Packings and tilings in dense graphs

by

Yanitsa Pehova

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Declarations

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- A. Blumenthal, B. Lidický, Y. Pehova, F. Pfender, O. Pikhurko, and J. Volec. Sharp bounds for decomposing graphs into edges and triangles. *Combinatorics, Probability and Computing*, page 1–17, doi: 10.1017/S0963548320000358
- R. Nenadov and Y. Pehova. On a Ramsey–Turán Variant of the Hajnal–Szemerédi Theorem. *SIAM J. Discrete Math.*, 34(2):1001–1010, doi: 10.1137/18M1211970
- A. Liebenau and Y. Pehova. An approximate version of Jackson's conjecture. *Combin. Probab. Comput.*, 29(6):886–899, doi: 10.1017/S0963548320000152
- T. F. N. Chan, D. Král', J. A. Noel, Y. Pehova, M. Sharifzadeh, and J. Volec. Characterization of quasirandom permutations by a pattern sum. *Random Structures Algorithms*, 57(4):920–939, doi: 10.1002/rsa.20956

Abstract

In this thesis we present results on selected problems from extremal graph theory, and discuss both known and new methods used to solve them.

In Chapter 1, we give an introductory overview of the regularity method, the flag algebra framework, and some probabilistic tools, which we use to prove our results in subsequent chapters.

In Chapter 2 we prove a new result on the packing density of triangles in graphs with given edge density. In doing so, we settle a few conjectures of Győri and Tuza on decompositions and coverings of graphs with cliques of bounded size.

In Chapter 3 we show that a famous conjecture on Hamilton decompositions of bipartite tournaments due to Jackson holds approximately, providing the first intermediate result towards a full proof of the conjecture.

In Chapter 4, we introduce a novel absorbing paradigm for graph tilings, which we apply in a few different settings to obtain new results. Using this method, we are able to extend a result on triangle-tilings in graphs with high minimum degree and sublinear independence number to clique-tilings of arbitrary size. We also strengthen an existing result on tilings in randomly perturbed graphs.

Finally, in Chapter 5, we consider a problem on quasi-randomness in permutations. We obtain simple density conditions for a sequence of permutations to be quasi-random, and give a full characterisation of all conditions of the same type that force quasi-randomness in the same way.

Chapter 1

Introduction

Graph theory is a rapidly developing area of mathematics, with a lot of research being closely related to fundamental problems in computer science. Within mathematics, the study of graph properties and parameters, and the relationships between them, has been of interest for decades, and results in the area have been successfully applied to problems in this and other fields of study. In this thesis, we consider several questions from extremal graph theory, which can be roughly described as the study of relationships between local and global properties of graphs.

A wide variety of classical problems in the area can be stated as asking the following question:

Given a collection of graphs $\mathcal{H} = \{H_i(n)\}$, can we decompose the edges of a given graph G on n vertices into copies of the graphs from $\mathcal{H}$?

For example, if $\mathcal{H} = \{K_3\}$, and G is the complete graph on 15 vertices, this is the famous Kirkman's schoolgirl problem. If $\mathcal{H} = \{C_n\}$, then we get the problem of decomposing the edges of G into Hamilton cycles. The case $\mathcal{H} = \{F(n)\}$, where $F(n)$ is a union of cycles of total length n is known as the Oberwolfach problem, and was resolved only recently [44, 69]. A natural restatement of the above question in the world of hypergraphs covers the fundamental combinatorial problem known as existence of designs: in 1853 Steiner asked whether a collection of of q-subsets of $[n]$ exists, such that every r-subset of n is in exactly λ sets in the collection. This corresponds to taking the multigraph given by λ copies of each hyperedge of the complete r-uniform hypergraph on n vertices $K_n^{(r)}$, and decomposing it into edge-disjoint copies of $K_q^{(r)}$. This problem was recently resolved by Keevash [67], and extended to general F-designs, where F is an arbitrary r-uniform hypergraph, by Glock, Kühn, Lo and Osthus [45].

While versions of the above question are known as *packing* or *decomposition* questions, a similar notion is the one of *graph tiling.* In graph tiling problems, instead of seeking to decompose the edges of G into graphs in $\mathcal{H}$, we seek a collection of vertex-disjoint copies of graphs in $\mathcal{H}$ which cover all vertices of G. In the literature this spanning structure is often called a *$\mathcal{H}$-factor* or a *perfect $\mathcal{H}$-tiling.* A classical example of a graph tiling problem is the well-known notion of a perfect matching, which corresponds to $\mathcal{H} = \{K_2\}$ (see, for example, [87]). Various other choices for $\mathcal{H}$ as a singleton have been studied since, one notable example being a result of Hajnal and Szemerédi on clique-tilings [55] (see Chapter 4). Another popular choice for $\mathcal{H}$ is a single graph on n vertices. For example, taking $\mathcal{H} = \{F(n)\}$ as in the Oberwolfach problem yields the problem of finding a collection of vertex-disjoint cycles of total length n in an n-vertex graph. Results in the area include a theorem by Komlós, Sárközy and Szemerédi [71] which guarantees the existence of the square of a Hamilton cycle[1] when the minimum degree of G is at least $2n/3$. In particular, C_n^2 contains any collection of vertex-disjoint cycles of total length n as a subgraph, and the degree threshold $2n/3$ is tight for this property as there exist graphs of minimum degree $2n/3 - 1$ which cannot be covered by triangles.

This thesis presents, among other things, conditions for the existence of tilings and decompositions as described above, for different families $\mathcal{H}$. In what follows, a *H-packing* is a collection of edge-disjoint copies of H in a host graph, a *perfect H-packing* or a *H-decomposition* is a packing covering all the edges of the host graph; similarly, a *H-tiling* is a collection of vertex-disjoint copies of H in a host graph, and a *perfect H-tiling* or a *H-factor* is a H-tiling covering all vertices of the host graph.

In this chapter, we introduce some notation and mathematical tools which we use later; in particular, we provide a description of the flag algebra framework developed by Razborov [98], the graph regularity method [73], and some standard probabilistic tools (see [2]).

In Chapter 2 we look at an old theorem of Erdős, Goodman and Pósa on decomposing graphs into copies of K_2 and K_3 [36]. In [36] the authors showed that

[1]The k-th power G^k of a graph G is the graph obtained by placing an edge between every two vertices of G which are at distance at most k.

the edges of every graph on n vertices can be decomposed into at most $n^2/4$ edges and triangles. We present an answer to the following related question of Győri and Tuza [11, 12, 74]: can every graph on n vertices be decomposed into edges and triangles whose total number of vertices is at most $n^2/2 + 2$? Note that, in essence, this is a triangle-packing problem: given an n-vertex graph G, minimising the total number of vertices in a decomposition of G into edges and triangles is the same as maximising the number of triangles used in the decomposition. The main result presented in Chapter 2 (Theorem 2.5) may be restated as a lower bound on the number of edge-disjoint triangles guaranteed in a graph with a given number of edges.

Chapter 3 explores another classical packing question in graph theory. We consider a regular graph with a given degree, and ask whether it can be decomposed into Hamilton cycles. One of the oldest results in this area is due to Walecki [3], who showed that the complete graph K_n can be decomposed into Hamilton cycles. It was later shown by Csaba, Kühn, Lo, Osthus and Treglown [27] that in fact any regular graph with minimum degree at least $n/2$ can be decomposed in this way. Analogous questions on directed and oriented graphs have also received significant attention in the literature, with a famous conjecture of Kelly having been resolved recently by Kühn and Osthus [78]. Kelly's conjecture posits that every regular tournament has a Hamilton decomposition. The authors in [78] showed that any regular n-vertex oriented graph whose vertex degrees are at least $(3/8 + o(1))n$ can be decomposed into Hamilton cycles (the degree bound is essentially tight). In Chapter 3, we prove an approximate version of the corresponding question for bipartite regular tournaments, which were conjectured to be decomposable into Hamilton cycles by Jackson [64].

In Chapter 4 we present a robust absorbing framework for constructing perfect tilings in a variety of settings. We then apply it to obtain a proof of a result on clique-tilings in graphs with high minimum degree and low independence number. A seminal result of Hajnal and Szemerédi [55] states that every n-vertex graph with minimum degree at least $\frac{r-1}{r}n$ has a K_r-tiling. Extremal graphs for this result are complete partite. In Chapter 4 we show that by forbidding this extremal configuration by imposing that the host graph has sublinear independence number, we can relax the minimum degree condition from Hajnal and Szemerédi's theorem. This result is an extension of a result due to Balogh, Molla and Sharifzadeh [7], which covers the case $r = 3$. Our result was subsequently improved by Knierim and Su [70] by a more careful application of our methods. Further in this chapter, we apply our absorbing framework to recover a known result of Balogh, Treglown and Wagner [8] on general H-tilings in randomly perturbed graphs.

In Chapter 5 we consider a problem on forcing quasi-randomness in permutations. A famous conjecture due to Sidorenko [102] and Erdős and Simonovits [39] states that the homomorphism density[2] of a fixed bipartite graph in a host graph is minimised when the host graph is quasi-random. This was shown to be true for $H = C_4$ in the 1980s by several groups of authors [108, 22, 100], and this constitutes the simplest known condition which forces a graph with given edge density to be quasi-random. This chapter considers the corresponding question for combinatorial permutations. In particular, we ask for sets S of 4-permutations for which the total density of elements of S in a host permutation is minimised or maximised when the host permutation is quasi-random.

1.1 Notation

We use largely standard graph theory notation. Below is a summary of the main definitions and basic notation used throughout.

Basic combinatorics. In this thesis, $\mathbb{N}$ denotes the set of natural numbers without zero, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{R}$ the set of real numbers. We write $[n]$ for the set $\{1, ..., n\}$. Given a set A, we denote by A^k the set of ordered k-tuples of elements in A, by $A^{(k)}$ the set of ordered k-tuples of pairwise distinct elements in A, and $\binom{A}{k}$ for the collection of k-subsets of A. All notation for set operations used in this text is standard: $A \cup B$ and $A \cap B$ respectively denote the union and intersection of two sets, and $A \,\dot\cup\, B$ denotes disjoint union. Both $A \backslash B$ and $A - B$ stand for the set of elements in A which are not in B, and $A \triangle B$ denotes symmetric difference, i.e., $A \triangle B = (A \backslash B) \cup (B \backslash A)$.
When choosing constants, we use the standardised notation $c \ll d$ as short-hand for "there exists a non-decreasing function $f : (0, 1] \to (0, \infty)$ such that the result holds for all c with $0 < c \leqslant f(d)$".

Graphs, edges and vertices. Throughout, a (simple) graph consists of a set V of vertices and a set $E \subseteq \binom{V}{2}$ of edges. In particular, simple graphs have no

[2]Given graphs H and G, the *homomorphism density* of H in G is defined as $t(H, G) = |Hom(H, G)|/|G|^{|H|}$, i.e., $t(H, G)$ is the probability that a mapping from $V(H)$ to $V(G)$ chosen uniformly at random is a homomorphism.

loops and no multiple edges. By convention, we write a graph as $G = (V, E)$. The vertex set of a graph G is denoted by $V(G)$, and the edge set by $E(G)$. We write $|G|$ or $v(G)$ for $|V(G)|$, and $e(G)$ for $|E(G)|$. If G is a bipartite graph, we write $V(G) = (A, B)$ to indicate that G has vertex classes A and B. Throughout, $\overline{G}$ denotes the complement of the graph G, that is, the graph with vertex set $V(G)$ and edge set $\binom{V}{2}\backslash E(G)$. With a slight abuse of notation, given a graph G and a set $W \subseteq V(G)$, we write $G\backslash W$ as short-hand for the graph with vertex set $V(G)\backslash W$ and edge set $\{\{x, y\} \in E(G) : x, y \notin W\}$. When referring to an edge $\{x, y\}$ in a graph, we often write it as simply xy.

Subgraphs. Given two graphs H and G, we say that H is a subgraph of G and write $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set of vertices $V' \subseteq V(G)$, the subgraph induced on V', denoted by $G[V']$, has vertex set V' and edge-set $E(G) \cap \binom{V'}{2}$.

Oriented and directed graphs. An *oriented graph* is an orientation of a simple graph, that is, its edge-set is obtained by ordering each edge of the simple graph. A *directed graph* (or *digraph*) is a pair (V, E), where E is a subset of $V^{(2)}$. Note that every oriented graph is a directed graph.

Degrees and neighbourhoods. For a vertex $v \in V(G)$, the neighbourhood of v in G is $N_G(v) = \{w \in V(G) : \{v, w\} \in E(G)\}$, and the degree of v is $d_G(v) = |N(v)|$. (We often omit the subscript G if the graph is clear from context.) For a set $S \subseteq V(G)$, the neighbourhood of S is $N(S) = \bigcup_{s \in S} N(s)$. Given two sets of vertices $X, Y \subseteq V(G)$, we denote by $E(X, Y)$ the set of edges with one endpoint in X and one endpoint in Y, and write $e(X, Y)$ for $|E(X, Y)|$. These notions carry over to the world of directed graphs as follows. For a vertex v in a directed graph G, the out-neighbourhood of v is $N^+(v) = \{w \in V(G) : vw \in E(G)\}$ and the in-neighbourhood of v is $N^-(v) = \{w \in V(G) : wv \in E(G)\}$. Each of these gives rise to the out- and in-degree of a vertex, denoted by $d^+(v)$ and $d^-(v)$, respectively. The in- and out-neighbourhood of a set of vertices are defined analogously to the simple graph case. Given $X, Y \subseteq V(G)$, $E(X, Y)$ denotes the edges (u, v) such that $u \in X$ and $v \in Y$.

We say that a digraph is *regular* if there exists an integer d such that each vertex in the graph has in- and out-degree d.

Minimum and maximum degrees. The minimum degree of a graph G is $\delta(G) = \min_{v \in V(G)} d_G(v)$, and the maximum degree is $\Delta(G) = \max_{v \in V(G)} d_G(v)$. The average degree $d(G)$ is $\frac{1}{|V(G)|}\sum_{v \in V(G)} d_G(v)$. For directed graphs, the minimum out-degree is $\delta^+(G) = \min_{v \in V(G)} d^+(v)$, and the maximum out-degree is $\Delta^+(G) = \max_{v \in V(G)} d^+(v)$. The minimum and maximum in-degrees δ^- and Δ^- are defined similarly. Finally, the minimum and maximum semi-degree of a directed graph are $\delta^0(G) = \min\{\delta^+(G), \delta^-(G)\}$ and $\Delta^0(G) = \max\{\Delta^+(G), \Delta^-(G)\}$.

1.2 Szemerédi's regularity lemma

A well-known and versatile method in extremal combinatorics is the so-called "regularity method", named after the celebrated Regularity Lemma of Szemerédi [107], which was used (in a weaker variant) to prove that for large positive integers N, every subset of $[N]$ of positive density contains arbitrarily long arithmetic progressions [106].

Loosely speaking, the Regularity Lemma states that the vertex set of any sufficiently large graph can be partitioned into finitely many sets, among which the edges are distributed somewhat randomly. We start with a precise definition of "somewhat randomly".

Definition 1.1. For disjoint vertex sets $X, Y \subseteq V(G)$, the *density* or *edge-density* of the pair (X, Y) is defined as

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

Definition 1.2. We say that a pair (A, B) of disjoint subsets of $V(G)$ is *μ-regular* if for all $X \subseteq A$, $|X| \geqslant \mu|A|$ and $Y \subseteq B$, $|Y| \geqslant \mu|B|$ we have

$$|d(X,Y) - d(A,B)| \leqslant \mu.$$

A partition $V(G) = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_k$ of the vertex set of a graph G is said to be regular if all sets have equal size, and most pairs of sets are regular as defined above. More formally,

Definition 1.3. A partition $V(G) = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_k$ of the vertex set of a graph G is said to be *μ-regular* if $||V_i| - |V_j|| \leqslant 1$ for all i, j, and all but $\mu\binom{k}{2}$ pairs (V_i, V_j) are μ-regular.

With these definitions in hand, the Regularity Lemma reads as follows.

Lemma 1.4 (Regularity Lemma [106]). *For every $\mu > 0$ and $m \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ such that every graph G on $n \geqslant n_0$ vertices has a μ-regular partition into k parts, where $m \leqslant k \leqslant M$.*

In practice, one often uses only the part of the graph formed by dense regular pairs, and discards all other edges. The following corollary of Lemma 1.4 can be found in [73].

Lemma 1.5 (Regularity Lemma, degree version). *For every $\mu > 0$ there is an $M = M(\mu)$ such that if G is a graph with n vertices and $d \in [0, 1]$ is any real number, there exists a partition $V(G) = V_0 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} V_k$ and a spanning subgraph $G' \subseteq G$ with the following properties:*

(a) $k \leqslant M$;

(b) $|V_0| \leqslant \mu n$;

(c) $|V_i| = n'$ *for all* $1 \leqslant i \leqslant k$, *where* $n' \leqslant \mu n$;

(d) $d_{G'}(v) > d_G(v) - (d + \mu)n$ *for all* $v \in V(G)$;

(e) all V_i *for* $i \in [k]$ *are independent sets in* G';

(f) all pairs (V_i, V_j) *for* $i, j \in [k]$ *are* μ*-regular in* G' *with density 0 or at least* d.

The graph G' obtained in Lemma 1.5, with V_0 removed, is often called a *pure* regular graph. Note that since V_0 is small, the degree and density conditions are not significantly affected.

Finally, let us consider the graph R_G on vertex set $[k]$ whose edge-set consists of pairs $\{i, j\}$ such that (V_i, V_j) is a dense μ-regular pair in G'. The graph R_G is called the *reduced graph* of the regularity partition, and usually finding a fixed subgraph $H \subseteq R_G$ translates to finding many copies, or a certain type of blow-up, of H in G. For example, in Chapter 4 (see Lemma 4.9) we find a clique-tiling in R_G, which translates to a clique-tiling of G. More generally, results such as the Blow-Up Lemma [72] (also [73, Theorem 6.5.]) and the Embedding Lemma [73, Theorem 2.1.] formalise the relationship between existence of substructures in R_G and in G.

1.3 Probabilistic tools

This section contains key probabilistic tools used throughout this thesis. A wonderful, and much more detailed, exposition of numerous applications of probability in combinatorics can be found in [2], a book which the author of this thesis thoroughly enjoyed leafing through during their early years as a doctoral student.

We start by recalling three basic probabilistic inequalities.

Lemma 1.6 (Markov's inequality)**.** *Let X be a non-negative random variable with mean μ, and let $\lambda > 0$. Then*

$$\mathbb{P}(X \geqslant \lambda) \leqslant \frac{\mu}{\lambda}.$$

Lemma 1.7 (Chebyshev's inequality)**.** *Let X be a random variable with finite mean μ and variance σ^2. Then for $\varepsilon > 0$,*

$$\mathbb{P}(|X - \mu| \geqslant \varepsilon) \leqslant \frac{\sigma^2}{\varepsilon^2}.$$

Lemma 1.8 (Chernoff bound [41])**.** *Let X be a binomially distributed random variable with mean μ. Then for $\varepsilon \in (0, 1)$,*

$$\mathbb{P}(|X - \mu| \geqslant \varepsilon\mu) \leqslant 2e^{-\varepsilon^2\mu/3}.$$

In Chapter 3 we make heavy use of a variant of Chernoff's inequality applied to a hypergeometric random variable. Recall that X is hypergeometrically distributed with parameters (n, n_1, r) if

$$\mathbb{P}(X = k) = \frac{\binom{n_1}{k}\binom{n-n_1}{r-k}}{\binom{n}{r}}$$

for $k = 0, ..., r$. This way X models the number of red balls obtained when drawing r balls from a bin containing n_1 red and $n - n_1$ blue balls. Hypergeometric random variables are concentrated around their mean similarly to binomial random variables (for details see [41, Section 21.5]). In particular, Lemma 1.8 still holds if X is hypergeometrically distributed with mean μ.

The following is a short application of this inequality, which we use in Chapter 3 to prove Lemma 3.8.

Lemma 1.9. *Let $0 < \alpha < 1$, $n, K, s \in \mathbb{N}$, and $s \leqslant n$. Let S be a set of size n and let $T \subseteq S$ be a subset of size αn.*

Suppose $U_1, ..., U_K$ are subsets of S chosen uniformly and independently at random among all s-size subsets of S. Then if we denote $U = \bigcup_{i=1}^K U_i \cap T$, we have that

- $\mathbb{E}(|U|) = \alpha np'$*, where* $p' = \big(1 - (1 - s/n)^K\big)$*;*
- $\mathbb{P}(||U| - \mathbb{E}(|U|)| \geqslant t) \leqslant 2(n+1)^K e^{-t^2/3\alpha np'}$ *for any* $t \leqslant \alpha np'$*.*

Proof. First, note that for each element $v \in T$, the probability that it lies in $\bigcup U_i$ is exactly $1 - (1 - s/n)^K = p'$, so $\mathbb{E}(|U|) = \alpha np'$.
Now, for each $1 \leqslant i \leqslant K$, let $U_i^* \subseteq S$ be obtained by selecting each element of S independently with probability s/n, and let $U^* = \bigcup_{i=1}^K U_i^* \cap T$. Then $|U^*| \sim Bin(\alpha n, p')$, so by Chernoff's inequality we have

$$\mathbb{P}(||U^*| - \mathbb{E}(|U^*|)| \geqslant t) \leqslant 2e^{-t^2/3\alpha np'}.$$

Also note that, conditioned on the event "$|U_i^*| = s$ for all $1 \leqslant i \leqslant K$", $|U^*|$ has the same distribution as $|U|$. Then,

$$\mathbb{P}(||U| - \mathbb{E}(|U|)| \geqslant t) \leqslant \frac{2e^{-t^2/3\alpha np'}}{\mathbb{P}(|U_i^*| = s \text{ for all } i)}.$$

Now, each $|U_i^*|$ is a binomial random variable with mean s, so $\mathbb{P}(|U_i^*| = k)$ is maximised when $k = s$, giving $\mathbb{P}(|U_i^*| = s) \geqslant 1/(n+1)$. We substitute this in the above inequality to obtain

$$\mathbb{P}(||U| - \mathbb{E}(|U|)| \geqslant t) \leqslant 2(n+1)^K e^{-t^2/3\alpha np'}.$$

□

In Chapter 4, we briefly touch upon results on random graphs. In particular, we prove a subgraph containment result for the Erdős-Rényi random graph.

Definition 1.10 (Erdős-Rényi random graph)**.** Let $n \in \mathbb{N}$ be an integer and $p \in [0, 1]$. The *Erdős-Rényi random graph* $G(n, p)$ is the probability distribution on the family of all graphs with vertex set $[n]$ obtained by placing each of the $\binom{n}{2}$ possible edges with probability p.

To do so, we use a corollary of Janson's inequality to show that sets of linear size in random graph models similar to $G(n, p)$ contain certain subgraphs.

Lemma 1.11 (Janson's Inequality, Corollary of (3.5) in [65]). *Let $\{A_i\}_{i \in I}$ be a collection of equal size subsets of a finite set X, and let $R \subset X$ be a random subset given by $\mathbb{P}(x \in R) = p$ for all $x \in X$. For each $i \in I$, let B_i be the event $\{A_i \subseteq R\}$. Denote*

$$\mu = \sum_{i \in I} \mathbb{P}(B_i)$$

and

$$\Delta = \sum \mathbb{P}(B_i \cap B_j),$$

where the second sum is taken over unordered pairs $\{i, j\}$ such that $A_i \cap A_j \neq \emptyset$.

Then

$$\mathbb{P}\left(\bigcap_{i \in I} \overline{B_i}\right) \leqslant e^{-\frac{\mu^2}{2\Delta}}.$$

1.4 The flag algebra method

The flag algebra method due to Razborov [98] is a rather general framework for estimating densities of substructures in combinatorial objects, most commonly subgraph densities in large graphs. Since it was introduced in 2007, this method has been applied to many long-standing open problems such as determining the minimal density of triangles in a graph with a given number of edges, maximum induced C_5 density, maximum number of rainbow triangles in a 3-edge-coloured K_n, and more [5, 6, 62, 75, 99]. The method is designed to analyse asymptotic behaviour of substructure densities and we now briefly describe its general philosophy. It may be applied to a range of suitable discrete structures, although here we focus our attention on graphs and permutations.

Let $\mathcal{C}$ be a class of combinatorial objects, and let $d : \mathcal{C} \times \mathcal{C} \to [0, 1]$ be a notion of substructure density, that is, $d(c, c')$ is a normalised number of instances of c as a substructure of c'. Then the flag algebra $\mathcal{F}_{\mathcal{C}}$ is the ring of formal linear combinations of elements in $\mathcal{C}$ with real coefficients, with additional multiplication and quotient structure which we explain below. The flag algebra models proofs of statements of the form

$$\sum_{i \in I} \alpha_i d(c_i, c) \geqslant \beta \text{ for all } c \in \mathcal{C},$$

where $\beta, \alpha_i \in \mathbb{R}$ and $c_i \in \mathcal{C}$ by "algebraically" proving statements of the form

$$\sum_{i \in I} \alpha_i c_i \geqslant \beta$$

in $\mathcal{F}_\mathcal{C}$.

One may draw an analogue between this and algebraic inequalities in the polynomial algebra $\mathbb{R}[X]$ such as $X^2 - 2X + 5 \geqslant 4$.

We usually work with $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$, where $\mathcal{C}_n$ is the collection of combinatorial structures of size n. Then the density function is such that $\sum_{c\in\mathcal{C}_j} d(c,c') = 1$ for all $c' \in \mathcal{C}$. We also have that $d(c,c') = 0$ if c is larger than c'. Normally $\mathcal{C}_1$ consists of only a single element which is the identity in $\mathcal{F}_\mathcal{C}$, as its density in any other combinatorial structure is always 1. The zero in $\mathcal{F}_\mathcal{C}$ is the empty linear combination. Note that by definition in $\mathcal{F}_\mathcal{C}$ we have $c \geqslant 0$ and $c \leqslant 1$ for all $c \in \mathcal{C}$, and these inequalities extend linearly.

So far, as a ring with no additional structure, $\mathcal{F}_\mathcal{C}$ models very simple proofs of density inequalities in $\mathcal{C}$. For example we have that

$$0.1c_1 + 0.2c_2 \leqslant 0.3$$

for any two $c_1, c_2 \in \mathcal{C}$, which corresponds to the somewhat trivial density inequality

$$0.1d(c_1, c) + 0.2d(c_2, c) \leqslant 0.3 \text{ for all } c \in \mathcal{C}.$$

Fixed-size densities sum to one. Let us add to $\mathcal{F}$ the law of total probability on $d(\cdot, c)$, which is a probability distribution on $\mathcal{C}_m$ for any c of size n and any $m \leqslant n$. Recall that $\sum_{c\in\mathcal{C}_j} d(c,c') = 1$ for any j and any c'. More formally, we take the ring quotient

$$\mathcal{F}'_\mathcal{C} = \mathcal{F}_\mathcal{C} \Big/ \langle \{1 - \textstyle\sum_{c\in\mathcal{C}_j} c\}_{j\in\mathbb{N}} \rangle .$$

With this structure addition, we can write slightly less trivial inequalities such as

$$\sum_{c\in\mathcal{C}_n} \alpha_c c \geqslant \min_{c\in\mathcal{C}_n} \alpha_c.$$

Multiplicative structure. To turn $\mathcal{F}'_\mathcal{C}$ into an algebra, we now endow it with a notion of multiplication. In order to maintain the modelling property, we wish to define our multiplication $c_1 \times c_2$ such that $d(\cdot, c)$ is an algebra homomorphism for each $c \in \mathcal{C}$:

$$d(c_1 \times c_2, c) = d(c_1, c)d(c_2, c).$$

Apart from some trivial pairs (c_1, c_2), there generally isn't an element c' independent of c such that $d(c_1, c)d(c_2, c) = d(c', c)$. We relax this to the asymptotic identity

$$d(c_1 \times c_2, c) = d(c_1, c)d(c_2, c) + o(1)$$

when $|c| \to \infty$. Then a universal element $c_1 \times c_2$ exists. This relaxation introduces errors into our inequalities, but as we write only finite proofs, the cumulative error still tends to zero as the size of c grows. (See (1.3) and (1.5) for the exact definitions for multiplication of graphs and permutations.)

Expansion. To complete $\mathcal{F}'_\mathcal{C}$ with the newly defined multiplication into the flag algebra we use in Chapters 2 and 5, we add the following expansion identity, which holds for any c of size $m \leqslant n$

$$c = \sum_{c' \in \mathcal{C}_n} d(c, c')c'$$

by considering conditional probability on the space $\mathcal{C}_n$ endowed with the probability function $d(\cdot, c'')$ for any other $c'' \in \mathcal{C}$. That is, we claim that for any c'' and any c, it holds that

$$d(c, c'') = \sum_{c' \in \mathcal{C}_m} d(c, c')d(c', c'').$$

So, finally, we define the flag algebra $\mathcal{F}''_\mathcal{C}$ as

$$\mathcal{F}''_\mathcal{C} = \mathcal{F}'_\mathcal{C} \Big/ \langle \{c - \textstyle\sum_{c' \in \mathcal{C}_m} d(c, c')c'\}_{c \in \mathcal{C}, m \in \mathbb{N}} \rangle \,.$$

Throughout this thesis, we drop the primes and write $\mathcal{F}_\mathcal{C}$ for the flag algebra of linear combinations of elements of $\mathcal{C}$ built in this way.

In our proofs in Chapters 2 and 5, we make use of the following consequence of the multiplicativity of our densitity function on $\mathcal{F}_\mathcal{C}$. Let $f = \sum_i \alpha_i c_i \in \mathcal{F}_\mathcal{C}$. Then

$$f^2 \geqslant 0 \tag{1.1}$$

since evaluating $d(f^2, c)$ for any $c \in \mathcal{C}$ yields

$$d(f^2, c) = d(f, c)^2 = \left(\sum_i \alpha_i d(c_i, c)\right)^2 \geqslant 0.$$

Or, more generally, if M is a positive semi-definite matrix, and $\mathbf{c}$ is a vector of elements of $\mathcal{C}$, then

$$\mathbf{c}^T M \mathbf{c} \geqslant 0. \tag{1.2}$$

In the following two sections we describe the structure of the flag algebras of graphs and permutations, the appropriate density functions and multiplications used to define the algebra.

1.4.1 The flag algebra of graphs

Let us denote by $\mathcal{G}$ the family of graphs, and by $\mathcal{G}_\ell$ the family of graphs with ℓ vertices. The density function we work with is based on counting induced subgraphs of a given size. For two graphs G and H, we define $d(G, H)$ as the probability that $|G|$ distinct vertices chosen uniformly at random among the vertices of H induce a graph isomorphic to G; if $|G| > |H|$, we set $d(G, H) = 0$.

As for the multiplicative structure, given two graphs G_1 and G_2 of sizes n and m, we define their product as

$$G_1 \times G_2 = \sum_{F \in \mathcal{G}_{n+m}} \mathbb{P}_W \left(F[W] \cong G_1,\ F \backslash W \cong G_2 \right) F, \tag{1.3}$$

where the probability is taken over a uniformly chosen $W \in \binom{V(F)}{n}$. As defined above, this notion of multiplication has the property that

$$d(G_1 \times G_2, H) = d(G_1, H) d(G_2, H) + o(1),$$

which can be seen by considering sampling an n-subset and an m-subset of $V(H)$ independently, versus the same experiment conditioned on the two subsets being disjoint, and evaluating the probability that the resulting two graphs are G_1 and G_2. The density $d(G_1 \times G_2, H)$ is also written in the literature as $d(G_1, G_2; H)$.

Similarly, a flag algebra whose multiplication allows for interaction of the two graphs which are being sampled, can be defined as follows. We call a (typically small) graph σ with its vertices labelled with $1, ..., |\sigma|$ a *type*. A *σ-flag* G^σ (or just G, if σ is clear from the context) is a graph G with a subset of $|\sigma|$ vertices labelled by $1, ..., |\sigma|$, such that the labelled vertices induce a copy of σ, preserving the vertex labels. Let $\mathcal{G}^\sigma$ denote the set of all σ-flags, and as before, let $\mathcal{G}^\sigma_n$ denote those on n vertices. We say that two σ-flags are isomorphic if they are isomorphic as graphs, and they have an isomorphism which identifies their labelled vertices as specified by the labels. Then we may define a "rooted density function" similarly to the one for graphs: given two σ-flags G and H, we define $d(G,H)$ as the probability that $|G| - |\sigma|$ vertices chosen uniformly at random among the unlabelled vertices of H, together with the labelled vertices in H, induce a σ-flag isomorphic to G. In slightly fewer words, $d(G,H)$ is the probability of the same event as in the definition of graph density, except conditioned on the labelled vertices mapping to each other as specified by their respective labellings. With this amended density function in mind, we can complete the flag algebra $\mathcal{F}_{\mathcal{G}^\sigma}$ of σ-flags by defining

$$G_1 \times G_2 = \sum_{H \in \mathcal{G}^\sigma_{n+m-|\sigma|}} \mathbb{P}_W \left(H[W \cup \sigma] \cong G_1,\ H[V(H) \backslash W] \cong G_2 \right) H,$$

where (with a slight abuse of notation) σ denotes the set of labelled vertices in each H in the sum, $\cong$ denotes σ-flag isomorphism, and $W \in \binom{V(H)\backslash\sigma}{n-|\sigma|}$ is chosen uniformly at random. As before, $d(\cdot, H)$ is (asymptotically) an algebra homomorphism for each σ-flag H.

Intuitively, the flag algebra of σ-flags allows for counting of subgraphs containing a given vertex, a given edge, a given pair of non-adjacent vertices, etc. Naturally, averaging these counts over the choice of vertex or pair of vertices recovers the standard subgraph counts. We formalise this idea as follows. We define the *averaging map* $[\![\cdot]\!]_\sigma : \mathcal{F}_{\mathcal{G}^\sigma} \to \mathcal{F}_{\mathcal{G}}$ (or simply $[\![\cdot]\!]$ if σ is clear from the context) as

$$[\![G^\sigma]\!] = \mathbb{P}_W(G[W] \cong \sigma)G,$$

where G denotes G^σ without the labelling, W is chosen uniformly among all σ-tuples of vertices in G, and $\cong$ stands for labelled isomorphism. Note that if for a given flag G^σ and an unlabelled graph H, we average $d(G^\sigma, H^{\sigma'})$ over all choices of labelled vertices σ' in H (not necessarily isomorphic to σ), we recover the definition of $[\![\cdot]\!]_\sigma$:

$$\mathbb{E}_{\sigma'} d(G^\sigma, H^{\sigma'}) = \mathbb{P}_W(G[W] \cong \sigma) d(G,H).$$

So, we have that

$$d([\![G^\sigma]\!], H) = \mathbb{E}_{\sigma'} d(G^\sigma, H^{\sigma'}). \tag{1.4}$$

Since $[\![\cdot]\!]_\sigma$ is simply an expectation, applying Jensen's inequality to (1.1) yields

$$[\![f^2]\!] \geqslant [\![f]\!]^2 \geqslant 0,$$

for any $f \in \mathcal{F}_{\mathcal{G}^\sigma}$. Similarly (1.2) yields

$$[\![\mathbf{F}^T M \mathbf{F}]\!] \geqslant 0$$

for any positive semi-definite matrix M and any vector $\mathbf{F}$ of elements of $\mathcal{F}_{\mathcal{G}^\sigma}$.

1.4.2 The flag algebra of permutations

In this thesis, we represent permutations by length-n sequences of distinct elements in $[n]$. Such a sequence $a_1...a_n$ corresponds to the permutation mapping i to a_i. We denote by $\mathcal{S}_n$ the set of permutations of length n, and by $\mathcal{S}$ the set of all permutations, that is, $\mathcal{S} = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$. For a permutation π, we use $|\pi|$ to denote the length of π. In the literature such permutations are also referred to as *patterns*.

