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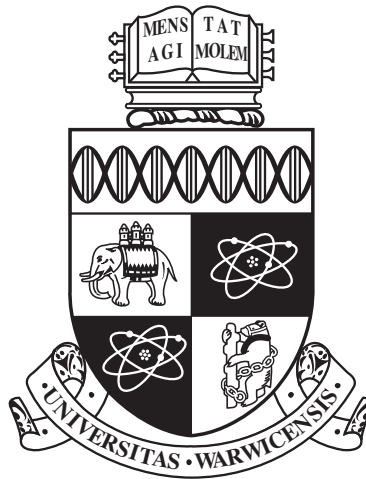
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Applications of Optimal Stopping in Behavioural Finance

by

Jonathan Muscat

Thesis

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Declarations

I declare that I have written and developed this PhD thesis entitled ”**Applications of optimal stopping in behavioural finance**” completely by myself, under the supervision of Prof. Vicky Henderson, for the degree of Doctor in Philosophy in Statistics. I have not used sources or means without declaration in the text. I also confirm that this thesis has not been submitted for a degree at any other university. During my PhD I have written the article: **Partial liquidation under reference-dependent preferences** (Henderson and Muscat [2020]), in collaboration with my supervisor Prof. Vicky Henderson. This article has been peer reviewed and published in the journal titled Finance and Stochastics.

Abstract

We study a set of optimal stopping problems arising from three branches from within the field of Behavioural Finance. We first consider a problem of an investor having S-shaped reference-dependent preferences who wishes to liquidate a divisible asset position at times of their choosing. We prove that it may be optimal for the investor to partially liquidate the asset at distinct price thresholds above the reference level rather than liquidate all the position in one block sale.

In the second part of our study we consider problems describing the behaviour of an investor who experiences realisation utility whenever they realise gains or losses after liquidating an asset. We build upon the work of Barberis and Xiong [2012] and propose two problems, which we solve by applying the methodology of Dayanik and Karatzas [2003]. The first part considers an agent whose preferences are described by the classical Cumulative Prospect Theory S-shaped Utility proposed by Tversky and Kahneman [1992]. The second problem extends upon the first, and we propose a new utility function under which the agent does not only compare their gains relative to the reference level linearly but also proportionally. As part of the solutions presented for these two problems, we provide explicit conditions differentiating between the optimal strategies arising under different parameter cases.

In the final part of our study, we consider models of optimal stopping with regret. We provide a continuous time re-formulation and extension to the dynamic model presented in Strack and Viefers [2015]. This model describes an agent whose preference structure incorporates a Regret term, where Regret is defined in the context of the work of Loomes and Sugden [1982].

Chapter 1

Introduction

Expected utility Theory remains a very common hypothesis when studying investor preferences. Whilst this hypothesis is at the basis of some prominent works in the field of Mathematical Finance, it is very well known that this theory fails to explain various behavioral phenomena which are observable when one considers real world data. Various theories have been proposed in recent decades with the motivation of explaining some of these behavioral anomalies. Our work in this thesis considers problems motivated by ideas proposed in two such theories; specifically Prospect Theory and Regret Theory. The problems we study as part of our work take the form of optimal stopping problems; where the aim is to find an optimal time to maximise an expected reward or minimise an expected cost. Such problems are well studied in the literature and there are various methods of solution which can be applied, depending on the problem. Peskir and Shiryaev [2006] brings together a number of these classical approaches previously described in literature.

The stochastic process at the base of each of the problems we consider is a one-dimensional time-homogeneous diffusion. In view of this, our solution approach for the problems described in Chapters 3 and 4 follows the methodology outlined in Dayanik and Karatzas [2003]. This approach extends the ideas of Dynkin and Yushkevich [1969] for solving optimal stopping problems driven by a Brownian Motion to cover the case of one-dimensional diffusion processes. The value of this solution method is predominantly the fact that the solution can be explained and derived geometrically in terms of concave majorants. An overview of the work presented in Dayanik and Karatzas [2003] is outlined in Chapter 2.

In Chapter 3 we consider the problem of an investor with S-shaped reference-dependent preferences who wants to sell a divisible asset at times of their choosing in the future. Utility is derived from gains and losses relative to a reference level. Our

main finding in this chapter is that under a certain model specification; specifically the assumptions considered by Kyle et al. [2006], the investor's optimal strategy takes the form of partial sales at different time points. This result is derived by applying a result discussed at the start of the chapter which allows for a multiple optimal stopping problem to be viewed as a sequence of standard optimal stopping problems. A version of the work presented in Chapter 3 has been published in the journal of Finance and Stochastics (see Henderson and Muscat [2020]).

Reference dependent preferences are also at the basis of the problems considered in Chapter 4. In this chapter we extend upon the framework first described in Barberis and Xiong [2012], where they propose the concept of Realisation Utility. This concept stems from the observation that investors do not solely derive utility from consumption or final wealth but also from the act of realising gains and losses when selling assets, where the amount of utility derived depends on the magnitude of the realised gain or loss. The model proposed by Barberis and Xiong [2012] considers an investor who invests all their wealth in a risky asset whose price dynamics are modelled by a Geometric Brownian Motion. The investor's objective is to decide when to sell the underlying, thus receiving realisation utility at the moment of sale. The agent then instantaneously re-invests all the proceeds after transaction costs in a risky asset with price dynamics equivalent to the asset they invested in a priori, thus essentially restarting the game. Whilst Barberis and Xiong [2012] assume that the investor's underlying S-shaped utility function to be a piecewise linear function, in Section 4.2 we consider a similar problem as the one discussed above for an investor whose preferences are described by the classical Cumulative Prospect Theory S-shaped Utility proposed in Tversky and Kahneman [1992]. In Section 4.3 we propose a new extension of the model described in Section 4.2. A new utility function is proposed in which gains and losses are compared to the reference level not only linearly but also proportionally. This utility was inspired by the structure of the optimal strategy obtained in Section 4.2.

Finally, in Chapter 5 we present a continuous time model inspired by the framework discussed in Strack and Viefers [2015]. The problem we consider describes an optimal liquidation problem for an investor whose preferences incorporate a regret term. Regret is formulated as a penalisation to the agent's utility which depends on the ex-post maximum of the risky asset's price process. In solving the arising optimal stopping problem, We adopt a different solution approach than that considered in Chapters 3 and 4. We start by proposing the structure of the optimal stopping time, inspired by the solution in discrete time and then verify the optimality of this stopping time.

For each of the chapters mentioned above, a review of the relevant literature is first discussed. A concise formulation of the problem is then described and the solution is then presented and analysed.

Chapter 2

Optimal Stopping for One-Dimensional Diffusions

2.1 Introduction

In this chapter we will review the methodology in Dayanik and Karatzas [2003] for solving optimal stopping problems for one dimensional diffusion processes. Dayanik and Karatzas [2003] characterize the value function geometrically in terms of concave majorants of an aptly defined transformation of the corresponding reward function.

This characterization of the value function is primarily due to Dynkin and Yushkevich [1969] where a framework is discussed to solve optimal stopping problems where the underlying process is a Brownian Motion. This work is extended by Dayanik and Karatzas [2003] to cover one-dimensional regular diffusion processes.

The connection between one-dimensional regular diffusions and a Brownian Motion through the diffusion's scale function is standard. This equivalence is in fact used by Dayanik and Karatzas [2003] to provide an alternative characterisation of the value function to the well-known one in terms of excessive (or harmonic) functions. This is then used to determine the value function in terms of a non-negative concave majorant of an aptly defined function. This result is very useful since concave-majorants are in general, geometrically easy to find.

Given that our work will consist of optimal stopping problems which are discounted, we will omit the authors' discussion of non-discounted problems. However we will briefly discuss how every discounted optimal stopping problem of a one-dimensional diffusion process can be re-formulated in terms of a non-discounted optimal stopping problem in terms of a Brownian Motion.

This approach is different from the Boundary-Value approach discussed in

detail in Peskir and Shiryaev [2006], where the value function is characterised as a solution to a system of differential equations.

2.2 Set-up and Some Important Results

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a Brownian Motion $(W_t : t \geq 0)$, and consider a one dimensional diffusion process X with state space $\mathcal{I} \subseteq \mathbb{R}$ and dynamics:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \quad (2.1)$$

for some Borel functions $\mu : \mathcal{I} \rightarrow \mathbb{R}$ and $\sigma : \mathcal{I} \rightarrow \mathbb{R}^+$. We assume that \mathcal{I} is an interval with endpoints $-\infty \leq a < b \leq +\infty$, and that (2.1) satisfies $X_0 = x$ and has a weak solution which is unique in the sense of the probability law. As discussed in Dayanik and Karatzas [2003], this is guaranteed if $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy:

$$\int_{(x-\epsilon, x+\epsilon)} \frac{1 + |\mu(y)|}{\sigma^2(y)} dy < \infty$$

for some $\epsilon > 0$, at every $x \in \text{int}(\mathcal{I})$. The existence of a weak solution to the SDE in (2.1) guarantees that X is regular in (a, b) ; that is, given any $x, y \in \mathcal{I}$, X hits y with positive probability when starting at x .

Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration of X , $\rho \in \mathbb{R}^+$ be a constant and let $h(\cdot)$ be a Borel function such that $\mathbb{E}_x[e^{-\rho\tau}h(X_\tau)]$ is well-defined for every \mathbb{F} -stopping time τ and $x \in \mathbb{F}$. We will refer to the function $h(\cdot)$ as the *reward function* and $\rho \geq 0$ as the *discount factor*. By convention we let $f(X_\tau(\omega)) = 0$ over the set $\{\tau = \infty\}$, for any Borel function f .

In this chapter we provide an overview of the approach studied in Dayanik and Karatzas [2003] for solving optimal stopping problems of one-dimensional regular diffusions of the form:

$$V(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_x \left[e^{-\rho\tau} h(X_\tau) \right], \quad x \in \mathcal{I} \quad (2.2)$$

where \mathcal{S} is the class of all \mathbb{F} -measurable stopping times. The function $V(\cdot)$ is referred to as the *value function*. We start by giving some important results and definitions and then discuss a general framework to solve (2.2) in Sections 2.3.1 and 2.3.2.

Let $H_r = \inf\{t \geq 0 : X_t = r\}$ be the first hitting time of a level $r \in \mathcal{I}$ by X . The regularity property of X has a few important consequences. Firstly, given $D = (l, r) \subset \mathcal{I}$, let τ_D be the first exit time of X from D . If $x \notin D$, then $\tau_D = 0$

\mathbb{P}_x -almost surely. If $x \in D$, then $\tau_D = H_l \wedge H_r$ \mathbb{P}_x -almost surely. We have the following two results:

Proposition 2.2.1. *If D is bounded, then $m_D(x) = \mathbb{E}_x[\tau_D]$ is bounded over D . In particular τ_D is a.s. finite.*

Proposition 2.2.2. *There exists a continuous, strictly increasing function $S(\cdot)$ on \mathcal{I} such that for any $l, r, x \in \mathcal{I}$ with $a \leq l < x < r \leq b$, we have:*

$$\mathbb{P}_x(\tau_r < \tau_l) = \frac{S(x) - S(l)}{S(r) - S(l)}, \quad \text{and} \quad \mathbb{P}_x(\tau_l < \tau_r) = \frac{S(r) - S(x)}{S(r) - S(l)}.$$

Any other function \tilde{S} with these properties is an affine transformation of S . The function S is unique in this sense and is called the "scale function" of X .

