

$$\lim_{m,L\to\infty} \mathbb{E}[|\xi_t(x) - \xi_t^*(x)|^2] = \lim_{m,L\to\infty} C_{t,\theta,\lambda,R}(1 + e^{\varepsilon|x|}) \int_0^t \mathbb{E}[|\xi_s(x) - \xi_s^*(x)|^2] ds.$$

Letting $\bar{\xi}_t(x) = \lim_{m,L\to\infty} \mathbb{E}[|\xi_t(x) - \xi_t^*(x)|^2]$ gives

$$\bar{\xi}_t(x) = C_{t,\theta,\lambda,R}(1 + e^{\varepsilon|x|}) \int_0^t \bar{\xi}_s(x) ds$$

whereupon Grönwall's inequality allows us to conclude that $\bar{\xi}_t(x) = 0$ and hence $\xi_t^*(x) \to \xi_t(x)$ almost surely as $m, L \to \infty$. □

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