Definition 1.12. For a permutation $\pi \in \mathcal{S}_n$ and a set I of indices $1 \leqslant i_1 \leqslant ... \leqslant i_k \leqslant n$, the *subpermutation* of π induced by I is the unique k-permutation σ such that $\sigma(j) < \sigma(\ell)$ if and only if $\pi(i_j) < \pi(i_\ell)$ for all $j, \ell \in [k]$.

Definition 1.13. Let σ and π be two permutations. The *(pattern) density* $d(\sigma, \pi)$ of σ in π is defined as the probability that a set of $|\sigma|$ distinct indices chosen uniformly at random among all $|\sigma|$-sized subsets of $|\pi|$ induces σ.
(If $|\sigma| > |\pi|$, we set $d(\sigma, \pi) = 0$.)

Given permutations π_1 and π_2, the product which defines the flag algebra $\mathcal{F}_{\mathcal{S}}$ is

$$\pi_1 \times \pi_2 = \sum_{\sigma \in \mathcal{S}_{|\pi_1|+|\pi_2|}} \mathbb{P}_I(\sigma[I] \cong \pi_1,\ \sigma[[n]\backslash I] \cong \pi_2)\sigma, \tag{1.5}$$

where $I \in \binom{[n]}{|\pi_1|}$ is chosen uniformly at random, and $\sigma[I] \cong \pi_1$ stands for the condition that the subpermutation of σ induced by I is π_1.

This product is also an (approximate) algebra homomorphism, meaning that

$$d(\pi_1 \times \pi_2, \sigma) = d(\pi_1, \sigma) d(\pi_2, \sigma) + o(1) \text{ as } |\sigma| \to \infty. \tag{1.6}$$

Analogously to the graph case, we now define a "rooted" flag algebra of permutations. Here, a type is just a permutation τ and a *τ-flag*, or a *τ-rooted permutation*, is a permutation with $|\tau|$ distinguished indices which induce τ. We denote the root by underlining its entries. The rooted permutation density, the permutation flag product and the averaging map are defined analogously to the graph case presented in Section 1.4.1. As in the graph case, we will use the fact that for any positive semi-definite M and any vector $\mathbf{F}$ of elements of $\mathcal{F}_{\mathcal{S}^\tau}$, the inequality

$$[\![\mathbf{F}^T M \mathbf{F}]\!] \geqslant 0$$

holds in $\mathcal{F}_{\mathcal{S}}$.

Example. The permutation 1243 has 2-pattern densities $d(12, 1243) = 5/6$ and $d(21, 1243) = 1/6$. The product of the two distinct 2-permutations is

$$\begin{aligned}
12 \times 21 = &\frac{1}{6} \times 1243 + \frac{1}{6} \times 1324 + \frac{2}{6} \times 1342 + \frac{2}{6} \times 1423 + \frac{3}{6} \times 1432 \\
&+ \frac{1}{6} \times 2134 + \frac{2}{6} \times 2314 + \frac{3}{6} \times 2341 + \frac{1}{6} \times 2413 + \frac{2}{6} \times 2431 \\
&+ \frac{2}{6} \times 3124 + \frac{1}{6} \times 3142 + \frac{3}{6} \times 3214 + \frac{2}{6} \times 3241 + \frac{1}{6} \times 3421 \\
&+ \frac{3}{6} \times 4123 + \frac{2}{6} \times 4132 + \frac{2}{6} \times 4213 + \frac{1}{6} \times 4231 + \frac{1}{6} \times 4312.
\end{aligned}$$

Let τ be the unique permutation of length 1. The rooted permutation densities of the four τ-flags of length 2 in $1\underline{2}43$ are

$$\begin{aligned}
d(\underline{1}2, 1\underline{2}43) &= 2/3 & \qquad d(1\underline{2}, 1\underline{2}43) &= 1/3 \\
d(\underline{2}1, 1\underline{2}43) &= 0 & \qquad d(2\underline{1}, 1\underline{2}43) &= 0.
\end{aligned}$$

The average of a τ-flag π^τ is always $[\![\pi^\tau]\!] = \binom{|\pi|}{|\tau|}^{-1} \pi$ as there is always only one way to obtain π^τ by rooting on $|\tau|$ entries of π. The averaging equivalence (1.4) holds as in graphs, for example for the 1-flag $1\underline{2}$ we have

$$\begin{aligned}
\mathbb{E}\, d(1\underline{2}, 1243) &= \frac{1}{4} \left(d(1\underline{2}, \underline{1}243) + d(1\underline{2}, 1\underline{2}43) + d(1\underline{2}, 12\underline{4}3) + d(1\underline{2}, 124\underline{3}) \right) \\
&= \frac{1}{4} \left(0 + \frac{1}{3} + \frac{2}{3} + \frac{2}{3} \right) = \frac{5}{12}
\end{aligned}$$

and $d([\![1\underline{2}]\!], 1243) = \frac{1}{2} d(12, 1243) = \frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$.

With these definitions at hand, the procedure outlined in Section 1.4.1 may also be carried out in the context of permutations. In the interest of brevity, we present a slightly different, "limit-language" outline of the flag algebra method for permutations, which we later use as-is in Chapter 5.

A *permuton* is a Borel probability measure μ on $[0,1]^2$ that has uniform marginals, i.e., $\mu\left([x, x'] \times [0,1]\right) = x' - x$ for every $0 \leqslant x < x' \leqslant 1$ and $\mu\left([0,1] \times [y, y']\right) = y' - y$ for every $0 \leqslant y < y' \leqslant 1$. A *$\mu$-random permutation of order k* is obtained as follows.

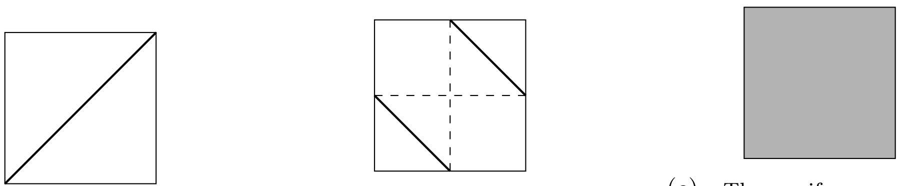
We first sample k points $(x_1, y_1), \ldots, (x_k, y_k)$ in $[0,1]^2$ according to the probability measure μ. Note that the probability that an x- or y-coordinate is shared by multiple points is zero because μ has uniform marginals. By renaming the points, we can assume that $x_1 < \cdots < x_k$. The μ-random permutation $\pi \in \mathcal{S}_k$ is then the unique permutation such that $\pi(i) < \pi(j)$ if and only if $y_i < y_j$ for every $i, j \in [k]$. We define the density of $\pi \in \mathcal{S}_k$ in the permuton μ to be the probability that a μ-random permutation of order k is π. A sequence $(\pi_i)_{i \in \mathbb{N}}$ of permutations is *convergent* if $|\pi_i|$ grows to infinity and the limit

$$\lim_{i \to \infty} d(\sigma, \pi_i)$$

exists for every permutation σ. It can be shown [63] that if $(\pi_i)_{i \in \mathbb{N}}$ is a convergent sequence of permutations, then there exists a unique permuton μ such that

$$\lim_{i \to \infty} d(\sigma, \pi_i) = d(\sigma, \mu)$$

for every permutation σ; the permuton μ is called the *limit* of the sequence $(\pi_i)_{i \in \mathbb{N}}$. In the other direction, if μ is a permuton, then a sequence of μ-random permutations with increasing orders converges with probability one and its limit is μ with probability one.



(a) Limit of the sequence $\pi_n = 12...n$. It holds that $d(12...k, \mu) = 1$ for all k and the densities of all non-increasing permutations are zero.

(b) Limit of the sequence $\pi_n = \lfloor n/2 \rfloor ... 1n ... (\lfloor n/2 \rfloor + 1)$. $d(12, \mu) = d(21, \mu) = 1/2$, $d(132, \mu) = 3/8$, $d(213, \mu) = 3/8$, and $d(321, \mu) = 1/4$.

(c) The uniform permuton. The sequence of random permutations of length n converges asymptotically almost surely to this permuton. $d(\sigma, \mu) = 1/|\sigma|!$ for all permutations σ.

Figure 1.1: Examples of permutons, sequences which converge to them, and sample densities.

Given a type $\tau \in S_\ell$, a *τ-rooted permuton* is an $(\ell+1)$-tuple

$$\mu^\tau = (\mu, (x_1, y_1), \ldots, (x_\ell, y_\ell))$$

such that μ is a permuton, $(x_1, y_1), \ldots, (x_\ell, y_\ell) \in \operatorname{supp}(\mu)$, $x_1 < \cdots < x_\ell$, and $\tau(i) < \tau(j)$ if and only if $y_i < y_j$ for all $i, j \in [\ell]$. The points $(x_1, y_1), \ldots, (x_\ell, y_\ell)$ are referred to as *roots*. If μ^τ is a τ-rooted permuton, then a *μ^τ-random permutation of order $k \geqslant \ell$* is a τ-rooted permutation obtained by sampling $k - \ell$ points in $[0,1]^2$

according to the measure μ, forming a permutation of order k using these $k-\ell$ points and the ℓ roots of μ^τ, and distinguishing the ℓ points corresponding to the roots of μ^τ to be the roots of the permutation. If σ^τ is a τ-rooted permutation, we write $d(\sigma^\tau, \mu^\tau)$ for the probability that a μ^τ-random permutation of order $|\sigma^\tau|$ is σ^τ.

The density and rooted density functions for permutons as defined above are still algebra homomorphisms on $\mathcal{F}_\mathcal{S}$. That is, the function $d(\cdot, \mu) : \mathcal{F}_\mathcal{S} \to \mathbb{R}$ which maps $\sum_i \alpha_i \sigma_i$ to $\sum_i \alpha_i d(\sigma_i, \mu)$ is an algebra homomorphism, much like its finite equivalent, and so is the rooted density $d(\cdot, \mu^\tau)$. By taking the limit as $|\sigma|$ tends to infinity in (1.6), we have that for permutons the homomorphism is exact:

$$d(\pi_1 \times \pi_2, \mu) = d(\pi_1, \mu) d(\pi_2, \mu)$$

for any $\pi_1, \pi_2 \in \mathcal{S}$ and μ a permuton. The same holds in the rooted setting.

For a fixed permutation τ and a permuton μ such that $d(\tau, \mu) > 0$, we can obtain a probability distribution on τ-rooted permutons μ^τ by taking the permuton μ and choosing $|\tau|$ points randomly according to the probability measure μ to be the roots (and sorting them according to their first coordinates) conditioned on the event that the chosen roots yield the permutation τ. In this way, the pair (τ, μ) yields a probability distribution on τ-rooted permutons. We write h^τ_μ for the homomorphism $d(\cdot, \mu^\tau)$, where μ^τ is a random τ-rooted permuton sampled as described above.

1.5 The absorbing method

In this section we give a general introduction to the *absorbing method*, together with a brief description of the way we apply it to prove the results presented in Chapter 4.

The *absorbing method* was pioneered by Rödl, Ruciński and Szemerédi [101], though it can be traced further back to Erdős, Gyárfás, and Pyber [37] and Krivelevich [77]. Broadly speaking, if we wish to build a spanning subgraph such as a Hamilton cycle or a perfect tiling, we might be able to do it greedily to start with. For example, in an n-vertex graph with minimum degree $2n/3$, we can find up to $n/9$ vertex-disjoint triangles by iterating neighbourhoods. By using the regularity method to construct tilings, we often cover all but an arbitrarily small proportion of vertices. But in either case, completing this to a perfect triangle-tiling can be difficult due to the little freedom remaining when trying to add the last few triangles. Instead, what we do to improve on this approach is to set aside a small set of special vertices at the beginning, often of high degree, high common neighbourhood, or "robustly tileable" as defined below. In this way the problem of finding a perfect tiling is reduced to finding an almost-perfect tiling, as we can use the special vertices to complete the tiling.

In this section, we describe a robust absorbing structure for graph tilings which we use to prove our results in Chapter 4. We build on ideas of Montgomery [90] on "robust matchability" by adapting them to the setting of general H-tilings. Similar ideas have been used in other areas, for example, by Kwan [81] to show that a typical Steiner triple system contains a perfect matching.

For the rest of this section, let H be a graph with h vertices and let G be a graph with n vertices, and suppose that we wish to show the existence of a perfect H-tiling in G.

The main idea behind the absorbing method for tilings is, given a fixed graph H on h vertices, to find an absorbing subset $A \subseteq V(G)$, such that for every sufficiently small "remainder set" $R \subseteq V(G)\backslash A$, the set A can be tiled together with R. More formally,

Definition 1.14. We say that a subset $A \subseteq V(G)$ is ξ-*absorbing* for some $\xi > 0$ if for every subset $R \subseteq V(G)\backslash A$ such that h divides $|A| + |R|$ and $|R| \leqslant \xi n$ the induced subgraph $G[A \cup R]$ contains a perfect H-tiling.

Having found A, the problem of finding a perfect H-tiling in G reduces to finding a H-tiling in $G\backslash A$ which covers all but at most ξn vertices, which is usually much simpler.

In our proofs in Chapter 4, we build A as a union of smaller absorbing structures called (S, t)-absorbers.

Definition 1.15. Given a subset $S \subseteq V(G)$ of size h and an integer $t \in \mathbb{N}$, we say that a subset $A_S \subseteq V(G)\backslash S$ is (S, t)-*absorbing* if $|A_S| = ht$ and both $G[A_S]$ and $G[A_S \cup S]$ contain a perfect H-tiling.

Widely used constructions of absorbers by Rödl, Ruciński and Szemerédi [101] and Hàn, Person, and Schacht [56] rely on the following *strong H-absorbing property* of G: for every subset $S \subseteq V(G)$ of size $v(H)$ there are $\Omega(n^{v(H)t})$ (S, t)-absorbers for some $t \in \mathbb{N}$. In many problems on finding H-factors such a property indeed holds (e.g. see [57, 86, 109, 110]). However, in some problem settings the conditions on G are not strong enough to guarantee this property. For example, as pointed out by the authors in [7], a graph G with minimum degree $(1/2+\varepsilon)|G|$ and independence number $\delta|G|$ for some $\delta \ll \varepsilon$ does not necessarily have this property when H is a triangle (and so the absorbing construction from [56, 101] fails), but can be shown to contain a perfect triangle-tiling [7, Theorem 1.2.].

Instead, to prove our main results from Chapter 4, we rely on the much weaker property that for every $v(H)$-subset S there exists a family of $\Theta(n)$ vertex-disjoint (S,t)-absorbers. In Section 4.1 we show that this property suffices to show the existence of an absorbing set.

Chapter 2

Decomposing graphs into cliques

Motivated by the following result of Erdős, Goodman and Pósa [36], we consider the problem of decomposing the edges of a given graph into edge-disjoint complete graphs.

Theorem 2.1 (Erdős, Goodman and Pósa [36])**.** *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ complete graphs.*

In fact, the following stronger statement is true.

Theorem 2.2 (Erdős, Goodman and Pósa [36])**.** *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ copies of K_2 and K_3.*

The bounds given in Theorems 2.1 and 2.2 are best possible as witnessed by complete bipartite graphs with parts of equal sizes.

Later, Chung [17], Győri and Kostochka [53], and Kahn [66], independently, proved a related conjecture of Katona and Tarján asserting that the edges of every n-vertex graph can be covered with complete graphs $C_1, \ldots, C_\ell$ such that the sum of their orders is at most $n^2/2$. In fact, the first two proofs yield a stronger decomposition statement, which implies Theorem 2.1 and which we next state as a separate theorem. Let us define $\pi_k(G)$ to be the minimum integer m such that the edges of a given graph G can be decomposed into complete graphs $C_1, \ldots, C_\ell$ of order at most k with $|C_1| + \cdots + |C_\ell| = m$, and we let $\pi(G) = \min_{k\in\mathbb{N}} \pi_k(G)$.

Theorem 2.3 (Chung [17]; Győri and Kostochka [53])**.** *Every n-vertex graph G satisfies $\pi(G) \leqslant n^2/2$.*

In [89, 88] McGuinness extended these results by showing that decompositions satisfying the upper bounds from Theorems 2.1 and 2.3 can be constructed by successively removing maximum cliques, which confirmed a conjecture of Winkler of this being the case in the setting of Theorem 2.1 (in fact, for Theorem 2.1 it is enough to remove maximal cliques).

In view of Theorem 2.2, it is natural to ask whether Theorem 2.3 also holds under the additional assumption that all complete graphs in the decomposition are copies of K_2 and K_3, i.e., whether $\pi_3(G) \leqslant n^2/2$. Note that $\pi_3(K_5) = 14 > 5^2/2$ but this example does not generalise to larger graphs. Győri and Tuza [53] provided a partial answer by proving that $\pi_3(G) \leqslant 9n^2/16$ and conjectured the following.

Conjecture 2.4 (Győri and Tuza [111, Problem 40])**.** *Every n-vertex graph G satisfies $\pi_3(G) \leqslant (1/2 + o(1))n^2$.*

In [74] we prove this conjecture. This result also solves [111, Problem 41], which we state as Corollary 2.10. Moreover, a careful stability argument building on the proof of Conjecture 2.4 gives an affirmative answer to a conjecture from [53] which claims that $\pi_3(G) \leqslant n^2/2 + O(n)$. In particular, we prove the following.

Theorem 2.5 (Blumenthal, Lidický, Pehova, Pfender, Pikhurko, Volec [12])**.** *For sufficiently large n every n-vertex graph G satisfies $\pi_3(G) \leqslant n^2/2 + 1$.*

A closely related variant of this problem with different costs was suggested by Erdős [111, Problem 43] and reads as follows. Let $\pi^-(G)$ be the minimum $m \in \mathbb{N}$ such that the edges of a graph G can be decomposed into complete graphs $C_1, \dots, C_\ell$ with $(|C_1| - 1) + \cdots + (|C_\ell| - 1) = m$. Erdős asked whether $\pi^-(G) \leqslant n^2/4$ for every n-vertex graph G. This problem remains open and was proved for K_4-free graphs only recently by Győri and Keszegh [52]. More specifically, they proved that every K_4-free graph with n vertices and $\lfloor n^2/4 \rfloor + k$ edges contains k edge-disjoint triangles.

Motivated by this setting, we study an extension of Theorem 2.5 which gives an upper bound on $\pi_3^\alpha(G)$, the minimum cost of a triangle-edge decomposition of G where each edge costs 2 and each triangle costs α for some real number $0 < \alpha < 6$ (in relation with our previous notation, we write $\pi_3(G)$ as short-hand for $\pi_3^3(G)$). In particular, we give the following characterisation of all π_3^α-extremal graphs. (Below, we denote by K_n^- the complete graph on n vertices with one edge removed, and by $K_n^=$ the complete graph with a matching of size two removed.)

Theorem 2.6 (Blumenthal, Lidický, Pehova, Pfender, Pikhurko, Volec [12])**.** *For every real α there exists $n_0 \in \mathbb{N}$ such that every π_3^α-extremal graph G with $n \geqslant n_0$ vertices satisfies the following (up to isomorphism).*

- *If $\alpha < 3$, then $G = T_2(n)$;*
- *if $\alpha = 3$ then Theorem 2.5 applies;*
- *if $3 < \alpha < 4$ and $n \equiv 0, 2, 4, 5 \mod 6$, then $G = K_n$;*
- *if $3 < \alpha < 4$ and $n \equiv 1, 3 \mod 6$, then $G = K_n^{=}$;*
- *if $\alpha = 4$ and $n \equiv 1, 3 \mod 6$, then $G \in \{K_n, K_n^-, K_n^{=}\}$ and, moreover, the three listed graphs are all π_3^α-extremal;*
- *if $\alpha = 4$ and $n \equiv 0, 2, 4, 5 \mod 6$, then $G = K_n$;*
- *if $4 < \alpha$, then $G = K_n$.*

The rest of this chapter is organised as follows. In Section 2.1 we prove Conjecture 2.4 (the asymptotic result). In Section 2.2 we prove the exact result (Theorem 2.5). In Section 2.3 we prove Theorem 2.6 as an extension of Theorem 2.5 to arbitrary triangle costs. Finally, in Section 2.4 we present some open problems and new research directions in this area.

2.1 Proof of the approximate result (Conjecture 2.4)

To prove Conjecture 2.4, we first consider the corresponding problem for fractional decompositions. A fractional triangle-edge decomposition of a graph G is an assignment of non-negative real weights to complete subgraphs of G of orders 2 and 3, such that the sum of the weights of the cliques containing any edge e is equal to one. More formally,

Definition 2.7. Let G be a graph, let $k \in \mathbb{N}$. A *fractional triangle-edge decomposition* of G is a function $w : \mathcal{T}(G) \cup E(G) \to [0, 1]$ such that for each $e \in E(G)$

$$w(e) + \sum_{\mathcal{T} \ni T \ni e} w(T) = 1.$$

We define the *cost* of a fractional triangle-edge decomposition $\mathcal{D}$ as

$$c(\mathcal{D}) = \sum_{e \in E(G)} 2w(e) + \sum_{T \in \mathcal{T}(G)} 3w(T).$$

Let

$$\pi_{3,f}(G) = \min_{\mathcal{D}} c(\mathcal{D})$$

be the minimum cost of a fractional triangle-edge decomposition of G. (Note that $\pi_{3,f}(G) \leqslant \pi_3(G)$ as every (integer) decomposition is a fractional decomposition.)

We start by proving the following lemma using the flag algebra method outlined in Section 1.4.

Lemma 2.8. *Let G be a weighted graph with all edges of weight one. It holds that*

$$\mathbb{E}_W \pi_{3,f}(G[W]) \leqslant 21 + o(1)$$

where W is a uniformly chosen random subset of seven vertices of G.

Proof. We work with flags on 4 vertices rooted at a single vertex $\sigma = K_1$. For the rest of this proof, we write $\bullet$ to denote σ.

We seek a positive semi-definite matrix M and a vector $\mathbf{F}$ of flags in $\mathcal{G}_4^\bullet$ such that for every $H \in \mathcal{G}_7$ the following quantity

$$c_H := \sum_{i,j} M_{ij} d([\![F_i \times F_j]\!], H)$$

satisfies

$$\pi_{3,f}(H) + c_H \leqslant 21.$$

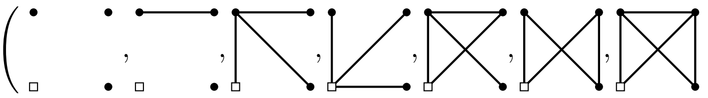
Note that $c_H = d([\![\mathbf{F}^T M \mathbf{F}]\!], H)$ and is hence always non-negative.

Then we can write out $\mathbb{E}_W \pi_{3,f}(G[W])$ in terms of $d(H, G)$ for each $H \in \mathcal{G}_7$ as follows.

$$\begin{aligned}
\mathbb{E}_W \pi_{3,f}(G[W]) &= \sum_{H \in \mathcal{G}_7} \pi_{3,f}(H) \cdot d(H, G) \\
&\leqslant \sum_{H \in \mathcal{G}_7} (\pi_{3,f}(H) + c_H) \cdot d(H, G) + o(1) \qquad (2.1)\\
&\leqslant \sum_{H \in \mathcal{G}_7} 21 \cdot d(H, G) + o(1) = 21 + o(1),
\end{aligned}$$

giving the desired inequality.

To do this, we consider the following vector $\mathbf{F}$ (the root is depicted by a white square and the remaining vertices by black circles).

$\mathbf{F} = \Bigg($  $\Bigg)$

Let M be the following 7×7-matrix[1].

$$M = \frac{1}{12 \cdot 10^9} \begin{pmatrix} 1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\ 2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\ 640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\ -1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\ 1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\ -732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\ -129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164 \end{pmatrix}.$$

The matrix M is a positive semidefinite matrix with rank six; the eigenvector corresponding to the zero eigenvalue is $(1,0,3,1,0,3,0)$.

By computing each c_H and $\pi_{3,f}(H)$ explicitly, it can be verified that this choice of $\mathbf{F}$ and M yields c_H satisfying $\pi_{3,f}(H) + c_H \leqslant 21$ and hence yields a proof of the lemma. $\square$

The second ingredient in the proof of Conjecture 2.4 is an immediate corollary of a result of Haxell and Rödl [59] on fractional triangle decompositions or from a more general result of Yuster [113].

Lemma 2.9. *For any n-vertex graph G it holds that $\pi_3(G) \leqslant \pi_{3,f}(G) + o(n^2)$.*

We now use Lemmas 2.8 and 2.9 to prove the main result of this section.

Proof of Conjecture 2.4. By Lemma 2.9, it is enough to show that $\pi_{3,f}(G) \leqslant (1/2 + o(1))n^2$.

Let $\mathcal{T}(G)$ denote the set of all triangles in G, and for each $H \in \mathcal{G}_7$, let $\mathcal{D}_H$ be an optimal fractional triangle-edge decomposition of H and let w_H be the weight function of this decomposition. Note that we have that $c(\mathcal{D}_H) = \pi_{3,f}(H)$ for every H.

We now construct a fractional triangle-edge decomposition of G given by

$$w(e) = \frac{1}{\binom{n-2}{5}} \sum_W w_{G[W]}(e),$$

[1]The computer programs used to generate this matrix and their outputs have been made available on arXiv:1710.08486 as ancillary files.

for each edge $e \in E(G)$ and

$$w(T) = \frac{1}{\binom{n-2}{5}} \sum_W w_{G[W]}(T)$$

for each triangle $T \in \mathcal{T}(G)$, where the sums are taken over all $W \subseteq V(G)$ of size 7 which contain both endpoints of e (or all three vertices of T).

It's straightforward to verify that this is a valid fractional triangle-edge decomposition of G. Indeed, for any $e \in E(G)$ we have

$$\begin{aligned} w(e) + \sum_{e \subset T} w(T) &= \frac{1}{\binom{n-2}{5}} \sum_{e \subset W} \left(w_{G[W]}(e) + \sum_{e \subset T} w_{G[W]}(T) \right) \\ &= \frac{1}{\binom{n-2}{5}} \sum_{e \subset W} 1 = 1 \end{aligned}$$

as there are exactly $\binom{n-2}{5}$ subsets W of 7 vertices which contain both endpoints of e. The cost of this decomposition is equal to

$$\begin{aligned} \frac{1}{\binom{n-2}{5}} \sum_{W \in \binom{V(G)}{7}} c(\mathcal{D}_{G[W]}) &= \frac{1}{\binom{n-2}{5}} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) \\ &\leqslant \frac{\binom{n}{7}}{\binom{n-2}{5}} (21 + o(1)) = n^2/2 + o(n^2) \ , \end{aligned}$$

where the inequality follows from Lemma 2.8. □

The next corollary follows directly from the proof of Conjecture 2.4.

Corollary 2.10. *Every n-vertex graph with $n^2/4 + k$ edges contains $2k/3 - o(n^2)$ edge-disjoint triangles.*

2.2 Stability analysis and a proof of the sharp result (Theorem 2.5)

In this section we prove Theorem 2.5 by building upon the proof of Conjecture 2.4. Moreover, we also show that the extremal graphs G which maximise $\pi_3(G)$ are the complete graph K_n and the bipartite Turán graph $T_2(n)$. Which of these two graphs is extremal is a matter of divisibility of n by 6. In the case of the Turán graph, $\pi_3(T_2(n)) = 2\lfloor n/2 \rfloor \lceil n/2 \rceil$, giving $n^2/2$ for even n and $(n^2-1)/2$ for odd n. In the case of the complete graph K_n, we can, in general, decompose all edges into copies of K_3 as long as K_n is *triangle-divisible*; that is, if each vertex has even degree and the total number of edges is divisible by three. In the cases when this is not true, as well as later in our proof, we use the following result.

Theorem 2.11 (Barber, Kuhn, Lo, Osthus [9] and Dross [30]). *For every $\varepsilon > 0$, if G is a triangle-divisible graph of large order n and minimum degree at least $(0.9+\varepsilon)n$, then G has a triangle decomposition.*

This theorem suits our proofs with 0.9 replaced by any constant $c < 1$. In particular, an earlier result of Gustavsson [48] asserts this with $c \approx 1 - 10^{-24}$. This was subsequently improved by a series of authors [112, 31, 32, 42], with the best known bounds to date being $c = 0.852$ due to Dukes and Horsley [33] and $c = 0.827327$ due to Delcourt and Postle [28]. These results are closely related to a well-known conjecture of Nash-Williams stating that the correct value of c is in fact $c = 3/4$.

Applying Theorem 2.11, we can easily derive the costs of the optimal triangle-edge decompositions of K_n for large n, for each residue class of n mod 6 (see Table 2.1).

n mod 6	optimal decomposition of K_n	$\pi_3(K_n)$	$\pi_3(T_2(n))$
0	triangle-divisible + perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
1	triangle-divisible	$\binom{n}{2}$	$\frac{n^2-1}{2}$
2	triangle-divisible + perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
3	triangle-divisible	$\binom{n}{2}$	$\frac{n^2-1}{2}$
4	triangle-divisible + perfect matching + $K_{1,3}$	$\frac{n^2}{2}+1$	$\frac{n^2}{2}$
5	triangle-divisible + C_4	$\binom{n}{2}+4$	$\frac{n^2-1}{2}$

Table 2.1: Values of $\pi_3(K_n)$ and $\pi_3(T_2(n))$ for large n.

The main result of this section states that the maximum of $\pi_3(G)$ among all n-vertex graphs G is attained by either K_n or $T_2(n)$. More specifically, we show the following, of which Theorem 2.5 is a direct corollary.

Theorem 2.12 (Blumenthal, Lidický, Pehova, Pfender, Pikhurko, Volec [12]). *There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all n-vertex graphs G, we have*

$$\pi_3(G) \leqslant \begin{cases} n^2/2 & \textit{for } n \equiv 0,2 \bmod 6 \quad \textit{attained only by } T_2(n) \textit{ and } K_n, \\ (n^2-1)/2 & \textit{for } n \equiv 1,3,5 \bmod 6 \quad \textit{attained only by } T_2(n), \\ n^2/2+1 & \textit{for } n \equiv 4 \bmod 6 \quad \textit{attained only by } K_n. \end{cases}$$

In this more detailed form, this result gives an affirmative answer to a question of Pyber [97] (see also [111, Problem 45]) for sufficiently large n.

Corollary 2.13. *For sufficiently large n, an edge set of every n-vertex graph be covered*[2] *with triangles of cost 3 and edges of cost 2 such that their total cost is at most $\lfloor n^2/2 \rfloor$.*

Proof. Theorem 2.12 directly implies the corollary unless $n \equiv 4 \bmod 6$ and the graph under consideration is K_n. Assume $n \equiv 4 \bmod 6$. Denote the vertices of K_n by $v_1, \ldots, v_n$. An optimal decomposition for K_n is obtained by taking edges v_1v_2, v_1v_3, v_1v_4 and v_iv_{i+1} for all odd $5 \leqslant i \leqslant n-1$. The rest of the graph becomes triangle-divisible and can be decomposed into triangles by Theorem 2.11. This gives a decomposition of total cost $n^2/2+1$. A covering of cost at most $\lfloor n^2/2 \rfloor$ can be obtained from this decomposition by replacing edges v_1v_2 and v_1v_3 by a triangle $v_1v_2v_3$. $\square$

To prove Theorem 2.12, we use the so-called *stability approach.* We start by describing the approximate structure of all almost π_3-extremal graphs of order $n \to \infty$. Namely, we first show (Lemma 2.14) that every such graph is close to K_n or $T_2(n)$. To complete the proof, we show that among all graphs close to $T_2(n)$, the Turán graph maximises π_3 (Lemma 2.15), and among all graphs close to K_n, the complete graph maximises π_3 (Lemma 2.16). Putting these three lemmas together gives a full proof of Theorem 2.12.

Lemma 2.14. *For every $\delta > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that if G is a graph of order $n \geqslant n_0$ with $\pi_3(G) \geqslant \left(\frac{1}{2} - \eta\right) n^2$, then G is δn^2-close*[3] *in edit distance to K_n or to $T_2(n)$.*

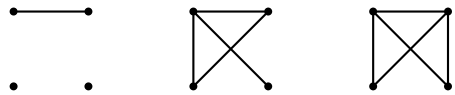
Proof. Let $1/n_0 \ll \eta \ll c \ll c' \ll \delta$ and let M and $\overrightarrow{F}$ be as in the proof of Lemma 2.8.

We start by showing that all almost π_3-extremal graphs contain almost no copies of the three graphs in Figure 2.1 (which are all unlabelled versions of flags in the vector $\mathbf{F}$ used in the proof of Lemma 2.8).

The rank of M is 6 with $\mathbf{v} = (1,0,3,1,0,3,0)$ being the only zero eigenvector. (Thus all others eigenvectors of M are strictly positive by $M \succeq 0$.)

[2]In a decomposition of a graph, every edge is used exactly once. In a covering, every edge is used at least once.

[3]We say that a graph G on vertex set $[n]$ is k-close to H on the same vertex set if there is a relabelling ϕ of the vertices of G such that $|E(\phi(G)) \triangle E(H)| \leqslant k$.



H_2 $\quad$ H_5 $\quad$ H_7

Figure 2.1: Graphs H_2, H_5, and H_7.

By the almost-optimality of G we have that

$$d([\![\mathbf{F}^T M \mathbf{F}]\!], G) = o_\eta(1) \tag{2.2}$$

as the first inequality in (2.1) follows from the above quantity being non-negative.

We now show that G must contain few copies of the graphs H_2, H_5 and H_7. Suppose, for contradiction, that G contains at least $c\binom{n}{4}$ copies of H_2. Then, by a simple double-counting argument we have that at least $cn/4$ vertices in G contain at least $c\binom{n}{3}/4$ copies of the flag F_2. In particular, $d(F_2, G^\bullet) \geqslant c/4$ for at least $cn/4$ choices of a root $\bullet$ in G. For each such choice of root, let $\mathbf{u} = d(\mathbf{F}, G^\bullet)$ be the vector of densities of elements of $\mathbf{F}$ in $G^\bullet$, and let $\mathbf{u}' = \mathbf{u}/\|d(\mathbf{F}, G^\bullet)\|_2$. Then $\mathbf{u}'$ has second coordinate at least $c/(4\sqrt{7})$, as $\|d(\mathbf{F}, G^\bullet)\|_2 \leqslant \sqrt{7}$. The scalar product of $\mathbf{u}'$ and the ℓ^2-normalised zero eigenvector $\mathbf{v}/\sqrt{20}$ (whose second coordinate is 0) is at most $\sqrt{1 - (c/4\sqrt{7})^2}$. Thus the projection of $\mathbf{u}$ on the orthogonal complement $L = \mathbf{v}^\perp$ of the zero eigenspace of M has ℓ^2-norm at least $c/4\sqrt{7}$. Thus $\mathbf{u}^T M \mathbf{u} \geqslant \lambda_2 (c/4\sqrt{7})^2$, where $\lambda_2 > 0$ is the smallest positive eigenvalue of M (in fact, one can check with computer that $\lambda_2 = 0.0005228...$). Thus, we have that the left-hand side of (2.2) is at least $(cn/4) \times \lambda_2 (c/4\sqrt{7})^2$, a contradiction with $\eta \ll c$.

An analogous argument shows that the densities of H_5 and H_7 in G are also at most c. Now it remains to show that this forces G to be close to K_n or $T_2(n)$.

By the Induced Removal Lemma [1], G can be made $\{H_2, H_5, H_7\}$-free by changing at most $c'n^2$ adjacencies. Denote this new graph by G' and note that $\pi_3(G') \geqslant \pi_3(G) - 2c'n^2$.

