European Journal of Operational Research 297 (2022) 579–591

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Search-and-rescue rendezvous

Pierre Leone[a,*], Julia Buwaya[a], Steve Alpern[b]

[a] Computer Science Department, University of Geneva, Route de Drize 7, Carouge 1227, Switzerland
[b] Warwick Business School, University of Warwick, Coventry CV4 7AL, UK

ARTICLE INFO

Article history:
Received 21 May 2020
Accepted 10 May 2021
Available online 23 May 2021

Keywords:
Linear programming
Rendezvous problem on the line
Search-and-rescue rendezvous

ABSTRACT

We consider a new type of asymmetric rendezvous search problem in which player II needs to give player I a 'gift' which can be in the form of information or material. The gift can either be transfered upon meeting, as in traditional rendezvous, or it can be dropped off by player II at a location he passes, in the hope it will be found by player I. The gift might be a water bottle for a traveller lost in the desert; a supply cache for Captain Scott in the Antarctic; or important information (left as a gift). The common aim of the two players is to minimize the time taken for I to either meet II or find the gift. We find optimal agent paths and drop off times when the search region is a line, the initial distance between the players is known and one or both of the players can leave gifts.

A novel and important technique introduced in this paper is the use of families of linear programs to solve this and previous rendezvous problems. Previously, the approach was to guess the answer and then prove it was optimal. Our work has applications to other forms of rendezvous on the line: we can solve the symmetric version (players must use the same strategy) with two gifts and we show that there are no asymmetric solutions to this two gifts problem. We also solve the GiftStart problem, where the gift or gifts must be dropped at the start of the game. Furthermore, we can solve the Minmax version of the game where the objective function is to minimize the maximum rendezvous time. This problem admits variations where players have $0, 1$ or 2 gifts at disposal. In particular, we show that the classical Wait For Mommy strategy is optimal for this setting.

1. Introduction

This paper introduces a new variation of the Asymmetric Rendezvous Problem on the Line (ARPL). That problem was the first asymmetric version of the continuous time rendezvous problem (Alpern & Gal, 1995): Two agents, I and II, are placed a distance D apart on a foggy road (real line) and faced equiprobably in one of the two possible directions. Each agent knows the distance D but not the direction to the other. That is, they do not know whether the other is in front of them (in the direction they are facing) or behind them. As it is foggy, they keep moving at unit speed until the first time T that they meet (bump into each other). Their common goal is to minimize the expected value $\hat{T}$ of their rendezvous time T. The minimum value of $\hat{T}$, calculated over all pairs of paths, is called the Rendezvous Value R of the problem.

A simple strategy pair is the so called Wait For Mommy (WFM) strategy. To illustrate this more simply we fix a particular value $D = 16$ and call the distance units miles and the time units hour. In WFM, agent I (Baby) stays still at his starting point. Mommy knows that Baby is either 16 miles in front of her, or 16 miles behind. So she goes to these two locations successively, with rendezvous times T of either 16 or $16 + 32 = 48$ and $\hat{T} = 24$. In Alpern and Gal (1995) Alpern and Gal showed that a small modification of WFM, called Modified Wait For Mommy (MWFM), is in fact optimal: Mommy moves as in WFM but Baby (I) tries to meet earlier by guessing the arrival direction of Mommy while returning to his start at times 16 and 48. If Baby guesses correctly, the meeting time of 16 reduces to 8 and of 48 reduces to 32, so that $\hat{T}$ is the average of 8,16,32 and 48, giving a Rendezvous Value of $(8 + 16 + 32 + 48)/4 = 26$, or more generally $R = 13D/8$ for the classical problem. The fact that the two agents must adopt different paths to achieve the Rendezvous Value is what makes this an *asymmetric* problem. If they must adopt the same mixed strategy, this is the symmetric problem, which is surprisingly unsolved. It is worth noting that simply by exhibiting this MWFM strategy and calculating the meeting times, we have shown that $R \leq 13D/8$. The

* Corresponding author.
E-mail addresses: Pierre.Leone@unige.ch (P. Leone), Julia.Buwaya@unige.ch (J. Buwaya), Steve.Alpern@wbs.ac.uk (S. Alpern).

https://doi.org/10.1016/j.ejor.2021.05.009
0377-2217/

difficult part of rendezvous problems is establishing the reverse inequality to prove optimality.

In this paper, we introduce a new version of the ARPL in which one or both agents have *gifts*, which they can drop anywhere along their path. The game ends, as before, if the agents meet. But it also ends at any earlier time when an agent finds (reaches) the gift dropped by the other. If only one player has a gift, this game is denoted G_1; and if both have a gift it is denoted G_2. (The original game without gifts is denoted G_0.) There are several interpretations of the gift games. For G_1, the agent with a gift might be a search party for Scott of Antarctica, with a gift of food and heating oil. Scott would have been saved if the search party reached him, or if he came across the gift. Gal (2019) has suggested the gift could be a fully charged mobile phone with which to call the rescue party. An interpretation of G_2 is of a boy and girl who want to meet up at a huge rock concert but have not specified a meeting point or exchanged phone numbers. Each has a business card with phone number which can be pinned to a bulletin board - if either finds the other's card, they can call the phone number and arrange to meet. If the agents are spies, the gift might be a memory card with the information they wish to transmit.

While the notion of gifts is new to this paper, an earlier notion of *markers* was studied by Baston and Gal (2001) and Leone and Alpern (2018a). In the former, the marker had to be dropped by the agent at his starting point, while in the latter the time and place of the drop is decided by the agents. The markers are worthless in that finding them is of no direct benefit to the finder, but they are useful in learning where the other agent has been and can reduce the rendezvous times. Obviously gifts are better for the finder than just markers, as finding them ends the game immediately. Baston and Gal (2001) found that when both agents have a marker which must be dropped at time 0, the game we call $M_2(t=0)$, the rendezvous value is $R = 3D/2$.Leone and Alpern (2018a) found that when one agent has a marker which can be dropped at any time, the game M_1, the rendezvous value is also $R = 3D/2$. The Rendezvous Value does not improve when both agents have a marker which can be dropped at any time, the game we call M_2. This may be summarized by the equation: 1 marker = 2 markers.

The main results of this paper are for the games where one player has a gift (which we call G_1) and where both have a gift (which we call G_2). The Rendezvous Values for theses games are respectively $21D/16$ and $20D/16$. For $D = 1$, one-gift reduces the rendezvous value by $(26-21)/26 = 19\%$, while the second gift (one for the other player) further reduces it by another $(21-20)/21 = 4.7\%$. Note that while a second marker was not of any benefit, the Rendezvous Values of the M_2 and M_1 games are the same, here a second gift lowers the Rendezvous Value. The proof method in Leone and Alpern (2018a) is based on simulation and is not related to what is presented here.

An important aspect of this paper is that we introduce a linear programming (LP) solution method to the rendezvous problems, which we use here to solve the gift games G_1 and G_2. This also gives an alternative method of obtaining a solution to the original problem G_0 of Alpern and Gal (1995). This method potentially brings solutions to more complex search regions like planar grids or networks, which are currently not covered by any methods of analysis.

Furthermore, our work has implications for other forms of rendezvous search on the line. We are able to solve the Symmetric Rendezvous Problem on the Line (both players must adopt same strategy) for two gifts, while the no-gift version remains unsolved after 25 years. We solve the analog of Baston and Gal (2001) Mark-Start problem (markers must be dropped at the start) for gifts. We are also able to show that G_2 has no asymmetric solution.

We apply the same solution tools to solve the Minmax problem, i.e. optimal strategies minimize the maximum rendezvous time. This setting admits variations where players have gifts at disposal or not. This leads to Minmax versions of G_0, G_1 and G_2. New solutions are shown for these problems. In particular the Wait For Mommy strategy is shown to be optimal for the G_0 Minmax problem.

2. Literature review

The rendezvous search problem was first informally proposed in Alpern (1976). A discrete version of the problem with a finite number of locations was analyzed in Anderson and Weber (1990). This problem was later solved for three locations by Weber (2012). Rendezvous-evasion on discrete locations was studied in Lim (1997) and solved for two locations (boxes) in Gal and Howard (2005).

The continuous form of the problem was introduced in Alpern (1995), for symmetric players who had to use the same mixed strategy when placed a known distance apart on the line. The player-asymmetric form of the problem (used in this paper), where players can adopt distinct strategies, was introduced in Alpern and Gal (1995).

The corresponding player-symmetric problem on the line was developed in Anderson and Essegaier (1995). Their results have been successively improved in Baston (1999), Gal (1999), and Han, Du, Vera, and Zuluaga (2008). These papers assumed that the initial distance between the players on the line was known. The version where the initial distance between the players is unknown was studied in Baston and Gal (1998), Alpern and Beck (1999), Alpern and Beck (2000) and Ozsoyeller, Beveridge, and Isler (2013).

Problems where players move on a circle share similarities with problems on the line (Di Stefano & Navarra, 2017; Flocchini, Kranakis, Krizanc, Santoro, & Sawchuk, 2004b; Kranakis, Krizanc, & Markou, 2010; Kranakis, Santoro, Sawchuk, & Krizanc, 2003). On the circle, symmetry breaking has to be solved as well to ensure rendezvous. Versions where tokens may be left by players are presented in Czyzowicz, Dobrev, Kranakis, and Krizanc (2008), Flocchini et al. (2004a) and Das, Mihalák, Šrámek, Vicari, and Widmayer (2008).

The continuous rendezvous problem has been studied on finite networks: the unit interval and circle in Howard (1999); arbitrary networks in Alpern (2002b); planar grids in Anderson and Fekete (2001) and Chester and Tutuncu (2004), the star graph in Di Stefano and Navarra (2017); Kikuta and Ruckle (2007).