The scale function also satisfies $\mathcal{A}S(\cdot) \equiv 0$ where the second-order differential operator:

$$\mathcal{A}f(\cdot) = \frac{1}{2}\sigma^2(\cdot)\frac{d^2f}{dx^2}(\cdot) + \mu(\cdot)\frac{df}{dx}(\cdot) = 0 \quad \text{over } \mathcal{I} \quad (2.3)$$

is the infinitesimal generator of X . The ordinary differential equation $\mathcal{A}f = \rho f$ has two linearly independent, positive solutions. These are uniquely determined up to multiplication by a scalar, if we require one of them to be strictly increasing and the other strictly decreasing. The increasing solution shall be denoted by $\psi(\cdot)$ and the corresponding decreasing solution shall be denoted by $\phi(\cdot)$. The respective boundary conditions are $\psi(a) = \phi(b) = 0$ (See Borodin and Salminen [2012] Chapter 2). We also define the function $F(\cdot)$ by:

$$F(x) = \frac{\psi(x)}{\phi(x)} \quad (2.4)$$

for $x \in \mathcal{I}$. Note that $F(\cdot)$ is strictly increasing over \mathcal{I} . Furthermore, note that the case $\rho = 0$; that is the non-discounted case, implies $\phi(x) = 1$ and $F(x) = S(x) = \psi(x)$ for all $x \in \mathcal{I}$.

In the general theory of optimal stopping, a well known characterisation of the value function $V(\cdot)$ is given in terms of ρ -excessive functions of X ; that is, the non-negative functions $f(\cdot)$ satisfying:

$$f(x) \geq \mathbb{E}_x[e^{-\rho\tau}f(X_\tau)], \quad \forall \tau \in \mathcal{S}, \forall x \in \mathcal{I}$$

The idea of excessive functions is closely related to the concept of *G-Concavity* which is defined as follows; let $G : [c, d] \rightarrow \mathbb{R}$ be a strictly increasing function. A

real-valued function u is called G -concave on $[c, d]$ if, for any $a \leq l < r \leq b$ and $x \in [l, r]$, we have:

$$u(x) \geq u(l) \frac{G(r) - G(x)}{G(r) - G(l)} + u(r) \frac{G(x) - G(l)}{G(r) - G(l)} \quad (2.5)$$

Lastly given the function $F(\cdot)$ defined in (2.4) we provide a definition of F -differentiability which is essential in the discussion of the smooth-fit principle in Section 2.3.3.

Definition 2.2.1. *Let $g : [c, d] \rightarrow \mathbb{R}$ be any function. Define:*

$$D_F^+ g(x) = \frac{d^+ g}{dF} = \lim_{y \downarrow x} \frac{g(x) - g(y)}{F(x) - F(y)}$$

and

$$D_F^- g(x) = \frac{d^- g}{dF} = \lim_{y \uparrow x} \frac{g(x) - g(y)}{F(x) - F(y)}$$

provided that they exist. If D_F^\pm exist and are equal, then $g(\cdot)$ is said to be F -differentiable at x .

2.3 Discounted Optimal Stopping

In this section we will consider Dayanik and Karatzas [2003]'s approach for problems of the form described in (2.2) with $\rho > 0$. A similar characterisation follows for the case when $\rho = 0$, also described in Dayanik and Karatzas [2003]. We also omit the proofs of all results outlined in this section as the aim of the section is to serve as a review of the methodology used in subsequent chapters.

In Section 2.3.1, we start by discussing Dayanik and Karatzas [2003]'s results for the problem in (2.2) when defined over a closed and compact subset of \mathcal{I} . In Section 2.3.2 this is generalised further to cover problems defined over subsets of \mathcal{I} which have natural boundaries.

2.3.1 Problems defined over a closed and bounded subset of \mathbb{R}

Suppose we start the diffusion X defined in (2.1) in a closed and bounded interval $[c, d] \subset \mathcal{I}$, and stop X as soon as it reaches one of the boundaries c or d . Let the reward function $h : [c, d] \rightarrow \mathbb{R}$ be a Borel-measurable, bounded function. In this section we discuss the general method of solution outlined in Dayanik and Karatzas

[2003] for the problem:

$$V(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_x \left[e^{-\rho\tau} h(X_\tau) \right], \quad x \in [c, d] \quad (2.6)$$

If $h \leq 0$ over $[c, d]$, then trivially $V = 0$ and $\tau^* = \infty$. Hence we shall assume throughout that $\sup_{x \in [c, d]} h(x) > 0$. It is well known in the general optimal stopping theory, that the value function $V(\cdot)$ is the smallest ρ -excessive function dominating $h(\cdot)$ (See Peskir and Shiryaev [2006]). Thus in order to characterise $V(\cdot)$ in terms of a concave majorant of a transformation of $h(\cdot)$, the first natural step is to characterise ρ -excessive functions as F -concave functions. This equivalence is presented in Proposition 2.3.1 below and then used to specify $V(\cdot)$ in terms of F -concave functions in Proposition 2.3.2.

Proposition 2.3.1 (Characterisation of ρ -excessive functions). *For a given function $U : [c, d] \rightarrow [0, \infty)$, the quotient $U(\cdot)/\phi(\cdot)$ is an F -concave function if and only if $U(\cdot)$ is ρ -excessive.*

Proposition 2.3.2 (Characterisation of the Value function). *The value function $V(\cdot)$ of the problem described in (2.6) is the smallest non-negative majorant of $h(\cdot)$ such that $V(\cdot)/\phi(\cdot)$ is F -concave on $[c, d]$.*

Proposition 2.3.2 fully characterizes the value function $V(\cdot)$ and if non-negative F -concave majorants of h were geometrically easy to find, this result would be enough. However this is not the case in general and hence further conditions are necessary. Note that the definition of F -concavity (see (2.5)) implies that the connection between F -concavity and concavity follows by changing the underlying space from \mathcal{I} to $F(\mathcal{I})$. This provides a way of determining $V(\cdot)$ geometrically, since concave majorants can be determined easily as seen in Figure 2.1.

Proposition 2.3.3. *Let $W(\cdot)$ be the smallest non-negative concave majorant of $H := (h/\phi) \circ F^{-1}$ on $[F(c), F(d)]$, where F^{-1} is the inverse of the strictly increasing function $F(\cdot)$ defined in (2.4). Then $V(x) = \phi(x)W(F(x))$, for every $x \in [c, d]$.*

Note that if $h(\cdot)$ is continuous on $[c, d]$, then $V(\cdot)$ is also continuous on $[c, d]$ since $\psi(\cdot)$, $\phi(\cdot)$ and $F(\cdot)$ are continuous on \mathcal{I} . Furthermore it is worth noting that since $H := (h/\phi) \circ F^{-1}$ is well defined everywhere over the closed and bounded set $[F(c), F(d)]$, the smallest non-negative concave majorant $W(\cdot)$ is well-defined. Define the stopping region Γ and the corresponding stopping time τ^* by:

$$\Gamma := \{x \in [c, d] : V(x) = h(x)\} \quad \text{and} \quad \tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}.$$

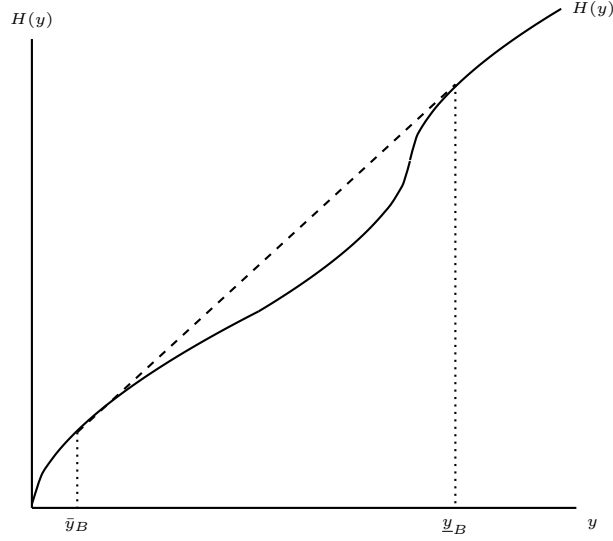


Figure 2.1: Stylized representation of the function $H(y)$ described in Proposition 2.3.3 as a function of $y = F(x)$. The corresponding non-negative concave majorant W is determined by the dashed chord joining $H(\bar{y}_B)$ to $H(\underline{y}_B)$.

The following proposition gives conditions verifying optimality.

Proposition 2.3.4. *If h is continuous on $[c, d]$, then τ^* is an optimal stopping rule.*

We have given an outline of the main results discussed in Dayanik and Karatzas [2003] for discounted optimal stopping problems of one-dimensional diffusions over sets of the form $[c, d] \subset \mathcal{I}$. It is worth noting however that even though we omit to include non-discounted optimal stopping here, a discounted problem for one dimensional diffusions can always be re-written as a non-discounted optimal stopping problem. Consider a standard Brownian Motion B on $[F(c), F(d)]$ and let W and H be defined as in Proposition 2.3.3. It can be shown that in fact we have:

$$W(y) = \sup_{\tau \geq 0} \mathbb{E}_y[H(B_\tau)] \quad y \in [F(c), F(d)], \quad (2.7)$$

and if h is continuous over $[c, d]$, then H is also continuous over $[F(c), F(d)]$. A similar result to Proposition 2.3.4 gives us that the problem in (2.7) also has an optimal stopping time $\tilde{\tau}^*$ and a corresponding optimal stopping region $\tilde{\Gamma}$. In fact $\tilde{\Gamma} = \{y \in [F(c), F(d)] : W(y) = H(y)\}$ giving $\Gamma = F^{-1}(\tilde{\Gamma})$.

2.3.2 Problems with Natural boundaries

In this section we provide an overview of how the results in Section 2.3.1 can be extended to cover problems over subsets $(a, b) \subseteq \mathcal{I}$ with natural boundaries. Consider

a reward function $h : (a, b) \rightarrow \mathbb{R}$ which is bounded on every compact subset of (a, b) and consider the optimal stopping problem:

$$V(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_x \left[e^{-\rho\tau} h(X_\tau) \right], \quad x \in (a, b). \quad (2.8)$$

A similar result as to the one discussed in Proposition 2.3.1 can be proved under this framework.

Proposition 2.3.5. *For a function $U : (a, b) \rightarrow [0, \infty)$, $U(\cdot)/\phi(\cdot)$ is F -concave on (a, b) , if and only if $U(x) \geq \mathbb{E}_x[e^{-\rho\tau} U(X_\tau)]$ for every $x \in (a, b)$ and $\tau \in \mathcal{S}$.*

By moving away from the assumptions of a problem defined over a closed and bounded set and the boundedness of h , further conditions that establish the *well-posedness* of $V(\cdot)$ are required. These are given in Proposition 2.3.6.

Proposition 2.3.6. *We have either $V = \infty$ on (a, b) , or $V(x) < \infty$ for all $x \in (a, b)$. Moreover, $V(x) < \infty$ for every $x \in (a, b)$, if and only if*

$$l_a := \limsup_{x \downarrow a} \frac{h^+(x)}{\phi(x)} \quad \text{and} \quad l_b := \limsup_{x \uparrow b} \frac{h^+(x)}{\psi(x)} \quad (2.9)$$

are both finite.