Let us show that G' is either triangle-free, or the disjoint union of at most two cliques (see Figure 2.2). Indeed, if some vertices u, v, w span a triangle in G' then, by the $\{H_5, H_7\}$-freeness of G, all the remaining vertices of G' have either no or three neighbours among $\{u, v, w\}$. Let A_0 be the set of vertices in $G' \setminus \{u, v, w\}$ which see none of $\{u, v, w\}$, and let A_3 be the set of vertices which see all of $\{u, v, w\}$. Then A_3 is a clique because G' is H_7-free. The set A_0 is also a clique because G' is H_2-free. Also, no pair xy in $A_3 \times A_0$ can be an edge as otherwise e.g. the 4-set $\{u, v, x, y\}$

spans a copy of H_5 in G. It follows that G is the disjoint union of the cliques on A_0 and $A_3 \cup \{u, v, w\}$, as required.

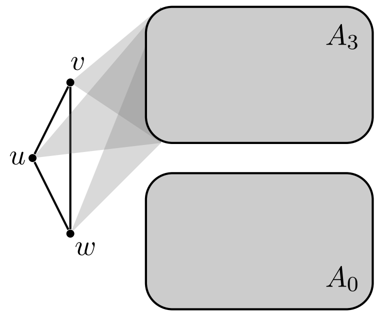


Figure 2.2: Structure of a $\{H_2, H_5, H_7\}$-free graph.

If G' is triangle-free, then

$$\begin{aligned} e(G') = \pi_3(G')/2 &\geqslant \frac{1}{2}\left(\pi_3(G) - 2c'n^2\right) \\ &\geqslant \frac{1}{2}\left(\frac{n^2}{2} - 3c'n^2\right) \geqslant e(T_2(n)) - 2c'n^2. \end{aligned}$$

Thus, by the stability result for Mantel's theorem by Erdős [34] and Simonovits [104], G must indeed be δn^2-close in edit distance to $T_2(n)$.

Otherwise, G' is the disjoint union of at most two cliques. Cliques are decomposable into triangles up to triangle-divisibility, we have $\pi_3(G') \leqslant e(G') + n/2 + 2$ (see Table 2.1). This gives

$$\begin{aligned} e(G) &\geqslant e(G') - c'n^2 \\ &\geqslant \pi_3(G') - n/2 - 2 - c'n^2 \\ &\geqslant n^2/2 - \eta n^2 - 3c'n^2 - n/2 - 2 \\ &\geqslant \binom{n}{2} - \delta n^2 \end{aligned}$$

so G is δn^2-close to K_n. □

Knowing that near-π_3-extremal graphs must be K_n-like or $T_2(n)$-like, it remains to prove that π_3-extremal graphs are K_n or $T_2(n)$, respectively. We start with the easier case near $T_2(n)$.

Lemma 2.15. *There exist constants $\delta > 0$ and $n_0 \in \mathbb{N}$ such that, among all graphs on $n \geqslant n_0$ vertices which are δn^2-close to $T_2(n)$, the maximiser of π_3 is $T_2(n)$.*

Proof. We start by choosing constants $1/n_0 \ll \delta \ll \varepsilon \ll 1$. Let G be an arbitrary graph with $n \geqslant n_0$ vertices which is δn^2-close to $T_2(n)$. We will show that $\pi_3(G) \leqslant \pi_3(T_2(n))$ with equality if and only if $G = T_2(n)$. In fact, this claim can be directly derived from a result of Győri [49, Theorem 1] that a graph with n vertices and $e(T_2(n)) + k$ edges, where $n \to \infty$ and $k = o(n^2)$, contains at least $k - O(k^2/n^2)$ edge-disjoint triangles. More specifically, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for large n every n-vertex graph with $t_2(n) + k$ edges where $k \leqslant \delta n^2$ has at least $k - \varepsilon k^2/n^2$ edge-disjoint triangles. (See also [50, Theorem 1] for an extension of this to r-cliques for any fixed $r \geqslant 3$.) Since G is δn^2-close to $T_2(n)$, it must have at most $t_2(n) + \delta n^2$ edges. From this and our choice of constants we have that, for $k := e(G) - t_2(n)$,

$$\pi_3(G) \leqslant 2(t_2(n)) + k) - 3(k - \varepsilon k^2/n^2) = 2t_2(n) - k(1 - 3\varepsilon k/n^2) \leqslant 2t_2(n).$$

Equality is achieved only when $k = 0$ and G is triangle-free, that is, when $G = T_2(n)$. □

Next, we consider graphs that are close to K_n. If $n \equiv 1, 3 \bmod 6$, let $\mathcal{E}_n$ be the set of graphs obtained from K_n by removing a matching of size $m \equiv 2 \bmod 3$; otherwise, let $\mathcal{E}_n := \{K_n\}$. Also, define

$$w(n) := \begin{cases} n/2, & n \equiv 0, 2 \bmod 6, \\ 2, & n \equiv 1, 3 \bmod 6, \\ n/2 + 1, & n \equiv 4 \bmod 6, \\ 4, & n \equiv 5 \bmod 6. \end{cases}$$

Using Theorem 2.11 (with the calculation for K_n appearing in Table 2.1), one can show that $\pi_3(G) = \binom{n}{2} + w(n)$ for all large n and every $G \in \mathcal{E}_n$. We are going to show that these are exactly the π_3-extremal graphs among those close to K_n.

Lemma 2.16. *There exist constants $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let G be a graph on $n \geqslant n_0$ vertices which is δn^2-close to K_n and $\pi_3(G) \geqslant \binom{n}{2} + w(n)$. Then $G \in \mathcal{E}_n$.*

Proof. Let $1/n_0 \ll \delta \ll c \ll 1$.

First, we show that we may assume that G has minimum degree at least $n/8$ by at most doubling n_0.

Claim 2.16.1. There exists an induced subgraph G' of G on $n' \geqslant n/2$ vertices such that $\delta(G') \geqslant n'/8$ and $\pi_3(G) \leqslant \pi_3(G') + \frac{1}{4}n(n-n')$.

In particular, it suffices to prove the lemma for G' as it is $4\delta n'^2$-close to $K_{n'}$, and so $\pi_3(G) \leqslant \frac{n'^2}{2} + 1 + \frac{n(n-n')}{4} < \binom{n}{2}$ for $n' \leqslant n-1$. This implies that G is either not extremal, or it is equal to G'.

Proof of claim. Suppose that the minimum degree of G is less than $n/8$ (otherwise we can take $G' = G$). Let $G_n := G$, and iteratively define a sequence of graphs $G_{n-1}, G_{n-2}, \ldots$ as follows. Given a graph G_i of order i, if it has a vertex x of degree less than $i/8$, let $G_{i-1} := G_i - x$ be obtained from G_i by removing the vertex x; otherwise stop. Note that the process does not reach $n/2$ iterations, for otherwise G has roughly at least $(n/2) \times (n/4)$ non-edges, which is a contradiction to G being δn^2-close to K_n.

Let G' with $|G'| = n' \geqslant n/2$ be the graph for which the above process terminates. By decomposing all edges in $E(G) \setminus E(G')$ as K_2's, we obtain that

$$\pi_3(G) \leqslant \pi_3(G') + (n-n') \cdot 2 \cdot \frac{n}{8}.$$

□

From now on, we write G for G' (and n for n'), and as shown by the claim, we can assume G has minimum degree at least $n/8$.

Let $U := \{v \in V(G) : d_G(v) \leqslant (1-c)n\}$, $W = V(G) \setminus U$, $S = \{v \in W : d_G(v) \text{ is odd}\}$. Take M to be a maximum matching in $G[S]$, and let $X = S \setminus V(M)$. Denote by Y_U the set of missing edges in G with at least one endpoint in U, and $Y_W = E(\overline{G[W]})$. Note that $e(G) = \binom{n}{2} - |Y_W| - |Y_U|$. See Figure 2.3 for an illustration of all these definitions. Then

$$\frac{|U|cn}{2} \leqslant e\left(\overline{G}\right) \leqslant \delta n^2,$$

and so $|U| \leqslant \frac{2\delta}{c}n$. We also have that by maximality of M, X is an independent set and thus $\binom{|X|}{2} \leqslant \delta n^2$, which implies that

$$|X| < cn. \tag{2.3}$$

Moreover, for every edge $yz \in M$ and any two distinct vertices $y', z' \in X$, at most one of yy' and zz' can be an edge of G (otherwise $y'yzz'$ is an augmenting path contradicting the maximality of M). It follows that, if $|X| \neq 1$, then for every edge

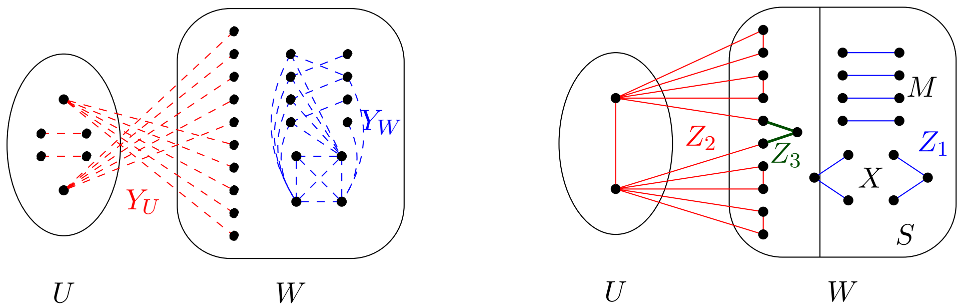
$yz \in M$ there are at least $|X|$ edges missing between $\{y, z\}$ and X. Thus, if $|X| \geqslant 2$, we have

$$|Y_W| \geqslant \binom{|X|}{2} + |M||X|, \tag{2.4}$$

Moreover, the remaining set Y_U of missing edges satisfies

$$|Y_U| \geqslant cn|U| - \binom{|U|}{2} \tag{2.5}$$

by the definition of U. Note that $e(G) = \binom{n}{2} - |Y_W| - |Y_U|$. See Figure 2.3 for a sketch ot Y_W and Y_U.



(a) The sets U, W, Y_U, and Y_W. Missing edges in Y_W are colored blue and edges in Y_U are red. (Note that this is a sketch and vertices in W can incident to both blue and red non-edges.)

(b) The three sets Z_1, Z_2, Z_3. Edges in Z_1 are colored blue, edges in Z_2 are red and in Z_3 green.

Figure 2.3

We now build a decomposition $\mathcal{D}$ of G into edges and triangles, starting with $\mathcal{D} = \emptyset$. If we add edges/triangles to $\mathcal{D}$, we regard them as removed from $E(G)$. It is convenient to split our argument into two cases.

Case 1: $U \neq \emptyset$ or $S = \emptyset$.

In this case, our procedure for constructing $\mathcal{D}$ is as follows. (See Figure 2.3b for some illustrations of the above steps.)

Step 1: Let Z_1 be the set of edges of M, and the edges of some $\lfloor |X|/2 \rfloor$ vertex-disjoint cherries whose degree-1 vertices are in X. Add Z_1 to $\mathcal{D}$.

Step 2: For each $u \in U$, one at a time, add to $\mathcal{D}$ a maximum set of edge-disjoint triangles containing u and two vertices from W. Let Z_2 consist of all remaining edges incident to vertices in U. Add Z_2 to $\mathcal{D}$.

Step 3: (a) Let $S' \subseteq V(G)$ be the set of vertices with odd degree after Step 2. Take Z_3 to be the collection of edges of some $|S'|/2$ vertex-disjoint cherries whose endpoints are in S'.

(b) If the number of remaining edges is not divisible by 3, then fix this by adding to Z_3 the edge set of some cycle of length 4 or 5.

Add Z_3 to $\mathcal{D}$.

Step 4: Add a perfect triangle decomposition of the remaining edges to $\mathcal{D}$.

For $i = 1, 2, 3$, let Z_i be the set of pairs that are added to $\mathcal{D}$ in Step i as copies of K_2.

Claim 2.16.2. The above steps can be carried out as stated. Moreover, the obtained decomposition $\mathcal{D}$ of G has at most $|M| + |X| + \binom{|U|}{2} + 2|U| + 6$ copies of K_2.

Proof of claim. In order to carry out Step 1 as stated, we can iteratively pick any two new vertices $x, y \in X$ and then an arbitrary vertex z which is suitable as the middle point for a cherry on xy. Note that the number of choices for z is at least $n-2-2cn$, the number of common neighbours of $x, y \in X \subseteq W$, minus $|X| - 1$, the number of vertices previously used as middle points. This is positive by (2.3) and $c \ll 1$, so we can always proceed. Note for future reference that every vertex is incident to at most 3 edges removed in Step 1. Also, Step 1 adds $|Z_1| = |M| + 2(\lfloor |X|/2 \rfloor) \leqslant |M| + |X|$ copies of K_2 to $\mathcal{D}$.

For Step 2, the maximum collection of triangles at vertices in U always exists. Consider the moment when we apply Step 2 to some $u \in U$. In the current graph, the induced subgraph $G[N(u) \cap W]$ has minimum degree at least $|N(u) \cap W| - cn - 3$, which is at least $|N(u) \cap W|/2$ since $|N(u)| \geqslant n/8 - 3$. So by Dirac's theorem, this subgraph has a matching covering all but at most one vertex, that is, all edges between u and W except at most one are decomposed as triangles in Step 2. Let U' be the set of those $u \in U$ for which an exceptional edge occurs. Thus we have $|U'| \leqslant |U|$ copies of K_2 connecting U to W that are added to $\mathcal{D}$ in Step 2. There are trivially at most $\binom{|U|}{2}$ edges with both endpoints in U. So Step 2 adds $|Z_2| \leqslant \binom{|U|}{2} + |U|$ copies of K_2 to $\mathcal{D}$. Note that all edges incident to U are decomposed after Step 2.

Since all vertices of W but at most one have even degree before Step 2, we have that S' has at most $|U'|+1 \leqslant |U|+1$ vertices. Similarly as in Step 1, a simple greedy algorithm finds all cherries as stated Step 3(a). (Note that S', as the set of all odd-degree vertices, has even size.)

The minimum degree of $G[W]$ after Step 3(a) is at least $0.99n$, since each $w \in W$ has at most $2|U|+6$ incident edges removed (at most $2|U|$ from Step 2 and at most 3 in each of Steps 1 and 3(a)). Thus, we can find the required 4- or 5-cycle in Step 3(b) by considering successive neighbourhoods.

In total, we add $|Z_3| \leqslant |S'|+5 \leqslant |U|+6$ copies of K_2 to $\mathcal{D}$ in Step 3.

Note that, at the end of Step 3, the graph $G[W]$ has minimum degree at least, say, $0.98n$ while all its degrees are even. By Theorem 2.11, all remaining edges can be decomposed using only triangles, so Step 4 indeed removes all remaining edges.

Step 4 adds no K_2's to the decomposition, so the total number of K_2's in $\mathcal{D}$ is

$$|Z_1|+|Z_2|+|Z_3| \leqslant |M|+|X|+\binom{|U|}{2}+2|U|+6,$$

finishing the proof of the claim. $\square$

Now we compute the cost of $\mathcal{D}$. Note that $\pi_3(G) \leqslant c(\mathcal{D}) = 2e(\mathcal{D})+3t(\mathcal{D})$, where $e(\mathcal{D})$ denotes the number of edges in $\mathcal{D}$ and $t(\mathcal{D})$ – the number of triangles. By substituting $e(G) = e(\mathcal{D})+3t(\mathcal{D})$, we have that $\pi_3(G) \leqslant e(G)+e(\mathcal{D})$. Using the notation from above, we have

$$\begin{aligned} w(n) &\leqslant \pi_3(G) - \binom{n}{2} \\ &\leqslant -|Y_U| - |Y_W| + |Z_1| + |Z_2| + |Z_3| \\ &\leqslant -|Y_U| - |Y_W| + |M| + |X| + \binom{|U|}{2} + 2|U| + 6. \end{aligned} \tag{2.6}$$

Substituting the bounds from (2.4) and (2.5) and rearranging the terms, we get

$$w(n) \leqslant \begin{cases} \left(2\binom{|U|}{2}+2|U|-cn|U|+6\right)+|M|+|X| & \text{if } |X|=0,1 \\ \left(2\binom{|U|}{2}+2|U|-cn|U|+6\right)-\left(\frac{|X|}{2}+|M|\right)(|X|-3)-2|M| & \text{if } |X| \geqslant 2 \end{cases} \tag{2.7}$$

We now claim that under the extremality assumption on G, U cannot contain any vertices.

Claim 2.16.3. $U = \emptyset$.

Proof of claim. Suppose, for contradiction, that $|U| > 0$. Then, since $|U| \leqslant 2\delta n/c$, the expression in the first bracket in (2.7) can be upper-bounded by $-cn/2$. Since $w(n) \geqslant 2$, we have that $|X| \leqslant 1$. For even n this immediately leads to a contradiction after substituting $|X| \leqslant 1$, $|M| \leqslant n/2$ and $w(n) \geqslant n/2$ in (2.7). So we may assume that n is odd and hence every vertex of degree $n-1$ has even degree. In particular, every vertex of S is in some pair from Y_W or Y_U, and so $2|M| \leqslant 2|Y_W| + |Y_U|$. Substituting this into the right-hand side of (2.6) and using our bound on $|Y_U|$ from (2.5), we obtain

$$2 \leqslant w(n) \leqslant -\frac{|Y_U|}{2} + \binom{|U|}{2} + 2|U| + 6 \leqslant \frac{3}{2}\binom{|U|}{2} + 2|U| - \frac{cn|U|}{2} + 6,$$

which again leads to a contradiction for large n. □

Thus $U = \emptyset$ and, by the assumption of Case 1, S is also empty (and so are X and M). This gives that the initial graph G has minimum degree at least $(1-c)n$, $|Z_1| = |Z_2| = 0$, $S' = \emptyset$, and no K_2's are added to $\mathcal{D}$ in Step 3(a).

If n is even, then every vertex of G has at least one missing edge, $e(G) \leqslant \binom{n}{2} - \frac{n}{2}$ and

$$\pi_3(G) \leqslant \binom{n}{2} - \frac{n}{2} + |Z_3| \leqslant \binom{n}{2} - \frac{n}{2} + 5,$$

which is strictly less than $\pi_3(K_n)$, a contradiction. Let n be odd and let $r := |Y_W| = \binom{n}{2} - e(G)$ be the number of missing edges in G. Note that either $r = 0$ and $G = K_n$, or $r \geqslant 3$, as otherwise G cannot have all even degrees. Let $\rho_r \in \{0, 4, 5\}$ be the number of edges added to $\mathcal{D}$ in Step 3(b). Note that $\binom{n}{2} - r - \rho_r \equiv 0 \bmod 3$ by definition. We also have that $\mathcal{D}$ costs $c(\mathcal{D}) = \binom{n}{2} - r + \rho_r \geqslant \binom{n}{2} + 2$. These observations combine into the following system

$$\begin{cases} r \geqslant 3 \\ \rho_r \in \{0,4,5\} \\ \rho_r - r \geqslant 2 \\ r + \rho_r \equiv 0, 1 \bmod 3 \end{cases}$$

which has no solutions. This contradiction completes Case 1.

Case 2: $U = \emptyset$ and $S \neq \emptyset$.

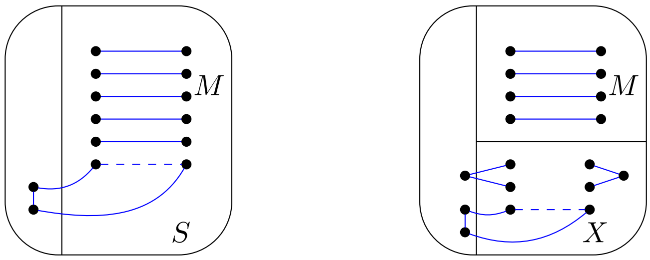
Some things simplify in this case, as we do not need to deal with U, but note that the non-complete extremal graphs (K_n minus a matching) arise here. We construct a decomposition $\mathcal{D}$ of G as in Case 1, except we replace the 4-cycle or 5-cycle removed in Step 3(b) with a carefully chosen "anchored path". Recall that M is a maximum matching in $G[S]$ and X is the set of vertices of S not matched by M. Also, $|S|$ is even, and in particular $|X| = |S| - 2|M|$ is also even.

Step 1: Add the following copies of K_2 to $\mathcal{D}$:

(a) If $X = \emptyset$, add all but one edge $xy \in M$ and a path with $\rho + 1 \in \{1, 2, 3\}$ edges whose endpoints are x and y.

(b) If $X \neq \emptyset$, add M and the edge sets of some $|X|/2 - 1$ cherries and one path of length $\rho + 2 \in \{2, 3, 4\}$ so that their degree-1 vertices partition X and their degree-2 vertices are distinct.

Step 2: Decompose the rest perfectly into triangles.



(a) If $X = \emptyset$. (b) If $X \neq \emptyset$.

Figure 2.4: Single edges in $\mathcal{D}$ when $U = \emptyset$ and $S \neq \emptyset$.

Since the minimum degree of G is at least $(1 - c)n$, a simple greedy algorithm achieves Step 1 (and Theorem 2.11 takes care of Step 2).

The decomposition $\mathcal{D}$ has exactly $|M| + |X| + \rho$ copies of K_2. Thus

$$w(n) \leqslant \pi_3(G) - \binom{n}{2} \leqslant -|Y_W| + |M| + |X| + \rho. \tag{2.8}$$

Also, $e(G) = \binom{n}{2} - |Y_W|$.

Claim 2.16.4. $|X| = 0$ or 2.

Proof of claim. Suppose, for contradiction, that $|X| \geqslant 4$. Since $|X|$ is even, this is sufficient for proving this claim.

Substituting the lower bound (2.4) on $|Y_W|$ into (2.8) gives

$$\begin{aligned} 2 \leqslant w(n) &\leqslant -\left(\frac{|X|}{2} + |M|\right)(|X| - 3) - 2|M| + \rho \\ &\leqslant \rho - 2 - 3|M| \leqslant -3|M|, \end{aligned} \tag{2.9}$$

which is a contradiction. □

Below we treat both possible sizes of X.

X **is empty.** First, for even n, every vertex in $W \backslash S$ is incident to at least one non-edge, so $Y_W \geqslant (n - |S|)/2$ and (2.8) simplifies to

$$\frac{n}{2} \leqslant w(n) \leqslant 2|M| - \frac{n}{2} + \rho.$$

Rearranging, we obtain $|S| \geqslant n - \rho \geqslant n - 2$, i.e., $|S| = n$ or $n - 2$. However, if $|S| = n - 2$, all inequalities above must be tight, so $\rho = 2$ and $n \equiv 0, 2 \bmod 6$ and $|Y_W| = 1$, giving $\binom{n}{2} - 1 - \frac{n-2}{2} - 2$ edges after Step 1, which is not divisible by 3, contradiction. So $|S|$ must be equal to n and $G = K_n$.
For odd n, since every vertex of S is incident to a missing edge of G, we have $|Y_W| \geqslant |S|/2 = |M|$ and (2.8) simplifies to $2 \leqslant w(n) \leqslant \rho \leqslant 2$. It follows that equality holds throughout, i.e., $|Y_W| = |M|$, $w(n) = \rho = 2$, $n \equiv 1, 3 \bmod 6$ and, for Step 2 to go through, $\binom{n}{2} - |M| - \rho \equiv 3 \bmod 3$. Thus G is K_n minus a matching M of size $|M| \equiv 2 \bmod 3$, as required.

X **contains 2 vertices.** Here, (2.9) simplifies to $w(n) \leqslant \rho - |M| + 1 \leqslant 3$. So $w(n) = 2$, $n \equiv 1, 3 \bmod 6$, and $|M| \leqslant 1$. A quick check of both cases for $|M|$ yields a contradiction with the extremality of G. This finishes Case 2 and the proof of the lemma. □

With Lemmas 2.14, 2.15 and 2.16 at hand, we can now prove our main result.

Proof of Theorem 2.12. Choose constants $1/n_0 \ll \delta \ll 1$. In particular, n_0 is sufficiently large to satisfy Lemma 2.14 for this δ as well as Lemmas 2.15 and 2.16. Let G be a graph of order $n \geqslant n_0$ such that $\pi_3(G) \geqslant \max\{\pi_3(K_n), \pi_3(T_2(n))\}$. By Lemma 2.14, G is δn^2-close to either $T_2(n)$ or K_n.

If G is close to $T_2(n)$ then it must be $T_2(n)$ by Lemma 2.15. If G is close to K_n then it must be in $\mathcal{E}_n$ by Lemma 2.16. By comparing $\pi_3(T_2(n))$ and $\pi_3(H)$ for

$H \in \mathcal{E}_n$ in each case, we conclude that G is either K_n or $T_2(n)$, as in the theorem statement. $\square$

2.3 Extension to arbitrary triangle costs

The goal of this section is to prove Theorem 2.6.

First, note that certain ranges of α are trivial. Indeed, if $\alpha \geq 6$, the cost of a triangle is not better than a cost of three edges. Thus for every graph G an optimal decomposition is to decompose all edges of G as K_2's. The unique graph maximising the number of edges is K_n, so it is also the unique maximiser of π_3^α for every $\alpha \geq 6$.

Next, let us make some easy general observations which we use to prove Theorem 2.6.

Proposition 2.17. *Let $0 < \alpha \leqslant \beta < 6$.*

(a) *For every G it holds that $\pi_3^\alpha(G) = 2e(G) - (6-\alpha)\nu(G)$, where $\nu(G)$ denotes the maximum number of edge-disjoint triangles contained in G.*

(b) *If for two graphs G_1 and G_2 it holds that $\pi_3^\alpha(G_1) < \pi_3^\alpha(G_2)$ and $\nu(G_1) \leqslant \nu(G_2)$, then $\pi_3^\beta(G_1) < \pi_3^\beta(G_2)$.*

(Note that part (b) imples that if K_n is a maximiser of π_3^α, then it is also a maximiser of π_3^β.)

Proof. For the first claim, it suffices to write out $\pi_3^\alpha(G)$ in terms of $\nu(G)$:

$$\pi_3^\alpha(G) = \alpha\nu(G) + 2(e(G) - 3\nu(G)) = 2e(G) - (6-\alpha)\nu(G).$$

For the second claim, we can use the first claim to rewrite the difference $\pi_3^\beta(G_2) - \pi_3^\beta(G_1)$ as

$$\pi_3^\beta(G_2) - \pi_3^\beta(G_1) = (\pi_3^\alpha(G_2) - \pi_3^\alpha(G_1)) + (\beta - \alpha)(\nu(G_2) - \nu(G_1)) > 0$$

as all three brackets above are non-negative and the first is positive. $\square$

We now proceed with the proof of our main theorem.

Proof of Theorem 2.6. We treat each range for α separately. Below, we assume that G is a π_3^α-extremal n-vertex graph and that n is large.

Case 1: $\alpha < 3$.

Since

$$\pi_3^3(G) \geqslant \pi_3^\alpha(G) \geqslant \pi_3^\alpha(T_2(n)) = \pi_3^3(T_2(n)) = (1/2 + o(1))n^2,$$

Lemma 2.14 gives that G is $o(n^2)$-close to K_n or $T_2(n)$. Since $\alpha < 3$, we have that $\pi_3^\alpha(T_2(n)) \geqslant (1 + \Omega(1))\pi_3^\alpha(K_n)$ and thus G is close to $T_2(n)$. Now, Lemma 2.15 implies that $\pi_3^\alpha(G) \leqslant \pi_3^3(G) \leqslant \pi_3^3(T_2(n)) = \pi_3^\alpha(T_2(n))$, with equality if and only if $G = T_2(n)$, giving the desired.

Case 2: $3 < \alpha < 4$.

First, let us show that G is either K_n or $K_n^=$.

Claim 2.17.1. $G \in \{K_n, K_n^=\}$.

Proof of claim. Suppose, for contradiction, that G is not K_n or $K_n^=$. By extremality $\pi_3^\alpha(G) \geq \pi_3^\alpha(K_n)$, and by Proposition 2.17 (b) we have that also $\pi_3^3(G) \geqslant \pi_3^3(K_n)$. Moreover, G must be close to K_n as $\pi_3^\alpha(G) \geqslant \pi_3^\alpha(K_n) \geqslant (1 + \Omega(1))\pi_3^\alpha(T_2(n))$, so G and $T_2(n)$ cannot be close in edit distance. In particular, as G is close, but not equal to, K_n, we must have that $n \equiv 1, 3 \bmod 6$. For such values of n we have that $\pi_3^\alpha(G) \geqslant \pi_3^\alpha(K_n^=)$ and $\pi_3^3(G) < \pi_3^3(K_n^=)$, so by Proposition 2.17 (b) $\nu(G) > \nu(K_n^=) = \frac{1}{3}\binom{n}{2} - 2$. So $\nu(G) = \frac{1}{3}\binom{n}{2} - 1$ and G must be K_n minus and edge, a path on two edges, or a triangle. Among these three graphs, $\pi_3^\alpha(K_n^-)$ is the largest, but still $\pi_3^\alpha(K_n^-) < \pi_3^\alpha(K_n^=)$, which contradicts the extremality of G. $\square$

It remains to compare K_n and $K_n^=$. Calculations based on Theorem 2.11 show that

$$\frac{\pi_3^\alpha(K_n^=) - \pi_3^\alpha(K_n) + 4}{6 - \alpha} = \nu(K_n) - \nu(K_n^=) = \begin{cases} 0, & n \equiv 0, 2, 4, 5 \bmod 6, \\ 2, & n \equiv 1, 3 \bmod 6. \end{cases}$$

Thus $\pi_3^\alpha(K_n) > \pi_3^\alpha(K_n^=)$ if $n \equiv 0, 2, 4, 5 \bmod 6$ and $\pi_3^\alpha(K_n^=) > \pi_3^\alpha(K_n)$ otherwise, as required.

Case 3: $4 \leq \alpha < 6$.

In this case we provide a direct proof. Let $\mathcal{D}$ be a decomposition of G with minimum weight consisting of t triangles and ℓ edges.

Assume, for contradiction, that G is not complete. Then G contains a non-edge $xy \notin E(G)$. Let G' be obtained from G by adding the edge xy. Let $\mathcal{D}'$ be an optimal decomposition of G' containing t' triangles and ℓ' edges. Recall that finding an optimal decomposition is equivalent to maximising a triangle packing, that is, $t' = \nu(G')$. Hence $t' \geq t$.

If xy is used as an edge in $\mathcal{D}'$, then removing xy from $\mathcal{D}'$ gives a decomposition of G with cost $\pi_3^\alpha(G') - 2$, contradicting the maximality of G. Therefore xy must appear in a triangle $xyz \in \mathcal{D}'$. We now construct a decomposition $\mathcal{D}^*$ of G by removing xyz from $\mathcal{D}'$ and adding the edges xz and yz. Since the total cost of $\mathcal{D}^*$ is $\alpha(t'-1) + 2(\ell'+2)$ we have

$$\pi_3^\alpha(G) \leq \text{cost}(\mathcal{D}^*) = \alpha(t'-1) + 2(\ell'+2) = \alpha t' + 2\ell' - \alpha + 4 \leq \alpha t' + 2\ell' = \pi_3^\alpha(G'),$$

which contradicts the maximality of $\pi_3^\alpha(G)$ if at least one of the inequalities is strict. Hence $\alpha = 4$, xy must be in a triangle in $\mathcal{D}'$ and $\pi_3^\alpha(G') = \pi_3^\alpha(n)$.

This means that it is possible to keep adding edges to G, which results in a sequence of graphs $G, G', \ldots, K_n$ where an optimal decomposition of each of these graphs has cost $\pi_3^\alpha(n)$, i.e., they all are π_3^α-extremal graphs.

Note that we can add missing edges to G in any order, always obtaining a sequence of extremal graphs.

This allows us to reverse the process and examine a sequence of edge removals from K_n.

Suppose that G is obtained from K_n by removing the edge xy, i.e., G' is K_n. Notice that if $\ell' > 0$, i.e., the optimal decomposition of K_n contains an edge, then there exist an option for $\mathcal{D}'$ that contains the edge xy, which was already ruled out. This means that K_n is triangle-divisible, which is the case if and only if $n \equiv 1, 3 \bmod 6$.

Now assume that G is missing more than one edge. Hence K_n^- must be also extremal. By the above, $n \equiv 1, 3 \bmod 6$, K_n is triangle-divisible, and $\pi_3^4(n) = 4\nu(K_n)$, where $\nu(K_n) = \frac{1}{3}\binom{n}{2}$.

Suppose that G is obtained from K_n by removing two edges uv and xy. First, suppose that $u = x$. Let $\mathcal{D}^\star$ be a decomposition of G into triangles and one edge vy. This gives

$$\pi_3^4(G) \leq \text{cost}(\mathcal{D}^\star) = 4(\nu(K_n) - 1) + 2 < 4\nu(K_n) = \pi_3^4(n),$$

contradicting the maximality of $\pi_3^4(G)$. Hence xy and uv form a matching. Notice that x, y, u, and v have odd degrees in G, so $\ell \geq 2$ for else we are unable to fix the parity of the vertices x, y, u, and v. Now $\binom{n}{2} - \ell - 2$ needs to be divisible by 3, so $\ell \geq 4$. There indeed exists a decomposition with $\ell = 4$ by taking edges xu, xv, yu, and yv and rest as triangles. This gives

$$\pi_3^4(G) = 4(\nu(K_n) - 2) + 2 \cdot 4 = \pi_3^4(n).$$

Therefore, G is extremal.

Suppose that G is obtained from K_n by removing three edges uv, xy, and zw. Since G' must be K_n without a matching, uv, xy, and zw also form a matching. Let $\mathcal{D}^\star$ be a decomposition of G into triangles and edges ux, yz, and vw. This gives

$$\pi_3^4(G) \leq \mathrm{cost}(\mathcal{D}^\star) = 4(\nu(K_n) - 2) + 6 < 4\nu(K_n) = \pi_3^4(n),$$

contradicting the maximality of $\pi_3^4(G)$. This implies that G cannot be obtained from K_n by deleting three or more edges, thus finishing the proof of this case and of Theorem 2.6.

□

2.4 Directions for future research

A related question of Erdős (see e.g., [35]) asks for the largest $t = t(n, m)$ such that every graph with n vertices and $t_2(n)+m$ edges has at least t edge-disjoint triangles. Of course, $t \leqslant m$. Győri [49] (see [51] for a correction) showed, for large n, that $t \geqslant m - O(m^2/n^2)$ if $m = o(n^2)$, and $t = m$ if n is odd and $m \leqslant 2n - 10$ or n is even and $m \leqslant 3n/2 - 5$. Moreover, the last two bounds on m are sharp.

More recently, Győri and Keszegh [52] proved that every K_4-free graph with $t_2(n)+m$ edges has m edge-disjoint triangles.

Theorem 2.5 shows that the maximum of $\pi_3(G)$ is attained for $G = T_2(n)$ or $G = K_n$. However, if we restrict the set of graphs under consideration to graphs of a particular edge density, the decomposition is perhaps cheaper. Note that if the optimal decomposition of a graph G contains t triangles and ℓ edges, then $\pi_3(G) = 2e(G) - 3t$. That is, we have that $\pi_3(G) = 2e(G) - 3\nu(G)$, where as before $\nu(G)$ denotes the maximum number of edge-disjoint triangles in G.

Then the proof of Conjecture 2.4 implies an inequality between the edge density of G and its *triangle packing density* which we denote by $\nu_d(G) := 3\nu(G)/\binom{n}{2}$:

Corollary 2.18 (of Conjecture 2.4)**.** *Let G be a graph with $d\binom{n}{2}$ edges. Then*

$$\nu_d(G) \geqslant 2d - 1 + o(1).$$

We also have that $\nu_d(G) \leqslant d$, which is tight for all graphs which are the union of edge-disjoint triangles.

A question reminiscent of the seminal result of Razborov on the minimal triangle density in graphs [99] (see also [85]) would be to determine the exact lower bound on $\nu_d(G)$ in terms of d (answering asymptotically the question of Erdős stated above).