The present paper is an application of rendezvous search to 'search-and-rescue' operations (Lidbetter, 2020). A different application of search theory to that area is in Alpern (2011) and Chrobak, Gąsieniec, Gorry, and Martin (2015), where the *searcher* must find the *hider* (injured person) and then bring him back to a specified first aid location. An application of rendezvous to robotic exploration is given in Roy and Dudek (2001). An application of rendezvous to the communications problem of finding a common channel is given in Chang, Liao, and Lien (2015). Using markers in communication networks to help matching publishers and consumers of information is suggested in Sarkar, Zhu, and Gao (2009), Shi, Zheng, Yang, and Zhao (2012), Leone and Muñoz (2013), Muñoz and Leone (2014), Kündig, Leone, and Rolim (2016) and Tang, Kuo, and Tsai (2017). These works have relevant applications to anonymous communication networks where the content of information is important (content based routing). It is observed that decentralized search strategies prove to be efficient in terms of congestion. A survey of the rendezvous search problem is given in Alpern (2002a).

It will be of interest to consider all of these problems in the two dimensional setting of a planar grid (Z^2), as initiated for asymmetric rendezvous (Anderson & Fekete, 2001) and studied in Chester and Tutuncu (2004); Zoroa, Zoroa, and Fernández-Sáez

(2009), and on arbitrary graphs as studied in Alpern (2002b) or bipartite graphs (Baston & Kikuta, 2019). The use of gifts could also be studied in the rendezvous contexts of Howard (1999), Lim (1997), Anderson and Essegaier (1995), Han et al. (2008) and in other settings discussed in the surveys (Alpern, 2002a; Pelc, 2019). Gifts might also be used in the discrete rendezvous problem discussed in Kikuta and Ruckle (2010); Weber (2012) or in the context of bounded resources, as in Alpern and Beck (1997).

It is useful to distinguish the dropping of gifts described and analyzed here with the dropping of markers introduced in Baston and Gal (2001) and further studied in Leone and Alpern (2018a). The earlier papers analyzed the situation where the rendezvous players arrive on the scene by parachutes, which cannot be moved. When one player comes upon the parachute of the other, he learns the initial location of the other player. He can then alter his intended movements accordingly. The game does not end, however, until the players eventually meet. Thus markers only help in meeting, whereas gifts end the game in themselves. When both players have a marker that must be dropped at time 0, Baston and Gal (2001) found the Rendezvous Value and optimal strategy.

3. Formalization of the problems

We begin by presenting the formalization of the problem G_0 when there are no-gifts, as given in Alpern and Gal (1995). This presentation is essentially the same as that given in Leone and Alpern (2018a). Two players, *I* and *II*, are placed a distance *D* apart on the real line, and faced in random directions. They are restricted to moving at maximum at unit speed, so their position, relative to their starting point, is given by a function $f(t) \in \mathcal{F}$ where

$$\mathcal{F} = \left\{ f : [0,T] \to R,\ f(0) = 0, \left| f(t) - f\left(t'\right) \right| \le \left| t - t' \right| \forall t, t' \in [0,T] \right\}, \quad (1)$$

for some *T* sufficiently large so that the rendezvous will have taken place. We will see in Proposition 4 that optimal paths are piecewise linear with slopes ±1 and so they can be specified by their turning points. Suppose *I* chooses path $f \in \mathcal{F}$ and *II* chooses path $g \in \mathcal{F}$. The meeting time depends on which way they are initially facing. If they are facing each other, the meeting time is given by

$$t^1 = t^{\rightarrow\leftarrow} = \min\{t : f(t) + g(t) = D\}.$$

If they are facing away from each other, the meeting time is given by

$$t^2 = t^{\leftarrow\rightarrow} = \min\{t : -f(t) - g(t) = D\}.$$

If they are facing the same way, say both left, and *I* is on the left, the meeting time is given by

$$t^3 = t^{\leftarrow\leftarrow} = \min\{t : -f(t) + g(t) = D\}.$$

If *I* is on the left and they are both facing right, the meeting time is given by

$$t^4 = t^{\rightarrow\rightarrow} = \min\{t : +f(t) - g(t) = D\}.$$

To summarize, the four meeting times when strategies (paths) *f* and *g* are chosen are given by the four values, see Fig. 1,

$$\min\{t : \pm f(t) \pm g(t) = D\}.$$

The rendezvous time for given strategies is their expected meeting time

$$R(f,g) = \frac{1}{4}\left(t^1 + t^2 + t^3 + t^4\right). \quad (2)$$

The Rendezvous Value $\bar{R}$ is the optimum expected meeting time,

$$\bar{R} = \min_{f,g \in \mathcal{F}} R(f,g) = R\left(\bar{f}, \bar{g}\right). \quad (3)$$

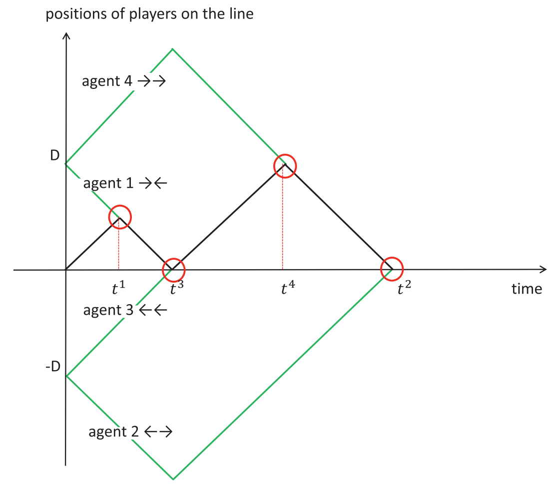

Fig. 1. Player I starts at position 0 and his path follows the black line. The paths of the four agents of player II starting at position ±1 with directions up or down are depicted with the green lines. The rendezvous of player I with the four agents, occurring at times t^1, t^2, t^3, t^4, are circled. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

There is a simple interpretation of Eq. (2) as being the average time for player *I* (whose position at time *t* is $f(t)$) to meet four agents of player *II*. We take as the origin of the line the starting point of Player *I* and we take his forward direction to be the positive direction on the line (up, if the line is depicted vertically). The four 'agents' of *II* start at $+D$ and $-D$ and face up or down, so their paths are $\pm D \pm g(t)$. The meeting times with these 'agents' are exactly the rendezvous times t^i, $i = 1, 2, 3, 4$, see Fig. 1.

It has been shown in Alpern and Gal (1995) for the 'no gift' case, that optimal paths are of the form

$$f = [f_1, \ldots, f_k], \quad (4)$$

where the times f_k are the turning points of the path *f*, namely

$$f'(t) = \begin{cases} +1, & \text{for } f_{2j} \le t \le f_{2j+1} \ \ (\text{where } f_0 \equiv 0), \text{ and} \\ -1 & \text{for } f_{2j-1} \le t \le f_{2j}. \end{cases}$$

If a player has a gift to drop off, we denote his strategy by

$$f = [\tau;\ f_1, \ldots, f_k], \quad (5)$$

where τ is the dropoff time and the f_j are as above. We are now in a position to state and illustrate the initial result in the field, for the case of no-gifts.

Theorem 1 (Alpern & Gal (1995)). ***(G_0-Game)*** *An optimal solution pair for the asymmetric rendezvous problem on the line, with initial distance D, is given, using the path notation of (4), by*

$$\bar{f} = [D/2, D/2, D],\ \ \bar{g} = [D],$$

or equivalently, see Figs. 2 and 3. The corresponding meeting times are

$$t_1 = t^1 = D/2,\ \ t_2 = t^4 = D,\ \ t_3 = t^3 = 2D,\ \ t_4 = t^2 = 3D, \quad (6)$$

with Rendezvous Value

$$\bar{R} = R\left(\bar{f}, \bar{g}\right) = (D/2 + D + 2D + 3D)/4 = 13D/8.$$

The optimal strategy is illustrated on Fig. 1.

Note that in (6) we have introduced the subscripted times t_j as the meeting times t^i given in increasing order. The duration of the strategy pair is the final meeting time t_4. We now illustrate the optimal strategies $\bar{f}, \bar{g}$ separately and then show how the solution can be seen by drawing the single path of *I* ($\bar{f}$) together with the

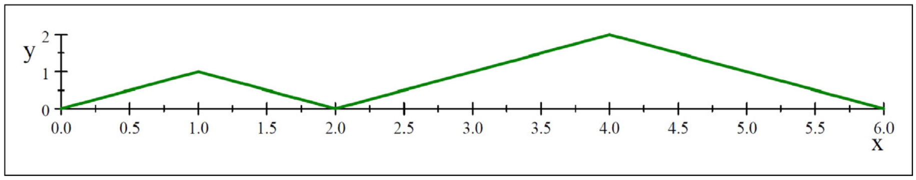

Fig. 2. Plot of $\bar{f}(t)$ in G_0 for $D = 2$.

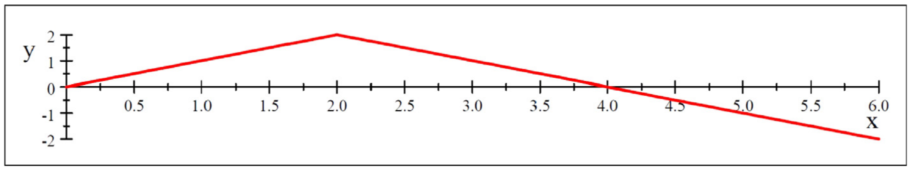

Fig. 3. Plot of $\bar{g}(t)$ in G_0 for $D = 2$.

paths of the four agents of player *II* ($\pm D \pm g(t)$). We take $D = 2$ and draw the paths up to time $t_4 = 3D = 6$, see Figs. 2 and 3.

$$\bar{f}(t) = \begin{cases} t & \text{if} \quad t < 1 \\ 1 - (t-1) & \text{if} \quad 1 \le t < 2 \\ t - 2 & \text{if} \quad 2 \le t < 4 \\ 2 - (t-4) & \text{if} \quad 4 \le t \le 6 \end{cases}$$

$$\bar{g}(x) = \begin{cases} x & \text{if} \quad x < 2 \\ 2 - (x-2) & \text{if} \quad x \ge 2 \end{cases}$$

4. Statement of results for the games G_1 and G_2.

In this section we state our theorems regarding the optimal solutions for the one and two gift games G_1 and G_2. The strategies that we define here immediately give upper bounds on the corresponding Rendezvous Values $\bar{R}_1$ and $\bar{R}_2$. The proofs that these strategies are optimal will be given later.