We assume in the remainder of the subsection that the quantities l_a and l_b are both finite. This implies that $\lim_{x \downarrow a} V(x)/\phi(x) = l_a$ and $\lim_{x \uparrow b} V(x)/\phi(x) = l_b$. Under these assumptions, similar results to those discussed in Section 2.3.1 can be stated.

Proposition 2.3.7. *The value function $V(\cdot)$ is the smallest non-negative majorant of $h(\cdot)$ on (a, b) such that $V(\cdot)/\phi(\cdot)$ is F -concave on (a, b) .*

Proposition 2.3.8. *Let $W : [0, \infty) \rightarrow \mathbb{R}$ be the smallest non-negative concave majorant of:*

$$H(y) = \begin{cases} \frac{h(F^{-1}(y))}{\phi(F^{-1}(y))}, & \text{if } y > 0 \\ l_a & \text{if } y = 0 \end{cases} \quad (2.10)$$

Then $V(x) = \phi(x)W(F(x))$ for every $x \in (a, b)$. Furthermore $W(0) = 0$ and $W(\cdot)$ is continuous at 0.

Define again:

$$\Gamma := \{x \in (a, b) : V(x) = h(x)\} \quad \text{and} \quad \tau^* := \inf\{t \geq 0 : X_t \in \Gamma\} \quad (2.11)$$

Theorem 2.3.9. *The value function $V(\cdot)$ is continuous on (a, b) . If $h : (a, b) \rightarrow \mathbb{R}$ is continuous, and $l_a = l_b = 0$, then τ^* of (2.11) is an optimal stopping time.*

The conditions necessary for Theorem 2.3.9 will be prevalent throughout our work. These conditions allow us to further characterise the optimal stopping time as a hitting time as discussed in the following result. The proof of this result is not part of the discussion in Dayanik and Karatzas [2003] and is relegated to the Appendix.

Corollary 2.3.10. *If $h : (a, b) \rightarrow \mathbb{R}$ is continuous, and $l_a = l_b = 0$, then the optimal stopping time τ^* is a hitting time.*

2.3.3 The Smooth-Fit Principle

In the previous sections we have characterized the stopping region Γ to be the subset of \mathcal{I} where $V = h$, and the continuation region \mathcal{C} as the region in \mathcal{I} where V majorizes h . A well known result in the general theory of optimal stopping characterizes the value function over the the boundary $\delta\mathcal{C}$; and is known as the *smooth-fit principle*. As the name implies this notion defines properties relating to the smoothness properties of V on $\delta\mathcal{C}$. Dayanik and Karatzas [2003] present an alternative but equivalent specification of this result using the idea of F -differentiation outlined in Definition 2.2.1. The main result discussed in Dayanik and Karatzas [2003] is stated below. The result is stated in-line with the framework of Section 2.3.2. However this can easily be re-stated for optimal stopping problems defined over closed and bounded subsets of \mathcal{I} .

Proposition 2.3.11. *At every $x \in \Gamma$ where $h(\cdot)/\phi(\cdot)$ is F -differentiable we have $V(\cdot)/\phi(\cdot)$ is also F -differentiable. Furthermore the F -derivatives of both functions agree at x :*

$$\frac{d}{dF} \frac{h}{\phi}(x) = \frac{d}{dF} \frac{V}{\phi}(x)$$

Note that Proposition 2.3.11 requires further assumptions on h than those required in Sections 2.3.1 and 2.3.2. Since the reward functions we consider throughout our work are smooth enough over \mathcal{I} , we do not need to discuss further generalisations of this result here. However it is worth noting that Dayanik and Karatzas [2003] give further conditions for the case when h is continuous but not differentiable (See Section 7).

Chapter 3

Partial Liquidation under Reference Dependent Preferences

3.1 Introduction

Prospect theory was proposed by Kahneman and Tversky [1979] and extended by Tversky and Kahneman [1992]. Under prospect theory, utility is reference-dependent so is defined over gains and losses relative to a reference level, rather than over final wealth. The utility function exhibits concavity in the domain of gains and convexity in the domain of losses, and so is S-shaped. It is steeper for losses than for gains, a feature known as loss aversion. Prospect theory was originally developed to better fit decision making behavior observed in experimental studies.

In recent years, optimal stopping models employing reference-dependent preferences have been developed in order to understand dynamic behavior of individuals with such preferences and to see to what extent the theory can be used to explain both experimental and empirically observed behavior. A strand of this literature, beginning with Kyle et al. [2006], has considered problems of optimal sale timing of risky assets under reference-dependent preferences. In this chapter we will extend the model of Kyle et al. [2006] to consider the question of partial liquidation of assets. Indeed Kyle et al. [2006] remark “...it would be of interest to incorporate partial liquidation in our model” (p284).

We propose an infinite horizon optimal stopping model whereby an investor with S -shaped reference-dependent preferences can sell their divisible asset position

at times of their choosing in the future. They derive utility from gains and losses relative to a reference level and utility is realized at the time when they sell their last tranche of asset.

We first give a general result which allows for a multiple stopping problem (where stopping times are allowed to coincide) to be viewed as a sequence of standard optimal stopping problems. This result is then applied to a model where utility is given by piece-wise exponential functions, steeper for losses than for gains, and the asset price follows a Brownian motion with drift. These explicit calculations enable us to compare to the paper of Kyle et al. [2006] who solve the block-sale case under similar modeling assumptions.

Our main finding is in showing that in the extended Kyle et al. [2006] model, the investor engages in partial sales. This represents the first time it has been shown that partial liquidation can occur under an S -shaped utility function. It is in contrast to the finding in Henderson [2012] where under the Kahneman-Tversky S -shaped utility function and exponential Brownian motion, the agent did not choose to partially sell an asset. Under our framework, if the agent sells, they will always sell at two distinct thresholds, thus partaking in partial liquidation. This is shown to be true under the assumption that the agent holds two units of claim of the same asset, but the result can be extended easily to a more general case with $N > 0$ units of claim. The agent's decisions on where to sell depend on the the price dynamics of the underlying asset and the value of the parameters determining the agent's utility function, particularly risk aversion.

By adopting a version of the model presented in Kyle et al. [2006] with the inclusion of a discount factor with respect to time, we recover tractable solutions for both the block sale problem and the partial liquidation problem.

Researchers are interested in modeling investor trading behavior under S -shaped reference-dependent preferences (of prospect theory) to see if it can better explain stylized facts in the empirical and experimental data. In particular, reference-dependence is a long standing explanation of why individual investors tend to sell winners too early and ride losers too long, a behavior called the disposition effect (Shefrin and Statman (1985)). In this vein, Kyle et al. [2006], Henderson [2012], Barberis and Xiong [2012] and Ingersoll and Jin [2013] contribute optimal stopping models for an investor with reference-dependent preferences under differing assumptions. Kyle et al. [2006] and Henderson [2012] treat one-shot or block sale optimal stopping problems under alternative assumptions on the S shaped utility and price processes. In particular, Henderson [2012] contributed a model whereby the investor sells at a loss voluntarily. This provided a better match to the disposition

effect (the tendency to sell more readily at a gain than at a loss, see Odean [1998]). Henderson [2012] also considers partial liquidation but finds under the Kahneman-Tversky S -shaped value function and exponential Brownian motion, the agent did not choose to partially sell.

Recent laboratory experiments of Magnani [2017] (also Lien and Zheng [2015], Magnani [2015]) have been designed to test predictions of S -shaped reference dependent preferences in a dynamic setting - that decision makers delay realizing disappointing outcomes but rush to realize outcomes that are better than expected. In their experiment, subjects choose when to stop an exogenous stochastic process and most tend to stop at a lower level than the risk-neutral upper threshold and delay capitulating until the process reaches a point significantly below the risk-neutral lower threshold. Imas [2016] studies how realized and paper losses affect behavior in an experiment where subjects make a sequence of investment decisions. In one of the treatments of this experiment, subjects decide whether to realize the outcome of the investment in the middle of the sequence and are found to be more likely to realize gains than losses.

Barberis and Xiong [2012], Ingersoll and Jin [2013] (and also He and Yang [2019]) consider realization utility models whereby investors treat their investing experience as a series of investment episodes, and receive utility from each individual sale at the time of sale. Mathematically, they sum up the utility of each individual sale and use a discount factor to model investors' tendency to realize gains early and losses late. Barberis and Xiong [2012] assume a piece-wise linear utility function and they find that the investors never voluntarily sell a stock at a loss. Ingersoll and Jin [2013] extend the model by assuming an S -shaped utility function and find that the investors voluntarily sell a stock both at a gain and at a loss. Recently, He and Yang [2019] extend to include an adaptive reference point which adapts to the stock's prior gain or loss. However, each of these models is separable, in that multiple identical units of assets would be sold simultaneously at the same threshold. None address the question of partial liquidation.

Our aim in this chapter is to give a simple, tractable optimal stopping model with S -shape reference-dependent preferences where partial sales *do* arise as an optimal solution. We employ the constructive potential-theoretic solution methods developed by Dayanik and Karatzas [2003] for optimal stopping of linear diffusions. This approach will be particularly useful for our problem as the smooth-fit principle does not apply everywhere because of the non-differentiability of the utility function, making the usual variational approach more challenging to apply. One-dimensional optimal stopping problems have been analysed by exploiting the rela-

tionship between functional concavity and r -excessivity (Dynkin [1965], Dynkin and Yushkevich [1969]) which has been applied by Dayanik and Karatzas [2003]. See also Alvarez [2001] and Alvarez et al. [2003] for related techniques. Carmona and Dayanik [2008] extend the methodology to consider an optimal multiple stopping problem for a regular diffusion process posed in the context of American options when the holder has a number of exercise rights. To make the problem non-trivial it is assumed that the holder chooses the consecutive stopping times with a strictly positive break period (otherwise the holder would use all his rights at the same time). It is difficult to explicitly determine the solution and Carmona and Dayanik [2008] describe a recursive algorithm. In contrast, here in our problem we do not wish to impose any breaks between stopping times, but rather, formulate a model setting where it may be *optimal* to have such breaks. Finally, direct methods for optimal stopping have also been used in stochastic switching problems (Bayraktar and Egami [2010]) and similar ideas are employed by Henderson and Hobson [2011] to solve a problem involving a perfectly divisible tranche of options on an asset with diffusion price process.

One strand of the recent literature has concerned itself with portfolio optimization (optimal control) under prospect theory and examples of this work include Jin and Yu Zhou [2008] and Carassus and Rasonyi [2015]. Another focus of the recent literature is on the probability weighting of prospect theory. However, probability weighting leads to a time-inconsistency and thus a difference in behaviour of naive and sophisticated agents, see Barberis [2012]. Henderson et al. [2017] (building on seminal work of Xu et al. [2013]) study agents who can pre-commit to a strategy and show that under some assumptions (satisfied by the models of interest including the Kahneman and Tversky [1979] and Tversky and Kahneman [1992] specification) it consists of a stop-loss threshold together with a continuous distribution on gains. However, recent results (Ebert and Strack [2015], also Henderson et al. [2017]) have shown that naive prospect theory agents never stop gambling. We focus in this chapter on reference-dependent S shaped preferences in the absence of probability weighting and extend the literature in the direction of holding a quantity of asset rather than just one unit.