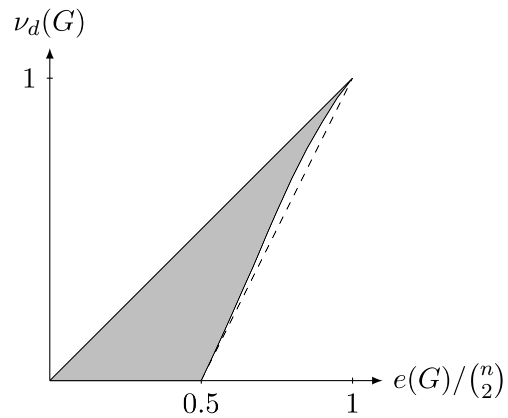


Figure 2.5: Asymptotic bounds on possible values of $\nu_d(G)$.

Some flag algebra computations yield numerical asymptotic lower bounds on $\nu_d(G)$ with different edge densities between 0.5 and 1. The result, depicted in Figure 2.5, suggests that the region $\{(d, \nu_d(G)) : 0 \leqslant d \leqslant 1, G \text{ graph}\}$ may indeed have a richer shape.

Chapter 3

Packing Hamilton cycles in bipartite directed graphs

Finding sufficient conditions for a graph to contain a Hamilton cycle, i.e., a cycle that contains every vertex of G, is one of the classical problems in graph theory. Dirac's theorem [29] states that every graph on n vertices with minimum degree at least $n/2$ contains a Hamilton cycle. Later, Ore [95] showed that it is enough if every pair of non-adjacent vertices has the sum of their degrees totalling at least n. A natural extension to the existence of one Hamilton cycle is then the existence of many edge-disjoint Hamilton cycles, or even of a decomposition into Hamilton cycles, i.e., a partition of the edges of a graph into Hamilton cycles. Clearly, if such a decomposition exists, say into d Hamilton cycles, then the graph must be $2d$-regular. A construction by Walecki (see, e.g., [3, 61]) shows that the complete graph K_{2d+1} admits such a decomposition for every $d \geqslant 1$. More generally, the complete r-partite graph $K(n;r)$ on rn vertices admits a decomposition into Hamilton cycles whenever $(r-1)n$ is even; and into Hamilton cycles and a perfect matching if $(r-1)n$ is odd [60, 82]. Some further graph classes have been shown to admit Hamilton decompositions, we refer the reader to the survey article by Alspach, Bermond and Sotteau [4].

Nash-Williams [92] extended Dirac's theorem by showing that every n-vertex graph with minimum degree at least $n/2$ contains at least $5n/224$ edge-disjoint Hamilton cycles, and conjectured that the minimum degree condition is sufficient to prove the existence of $\lfloor \frac{n+1}{4} \rfloor$ edge-disjoint Hamilton cycles. Babai (see [91]) provided a construction showing that this is false. However, Csaba, Kühn, Lo, Osthus and Treglown [27] proved that regular graphs satisfying the above minimum degree condition can be decomposed into Hamilton cycles and at most one perfect matching.

These problems naturally extend to the setting of *oriented graphs* that are obtained from simple graphs by endowing every edge with an orientation. Keevash, Kühn and Osthus [68] show that for n large enough, every oriented graph G on n vertices with minimum semi-degree at least $\frac{3n-4}{8}$ contains a Hamilton cycle. A construction due to Häggkvist [54] shows that this is best possible. Kühn and Osthus [78] prove that if $c > 3/8$, then every large cn-regular oriented graph G on n vertices has a Hamilton cycle decomposition. In particular, this establishes Kelly's conjecture which states that every regular tournament has a Hamilton cycle decomposition. The result in [78] builds on earlier work by Kühn, Osthus and Treglown [80] which includes an approximate version of Kelly's conjecture.

How many disjoint Hamilton cycles can one guarantee when the (oriented) graph is not regular? As the union of disjoint Hamilton cycles forms a regular spanning subgraph, the maximal r for which G contains an r-regular spanning subgraph is an upper bound for this quantity. Ferber, Long and Sudakov [40] show that this upper bound is asymptotically correct for oriented graphs of large enough minimum semi-degree.

Theorem 3.1 (Ferber, Long, Sudakov [40])**.** *Let $c > 3/8$, $\varepsilon > 0$ and let n be sufficiently large. Let G be an oriented graph on n vertices with $\delta^0(G) \geqslant cn$. Then G contains $(1-\varepsilon)r$ edge-disjoint Hamilton cycles, where r is the maximum integer such that G contains an r-regular spanning subgraph.*

In this chapter, we consider the corresponding degree conditions for regular bipartite oriented graphs. An obvious necessary condition for a bipartite (oriented) graph to contain a Hamilton cycle is that both parts of the bipartition have equal size, in which case the graph is called *balanced*. Note that the minimum semi-degree of a bipartite oriented graph G can be at most $\lfloor v(G)/4 \rfloor$, where $v(G)$ denotes the number of vertices of G. Graphs which attain this bound have $v(G)$ divisible by 4, and are necessarily balanced and every vertex has in- and out-degree $(v(G)/4)$. We call such graphs *regular bipartite tournaments.* Jackson [64] showed that regular bipartite tournaments are Hamiltonian, and he conjectured the following.

Conjecture 3.2 (Jackson [64])**.** *Every regular bipartite tournament is decomposable into Hamilton cycles.*

The main results of this chapter are two approximate versions of this conjecture for large graphs, the first of which shows that directed graphs whose vertex degrees are slightly above what we see in a bipartite tournament, are almost-decomposable into Hamilton cycles. More formally,

Theorem 3.3 (Liebenau, Pehova [84]). *Let $c > 1/2$, $\varepsilon > 0$, and let n be sufficiently large. Then every cn-regular bipartite digraph G on $2n$ vertices contains at least $(1-\varepsilon)cn$ edge-disjoint Hamilton cycles.*

Note that this result is more of a step sideways from Conjecture 3.2, as Conjecture 3.2 does not imply it. To the best of the author's knowledge, no other intermediate results towards Conjecture 3.2 are known.

Our second main result considers what is the smallest vertex degree in a regular bipartite oriented graph that forces the existence of a Hamilton decomposition. We provide strong evidence that in fact this degree might be as low as half of what we see in the regular bipartite tournament, by showing the following:

Theorem 3.4 (Liebenau, Pehova [84]). *Let $c > 1/4$, $\varepsilon > 0$, and let n be sufficiently large. Then every cn-regular bipartite oriented graph G on $2n$ vertices contains at least $(1-\varepsilon)cn$ edge-disjoint cycles of length at least $2n - O(n/\log^2 n)$.*

In particular, we can almost-decompose the edge set of every regular bipartite tournament into almost-spanning cycles.

We note that the constants $1/2$ and $1/4$ in Theorems 3.3 and 3.4 are optimal for such statements. Indeed, a d-regular bipartite digraph may be disconnected if $d = n/2$, as may be a d-regular oriented graph if $d = n/4$.

The rest of this chapter is organised as follows. In Section 3.1 we outline some preliminary results which we use in our proofs later on. The proofs of Theorems 3.3 and 3.4 are similar, apart from an intermediate connecting lemma which we prove differently in both settings. In Section 3.2 we prove a partition lemma which reduces the problem of finding an approximate Hamilton decomposition in either case to finding an approximate decomposition in a sparse near-regular graph, together with an absorbing set of high-degree vertices. In Section 3.3 we show that the near-regular graph obtained in this way has an approximate decomposition into pieces of a Hamilton cycle (which we refer to as *path covers*). In Section 3.4 we prove a connecting lemma for each of Theorems 3.3 and 3.4, which tells us how to complete our edge-disjoint path covers to Hamilton/long cycles. In Section 3.5, we prove both of our main results. Finally, in Section 3.6 we present two natural directions for future work on Conjecture 3.2.

3.1 Some preliminaries

In this section we introduce notation and present lemmas that we later use in the proof of our main result.

In the proofs presented in the rest of this chapter, given a graph or digraph with bipartition (V_1, V_2) and a subset $W \subseteq V(G)$, we will write W^{V_1} and W^{V_2} for $W \cap V_1$ and $W \cap V_2$, respectively. We also omit floor and ceiling signs for clarity of presentation.

The following provides a sufficient minimum semi-degree condition for a digraph to contain a Hamilton cycle.

Theorem 3.5 (Ghouila-Houri [43]). *Every strongly connected digraph G on n vertices with $\delta^+(G) + \delta^-(G) \geqslant n$ contains a Hamilton cycle. In particular, if $\delta^0(G) \geqslant n/2$, then G contains a Hamilton cycle.*

Let $D_{n,n}$ denote the complete bipartite balanced digraph in which both vertex classes have size n and every vertex has in- and out-degree n. A result by Ng [94] implies that the edge set of $D_{n,n}$ can be decomposed into Hamilton cycles. We use this to prove the following.

Lemma 3.6. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ the complete bipartite digraph $D_{n,n}$ contains n disjoint Hamilton paths starting in the same vertex class of the bipartition. Moreover, every vertex of $D_{n,n}$ is an endpoint of at most $2\sqrt{\log n}$ of these paths.*

Proof. Let A and B denote the vertex classes of $D_{n,n}$. It follows from Ng [94] that there is a decomposition of $D_{n,n}$ into n Hamilton cycles, say $C_1, ..., C_n$. For every $i \in [n]$ choose an edge $e_i = (a_i, b_i)$ of C_i with $a_i \in A$ uniformly at random among all n such edges, all choices being independent. Denote their union by H. We claim that with positive probability $\Delta^0(H)$ is at most $2\sqrt{\log n}$.

Fix a vertex $v \in A$. Then for each vertex $w \in B$, the edge (v, w) is in H with probability $1/n$. Moreover, the events $E_w = \{\text{the edge } (v, w) \text{ is in } H\}$ are independent since for any two distinct vertices $w, w' \in B$ the edges (v, w) and (v, w') are in different cycles of the decomposition. Therefore, the out-degree of v in H has a binomial distribution with parameters n and $1/n$. Similarly, the in-degree of w in H has a binomial distribution with parameters n and $1/n$ for every $w \in B$. Therefore, the probability that there exists $v \in A$ with $d^+_H(v) > 2\sqrt{\log n}$ or $w \in B$ with $d^-_H(w) > 2\sqrt{\log n}$ is at most $4ne^{-4\log n/3} = o(1)$, by Chernoff's inequality (Lemma 1.8) and the union bound. It follows that with positive probability H has maximum semi-degree at most $2\sqrt{\log n}$. The claim follows by taking $\{C_i - e_i\}_{i\in[n]}$, as the collection of Hamilton paths. By the choice of e_i's all these paths start in B. □

Finally, we state the following Lemma from [40], which we use as a building block in proving our main results.

Lemma 3.7 (Lemma 24 in [40]). *Let $\varepsilon > 0$ and $m, r \in \mathbb{N}$ with m sufficiently large and $2m^{24/25} \leqslant r \leqslant (1-\varepsilon)m/2$. Suppose that $G = (A \cup B, E)$ is a bipartite graph with $|A| = |B| = m$ and $r \leqslant \delta(G) \leqslant \Delta(G) \leqslant r + r^{2/3}$. Then G contains a collection of $r - m^{24/25}$ edge-disjoint matchings, each of which has size at least $m - m^{7/8}$, and whose union has minimum degree at least $r - m^{24/25} - 2m^{5/6}$.*

Remark 1. Note that practically the same assertion holds when $|A| = m = |B| + 1$, up to an additive constant of 1 which we neglect due to the asymptotic nature of the statement. To see this, apply the lemma to the graph obtained by adding an auxiliary vertex v to B and $\delta(G)$ edges between v and A.

3.2 A partition lemma for regular digraphs

We can now start building up to the proofs of Theorems 3.3 and 3.4.

Let $c > \varepsilon > 0$ where we may assume for the proof that ε is sufficiently small. Let n be a sufficiently large integer. Let $d = cn$ and assume that G is a balanced d-regular bipartite digraph on $2n$ vertices. (In particular, this setup covers both types of graphs considered in our main results.)

The next lemma asserts that we can split G into roughly $(\log n)^3$ spanning subgraphs, each consisting of a near-regular sparse graph ($H_i[U_i]$), together with a dense absorbing set (W_i).

Lemma 3.8. *Let $c > \varepsilon > 0$ be constants, let n be sufficiently large. Let D be a d-regular bipartite digraph with bipartition (A, B) such that $|A| = |B| = n$, where $d = cn$. Then for $K = \log n$ there are K^3 edge-disjoint spanning subdigraphs $H_1, ..., H_{K^3}$ of D with the following properties.*

(P1) For each $1 \leqslant i \leqslant K^3$ there is a partition $V(G) = U_i \cup W_i$ with $|W_i^A| = |W_i^B| = n/K^2 \pm 1$;

(P2) For some $r = (1 \pm \varepsilon)d/K^3$ and all $1 \leqslant i \leqslant K^3$, the induced subgraph $H_i[U_i]$ satisfies

$$\delta^0(H_i[U_i]), \Delta^0(H_i[U_i]) = r \pm r^{3/5};$$

(P3) For all $1 \leqslant i \leqslant K^3$ and all $u \in U_i$ we have that $d^{\pm}_{H_i}(u, W_i) \geqslant \varepsilon c|W_i|/8K$;

(P4) Each induced subgraph $H_i[W_i]$ has minimum semi-degree at least $(c-\varepsilon)|W_i|/2$.

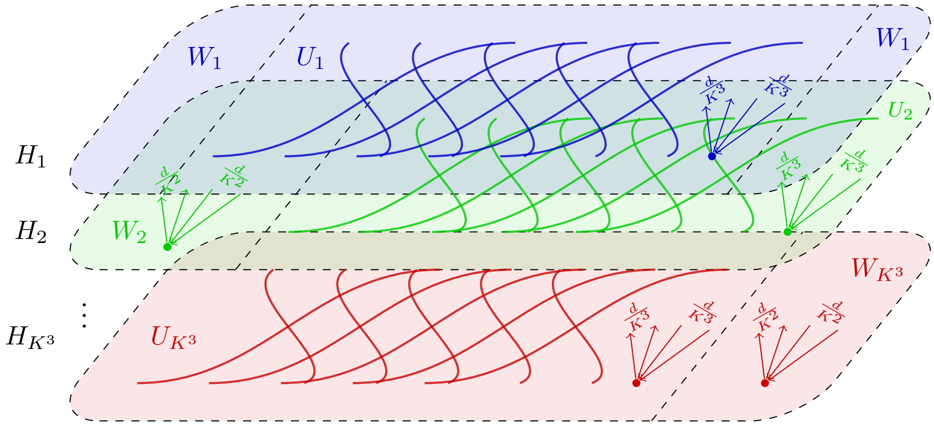


Figure 3.1: The partition given by Lemma 3.8. Colours represent edge-disjoint subgraphs of D.

The proof of the lemma is a straightforward adaptation of the proof of Lemma 27 in [40] to the bipartite setting. We include it for completeness.

Proof. Select at random K equipartitions of A and K equipartitions of B, each into K^2 sets: for each $i \in [K]$ let $\{S^A_{i,k}\}_{k=1}^{K^2}$ be the i^{th} partition of A and let $\{S^B_{i,k}\}_{k=1}^{K^2}$ be the i^{th} partition of B. Note that all parts of all partitions have size either $\lfloor n/K^2 \rfloor$ or $\lceil n/K^2 \rceil$, and for each index i and each vertex $v \in A$ (respectively B) there exists a unique index $k(i, v)$ such that $v \in S^A_{i,k(i,v)}$ (respectively $S^B_{i,k(i,v)}$). Denote by $S_{i,k}$ the union of $S^A_{i,k}$ and $S^B_{i,k}$.

Consider the following random sets. For $v \in V(D)$, $i \in [K]$, let $X^{\pm}(v, i)$ be the set of vertices $u \in N^{\pm}_D(v) \cap S_{i,k(i,v)}$ such that $u, v \in S_{j,\ell}$ for some $j \neq i$ and some ℓ. Further, let $Y^{\pm}(v)$ be the set of vertices $w \in N^{\pm}_D(v)$ such that both v and w are in the same set $S_{i,k}$ for some i, k. In other words, if we colour the edges of all induced subgraphs $D[S_{i,k}]$ in colour i (allowing multiple colours), $X^{\pm}(v, i)$ is the set of all vertices w such that the edge (v, w) (or (w, v), respectively) received colour i and at least one other colour, and $Y^{\pm}(v)$ is the set of vertices w such that the edge (v, w) (or (w, v), respectively) received at least one colour. Set $s = n/K^2$ and

$b = \mathbb{E}(|Y^{\pm}(v)|)$ where we note that b is independent of v since all degrees in D are equal and since the partitions were chosen uniformly. We claim that all of the following properties hold with high probability:

(a) For all $v \in V(D)$ and all sets $S_{i,k}$: $d^{\pm}_D(v, S_{i,k}) = \frac{d|S_{i,k}|}{2n} \pm 2\sqrt{s \log n}$;

(b) for all $v \in V(D)$ and $i \in [K]$: $|X^{\pm}(v,i)| = o(s)$;

(c) for all $v \in V(D)$, $|Y^{\pm}(v)| = b \pm 2\sqrt{K^2 s \log n}$.

For Property (a) note that for fixed $v \in V(D)$, $i \in [K]$, and $k \in [K^2]$, both $d^{+}_D(v, S_{i,k})$ and $d^{-}_D(v, S_{i,k})$ are hypergeometric random variables, each with parameters $(n, d, |S_{i,k}|/2)$. Hence, it follows that (a) holds with probability at least $1 - 16nK^3 e^{-4\log n/3} = 1 - o(1)$, by Lemma 1.8 and the union bound.

For fixed $u, v \in V(D)$ and $i \in [K]$, the event $\{u \in X^{\pm}(v,i)\}$ implies that $\bigcup_{j \neq i}\{u \in S_{i,k(i,v)}, u \in S_{j,k(j,v)}\}$. So, summing over all $u \in V(D)$, we get

$$|X^{\pm}(v,i)| = \sum_{u} \mathbb{1}_{\{u \in X^{\pm}(v,i)\}} \leqslant \sum_{u,j} \mathbb{1}_{\{u \in S_{i,k(i,v)}, u \in S_{j,k(j,v)}\}} \sim Bin(nK, (1/K^2)^2).$$

Thus $\mathbb{E}(|X^{\pm}(v,i)|) \leqslant \left(\frac{1}{K^2}\right)^2 nK = o(s)$ and (b) follows from a straightforward application of Chernoff's inequality (Lemma 1.8).

For Property (c) fix a vertex $v \in A$ and note that

$$|Y^{\pm}(v)| = \left| N^{\pm}_D(v) \cap \bigcup_{i=1}^{K} S^B_{i,k(i,v)} \right|.$$

By Lemma 1.9 applied with $S = B$, $T = N^{\pm}_D(v)$, $U_i = S^B_{i,k(i,v)}$, and $s = n/K^2$, we have that

$$b = \mathbb{E}(|Y^{\pm}(v)|) = d\left(1 - (1 - 1/K^2)^K\right), \text{ and}$$
$$\mathbb{P}\left(\big||Y^{\pm}(v)| - b\big| > t\right) \leqslant 2e^{-t^2/3b}(n+1)^K$$

for all $t \leqslant b$. If we take $t = 2\sqrt{n \log n}$, the right-hand side of the above inequality is of order $o(1/n)$, where we use that $b = d\left(1 - (1 - 1/K^2)^K\right) \sim cn/K$. The same inequality holds for all vertices $v \in B$, so (c) follows by taking the union bound over all $v \in V(D)$.

Now fix K partitions $\{S^A_{i,k}\}_{k=1}^{K^2}$ of A, and K partitions $\{S^B_{i,k}\}_{k=1}^{K^2}$ of B, such that properties (a), (b) and (c) are satisfied.

Let D' be the digraph consisting of all edges of D which are not contained in any $D[S_{i,k}]$. It follows directly from (c) that

$$d^{\pm}_{D'}(v) = d - b \pm 2\sqrt{K^2 s \log n} \tag{3.1}$$

for every $v \in V(D)$.

Relabel the sets $\{S_{i,k}\}_{(i,k)\in[K]\times[K^2]}$ as $W_1, \ldots, W_{K^3}$ and define the digraphs H_j on vertex sets W_j to be the edges of $D[W_j]$ that are not in $D[W_{j'}]$ for any $j' \neq j$. Finally, let $U_i = V(D)\backslash W_i$.

Property $(P1)$ of the lemma statement is trivially satisfied by definition. Furthermore, for every $1 \leqslant i \leqslant K^3$ and every $v \in W_i$ we have that

$$d^{\pm}_{H_i}(v, W_i) = \frac{d|W_i|}{2n} \pm \left(2\sqrt{s\log n} + o(s)\right),$$

by (a) and (b). Hence, Property $(P4)$ follows since $d = cn$ and $|W_i| = n/K^2$.

It remains to choose edge sets $E_{H_i}(U_i, W_i)$, $E_{H_i}(W_i, U_i)$ and $E_{H_i}(U_i)$ such that properties $(P2)$ and $(P3)$ are satisfied. For a vertex $u \in V(D)$, let I_u denote the set of indices i such that $u \in W_i$, and note that by construction $|I_u| = K$. Furthermore, for an edge $e = (u, v) \in D'$ we have $I_u \cap I_v = \emptyset$ by definition of D'. Define random edge sets $E_1, \ldots, E_{K^3}$ and $D_1, \ldots, D_{K^3}$ as follows. For every edge $e = (u, v) \in D'$, add e to exactly one of $E_1, \ldots, E_{K^3}, D_1, \ldots, D_{K^3}$ with the following probabilities. For each $i \in [K^3]$

- add e to E_i with probability $\frac{\varepsilon}{2K}$ if $i \in I_u \cup I_v$;
- add e to D_i with probability $\frac{1-\varepsilon}{K^3-2K}$ if $i \notin I_u \cup I_v$,

choices being independent for distinct edges. Note that the probabilities indeed add up to 1. Now for all $i \in [K^3]$ and all $v \in U_i$,

$$\mathbb{E}(d^{\pm}_{D_i}(v)) = d^{\pm}_{D'}(v)\frac{1-\varepsilon}{K^3-2K}$$

and

$$\mathbb{E}(d^{\pm}_{E_j}(v, W_j)) = d^{\pm}_{D}(v, W_j)\frac{\varepsilon}{2K}.$$

Hence by (3.1), Chernoff's inequality (Lemma 1.8) and the union bound, with probability at least $1 - 8nK^3e^{-\omega(\log n)} = 1 - o(1)$ we have that $d^{\pm}_{D_i}(v) = r \pm r^{3/5}$ for all $i \in [K^3]$ and all $v \in U_i$, for some suitable $r = (1 \pm \varepsilon)d/K^3$. Similarly we obtain that with probability at least $1 - 4nK^3e^{-\omega(\log n)} = 1 - o(1)$, we have that for all $i \in [K^3]$ and all $v \in U_i$,

$$d^{\pm}_{E_i}(v, W_i) \geqslant \frac{\varepsilon}{2K}\left(\frac{d|W_i|}{2n} - 2\sqrt{n/\log n}\right) \geqslant \varepsilon c|W_i|/8K,$$

by (a), Chernoff's inequality, the union bound, and where we use in the last inequality that $d = cn$ and $|W_i| \gg \sqrt{n \log n}$.

Finally, fix choices of E_i and D_i that satisfy $d^{\pm}_{D_i}(v) = r \pm r^{3/5}$ and $d^{\pm}_{E_i}(v, W_i) \geqslant \varepsilon c|W_i|/8K$ for all $i \in [K^3]$ and all $v \in U_i$, and set $H_i = E_i \cup D_i \cup H_i[W_i]$.

□

3.3 Path covers in almost-regular graphs

We now prove that each $H_i[U_i]$ as given by Lemma 3.8 has many edge-disjoint "almost-Hamilton-cycles" called *path covers*.

Definition 3.9. A *path cover of size k* in a directed graph H is a set $\mathcal{P}$ of k directed paths in H such that every vertex is contained in exactly one path of $\mathcal{P}$.

Note that every digraph H contains a *trivial path cover* in which every path consists of exactly one vertex of H, whereas a Hamilton path, if existent, is a path cover of size one. So, it is really a path cover of "small size" that approximates a Hamilton cycle, rather than any path cover.

Given a set of path covers $\mathbf{P}$ of a digraph H, we denote by $G_{\mathbf{P}}$ the graph whose edge set is formed by taking the union of all sets $E(P)$, for all paths $P \in \mathcal{P}$, for all path covers $\mathcal{P} \in \mathbf{P}$.

Now we are ready to state our lemma positing that the graphs $H_i[U_i]$ obtained in Lemma 3.8 contain many edge-disjoint path covers of small size.

Lemma 3.10. *There exists a positive integer $m_0 \in \mathbb{N}$, such that for $m \geqslant m_0$ and $m^{49/50} \leqslant r \leqslant m/3$ the following is true. Let H be a balanced bipartite digraph on $2m$ vertices such that $d^{\pm}(v) = r \pm r^{3/5}$ for every vertex v of H. Then H contains a collection $\mathbf{P}$ of at least $r - m^{24/25} \log m$ edge-disjoint path covers, each of size at most $m/\log^4 m$. Moreover, $\delta^0(G_{\mathbf{P}}) \geqslant r - m/(\log m)^{39/10}$.*

Proof. Let (A, B) be a bipartition of H such that $|A| = |B| = m$, and let $b = 2\log^4 m$. Let $V^A_1, \dots, V^A_b$ and $V^B_1, \dots, V^B_b$ be partitions of A and B

respectively, chosen independently and uniformly at random among all partitions such that $|V_i^A| = |V_i^B| = m/b$ for all i. For a fixed $i \in [b]$ and a fixed vertex $v \in A$, the random variable $d^+(v, V_i^B)$ has a hypergeometric distribution with parameters $(m, d^+(v), m/b)$. Therefore, the probability that $|d^+(v, V_i^B) - r/b| > (r/b)^{3/5}$ is at most $\exp(-(r/b)^{1/5}/6)$, by Lemma 1.8 and since $d^+(v, B) = r \pm r^{3/5}$ by assumption. A similar concentration argument applies to $d^-(v, V_i^B)$ as well as to $d^\pm(w, V_j^A)$ for every vertex $w \in B$ and $j \in [b]$. It follows by the union bound that with probability at least $1 - 8mb\exp(-(r/b)^{1/5}/6) = 1 - o(1)$ we have that

$$d^\pm(v, V_i^B) = \frac{r}{b} \pm \left(\frac{r}{b}\right)^{3/5} \quad \text{for all } v \in A,\ i \in [b], \text{ and} \tag{3.2}$$

$$d^\pm(w, V_j^A) = \frac{r}{b} \pm \left(\frac{r}{b}\right)^{3/5} \quad \text{for all } w \in B,\ j \in [b]. \tag{3.3}$$

Fix partitions of A and B that satisfy (3.2) and (3.3).

Let (W^A, W^B) denote a bipartition of the complete bipartite digraph $D_{b,b}$, where the elements of the two sets are labelled $W^A = \{w_j^A \mid 1 \leqslant j \leqslant b\}$ and $W^B = \{w_j^B \mid 1 \leqslant j \leqslant b\}$. Then $D_{b,b}$ contains b edge-disjoint Hamilton paths, say $P_1, \dots, P_b$, all of which have their start vertex in W^A, and such that no vertex in $W^A \cup W^B$ is the endpoint of more than $2\sqrt{\log b}$ of these paths, by Lemma 3.6.

Let $P_1 = w_{i_1}^A \dots w_{i_{2b}}^B$ and let $F_1, \dots, F_{2b-1}$ be the corresponding bipartite subgraphs of H having edge sets

$$E(V_{i_1}^A, V_{i_2}^B), E(V_{i_2}^B, V_{i_3}^A), \dots, E(V_{i_{2b-1}}^A, V_{i_{2b}}^B),$$

respectively (recall that $E(V, W)$ denotes the set of all edges of a digraph that are oriented *from* V *to* W).

For each $j \in [2b-1]$, we apply Lemma 3.7 to the digraph F_j (and keep Remark 1 in mind in case $|V_{i_j}^A|$ and $|V_{i_{j+1}}^B|$, say, differ by 1). Note that the assumptions are satisfied with slack for $m' = m/b$ and $r' = r/b - (r/b)^{3/5}$, by (3.2) and (3.3). We conclude that F_j contains at least

$$\frac{r}{b} - \left(\frac{r}{b}\right)^{3/5} - \left(\frac{m}{b}\right)^{24/25} \geqslant \frac{r}{b} - 2\left(\frac{m}{b}\right)^{24/25}$$

edge-disjoint matchings, each of size at least $(m/b) - (m/b)^{7/8}$. Moreover, every vertex in $V_{i_j}^A \cup V_{i_{j+1}}^B$ (or $V_{i_j}^B \cup V_{i_{j+1}}^A$, respectively) is contained in at least

$$\frac{r}{b} - \left(\frac{r}{b}\right)^{3/5} - \left(\frac{m}{b}\right)^{24/25} - 2\left(\frac{m}{b}\right)^{5/6} \geqslant \frac{r}{b} - 2\left(\frac{m}{b}\right)^{24/25}$$

of these matchings.

Note that, for each $j \in [2b-1]$, all edges of F_j are oriented from $V_{i_j}^A$ to $V_{i_{j+1}}^B$ if j is odd, and from $V_{i_j}^B$ to $V_{i_{j+1}}^A$ if j is even. Therefore, we may pick an arbitrary such matching from F_j for every $j \in [2b-1]$ and concatenate those matchings to form a path cover $\mathcal{P}$ of H.

Then $\mathcal{P}$ contains at least $(2b-1)(m/b-(m/b)^{7/8})$ edges and so it must be of size at most $m/b+(2b-1)(m/b)^{7/8} \leqslant m/\log^4 m$, since each of the $2m$ vertices of H is in exactly one of the paths of $\mathcal{P}$.

Iteratively picking distinct matchings for each F_j, we obtain $r/b-2(m/b)^{24/25}$ such path covers for P_1. We do the same for all b Hamilton paths $P_1,\dots,P_b$ of $D_{b,b}$. Denote the union of all path covers obtained this way by $\mathbf{P}$, and note that $\mathbf{P}$ contains at least $b\left(r/b-2(m/b)^{24/25}\right) \geqslant r-m^{24/25}\log m$ path covers since m is large enough. Since the paths $P_1,\dots,P_b$ are pairwise edge-disjoint it follows that the path covers in $\mathbf{P}$ are pairwise edge-disjoint.

It remains to show that the graph $G_{\mathbf{P}}$ has minimum semi-degree at least $r-m/(\log m)^{39/10}$. As noted above, for every bipartite graph F_j of P_1, $1 \leqslant j < 2b-1$, every vertex in $V_{i_j}^{A/B}$ is in at least $r/b-2(m/b)^{24/25}$ matchings. That is, every such v has $d^+(v, V_{i_{j+1}}^{B/A})$ at least $r/b-2(m/b)^{24/25}$ in the graph formed by the union of those matchings. The same lower bound holds for every path P_j and every v that is not in the vertex class of the endpoint of P_j. Since a particular $V_{i_j}^{A/B}$ is the "endpoint" of at most $2\sqrt{\log b}$ of the paths $P_1,\dots,P_b$ we get that for all $v \in V(H)$

$$d^+(v) \geqslant (b-2\sqrt{\log b})\cdot\left(\frac{r}{b}-2\left(\frac{m}{b}\right)^{24/25}\right) \geqslant r-\frac{m}{(\log m)^{39/10}}$$

in the graph formed by the union $\bigcup \mathcal{P}_i$ of all path covers. A similar argument applies to $d^-(v)$ in $G_{\mathbf{P}}$, which finishes the proof the lemma. $\square$

3.4 Two connecting lemmas

In the proofs of Theorems 3.3 and 3.4, respectively, we apply Lemma 3.10 to each $H=H_i[U_i]$. The strategy is then to connect the paths of each path cover in $H_i[U_i]$ to a Hamilton cycle (for Theorem 3.3) or to a long cycle (for Theorem 3.4) using the vertices in W_i in such a way that the cycles corresponding to distinct path covers are edge disjoint. We make this precise in the following Lemmas 3.11 and 3.12.

Lemma 3.11 (Connecting to a Hamilton cycle). *Let $c' > 1/2$, and let a, n' be positive integers such that $a = o(n'/\log n')$. Let F be a balanced bipartite digraph on $2n'$ vertices such that $\delta^0(F) \geqslant c'n'$. Then, given a balanced set of distinct vertices $s_1, t_1, ..., s_a, t_a \in V(F)$ with respect to a balanced[1] bipartition of F, there exists a path cover $\mathcal{P} = \{P_1, ..., P_a\}$ of F such that each path P_i starts at s_i and ends at t_i.*

Proof. Let (A, B) be a bipartition of F such that $|A| = |B| = n'$. Choose a partition $W_1 \dot\cup \ldots \dot\cup W_a$ of $A \cup B$ uniformly at random from all partitions that satisfy

(a) $s_i, t_i \in W_i$ for all i,

(b) $||W_i| - |W_j|| \leqslant 2$ for all i, j,

(c) $|W_i^A| - |W_i^B| = |\{s_i, t_i\} \cap A| - 1$.

To see that such a partition exists let $S = \{s_1, t_1, \ldots, s_a, t_a\}$, let $I_A \subseteq [a]$ be the set of indices such that $s_i, t_i \in A$, let $I_B \subseteq [a]$ be the set of indices such that $s_i, t_i \in B$, and let $I_m = [a] \backslash (I_A \cup I_B)$. Since S is balanced, $|I_A| = |I_B|$ which we denote by a'. Let $A' = A \backslash S$, $B' = B \backslash S$ and assume first that $x = (n' - a - a')/a$ is an integer. Let $W_1' \dot\cup \ldots \dot\cup W_a'$ be a partition of $A' \cup B'$ such that $|W_i' \cap A| = x$ if $i \in I_A \cup I_m$, $|W_i' \cap A| = x+1$ if $i \in I_B$, and similarly, $|W_i' \cap B| = x$ if $i \in I_B \cup I_m$, $|W_i' \cap B| = x+1$ if $i \in I_B$. Note that this is possible by choice of x and since $|I_A| = |I_B|$. Then the partition $W_1 \dot\cup \ldots \dot\cup W_a$ is a partition as desired if we let $W_i = W_i' \cup \{s_i, t_i\}$ for all $i \in A$. In this case the bound in (b) is even 1. When x is not an integer then a similar construction works (some occurrences of x replaced by $\lfloor x \rfloor$ and some by $\lceil x \rceil$), in which case the set sizes may differ by 2.

Fix $v \in V(F)$ and $i \in [a]$. Note that $d^+(v, W_i \backslash \{s_i, t_i\})$ has a hypergeometric distribution with parameters $(n', d^+(v, V(F) \backslash S), m)$, where $m = n'/a \pm 1$ and $d^+(v, V(F) \backslash S) \geqslant d^+(v) - a$. Therefore, for all $\varepsilon > 0$ the probability that $d^+(v, W_i) < (c' - \varepsilon)n'/a$ is at most $\exp(-\varepsilon^2 n'/12a)$, since $d^+(v) \geqslant c'n'$ and by Lemma 1.8. A similar bound holds for $d^-(v, W_i)$. Taking the union bound we deduce that with probability $1 - 4n'a \exp(-\varepsilon^2 n'/12a) = 1 - o(1)$

$$d^{\pm}(v, W_i) \geqslant (c' - \varepsilon)\frac{n'}{a} > \frac{m' + 3}{2} \text{ for all } v \in V(F),\ i \in [a], \tag{3.4}$$

where $m' = \min\{|W_i^A|, |W_i^B|\}$, ε satisfies $0 < \varepsilon < c' - 1/2$, and we use that $a = o(n'/\log n')$.