In this section we have to extend the definition of the t^i so that they represent the first of three events: player I meets agent i, player I finds the gift dropped by agent i, agent i finds the gift dropped by player I (in the G_2 game only).

4.1. The game G_1

In the game G_1, only player II has a gift. The game ends when the two players meet or when player I finds the gift dropped by player II, whichever comes first. It is optimal for player II to drop the gift at time $D/4$, where D denotes the initial distance between the players. The full optimal strategies are given in the following result. Recall that in our notation (5) the strategy $\bar{g}$ indicates that player II drops the gift at time $D/4$ and turns at times $D/4$ and $3D/2$.

Theorem 2. ***(G_1-game)*** *An optimal solution for the asymmetric rendezvous problem on the line when one player has a gift is given, using the path notation of (5), by*

$$f = [3D/4], \;\; g = [D/4;\; D/4, 3D/2]. \tag{7}$$

The corresponding times are

$$t_1 = t^1 = 3D/4, \;\; t_2 = t^4 = 3D/4, \;\; t_3 = t^2 = 3D/2, \;\; t_4 = t^3 = 9D/4, \tag{8}$$

with Rendezvous Value

$$\bar{R}_1 = R(f, g) = (3D/4 + 3D/4 + 3D/2 + 9D/4)/4 = 21D/16.$$

The proof of this Theorem is deferred to Section 6.2.3.

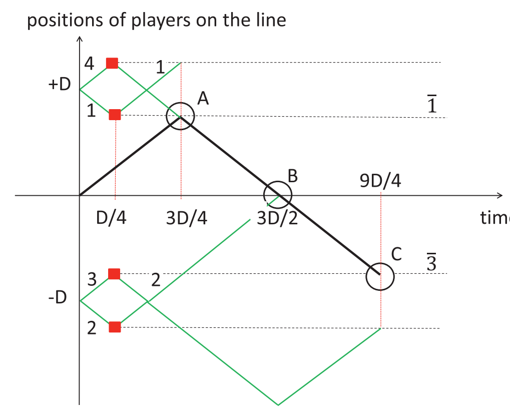

Fig. 4. Solution of the rendezvous problem on the line with one gift (G_1-game). In the figure, the gift is dropped off at time $D/4$ by player II. Each green line has a label $1, 2, 3, 4$ that is the identifier of the agent of player II that follows the trajectory. The labels $\bar{1}$ and $\bar{3}$ refer to the position of the gift of the agents 1 and 3. The circles are meant to highlight the times t^i when rendezvous occur or when the gift is found by player I. The red squares highlight the positions of the gifts dropped off by the agents of player II. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4 illustrates the optimal strategy for G_1 outlined in Theorem 2. The path of player I is indicated by a thick solid line, starting at height (location) 0. The paths of the four agents of player II, starting each at $+D$ and $-D$, are labeled with their agent numbers. At time $D/4$, indicated by four solid squares, these agents drop a gift which is indicated by a dashed horizontal line from time $D/4$. These paths are labeled by the agent number with an upper bar, say $\bar{1}$ for agent 1. The four meeting times are marked as the circles A, B and C. There are two meetings at A (time $3D/4$), player I meets agent 4 and also finds the gift dropped earlier by agent 1 (crosses the line $\bar{1}$): Thus $t^1 = t^4 = 3D/4$. The meeting point B at time $3D/2$ denotes the meeting of player I with agent 2, so $t^2 = 3D/2$. Finally, the meeting at C at time $9D/4$ occurs when player I finds the gift dropped by agent 1 (crosses the line $\bar{3}$); so $t^3 = 9D/4$. The average of these generalized meeting times is $21D/16$. Thus this number is an upper bound on the Rendezvous Value $\bar{R}_1$. What remains to do (the harder part) is to show that no strategy has a smaller Rendezvous Value. This will be shown in Section 6. The computation of the optimal solution of G_1 is presented in Section 6.2.1 while suboptimal solutions are presented in Section 6.2.1. There, Table 2 summarizes our findings for G_1. In Section 6.2.2, we use the linear programs to compute the Rendezvous Value as a function of the dropping time z.

Table 1
Rendezvous Values with markers and gifts.

G, M	R; with $D = 16$	where
G_0	$13D/8 = 26$	[9], Theorem 3.2
M_1	$3D/2 = 24$	Leone and Alpern (2018a), Theorem 2
M_2	$3D/2 = 24$	Leone and Alpern (2018a), Theorem 9
G_1	$21D/16 = 21$	Theorem 2
G_2	$20D/16 = 20$	Theorem 3

Table 2
Summary of the Rendezvous Values for the G_1 game.

Linear programs	Dropping time	Rendezvous Value; D = 16	Figure
(13)	$z \le t_1$	$21D/16 = 21$	4
(−−)	$t_1 \le z \le t_2$	$11D/8 = 22$	8
(−−)	$t_2 \le z \le t_3$	$13D/8 = 26$	9

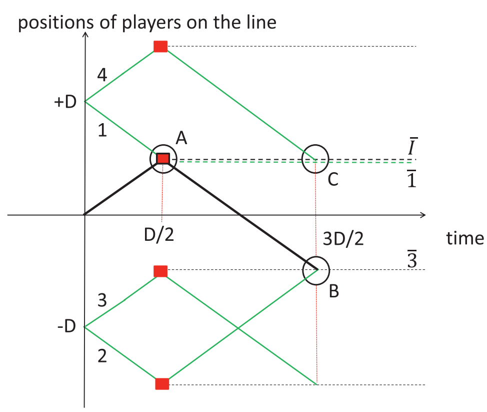

Fig. 5. Solution of the G_2-game. The gifts are dropped off at time $D/2$. In point A, the gifts of player I and agent 1 are dropped simultaneously. Each green line has a label $1, 2, 3, 4$ that is the identifier of the agent of player II that follows the trajectory. The labels $\bar{1}$ and $\bar{3}$ and $\bar{I}$ refer to the position of the gift of the agents 1 and 3 and of player I. The circles are meant to highlight the times t^i when rendezvous occur or when a gift is found. The red squares highlight the positions of the gifts dropped off by the agents of player II. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4.2. The game G_2

In the G_2 game, each player has one-gift to drop. The game ends in three possible ways, whichever comes first: the players meet, player I finds the gift dropped by player II, player II finds the gift dropped by player I. The optimal solution to G_2 is given in the following.

Theorem 3. *(G_2-game) An optimal solution for the asymmetric rendezvous problem on the line when both players have a gift is given, using the path notation of (5), by*

$$f^* = [D/2; D/2], \ \ g^* = [D/2; \ D/2]. \tag{9}$$

The corresponding meeting times are

$$t_1 = t^1 = D/2, \ \ t_2 = t^4 = 3D/2, \ \ t_3 = t^2 = 3D/2, \ \ t_4 = t^3 = 3D/2, \tag{10}$$

with Rendezvous Value

$$\bar{R}_2 = R(f^*, g^*) = (D/2 + 3D/2 + 3D/2 + 3D/2)/4 = 20D/16.$$

The proof of this Theorem follows the structure of the proof of Theorem 2 given in Section 6.2.3. It is detailed in Section 6.3.

Fig. 5 illustrates the optimal strategy for G_2 outlined in Theorem 3. The path of player I is indicated by a thick solid line, starting at height (location) 0. The paths of the four agents of player II, starting each at $+D$ and $-D$, are labeled with their agent numbers. At time $D/2$, indicated by four solid squares, these agents and player I drop a gift which is indicated by a dashed horizontal line from time $D/2$. These paths are labeled by the agent number with an upper bar, $\bar{3}$ indicates the position of the gift of agent 3 while $\bar{I}$ the gift of player I.

The four generalized meeting times are marked as the circles A, B and C. At A (time $D/2$), player I and agent 1 meet, the gift of agent 1 is no longer relevant after that time. Thus $t^1 = D/2$. The meeting points B and C at time $3D/2$ denotes the meeting of player I with agent 2, so $t^2 = 3D/2$. At the same time, player I finds the gift of agent 3 (crosses the line $\bar{3}$) and agent 4 finds the gift of player I (crosses the line $\bar{I}$). Thus $t^3 = t^4 = 3D/2$. The average of these generalized meeting times is $20D/16$. Thus this number is an upper bound on the Rendezvous Value $\bar{R}_2$. We see in Section 6 that this is the optimal solution. The solution of the family of linear programs (16) corresponds to this optimal strategy.

5. Properties of optimal strategies

In this section we prove Proposition 4 that is the tool that makes possible the reduction of optimal strategies from the set $\mathcal{F}$ defined by Eq. (1) to the much smaller set defined by strategies of the form of Eq. (4). More precisely, it proves that the turning points f_i must occur only at the times where player I meets one of player II's agents or at the time of dropping the gift. This proposition is a generalization of Lemma 5.1 of Alpern and Gal (1995), Theorem 1 of Alpern (2002b) and Theorem 16.10 of Alpern and Gal (2006).

Note that in Fig. 1 (with no gifts), each player moves with slope ± 1 in each of the time intervals $[0, t^1] = [0, D/2]$, $[t^1, t^2] = [D/2, D]$, $[t^3, t^4] = [D, 2D]$ and $[t^4, t^2] = [2D, 3D]$. Proposition 4 states that this also holds when players have a gift at disposal.