3.2 General Framework

3.2.1 The Partial Liquidation Problem

Consider an investor who is holding $N \geq 1$ units of claim on an asset with current price Y_t . The investor is able to liquidate or sell the position in the asset at any

time in the future. They can choose times τ_i ; $i = 1, \dots, N$ at which to liquidate their N units of the claim, and hence is able to partially liquidate their divisible position. We will write $\tau_1 \geq \dots \geq \tau_N$ so τ_i denotes the sale time when there are i units *remaining* in the portfolio. It is worth noting that this formulation allows for units i and $i + 1$ to be liquidated at the same time point by setting $\tau_{N-i} = \tau_{N-i-1}$. It is however assumed that there is no terminal time to this problem; that is, the problem is formulated over an infinite time horizon¹.

For each unit i , the investor receives payoff $h^i(Y_{\tau_i})$ where the $h^i(\cdot)$ are non-decreasing functions, and compares this amount to a corresponding reference level h_R^i . As is often the case in the literature, an interpretation of h_R^i is the break-even level or the amount paid for the claim on the asset itself, and we will later specialize to this choice. h_R^i can be assumed to be given constants since by definition they are known a priori.

We would like a formulation in which the potential partial sales are not independent (so delaying a partial sale will impact on future sales) and so our investor considers their position as an investment episode which is closed and valued once the final partial sale takes place. This might be appropriate for institutional investors who are more likely to view investments in terms of overall portfolio position. Under this interpretation, the investor's problem can be written as:

$$V_N(y, 0) = \sup_{\tau_1 \geq \dots \geq \tau_N} \mathbb{E} \left[e^{-\rho \tau_1} U \left(\sum_{i=1}^N (h^i(Y_{\tau_i}) - h_R^i) \right) | Y_0 = y \right] \quad (3.1)$$

where utility function U is an increasing function. Later we will specialize to the reference-dependent S -shaped U given in the next section. Note that the formulation in (3.1) assumes that the investor receives no interest for cash flow i between the time of liquidation τ_{N-i} and the time of the last liquidation τ_1 . Whilst this is a possible improvement to the current formulation, in the spirit of being able to compare our results with those presented in Kyle et al. [2006] this extension is omitted.

Whilst later we will assume a linear payoff function for each partial sale i.e. $h^i(y) = y$ for all i , the methodology can be used to treat more complex payoffs. For example, take $N = 2$, and call option payoffs $h^1(y) = (y - k_1)^+$, $h^2(y) = (y - k_2)^+$ with strikes $k_1 > k_2$. Denote by h_R^1, h_R^2 two different reference levels, with one interpretation being the price paid for each option. Using a general ordering result in Henderson et al. [2014], we know the options are exercised in increasing strike

¹While it is possible to introduce a terminal time to a similar setting, the problem will need to be approached differently since the methodology in Dayanik and Karatzas [2003] assumes an infinite time horizon.

order, and hence our solution method applies.

Whilst the inclusion of the discount term makes the one-dimensional problem slightly different than that described in Kyle et al. [2006], this specification will only make the solution of the multiple optimal stopping problem described in Section 3.3.1 more comprehensible. In fact, it is worth noting that similar results can also be derived in the absence of discounting. The discounting with respect to τ_1 also captures the idea that there is an inter-dependency between the N partial sales. This dependency between liquidations is emphasised by the fact that the agent will consider the *game* to be terminated only once all liquidations have occurred, hence the discounting with respect to τ_1 . Contrastingly, as shown in Henderson [2012] and Barberis and Xiong [2012], if the investor instead considered each partial sale as an independent investment episode then she would optimize:

$$\sup_{\tau_1 \geq \dots \geq \tau_N} \mathbb{E} \left[\sum_{i=1}^N e^{-\rho \tau_i} U(h^i(Y_{\tau_i}) - h_R^i) | Y_0 = y \right] \quad (3.2)$$

Whilst this captures the spirit of realization utility in Barberis and Xiong [2012], whereby investors consider a series of investing “episodes”, mathematically, this formulation splits into N independent stopping problems and thus does not capture the inter-dependency we desire.

3.2.2 Reference-Dependent Preferences

When we present results for a specific model, we shall take the two-piece exponential utility function used by Kyle et al. [2006]:

$$U(y) = \begin{cases} \phi_1(1 - e^{-\gamma_1 y}), & \text{if } y \geq 0 \\ \phi_2(e^{\gamma_2 y} - 1), & \text{if } y < 0 \end{cases} \quad (3.3)$$

where $\phi_1, \phi_2, \gamma_1, \gamma_2 > 0$. Above the reference point, the agent’s utility function is a concave exponential function, with γ_1 measuring the local absolute risk aversion. Below the reference point, the value function is a convex exponential function, with γ_2 measuring the local absolute risk loving level. In addition, we assume $\phi_1 \gamma_1 < \phi_2 \gamma_2$ to ensure that the agent is loss averse, that is, more sensitive to losses than to gains around the reference point, i.e. $U'(0-) > U'(0+)$.

3.2.3 The Price Process

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting a Brownian Motion $W = (W_t)_{t \geq 0}$ and let $Y = (Y_t)_{t \geq 0}$ be a one dimensional time-homogeneous diffusion process solving:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \quad (3.4)$$

for Borel functions² $\mu : \mathcal{I} \rightarrow \mathbb{R}$ and $\sigma : \mathcal{I} \rightarrow \mathbb{R}^+$ where $\mathcal{I} = (a_{\mathcal{I}}, b_{\mathcal{I}}) \subseteq \mathbb{R}$ is the state space of Y_t with endpoints $-\infty \leq a_{\mathcal{I}} < b_{\mathcal{I}} \leq \infty$. Consider the infinitesimal generator of Y on \mathcal{I} , given by the second-order differential operator:

$$\mathcal{L}f(y) = \frac{1}{2}\sigma^2(y)\frac{d^2f}{dy^2}(y) + \mu(y)\frac{df}{dy}(y) = 0 \quad y \in \mathcal{I} \quad (3.5)$$

Then as discussed in Itô et al. [2012] and Dayanik and Karatzas [2003], given $\rho > 0$, the second order differential equation $\mathcal{L}f = \rho f$ has two linearly independent positive solutions $\psi(\cdot)$ and $\phi(\cdot)$ on \mathcal{I} . These are uniquely determined up to multiplication by a scalar factor, if we require one of them to be strictly increasing and the other to be strictly decreasing. We will denote the increasing solution by $\psi(\cdot)$ and the decreasing solution by $\phi(\cdot)$. We shall also define the function $F(\cdot)$ by:

$$F(y) = \frac{\psi(y)}{\phi(y)} \quad (3.6)$$

which is well-defined and strictly increasing on \mathcal{I} . The function $F(\cdot)$ is essential for solving (3.1) as outlined in Proposition 2.3.8.

Specifically we specialize to the model used in Kyle et al. [2006] and hence take:

$$dY_t = \mu dt + \sigma dW_t \quad (3.7)$$

where μ and $\sigma > 0$ are constants and $\mathcal{I} = (-\infty, \infty)$. Given that Y_t can be negative, it is possible to think of Y_t as a log-price process. Note that a similar problem considering different price dynamics than the above is outlined in Henderson [2012], where the Y_t is assumed to be given by a Geometric Brownian Motion.

Given that the state space \mathcal{I} has natural boundaries, the linearly independent solutions to the differential equation $\mathcal{L}f = \rho f$ assume the boundary conditions discussed in Section 2.3.2. They are given by $\psi(y) = e^{\beta_1 y}$ and $\phi(y) = e^{\beta_2 y}$ with

²We assume that $\mu(\cdot)$ and $\sigma(\cdot)$ are sufficiently regular so there exists a weak solution to the SDE. See Revuz and Yor [2013].

$\beta_2 < 0 < \beta_1$ given by:

$$\beta_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2} \quad \text{and} \quad \beta_2 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2} \quad (3.8)$$

This gives $F(y) = e^{\beta y}$ with $\beta = \beta_1 - \beta_2 = \frac{2\sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2} > 0$.

3.3 Solution to the Partial Liquidation Problem

3.3.1 The General Problem

An approach towards solving the optimal stopping problem outlined in (3.1) is outlined in Kobylanski et al. [2011]. This approach breaks down the original optimal stopping problem into N sub-problems. In Proposition 3.3.1 below we provide an alternative construction and proof of how such a decomposition can be achieved. It is worth noting that the result is in the same spirit of the discussion presented in Kobylanski et al. [2011], particularly Theorem 3.1.

Denote by x the total gains or losses from previous sales, if any; sales which are considered by the investor to persist in the current investment episode. Define:

$$\begin{aligned} V_N(y, x) &= \sup_{\tau_1 \geq \dots \geq \tau_N} \mathbb{E} \left[e^{-\rho\tau_1} U \left(x + \sum_{i=1}^N (h^i(Y_{\tau_i}) - h_R^i) \right) | Y_0 = y \right] \\ &= \sup_{\tau_1 \geq \dots \geq \tau_N} \mathbb{E} \left[e^{-\rho\tau_N} \mathbb{E} \left[e^{-\rho(\tau_1 - \tau_N)} U \left(x + \sum_{i=1}^N (h^i(Y_{\tau_i}) - h_R^i) \right) | \mathcal{F}_{\tau_N} \right] | Y_0 = y \right] \end{aligned} \quad (3.9)$$

We are primarily interested in (3.1), i.e. $x = 0$. The following result will facilitate the decomposition of (3.9) into N sub-problems.

In order to be able to solve the problem in (3.9) we assume that the problem satisfies the usual integrability condition³:

$$\mathbb{E} \left[\sup_{0 \leq t_N \leq \dots \leq t_1 < \infty} \left| U \left(x + \sum_{i=1}^N (h^i(Y_{t_i}) - h_R^i) \right) \right| \right] < \infty \quad (3.10)$$

Proposition 3.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Y be an Ito diffusion process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and $f(\cdot)$ is a strictly increasing continuous function.*

³Note that in our case this is obvious since $U(\cdot)$ is bounded.

Given:

$$\mathbb{E} \left[\sup_{0 \leq t_n \leq \dots \leq t_1 < \infty} \left| f \left(\sum_{i=1}^n h^i(Y_{t_i}) \right) \right| \right] < \infty \quad (3.11)$$

then it follows that:

$$\begin{aligned} & \sup_{\tau_n \leq \dots \leq \tau_1} \mathbb{E} \left[e^{-\rho \tau_1} f \left(\sum_{i=1}^n h^i(Y_{\tau_i}) \right) | Y_0 = y \right] \\ &= \sup_{\tau_n} \mathbb{E} \left[e^{-\rho \tau_n} \left(\operatorname{ess\,sup}_{\substack{\tau_{n-1} \leq \dots \leq \tau_1 \\ \tau_{n-1} \geq \tau_n}} \mathbb{E} \left[e^{-\rho(\tau_1 - \tau_n)} f \left(\sum_{i=1}^n h^i(Y_{\tau_i}) \right) | \mathcal{F}_{\tau_n} \right] \right) | Y_0 = y \right] \end{aligned}$$