[1] A subset S of the vertices of a bipartite digraph F with bipartition (A, B) is called *balanced* if $|S^A| = |S^B|$.

Fix a partition that satisfies (3.4). We claim that this is sufficient to find a Hamilton s_i-t_i-path in $F[W_i]$, for every $i \in [a]$. The following implies this already when $s_i \in A$, $t_i \in B$ (or vice versa), when, by c, we have $|W_i^A| = |W_i^B|$.

Claim 3.11.1. Let m' be a non-negative integer and let $G = (A, B)$ be a bipartite digraph such that $|A| = |B| = m'$. Let $x \in A$, $y \in B$. If $\delta^0(G) \geqslant m'/2 + 1$ then G contains a Hamilton path from x to y.

Proof of claim. Let $A' = A \backslash \{x\}$ and $B' = B \backslash \{y\}$, and let G' be the (undirected) bipartite graph with vertex set $V' = A' \cup B'$ and edge set $E' = \{ab : (b, a) \in E(G)\}$.

We claim that G' contains a perfect matching. Note that $d_{G'}(a) \geqslant d_G^-(a) - 1 \geqslant (m' - 1)/2$ for all $a \in A'$ and $d_{G'}(b) \geqslant d_G^+(b) - 1 \geqslant (m' - 1)/2$ for all $b \in B'$. Let now $X \subseteq A'$ be non-empty and assume that $|N_{G'}(X)| < |X|$. Since every vertex in X has at least $(m' - 1)/2$ neighbours in G' it follows that $|X| > (m' - 1)/2$. Moreover, the set $B' \backslash N_{G'}(X)$ is non-empty, so for any vertex $v \in B' \backslash N_{G'}(X)$ we have $N_{G'}(v) \subseteq A' \backslash X$. This, however, implies that $d_{G'}(v) \leqslant |A' \backslash X| < (m' - 1)/2$, a contradiction. Thus, $|N_{G'}(X)| \geqslant |X|$ for all $X \subseteq A'$, which implies that G' contains a perfect matching, by Hall's Theorem.

Let $\{(v_1, w_1), ..., (v_{m'-1}, w_{m'-1})\}$ denote the corresponding matching of directed edges in G such that $v_i \in B'$ and $w_i \in A'$ for all $1 \leqslant i \leqslant m' - 1$, and let $w_{m'} = x$ and $v_{m'} = y$. Consider now the following auxiliary digraph H on vertex set $V(H) = \{z_1, ..., z_{m'}\}$. For each pair (i, j) let (z_i, z_j) be an edge of H if (w_i, v_j) is an edge of G. Note that H satisfies $\delta^0(H) \geqslant \delta^0(G) - 1 \geqslant m'/2$. Therefore, H contains a Hamilton cycle, say with edge set C, by Theorem 3.5. Now, this Hamilton cycle corresponds to a Hamilton path from x to y in G which can be obtained by replacing each edge (z_i, z_j) in C by the edges (w_i, v_j) and (v_j, w_j) (the latter only if $j \neq m'$) in G. □

Clearly this implies that every $F[W_i]$ has a Hamilton s_i-t_i-path in the case when $s_i \in W_i^A$ and $t_i \in W_i^B$, or vice versa. Assume now that both s_i and t_i are on the same side of the bipartition, say without loss of generality in W_i^A. In that case $|W_i^A| = |W_i^B| + 1$ by c. The balanced bipartite digraph $F[(W_i^A \cup W_i^B) \backslash \{s_i\}]$ satisfies the assumptions of the claim and thus contains a Hamilton path from u to t_i for any out-neighbour u of s_i. Adding the edge (s_i, u) to that path yields a Hamilton path from s_i to t_i in $F[W_i]$, as required. □

Lemma 3.12 (Connecting to a long cycle)**.** *Let $c' > 1/4$, and let a, n' be positive integers such that, $a = o(n'/\log n')$. Let F be a balanced bipartite oriented graph* *on $2n'$ vertices such that $\delta^0(F) \geqslant c'n'$. Then, given a set of distinct vertices $s_1, t_1, ..., s_a, t_a \in V(F)$, there exists a collection of pairwise vertex disjoint paths $\{P_1, ..., P_a\}$ of F such that each path P_i starts at s_i and ends at t_i.*

Proof. Let (A, B) be a bipartition of F such that $|A| = |B| = n'$. Similarly to the proof of Lemma 3.11 we choose a partition $W_1 \dot\cup \dots \dot\cup W_a$ of $A \cup B$ uniformly at random from all partitions that satisfy

(a) $s_i, t_i \in W_i$ for all i,

(b) $||W_i| - |W_j|| \leqslant 1$ for all i, j,

(c) $|W_i^A| = |W_i^B|$.

Analogously to (3.4) we deduce that with probability $1 - o(1)$

$$d^{\pm}(v, W_i) \geqslant (c' - \varepsilon)\frac{n'}{a} > \frac{m'}{4} \text{ for all } v \in V(F),\ i \in [a], \tag{3.5}$$

where $m' = |W_i^A|$. Fix a partition such that (3.5) is satisfied. We now find an s_i-t_i-path in $F[W_i]$ using the following.

Claim 3.12.1. Let G be a balanced bipartite oriented graph on $2m'$ vertices. Assume that the minimum semi-degree of G satisfies $\delta^0(G) > m'/4$. Then G is strongly connected.

Proof. Let v be an arbitrary vertex in G and let $R^+(v)$ be the set of vertices w such that there is a v-w-path in G. We first show that $|R^+(v)| > m'$.

Suppose not. Let $G' = G[R^+(v)]$. Then $\delta^+(G') > m'/4$ as all out-neighbours of all $w \in R^+$ are elements of $R^+(v)$, by definition. Since G is bipartite, so is G'. Let $A \cup B$ be some bipartition of G'. By the minimum degree assumption, the set $E(A, B)$ has size greater than $|A|m'/4$, and so there is a vertex b in B of in-degree greater than $|A|m'/4|B|$. As the in- and out-neighbours of b are distinct elements of A (since G' is an oriented graph) we obtain that

$$|A| > \frac{m'}{4}\left(\frac{|A|}{|B|} + 1\right).$$

Counting the edges in $E(B, A)$ gives analogously that

$$|B| > \frac{m'}{4}\left(\frac{|B|}{|A|} + 1\right).$$

Combining the two inequalities implies that

$$|R^+(v)| = |A| + |B| > \frac{m'}{4}\left(\frac{|A|}{|B|} + \frac{|B|}{|A|} + 2\right) \geqslant m',$$

where the last step follows from the AM-GM inequality.

Analogously one can show that the set $R^-(v)$ of vertices w such that there is a w-v-path in G has size greater than m'. Since this is true for any $v \in V(G)$, it follows that for any two vertices v and v' of G, the sets $R^+(v)$ and $R^-(v')$ intersect, that is, there is a path from v to v'. □

This finishes the proof of the lemma since all graphs $F[W_i]$ are balanced bipartite oriented graphs and satisfy the degree condition (3.5). □

3.5 Proofs of Theorems 3.3 and 3.4

With the results of the previous four sections at hand, we are now ready to prove our main theorems.

Proof of Theorem 3.3. Let $c > 1/2$, $\varepsilon > 0$ where we may assume for the proof that ε is sufficiently small. Let n be a sufficiently large integer. Let $d = cn$ and assume that G is a balanced d-regular bipartite digraph on $2n$ vertices. Let $K = \log n$ and let $H_1, \dots, H_{K^3}$ be the subdigraphs given by Lemma 3.8 satisfying the properties 1–4.

For each $i \in [K^3]$ we apply Lemma 3.10 with $m = |U_i^A| = |U_i^B| = n - n/K^2 \pm 1$ and r given by 2. Note that $r = (1 \pm \varepsilon)d/K^3 = \Theta(n/K^3)$ and $H_i[U_i]$ is balanced so that the assumptions of Lemma 3.10 are satisfied for $H = H_i[U_i]$. Therefore, for every $i \in [K^3]$, we obtain a collection $\mathbf{P}^{(i)}$ of at least $r' = r - n^{24/25}\log n$ edge-disjoint path covers of $H_i[U_i]$, each of size at most $a = n/\log^4 n$, and such that

$$\delta^0(G_{\mathbf{P}^{(i)}}) \geqslant r - n/(\log n)^{39/10}. \tag{3.6}$$

Now fix $i \in [K^3]$ and let $\mathcal{P}_1^{(i)}, \dots, \mathcal{P}_{r'}^{(i)}$ be r' path covers of $\mathbf{P}^{(i)}$ as above. We iteratively find r' edge-disjoint Hamilton cycles $C_1^{(i)}, \dots, C_{r'}^{(i)}$ in H_i such that $C_k^{(i)}[U_i]$ consists exactly of the edges in $\mathcal{P}_k^{(i)}$, for all $1 \leqslant k \leqslant r'$. In other words, the paths in $\mathcal{P}_k^{(i)}$ are connected to a cycle $C_k^{(i)}$ via edges in $E(U_i, W_i) \cup E(W_i, U_i) \cup E(W_i)$. For $1 \leqslant k \leqslant r'$ suppose that we have obtained such $k-1$ edge disjoint Hamilton cycles $C_1^{(i)}, \ldots, C_{k-1}^{(i)}$. Let F_k be the graph obtained from H_i by removing the edges of those $k-1$ cycles. Let $(x_1, y_1), \ldots, (x_\ell, y_\ell)$ be the pairs of start and end points of the paths in $\mathcal{P}_k^{(i)}$, and note that $\ell \leqslant n/\log^4 n$. We now greedily pick pairwise distinct vertices $s_1, t_1, \ldots, s_\ell, t_\ell \in W_i$ such that

$$(y_1, s_1), (t_1, x_2), \ldots, (y_\ell, s_\ell), (t_\ell, x_1) \in E(F_k). \tag{3.7}$$

We verify briefly that this is indeed possible. For a vertex $v \in \{x_1, y_1, \ldots, x_\ell, y_\ell\} \subseteq U_i$ we have that $d^{\pm}_{H_i}(v, W_i) \geqslant \varepsilon|W_i|/16K$, by 3 and since $c > 1/2$. An edge in $E(v, W_i)$ (or $E(W_i, v)$, respectively) is removed from H_i only if v is the endpoint (or startpoint, respectively) of a path in $\bigcup_{j=1}^{k-1} \mathcal{P}_j^{(i)}$ (and in this case, at most one edge is removed from H_i). Since $\delta^0(G_{\mathbf{P}^{(i)}}) \geqslant r - n/(\log n)^{39/10} \geqslant r' - n/(\log n)^{39/10}$ by (3.6), it follows that every $v \in U_i$ is the start (or end) point of at most $n/(\log n)^{39/10}$ paths in $\bigcup_{j=1}^{r'} \mathcal{P}_j^{(i)}$. Thus,

$$d^+_{F_k}(v, W_i) \geqslant d^+_{H_i}(v, W_i) - n/(\log n)^{39/10} > 0$$

at each step, and we can indeed pick $s_1, t_1, \ldots, s_\ell, t_\ell$ greedily in W_i such that (3.7) holds.

We verify that $F_k[W_i]$, together with the set $\{s_1, t_1, s_2, t_2, \ldots, t_\ell\}$ satisfies the assumptions of Lemma 3.11. Note that $n' = |W_i^A| = n/K^2 \pm 1$. Furthermore, the path cover $\mathcal{P}_k^{(i)}$ has size at most $n/\log^4 n$, hence $\ell \leqslant n/\log^4 n = o(n'/\log n')$. Now, $\delta^0(F_k[W_i]) \geqslant (c - \varepsilon)n' - (k-1)$ by 4 and since the only edges incident to vertices in W_i that were removed from H_i are those belonging to the Hamilton cycles $C_1^{(i)}, \ldots, C_{k-1}^{(i)}$. This implies that $\delta^0(F_k[W_i]) \geqslant c'n'$ for some $c' > 1/2$, since $c > 1/2$, $\varepsilon > 0$ is small enough, and $k = o(n')$. Finally, the set of vertices $s_1, t_1, s_2, t_2, \ldots, t_\ell$ is balanced because the set $x_1, y_1, ..., x_\ell, y_\ell$ of endpoints of paths in $\mathcal{P}$ is also balanced.

Therefore, by Lemma 3.11, $F_k[W_i]$ contains a path cover $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ such that P_j is an s_j-t_j-path for $1 \leqslant j \leqslant \ell$. These paths, together with the paths in $\mathcal{P}_k^{(i)}$ and the edges in (3.7) form a Hamilton cycle $C_k^{(i)}$ in $F_k \subseteq H_i$ that is edge-disjoint from $C_1^{(i)}, \ldots, C_{k-1}^{(i)}$ and from the paths in $\mathcal{P}_{k+1}^{(i)}, \ldots, \mathcal{P}_{r'}^{(i)}$.

Thus, after r' iterations, we obtain the desired edge-disjoint Hamilton cycles $C_1^{(i)}, \ldots, C_{r'}^{(i)}$ of H_i. Treating all K^3 subgraphs H_i in parallel (recall that they were edge-disjoint), we obtain $K^3 r' \geqslant (1-2\varepsilon)d$ edge-disjoint Hamilton cycles of G. □

Proof of Theorem 3.4. The proof is similar to the proof of Theorem 3.3 and so we

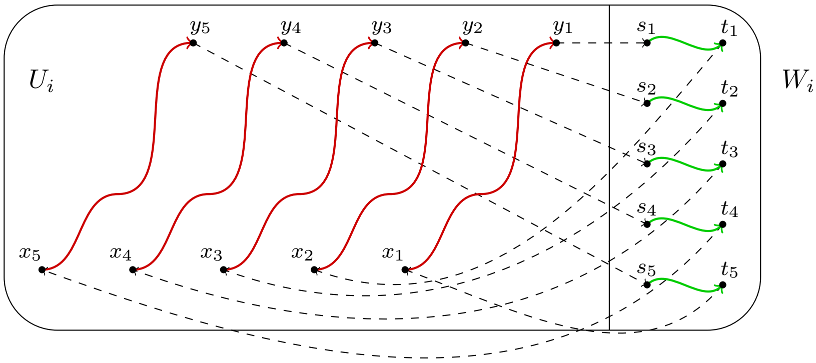


Figure 3.2: Completing a (red) path cover in $H_i[U_i]$ to a Hamilton cycle using (green) paths in $H_i[W_i]$. We do this r' times for each H_i. For Theorem 3.4 we do the same, except the green paths do not necessarily span W_i.

merely sketch it and point out the differences.

Let $c > 1/4$, $\varepsilon > 0$ where we may assume for the proof that ε is sufficiently small. Let n be a sufficiently large integer. Let $d = cn$ and assume that G is a balanced d-regular bipartite oriented graph on $2n$ vertices. Obviously, an oriented graph is a digraph, and so Lemmas 3.8 and 3.10 apply to this case just as above. Thus we obtain $K^3 = \log^3 n$ oriented subgraphs $H_1, \dots, H_{K^3}$ satisfying the properties 1–4 as in the previous proof. Furthermore, for every $i \in [K^3]$, we obtain a collection $\mathbf{P}^{(i)}$ of at least $r' = r - n^{24/25} \log n$ edge-disjoint path covers of $H_i[U_i]$, each of size at most $a = n/\log^4 n$, and such that (3.6) holds.

Now fix $i \in [K^3]$ and let $\mathcal{P}_1^{(i)}, \dots, \mathcal{P}_{r'}^{(i)}$ be r' of those path covers of $\mathbf{P}^{(i)}$. We iteratively find r' edge-disjoint cycles $C_1^{(i)}, \dots, C_{r'}^{(i)}$ in H_i such that $C_k^{(i)}[U_i]$ consists exactly of the edges in $\mathcal{P}_k^{(i)}$, for all $1 \leqslant k \leqslant r'$. That is, again, the paths in $\mathcal{P}_k^{(i)}$ are connected to a cycle $C_k^{(i)}$ via edges in $E(U_i, W_i) \cup E(W_i, U_i) \cup E(W_i)$. For $1 \leqslant k \leqslant r'$ suppose that we have obtained such $k-1$ edge disjoint cycles $C_1^{(i)}, \dots, C_{k-1}^{(i)}$. Let F_k be the graph obtained from H_i by removing the edges of those $k-1$ cycles. The argument why we can greedily pick pairwise distinct vertices $s_1, t_1, \dots, s_\ell, t_\ell \in W_i$ satisfying (3.7) only differs in the constant factor in the lower bound $d_{H_i}^{\pm}(v, W_i) \geqslant \varepsilon |W_i|/32K$, but the rest of the argument is essentially the same.

Similarly, we obtain analogously to above that $\delta^0(F_k[W_i]) \geqslant c'n'$ for some $c' > 1/4$.

Now instead of Lemma 3.11 we use Lemma 3.12 to find a collection $\{P_1, \dots, P_\ell\}$ of pairwise vertex disjoint paths in $F_k[W_i]$ such that P_j is an s_j-t_j-path for $1 \leqslant j \leqslant \ell$. These paths, together with the paths in $\mathcal{P}_k^{(i)}$ and the edges in (3.7) form a cycle $C_k^{(i)}$ in $F_k \subseteq H_i$ that is edge-disjoint from $C_1^{(i)}, \dots, C_{k-1}^{(i)}$ and from the paths in

$\mathcal{P}_{k+1}^{(i)}, \dots, \mathcal{P}_{r'}^{(i)}$. Since $C_k^{(i)}$ covers all the vertices of U_i this implies that the length of $C_k^{(i)}$ is at least $|U_i| = n - O(n/\log^2 n)$. The rest is analogous to the proof above. $\square$

3.6 Towards a proof of Jackson's conjecture

In this final section we present some thoughts on further research directions leading up to a proof of Conjecture 3.2 and related problems. The following two would each constitute a significant step towards Conjecture 3.2.

Conjecture 3.13. *Let $c > 1/2$ and let n be sufficiently large. Then every cn-regular bipartite digraph G on $2n$ vertices has a Hamilton cycle decomposition.*

Note that this is a bipartite analogue of [78, Theorem 1.4]: an cn-regular digraph on n vertices for $c > 1/2$ has a Hamilton decomposition, provided that $n \geqslant n_0(c)$.

Conjecture 3.14. *Let $\varepsilon > 0$, let n be sufficiently large, and let $d > n/4$ be an integer. Then every d-regular bipartite oriented graph on $2n$ vertices contains at least $(1-\varepsilon)dn$ edge-disjoint Hamilton cycles.*

The condition $d > n/4$ would be best possible since the oriented graph may be disconnected otherwise. In particular, taking $d = n/2$ would be a direct approximation of Jackson's conjecture as proved in [80] before the full proof of Kelly's conjecture in [78].

A further direction for exploration may be multi-partite tournaments. For $r \geqslant 2$, we consider *regular r-partite tournaments*, that is, regular orientations of the complete r-partite graph with equal size vertex classes. In [78], Kühn and Osthus not only prove Kelly's conjecture, but more generally, that every sufficiently large regular digraph G on n vertices whose degree is linear in n and which is a *robust outexpander* contains a Hamilton cycle decomposition. In [79, Section 1.6] they then argue that, for $r \geqslant 4$, every sufficiently large r-partite tournament is a robust outexpander, and thus, has a Hamilton cycle decomposition. The approach via robust outexpanders does not cover the bipartite nor the tripartite case. Yet it is conjectured in [79], additionally to Jackson's conjecture, that every regular tripartite tournament has a Hamilton cycle decomposition.

A possible approximate version of the conjecture for tripartite tournaments could be the following.

Conjecture 3.15. *Let $\varepsilon > 0$, $c > 1$ and let n be sufficiently large. Let G be a cn-regular tripartite digraph with vertex classes each of size n. Then G contains at least $(1-\varepsilon)cn$ edge-disjoint Hamilton cycles.*

Parts of our arguments do work for such an approximate version. The equivalent of Claim 3.11.1, however, does not seem to easily transfer. In fact, assuming just a lower bound of roughly n on the minimum semidegree of a balanced tripartite digraph on $3n$ vertices does not necessarily imply that the graph is Hamiltonian.

Chapter 4

A robust absorbing strategy for graph tilings and applications

Recall from Chapter 1 that an H-tiling in a graph G is a collection of vertex-disjoint copies of H in G, and this tiling is perfect if this collection is spanning. Note that in order for a perfect H-tiling in a graph G to exist, it is necessary that $|H| = h$ divides $|G|$, so we assume this for results on perfect tilings throughout this chapter.

Determining sufficient conditions for the existence of a perfect H-tiling is one of the fundamental lines of research in extremal graph theory. In particular, the case $H = K_2$ corresponds to finding a perfect matching, and as such, the perfect H-tiling problem is a natural generalisation of the problem of finding a perfect matching in a graph.

In this chapter we present a general absorbing lemma for tilings, and apply it to the following two problems.

Clique-tilings in graphs with sublinear independence number. A seminal result of Hajnal and Szemerédi [55] states that if a graph G with n vertices has minimum degree $\delta(G) \geqslant (r-1)n/r$ for some integer $r \geqslant 2$, then G contains a perfect K_r-tiling, assuming that r divides n. Extremal examples which show optimality of the bound on $\delta(G)$ are very structured and, in particular, contain large independent sets. In [7] Balogh, Molla, and Sharifzadeh initiated the study of how the absence of such large independent sets influences sufficient minimum degree, and show that in the case of triangle-tilings, the minimum degree of $2n/3$ given by Hajnal and Szemerédi's theorem (Theorem 4.4) can be improved to $(1/2 + \varepsilon)n$ for any

$\varepsilon > 0$, under the additional assumption that the host graph contains no independent sets of linear size. We extend their result to general K_r-tilings and a more general notion of independence number. Our minimum degree threshold has since been improved through a more careful proof using our absorbing lemma by Knierim and Su [70], to match the construction given in Proposition 4.10.

Tilings in randomly perturbed graphs. In [13] Bohman, Frieze and Martin introduce the randomly perturbed graph model, in which one adds random edges with probability p to a dense base graph until it asymptotically almost surely satisfies a certain property. Balogh, Treglown and Wagner [8] gave the correct p-threshold when the property considered is the existence of a perfect H-tiling. We give a new short proof of a strengthening of their result.

As described in Section 1.5, we prove the results of this chapter using the following absorbing strategy:

(i) Find a ξ-absorbing set $A \subseteq V(G)$ as in Definition 1.14.

(ii) Find an H-tiling in $V(G)\backslash A$ covering all but at most ξn vertices.

In executing this strategy for each of the problems described above, we do step (ii) using classical methods specific to the problem. Our main improvement comes in the mechanism used to find the absorbing set A for step (i), which we reduce to simply verifying the existence of linearly many smaller and simpler "locally absorbing" structures for any finite set of vertices S, which we call *the weak absorbing property.* Subsequently, to prove each of the results outlined above, we show two things about the host graph G.

(TP1) It has the weak absorbing property.

(TP2) It has a near-perfect H-tiling.

In the problems considered in this chapter, (TP2) guarantees success in step (ii) above, as $G\backslash A$ inherits the properties of G.

With this in mind, the rest of this chapter is organised as follows. In Section 4.1 we show that the existence of a ξ-absorbing set A can be reduced to the aforementioned weak absorbing property. In Section 4.2 we prove a result on clique-tilings in graphs with sublinear independence number which extends the main result from [7] using our new method. In Section 4.3 we give a short proof of the threshold for the existence of tilings in randomly perturbed graphs derived in [8]. In each of Sections 4.2 and 4.3 we prove our main result by verifying that in the setting considered, both (TP1) and (TP2) hold.

4.1 From the weak absorbing property to an absorbing set for tilings

In this section, we prove the following lemma which gives a sufficient condition for the existence of ξ-absorbing sets based on the following weak absorbing property. Recall from Definition 1.15 that for $S \subseteq V(G)$, a subset $A_S \subseteq V(G)\backslash S$ is (S,t)-*absorbing* if $|A_S| = t|S|$ and both $G[A_S]$ and $G[A_S \cup S]$ contain a perfect H-tiling.

Definition 4.1 (Weak absorbing property)**.** We say that an n-vertex graph G has the *weak* (γ,t)*-absorbing property* if for every $S \in \binom{V(G)}{h}$, G contains a family of at least γn vertex-disjoint (S,t)-absorbing sets.

We now show that for each $\gamma > 0$ and $t \in \mathbb{N}$ this weak absorbing property guarantees the existence of a ξ-absorbing set $A \subseteq V(G)$ for some $\xi > 0$.

Lemma 4.2. *Let H be a graph with h vertices and let $\gamma > 0$ and $t \in \mathbb{N}$ be constants. Then there exist $n_0 \in \mathbb{N}$ and $\xi > 0$ such that the following holds.*
Let G be a graph with $n \geqslant n_0$ vertices which satisfies the weak (γ,t)-absorbing property. Then G contains a ξ-absorbing set of size at most γn.

The proof of Lemma 4.2 is based on ideas of Montgomery [90] and relies on the existence of "robustly matchable" sparse bipartite graphs given by the following lemma.

Lemma 4.3 (Corollary of Lemma 3.43 in [90])**.** *For every $0 < \beta < 1$ there exists $m_0 \in \mathbb{N}$ such that for any $m \geqslant m_0$ there exists a bipartite graph B_m such that*

- *$V(B_m) = (X_m \cup Y_m, Z_m)$, where $|X_m| = m + \beta m$, $|Y_m| = 2m$ and $|Z_m| = 3m$;*
- *$\Delta(B_m) \leqslant 100$;*
- *for every subset $X'_m \subseteq X_m$ of size m, the induced graph $B_m[X'_m \cup Y_m \cup Z_m]$ contains a perfect matching.*

Proof of Lemma 4.2. From the assumption that for every $S \in \binom{V(G)}{h}$ there are γn disjoint (S,t)-absorbing sets, it follows that for every vertex $v \in V(G)$ there is a family of at least γn copies of H which contain v and are otherwise vertex-disjoint. Let us denote the family of sets of vertices of each such copy (without the vertex v) by $\mathcal{H}_v$.

Choose a subset $X \subseteq V(G)$ by including each vertex of G with probability $q = \gamma/(2000ht)$. By Chernoff's inequality and a union bound, we have that with high probability $|X| \leqslant 2nq$ and for each vertex $v \in V(G)$ at least $q^{h-1}|\mathcal{H}_v|/2$ sets from $\mathcal{H}_v$ are contained in X. Fix one such choice for X, and denote the family of sets from $\mathcal{H}_v$ completely contained in X by $\mathcal{H}'_v$. (So, we have $|\mathcal{H}'_v| \geqslant q^{h-1}|\mathcal{H}_v|/2$.)

Set $\beta = q^{h-1}\gamma/4$ and $m = |X|/(1+\beta)$. Let B_m be a graph given by Lemma 4.3. Choose disjoint subsets $Y, Z \subseteq V(G)\backslash X$ of size $|Y| = 2m$ and $|Z| = 3m(h-1)$ and arbitrarily partition Z into subsets $\mathcal{Z} = \{Z_i\}_{i\in[3m]}$ of size $h-1$. Take any injective mapping $\phi_1 \colon X_m \cup Y_m \to X \cup Y$ such that $\phi_1(X_m) = X$, and any injective $\phi_2 \colon Z_m \to \mathcal{Z}$. We claim that there exists a family $\{A_e\}_{e\in B_m}$ of pairwise disjoint (ht)-subsets of $V(G)\backslash(X \cup Y \cup Z)$ such that for each $e = \{w_1, w_2\} \in B_m$, where $w_1 \in X_m \cup Y_m$ and $w_2 \in Z_m$, the set A_e is $(\phi_1(w_1) \cup \phi_2(w_2), t)$-absorbing.

Indeed, such a family can be chosen greedily. Suppose we have already found desired subsets for all the edges in some $E' \subseteq B_m$. These sets, together with $X \cup Y \cup Z$, occupy at most

$$\begin{aligned}|X| + |Y| + |Z| + ht|E'| &< 4m + 3m(h-1) + ht \cdot 100|Z_m| \\ &\leqslant 4hm + 300htm \leqslant 304htm < 608htnq \leqslant \gamma n/2\end{aligned}$$

vertices in G. Choose arbitrary $e = \{w_1, w_2\} \in B_m\backslash E'$. As there are γn disjoint $(\phi_1(w_1)\cup\phi_2(w_2), t)$-absorbing sets, there are at least $\gamma n/2$ ones which do not contain any of the previously used vertices. Pick any and proceed.

It remains to show that the set

$$A = X \cup Y \cup Z \cup \left(\bigcup_{e\in B_m} A_e\right)$$

depicted in Figure 4.1 has the ξ-absorbing property for $\xi = \beta/(h-1)$.

Consider some subset $R \subseteq V(G)\backslash A$ such that $|R|+|A|$ is divisible by h and $|R| \leqslant \xi n$. As

$$|\mathcal{H}'_v| \geqslant q^{h-1}\gamma n/2 \geqslant 2\beta n = 2(h-1)\xi n \geqslant 2(h-1)|R|,$$

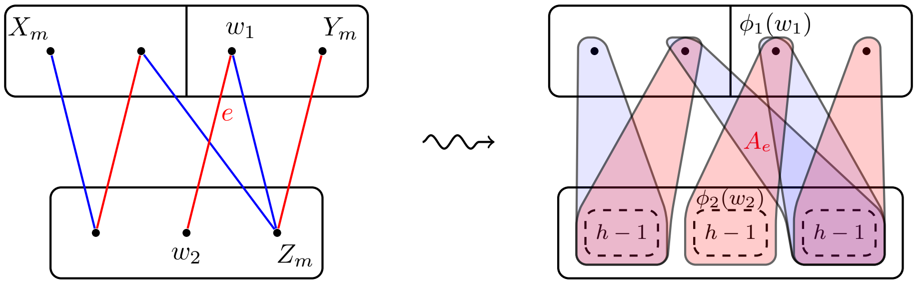


Figure 4.1: Obtaining an absorber from the robustly matchable bipartite graph B_m by replacing each edge e with an (S,t)-absorbing set A_e. For each red (matching) edge w_1w_2 in B_m, we take a tiling of $A_e \cup \phi_1(w_1) \cup \phi_2(w_2)$, and for each blue (non-matching) edge e in B_m, we take a tiling of A_e.

we can greedily choose a subset $A_v \in \mathcal{H}'_v$ for each $v \in R$ such that all these sets are pairwise disjoint (recall that each set in $\mathcal{H}'_v$ is of size $h-1$ and forms a copy of H with v). This takes care of vertices from R and uses exactly $|R|(h-1) \leqslant \beta m$ vertices from X. Denote the collection of $|R|$ copies of H obtained by $\mathcal{F}_1$. If $|R|(h-1) < \beta m$ then as h divides $|A|+|R|$, we have that h also divides $\beta m - |R|(h-1)$, thus we can cover the remaining vertices from X with disjoint copies of H such that there are *exactly* m vertices remaining. Again, $|\mathcal{H}'_v| \geqslant 2\beta n > 2\beta m$ implies that such copies of H can be found in a greedy manner. Denote this set of copies of H by $\mathcal{F}_2$.

Let X' be the remaining vertices from X and set $X'_m = \phi_1^{-1}(X')$. By Lemma 4.3 there exists a perfect matching M in B_m between $X'_m \cup Y_m$ and Z_m. For each edge $e = \{w_1, w_2\} \in M$, take $\mathcal{F}_e$ to be a perfect H-tiling in $G[\phi_1(w_1) \cup \phi_2(w_2) \cup A_e]$ and for each $e \in E(B_m) \backslash M$, take $\mathcal{F}_2$ to be a perfect H-tiling in $G[A_e]$. The union $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \left(\bigcup_{e \in E(B_m)} \mathcal{F}_e\right)$ is a perfect H-tiling of $G[A \cup R]$. $\square$

4.2 Clique-tilings in graphs with sublinear independence number

Recall the following theorem by Hajnal and Szemerédi which determines the minimum degree threshold for the existence of clique-tilings.

Theorem 4.4 (Hajnal, Szemerédi [55]; $r = 3$ by Corradi, Hajnal [26])**.** *Let $r \geqslant 3$ be an integer. Every n-vertex graph with minimum degree at least $\frac{r-1}{r}n$ contains a perfect K_r-tiling.*

In [7] Balogh, Molla and Sharifzadeh proved the following result.

Theorem 4.5. *For every $\varepsilon > 0$, there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ the following holds.*
Let G be an n-vertex graph such that $\delta(G) \geqslant (1/2 + \varepsilon)n$ and $\alpha(G) \leqslant \eta n$. Then G has a perfect triangle-tiling.

This theorem shows that the minimum degree threshold of $2n/3$ required in Theorem 4.4 in the case of triangle tilings drops down to (essentially) $n/2$ when the extremal example of a slightly imbalanced Turán graph is forbidden by forcing the independence number of G to be sublinear. The degree bound in Theorem 4.5 is tight, as shown in [7]. In this section we show that this pattern extends to general cliques and generalised independence number.

Definition 4.6. The *ℓ-independence number* $\alpha_\ell(G)$ of a graph G is the size of the largest vertex set in G which contains no copy of K_ℓ.
(In particular, $\alpha_2(G)$ is equal to the classical independence number $\alpha(G)$.)

As mentioned in Section 1.5, in the case of triangles the strong triangle-absorbing property fails, and hence previously known methods for constructing tiling absorbers cannot be applied. This turns out to be the case for an arbitrary clique of size r and bounded ℓ-independence number. Consider a graph G obtained by taking an r-partite complete graph with vertex classes $V_1, \dots, V_r$ and in each V_i place a graph F with $|V_i|$ vertices such that $\alpha_\ell(F) = o(n)$ and $\Delta(F) = o(n)$ (see the proof of Proposition 4.10 for the existence of such graphs). Take an arbitrary independent set $S \subseteq V_1$ of size r and consider some fixed $t \in \mathbb{N}$. Any S-absorber $A_S \subseteq V(G)\backslash S$ of size $|A_S| = rt$ which does not contain edges of F needs to intersect each V_i equally. However, any K_r-tiling of $A_S \cup S$ has to be traversing (that is, each copy of K_r contains exactly one vertex from each V_i), which leaves at least r vertices of $(A_S \cup S) \cap V_1$ unmatched. Therefore, A_S needs to contain an edge from some V_i, which implies an upper bound of order $o(n^{rt})$ on the number of such sets. To summarise, when looking to construct a K_r-tiling in a graph with minimum degree $\delta(G) \leqslant (r-1)n/r$, we cannot use constructions from [56, 101].

Having noted the limitations of previous absorbing methods in this setting, we now state our main result.

Theorem 4.7 (Nenadov, Pehova [93])**.** *Let $r > \ell \geqslant 2$ be integers. For any $\varepsilon > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds.*
Let G be an n-vertex graph such that $\delta(G) \geqslant \left(\frac{r-\ell}{r-\ell+1} + \varepsilon\right) n$ and $\alpha_\ell(G) \leqslant \eta n$. Then G contains a perfect K_r-tiling.

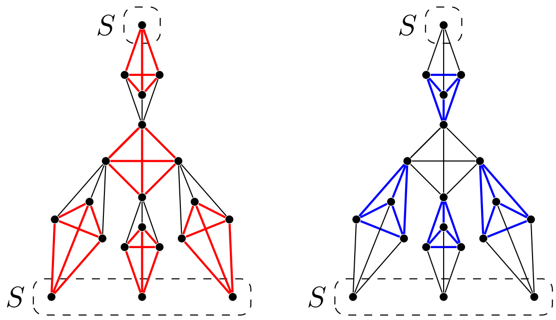


Figure 4.2: An $(S,4)$-absorbing set A_S for K_4-tilings. The red tiling covers $A_S \cup S$, the blue tiling covers only A_S.

The construction providing a lower bound on the degree threshold in the case when $\ell = 2$ and $r = 3$ given in [7] can be extended to give a lower bound in the more general setting of Theorem 4.7 as well. We refer the reader to Proposition 4.10 proved at the end of this section.