The proof of Proposition 4 is based on a method of improving (reducing) one of the four meeting times of a strategy pair if it does not satisfy the property that players must move at unit speed in a fixed direction on time intervals between meeting times (or time of dropping or finding a gift), i.e. a player turns inside the time interval or moves more slowly. We have already seen an example of this method of improving the Wait For Mommy strategy in the Introduction. In WFM, Baby stays still (so not at unit speed) on the time interval $J[0, D]$ until a first possible meeting with Mommy, see the left of Fig. 6. Suppose we modify Baby's motion during $J = [0, D]$ only, so that he moves in some direction at unit speed during $[0, D/2]$ and goes back to his starting point during time $[D/2, D]$. After time D, he moves as in WFM (in this case stays still). This has the effect of bringing one of the meetings at time D (those where Mommy starts by moving towards him) forward to time $D/2$. So instead of meeting Mommy equiprobably at times D and $3D$ with average time $2D = 16D/8$, he meets her with probability $1/4$ at each time $D/2$ and D, and (as before) with probability $1/2$ at the $3D$, hence with average time $15D/8$. The modified strategy (on the way to Modified Wait For Mommy, see right of Fig. 6) is not yet optimal, but it shows that the original WFM strategy was not optimal. In a similar manner it can be shown (see proof of Proposition 4) that a strategy pair in which one of the players does not move at unit speed in a single direction on some such time interval J cannot be optimal.

Proposition 4. *Let G be any asymmetric rendezvous game on the line where each player has at most one-gift, G_0, G_1, G_2. Then in any optimal strategy pair each player moves at unit speed in a fixed direction (no turns) on each of the time intervals $[c_1, c_2]$ where c_1, c_2 are successive times of events of the form:*

1. *The times t^i when player I and agent i of player II meet,*

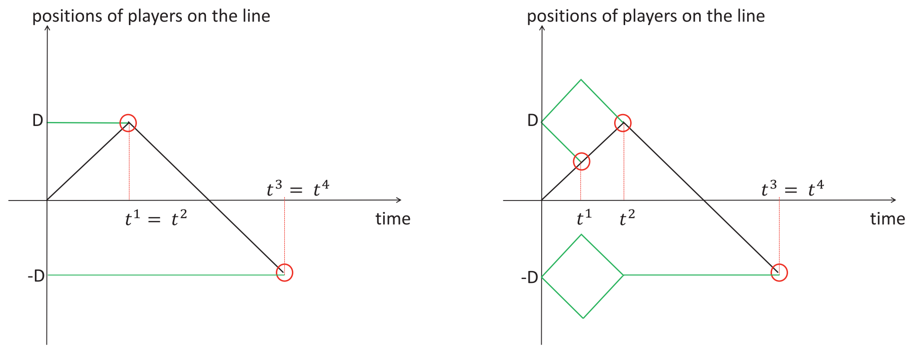

Fig. 6. Illustration of Wait For Mommy strategy (WFM) on the left and the partly modified strategy on the right.

2. *The times t^i when player I finds the gift dropped by agent i of player II, or agent i of player II finds the gift dropped by player I (in the two-gift game G_2),*
3. *The time τ when one player drops off a gift.*
4. *The starting time of the game, $c_1 = 0$.*

The intervals are then of the form $[0, t^i]$, $[0, \tau]$, $[t^i, t^{i+1}]$, $[\tau, t^i]$, $[t^i, \tau]$.

Proof. Assume on the contrary that for some optimal strategy pair (f, g), the condition fails on some time interval $J = [c_1, c_2]$, i.e. suppose player I, whose path is given by $f(t)$, does not move at (maximal) unit speed on J. There are three cases, depending on what happens at time c_2.

1. **At time $c_2 = t^i$ when player I and agent i of player II meet.** By increasing his speed, player I arrives at the meeting location $f(c_2)$ at an earlier time $c_2 - e$ ($< c_2$) with $e > 0$. At time $c_2 - e$, agent i of player II is either at location $f(c_2)$ or lies in some direction (call this i's direction) from $f(c_2)$.
In the former case the meeting with i is moved forward to time $c_2 - e$. So player I can stay there until time c_2 and then resumes his original strategy, so all other meeting times are unchanged. Otherwise, player I goes from $f(c_2)$ in agent i's direction at unit speed on interval $[c_2 - e, c_2 - e/2]$ and then back to $f(c_2)$ at time c_2, when he resumes his original strategy. This brings the meeting time with i no later than $c_2 - e/2$, without changing any other meeting times. It may also be that player I meets agent i on his way to $f(c_2)$, before time $c_2 - e$, reducing again the meeting time. After the meeting time player I goes to $f(c)$, waits for time c_2 and resumes his strategy letting unchanged the other meeting times.
In either case the expected rendezvous time is lower, contradicting the assumption that (f, g) is an optimal strategy pair. The arguments are the same if it is player II that is not moving at maximal speed.
2. **At time $c_2 = t^i$,player *I*first finds the gift dropped by agent** i (or vice-versa). If the gift has just been dropped off at time c_2, then *I* also meets agent i at time c_2, so the previous case applies. Similarly, the previous case applies if the gift is not present at time $c_2 - e$ when player I can reach the position $f(c_2)$. Otherwise, player *I* finds the gift at time $c_2 - e$, waits there until time c_2, and then resumes his original strategy f. The meeting time is then reduced while others are unchanged. This contradicts the assumption that (f, g) is an optimal strategy pair. The arguments are the same if it is player II that is not moving at maximal speed.
3. **At time $c_2 = \tau$,when one player drops off a gift.** Suppose first that it is the player that drops off the gift that does not move at maximal speed. His speed can be increased to get earlier to the dropoff location $f(c_2)$ at time $c_2 - e$, drop off the gift, then stay still until time c_2, and resume with the original strategy.
In the next time interval, of the form $[\tau, t^i]$, the player is not moving at maximal speed and case 1. or 2. occurs. Hence, the strategy can again be modified to decrease the average meeting times.
Notice, that there is still the particular case where the game stops after the other player finds the gift. In this case the speed of the player that dropped the gift is immaterial. We do not observe such optimal solutions.

In the three cases The assumption that (f, g) is an optimal strategy pair is contradicted. The arguments are the same if it is player II that is not moving at maximal speed. □

Corollary 5. *Optimal strategy pairs (f, g) in the games G_0, G_1, G_2 admit a representation as in (4),(5). In particular, there are only a finite set of strategies candidate for optimality.*

Proof. By proposition 4 players move at full speed and turning points must coincide with locations where the player drops/finds a gift or meets the other player. At the start, the configuration is the one displayed in Fig. 1. If there is no-gift, trying all possible NE amounts to check all strategies that can be enumerated in the following way. Let ξ a permutation of $\{1, \ldots, 4\}$ player I rendezvous with agents $\xi(1)$, then $\xi(2)$, then $\xi(3)$, and finally $\xi(4)$. Given a permutation each rendezvous time is deterministic and leads to the definition of f_i in (4), (5). In the case where there is a gift, the number possible paths are doubled since player II can change direction. □

6. Reduction of the G_0, G_1, G_2 games to families of linear programs (LPs) for solutions

In this section, we show how the G_0, G_1, G_2 games can be reduced to families of LPs. For the computations, we set $D = 1$. For the statements of the results, D is reintroduced in Tables and Figures.

Proposition 4 teaches us that for a strategy to be optimal, players must move at maximal speed and that the turning points occur only at specified times. This makes it possible to reduce the problem to a family of linear programs. As mentioned in the Introduction, this is an important new idea for solving rendezvous problems. For the original G_0 game, the family is given by (11). For the G_1 game, we have 4 families to write down and solve. One family for the dropping time z constrained to $z \le t_1$, see (13), and three others when $t_1 \le z \le t_2$, $t_2 \le z \le t_3$ or $t_3 \le z \le t_4$. By solving all the LP constituting the families we prove that the strategies given in Theorems 2 and 3 are optimal.

We first illustrate the technique by reproving Theorem 1.

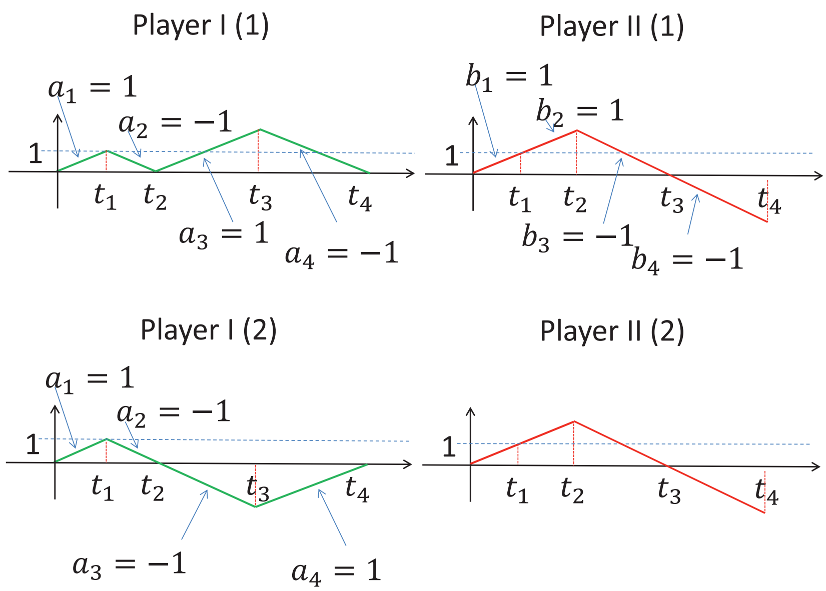

Fig. 7. Plot of the two optimal solutions of the G_0 games with $D = 1$. The vertical axis is the position of the players on the line and the horizontal axis is the time. On the left are plotted the strategies of player I and on the right of player II. The only difference between the two strategy pairs is that player I reverses the direction of motion at times t_2 and t_3.