Proof. The result follows if we show:

$$\begin{aligned} & \sup_{\tau_n \leq \dots \leq \tau_1} \mathbb{E} \left[e^{-\rho \tau_1} f \left(\sum_{i=1}^n h^i(Y_{\tau_i}) \right) | Y_0 = y \right] \\ & \geq \sup_{\tau_n} \mathbb{E} \left[e^{-\rho \tau_n} \left(\operatorname{ess\,sup}_{\substack{\tau_{n-1} \leq \dots \leq \tau_1 \\ \tau_{n-1} \geq \tau_n}} \mathbb{E} \left[e^{-\rho(\tau_1 - \tau_n)} f \left(\sum_{i=1}^n h^i(Y_{\tau_i}) \right) | \mathcal{F}_{\tau_n} \right] \right) | Y_0 = y \right] \quad (3.12) \end{aligned}$$

since the reverse inequality is trivial. Given an arbitrary stopping time τ_n consider the random variable:

$$Z^{(\tau_{n-1}, \dots, \tau_1)} = \mathbb{E} \left[e^{-\rho(\tau_1 - \tau_n)} f \left(\sum_{i=1}^n h^i(Y_{\tau_i}) \right) | \mathcal{F}_{\tau_n} \right]$$

and consider the family $\Gamma = \{Z^\alpha : \alpha \in \mathcal{I}\}$ where \mathcal{I} is the set of all $(n-1)$ tuples of $\{F_t\}$ -measurable stopping times $(\xi_{n-1}, \dots, \xi_1)$ satisfying $\tau_n \leq \xi_{n-1} \leq \dots \leq \xi_1$ almost surely. As shown in Lemma B.1.1 in Appendix B.1, the family Γ has the lattice property and hence there exists a countable subset $\mathcal{J} \subseteq \mathcal{I}$ where $\mathcal{J} = \{\alpha_j : j \in \mathbb{N}\}$ and:

$$Z^* = \operatorname{ess\,sup}_{\alpha \in \mathcal{I}} Z^\alpha = \lim_{j \rightarrow \infty} Z^{\alpha_j} \quad \text{with} \quad |Z^{\alpha_1}| \leq |Z^{\alpha_2}| \leq \dots \quad \mathbb{P} - a.s.$$

Using (3.11) and Jensen's inequality we get $\mathbb{E}[|Z^*|] < \infty$ and hence by the

Dominated Convergence Theorem the right hand side of (3.12) becomes:

$$\begin{aligned} \sup_{\tau_n} \mathbb{E} \left[e^{-\rho\tau_n} \left(\operatorname{ess\,sup}_{\alpha \in \mathcal{I}} Z^\alpha \right) \right] &= \sup_{\tau_n} \lim_{j \uparrow \infty} \mathbb{E} \left[e^{-\rho\tau_n} Z^{\alpha_j} \right] \\ &\leq \sup_{\tau_n} \sup_{\tau_{n-1} \leq \dots \leq \tau_1 : \tau_{n-1} \geq \tau_n} \mathbb{E} \left[e^{-\rho\tau_n} Z^{(\tau_{n-1}, \dots, \tau_1)} \right] \end{aligned}$$

□

Assuming the condition in (3.10) applies, from Proposition 3.3.1 it follows that for $1 \leq n \leq N$:

$$\begin{aligned} V_n(y, x) &= \sup_{\tau_n \leq \dots \leq \tau_1} \mathbb{E} \left[e^{-\rho\tau_1} U \left(x + \sum_{i=1}^n (h^i(Y_{\tau_i}) - h_R^i) \right) | Y_0 = y \right] \\ &= \sup_{\tau_n} \mathbb{E} \left[\sup_{\tau_{n-1} \leq \dots \leq \tau_1 : \tau_{n-1} \geq \tau_n} \mathbb{E} \left[e^{-\rho\tau_1} U \left(x + \sum_{i=1}^n (h^i(Y_{\tau_i}) - h_R^i) \right) | \mathcal{F}_{\tau_n} \right] | Y_0 = y \right] \\ &= \sup_{\tau_n} \mathbb{E} \left[e^{-\rho\tau_n} V_{n-1}(Y_{\tau_n}, x + h^n(Y_{\tau_n}) - h_R^n) | Y_0 = y \right] \end{aligned} \quad (3.13)$$

where $V_0(y, x) = U(x)$.

As discussed in Dayanik and Karatzas [2003], given the time-homogeneity of the problem, the structure of the solution must be to stop when the price process Y exits some sub-interval of \mathcal{I} . Thus, the approach is to consider stopping times of this form and choose the “best” such interval. We employ the theory in Dayanik and Karatzas [2003], which is summarised and discussed in Chapter 2. In fact by using the transformation of the reward function described in (2.10) and letting $\theta = F(y)$, we can define:

$$g_n(\theta, x) = \frac{V_{n-1}(F^{-1}(\theta), x + h^n(F^{-1}(\theta)) - h_R^n)}{\phi(F^{-1}(\theta))}. \quad (3.14)$$

The solution of (3.13) is obtained by applying the following Proposition, which is in the spirit of the methodology of Dayanik and Karatzas [2003], specifically Proposition 2.3.8.

Proposition 3.3.2. *Let $\theta = S(y)$ where $y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$ and let $g_n(\theta, x)$ be defined as in (3.14) with:*

$$l_{a_{\mathcal{I}}}^{n-1} = \limsup_{y \downarrow a_{\mathcal{I}}} \frac{V_{n-1}(y, x + h^n(y) - h_R^n)}{\phi(y)} = 0$$

and

$$l_{b_{\mathcal{I}}}^{n-1} = \limsup_{y \uparrow b_{\mathcal{I}}} \frac{V_{n-1}(y, x + h^n(y) - h_R^n)}{\psi(y)} = 0.$$

Furthermore let $\bar{g}_n(\theta, x)$ be the smallest non-negative concave majorant of:

$$G(\theta, x) = \begin{cases} g_n(\theta, x), & \text{for } \theta > 0 \\ l_{a_I}^{n-1}, & \text{for } \theta = 0 \end{cases}$$

Then $V_n(y, x) = \phi(y)\bar{g}_n(F(y), x)$ for $y \in (a_I, b_I)$. Furthermore, defining $\Gamma = \{y \in (a_I, b_I) : V_n(y, x) = V_{n-1}(y, x + h^n(y) - h_R)\}$, the corresponding optimal stopping time is given by $\tau_n^* = \inf\{t \geq 0 : Y_t \in \Gamma\}$.

Note that by Propositions 2.3.6 and 2.3.8, the assumption $l_{\alpha_I}^{n-1} = l_{\beta_I}^{n-1} = 0$ in Proposition 3.3.2, yields that a minimal non-negative concave majorant $\bar{g}_n(\theta, x)$ of $g_n(\theta, x)$ exists. This means that as $\theta \uparrow \infty$, $g_n(\theta, x)$ cannot be convex.

3.3.2 Piece-wise exponential utility and drifting Brownian motion

Having obtained such a characterization for the value function under partial liquidation, we shall apply the above methodology to the price process and preference function defined in Sections 3.2.2 and 3.2.3 respectively. We shall limit our discussion to the case when $N = 2$. The solutions for $N > 2$ can then be obtained through the same approach but become slightly more unwieldy. Since our aim is to show that the investor may partially liquidate, we only need consider $N = 2$ to show this.

We specialize to the case when the investor is selling or liquidating the asset itself, so consider $h^i(y) = y$ for $i = 1, \dots, N$, with the common reference price $h_R^i = y_R$ for $i = 1, \dots, N$. We also interpret the reference price y_R as the price at which the asset was purchased in the past.

Before stating the main result described above, we shall first re-state a version of the results obtained by Kyle et al. [2006] and Henderson [2012] for the case of $N = 1$ where we have included discounting; that is, when only block sales are allowed.

Proposition 3.3.3. *Consider the optimal liquidation problem in (3.1) with $N = 1$, $h^1(y) = y$ and $h_R^1 = y_R$ and suppose that the price process $(Y_t)_{t \geq 0}$ is given by a Brownian Motion with drift $dY_t = \mu dt + \sigma dW_t$ (see (3.7)) and the utility function U is the S-shaped piece-wise exponential given by (3.3). If the agent stops, the stopping level is $\bar{y}_1 > y_R$; defined by:*

$$\bar{y}_1 = y_R + \frac{1}{\gamma_1} \ln \left(\frac{\gamma_1 + \beta_1}{\beta_1} \right). \quad (3.15)$$

The proof is given in Section B.2 of the Appendix. We see from the above Proposition for the block sale problem that, if the agent sells, they will always sell

at a gain relative to the reference level. When one compares this solution to that of the equivalent non-discounted problem discussed in Henderson [2012], apart from the exclusion of the degenerate⁴ cases mentioned earlier, we also see that under our framework the agent also never liquidates at break-even. The selling threshold \bar{y}_1 in (3.15), still depends on the parameters determining the price dynamics of the underlying asset and the agent's preference structure, particularly risk aversion. As expected, from (3.15), we see that \bar{y}_1 decreases with an increase in γ_1 . An increase in the expected rate of return μ or the volatility parameter σ pushes the selling threshold higher. The selling threshold \bar{y}_1 however decreases in ρ . This is because with a higher discount rate, the agent becomes less interested in long term gains as it is more advantageous to sell sooner. This is in fact the same reason why under our formulation with discounting as opposed to Kyle et al. [2006] and Henderson [2012] we do not obtain the degenerate case of never stopping, even when μ is very large. Note that the aforementioned observations about \bar{y}_1 follow from the fact that β_1 as defined in (3.8) is positive, decreasing in μ and σ and increasing in ρ . Discounting is also the reason why there is no reason where the agent "sells immediately" at all prices, even when μ is large and negative.

Note that in their approach, Kyle et al. [2006] use a variational approach which is challenging due to the S -shaped utility function. In fact, in their solution they omit the case where the agent stops above break-even and only give solutions to the parameter combinations leading to other, simpler cases.

We now consider the partial liquidation problem for an agent with the same preference structure and holding a divisible asset with the same price dynamics as the one considered in the block sale problem described in Proposition 3.3.3.

Proposition 3.3.4. *Consider the optimal partial liquidation problem in (3.1) with $N = 2$, $h^2(y) = h^1(y) = y$ and $h_R^2 = h_R^1 = y_R$ and suppose that the price process $(Y_t)_{t \geq 0}$ is given by a Brownian Motion with drift $dY_t = \mu dt + \sigma dW_t$ (see (3.7)) and the utility function U is the S -shaped piece-wise exponential given by (3.3). If the agent stops, they will first sell at \bar{y}_2 and then at \bar{y}_1 where $\bar{y}_2 < \bar{y}_1$ and:*

$$\bar{y}_1 = y_R + \frac{1}{\gamma_1} \ln \left(\frac{\gamma_1 + \beta_1}{\beta_1} \right) \quad (3.16)$$

$$\bar{y}_2 = y_R + \frac{1}{2\gamma_1} \ln \left(\frac{2\gamma_1 + \beta_1}{\beta_1} \right) \quad (3.17)$$

The proof is given in Section B.2 of the Appendix. Similar to the case when

⁴By degenerate cases we mean, scenarios when it is optimal for the agent to either never sell or stop right away; that is, $\tau^* = \infty$ or $\tau^* = 0$.

only block sales are allowed (Proposition 3.3.3), the above proposition shows that under partial liquidation, the behaviour of the investor still depends on the value of β_1 ; which could be viewed as an adjusted Sharpe ratio for the underlying risky asset, and the agent's risk aversion parameter γ_1 . Since $\bar{y}_2 < \bar{y}_1$, if the price reaches \bar{y}_2 , one unit of asset is sold. If the price then reaches \bar{y}_1 , the final unit of asset will be sold. Both \bar{y}_1 and \bar{y}_2 are decreasing with γ_1 ; and via β_1 , increasing in μ and σ . As seen in the proof of Proposition 3.3.4, the thresholds \bar{y}_1 and \bar{y}_2 are determined from the transformed reward functions $g_1(\theta)$ and $g_2(\theta)$ and the corresponding non-negative concave majorants $\bar{g}_1(\theta)$ and $\bar{g}_2(\theta)$, depicted in Figures 3.1a, 3.1b and 3.1c.