To prove Theorem 4.7, we verify (TP1) and (TP2) as promised in the introduction to this chapter.

Lemma 4.8 ((TP1) holds)**.** *Let $r > \ell \geqslant 2$ be integers. For any $\varepsilon > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that every graph G on $n \geqslant n_0$ vertices satisfying $\delta(G) \geqslant \left(\frac{r-\ell}{r-\ell+1} + \varepsilon\right) n$ and $\alpha_\ell(G) \leqslant \eta n$ has the weak $(\varepsilon/4r(r+1), r)$-absorbing property.*

Proof. Let $1/n_0 \ll \eta \ll \varepsilon$ and consider some $S \in \binom{V(G)}{r}$. Partition randomly $V(G)\backslash S$ into $r+1$ sets denoted by $V_1, \dots, V_{r+1}$. Each V_i is of size $(n-r)/(r+1)$ and by Chernoff's inequality and union bound, with high probability every vertex in G has at least

$$\left(\frac{r-\ell}{r-\ell+1} + \frac{\varepsilon}{2}\right) \frac{n}{r+1}$$

neighbours in each V_i. Fix a partition $V_1, \dots, V_{r+1}$ for which this holds.

Let us enumerate the vertices in S as $v_1, \dots, v_r$. We show that for every $X_i \subseteq V_i$ of size at most $\varepsilon n/4(r+1)$ there exists a copy of K_r in $V_{r+1}\backslash X_{r+1}$, with vertices labelled $w_1, \dots, w_r$, and a copy of K_{r-1} in $N_G(v_i) \cap N_G(w_i) \cap (V_i \backslash X_i)$ for every $i \in \{1, \dots, r\}$. Note that such copies of K_{r-1} together with the copy of K_r in V_{r+1} form an (S, r)-absorbing set (see Figure 4.2). This allows us to greedily form a family of $\varepsilon n/4(r+1)r$ vertex-disjoint (S, r)-absorbing sets, which finishes the proof.

The previous claim follows from the bound on the minimum degree and $\alpha_\ell(G) \leqslant \eta n$.

Indeed, note that each vertex has at least

$$\left(\frac{r-\ell}{r-\ell+1}+\frac{\varepsilon}{2}\right)\frac{n}{r+1}-\frac{\varepsilon n}{4(r+1)}=\left(\frac{r-\ell}{r-\ell+1}+\frac{\varepsilon}{4}\right)\frac{n}{r+1}$$

neighbours in each $V_i \backslash X_i$. As $|V_i \backslash X_i| \leqslant |V_i| \leqslant n/(r+1)$, by taking a union bound over complements of neighbourhoods, we obtain that any set $\{u_1, ..., u_{r-\ell+1}\}$ of $r-\ell+1$ vertices has a common neighbourhood of size at least $\varepsilon n/4(r+1)$ in $V_i \backslash X_i$.

This means we can start with an arbitrary vertex $w_1 \in V_{r+1} \backslash X_{r+1}$ and iteratively for $2 \leqslant i \leqslant r-\ell$ pick a vertex $w_i \in V_{r+1} \backslash X_{r+1}$ which is in the common neighbourhood of $w_1, \ldots, w_{i-1}$. Such vertices form $K_{r-\ell}$ and, as $\alpha_\ell(G) \leqslant \varepsilon n/4(r+1)$, there exists a copy of K_ℓ in their common neighbourhood in $V_{r+1} \backslash X_{r+1}$. This gives us a copy of K_r in $V_{r+1} \backslash X_{r+1}$. Now for each $i \in [r]$ repeat a similar argument in order to find a copy of K_{r-1} in $N_G(v_i) \cap N_G(w_i) \cap (V_i \backslash X_i)$. $\square$

Lemma 4.9 ((TP2) holds). *Let $r > \ell \geqslant 2$ be integers. For any $\varepsilon, \xi > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds.*
Let G be an n-vertex graph such that $\delta(G) \geqslant \left(\frac{r-\ell}{r-\ell+1}+\varepsilon\right)n$ and $\alpha_\ell(G) \leqslant \eta n$, and let $A \subseteq V(G)$ be of size at most $\varepsilon n/2$. Then $G \backslash A$ contains a K_r-tiling covering all but at most ξn vertices.

Proof. Let $\mu \ll \varepsilon\xi$, and set $d = \varepsilon/4$. Apply the Regularity Lemma (Lemma 1.5) to $G \backslash A$ with parameters μ and d to obtain a partition $V_0, ..., V_k$ of $V(G \backslash A)$ and a spanning subgraph $G' \subseteq G \backslash A$ with the properties (a)–(f) as stated. Let $R_{G\backslash A}$ be the reduced graph of this partition. Recall that $R_{G\backslash A}$ has vertex set $\{1, \ldots, k\}$ and there is an edge between i and j if and only if the pair (V_i, V_j) has density at least d in G'.

Claim 4.9.1. $\delta(R_{G\backslash A}) \geqslant \frac{r-\ell}{r-\ell+1}k$.

Proof of claim. Suppose, for contradiction, that V_1 has density at least d in G' with less than $\frac{r-\ell}{r-\ell+1}k$ other vertex classes of the regular partition. Then V_1 can have as many as n'^2 edges to V_j's such that (V_1, V_j) is dense, but at most $\mu n'^2$ edges to V_j's such that $1j \notin E(R_{G\backslash A})$. We have

$$e_{G'}(V_1, G' \backslash V_1) < \left(\frac{r-\ell}{r-\ell+1}k\right) \times n'^2 + \left(1-\frac{r-\ell}{r-\ell+1}\right)k \times \mu n'^2.$$

Since $kn' = \sum_{j=1}^{k} |V_i| \leqslant n$, the above quantity can be upper-bounded by

$$e_{G'}(V_1, G' \backslash V_1) < \frac{r-\ell}{r-\ell+1} nn' + \mu nn'.$$

On the other hand,

$$e_{G'}(V_1, G' \backslash V_1) = \sum_{v \in V_1} d_{G'}(v_1) \geqslant \sum_{v \in V_1} (d_G(v_1) - (\varepsilon/4 + \mu)n) \geqslant \left(\frac{r-\ell}{r-\ell+1} + \frac{\varepsilon}{2} \right) nn',$$

which contradicts the above upper bound when $\mu \ll \varepsilon$. □

Thus, by the Hajnal-Szemerédi theorem (Theorem 4.4), $R_{G \backslash A}$ contains a $K_{r-\ell+1}$-tiling which covers all but at most $r-\ell$ vertices (in case k is not divisible by $r-\ell+1$). For the rest of the proof we ignore V_0 and V_i's for $i \in [k]$ which correspond to vertices not covered by such a tiling. This way we ignore at most $\mu n + (r-\ell)\mu n < \xi n/2$ vertices.

Consider one of the copies of $K_{r-\ell+1}$ in the obtained tiling in $R_{G \backslash A}$. Without loss of generality we may assume that it corresponds to vertex classes $V_1, \ldots, V_{r-\ell+1}$. We show that we can find a K_r-tiling in $G[V_1 \cup \ldots \cup V_{r-\ell+1}]$ which covers all but at most $\xi m/2$ vertices in each V_j. Applying this to every copy of $K_{r-\ell+1}$ from the tiling of R we find a K_r-tiling of $G \backslash A$ covering all but at most ξn vertices, as desired.

To show that there exists a K_r-tiling in $G[V_1 \cup \ldots \cup V_{r-\ell+1}]$ which covers all but at most $\xi n'/2$ vertices in each V_j, it suffices to show the following claim.

Claim 4.9.2. For any $z \in [r-\ell+1]$ and any choice of subsets $V_j' \subseteq V_j$ of size $|V_j'| \geqslant \xi n'/8$ for $j \in [r-\ell+1]$, there exists a copy of K_r in $G[V_1' \cup \ldots \cup V_{r-\ell+1}']$ with exactly one vertex in each V_j' for $j \in [r-\ell+1] \backslash z$ and ℓ vertices in V_z'.

By repeatedly applying this $(1-\xi/4)n'/r$ times for each $z \in [r-\ell+1]$, each time removing vertices from the obtained K_r, we obtain the desired K_r-tiling.

Proof of claim. Consider some subsets $V_j' \subseteq V_j$ for $j \in [r-\ell+1]$ such that $|V_j'| \geqslant \xi n'/8$. By the Slicing Lemma (see [73, Fact 1.5]) each pair (V_i', V_j') is μ'-regular with density at least $d-\mu$ for $\mu' = 8\mu/\xi$. Without loss of generality we may assume $z = r-\ell+1$. Our goal is to find a vertex $w_j \in V_j'$ for each $1 \leqslant j \leqslant r-\ell$ such that these vertices form $K_{r-\ell}$ and their common neighbourhood $N_z \subseteq V_z'$ in V_z' is of size at least ηn. Indeed, by considering successive common neighbourhoods within each V_j', we get that we may choose $w_1, ..., w_{r-\ell}$ such that their common neighbourhood

N_z in V'_z satisfies

$$|N_z| \geqslant (d/2)^{r-\ell}|V'_z| \geqslant (d/2)^{r-\ell}\xi m/8 \geqslant (d/2)^{r-\ell}\xi n/16N > \eta n,$$

for sufficiently small η (recall that N is a constant). Finally, as $\alpha_\ell(G) \leqslant \eta n$ we can find a copy of K_ℓ in $G[N_z]$ which completes the desired copy of K_r. □

By applying the procedure from Claim 4.9.2 to every clique from the $K_{r-\ell+1}$-tiling of $R_{G\backslash A}$, we obtain a K_r-tiling of $G\backslash A$ covering all but at most $\xi n/2 + k\xi n'/8 \leqslant \xi n$ vertices, as required. □

Now it remains to combine Lemmas 4.8 and 4.9 to give a proof of our main result, which is now very short.

Proof of Theorem 4.7. First, by Lemma 4.8 we have that G satisfies the weak $(\varepsilon/4r(r+1), r)$-absorbing property. So by Lemma 4.2 G contains a ξ-absorbing set $A \subseteq V(G)$ of size at most $\varepsilon n/4r(r+1) \leqslant \varepsilon n/2$ for some $\xi > 0$.
Now, by Lemma 4.9 (reducing η if necessary) there is a K_r-tiling of $G\backslash A$ covering all but a set $R \subseteq G\backslash A$ of at most ξn vertices. Note that since r divides $|G|$, it must hold that r divides $|A| + |R|$. Since A is ξ-absorbing, $A \cup R$ has a perfect K_r-tiling, which together with the near-perfect tiling of $G\backslash A$ gives the required perfect K_r-tiling of G. □

We conclude this section by giving a lower bound on the degree threshold that forces a perfect K_r-tiling in the setting of Theorem 4.7.

Proposition 4.10. *If $r > \ell \geqslant 2$ then there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ there is a graph G on n vertices such that*

$$\delta(G) \geqslant \begin{cases} \frac{r-\ell}{r}n - 1, & \text{if } \ell < r/2 \\ n/2 - 2, & \text{otherwise,} \end{cases}$$

and $\alpha_\ell(G) = o(n)$ which does not contain a K_r-factor.

Proof. We first treat the case when $\ell \geqslant r/2$. Let G be the disjoint union of two cliques of sizes $\lceil n/2 \rceil - 1$ and $\lfloor n/2 \rfloor + 1$, respectively. Then G has minimum degree $\lceil n/2 \rceil - 2 \geqslant n/2 - 2$ but no K_r-factor for any $r \geqslant 3$ as at least one of $\lceil n/2 \rceil - 1$ and $\lfloor n/2 \rfloor + 1$ is not divisible by r.

Otherwise, consider some $2 \leqslant \ell < r/2$. Let $\Gamma_\ell(n)$ be a $K_{\ell+1}$-free graph with $n \geqslant n_0$ vertices and $\alpha_\ell(G) = o(n)$. Such graphs have been shown to exist by Erdős and Rogers [38] (one can also construct it directly by considering $G(n,p)$ with edge probability $p = n^{-2/(\ell+1)}$).

Having graphs $\Gamma_\ell(n)$ at hand, we can finish the proof of the claim. Let $r = x\ell + y$ for some $x, y \in \mathbb{N}$ and $1 \leqslant y \leqslant \ell$. We create a graph G by taking an $(x+1)$-partite complete graph with one set V_1 of size $yn/r - 1$, one set V_2 of size $\ell n/r + 1$ and all other sets $V_3, \dots, V_{x+1}$ of size $\ell n/r$, and within each set V_i put the graph $\Gamma_\ell(|V_i|)$. Such a graph has minimum degree at least $yn/r - 1 + (x-1)\ell n/r = (r-\ell)n/r - 1$. Because V_i does not contain $K_{\ell+1}$, any K_r in such a graph G has to contain at least y vertices from V_1 and cannot contain more than ℓ from any other set. Therefore, a K_r-tiling can have at most $\lfloor |V_1|/y \rfloor < n/r$ copies of K_r, which is not enough to cover all the vertices in V_2. □

In [70], Knierim and Su show that for $\ell = 2$ the above minimum degree condition is sufficient for the existence of a perfect K_r-tiling in a graph with sublinear independence number. Their result readily extends to general ℓ.

4.3 Tilings in randomly perturbed graphs

In this section we consider a variant of the classical property threshold question for the Erdős-Rényi random graph $G(n,p)$. Given a property $\mathcal{P}$, we seek to determine a *threshold* function $p(n)$ such that if $p \ll p(n)$, $G(n,p)$ a.a.s. doesn't have $\mathcal{P}$, and if $p = \omega(p(n))$, $G(n,p)$ a.a.s. has $\mathcal{P}$. For example, the threshold for Hamiltonicity in $G(n,p)$ was shown to be $p(n) = \frac{\log n}{n}$ by Pósa [96]. Often, however, in showing that $G(n,p)$ doesn't have property $\mathcal{P}$ when $p = o(p(n))$, we show that a much simpler, often local, property is violated. For example, $\log n/n$ is also the threshold for the existence of an isolated vertex, and in particular, for $p = o(\log n/n)$ the random graph $G(n,p)$ contains an isolated vertex and hence cannot be Hamiltonian. This suggests that Hamiltonicity in $G(n,p)$ at least for some range of $p = o(\log n/n)$ may be prevented by the existence of an isolated vertex, and in fact the "essential" threshold, modulo this local restriction, may lie elsewhere. To capture this essential threshold, we consider the randomly perturbed graph model introduced by Bohman, Frieze and Martin [13], which consists of the union of a graph G of linear minimum degree with a random set of m edges on n vertices, also known as $G(n,m)$ in the literature. In [13] the authors showed that for every $\alpha > 0$ there exists $c(\alpha) > 0$

such that the union of any n-vertex G of minimum degree αn with $G(n, cn)$ a.a.s. contains a Hamilton cycle. This result can be interpreted as "forbidding isolated vertices pushes the threshold for Hamiltonicity down from $\log n/n$ to $1/n$".

The second application of our absorbing method is in determining the threshold for existence of H-tilings in randomly perturbed graphs of the form $G \cup G(n,p)$. This threshold is related to known thresholds for perfect H-tilings in $G(n,p)$, and can be expressed in terms of the *maximum 1-density* of H defined as

$$m_1(H) = \max\left\{\frac{e(H')}{|H'|-1} : H' \subseteq H, |H'| \geqslant 2\right\}$$

In particular, we give a short proof of (a strengthening of) the following theorem.

Theorem 4.11 (Balogh, Treglown, Wagner [8])**.** *Let H be a fixed graph with h vertices and at least one edge. For every $\alpha > 0$ there exists $c(\alpha, H) > 0$ such that if G is an n-vertex graph with $\delta(G) \geqslant \alpha n$, and $p \geqslant cn^{-1/m_1(H)}$, then a.a.s. $G \cup G(n,p)$ contains a perfect H-tiling.*

Using our absorbing strategy, we show that above the same range of p universally guarantees a perfect H-tiling *for all* dense graphs G. Some advantages of our approach over the proof from [8] are that our argument is significantly shorter, simpler, and avoids using the Regularity Lemma.

Theorem 4.12 (Nenadov, Pehova [93])**.** *Let H be a fixed graph with h vertices and at least one edge. For every $\alpha > 0$ there exists $c(\alpha, H) > 0$ such that if $p \geqslant cn^{-1/m_1(H)}$, then a.a.s. $G \cup G(n,p)$ contains a perfect H-tiling for every n-vertex graph G with $\delta(G) \geqslant \alpha n$.*

Before we prove this theorem, let us briefly discuss the excluded case $r = \ell$ in Theorem 4.7. The required minimum degree if $\alpha_r(G) = o(n)$ is clearly at most as large as if we would only know $\alpha_{r-1}(G) = o(n)$. However, the disjoint union of two cliques $K_{n/2-1} \cup K_{n/2+1}$ for even n or $K_{\lfloor n/2 \rfloor} \cup K_{\lceil n/2 \rceil}$ for odd n doesn't contain a perfect K_r-tiling for any $r \geqslant 3$ and has independence number $\alpha_\ell = 2\ell - 2$ for all constant $\ell \geqslant 2$, so the minimum degree threshold for $r = \ell$ in Theorem 4.7 must be at least $n/2$. In spirit, however, the minimum degree required from G whenever $\alpha_r(G) = o(n)$ should also be $o(n)$. To capture this behaviour, we turn to a slightly stronger notion of independence number.

Definition 4.13. Let H be a graph on h vertices. The r-partite H-independence number $\alpha_H^*(G)$ denotes the smallest m such that for any h pairwise disjoint vertex sets $V_1, ..., V_h \subseteq V(G)$, each of size m, there is a copy of H with one vertex in each V_i.

Note that, for example, $\alpha_\ell(G) + 1 \leqslant \ell\alpha^*_{K_\ell}(G)$.

In the proof of Theorem 4.12 we show that under the stronger assumption that $\alpha^*_{K_r}(G) = o(n)$, one can take arbitrarily small minimum degree and still be guaranteed a perfect K_r-tiling. More generally, we prove such a statement for an arbitrary graph H.

Lemma 4.14. *Let H be a fixed graph with h vertices. For any $\alpha > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that if G is a graph on $n \geqslant n_0$ vertices such that $\delta(G) \geqslant \alpha n$ and $\alpha^*_H(G) \leqslant \eta n$, then G contains a perfect H-tiling.*

Proof. As in the previous section, we need to verify (TP1) and (TP2).

Claim 4.14.1. G has the weak $(\alpha/8h^2, h)$-absorbing property.

Proof of claim. Let $S \in \binom{V(G)}{h}$ be chosen arbitrarily. We show that G contains at least $\alpha n/8h^2$ vertex-disjoint (S, h)-absorbing sets.

First, for each $s_i \in S$ choose a subset $N_i \subseteq N_G(s_i) \setminus S$ of size $\alpha n/(2h)$ such that all these sets are pairwise disjoint. From $h\alpha^*_H(G) < \alpha n/4h$ we have that $G[N_i]$ contains a family $\mathcal{H}_i$ of $\alpha n/4h^2$ vertex-disjoint copies of H. Let $V_i \subseteq N_i$ contain one vertex from each copy of H obtained in this way. In particular, $|V_i| = \alpha n/4h^2$. Note that since $V_i \subseteq N_G(s_i)$, s_i forms a copy of H with any $h-1$ vertices in any copy of H in the family $\mathcal{H}_i$.

Any copy H^t of H in G with vertex set $\{v_1, ..., v_h\}$ such that $v_i \in V_i$ for each $i \in [h]$, forms an (S, h)-absorbing set with the corresponding copies $H_i \in \mathcal{H}_i$ such that $v_i \in H_i$. Indeed, $\{H_i\}_{i\in[h]}$ (somewhat trivially) form a perfect H-tiling of $\bigcup_{i\in[h]} V(H_i)$, and H^t together with a copy of H on each of $\{s_i\} \cup (V(H_i)\setminus\{v_i\})$ form a perfect H-tiling of $\bigcup_{i\in[h]} V(H_i) \cup S$ (see Figure 4.3).

Greedily pick such disjoint traversing copies H^t of H. As long as we have at least $|V_i|/2 \geqslant \alpha^*_H(G)$ unused vertices in each V_i, that is, we have found less than $|V_i|/2$ traversing copies of H so far, the process continues. This way we construct a family of at least $|V_i|/2 \geqslant \alpha n/8h^2$ vertex-disjoint (S, h)-absorbing sets. □

From the claim and Lemma 4.2, G contains a ξ-absorbing set A of size at most $\alpha n/8h^2$ for some $\xi > 0$.

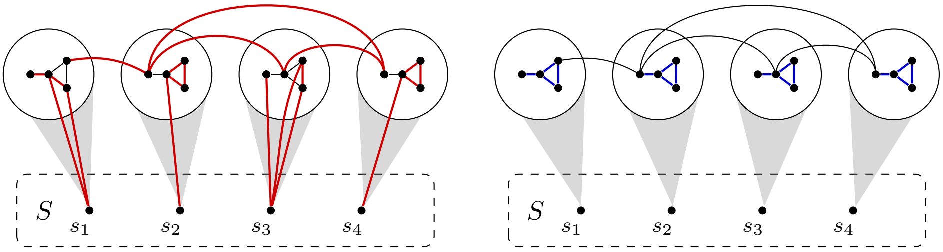


Figure 4.3: An $(S, 4)$-absorbing set for H being a triangle with a pendant edge. The red tiling covers $A_S \cup S$ and the blue tiling covers only A_S.

It remains to note that for (TP2) we may simply pick vertex-disjoint copies of H in $V(G) \backslash A$ greedily for as long as possible. The remaining set R is H-free thus it has to be smaller than $h\alpha^*_H(G) \leqslant \xi n$ (reducing η, if necessary) and such that h divides $|A| + |R|$. Since A is ξ-absorbing, $G[A \cup R]$ has a perfect H-tiling, which together with the greedy tiling constructed above, gives a perfect H-tiling of G. □

Now to prove Theorem 4.12, it suffices to show that $\alpha^*_H(G \cup G(n,p)) = o(n)$.

Proof of Theorem 4.12.

Claim 4.14.2. For any $\eta > 0$, $\alpha^*_H(G(n,p)) \leqslant \eta n$ with high probability.

Proof of claim. Let $m = \lfloor \eta n \rfloor$, and fix a collection $V_1, ..., V_h$ of pairwise disjoint subsets of $[n]$. We will use Janson's inequality (Lemma1.11) to give an upper bound on the probability that the h-partite subgraph of $G(n,p)$ induced on these sets contains no copy of H.

Let X be the set of all pairs $\{v_i, v_j\}$ of vertices such that $v_i \in V_i$ and $v_j \in V_j$ for $1 \leqslant i < j \leqslant h$. Then the h-partite subgraph $G'(m, h; p)$ of $G(n,p)$ induced on $V_1, ..., V_h$ is a random subset of X, where each element of X is included with probability p. Take A_i to be the edge-sets of all copies of H in the complete multipartite graph on vertex set $V_1 \cup ... \cup V_h$. Then, following the notation laid out in Lemma 1.11, we have

$$\mu = \Theta_H(m^h p^{e(H)}), \text{ and}$$

$$\Delta = \sum_{A_i \cap A_j \neq \emptyset} p^{2e(H) - |A_i \cap A_j|}.$$

Splitting the sum according to the graph we see in $A_i \cap A_j$, we get

$$\begin{aligned}
\Delta &= \sum_{\substack{H' \subseteq H \\ e(H') \geqslant 1}} \sum_{A_i \cap A_j = H'} p^{2e(H)-e(H')} \\
&= \sum_{\substack{H' \subseteq H \\ e(H') \geqslant 1}} O_H\left(m^{2h-|H'|} p^{2e(H)-e(H')}\right) \\
&= O_H\left(\max_{\substack{H' \subseteq H \\ e(H') \geqslant 1}} m^{2h-|H'|} p^{2e(H)-e(H')}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\mu^2}{2\Delta} &= \Omega_H\left(\frac{m^{2h} p^{2e(H)}}{\max_{\substack{H' \subseteq H \\ e(H') \geqslant 1}} m^{2h-|H'|} p^{2e(H)-e(H')}}\right) \\
&= \Omega_H\left(\min_{\substack{H' \subseteq H \\ e(H') \geqslant 1}} m^{|H'|} p^{e(H')}\right).
\end{aligned}$$

(Note that we can replace the assumption $e(H') \geqslant 1$ with $|H'| \geqslant 2$ as no edgeless H' will attain the minimum.)

Now for any $H' \subseteq H$ on at least two vertices

$$\begin{aligned}
m^{|H'|} p^{e(H')} &\geqslant m^{|H'|}\left(cn^{-1/m_1(H)}\right)^{e(H')} \\
&\geqslant m^{|H'|}\left(cn^{-\frac{|H'|-1}{e(H')}}\right)^{e(H')} \\
&\geqslant c^{e(H')}\left(\frac{\eta}{2}\right)^{|H'|} n,
\end{aligned}$$

so by choosing c appropriately, we can ensure that $\mu^2/2\Delta \geqslant hn$.

Finally, by Lemma 1.11 and the union bound we have that

$$\mathbb{P}(\alpha_H^*(G(n,p)) > \eta n) \leqslant 2^{hn}\mathbb{P}(H \not\subseteq G'(m,h;p)) \leqslant 2^{hn} e^{-\mu^2/2\Delta} \leqslant 2^{hn} e^{-hn} \to 0,$$

as required. $\square$

With the above claim at hand, it remains to note that $\alpha_H^*(G \cup G(n,p)) \leqslant \alpha_H^*(G(n,p)) \leqslant \eta n$ and $\delta(G \cup G(n,p)) \geqslant \delta(G) \geqslant \alpha n$, so with high probability $G \cup G(n,p)$ satisfies the assumptions of Lemma 4.14 for all G such that $\delta(G) \geqslant \alpha n$. The theorem now follows directly from Lemma 4.14. $\square$

Chapter 5

A Sidorenko-type condition for quasi-randomness in permutations

A combinatorial object is said to be quasi-random if it looks as if it was generated at random in a fundamental way. The theory of *quasirandom graphs* can be traced back to the work of Rödl [100], Thomason [108] and Chung, Graham and Wilson [22] from the 1980s, who showed that several properties of random graphs involving subgraph density, edge distribution and eigenvalues of the adjacency matrix are equivalent. In particular, the following $G(n,p)$-like properties in a graph with edge density p are equivalent, and can be used to define a quasi-random graph.

Theorem 5.1 (Theorem 1 in [22], simplified)**.** *Let $s \geqslant 4$ be an integer, $p \in (0,1)$, and let G be an n-vertex graph. The following are equivalent*

- $e(G) \geqslant (p+o(1))\binom{n}{2}$ *and G contains at most $(1+o(1))p^4n^4$ labelled copies of C_4.*
- *G contains $(1+o(1))n^{v(H)}p^{e(H)}$ labelled copies of every graph H.*
- $e(G) \geqslant (1+o(1))p\frac{n^2}{2}$, $\lambda_1 = (1+o(1))pn$ *and* $\lambda_2 = o(n)$, *where λ_1 and λ_2 are the two largest in absolute value eigenvalues of G.*
- *For each $S \subseteq V(G)$, $e(G[S]) = p\binom{|S|}{2} + o(n^2)$.*

The most interesting implication in Theorem 5.1 is that containing $p\binom{n}{2}$ edges and p^4n^4 labelled cycles forces a graph to contain $p^{e(H)}n^{v(H)}$ labelled copies of *any* other graph H. This shows that quasi-randomness in graphs can be captured by the densities of a finite set of subgraphs, in particular even K_2 and C_4. In fact, from Theorem 5.1 we know that among all graphs of edge-density p, the density of C_4 is minimised by $G(n,p)$. A famous conjecture of Sidorenko posits that this is the case for all bipartite graphs.

Conjecture 5.2 (Sidorenko's conjecture [39, 102])**.** *Let H be a bipartite graph. Then for all G,*

$$t(H,G) \geqslant t(K_2,G)^{e(H)}.$$

Above, $t(H,G)$ denotes the homomorphism density of H in G, as defined in Chapter 1. Sidorenko's conjecture is known to be true for trees, even cycles and complete bipartite graphs [103], the hypercube [58], and more, as well as a few more general settings such as [23, 24, 83], but a full proof is currently out of reach.

A stronger version of Sidorenko's conjecture, due to Skokan [105], suggests that the random graph is the unique graph which attains this bound.

Conjecture 5.3 (Forcing conjecture [105])**.** *Let G, H be graphs such that H is bipartite but not a tree. Then for every $p \in (0,1)$, if $t(K_2,G) = (1+o(1))p$ and $t(H,G) = (1+o(1))p^{e(H)}$, then G is quasi-random in the sense that $t(F,G) = (1+o(1))p^{e(F)}$ for any graph F.*

In light of Theorem 5.1 for general p, this is equivalent to saying roughly that among graphs G with at least $(1+o(1))p\binom{n}{2}$ edges, the homomorphism density $t(H,G)$ for bipartite non-trees H is minimised when G is quasi-random.

Results similar to Theorem 5.1 have been obtained for other types of combinatorial objects, for example groups [47], hypergraphs [18, 46], set systems [19], subsets of integers [21] and tournaments [14, 20, 25]. A lot of work in the area was carried out in the late 80s and early 90s. For permutations, however, it was not known until much later whether quasi-randomness can be characterised by a finite set of densities. Recall the appropriate definitions of permutation density from Section 1.4.2.

Theorem 5.4 (Král', Pikhurko [76])**.** *Let $\{\pi_n\}$ be a sequence of permutations such that $|\pi_n| \to \infty$. If*

$$d(\sigma,\pi_i) \to \frac{1}{4!} \text{ for all } \sigma \in \mathcal{S}_4,$$

then

$$d(\sigma,\pi_i) \to \frac{1}{|\sigma|!} \text{ for all } \sigma \in \mathcal{S}.$$

The above theorem asserts that if the limit densities of all 4-permutations in a sequence are equal to $1/4!$, then the sequence is quasi-random in the stronger sense that all other permutations eventually become equally likely. Hence, it is natural to ask whether it is possible to replace the set of all 4-permutations in the statement of Theorem 5.4 with a smaller set.

Definition 5.5. A sequence $\{\pi_n\}$ of permutations such that $|\pi_n| \to \infty$ is said to be *quasi-random* if

$$d(\sigma, \pi_i) \to \frac{1}{|\sigma|!} \text{ for all } \sigma \in \mathcal{S}.$$

In light of the convergence and permutation limit definitions given in Section 1.4.2, we note that in our context it makes sense to talk of *the* quasi-random permutation sequence as every quasi-random permutation sequence converges to the uniform measure. In fact, Definition 5.5 is equivalent to

Definition 5.5*. A sequence $\{\pi_n\}$ of permutations such that $|\pi_n| \to \infty$ is said to be *quasi-random* if it converges to the uniform measure.

Definition 5.6. Let $k \in \mathbb{N}$. A set $S \subseteq \mathcal{S}_k$ is said to be *quasirandom-forcing* if any sequence $\{\pi_n\}$ of permutations with $|\pi_n| \to \infty$ is quasi-random if and only if

$$d(\sigma, \pi_n) \to \frac{1}{k!} \text{ for all } \sigma \in S.$$

In limit language, this definition is equivalent to

Definition 5.6*. Let $k \in \mathbb{N}$ and let μ be a permuton. A set $S \subseteq \mathcal{S}_k$ is said to be *quasirandom-forcing* if

$$d(\sigma, \mu) = \frac{1}{k!} \text{ for all } \sigma \in S$$

if and only if μ is the uniform measure.

In this language, Theorem 5.4 can be restated as "$\mathcal{S}_4$ is quasirandom-forcing". In this chapter, we show that several 8-element subsets of $\mathcal{S}_4$ have this property. In fact, the sets $S \subseteq \mathcal{S}_4$ that we identify have the stronger property that fixing the sum of densities of elements of S is enough to force quasi-randomness; i.e., it is not necessary to fix the density of each individual element of S.

Definition 5.7. Let $k \in \mathbb{N}$. A set $S \subseteq \mathcal{S}_k$ is said to be *sum-forcing* if any sequence $\{\pi_n\}$ of permutations with $|\pi_n| \to \infty$ is quasi-random if and only if

$$d(S, \pi_n) := \sum_{\sigma \in S} d(\sigma, \pi_n) \to \frac{|S|}{k!}.$$

The corresponding limit version of this definition is

Definition 5.7*. Let $k \in \mathbb{N}$ and let μ be a permuton. A set $S \subseteq \mathcal{S}_k$ is said to be *sum-forcing* if

$$d(S,\mu) := \sum_{\sigma \in S} d(\sigma,\mu) = \frac{|S|}{k!}$$

if and only if μ is the uniform measure.

This stronger property was studied in the context of statistics by Bergsma and Dassios [10] who also identified the first of the 8-element sets listed in Theorem 5.8 below. We take this further and give a complete characterisation of all sum-forcing sets of 4-permutations.

Theorem 5.8 (Chan, Král', Noel, Pehova, Sharifzadeh, Volec [15])**.** *The sum-forcing subsets of $\mathcal{S}_4$ are*

- $\{1234, 1243, 2134, 2143, 3412, 3421, 4312, 4321\}$,
- $\{1234, 1432, 2143, 2341, 3214, 3412, 4123, 4321\}$,
- $\{1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231\}$,
- $\{1324, 1423, 2314, 2413, 3142, 3241, 4132, 4231\}$,
- $\{1234, 1243, 1432, 2134, 2143, 2341, 3214, 3412, 3421, 4123, 4312, 4321\}$, *and*

their complements.

Note that any sum-forcing set is quasirandom-forcing, so our main theorem implies that each of the above sets is quasirandom-forcing. Unfortunately, our characterisation does not give information on the existence (or lack thereof) of smaller quasirandom-forcing sets.

In the process of characterising the sum-forcing sets listed in Theorem 5.8, we prove stronger, Sidorenko-type statements on the forcing properties of the listed sets (see Conjectures 5.2 and 5.3). More specifically, for each S listed in Theorem 5.8, we prove that the quasi-random permutation sequence is the unique minimiser or maximiser of the limit $\lim_{n\to\infty} \sum_{\sigma \in S} d(\sigma, \pi_n)$, thus showing that phenomena such as the widely-believed-to-be-true Sidorenko's conjecture appear in the world of combinatorial permutations as well.

The rest of this chapter is organised as follows. In Section 5.1 we show that the sets listed in Theorem 5.8 are sum-forcing. In Section 5.2 we give a simple combinatorial condition which implies that a set S is not sum-forcing. In Section 5.3 we tie the loose ends by discussing the classification of all remaining subsets of $\mathcal{S}_4$, thus completing the proof of Theorem 5.8.

5.1 Sum-forcing sets

In this section, we prove that the sets listed in Theorem 5.8 are sum-forcing. Note that if $S \subseteq \mathcal{S}_4$ is sum-forcing, then so is its complement, so it suffices to give proofs only for the five sets we have explicitly stated.

The proof is based on flag algebra calculations, which we present further in this section. We start with the following lemma, which says that a permuton is uniform if and only if all rectangles with endpoints in its support have the correct measure.

Lemma 5.9. *Let μ be a permuton. If it holds that*

$$\mu\left([\min\{x_1, x_2\}, \max\{x_1, x_2\}] \times [\min\{y_1, y_2\}, \max\{y_1, y_2\}]\right) = |x_2 - x_1| \cdot |y_2 - y_1|$$

for all points $(x_1, y_1), (x_2, y_2) \in supp\,(\mu)$, then μ is the uniform measure.

Proof. First, let us show that $\operatorname{supp}(\mu) = [0, 1]^2$.

Claim 5.9.1. $[0, 1]^2 \backslash (0, 1)^2 \subseteq \operatorname{supp}(\mu)$.

Proof of claim. Suppose that $\operatorname{supp}(\mu)$ does not contain the whole boundary of $[0, 1]^2$. Since $\operatorname{supp}(\mu)$ is closed, it is enough to consider the points distinct from the four corners. By symmetry, it suffices to consider the following two cases.