6.1. Reduction of the G_0 game to a family of LPs and optimal solutions

A strategy amounts to selecting the order in which player I meets with the agents of player II. Hence, given any of the $4! = 24$ permutations π of $\{1, 3, 4, 2\}$, a strategy amounts to the successive meetings of player I with the agents $\pi(1), \pi(2), \pi(3)$, and $\pi(4)$. For the G_0 game, there are two permutations that are optimal $\pi_1 = \{1, 3, 4, 2\}$ (so that $t_1 = t^1$, $t_2 = t^3$, $t_3 = t^4$, $t_4 = t^2$) and $\pi_2 = \{1, 3, 2, 4\}$ (so that $t_1 = t^1$, $t_2 = t^3$, $t_3 = t^2$, $t_4 = t^4$). The optimal solutions are plotted in Fig. 7.

To represent the agents of player II in the LPs, we use two variables o and d that may take the values ± 1. The variable o is used to represent the initial position of the agent while d represents the forward direction of the agent, i.e. if $d = 1$ ($d = -1$) the agent forward direction is along the positive (negative) axis. According to the numbering of the agents of player II in Fig. 1, we have $o = 1, d = 1$ for agent 4, $o = 1, d = -1$ for agent 1, $o = -1, d = 1$ for agent 3, $o = -1, d = -1$ for agent 2.

The strategies are given by direction variables a_1, a_2, a_3, a_4 for player I and b_1, b_2, b_3, b_4 for player II. By Proposition 4, these variables are restricted to be ± 1 to indicate that a player moves in the forward (backward) direction at maximal speed.

Variable a_i, b_i indicates the direction of the motion of the two players during the time interval $[t_{i-1}, t_i]$, with $t_0 = 0$ (remember that t_i is the time of the ith rendezvous). To reduce the computations, we always have $a_1 = b_1 = 1$ because there is nothing essential to consider the other cases by the symmetry of the problem. This reduces from 24 to 6 the number of permutations to be considered. With $a_1 = b_1 = 1$ the first rendezvous occurs always with player I and agent 1.

In the rendezvous search problem, we search for minimizing the average meeting time, hence the cost function to minimize is given by

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= 4t_1 + 3(t_2 - t_1) + 2(t_3 - t_2) + (t_4 - t_3) \\ &= 4\Delta_1 + 3\Delta_2 + 2\Delta_3 + \Delta_4, \end{aligned} \tag{11a}$$

where $\Delta_1 = t_1$ and $\Delta_i = t_1 - t_{i-1}$, $i = 2, 3, 4$ (to simplify the notation the division by 4 is not included).

The first rendezvous occurs between player I and the agent represented by (o_1, d_1) where the particular values of o_1 and d_1 are ± 1 depending on the order on which the rendezvous occur with the agents. The first meeting occurs at time t_1 defined by the equation ($a_1 = b_1 = 1$ are not included because not essential by symmetry of the problem)

$$t_1 = o_1 + d_1 t_1. \tag{11b}$$

The left side of (11b) accounts for the motion of player I that starts at the origin and moves for a duration t_1. The right side accounts for the motion of the agent of player II that starts at position $o_1 (= \pm 1)$ and moves in direction $d_1 (= \pm 1)$ for a duration t_1. The equality between the two sides accounts for the rendezvous.

Next, player I meets with the agent whose initial position is o_2 and forward direction d_2 at time t_2. The occurrence of a rendezvous is stated by equation

$$t_1 + a_2(t_2 - t_1) = o_2 + d_2\big(t_1 + b_2(t_2 - t_1)\big), \tag{11c}$$

where a_2 is introduced to indicate whether player I continues in the forward direction $a_2 = 1$ or backward $a_2 = -1$. The left hand side of the equation is the position of player I at time t_2, while the right hand side is the position of agent (o_2, d_2) at time t_2.

The third meeting occurs with agent (o_3, d_3) at time t_3 defined by

$$\begin{aligned} &t_1 + a_2(t_2 - t_1) + a_3(t_3 - t_2) \\ &\quad = o_3 + d_3\big(t_1 + b_2(t_2 - t_1) + b_3(t_3 - t_2)\big). \end{aligned} \tag{11d}$$

Finally, the fourth and last meeting occurs with agent (o_4, d_4) and occurs at time t_4 defined by

$$\begin{aligned} &t_1 + a_2(t_2 - t_1) + a_3(t_3 - t_2) + a_4(t_4 - t_3) \\ &\quad = o_4 + d_4\big(t_1 + b_2(t_2 - t_1) + b_3(t_3 - t_2) + b_4(t_4 - t_3)\big). \end{aligned} \tag{11e}$$

The problem can be stated as the following family of LPs where a_i, b_i, o_i, d_i are parameters and an LP is solved for Δ_i:

$$\min_{\Delta_i} \; 4\Delta_1 + 3\Delta_2 + 2\Delta_3 + \Delta_4 \tag{11a}$$

$$\Delta_1 = o_1 + d_1 \Delta_1 \tag{11b}$$

$$\Delta_1 + a_2\Delta_2 = o_2 + d_2\big(\Delta_1 + b_2\Delta_2\big) \tag{11c}$$

$$\Delta_1 + a_2\Delta_2 + a_3\Delta_3 = o_3 + d_3\big(\Delta_1 + b_2\Delta_2 + b_3\Delta_3\big) \tag{11d}$$

$$\begin{aligned}&\Delta_1 + a_2\Delta_2 + a_3\Delta_3 + a_4\Delta_4\\ &\quad = o_4 + d_4\big(\Delta_1 + b_2\Delta_2 + b_3\Delta_3 + b_4\Delta_4\big)\\ &\quad \Delta_i \geq 0.\end{aligned} \tag{11e}$$

The variables o_i, d_i, a_i and b_i are the parameters of the problem. Their values are constraint to

$$a_i, b_i, o_i, d_i \in \{-1, 1\}, \sum o_i = 0, \sum d_i = 0, \;\; o_i = o_j \Rightarrow d_i \neq d_j, \tag{12}$$

$\sum o_i = 0$ ensures that there are two agents located at the $+1$ and two at the -1 initial positions. $\sum d_i = 0$ ensures that two of them have direction $+1$ and two have direction -1. $o_i = o_j \Rightarrow d_i \neq d_j$ ensures that two agents located at the same position have opposite forward directions. The other ones are used to generate the family of LPs that has to be solved. We found 1536 parameter tuples that satisfy the constraints (12) each tuple leading to an LP to be solved. Among the solved tuples 6 are feasible.

In (11a), the decreasing arithmetic sequence 4,3,2,1 is easily explained. Any delay in the first meeting delays all the subsequence meetings as well, so four equiprobably meetings. A delay to the second meeting affects all but the first meeting, so three meetings, and so on. Note that we minimize the sum of the meeting times, which is equivalent to the mean of these equally likely meeting times.

The optimal solutions found are displayed in Fig. 1. The first solution corresponds to the strategy given in Theorem 1, where player I's strategy is $\bar{f} = [D/2, D/2, D]$ and player II's strategy $\bar{g} = [D]$. In the second solution, player I's strategy is $\bar{f} = [D/2, 3D/2]$ and player II's strategy $\bar{g} = [D]$, see the plots on Fig. 7.

6.2. Reduction of the G_1 game to families of LPs and optimal solutions

The G_1 game stops when player I meets with agent i or finds the gift of agent i. Independently of which event occurs, we will speak of a generalized rendezvous time. Similarly to the G_0 game, Proposition 4 says that an optimal strategy is given by a permutation π of $\{1, 2, 3, 4\}$ such that the generalized rendezvous times are of the form $t_1 = t^{\pi(1)}$, $t_2 = t^{\pi(2)}$, $t_3 = t^{\pi(3)}$, $t_4 = t^{\pi(4)}$.

The solution of the G_1 game is obtained by writing the problem in a form similar to (11). However, we have four more cases depending on when the gift is dropped off. If the dropping time is written z, the four possibilities are: $z \in [0, t_1]$, $z \in [t_1, t_2]$, $z \in [t_2, t_3]$, $z \in [t_3, t_4]$. The optimal solution for each case is computed by solving the families of linear programs given by (13) when the dropping time z is constrained by $z \leq t_1$, see (13), and three others when $t_1 \leq z \leq t_2$, $t_2 \leq z \leq t_3$ or $t_3 \leq z \leq t_4$.

The optimal solution for the G_1 game is found for $z \in [0, t_1]$. The family of LPs to be solved in this case is written in (13). The optimal solution is drawn in Fig. 4.

$$\begin{aligned}
&\min_{\Delta_i, z} \;\; 4\Delta_1 + 3\Delta_2 + 2\Delta_3 + \Delta_4 \qquad (\mathbf{z \leq t_1})\\
&a_0 z + a_1(\Delta_1 - z) = o_1 + k_1 d_1\big(b_0 z + b_1(\Delta_1 - z)\big) + (1 - k_1)d_1 b_0 z\\
&a_0 z + a_1(\Delta_1 - z) + a_2\Delta_2 =\\
&o_2 + k_2 d_2\big(b_0 z + b_1(\Delta_1 - z) + b_2\Delta_2\big) + (1 - k_2)d_2 b_0 z\\
&a_0 z + a_1(\Delta_1 - z) + a_2\Delta_2 + a_3\Delta_3 =\\
&o_3 + k_3 d_3\big(b_0 z + b_1(\Delta_1 - z) + b_2\Delta_2 + b_3\Delta_3\big) + (1 - k_3)d_3 b_0 z\\
&a_0 z + a_1(\Delta_1 - z) + a_2\Delta_2 + a_3\Delta_3 + a_4\Delta_4 =\\
&o_4 + k_4 d_4\big(b_0 z + b_1(\Delta_1 - z) + b_2\Delta_2 + b_3\Delta_3 + b_4\Delta_4\big)\\
&\quad + \;(1 - k_4)d_4 b_0 z\\
&0 \leq z \leq \Delta_1, \quad \Delta_i \geq 0
\end{aligned} \tag{13}$$

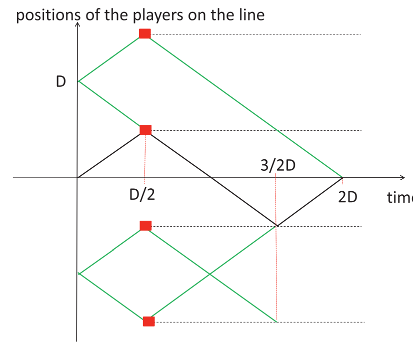

Fig. 8. Suboptimal solution when the dropping time is constraint to $z \in [t_1, t_2]$, $R(f, g) = 11D/8$.