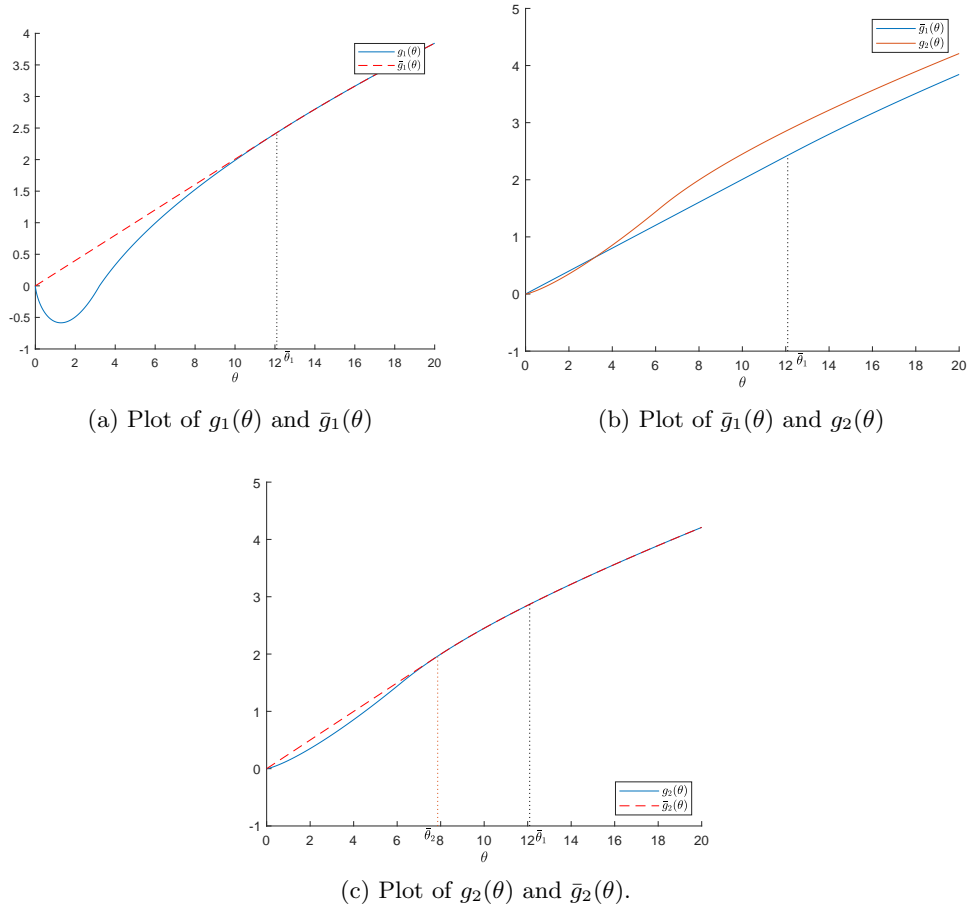


Figure 3.1: Plots depicting the functions $g_1(\theta)$, $\bar{g}_1(\theta)$, $g_2(\theta)$ and $\bar{g}_2(\theta)$. (Parameter Values: $y_R = 0.5$, $\beta_1 = 0.67$, $\beta_2 = -1.67$, $\gamma_1 = 3$, $\gamma_2 = 2$, $\phi_1 = 0.5$ and $\phi_2 = 0.9$.)

It is also evident from (3.16) and (3.17) that after the agent sells the first unit of asset at \bar{y}_2 , as either μ or σ increases, they are willing to wait further to sell the second unit of asset. This is because as either of these parameters increases, the

distance between the two price thresholds \bar{y}_1 and \bar{y}_2 increases. The opposite is true for the discount rate ρ , and while the agent still chooses to liquidate at distinct prices even for very large values of ρ , the distance between the two thresholds decreases to 0.

Somewhat surprisingly, we see that for a fixed value of β_1 , the distance between the two thresholds does not always decrease with a higher value of γ_1 ; that is, after selling the first unit of claim at $\bar{y}_2(\gamma_1)$, an agent with a higher risk aversion might choose to sell their second unit at a price which is much further away from $\bar{y}_2(\gamma_1)$ than an agent with lower risk aversion. This means that after selling the first unit of claim, the allowance for partial liquidation can make an agent with higher risk aversion employ *more risk* than an agent with lower risk aversion, even when the expected rate of return μ is negative. This relation can be seen more clearly in Figure 3.2 below.

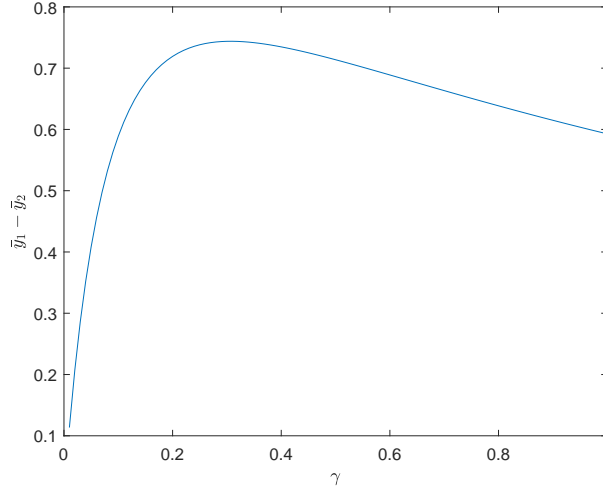


Figure 3.2: The distance between the two thresholds $\bar{y}_1 - \bar{y}_2$ for $\gamma_1 \in (0, 1)$ and $\beta_1 = 0.2$

Whilst the inclusion of the discount factor does change the overall structure of the solution when compared to that given in Kyle et al. [2006] and Henderson [2012] for the case of $N = 1$, the solution still captures the same essence of the solution given in Kyle et al. [2006] and Henderson [2012] for the cases that matter. In fact, the inclusion of the discount factor removes from the solution the degenerate cases where the agent either sells right away or never sells; both present in the solution of the equivalent non-discounted problem.

Unlike the result obtained in Henderson [2012] for liquidation with a divisible asset under a Cumulative Prospect Theory S -shaped utility function and exponen-

tial Brownian motion, our solution does not split into several cases depending on where the underlying parameters lie relative to each other. In fact as described in Proposition 3.3.4, under our framework, the agent will always choose to split their asset and liquidate first at price level \bar{y}_2 and then at \bar{y}_1 .

3.4 Discussion and Conclusions

Researchers have studied multiple optimal stopping problems under standard concave utility functions in other settings. For example, Grasselli and Henderson [2009], Leung and Sircar [2009] and Henderson and Hobson [2011] consider the exercise of American options under concave utilities and demonstrate that the optimal solution involves exercising a tranche of (identical) options over different asset price thresholds. Intuitively, a risk averse investor wants to spread the risk of continuing to hold the options by exercising them separately. Similarly, intuition would tell us that an investor who is risk seeking with convex utility, would prefer to engage in a block sale. What might we expect from an S -shaped reference dependent utility? Since there are concave and convex parts to the utility, we could reasonably expect that either might be dominant, depending on parameters. Somewhat surprisingly, Henderson [2012] showed that under Tversky and Kahneman [1992] S shaped function and exponential Brownian motion, the investor's optimal strategy, when not degenerate, *always* involved selling both units of asset together. In this chapter we demonstrate that it is indeed possible to obtain a situation whereby the investor chooses to sell her asset gradually rather than in a block.

Our results suggest that it would be worthwhile for experimental tests of optimal stopping under reference-dependent preferences to extend their focus to consider the question of how individuals sell a divisible quantity of asset. For example, in the context of Magnani [2017]'s laboratory test, do subjects with a quantity of asset still stop once (before the risk neutral upper threshold B^*) or do they sometimes stop more than once (and where in relation to B^*)?

Potential further theoretical work may examine the additional feature of an exogenous end-of-game whereby the asset is liquidated upon arrival of the first jump of a Poisson process (see Kyle et al. [2006], Barberis and Xiong [2012] for examples). Whilst injecting realism, this addition would be at the expense of the tractability of the solution method and for this reason, we do not pursue it here.

Chapter 4

Realisation Utility

4.1 Introduction

The concept of utility is classically related to the ideas of consumption or final wealth. However a recent strand of literature, formalised primarily in Barberis and Xiong [2012] suggests that investors also derive utility from realising gains and losses when selling assets, where the amount of utility derived depends on the magnitude of the realised gain and loss. They argue that realisation utility is principally the result of two cognitive processes. Firstly, some investors tend to think of their investments as a series of investment episodes wherein the purchase price and selling price play a very principal role in how they think of each individual investment distinctively. Secondly, they argue that some investors are predominantly driven to think of their investments by a very simple idea: Selling each individual investment at a gain is *good* whilst selling at a loss is *bad*. They argue that these ideas suggest that some investors experience bursts in utility when realising gains and losses. Furthermore it is worth noting that these ideas tend to naturally be more pronounced in individual investors than in institutional (more sophisticated) investors since the latter tend to view their investments in terms of the overall portfolio performance rather than separate investment episodes.

Barberis and Xiong [2012] argue that realisation utility together with another key ingredient provide an explanation to why various behavioural phenomena occur when dealing with risk. One such anomaly is the Disposition effect which describes the tendency of some investors to sell well-performing assets too early and sell under-performing assets too late. They argue that the missing ingredient which ensures that an agent realises a gain today instead of tomorrow and a loss tomorrow rather than today, is an S-shaped utility function.

In order to better understand the idea behind realisation utility, Barberis and Xiong [2012] formulate a model where an investor invests all their wealth in a risky asset whose price dynamics are modelled by a Geometric Brownian Motion and has to decide when to sell the underlying, thus receiving realisation utility at the moment of sale. The agent then instantaneously re-invests their proceeds after transaction costs in a risky asset with price dynamics equivalent to the asset they invested in a priori, thus essentially restarting the game. They approximate the investor’s underlying S-shaped utility function with a piece-wise linear function and find that under this framework, the investor never sells at a loss unless forced by a market shock modelled by an exponential random time.

This model has been revisited and expanded upon primarily by Ingersoll and Jin [2013] and He and Yang [2019]. In Ingersoll and Jin [2013] the authors consider a very similar framework to that considered by Barberis and Xiong [2012] whilst generalising the underlying utility function to a more general S-shaped utility function. By making use of the homogeneity of the underlying reward function and the Dynamic Programming Principle, the multiple optimal stopping problem is re-written as a one-dimensional optimal stopping problem which they solve by employing a PDE approach.

This framework is further generalised in He and Yang [2019]. The first difference from the aforementioned works is that besides realisation utility, He and Yang [2019] suppose that the agent also derives utility from consuming their terminal wealth. The agent’s reference level is also assumed to be non-constant in that it adapts to the stock’s prior gains and losses. Thirdly, they consider a general functional form for the agent’s realisation utility. Finally, they also assume that between investment episodes, the investor is allowed to put all their wealth in a bank account and then re-invest in the risky asset at some other time (i.e. the time between sale and re-purchase of the risky asset is allowed to be not instantaneous). The solution of this problem is expressed as a solution in the viscosity sense to the underlying variational inequality. He and Yang [2019] show that two cases arise depending on the value of the underlying parameters: it is either optimal for the agent to ignore the bank account completely and re-invest in the risky asset instantaneously after each investment episode, or to always only invest their wealth in the bank account. Note that the problem described in He and Yang [2019] is not a portfolio allocation problem similar to the classic Merton portfolio allocation problem. This is because under this formulation, at every time point all of the agent’s wealth has to be invested solely in one of the two assets.