Case 1. There exists $x \in (0, 1)$ such that $(x, 0) \notin \operatorname{supp}(\mu)$ but $(x, 1) \in \operatorname{supp}(\mu)$.

As $\operatorname{supp}(\mu)$ is closed

$$\mu\left([x - \varepsilon, x + \varepsilon] \times [0, \varepsilon]\right) = 0$$

for some $\varepsilon > 0$. Let $y' \in [\varepsilon, 1]$ be the infimum among all reals such that $(x', y') \in \operatorname{supp}(\mu)$ for some $x' \in (x - \varepsilon, x)$. If there was no such y', then the measure of the rectangle $[x - \varepsilon, x] \times [0, 1]$ would be zero, which contradicts uniform marginals. Since $\operatorname{supp}(\mu)$ is a closed set, there exists $x' \in [x - \varepsilon, x]$ such that $(x', y') \in \operatorname{supp}(\mu)$.

If $x' < x$, the rectangle $[x', x] \times [y', 1]$ has measure $(x - x')(1 - y')$, while our choice of y' implies that $\mu([x', x] \times [0, y']) = 0$. Consequently, the measure of the rectangle $\mu([x', x] \times [0, 1]) = (x - x')(1 - y') < x - x'$, contradiction.

Otherwise, if $x' = x$, the choice of y' implies that there exist $y'' \in (y', 1]$ and $x'' \in (x-\varepsilon, x)$ such that $(x'', y'') \in \operatorname{supp}(\mu)$. Since $\mu([x'', x] \times [y'', 1]) = (x-x'')(1-y'')$ and $\mu([x'', x] \times [y', y'']) = (x-x'')(y''-y')$, we have that $\mu([x'', x] \times [y', 1]) = (x-x'')(1-y')$. On the other hand, the choice of y' implies that $\mu([x'', x] \times [0, y']) = 0$, which yields that the measure of the rectangle $[x'', x] \times [0, 1]$ is less than $x - x''$, contradiction.

Case 2. There exists $x \in (0, 1)$ such that $(x, 0) \notin \operatorname{supp}(\mu)$ and $(x, 1) \notin \operatorname{supp}(\mu)$. As in Case 1, we have that

$$\mu([x-\varepsilon, x+\varepsilon] \times [0, \varepsilon]) = 0 \text{ and } \mu([x-\varepsilon, x+\varepsilon] \times [1-\varepsilon, 1]) = 0$$

for some $\varepsilon > 0$. Let $y_1 \in [0, 1]$ be the infimum among all reals such that $(x_1, y_1) \in \operatorname{supp}(\mu)$ for some $x_1 \in (x-\varepsilon, x+\varepsilon)$. If there was no such y_1, then the measure of the rectangle $[x-\varepsilon, x+\varepsilon] \times [0, 1]$ would be zero, which contradicts uniform marginals. Since $\operatorname{supp}(\mu)$ is a closed set, there exists $x_1 \in [x-\varepsilon, x+\varepsilon]$ such that $(x_1, y_1) \in \operatorname{supp}(\mu)$. Note that $y_1 \in [\varepsilon, 1-\varepsilon]$. Similarly, let $y_2 \in [0, 1]$ be the supremum among all reals such that $(x_2, y_2) \in \operatorname{supp}(\mu)$ for some $x_2 \in (x-\varepsilon, x+\varepsilon)$ (again note that $y_2 \in [\varepsilon, 1-\varepsilon]$), and let $x_2 \in [x-\varepsilon, x+\varepsilon]$ be such that $(x_2, y_2) \in \operatorname{supp}(\mu)$.

We first consider the case that $x_1 \neq x_2$; without loss of generality, we can assume that $x_1 < x_2$. Again we have that $\mu([x_1, x_2] \times [y_1, y_2]) = (x_2 - x_1)(y_2 - y_1)$, and the choices of y_1 and y_2 imply that the measure of each of the rectangles $[x_1, x_2] \times [0, y_1]$ and $[x_1, x_2] \times [y_2, 1]$ is zero. It follows that the measure of the rectangle $[x_1, x_2] \times [0, 1]$ is $(x_2 - x_1)(y_2 - y_1) < x_2 - x_1$, which is a contradiction.

And finally, if $x_1 = x_2$, since the rectangle $[x-\varepsilon, x+\varepsilon] \times [0, 1]$ has measure 2ε by uniform marginals, there exists $x_3 \in [x-\varepsilon, x+\varepsilon]$, $x_3 \neq x_1$, and $y_3 \in (y_1, y_2)$ such that $(x_3, y_3) \in \operatorname{supp}(\mu)$. By symmetry, we can assume that $x_1 < x_3$. We have that $\mu([x_1, x_3] \times [y_1, y_3]) = (x_3 - x_1)(y_3 - y_1)$ and $\mu([x_1, x_3] \times [y_3, y_2]) = (x_3 - x_1)(y_2 - y_3)$, and since both $[x_1, x_3] \times [0, y_1]$ and $[x_1, x_3] \times [y_2, 1]$ have measure zero, we conclude that $\mu([x_1, x_3] \times [0, 1]) = (x_3 - x_1)(y_2 - y_1) < x_3 - x_1$, a contradiction. $\square$

Now we can use the boundary to fill the whole of $[0, 1]^2$.

Claim 5.9.2. $(0, 1)^2 \subseteq \operatorname{supp}(\mu)$.

Proof of claim. Suppose, for contradiction, that $(x, y) \in (0, 1)^2$ is not in $\operatorname{supp}(\mu)$. Then $[x-\varepsilon, x+\varepsilon] \times [y-\varepsilon, y+\varepsilon] \subseteq \overline{\operatorname{supp}(\mu)}$ for some $\varepsilon > 0$.

Let y_1 be the supremum among all reals in $[0, y-\varepsilon]$ such that $(x_1, y_1) \in \operatorname{supp}(\mu)$ for some $x_1 \in (x-\varepsilon, x+\varepsilon)$, and let y_2 be the infimum among all reals in $[y+\varepsilon, 1]$ such that $(x_2, y_2) \in \operatorname{supp}(\mu)$ for some $x_2 \in (x-\varepsilon, x+\varepsilon)$. Further, let $x_1, x_2 \in [x-\varepsilon, x+\varepsilon]$ be such that $(x_1, y_1) \in \operatorname{supp}(\mu)$ and $(x_2, y_2) \in \operatorname{supp}(\mu)$. Note that y_1 can be 0 and y_2 can be 1, and $y_2 - y_1 \geqslant 2\varepsilon$.

We first consider the case that $x_1 \neq x_2$; without loss of generality, we can assume that $x_1 < x_2$. Since the boundary of the square $[0,1]^2$ is contained in $\operatorname{supp}(\mu)$, the measures of the rectangles $[x_1, x_2] \times [0, y_1]$ and $[x_1, x_2] \times [y_2, 1]$ are $(x_2 - x_1)y_1$ and $(x_2 - x_1)(1 - y_2)$, respectively. On the other hand, the choice of y_1 and y_2 implies that $\mu([x_1, x_2] \times [y_1, y_2]) = 0$. Consequently, the measure of the rectangle $[x_1, x_2] \times [0, 1]$ is $(x_2 - x_1)(1 - y_2 + y_1) < x_2 - x_1$, contradiction.

On the other hand, if $x_1 = x_2$, take x_3 to be any point in the interval $[x - \varepsilon, x + \varepsilon]$ distinct from $x_1 = x_2$. By symmetry, we can assume that $x_1 < x_3$. Again, since the boundary of the square $[0,1]^2$ is contained in $\operatorname{supp}(\mu)$, it follows that the measures of the rectangles $[x_1, x_3] \times [0, y_1]$ and $[x_1, x_3] \times [y_2, 1]$ are $(x_3 - x_1)y_1$ and $(x_3 - x_1)(1 - y_2)$, respectively, and the choice of y_1 and y_2 yields that $\mu([x_1, x_3] \times [y_1, y_2]) = 0$. We obtain that the measure of the rectangle $[x_1, x_3] \times [0, 1]$ is $(x_3 - x_1)(1 - y_2 + y_1) < x_3 - x_1$, which is again a contradiction. $\square$

Consequently the measure of each set $[x, x'] \times [y, y']$ is equal to $(x' - x)(y' - y)$, which implies that the measure μ is the uniform measure on $[0,1]^2$. $\square$

For the rest of the section, we fix the following elements $A_1 \in \mathcal{F}_{\mathcal{S}^{12}}$ and $A_2 \in \mathcal{F}_{\mathcal{S}^{21}}$.

$$\begin{aligned}
A_1 &= (\underline{1}2\underline{3}4 - \underline{1}4\underline{3}2) + (1\underline{2}3\underline{4} - 3\underline{2}1\underline{4}) + (\underline{2}3\underline{4}1 - \underline{2}1\underline{4}3) + (4\underline{1}2\underline{3} - 2\underline{1}4\underline{3}) \\
A_2 &= (\underline{3}2\underline{1}4 - \underline{3}4\underline{1}2) + (1\underline{4}3\underline{2} - 3\underline{4}1\underline{2}) + (\underline{4}3\underline{2}1 - \underline{4}1\underline{2}3) + (4\underline{3}2\underline{1} - 2\underline{3}4\underline{1})
\end{aligned}$$

We next show that if $h^{12}_\mu(A_1) = h^{21}_\mu(A_2) = 0$ with probability one, then then μ satisfies the assumptions of Lemma 5.9. In particular, this implies that in order to show that a set $S \subseteq \mathcal{S}_4$ is sum-forcing, it suffices to prove that $h^{12}_\mu(A_1) = h^{21}_\mu(A_2) = 0$ for permutons μ satisfying $d(S, \mu) = |S|/24$.

Lemma 5.10. *Let μ be a permuton. If $h^{12}_\mu(A_1) = h^{21}_\mu(A_2) = 0$ with probability one, then*

$$\mu\left([\min\{x_1, x_2\}, \max\{x_1, x_2\}] \times [\min\{y_1, y_2\}, \max\{y_1, y_2\}]\right) = |x_2 - x_1| \cdot |y_2 - y_1|$$

for all points $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\mu)$.

Proof. We split the proof into two parts. First we show that zeroing the density of A_1

implies that the lemma holds for all points $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\mu)$ such that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. A symmetric argument (which we omit) then shows that zeroing the density of A_2 implies that the lemma holds for all points $(x_1, y_2), (x_2, y_1) \in \operatorname{supp}(\mu)$ such that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

Fix $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\mu)$ such that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ and such that $d(A_1, (\mu, (x_1, y_1), (x_2, y_2))) = 0$. Further let $(x_0, y_0) = (0, 0)$ and $(x_3, y_3) = (1, 1)$, and let

$$a_{ij} = \mu\left([x_{i-1}, x_i] \times [y_{j-1}, y_j]\right)$$

for $i, j \in [3]$ (see Figure 5.1).

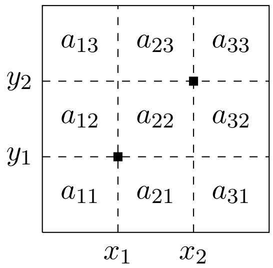


Figure 5.1: Notation from the proof of Lemma 5.10

Since $d(A_1, (\mu, (x_1, y_1), (x_2, y_2))) = 0$, we have that

$$a_{22}a_{33} - a_{23}a_{32} + a_{22}a_{11} - a_{12}a_{21} + a_{22}a_{31} - a_{21}a_{32} + a_{22}a_{13} - a_{12}a_{23} = 0 \,.$$

As μ has uniform marginals, we can rewrite this as

$$\begin{aligned}
0 &= a_{22}a_{33} - a_{23}a_{32} + a_{22}a_{11} - a_{12}a_{21} + a_{22}a_{31} - a_{21}a_{32} + a_{22}a_{13} - a_{12}a_{23} \\
&= a_{22}(a_{11} + a_{13} + a_{31} + a_{33}) - (a_{21} + a_{23})(a_{12} + a_{32}) \\
&= a_{22}(1 - (x_2 - x_1) - (y_2 - y_1) + a_{22}) - (x_2 - x_1 - a_{22})(y_2 - y_1 - a_{22}) \\
&= a_{22} - (x_2 - x_1)(y_2 - y_1)
\end{aligned}$$

Therefore, for almost all $(x_1, y_1), (x_2, y_2)$ as above, the measure of the rectangle $[x_1, x_2] \times [y_1, y_2]$ is $|x_2 - x_1| \cdot |y_2 - y_1|$, as required. It remains to show that this holds for all pairs of points in the support of μ which induce the permutation 12.

Fix $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\mu)$ such that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. If $x_1 = x_2$ or $y_1 = y_2$, then the equality holds trivially. Otherwise, let $\varepsilon_0 = \min\{x_2 - x_1, y_2 - y_1\}$ and consider $\varepsilon \in (0, \varepsilon_0/2)$. By our choice of ε, almost all pairs of points $((x_1', y_1'), (x_2', y_2')) \in B_\varepsilon((x_1, y_1)) \times B_\varepsilon((x_2, y_2))$ satisfy the equality from the statement of the lemma. It also holds that

$$\left|\mu\left([x_1, x_2] \times [y_1, y_2]\right) - \mu\left(\left[x_1', x_2'\right] \times \left[y_1', y_2'\right]\right)\right| \leqslant 4\varepsilon$$

because the measure μ has uniform marginals. Thus, by the triangle inequality

$$\left|\mu\left([x_1, x_2] \times [y_1, y_2]\right) - |x_2 - x_1| \cdot |y_2 - y_1|\right| \leqslant 8\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0/2)$, giving that $\mu([x_1, x_2] \times [y_1, y_2]) = |x_2 - x_1| \cdot |y_2 - y_1|$ for all $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\mu)$ inducing the permutation 12, as required.

A symmetric argument gives that $h_\mu^{21}(A_2) = 0$ implies the statement of the lemma for all $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\mu)$ which induce 21. $\square$

Now we are ready to prove that each of the five sets listed in Theorem 5.8 is sum-forcing.

Theorem 5.11. *Each of the sets*

$$S_1 = \{1234, 1243, 2134, 2143, 3412, 3421, 4312, 4321\},$$

$$S_2 = \{1234, 1432, 2143, 2341, 3214, 3412, 4123, 4321\},$$

$$S_3 = \{1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231\},$$

$$S_4 = \{1324, 1423, 2314, 2413, 3142, 3241, 4132, 4231\}, \text{ and}$$

$$S_5 = \{1234, 1243, 1432, 2134, 2143, 2341, 3214, 3412, 3421, 4123, 4312, 4321\}$$

is sum-forcing.

Proof. By Lemmas 5.9 and 5.10 it suffices to show that for each i, if $d(S_i, \mu) = |S_i|/4!$, then $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one. We show this in four separate claims.

Claim 5.11.1. If $d(S_1, \mu) = |S_1|/4!$, then $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one.

Proof of claim. Let B_1, C_1, D_1 and E_1 be the following four elements of $\mathcal{F}_{\mathcal{S}^{12}}$.

$$\begin{aligned}
B_1 &= (1\underline{2}3\underline{4} - 3\underline{2}1\underline{4}) + (1\underline{23}4 - 4\underline{23}1) + (1\underline{24}3 - 3\underline{24}1) + (1\underline{2}4\underline{3} - 4\underline{2}1\underline{3}) \\
C_1 &= (\underline{1}2\underline{3}4 - \underline{1}4\underline{3}2) + (1\underline{23}4 - 4\underline{23}1) + (\underline{2}1\underline{3}4 - \underline{2}4\underline{3}1) + (2\underline{13}4 - 4\underline{13}2) \\
D_1 &= (2\underline{1}4\underline{3} - 4\underline{1}2\underline{3}) + (1\underline{23}4 - 4\underline{23}1) + (2\underline{13}4 - 4\underline{13}2) + (1\underline{2}4\underline{3} - 4\underline{2}1\underline{3}) \\
E_1 &= (\underline{2}1\underline{4}3 - \underline{2}3\underline{4}1) + (1\underline{23}4 - 4\underline{23}1) + (\underline{2}1\underline{3}4 - \underline{2}4\underline{3}1) + (1\underline{24}3 - 3\underline{24}1)
\end{aligned}$$

Further, let B_2, C_2, D_2 and E_2 be the corresponding four elements of $\mathcal{F}_{\mathcal{S}^{21}}$, e.g., B_2 is the following element:

$$B_2 = (1\underline{4}3\underline{2} - 3\underline{4}1\underline{2}) + (1\underline{32}4 - 4\underline{32}1) + (1\underline{42}3 - 3\underline{42}1) + (1\underline{3}4\underline{2} - 4\underline{3}1\underline{2}).$$

Finally, let M_1 be the following (positive definite) matrix.

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

A direct computation yields that

$$\begin{aligned}
d\left(\llbracket v_1 M_1 v_1^T \rrbracket_{12} + \llbracket w_1 M_1 w_1^T \rrbracket_{21}, \mu\right) &= d\left(\frac{8}{9}\sum_{\pi \in S_1} \pi - \frac{2}{9}\sum_{\pi \in \mathcal{S}_4 \setminus S_1} \pi, \mu\right) \\
&= \frac{2}{3}\left(d(S_1, \mu) - \frac{1}{3}\right)
\end{aligned}$$

where $v_1 = (B_1, C_1, D_1, E_1)$ and $w_1 = (B_2, C_2, D_2, E_2)$. Since the matrix M_1 is positive semi-definite, it holds that $d\left(\llbracket v_1 M_1 v_1^T \rrbracket_{12}, \mu\right) \geqslant 0$ and $d\left(\llbracket w_1 M_1 w_1^T \rrbracket_{21}, \mu\right) \geqslant 0$, which implies that

$$d(S_1, \mu) \geqslant \frac{1}{3}.$$

Moreover, if equality holds, then $h_\mu^{12}(v_1 M_1 v_1^T) = h_\mu^{21}(w_1 M_1 w_1^T) = 0$ with probability one. Since all the eigenvalues of the matrix M_1 are positive, this implies that $h_\mu^{12}(B_1) = 0$, $h_\mu^{12}(C_1) = 0$, $h_\mu^{12}(D_1) = 0$, $h_\mu^{12}(E_1) = 0$, $h_\mu^{12}(B_2) = 0$, $h_\mu^{12}(C_2) = 0$, $h_\mu^{12}(D_2) = 0$ and $h_\mu^{12}(E_2) = 0$. Since $A_1 = B_1 + C_1 - D_1 - E_1$, and $A_2 = B_2 + C_2 - D_2 - E_2$, we have that $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one, as required. $\square$

Claim 5.11.2. If $d(S_2, \mu) = |S_2|/4!$, then $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one.

Proof of claim. Let F_1 and G_1 be the following two elements of $\mathcal{F}_{\mathcal{S}^{12}}$.

$$\begin{aligned}
F_1 &= (1\underline{24}3 - 3\underline{24}1) + (4\underline{13}2 - 2\underline{13}4) + (\underline{1}24\underline{3} - \underline{1}42\underline{3}) + (\underline{2}31\underline{4} - \underline{2}13\underline{4}) \\
&\quad + (\underline{13}24 - \underline{13}42) + (\underline{24}31 - \underline{24}13) + (31\underline{24} - 13\underline{24}) + (24\underline{13} - 42\underline{13}) \\
G_1 &= (\underline{12}43 - \underline{12}34) + (\underline{34}21 - \underline{34}12) + (\underline{14}32 - \underline{14}23) + (\underline{23}14 - \underline{23}41) \\
&\quad + (43\underline{12} - 34\underline{12}) + (21\underline{34} - 12\underline{34}) + (32\underline{14} - 23\underline{14}) + (14\underline{23} - 41\underline{23}) \\
&\quad + (\underline{1}43\underline{2} - \underline{1}34\underline{2}) + (\underline{3}21\underline{4} - \underline{3}12\underline{4}) + (\underline{1}32\underline{4} - \underline{1}23\underline{4}) + (\underline{2}14\underline{3} - \underline{2}41\underline{3}) \\
&\quad + (3\underline{12}4 - 4\underline{12}3) + (1\underline{34}2 - 2\underline{34}1) + (2\underline{14}3 - 3\underline{14}2) + (4\underline{23}1 - 1\underline{23}4)
\end{aligned}$$

Further, let F_2 and G_2 be the corresponding elements of $\mathcal{F}_{\mathcal{S}^{21}}$ as in Claim 5.11.1. Finally, let M_2 be the following (positive definite) matrix.

$$M_2 = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 9 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

A direct computation yields that

$$\begin{aligned}
d\left(\llbracket v_2 M_2 v_2^T \rrbracket_{12} + \llbracket w_2 M_2 w_2^T \rrbracket_{21}, \mu\right) &= d\left(\frac{8}{9}\sum_{\pi \in S_2} \pi - \frac{2}{9}\sum_{\pi \in \mathcal{S}_4 \setminus S_2} \pi, \mu\right) \\
&= \frac{2}{3}\left(d(S_2, \mu) - \frac{1}{3}\right)
\end{aligned}$$

where $v_2 = (A_1, F_1, G_1)$ and $w_2 = (A_2, F_2, G_2)$. Since the matrix M_2 is positive semi-definite, it holds that $d\left(\llbracket v_2 M_2 v_2^T \rrbracket_{12}, \mu\right) \geqslant 0$ and $d\left(\llbracket w_2 M_2 w_2^T \rrbracket_{21}, \mu\right) \geqslant 0$, which implies that

$$d(S_2, \mu) \geqslant \frac{1}{3}.$$

Moreover, if equality holds, then $h_\mu^{12}(v_2 M_2 v_2^T) = h_\mu^{21}(w_2 M_2 w_2^T) = 0$ with probability one. Since all the eigenvalues of the matrix M_2 are positive, this implies that $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one. $\square$

Claim 5.11.3. If $d(S_3, \mu) = |S_3|/4!$, then $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one.

Proof of claim. For S_3 it proves more convenient to consider its complement $\overline{S_3} = \mathcal{S}_4 \setminus S_3$. (Recall that a set $S \subseteq \mathcal{S}_4$ is sum-forcing if and only if its complement is sum-forcing.)

Let H_1, I_1, J_1 and K_1 be the following four elements of $\mathcal{F}_{\mathcal{S}^{12}}$.

$$
\begin{aligned}
H_1 &= (1\underline{2}3\underline{4} - 3\underline{2}1\underline{4}) + (\underline{2}3\underline{4}1 - \underline{2}1\underline{4}3) + (1\underline{2}4\underline{3} - 4\underline{2}1\underline{3}) + (\underline{2}4\underline{3}1 - \underline{2}1\underline{3}4) \\
I_1 &= (2\underline{1}4\underline{3} - 4\underline{1}2\underline{3}) + (\underline{1}4\underline{3}2 - \underline{1}2\underline{3}4) + (1\underline{2}4\underline{3} - 4\underline{2}1\underline{3}) + (\underline{2}4\underline{3}1 - \underline{2}1\underline{3}4) \\
J_1 &= (\underline{2}13\underline{4} - \underline{2}31\underline{4}) + (13\underline{24} - 31\underline{24}) + (3\underline{24}1 - 1\underline{24}3) + (\underline{24}13 - \underline{24}31) \\
&\quad + (4\underline{23}1 - 1\underline{23}4) + (14\underline{23} - 41\underline{23}) + (\underline{23}14 - \underline{23}41) + (\underline{2}14\underline{3} - \underline{2}41\underline{3}) \\
K_1 &= (24\underline{13} - 42\underline{13}) + (4\underline{13}2 - 2\underline{13}4) + (\underline{1}24\underline{3} - \underline{1}42\underline{3}) + (\underline{13}24 - \underline{13}42) \\
&\quad + (4\underline{23}1 - 1\underline{23}4) + (14\underline{23} - 41\underline{23}) + (\underline{23}14 - \underline{23}41) + (\underline{2}14\underline{3} - \underline{2}41\underline{3})
\end{aligned}
$$

Further, let H_2, I_2, J_2 and K_2 be the corresponding four elements of $\mathcal{F}_{\mathcal{S}^{21}}$, as in Claim 5.11.1. Finally, let M_3 be the following (positive definite) matrix.

$$
M_3 = \begin{pmatrix} 35 & 0 & 12 & 0 \\ 0 & 35 & 0 & -12 \\ 12 & 0 & 37 & 0 \\ 0 & -12 & 0 & 37 \end{pmatrix}
$$

A direct computation yields that

$$
d\left(\llbracket v_3 M_3 v_3^T \rrbracket_{12} + \llbracket w_3 M_3 w_3^T \rrbracket_{21}, \mu\right) = 16\left(d(\overline{S_3}, \mu) - \frac{2}{3}\right)
$$

where $v_3 = (H_1, I_1, J_1, K_1)$ and $w_3 = (H_2, I_2, J_2, K_2)$. Since the matrix M_3 is positive semi-definite, it holds that $d\left(\llbracket v_3 M_3 v_3^T \rrbracket_{12}, \mu\right) \geqslant 0$ and $d\left(\llbracket w_3 M_3 w_3^T \rrbracket_{21}, \mu\right) \geqslant 0$, which implies that

$$
d(\overline{S_3}, \mu) \geqslant \frac{2}{3},
$$

and, equivalently,

$$
d(S_3, \mu) \leqslant \frac{1}{3}.
$$

Moreover, if equality holds, then $h_\mu^{12}(v_3 M_3 v_3^T) = h_\mu^{21}(w_3 M_3 w_3^T) = 0$ with probability one. Since all the eigenvalues of the matrix M_3 are positive, this implies that $h_\mu^{12}(H_1) = 0$, $h_\mu^{12}(I_1) = 0$, $h_\mu^{12}(H_2) = 0$ and $h_\mu^{12}(I_2) = 0$. Since $A_1 = H_1 - I_1$ and $A_2 = H_2 - I_2$, we have that $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one, as required. $\square$

Claim 5.11.4. If $d(S_5, \mu) = |S_5|/4!$, then $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one.

Proof of claim. Let L_1, M_1, N_1 and O_1 be be the following four elements of $\mathcal{F}_{\mathcal{S}^{12}}$.

$$\begin{aligned}
L_1 &= (4\underline{2}1\underline{3} - 1\underline{2}4\underline{3}) + (4\underline{1}2\underline{3} - 2\underline{1}4\underline{3}) + (\underline{2}3\underline{4}1 - \underline{2}1\underline{4}3) \\
&\quad + (4\underline{23}1 - 1\underline{23}4) + (\underline{1}2\underline{3}4 - \underline{1}4\underline{3}2) + (3\underline{24}1 - 1\underline{24}3) \\
M_1 &= (\underline{2}1\underline{3}4 - \underline{2}4\underline{3}1) + (1\underline{23}4 - 4\underline{23}1) + (\underline{1}2\underline{3}4 - \underline{1}4\underline{3}2) \\
&\quad + (1\underline{24}3 - 3\underline{24}1) + (2\underline{13}4 - 4\underline{13}2) + (3\underline{24}1 - 1\underline{24}3) \\
N_1 &= (\underline{12}43 - \underline{12}34) + (21\underline{34} - 12\underline{34}) + (\underline{1}32\underline{4} - \underline{1}23\underline{4}) + (\underline{2}14\underline{3} - \underline{2}41\underline{3}) \\
&\quad + (2\underline{14}3 - 3\underline{14}2) + (\underline{23}14 - \underline{23}41) + (32\underline{14} - 23\underline{14}) + (\underline{1}43\underline{2} - \underline{1}34\underline{2}) \\
&\quad + (1\underline{34}2 - 2\underline{34}1) + (\underline{3}21\underline{4} - \underline{3}12\underline{4}) + (3\underline{12}4 - 4\underline{12}3) + (\underline{34}21 - \underline{34}12) \\
&\quad + (43\underline{12} - 34\underline{12}) + (\underline{14}32 - \underline{14}23) + (14\underline{23} - 41\underline{23}) + (4\underline{23}1 - 1\underline{23}4) \\
O_1 &= (\underline{1}42\underline{3} - \underline{1}24\underline{3}) + (\underline{13}42 - \underline{13}24) + (13\underline{24} - 31\underline{24}) + (\underline{24}13 - \underline{24}31) \\
&\quad + (42\underline{13} - 24\underline{13}) + (\underline{2}13\underline{4} - \underline{2}31\underline{4}) + (2\underline{13}4 - 4\underline{13}2) + (3\underline{24}1 - 1\underline{24}3)
\end{aligned}$$

Further, let L_2, M_2, N_2 and O_2 be the corresponding four elements of $\mathcal{F}_{\mathcal{S}^{21}}$, as in Claim 5.11.1. Finally, let M_4 be the following (positive definite) matrix.

$$M_5 = \begin{pmatrix} 1132 & -652 & -638 & 197 & 326 \\ -652 & 774 & 516 & -68 & -326 \\ -638 & 516 & 774 & 68 & -326 \\ 197 & -68 & 68 & 172 & 0 \\ 326 & -326 & -326 & 0 & 516 \end{pmatrix}$$

A direct computation yields that

$$d\left(\llbracket v_5 M_5 v_5^T \rrbracket_{12} + \llbracket w_5 M_5 w_5^T \rrbracket_{21}, \mu\right) = 172\left(d(S_5, \mu) - \frac{1}{2}\right)$$

where $v_5 = (A_1, L_1, M_1, N_1, O_1)$ and $w_5 = (A_2, L_2, M_2, N_2, O_2)$. Since the matrix M_5 is positive semi-definite, it holds that $d\left(\llbracket v_5 M_5 v_5^T \rrbracket_{12}, \mu\right) \geqslant 0$ and $d\left(\llbracket w_5 M_5 w_5^T \rrbracket_{21}, \mu\right) \geqslant 0$, which implies that

$$d(\overline{S_5}, \mu) \geqslant \frac{1}{2}.$$

Moreover, if equality holds, then $h_\mu^{12}(v_5 M_5 v_5^T) = h_\mu^{21}(w_5 M_5 w_5^T) = 0$ with probability one. Since all the eigenvalues of the matrix M_5 are positive, we again have that $h_\mu^{12}(A_1) = h_\mu^{21}(A_2) = 0$ with probability one. $\square$

These four claims, together with Lemma 5.9 and Lemma 5.10, show that S_1, S_2, S_3 and S_5 are sum-forcing.

Finally, note that S_4 can be obtained from S_3 by a rotation. In particular, if S_4 is not sum-forcing, then any non-uniform permuton μ such that $d(S_4, \mu) = 1/3$ can be rotated in the same way to obtain a (non-uniform) permuton μ' such that $d(S_3, \mu') = 1/3$, a contradiction. Therefore S_4 is also sum-forcing. $\square$

5.2 A condition for non-sum-forcing sets

In this section, we give a combinatorial condition for a set $S \subseteq \mathcal{S}_4$ to be non-sum-forcing. We do this by showing that if S is not symmetric in a certain sense, then we can perturb the uniform permuton to obtain a (non-uniform) permuton μ which satisfies $d(S, \mu) = |S|/24$.

Definition 5.12. For $S \subseteq \mathcal{S}_4$, we define the *cover matrix* C^S of S to be the 4×4 matrix C^S such that C^S_{ij} is the number of permutations $\pi \in S$ such that $\pi(j) = i$. (We omit the superscript if the set S is clear from the context.)
We say that S has *constant cover matrix* if $C^S_{ij} = k$ for all $i, j \in [4]^2$ and some $k \in \mathbb{N}$.

Theorem 5.13. *Let $S \subseteq \mathcal{S}_4$ be a set whose cover matrix is not constant. Then S is not sum-forcing.*

Before we prove this main theorem, we state and prove a few lemmas which allow us to construct the witnessing permuton μ, using the fact that C^S is not constant. In outline, we parametrise a family of permutons among which we will find μ which witnesses that S is not sum forcing. We then express the gradient $\nabla d(S, \mu)$ evaluated at the uniform permuton, with respect to these parameters (Equation (5.1)). In particular, we show that if C^S is not constant, then this gradient is non-zero (Lemma 5.15. Thus, perturbing the uniform permuton in opposite directions along the non-zero coordinate of the gradient $\nabla d(S, \mu)$ yields two permutons μ_1 and μ_2 such that $d(S, \mu_1) < |S|/24 < d(S, \mu_2)$. Finally, we prove an intermediate value lemma (Lemma 5.16) which shows the existence of a permuton μ such that $d(S, \mu) = |S|/24$.

We start with the definition of a step permuton. Let A be a non-negative doubly stochastic square matrix of order n, i.e., each row sum and each column sum of A

$$\begin{pmatrix} 0.3 & 0.4 & 0.2 & 0.1 \\ 0.2 & 0 & 0.3 & 0.5 \\ 0 & 0.6 & 0.2 & 0.2 \\ 0.5 & 0 & 0.3 & 0.2 \end{pmatrix} \rightsquigarrow$$

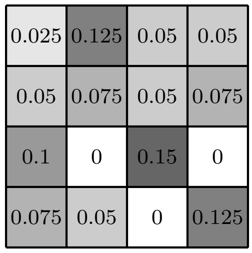


Figure 5.2: A 4×4 matrix A and the associated measures of the sixteen regions of $[0,1]^2$ where $\mu[A]$ is uniform.

is equal to one. We can associate with it a permuton $\mu[A]$ by setting

$$\mu[A](X) := \sum_{i,j\in[n]} A_{ij} \cdot n \cdot \left| X \cap \left[\frac{i-1}{n}, \frac{i}{n}\right) \times \left[\frac{j-1}{n}, \frac{j}{n}\right) \right|$$

for every Borel set $X \subseteq [0,1]^2$. We refer to permutons that can be obtained in this way from a doubly stochastic square matrix as *step permutons* (see Figure 5.2). For a step permuton, the density of a k-permutation π in $\mu[A]$ is expressible combinatorially in terms of the entries of A. Below, we use $f : [k] \nearrow [n]$ to denote a non-decreasing function $f : [k] \to [n]$.

Lemma 5.14. *Let A be a doubly stochastic square matrix of order n, and π a k-permutation. It holds that*

$$d(\pi, \mu[A]) = \frac{k!}{n^k} \sum_{f,g:[k]\nearrow[n]} \frac{1}{\prod\limits_{i\in[n]} |f^{-1}(i)|! \cdot |g^{-1}(i)|!} \times \prod_{i\in[k]} A_{f(i),g(\pi(i))}.$$

The above lemma indicates that we may associate step permutons with doubly stochastic matrices, and in particular, we can take the following combinatorial basis for the space of doubly stochastic matrices as a basis for the space of step permutons as well.

For $i, j \in [n-1]$, let B^{ij} be the matrix such that

$$B^{ij}_{i'j'} = \begin{cases} +1 & \text{if either } i' = i \text{ and } j' = j \text{ or } i' = i+1 \text{ and } j' = j+1, \\ -1 & \text{if either } i' = i \text{ and } j' = j+1 \text{ or } i' = i+1 \text{ and } j' = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

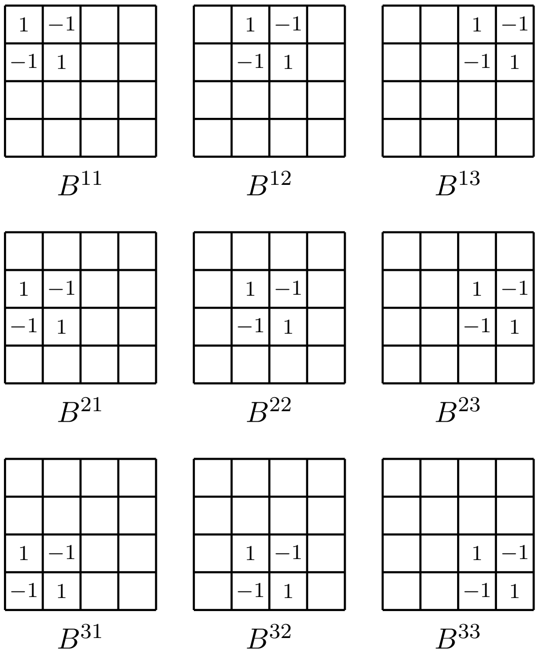


Figure 5.3: The matrices B^{ij} for $n = 4$.