In the LP family (13), the variable z denotes the dropping time, the variables k_i are used to distinguish the cases where player I meets with the agent (o_i, d_i) (if $k_i = 1$) or finds the gift of agent (o_i, d_i) (if $k_i = 0$). The gift is dropped of at position $d_i b_0 z$. The parameters are constrained by

$$\begin{aligned}&k_i \in \{0, 1\}, \quad a_i, b_i, o_i, d_i \in \{-1, 1\},\\ &\sum o_i = 0, \quad \sum d_i = 0, \quad o_i = o_j \Longrightarrow d_i \neq d_j,\end{aligned} \tag{14}$$

which have the same meaning than in (11) with the addition of the constraint on the dropping time z, $z \leq \Delta_1$, to fix $z \in [0, t_1]$.

Each particular LP in the family is determined by the direction and origin variables a_i, b_i and o_i, d_i. Once these are fixed, we have the LP of (13) to solve for Δ_i and z. It is more convenient to use the differences Δ_i instead of the rendezvous time t_i. The objective function, which we minimize, is the average meeting time (multiplied by 4 to remove fractions). The top four constraints say that

1. At time $t_1 = \Delta_1$ player I (at position $a_0 z\Delta_1 + a_1(\Delta_1 - z)$) meets agent (o_1, d_1) (at position $o_1 + d_1\big(b_0 z + (\Delta_1 - z)b_1\big)$) if $k_1 = 1$, or finds the gift at position $d_1 b_0 z$ if $k_1 = 0$,
2. At time $t_2 = \Delta_1 + \Delta_2$ player I meets agent (o_2, d_2) if $k_2 = 1$, or finds the gift at position $d_2 b_0 z$ if $k_1 = 0$,
3. At time $t_3 = \Delta_1 + \Delta_2 + \Delta_3$ player I meets agent (o_3, d_3) if $k_3 = 1$, or finds the gift at position $d_3 b_0 z$ if $k_1 = 0$,
4. At time $t_4 = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ player I meets agent (o_4, d_4) if $k_4 = 1$, or finds the gift at position $d_4 b_0 z$ if $k_1 = 0$.

6.2.1. Optimal solutions when the dropping time is constraint to $t_1 \leq z \leq t_2$, or $t_2 \leq z \leq t_3$

To solve the cases when $t_1 \leq z \leq t_2$, or $t_2 \leq z \leq t_3$ two families of LPs similar to (13) are written down and solved. The Rendezvous Values obtained in each case are presented in Table 2. Strategies leading to suboptimal Rendezvous Values are drawn in Figs. 8 and 9.

6.2.2. Optimal solution as a function of the dropping time z

In Fig. 10, we observe that the optimal solution as a function of the dropping time z is piecewise linear. This follows from the theory of parametric linear programming pioneered in Gass and Saaty (1955).

By computing the optimal solutions for z in a mesh of values, we obtain Fig. 10 where the solution value is linear on the

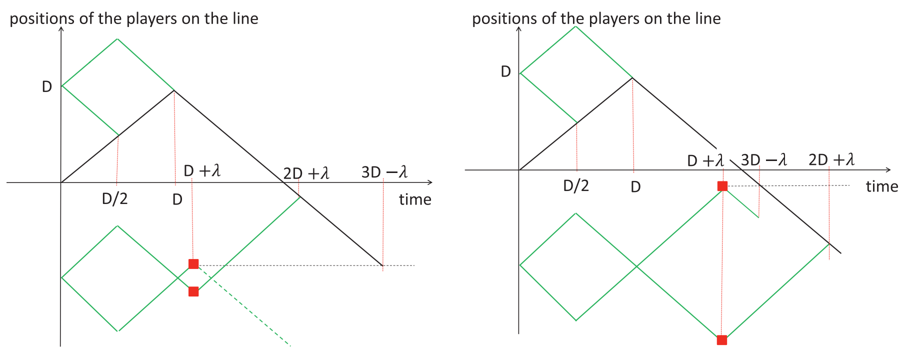

Fig. 9. Suboptimal solution when the dropping time $z = D + \lambda$ is constraint to $z \in [t_2, t_3]$, $R(f, g) = 13D/8$. **Left:** $\lambda \leq \frac{1}{2}$. **Right:** $\lambda \geq \frac{1}{2}$.

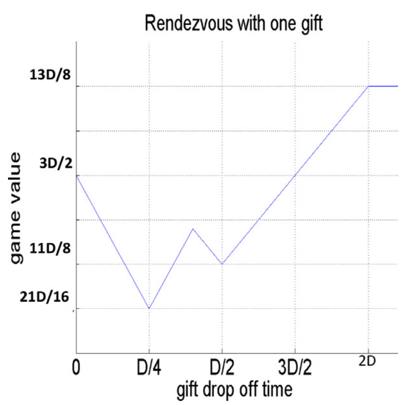

Fig. 10. Optimal solution value of the G_1 game as a function of the gift dropping time z.

segments $[0, D/4]$, $[D/4, 2D/5]$, $[2D/5, D/2]$, $[D/2, 2D]$. We numerically found the optimal strategies on these intervals as plotted in Figs. 11 and 12 with the algebraic expression of the Rendezvous Value as a function of the dropping time z summarized in Table 3. Fig. 12 is interesting in providing an example where the optimal solution (for fixed dropping time z) does not turn at time of dropping the gift.

6.2.3. Proof of Theorem 2

Theorem 2 (G_1 **-game)** An optimal solution for the asymmetric rendezvous problem on the line when one player has a gift is given, using the path notation of (5), by

$$f = [3D/4], \;\; g = [D/4;\; D/4, 3D/2].$$

The corresponding times are

$$t_1 = t^1 = 3D/4, \;\; t_2 = t^4 = 3D/4, \;\; t_3 = t^2 = 3D/2, \;\; t_4 = t^3 = 9D/4, \tag{15}$$

with Rendezvous Value

$$\bar{R}_1 = R(f, g) = (3D/4 + 3D/4 + 3D/2 + 9D/4)/4 = 21D/16.$$

Proof. The proof of this Theorem is in three steps. The first step consists in the reduction of the space the optimal solution belongs to. In the problem formulation, the optimal solution are Lipschitz functions as defined in (1). Proposition 4 shows that the optimal solutions are piecewise linear, i.e. the players move at constant maximal speed between events corresponding to rendezvous or dropping of the gift. Hence, the player's strategy is given by (4) if the player has no gift, or (5) if a gift is at disposal. This reduction is stated by Corollary 5.

In the second step of the proof, the problem is formulated as a parametric LP, i.e. a family of linear programs that enumerates all the permutations of the rendezvous or gift dropping time. More formally, the family of LPs enumerates the linear orderings of the four rendezvous time t^1, t^2, t^3, t^4 and the dropping time z.

In the third and last step, an LP solver finds the optimal solution (if any) of each LP belonging to the family of LPs (13) if the dropping time is restricted to $z \leq t_1$ and similar LP families when $t_1 \leq z \leq t_2$ and $t_3 \leq z \leq t_4$. The optimal strategies are discovered by inspection of the results. □

6.3. Reduction of the G_2 game to families of LPs and optimal solutions

For the G_2 game, each player has a gift, and we denote by z_1, z_2 the dropping times of player I and II respectively. We have now 16 different families to write down corresponding to the different dropping times: $z_1, z_2 \in [0, t_1]$, $z_1, z_2 \in [t_1, t_2]$, $z_1, z_2 \in [t_2, t_3]$, $z_1, z_2 \in [t_3, t_4]$. We find that the optimal solution occurs for $z_1 = z_2 = D/2$ and for $z_1 = z_2 = 0$.

Around the dropping time $D/2$ there are only 3 possibilities: $z_1, z_2 \leq t_1$, $z_1 \leq t_1 \leq z_2 \leq t_2$, $t_1 \leq z_1, z_2 \leq t_2$.

In the first case, $z_1, z_2 \leq t_1 = \Delta_1$, the system to solve is given by (16). This system is very similar to the systems solving G_1. We have introduced the supplementary variables k_i, l_i to indicate that player I meets player II ($k_i = l_i = 1$), player I finds the gift of player II ($k_i = 1, l_i = 0$) or player II finds the gift of player I ($k_i = 0, l_i = 1$). The case $k_i = l_i = 0$ is not meaningful.