In this chapter, we revisit the problem first proposed in Barberis and Xiong

[2012]. In Section 4.2 we first return to a model similar to that described in Ingersoll and Jin [2013] and solve the problem for an agent whose preferences are described by the classical Cumulative Prospect Theory S-shaped Utility proposed in Tversky and Kahneman [1992] as opposed to a scaled version of this utility function considered in Ingersoll and Jin [2013]. We solve this problem by utilising a different approach to that in Ingersoll and Jin [2013]. We show that whilst under this set of assumptions, the agent either waits and always sells at a profit, or they adopt a strategy where besides selling at a profit they also can sell at a loss. The optimality of one strategy over the other is shown to depend on the agent’s loss aversion.

In Section 4.3, we propose a new extension to the model described in Section 4.2. We propose a new utility function in which the agent does not compare their gains relative to the reference level only linearly, but also proportionally. The inclusion of the proportional term was inspired by the structure of the optimal strategy obtained in Section 4.2 which depends entirely on the value of the proportional gains or losses made by the agent. The newly introduced proportional term also imposes an additional property in the agent’s preferences. We see that the closer the agent gets to losing everything, the utility starts decreasing drastically and hence the agent is penalised for big losses much more than under any other preference function previously mentioned.

The two problems in this chapter are both multiple optimal stopping problems. This is also the case for the model considered in Chapter 3 and it is worth mentioning here the key differences between these two optimal stopping formulations. In this chapter we address the idea of realisation utility under the framework introduced in Barberis and Xiong [2012]. For this model to make sense, the agent adopts a ‘narrow-framing’ viewpoint, since otherwise they would not consider realisation utility derived from each asset independently. In Chapter 3 we address the problem of partial liquidation for an agent with reference dependent preferences, and thus the agent is assumed to adopt a ‘non-narrow-framing’ strategy. This is the case since otherwise each partial sale would be considered as an independent investment episode resulting in the agent never partaking in partial liquidation.

Decision framing was introduced by Tversky and Kahneman [1981] and refers to the idea that how a person subjectively frames a transaction in their mind will determine the utility they expect to receive. Narrow framing (see Barberis et al. [2006]) occurs when an agent who is offered a new gamble evaluates it in isolation, separately to their other risks.

While the models for realisation utility discussed in Barberis and Xiong [2012], Ingersoll and Jin [2013], He and Yang [2019] and Sections 4.2 and 4.3 below

provide good insights into the optimal behaviour of an investor experiencing a burst in utility when realising gains and losses, they have some shortcomings. One of the main drawbacks is the idea that after each investment episode the investor re-invests all their wealth in an asset with the same price dynamics. This assumption is used because it allows the underlying multiple optimal stopping problem to be re-written as a one dimensional optimal stopping problem. However Barberis and Xiong [2012] argue that this is also a reasonable assumption for an investor who thinks of each individual investment independently, since for such investors utility is derived separately from the gains and losses of each individual stock.

Another shortcoming of this model is the idea that at the start of each investment episode the investor *has to* invest all their wealth in the risky asset (or the riskless asset, when considering the formulation of He and Yang [2019]) and cannot opt to allocate their holdings optimally between the two assets. While this idea makes sense when one compares this model with the classical Merton-style portfolio allocation problem, careful consideration must be made when treating realisation utility. If one allows the investor to re-balance their portfolio infinitesimally often as is the case under the assumptions of Merton [1969], then the idea of realisation utility does not make much sense as the agent is then also possibly realising gains and losses infinitesimally often. Furthermore, in order to integrate the bank account as part of the model one has to integrate into the model some form of consumption or final wealth term similar to the formulation in He and Yang [2019]. This problem can be expressed in terms of a stochastic impulse control problem and is currently a work in progress.

4.2 Realisation Utility and Cumulative Prospect Theory

Consider an agent who starts at $t = 0$ with initial wealth $W_0 = w$, which they invest into a risky asset with price process $(X_t)_{t \geq 0}$ following a geometric Brownian Motion with constant parameters μ and σ :

$$dX_t = \mu X_t dt + \sigma X_t dB_t \quad (4.1)$$

where $(B_t)_{t \geq 0}$ is a Brownian Motion adapted to the underlying filtration $(\mathcal{F}_t)_{t \geq 0}$. Assume that the investor is constrained to invest all their wealth in the risky asset and their first objective is to choose a stopping time τ_1 at which to liquidate their position. Upon liquidation the agent derives realisation utility $U(W_{\tau_1-}, R_{\tau_1-})$

(suitably discounted). Their burst of realisation utility depends on their wealth at the liquidation time; whose dynamics are described by the process $(W_t)_{t \geq 0}$, and an appropriate reference level described by the process $(R_t)_{t \geq 0}$ which will be defined further on. The agent then re-invests their wealth in the same risky asset. Upon purchasing the risky asset the agent incurs a transaction cost proportional to their total wealth and the game essentially restarts at τ_1 with suitably defined wealth W_{τ_1} and reference level R_{τ_1} .

Given that the agent invests all their wealth in the underlying risky asset, the agent's wealth process $(W_t)_{t \geq 0}$ satisfies:

$$W_t = \begin{cases} w \frac{X_t}{X_0} & \text{for } t \in [0, \tau_1) \\ W_{\tau_n} \frac{X_t}{X_{\tau_n}} & \text{for } t \in [\tau_n, \tau_{n+1}) \text{ and } n \geq 1 \\ KW_{\tau_n-} & \text{for } t = \tau_n, n \geq 1 \end{cases} \quad (4.2)$$

where $W_{t-} = \lim_{s \uparrow t} W_s$ and $K \in (0, 1]$ is the proportion of wealth remaining after transaction cost. As discussed in both Ingersoll and Jin [2013] and He and Yang [2019], one interpretation of the constant K can be $K = (1 - k_s)/(1 + k_p)$ where $k_s \in [0, 1)$ is a proportional transaction cost the agent pays when selling the asset and $k_p \in [0, 1)$ determines another proportional transaction cost paid by the investor when re-purchasing the asset.

Furthermore by definition, the reference level $(R_t)_{t \geq 0}$ captures the price level against which the agent compares upon liquidation in order to calculate whether they made a gain or loss. Thus it makes sense for $(R_t)_{t \geq 0}$ to be a piece-wise constant, right-continuous stochastic process since the agent will only change their reference point every time they re-purchase the risky asset, and will keep it constant until their next liquidation time. Hence a reasonable choice for the reference level R_t would be the wealth level at the previous stopping time chosen by the agent, defined as follows:

$$R_t = \begin{cases} W_{\tau_n}, & \text{for } t \in (\tau_n, \tau_{n+1}) \text{ and } n \geq 1 \\ r, & \text{for } t \in [0, \tau_1) \end{cases} \quad (4.3)$$

and let $R_{t-} = \lim_{s \uparrow t} R_s$. Alternative formulations of the reference level are also discussed in He and Yang [2019] where they consider a reference level which changes continuously depending on the value of W_t .

The agent's objective is to choose stopping times $0 \leq \tau_1 \leq \tau_2 \leq \dots$ at which to realize gains and losses in order to maximize the value of the game described

above¹. This can be described by the following Optimal Stopping Problem:

$$Z(w, r) = \sup_{0 \leq \tau_1 \leq \tau_2 \leq \dots} \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho \tau_n} U(W_{\tau_n-}, R_{\tau_n-}) \mathbb{I}_{\{\tau_n < \infty\}} \right] \quad (4.4)$$

where we include discounting with respect to time through a constant discount rate $\rho > 0$. For ease of notation, we write $\mathbb{E}_{w,r}[\cdot]$ instead of the conditional expectation $\mathbb{E}[\cdot | W_0 = w, R_0 = r]$.

Note that each time the agent liquidates their asset, they derive realisation utility by comparing the gross value from sales to their reference level. In this section, the S-shaped utility function $U(w, r)$ centred around the reference level r , which was first proposed by Tversky and Kahneman [1992], is imposed:

$$U(w, r) = \begin{cases} -\lambda(r - w)^{\gamma_1}, & \text{for } w \leq r \\ (w - r)^{\gamma_2}, & \text{for } w > r \end{cases} \quad (4.5)$$

with $0 < \gamma_1 < 1$, $0 < \gamma_2 < 1$, $\lambda \geq 1$ and $U_w(r-, r) = U_w(r+, r) = \infty$. The parameters γ_1 and γ_2 capture relative risk aversion over losses and gains respectively and λ captures the agent's level of loss aversion. Barberis and Xiong [2012] use a piece-wise linear function which is the special case with $\gamma_1 = \gamma_2 = 1$. A scaled version of this utility function is considered in Ingersoll and Jin [2013] where preferences are described by the function $U_{IJ}(w, r)$ defined by:

$$U_{IJ}(w, r) = \begin{cases} -\lambda r^{\zeta} (1 - \frac{w}{r})^{a_L}, & \text{for } w \leq r \\ r^{\zeta} (\frac{w}{r} - 1)^{a_G} & \text{for } w > r \end{cases} \quad (4.6)$$

where $0 < a_G$, $a_L < 1$ and $0 < \zeta \leq \min\{a_G, a_L\}$. (4.5) is equivalent to (4.6) under the case when $\gamma_1 = \gamma_2 = a_G = a_L = \zeta$. In this section we will solve the problem described in (4.4) by using a different methodology to that adopted in Ingersoll and Jin [2013]; specifically the methodology outlined in Dayanik and Karatzas [2003] summarised in Chapter 2. By focusing solely on the classical Cumulative Prospect Theory utility function in (4.5) we are also able to distinguish some features unique to this problem.

We take $\gamma_1 = \gamma_2 = \gamma$ where $\gamma \in (0, 1)$. This assumption is essential for our solution as this implies that $U(w, r)$ is homogeneous in w and r of degree γ . It is also worth noting that through an experiment, Kahneman and Tversky [2013]

¹Note that we assume that the agent records utility solely from the act of realising gains or losses. In Section 4.4 we briefly describe an extension of this problem whereby an extra term is added for utility derived from final wealth

estimated the value of the parameters γ_1 , γ_2 and λ as $\gamma_1 = \gamma_2 = 0.88$ and $\lambda = 2.25$. Thus this assumption on the parameter γ is still in line with the original findings in Kahneman and Tversky [2013].

4.2.1 Well-Posedness Conditions and the Dynamic Programming Principle

In solving the problem formulated in (4.4) we first provide a necessary and sufficient condition for $Z(w, r)$ to be finite in Proposition 4.2.2 below.

Remark 4.2.1. *In proving the well-posedness conditions and the dynamic Programming principle for (4.4), we assume that the family of stopping times $\{\tau_n : n \in \mathbb{N}\}$ is such that for every $n \in \mathbb{N}$ there exists a constant $c_n \in \mathbb{R}^+$ giving $|W_{t \wedge \tau_n}| \leq c_n$ almost surely. It is shown later on as part of our work in Section 4.2.2 that the optimal stopping times $\{\tau_n : n \in \mathbb{N}\}$ are in fact hitting times of W_t , and hence this assumption is satisfied.*

Proposition 4.2.2. *Consider the problem defined in (4.4). Then $Z(w, r) < \infty \iff \rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$.*

The proof of this result is relegated to Appendix C.2. Note that $\gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$ can be understood as the agent's expected growth rate of realisation utility. The condition in the above Proposition hence restricts this growth rate to be less than the underlying discount rate, ρ . Otherwise, as we see in the second part of the proof, it would always be optimal for the agent to postpone selling their asset.