In particular, there are $(n-1)^2$ matrices B^{ij}, which we may use to parametrise small perturbations of the uniform $n \times n$ step permuton. We are interested in the density of a given permutation π in a perturbation of this kind, so let us define the function $h_{\pi,n} : \mathcal{B}_n \to \mathbb{R}$ on the cube $\mathcal{B}_n := \{\mathbf{x} \in \mathbb{R}^{[n-1]^2} : \|\vec{x}\|_\infty \leqslant 1/4n\}$ around the origin as

$$h_{\pi,n}(x_{1,1}, \ldots, x_{n-1,n-1}) := d\left(\pi, \mu\left[U + \sum_{i,j\in[n-1]} x_{ij}B^{ij}\right]\right)$$

where U is the $n \times n$ matrix with all entries equal to $1/n$. In particular, every step permuton arising from a matrix with entries in $[3/4n, 5/4n]$ can be written as $\mu\left[U + \sum_{i,j\in[n-1]} x_{ij}B^{ij}\right]$ for some vector $\mathbf{x} \in \mathcal{B}_n$, and $h_{\pi,n}$ simply records the density of π in this permuton.

And finally, for a set of permutations S, we define $h_{S,n} : \mathcal{B}_n \to \mathbb{R}$ as

$$h_{S,n}(\mathbf{x}) := \sum_{\pi\in S} h_{\pi,n}(\mathbf{x}).$$

We now show that the gradient of $h_{S,n}(\mathbf{x})$ at zero can be expressed in terms of the cover matrix C^S. In particular, if C^S is non-constant, then $\nabla h_{S,n}(\mathbf{0}) \neq \mathbf{0}$ for sufficiently large n.

Lemma 5.15. *There exists $n_0 \in \mathbb{N}$ such that for any $S \subseteq \mathcal{S}_4$ and $n \geqslant n_0$ the cover matrix C^S is constant if and only if*

$$\nabla h_{S,n}(\mathbf{0}) = \mathbf{0}.$$

Proof. First, let us note that $\nabla h_{S,n}(\mathbf{0})$ can be expressed combinatorially in terms of the cover matrix $C = C^S$. It holds that

$$\begin{aligned}\frac{\partial}{\partial x_{ij}} h_{S,n}(\mathbf{0}) =& \frac{4!}{n^7} \sum_{f,g:[4]\nearrow[n]} \frac{1}{\prod\limits_{m\in[n]} |f^{-1}(m)|! \cdot |g^{-1}(m)|!} \times \left(\sum_{\substack{k\in f^{-1}(i)\\ \ell\in g^{-1}(j)}} C_{k,\ell} \right.\\ & \left. - \sum_{\substack{k\in f^{-1}(i+1)\\ \ell\in g^{-1}(j)}} C_{k,\ell} - \sum_{\substack{k\in f^{-1}(i)\\ \ell\in g^{-1}(j+1)}} C_{k,\ell} + \sum_{\substack{k\in f^{-1}(i+1)\\ \ell\in g^{-1}(j+1)}} C_{k,\ell} \right) \end{aligned} \tag{5.1}$$

for every $i, j \in [n-1]$. This follows directly from Lemma 5.14.

We start by quickly showing that C^S being constant implies that $\nabla h_{S,n}(\mathbf{0}) = \mathbf{0}$. We start by defining for each $k \in [n-1]$ the operator $\tilde{f}^{(k)}$ on non-decreasing functions $f : [4] \nearrow [n]$. Let Z be the image of f viewed as a multiset with every k replaced with $k+1$ and every $k+1$ replaced with k. Then $\tilde{f}^{(k)}$ is the unique non-decreasing function from $[4]$ to $[n]$ whose image is Z. Informally speaking, we switch the values k and $k+1$ and reorder to obtain a non-decreasing function. Note that $f = \widetilde{(\tilde{f}^{(k)})}^{(k)}$ for all f and k, and $f = \tilde{f}^{(k)}$ if and only if $|f^{-1}(k)| = |f^{-1}(k+1)|$.

We now analyse individual summands in (5.1). Fix two indices i and j, and a function $g : [4] \nearrow [n]$. If $f = \tilde{f}^{(i)}$, then the expression in the parenthesis evaluates to zero. If $f \neq \tilde{f}^{(i)}$, then the expressions for f and $\tilde{f}^{(i)}$ have opposite signs, in particular their contributions cancel out. Therefore, if C^S is constant, we have $\nabla h_{S,n}(\mathbf{0}) = \mathbf{0}$.

The remainder of this proof is devoted to showing the contrary. Suppose, for contradiction, that C^S is not constant and $\nabla h_{S,n}(\mathbf{0}) = \mathbf{0}$.

Claim 5.15.1. $C_{k,\ell} - C_{k+1,\ell} - C_{k,\ell+1} + C_{k+1,\ell+1} = 0$ for all $k, \ell \in [3]$.

Proof of claim. We start by analysing $\frac{\partial}{\partial x_{11}} h_{S,n}(\mathbf{0})$.
If $|\mathrm{Im}(f) \cap \{1,2\}| \leqslant 1$ or $|\mathrm{Im}(g) \cap \{1,2\}| \leqslant 1$, then the summands in (5.1)

corresponding to (f,g), $(\tilde{f}^{(1)},g)$, $(f,\tilde{g}^{(1)})$ and $(\tilde{f}^{(1)},\tilde{g}^{(1)})$ sum to zero. Hence, it suffices to consider summands such that $\{1,2\}\subseteq \mathrm{Im}(f)$ and $\{1,2\}\subseteq \mathrm{Im}(g)$. Note that the number of summands such that f or g is not injective is $O(n^3)$, which yields the following.

$$\begin{aligned}\frac{\partial}{\partial x_{11}}h_{S,n}(\mathbf{0}) &= \frac{4!}{n^7}\left(\sum_{\substack{f,g:[4]\nearrow[n]\\ f(1)=1,f(2)=2,|\mathrm{Im}(f)|=4\\ g(1)=1,g(2)=2,|\mathrm{Im}(g)|=4}}(C_{11}-C_{12}-C_{21}+C_{22})+O(n^3)\right)\\ &= \frac{4!}{n^7}\binom{n-2}{2}^2(C_{11}-C_{12}-C_{21}+C_{22})+O\left(\frac{1}{n^4}\right).\end{aligned}$$

If n is sufficiently large, the above expression can be zero only if $C_{11}-C_{12}-C_{21}+C_{22}=0$. An analogous argument for $i=1$ and $j=n-1$ yields that $C_{13}-C_{14}-C_{23}+C_{24}=0$, for $i=n-1$ and $j=1$ that $C_{31}-C_{32}-C_{41}+C_{42}=0$, and for $i=n-1$ and $j=n-1$ that $C_{33}-C_{34}-C_{43}+C_{44}=0$.

Next, we do a similar analysis of $\frac{\partial}{\partial x_{1\lfloor n/2\rfloor}}h_{S,n}(\mathbf{0})$.
If $|\mathrm{Im}(f)\cap\{1,2\}|\leqslant 1$ or $|\mathrm{Im}(g)\cap\{\lfloor n/2\rfloor,\lfloor n/2\rfloor+1\}|\leqslant 1$, then the summands in (5.1) corresponding to (f,g), $(\tilde{f}^{(1)},g)$, $(f,\tilde{g}^{(\lfloor n/2\rfloor)})$ and $(\tilde{f}^{(1)},\tilde{g}^{(\lfloor n/2\rfloor)})$ sum to zero. Hence, it suffices to consider summands given by f,g such that $|\mathrm{Im}(f)\cap\{1,2\}|=2$ and $|\mathrm{Im}(g)\cap\{\lfloor n/2\rfloor,\lfloor n/2\rfloor+1\}|=2$. Again, we may ignore summands corresponding to f,g which are not injective, as there are only $O(n^3)$ of them. Hence, we obtain

that

$$\frac{\partial}{\partial x_{1,\lfloor n/2\rfloor}} h_{S,n}(\mathbf{0}) = \frac{4!}{n^7}\left(\sum_{\substack{f,g:[4]\nearrow[n] \\ f(1)=1, f(2)=2, |\mathrm{Im}(f)|=4 \\ g(1)=\lfloor n/2\rfloor, g(2)=\lfloor n/2\rfloor+1, |\mathrm{Im}(g)|=4}} (C_{11}-C_{12}-C_{21}+C_{22}) \right.$$
$$+ \sum_{\substack{f,g:[4]\nearrow[n] \\ f(1)=1, f(2)=2, |\mathrm{Im}(f)|=4 \\ g(2)=\lfloor n/2\rfloor, g(3)=\lfloor n/2\rfloor+1, |\mathrm{Im}(g)|=4}} (C_{12}-C_{13}-C_{22}+C_{23})$$
$$\left. + \sum_{\substack{f,g:[4]\nearrow[n] \\ f(1)=1, f(2)=2, |\mathrm{Im}(f)|=4 \\ g(3)=\lfloor n/2\rfloor, g(4)=\lfloor n/2\rfloor+1, |\mathrm{Im}(g)|=4}} (C_{13}-C_{14}-C_{23}+C_{24}) \right)$$
$$+ O\left(\frac{1}{n^4}\right).$$

Since the first and the third sum are equal to zero, we obtain that

$$\frac{\partial}{\partial x_{1,\lfloor n/2\rfloor}} h_{S,n}(\mathbf{0}) = (C_{12}-C_{13}-C_{22}+C_{23}) \cdot \Theta\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n^4}\right).$$

Hence, if n is large enough and this partial derivative is zero, it must hold that $C_{12}-C_{13}-C_{22}+C_{23} = 0$. An analogous argument for $i = \lfloor n/2\rfloor$ and $j = 1$ yields that $C_{21}-C_{22}-C_{31}+C_{32} = 0$, for $i = n-1$ and $j = \lfloor n/2\rfloor$ that $C_{32}-C_{33}-C_{42}+C_{43} = 0$, and for $i = \lfloor n/2\rfloor$ and $j = n-1$ that $C_{23}-C_{24}-C_{33}+C_{34} = 0$.

Finally, we analyse $\frac{\partial}{\partial x_{\lfloor n/2\rfloor \lfloor n/2\rfloor}} h_{S,n}(\mathbf{0})$. As in the preceding two cases, we consider the functions $\tilde{f}^{(\lfloor n/2\rfloor)}$ and $\tilde{g}^{(\lfloor n/2\rfloor)}$ to conclude that the summands with $|\mathrm{Im}(f) \cap \{\lfloor n/2\rfloor, \lfloor n/2\rfloor+1\}| \leqslant 1$ or $|\mathrm{Im}(g) \cap \{\lfloor n/2\rfloor, \lfloor n/2\rfloor+1\}| \leqslant 1$ sum to zero. We next express the partial derivative as the sum of nine terms corresponding to injective mappings f and g with $\{\lfloor n/2\rfloor, \lfloor n/2\rfloor+1\} \subseteq \mathrm{Im}(f)$ and $\{\lfloor n/2\rfloor, \lfloor n/2\rfloor+1\} \subseteq \mathrm{Im}(g)$ (the terms are determined by the preimages of $\lfloor n/2\rfloor$ and $\lfloor n/2\rfloor+1$). Eight of these terms correspond to the sums of the entries of the cover matrix that we have already shown to be zero, which leads to the following expression for the considered partial derivative:

$$\frac{\partial}{\partial x_{\lfloor n/2\rfloor,\lfloor n/2\rfloor}} h_{S,n}(\mathbf{0}) = (C_{22}-C_{23}-C_{32}+C_{33}) \cdot \Theta\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n^4}\right).$$

Hence, if n is large enough and the partial derivative is zero, it must hold that $C_{22} - C_{23} - C_{32} + C_{33} = 0$. $\square$

From this claim it follows that C is of the form

$$C = \begin{pmatrix} a & b & c & d \\ e & b+e-a & c+e-a & d+e-a \\ f & b+f-a & c+f-a & d+f-a \\ g & b+g-a & c+g-a & d+g-a \end{pmatrix}$$

for some $a, \dots, g \in \mathbb{N}$. Since C is a cover matrix for a set S of 4-permutations, the row and column sums of C are all equal to $|S|$.
It follows that $b = c = d$ and $e = f = g$, so

$$C = \begin{pmatrix} a & b & b & b \\ e & b+e-a & b+e-a & b+e-a \\ e & b+e-a & b+e-a & b+e-a \\ e & b+e-a & b+e-a & b+e-a \end{pmatrix}.$$

Further, $b = e$ as otherwise the sum of the second row and the second column would differ. So C is be of the form

$$C = \begin{pmatrix} a & b & b & b \\ b & 2b-a & 2b-a & 2b-a \\ b & 2b-a & 2b-a & 2b-a \\ b & 2b-a & 2b-a & 2b-a \end{pmatrix}.$$

Hence, we get that $a + 3b = 7b - 3a$ and that C is constant, a contradiction. $\square$

Finally, we give a short intermediate value lemma which reduces the problem of finding a non-uniform permuton such that $d(S, \mu) = |S|/24$ to finding two permutons whose S-densities are below and above this value.

Lemma 5.16. *Let $S \subseteq \mathcal{S}_4$ and let μ_1, μ_2 be permutons such that $d(S, \mu_1) < |S|/24 < d(S, \mu_2)$. Then there exists a non-uniform permuton μ^* such that $d(S, \mu^*) = |S|/24$.*

Proof. Define a permuton μ_a for $a \in [1,2]$ as follows:

$$\mu_a(X) = (2-a)\cdot\mu_1\left(\frac{1}{2-a}\times\left(X\cap[0,2-a]^2\right)\right)$$
$$+ (a-1)\cdot\mu_2\left(\frac{1}{a-1}\times\left(X\cap[2-a,1]^2-(2-a,2-a)\right)\right),$$

where $\alpha \times X := \{\alpha\cdot x : x \in X\}$ and $X - v := \{x - v : x \in X\}$. The definition of a permuton μ_a is illustrated in Figure 5.4. Note that μ_a is μ_1 for $a = 1$ and μ_2 for

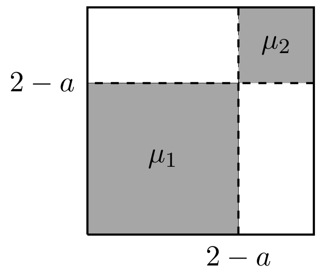


Figure 5.4: The permuton μ_a.

$a = 2$, and μ_a is not uniform for any $a \in (1,2)$.

Finally, the function $a \mapsto d(S,\mu_a)$ is continuous on the interval $[1,2]$, so by the intermediate value theorem, there exists $a^* \in (1,2)$ such that $d(S,\mu_{a^*}) = |S|/24$, as required. □

Proof of Theorem 5.13. Let $S \subseteq \mathcal{S}_4$ be such that C^S is not constant. By Lemma 5.15 there exists n such that $\nabla h_{S,n}(\mathbf{0}) \neq \mathbf{0}$. Then there exists $\varepsilon > 0$ such that for $\mathbf{x} = -\varepsilon\nabla h_{S,n}(\mathbf{0})$ and $\mathbf{y} = \varepsilon\nabla h_{S,n}(\mathbf{0})$. Let $\mu_1 = \mu\left[U + \sum_{i,j\in[n-1]} x_{ij}B^{ij}\right]$ and $\mu_2 = \mu\left[U + \sum_{i,j\in[n-1]} y_{ij}B^{ij}\right]$ with $\mathbf{x}$ and $\mathbf{y}$ as above. Then as the vectors $\mathbf{x}$ and $\mathbf{y}$ point in a direction of non-zero slope, we have that (without loss of generality)

$$d(S,\mu_1) < |S|/24 < d(S,\mu_2).$$

So, by Lemma 5.16, we can find a non-uniform μ^* such that $d(S,\mu^*) = |S|/24$. □

5.3 Classification of all sum-forcing sets

In this section, we complete the proof of Theorem 5.8 by classifying all sets $S \subseteq \mathcal{S}_4$ not covered by Theorems 5.11 and 5.13.

Proof of Theorem 5.8. By Theorem 5.11, it suffices to show that for all sets $S \subseteq \mathcal{S}_4$ not listed in the theorem statement, there exists a non-uniform permuton μ_S such that $d(S, \mu_S) = |S|/24$. Moreover, by taking complements, it's enough to consider sets of size at most 12.

By Theorem 5.13, any set S with a non-constant cover matrix is not sum-forcing, so it suffices to consider sets $S \subseteq \mathcal{S}_4$ of size at most 12 whose cover matrix is constant. By Lemma 5.15, for such S it holds that $\nabla h_{S,n}(\mathbf{0}) = \mathbf{0}$. However, if for some n the Hessian matrix of mixed partial derivatives of $h_{S,n}$ at zero has a positive and a negative eigenvalue, then we can still perturb the uniform permuton as in the proof of Theorem 5.13 to obtain the desired permuton μ_S. A list of all Hessian matrices for $n = 5$ and their eigenvalues can be found as an ancillary file in [16].

We can now simply list the remaining sets S we need to consider (up to symmetry):

- $\{1234, 2143, 3412, 4321\}$,
- $\{1342, 1423, 2314, 2431, 3124, 3241, 4132, 4213\}$,
- $\{1234, 1243, 1324, 2134, 2143, 2413, 3142, 3412, 3421, 4231, 4312, 4321\}$,
- $\{1234, 1243, 1342, 2134, 2143, 2431, 3124, 3412, 3421, 4213, 4312, 4321\}$,
- $\{1234, 1243, 1342, 2134, 2143, 2431, 3214, 3412, 3421, 4123, 4312, 4321\}$,
- $\{1234, 1243, 1432, 2134, 2341, 2413, 3142, 3214, 3421, 4123, 4312, 4321\}$,
- $\{1234, 1243, 1432, 2143, 2314, 2341, 3214, 3412, 3421, 4123, 4132, 4321\}$,
- $\{1234, 1342, 1423, 2143, 2314, 2431, 3124, 3241, 3412, 4132, 4213, 4321\}$,
- $\{1234, 1342, 1423, 2314, 2413, 2431, 3124, 3142, 3241, 4132, 4213, 4321\}$, or their complements.

All sets listed above but $\{1342, 1423, 2314, 2431, 3124, 3241, 4132, 4213\}$ contain the permutation 1234. For all those we can take μ_2 to be the limit of the sequence $\pi_n = 12 \ldots n$ (the monotone increasing permuton), and μ_1 as the step permuton given by the doubly stochastic matrices listed in [15, Appendix 5]. By Lemma 5.16 none of these sets are sum-forcing.

Finally, for $S = \{1342, 1423, 2314, 2431, 3124, 3241, 4132, 4213\}$ take μ_1 to be the limit of the sequence $\pi_n = 12 \ldots n$ (we have $d(S, \mu_1) = 0$), and μ_2 to be the step permuton given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can be shown that $d(S, \mu[A]) = \frac{25}{72} > \frac{1}{3}$, so again by Lemma 5.16 this set is also not sum-forcing. $\square$

Bibliography

[1] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy. Efficient testing of large graphs. *Combinatorica*, 20(4):451–476, 2000.

[2] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, fourth edition, 2016.

[3] B. Alspach. The wonderful Walecki construction. *Bull. Inst. Combin. Appl.*, 52:7–20, 2008.

[4] B. Alspach, J.-C. Bermond, and D. Sotteau. Decomposition into cycles. I. Hamilton decompositions. In *Cycles and rays (Montreal, PQ, 1987)*, volume 301 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 9–18. Kluwer Acad. Publ., Dordrecht, 1990.

[5] J. Balogh, P. Hu, B. Lidický, and F. Pfender. Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle. *European J. Combin.*, 52:47–58, 2016.

[6] J. Balogh, P. Hu, B. Lidický, F. Pfender, J. Volec, and M. Young. Rainbow triangles in three-colored graphs. *J. Combin. Theory Ser. B*, 126:83–113, 2017.

[7] J. Balogh, T. Molla, and M. Sharifzadeh. Triangle factors of graphs without large independent sets and of weighted graphs. *Random Structures Algorithms*, 49(4):669–693, 2016.

[8] J. Balogh, A. Treglown, and A. Z. Wagner. Tilings in randomly perturbed dense graphs. *Combin. Probab. Comput.*, 28(2):159–176, 2019.

[9] B. Barber, D. Kühn, A. Lo, and D. Osthus. Edge-decompositions of graphs with high minimum degree. *Adv. Math.*, 288:337–385, 2016.

[10] W. Bergsma and A. Dassios. A consistent test of independence based on a sign covariance related to Kendall's tau. *Bernoulli*, 20(2):1006–1028, 2014.

[11] A. Blumenthal, B. Lidický, Y. Pehova, F. Pfender, O. Pikhurko, and J. Volec. Sharp bounds for decomposing graphs into edges and triangles. *Acta Math. Univ. Comenian. (N.S.)*, 88(3):463–468, 2019.

[12] A. Blumenthal, B. Lidický, Y. Pehova, F. Pfender, O. Pikhurko, and J. Volec. Sharp bounds for decomposing graphs into edges and triangles. *Combinatorics, Probability and Computing*, page 1–17, doi: 10.1017/S0963548320000358.

[13] T. Bohman, A. Frieze, and R. Martin. How many random edges make a dense graph Hamiltonian? *Random Structures Algorithms*, 22(1):33–42, 2003.

[14] M. Bucić, E. Long, A. Shapira, and B. Sudakov. Tournament quasirandomness from local counting, Oct. 2019, arXiv:1910.09936.

[15] T. F. N. Chan, D. Kráľ, J. A. Noel, Y. Pehova, M. Sharifzadeh, and J. Volec. Characterization of quasirandom permutations by a pattern sum. *Random Structures Algorithms*, 57(4):920–939, doi: 10.1002/rsa.20956.

[16] T. F. N. Chan, D. Kral, J. A. Noel, Y. Pehova, M. Sharifzadeh, and J. Volec. Characterization of quasirandom permutations by a pattern sum, Sept. 2020, arXiv:1909.11027.

[17] F. R. K. Chung. On the decomposition of graphs. *SIAM J. Algebraic Discrete Methods*, 2(1):1–12, 1981.

[18] F. R. K. Chung and R. L. Graham. Quasi-random hypergraphs. *Random Structures Algorithms*, 1(1):105–124, 1990.

[19] F. R. K. Chung and R. L. Graham. Quasi-random set systems. *J. Amer. Math. Soc.*, 4(1):151–196, 1991.

[20] F. R. K. Chung and R. L. Graham. Quasi-random tournaments. *J. Graph Theory*, 15(2):173–198, 1991.

[21] F. R. K. Chung and R. L. Graham. Quasi-random subsets of Z_n. *J. Combin. Theory Ser. A*, 61(1):64–86, 1992.

[22] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[23] D. Conlon, J. Fox, and B. Sudakov. An approximate version of Sidorenko's conjecture. *Geom. Funct. Anal.*, 20(6):1354–1366, 2010.

[24] D. Conlon and J. Lee. Sidorenko's conjecture for blow-ups, Sept. 2018, arXiv:1809.01259.

[25] L. N. Coregliano and A. A. Razborov. On the density of transitive tournaments. *J. Graph Theory*, 85(1):12–21, 2017.

[26] K. Corradi and A. Hajnal. On the maximal number of independent circuits in a graph. *Acta Math. Acad. Sci. Hungar.*, 14:423–439, 1963.

[27] B. Csaba, D. Kühn, A. Lo, D. Osthus, and A. Treglown. Proof of the 1-factorization and Hamilton decomposition conjectures. *Mem. Amer. Math. Soc.*, 244(1154):v+164, 2016.

[28] M. Delcourt and L. Postle. Progress towards Nash-Williams' Conjecture on Triangle Decompositions, Sept. 2019, arXiv:1909.00514.

[29] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc. (3)*, 2:69–81, 1952.

[30] F. Dross. Fractional triangle decompositions in graphs with large minimum degree. *SIAM J. Discrete Math.*, 30(1):36–42, 2016.

[31] P. J. Dukes. Rational decomposition of dense hypergraphs and some related eigenvalue estimates. *Linear Algebra Appl.*, 436(9):3736–3746, 2012.

[32] P. J. Dukes. Corrigendum to "Rational decomposition of dense hypergraphs and some related eigenvalue estimates" [Linear Algebra Appl. 436 (9) (2012) 3736–3746] [mr2900749]. *Linear Algebra Appl.*, 467:267–269, 2015.

[33] P. J. Dukes and D. Horsley. On the minimum degree required for a triangle decomposition, Aug. 2019, arXiv:1908.11076.

[34] P. Erdős. Some recent results on extremal problems in graph theory. Results. In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pages 117–123 (English); pp. 124–130 (French). Gordon and Breach, New York; Dunod, Paris, 1967.

[35] P. Erdős. Some unsolved problems in graph theory and combinatorial analysis. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 97–109. Academic Press, London, 1971.

[36] P. Erdős, A. W. Goodman, and L. Pósa. The representation of a graph by set intersections. *Canadian J. Math.*, 18:106–112, 1966.

[37] P. Erdős, A. Gyárfás, and L. Pyber. Vertex coverings by monochromatic cycles and trees. *J. Combin. Theory Ser. B*, 51(1):90–95, 1991.

[38] P. Erdős and C. A. Rogers. The construction of certain graphs. *Canadian J. Math.*, 14:702–707, 1962.

[39] P. Erdős and M. Simonovits. Supersaturated graphs and hypergraphs. *Combinatorica*, 3(2):181–192, 1983.

[40] A. Ferber, E. Long, and B. Sudakov. Counting Hamilton decompositions of oriented graphs. *Int. Math. Res. Not. IMRN*, 2018(22):6908–6933, 2018.

[41] A. Frieze and M. Karoński. *Introduction to random graphs.* Cambridge University Press, Cambridge, 2016.

[42] K. Garaschuk. *Linear methods for rational triangle decompositions.* PhD thesis, University of Victoria, 2014.

[43] A. Ghouila-Houri. Une condition suffisante d'existence d'un circuit hamiltonien. *C. R. Acad. Sci. Paris*, 251:495–497, 1960.

[44] S. Glock, F. Joos, J. Kim, D. Kühn, and D. Osthus. Resolution of the Oberwolfach problem. *Acta Math. Univ. Comenian. (N.S.)*, 88(3):735–741, 2019.

[45] S. Glock, D. Kühn, A. Lo, and D. Osthus. The existence of designs via iterative absorption: hypergraph F-designs for arbitrary F, Nov. 2016, arXiv:1611.06827.

[46] W. T. Gowers. Quasirandomness, counting and regularity for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):143–184, 2006.

[47] W. T. Gowers. Quasirandom groups. *Combin. Probab. Comput.*, 17(3):363–387, 2008.

[48] T. Gustavsson. *Decompositions of Large Graphs and Digraphs with High Minimum Degree.* Matematiska institutionen, Stockholms universitet, 1991.

[49] E. Győri. On the number of edge-disjoint triangles in graphs of given size. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 267–276. North-Holland, Amsterdam, 1988.

[50] E. Győri. On the number of edge disjoint cliques in graphs of given size. *Combinatorica*, 11(3):231–243, 1991.

[51] E. Győri. Edge disjoint cliques in graphs. In *Sets, graphs and numbers (Budapest, 1991)*, volume 60 of *Colloq. Math. Soc. János Bolyai*, pages 357–363. North-Holland, Amsterdam, 1992.

[52] E. Győri and B. Keszegh. On the number of edge-disjoint triangles in K_4-free graphs. *Combinatorica*, 37(6):1113–1124, 2017.

[53] E. Győri and A. V. Kostochka. On a problem of G. O. H. Katona and T. Tarján. *Acta Math. Acad. Sci. Hungar.*, 34(3-4):321–327 (1980), 1979.

[54] R. Häggkvist. Hamilton cycles in oriented graphs. *Combin. Probab. Comput.*, 2(1):25–32, 1993.

[55] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623, 1970.

[56] H. Hàn, Y. Person, and M. Schacht. On perfect matchings in uniform hypergraphs with large minimum vertex degree. *SIAM J. Discrete Math.*, 23(2):732–748, 2009.

[57] J. Han, A. Lo, A. Treglown, and Y. Zhao. Exact minimum codegree threshold for K_4^--factors. *Combin. Probab. Comput.*, 26(6):856–885, 2017.

[58] H. Hatami. Graph norms and Sidorenko's conjecture. *Israel J. Math.*, 175:125–150, 2010.

[59] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, 2001.

[60] G. Hetyei. On the 1-factors and Hamilton circuits of complete n-colorable graphs. *Acta Acad. Paedagog. Civitate Pecs Ser. 6 Math. Phys. Chem. Tech*, 19:5–10, 1975.

[61] A. J. W. Hilton. Hamiltonian decompositions of complete graphs. *J. Combin. Theory Ser. B*, 36(2):125–134, 1984.

[62] J. Hladký, D. Král', and S. Norin. Counting flags in triangle-free digraphs. *Combinatorica*, 37:49–76, 2017.

[63] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Ráth, and R. Menezes Sampaio. Limits of permutation sequences. *J. Combin. Theory Ser. B*, 103(1):93–113, 2013.

[64] B. Jackson. Long paths and cycles in oriented graphs. *J. Graph Theory*, 5(2):145–157, 1981.

[65] S. Janson, T. Ł uczak, and A. Ruciński. An exponential bound for the probability of nonexistence of a specified subgraph in a random graph. In *Random graphs '87 (Poznań, 1987)*, pages 73–87. Wiley, Chichester, 1990.

[66] J. Kahn. Proof of a conjecture of Katona and Tarján. *Period. Math. Hungar.*, 12(1):81–82, 1981.

[67] P. Keevash. The existence of designs, Jan. 2014, arXiv:1401.3665.

[68] P. Keevash, D. Kühn, and D. Osthus. An exact minimum degree condition for Hamilton cycles in oriented graphs. *J. Lond. Math. Soc. (2)*, 79(1):144–166, 2009.

[69] P. Keevash and K. Staden. The generalised Oberwolfach problem, Apr. 2020, arXiv:2004.09937.

[70] C. Knierim and P. Su. K_r-Factors in Graphs with Low Independence Number, Nov. 2019, arXiv:1912.00230.

[71] J. Komlós, G. N. Sárközy, and E. Szemerédi. On the square of a Hamiltonian cycle in dense graphs. In *Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995)*, volume 9, pages 193–211, 1996.

[72] J. Komlós, G. N. Sárközy, and E. Szemerédi. Blow-up lemma. *Combinatorica*, 17(1):109–123, 1997.

[73] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 295–352. János Bolyai Math. Soc., Budapest, 1996.

[74] D. Král', B. Lidický, T. L. Martins, and Y. Pehova. Decomposing graphs into edges and triangles. *Combin. Probab. Comput.*, 28(3):465–472, doi: 10.1017/S0963548318000421.

[75] D. Král', L. Mach, and J.-S. Sereni. A new lower bound based on Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 48:487–498, 2012.

[76] D. Král' and O. Pikhurko. Quasirandom permutations are characterized by 4-point densities. *Geom. Funct. Anal.*, 23(2):570–579, 2013.

[77] M. Krivelevich. Triangle factors in random graphs. *Combin. Probab. Comput.*, 6(3):337–347, 1997.

[78] D. Kühn and D. Osthus. Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments. *Adv. Math.*, 237:62–146, 2013.

[79] D. Kühn and D. Osthus. Hamilton decompositions of regular expanders: applications. *J. Combin. Theory Ser. B*, 104:1–27, 2014.

[80] D. Kühn, D. Osthus, and A. Treglown. Hamilton decompositions of regular tournaments. *Proc. Lond. Math. Soc. (3)*, 101(1):303–335, 2010.

[81] M. Kwan. Almost all Steiner triple systems have perfect matchings, Nov. 2016, arXiv:1611.02246.

[82] R. Laskar and B. Auerbach. On decomposition of r-partite graphs into edge-disjoint Hamilton circuits. *Discrete Math.*, 14(3):265–268, 1976.

[83] J. L. X. Li and B. Szegedy. On the logarithimic calculus and Sidorenko's conjecture, July 2011, arXiv:1107.1153.

[84] A. Liebenau and Y. Pehova. An approximate version of Jackson's conjecture. *Combin. Probab. Comput.*, 29(6):886–899, doi: 10.1017/S0963548320000152.

[85] H. Liu, O. Pikhurko, and K. Staden. The exact minimum number of triangles in graphs with given order and size. *Forum Math. Pi*, 8:e8, 144, 2020.

[86] A. Lo and K. Markström. F-factors in hypergraphs via absorption. *Graphs Combin.*, 31(3):679–712, 2015.

[87] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam; North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29.

[88] S. McGuinness. The greedy clique decomposition of a graph. *J. Graph Theory*, 18(4):427–430, 1994.

[89] S. McGuinness. Greedy maximum-clique decompositions. *Combinatorica*, 14(3):335–343, 1994.

[90] R. Montgomery. Spanning trees in random graphs. *Adv. Math.*, 356:106793, 92, 2019.

[91] C. S. J. A. Nash-Williams. Hamiltonian lines in graphs whose vertices have sufficiently large valencies. In *Combinatorial theory and its applications, III (Proc. Colloq., Balatonfüred, 1969)*, pages 813–819, 1970.

[92] C. S. J. A. Nash-Williams. Edge-disjoint Hamiltonian circuits in graphs with vertices of large valency. In *Studies in Pure Mathematics (Presented to Richard Rado)*, pages 157–183. Academic Press, London, 1971.

[93] R. Nenadov and Y. Pehova. On a Ramsey–Turán Variant of the Hajnal–Szemerédi Theorem. *SIAM J. Discrete Math.*, 34(2):1001–1010, doi: 10.1137/18M1211970.

[94] L. L. Ng. Hamiltonian decomposition of complete regular multipartite digraphs. *Discrete Math.*, 177(1-3):279–285, 1997.

[95] O. Ore. Note on Hamilton circuits. *Amer. Math. Monthly*, 67:55, 1960.

[96] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Math.*, 14(4):359–364, 1976.

[97] L. Pyber. Covering the edges of a graph by . . .. In *Sets, graphs and numbers (Budapest, 1991)*, volume 60 of *Colloq. Math. Soc. János Bolyai*, pages 583–610. North-Holland, Amsterdam, 1992.

[98] A. A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[99] A. A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17(4):603–618, 2008.

[100] V. Rödl. On universality of graphs with uniformly distributed edges. *Discrete Math.*, 59(1-2):125–134, 1986.

[101] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *J. Combin. Theory Ser. A*, 116(3):613–636, 2009.

[102] A. Sidorenko. A correlation inequality for bipartite graphs. *Graphs Combin.*, 9(2):201–204, 1993.

[103] A. F. Sidorenko. Inequalities for functionals generated by bipartite graphs. *Diskret. Mat.*, 3(3):50–65, 1991.

[104] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

[105] J. Skokan and L. Thoma. Bipartite subgraphs and quasi-randomness. *Graphs Combin.*, 20(2):255–262, 2004.

[106] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arith.*, 27:199–245, 1975.

[107] E. Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[108] A. Thomason. Pseudorandom graphs. In *Random graphs '85 (Poznań, 1985)*, volume 144 of *North-Holland Math. Stud.*, pages 307–331. North-Holland, Amsterdam, 1987.

[109] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs. *J. Combin. Theory Ser. A*, 119(7):1500–1522, 2012.

[110] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II. *J. Combin. Theory Ser. A*, 120(7):1463–1482, 2013.

[111] Z. Tuza. Unsolved Combinatorial Problems, Part I. BRICS Lecture Series LS-01-1, 2001.

[112] R. Yuster. Asymptotically optimal K_k-packings of dense graphs via fractional K_k-decompositions. *J. Combin. Theory Ser. B*, 95(1):1–11, 2005.

[113] R. Yuster. Integer and fractional packing of families of graphs. *Random Structures Algorithms*, 26(1-2):110–118, 2005.