In (16), a particular linear program results from the choices of the direction and origin variables variables a_i, b_i and o_i, d_i as well as the variables k_i, l_i that decide for each i which event occurs among the following: player I meets agent (o_i, d_i) $(k_i = 1, l_i = 1)$, player I finds the gift of agent (o_i, d_i) $(k_i = 1, l_i = 0)$, agent (o_i, d_i) finds the gift of player I $(k_i = 0, l_i = 1)$. The linear program is solved for the optimal Δ_i (leading to the generalized rendezvous times t_i) and the dropping times z_1, z_2. The optimal solutions are plotted in Figs. 5 and 14. The 15 other cases are not included here and do not lead to optimal solutions.

$$\min_{\Delta_i, z_1, z_2} \quad 4\Delta_1 + 3\Delta_2 + 2\Delta_3 + \Delta_4 \qquad (\mathbf{z_1 \leq t_1, z_2 \leq t_1})$$

$$\begin{aligned}
&(1-k_1)l_1 a_0 z_1 + k_1\big(a_0 z_1 + a_1(\Delta_1 - z_1)\big) = \\
&o_1 + l_1 d_1\big(b_0 z_2 + b_1(\Delta_1 - z_2)\big) + k_1(1-l_1)d_1 b_0 z_2 \\
&(1-k_2)l_2 a_0 z_1 + k_2\big(a_0 z_1 + a_1(\Delta_1 - z_1) + a_2\Delta_2\big) = \\
&o_2 + l_2 d_2\big(b_0 z_2 + b_1(\Delta_1 - z_2) + b_2\Delta_2\big) + k_2(1-l_2)d_2 b_0 z_2 \\
&(1-k_3)l_3 a_0 z_1 + k_3\big(a_0 z_1 + a_1(\Delta_1 - z_1) + a_2\Delta_2 + a_3\Delta_3\big) = \\
&o_3 + l_3 d_3\big(b_0 z_2 + b_1(\Delta_1 - z_2) + b_2\Delta_2 + b_3\Delta_3\big) + k_3(1-l_3)d_3 b_0 z_2 \\
&(1-k_4)l_4 a_0 z_1 + k_4\big(a_0 z_1 + a_1(\Delta_1 - z_1) + a_2\Delta_2 + a_3\Delta_3 + a_4\Delta_4\big) = \\
&o_4 + l_4 d_4\big(b_0 z_2 + b_1(\Delta_1 - z_2) + b_2\Delta_2 + b_3\Delta_3 + b_4\Delta_4\big)
\end{aligned}$$

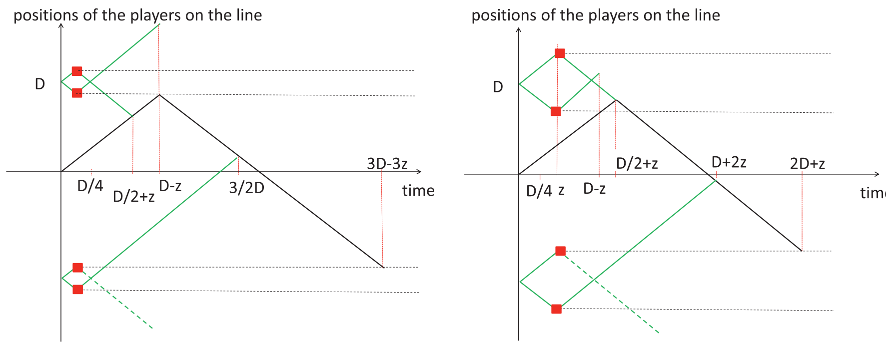

Fig. 11. Optimal solutions as a function of the dropping time z. **Left:** $0 \le z \le D/4$, $R(f,g) = (6D - 3z)/4$. **Right:** $D/4 \le z \le 2D/5$, $R(f,g) = (9D/2 + 3z)/4$.

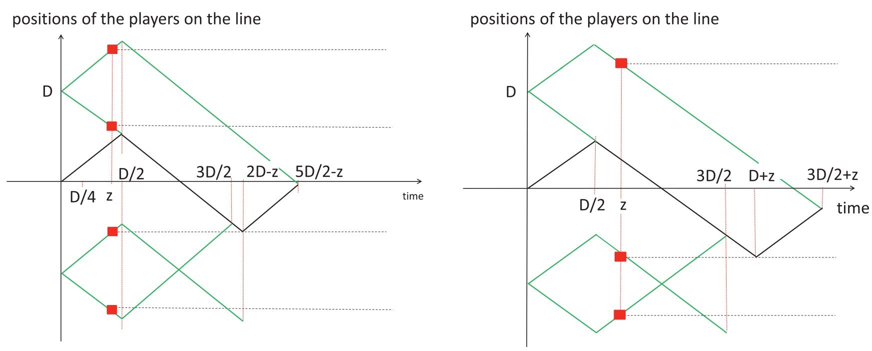

Fig. 12. Optimal solutions as a function of the dropping time z. **Left:** $2D/5 \le z \le D/2$, $R(f,g) = (13D/2 - 2z)/4$. **Right:** $D/2 \le z \le 2D$, $R(f,g) = (9D/2 + 2z)/4$.

Table 3
Optimal play for fixed dropping time z.

Dropping time	Rendezvous Value	Figure
$0 \le z \le D/4$	$(6D - 3z)/4$	11 Left
$D/4 \le z \le 2D/5$	$(9D/2 + 3z)/4$	11 Right
$2D/5 \le z \le D/2$	$(13D/2 - 2z)/4$	12 Left
$D/2 \le z \le 2D$	$(9D/2 + 2z)/4$	12 Right

Table 4
Rendezvous Value R when markers or gifts must be dropped at start.

G, M	R; with $D = 16$	where
MS_1	$13D/8 = 26$	[36], Fig. 7
MS_2	$3D/2 = 24$	[16], Theorem 2
GS_1	$3D/2 = 24$	Fig. 14 left
GS_2	$5D/4 = 20$	Fig. 14 right

$$+ \; k_4(1 - l_4)d_4 b_0 z_2$$

$$0 \le z_1 \le \Delta_1, \quad 0 \le z_2 \le \Delta_1, \quad \Delta_i \ge 0 \tag{16}$$

The variables $o_i, d_i, a_i, b_i, k_i, l_i$ are parameters of the problem to be solved. Their values are constraint to

$$k_i, l_i \in \{0, 1\}, \quad a_i, b_i, o_i, d_1 \in \{-1, 1\},$$

$$\sum o_i = 0, \quad \sum d_i = 0, \quad o_i = o_j \Longrightarrow d_i \ne d_j, \quad k_i + l_i \ge 1$$

The constraints are similar than the constraints of the G_1 game given in Eq. (14). Here, variables l_i are added to deal with the gift hold by player I. Two new cases may occur. Agent i finds the gift of player I ($l_i = 0$) or agent i rendezvous with player I ($l_i = 1$).

In Fig. 13 we plot the Rendezvous value of G_2 as a function of the dropping times z_1 and z_2 with initial inter distance $D = 1$.

7. GiftStart, symmetric and minmax rendezvous

This section looks more closely at optimal solutions to the relevant LPs and gives some qualitative results when the gifts in problem G_2 must be dropped at the start and when players must play identically.

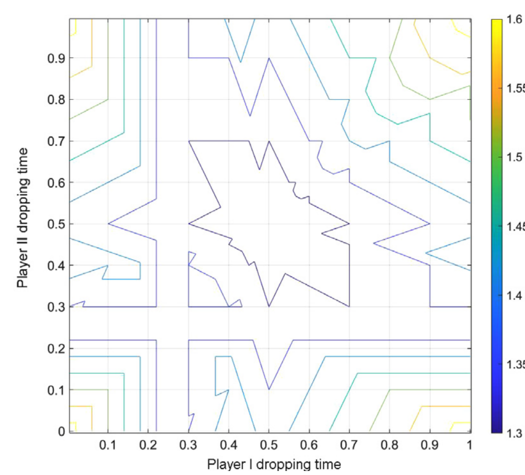

Fig. 13. Contour plot of the Rendezvous value of G_2 as a function of the dropping times z_1, z_2 of players I and II.

7.1. GiftStart rendezvous

The form of rendezvous search where the players had to drop markers at their starting points was termed MarkStart Rendezvous by Baston and Gal (2001). When both players have markers, we denote this by MS_2 if only one has a marker as in Leone and Alpern (2018b), we denote it as MS_1. By analogy, we call the problems when one or both players have gifts that must be dropped at the start as GiftStart Rendezvous, denoted by GS_1 and GS_2. The solution to GS_1 has already been shown: the Rendezvous Value of $3D/2$ for GS_1 is given in Fig. 10 and the optimal strategy can be seen as an extreme case of the left side of Fig. 13. This optimal strategy for

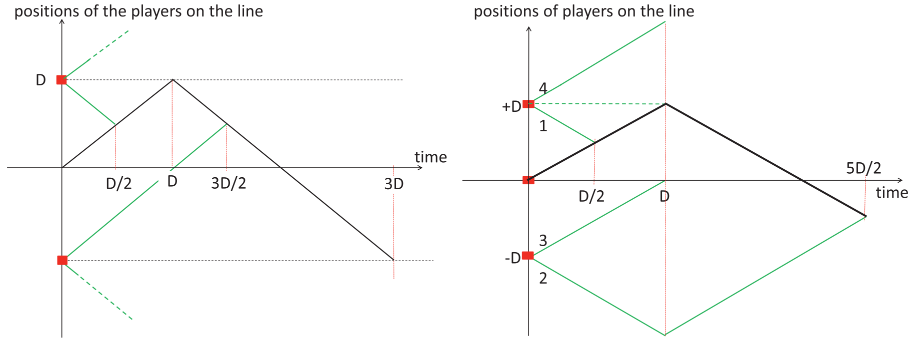

Fig. 14. Optimal strategy for the GS_1 and GS_2 game on the left and right respectively.

GS_1 is $[D]$ for the player without a gift and $[0;\ 3D/2]$ for the player who drops the gift at time 0.

To find an optimal strategy pair for GS_2, we added the constraint $z \leq 0$ to our LP system (16) and found the following optimal solution

$$f^{**} = g^{**} = [0;\ D], \tag{17}$$

they both drop their gift at the start and turn after time D, as displayed on the right of Fig. 14.

This gives a rendezvous time of $5D/4$. Note that by Theorem 3 this is the Rendezvous Value of the larger strategy G_2 game (with no restriction on dropping times), which proves it is also optimal in GS_2. This is a good point to compare the MarkStart and GiftStart Rendezvous Values. When dropping at the start:

- one gift is as good as two markers (both give 24)
- one marker is no better than none (both give 26, compare with Table 1)
- two markers are better than one marker ($20 < 24$)
- two gifts at the start are as good as two gifts which can be dropped at any time, (both give 2, compare with Table 1 or Theorem 3)

7.2. Symmetry and asymmetry of strategy pairs

A strategy pair (f, g) is called symmetric if $f = g$. Similarly we talk about the player-symmetric (or just symmetric) or player-asymmetric forms of the rendezvous problem. So far in this paper we have been considering the asymmetric rendezvous problem. If one is to say write a best selling book for hikers saying what to do if you get separated from your partner, then if both hikers read it, this is the symmetric form. If however it says what the taller and shorter hiker should each do, then it is the asymmetric form. For the line with a given distance between the players, the asymmetric problem (without gifts or markers) was posed and solved in Alpern and Gal (1995). Surprisingly, the equivalent symmetric problem posed earlier in Alpern (1995) is still unsolved. Progress has been made in successively reducing the upper bound on the Rendezvous Value in Han et al. (2008).

A simple but important observation about symmetric rendezvous on the line is the following. In the traditional version, without gifts or markers, a symmetric pure strategy pair (f, f) has infinite expected meeting time. This is because if the two players happen to start facing the same direction, following the same path f will preserve their initial distance, so they will never meet. For this reason all the work on the symmetric rendezvous problem on the line has considered common mixed strategies for the players. However for the symmetric problem G_2 where each player has a gift, we have already exhibited two symmetric strategies (the one in Theorem 3 and the one in Fig. 14 left) which are optimal (rendezvous time $5D/4$). This means that they are necessarily still optimal in the smaller symmetric rendezvous problem, so that problem is solved in the presence of gifts, as follows.

Table 5
Minmax values of the games G_0, G_1, G_2, GS_1 and GS_2.

G	Minmax Value; D=16	Dropping Time	where
G_0 Minmax	$3D = 48$	$--$	Figs. 1, 15 left, 6
G_1 Minmax	$2D = 32$	$z = D/2$	Fig. 15 right
G_2 Minmax	$3D/2 = 24$	$z_1 = z_2 = D/2$	Fig. 5
GS_1 Minmax	$5D/2 = 40$	$z = 0$	Fig. 16 left
GS_2 Minmax	$2D = 32$	$z_1 = z_2 = 0$	Fig. 16 right

Theorem 6. *Consider the Symmetric Rendezvous Problem on the Line, where two players are placed a distance D apart on the line and faced in random directions. They each have a gift which they may drop at a chosen time. They must follow the same mixed strategy based on unit speed paths and arbitrary dropping times. The game ends at the first time T when the players meet or one finds a gift dropped earlier by the other player. An optimal mixed strategy is the pure (atomic) strategy where each drops their gift at the starting time, moves at unit speed in the direction they are facing and turns at time D. They will meet with probability* $1/4$ *at times* $D/2$ *and* $5D/2$ *and each will each find the other's gift at time D with disjoint probability* $1/4$*. So the Rendezvous Value (the expected value of* T*) is* $5D/4$*.*

The reason our earlier argument (about the non existence of symmetric optimal strategies which are pure) fails is that when faced in the same direction the player who starts behind will reach the gift left by the other at time D.

To review, the only optimal strategies in the no-gift game G_0 are asymmetric. In the G_2 game, we found two symmetric optimal strategies. So a natural question is whether there are any asymmetric optimal strategies in G_2. To answer this question, we ran our LPs in a way to find all optimal solutions. We determined that there were only two, namely the symmetric ones from Theorem 3 and (17). To summarize, we have found the following.

Theorem 7. *The two gift games which are symmetric in definition are* G_0 *and* G_2*. For these,*

1. *The game* G_0 *has only asymmetric solutions.*
2. *The game* G_2 *has only symmetric solutions.*

While the first result is perhaps widely known (but unpublished), the second is new to this paper.

7.3. Minmax rendezvous

An anonymous referee has suggested that our techniques allow easily changing the objective function, for example to minimizing the maximum rendezvous time (t^4). If Captain Scott in the Antarctic can go without food for at most three days, it is natural to

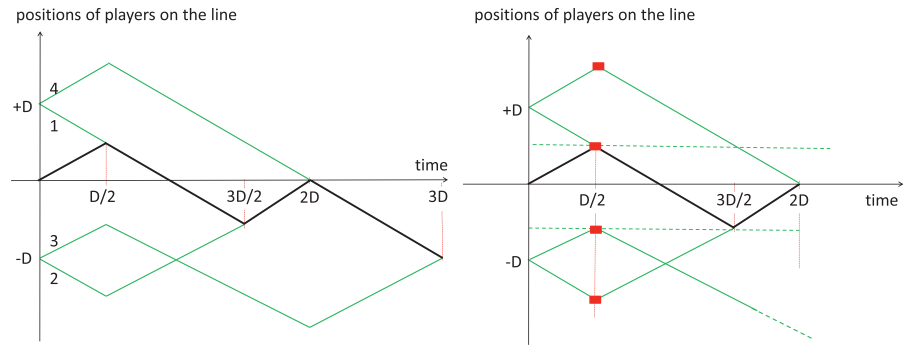

Fig. 15. Optimal strategies for the G_0 Minmax Rendezvous problem on the left and the G_1 Minmax Rendezvous problem on the right.

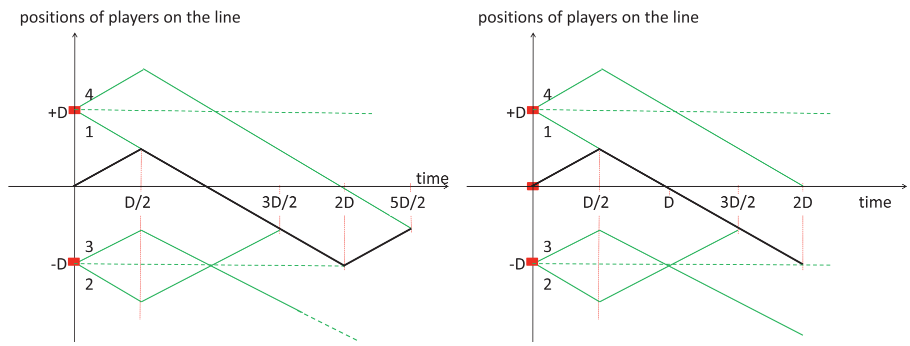

Fig. 16. Minmax strategies for the Rendezvous problem when gifts are dropped at the start time. The solution to the problem with one gift on the left (GS_1) and two gifts on the right (GS_2).

see if there is a rendezvous strategy which ensures rendezvous before this, even if it does not minimize the expected time for him to get food. We would want to know if the Minmax rendezvous time (possibly with gifts) is less than this. In this case the objective function for our LPs becomes $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$. The strategy reduction obtained by our use of Proposition 4 still holds, as given any Minmax strategy, the proof creates one with unit speed paths without increasing any of the meeting times, in particular it doesn't increase the last meeting time. The results of changing the objective function to Minmax are listed in Table 5. Note that, unlike earlier results for Rendezvous Values (expected time), the Minmax values strictly decrease, in both regular or Giftstart versions, as the number of gifts increase. The new Minmax strategies for G_i and for GS_i are shown in Figs. 15 and 16.

For the original game G_0 with no gifts, the optimal solution (Modified Wait For Mommy MWFM) of Fig. 1 (or Theorem 1) is also Minmax. However the simpler (unmodified) Wait For Mommy is also a Minmax strategy, although it is not (expected time) optimal because the second meeting for example takes place later than that of MWFM. The Minmax problem for G_0 is the only one which has previously been studied, see Alpern and Lim (2002).

For the one gift problem G_1, there is no strategy which is both optimal and Minmax. A Minmax solution is shown on the right side of Fig. 15, with all agents meeting player I by time $2D$. However for the two gift problem G_2 there the symmetric strategy $f^* = g^*$ stated in Theorem 3 (and drawn in Fig. 5) is both optimal and Minmax, with last meeting at time $3D/2$. Thus we have also found a Minmax solution for the symmetric problem with two gifts. It is worth observing that the optimal Giftstart solution of G_2, shown on the right side of Fig. 14, while also an optimal solution for G_2, is not also a Minmax solution to that problem, as the last meeting is at time $5D/2$.

Finally, we consider the Minmax problem for one or two gifts, when they must be dropped at the start (Minmax Giftstart). Here the Minmax solutions are different from the optimal solutions. The Minmax solutions are shown in Fig. 16. Our techniques could easily be adapted to finding the best average meeting time subject to a maximum meeting time, a useful objective for search and rescue.

8. Conclusions

This paper introduced a new variation of the rendezvous search problem tailored to the context of search-and-rescue. In the basic case (one-gift), we distinguish one of the players as the *lost* one and the other player as the *rescuer*. We assume that the lost player requires something (e.g. water) which he can obtain from the rescue player either on meeting or by finding it in the form of a *gift* (in this case a canteen of water) dropped earlier by the rescuer. The gift could also consist of information, in the form of a dropped message, which is useful for the lost player (perhaps telling him to go downhill to a lake). This form of rendezvous search involves both optimal movement patterns and optimal times to drop a gift. We solve the problem where only one player has a gift and also when both have gifts and only one of them needs to be found. Unlike the qualitative solution for the related marker game, where two markers are no better than one, we find that two-gift are better than one.

In addition to solving the problems with one or two gifts, we demonstrate a new solution method which can hopefully be used both retrospectively to give derivations for previous forms of rendezvous and in a forward looking manner to solve two dimensional rendezvous problems which have been considered too difficult to attack. These applications of the method will be the subject of future work. The idea of the method is to use a result on when the players can turn (our Proposition 4) to reduce the general problem to a finite number of linear programs. When we compare the rendezvous times for each of these, the minimum represents a solution to the original problem. Previous rendezvous problems have generally been solved by guessing the solution and then proving it is optimal. Ours is a more algorithmic approach.

We have also introduced the GiftStart Rendezvous problem where both players have a gift that must be dropped at the start. This game is a gift version of the MarkStart problem known in the literature. We show that this symmetric problem admits symmetric solution. This result contrasts with the G_1 problem where we showed that solutions must be asymmetric.

Finally, we show how our methods are easily adaptable to minimizing the maximum rendezvous time. This is important when there is a deadline for rendezvous.

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