Remark 4.2.3. *Assume $\rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$.*

An important implication which follows from the homogeneity property of the underlying S-shaped Utility function in (4.5) is that the value function of the problem defined in (4.4) is also homogeneous in r of degree γ , as stated in Lemma 4.2.4 below. The proof is omitted since this result follows directly from the homogeneity of U .

Lemma 4.2.4. *$Z(w, r)$ is homogeneous in r of degree γ .*

Given the above well-posedness conditions we can show that the dynamic programming principle holds for $Z(w, r)$ as defined in (4.4).

Proposition 4.2.5. *The following Dynamic Programming Principle holds:*

$$Z(w, r) = \sup_{\tau} \mathbb{E}_{w, r} \left[e^{-\rho\tau} \left(U(W_{\tau-}, R_{\tau-}) + Z(KW_{\tau-}, KW_{\tau-}) \right) \mathbb{I}_{\{\tau < \infty\}} \right] \quad (4.7)$$

A proof is given in Appendix C.2. The above dynamic programming principle captures the idea laid out in the description of the problem, that the agent essentially restarts the same game at every liquidation point.

4.2.2 Solution

Having established the conditions outlined in Propositions 4.2.2 and 4.2.5 above, in this section we will approach the problem described in (4.7) following the methodology outlined in Dayanik and Karatzas [2003], summarised in Chapter 2.

For a general \mathbf{C}^2 function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, consider the infinitesimal generator of geometric Brownian Motion in (4.1) described by the second order differential operator:

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2 x^2 \frac{d^2 f}{dx^2}(x) + \mu x \frac{df}{dx}(x)$$

Given the underlying state space $\mathcal{I} = (0, \infty)$, the discussion in Chapter 2 outlines that the ordinary differential equation $\mathcal{A}u = \rho u$ on \mathcal{I} has two linearly independent solutions $\psi(\cdot)$ (increasing) and $\phi(\cdot)$ (decreasing) which are uniquely determined up to multiplication by a scalar and satisfy the boundary conditions $\lim_{x \uparrow \infty} \phi(x) = \psi(0) = 0$. this gives $\psi(x) = x^\beta$ and $\phi(x) = x^\alpha$, where $\alpha < 0 < \beta$ satisfy:

$$\alpha = \sigma^{-2} \left[-\left(\mu - \frac{1}{2}\sigma^2\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2} \right] \quad (4.8)$$

$$\beta = \sigma^{-2} \left[-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2} \right] \quad (4.9)$$

Note that for $x \in \mathcal{I}$, $\psi(x)$ is monotonically increasing and $\phi(x)$ is monotonically decreasing. Lastly define the function $F(x) = \psi(x)/\phi(x) = x^{\beta-\alpha}$. The function $F(\cdot)$ is increasing over \mathcal{I} . We hence let $y = F(x)$ for $x \in \mathcal{I}$ with inverse $x = F^{-1}(y) = y^{\frac{1}{\beta-\alpha}}$ for $y \in (0, \infty)$. As seen in our discussion in Chapter 2, these functions will play an important role in solving the optimal stopping problem in (4.7). At this point, we observe that the condition in Proposition 4.2.2, $\rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2$ can equivalently be expressed in terms of β as: $\gamma \leq \beta$.

Define $V(w) = Z(w, w)$ to be the value of the game when both initial wealth and reference level are equal to w . Then:

$$V(w) = \sup_{\tau} \mathbb{E}_w \left[e^{-\rho\tau} \left(U(W_{\tau-}, w) + V(W_{\tau}) \right) \mathbb{I}_{\{\tau < \infty\}} \right]. \quad (4.10)$$

We will first solve the optimal stopping problem in (4.10) above. The solution of (4.7) will follow from the solution of this problem. By the homogeneity property of

the value function given in Lemma 4.2.4 above, we have:

$$w^\gamma V(1) = \sup_{\tau} \mathbb{E}_w \left[e^{-\rho\tau} (w^\gamma U(W_{\tau-}/w, 1) + (KW_{\tau-})^\gamma V(1)) \mathbb{I}_{\{\tau < \infty\}} \right] \quad (4.11)$$

and hence the constant $V(1)$ solves:

$$V(1) = \sup_{\tau} \mathbb{E}_w \left[e^{-\rho\tau} (U(W_{\tau-}/w, 1) + (KW_{\tau-}/w)^\gamma V(1)) \mathbb{I}_{\{\tau < \infty\}} \right]. \quad (4.12)$$

Substituting for the asset price X_t we get:

$$V(1) = \sup_{\tau} \mathbb{E} \left[e^{-\rho\tau} (U(X_{\tau}, 1) + (KX_{\tau})^\gamma V(1)) \mathbb{I}_{\{\tau < \infty\}} | X_0 = 1 \right]. \quad (4.13)$$

Temporarily fix $V(1) = v$. This gives the following optimal stopping problem:

$$\mathcal{H}(v, \tau) = \mathbb{E} \left[e^{-\rho\tau} (U(X_{\tau}, 1) + (KX_{\tau})^\gamma v) \mathbb{I}_{\{\tau < \infty\}} | X_0 = 1 \right] \quad (4.14)$$

$$\mathcal{H}(v) = \sup_{\tau} \mathcal{H}(v, \tau) \quad (4.15)$$

By solving the optimal stopping problem in (4.15) for fixed v , we can then determine the solution of (4.13) by determining v^* satisfying $\mathcal{H}(v^*) = v^*$. A uniqueness result to this fixed point problem is provided in the following Lemma.

Lemma 4.2.6. *Let $\mathcal{H}(v)$ be as defined in (4.14) and (4.15). The optimal stopping time τ^* is a hitting time. Furthermore if a solution to the fixed point problem $\mathcal{H}(v^*) = v^*$ exists, then it is unique.*

The proof of Lemma 4.2.6 is relegated to Appendix C.2. Some time was dedicated to try and obtain a result on the existence of a solution the fixed point problem $\mathcal{H}(v^*) = v^*$. While numerically we could not determine cases when such a solution doesn't exist, we were unsuccessful in deriving a general result at this stage. However as part of our work we will later discuss a result (Proposition 4.2.12) which is closely related to the characterisation of the solution of the fixed point problem $\mathcal{H}(v^*) = v^*$.

Remark 4.2.7. *Suppose that the optimal stopping strategy of (4.15) is a one threshold strategy; that is, $a = 0$ where a is as defined in the proof of Lemma 4.2.6. Then a solution to the fixed point problem $\mathcal{H}(v^*) = v^*$ always exists. This follows from the proof of Lemma 4.2.6 by noting that this yields:*

$$\mathcal{H}(v) = \mathcal{H}(0) + \tilde{C}v$$

and when $\tau^* = H_b$ with $b \geq 1$, we have $\mathcal{H}(0) > 0$ and $\tilde{C} = \mathbb{E}_1[e^{-\rho H_b}(KX_{H_b})^\gamma] < 1$.

In solving the problem described in (4.14) and (4.15) above, we shall consider the same problem with a non-fixed starting value X_0 . The solution of the above problem will then follow as a special case of this problem with $X_0 = 1$. Thus for $x \in (0, \infty)$, consider the following complimentary problem:

$$\mathcal{H}(v, x, \tau) = \mathbb{E} \left[e^{-\rho\tau} (U(X_\tau, 1) + (KX_\tau)^\gamma v) \mathbb{I}_{\{\tau < \infty\}} | X_0 = x \right] \quad (4.16)$$

$$\mathcal{H}(v, x) = \sup_{\tau} \mathcal{H}(v, x, \tau) \quad (4.17)$$

Denote the corresponding reward function of the problem in (4.16) and (4.17) by $h_v(x)$, given by:

$$h_v(x) = K^\gamma v x^\gamma + U(x, 1) \quad (4.18)$$

where $h_v : (0, \infty) \rightarrow \mathbb{R}$ is bounded on every compact subset of $\mathbb{R}^+/\{0\}$. Letting $y = F(x)$ we define the corresponding transformed reward function $g_v(y) = h_v(F^{-1}(y))/\phi(F^{-1}(y))$, given by:

$$g_v(y) = \begin{cases} K^\gamma v y^{\frac{\gamma-\alpha}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y \geq 1 \\ K^\gamma v y^{\frac{\gamma-\alpha}{\beta-\alpha}} - \lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y < 1 \end{cases}$$

This will allow us to follow the methodology outlined in Chapter 2 to solve the above problem.

The discussion of the geometric structure of the function $g_v(y)$ over \mathbb{R}^+ in Appendix C.1 infers that the solution to the optimal stopping problem described in (4.16) and (4.17) can take two general forms. The first class of solutions contains strategies wherein the agent only sells at a profit. We will refer to this type of solution as a *one threshold strategy*. The second class of solutions contains strategies where the continuation region is comprised of two disconnected neighbourhoods; one containing 0 and another neighbourhood containing the break-even point 1. This means that the agent will either continue if they start at a deep loss or if they start at a relatively small gain or loss. However since the starting value of X in (4.14) and (4.15) is always 1, the neighbourhood around 0 is of no interest to us. Thus we refer to the strategies contained in this class as *two-threshold strategies*.

In Proposition 4.2.8 below, the characterisation of the optimal stopping time, the value function and the selling threshold for the case when the one-threshold strategy is optimal is given. A similar characterisation is also given for the two threshold strategy in Proposition 4.2.10. The proof of Proposition 4.2.10 is omitted

as it relies on similar arguments to that used in Proposition 4.2.8.

Proposition 4.2.8. *Consider the problem defined in (4.16) and (4.17) and let $v \in \mathbb{R}^+$ be fixed. Suppose that the one-threshold strategy is optimal. Then the optimal stopping rule τ_v^* takes the form $\tau_v^* = \inf\{t \geq 0 : X_t \notin \mathcal{C}_1\}$ where $\mathcal{C}_1 = (0, \bar{x}_v)$ and $\bar{x}_v > 1$ solves:*

$$K^\gamma v \bar{x}_v = \left(\left(\frac{\beta}{\beta - \gamma} \right) - \bar{x}_v \right) (\bar{x}_v - 1)^{\gamma-1}. \quad (4.19)$$

Furthermore we have:

$$\mathcal{H}(v, x) = \begin{cases} K^\gamma v x^\gamma + (x - 1)^\gamma & \text{for } x \geq \bar{x}_v \\ (K^\gamma v \bar{x}_v^{\gamma-\beta} + (\bar{x}_v - 1)^\gamma \bar{x}_v^{-\beta}) x^\beta & \text{for } x < \bar{x}_v \end{cases} \quad (4.20)$$

The proof is given in Appendix C.2. The following corollary follows from the above Proposition and offers a characterisation of the constant $V(1) = v$ when the one-threshold strategy is optimal. This result follows directly from the fact that v satisfies $v = \mathcal{H}(v)$, where $\mathcal{H}(v) = \mathcal{H}(v, 1)$.

Corollary 4.2.9. *Under the assumptions of Proposition 4.2.8, $V(1) = v$ satisfying (4.13) solves:*

$$V(1) = \frac{\bar{x}_v^{-\beta} (\bar{x}_v - 1)^\gamma}{(1 - K^\gamma \bar{x}_v^{\gamma-\beta})} \quad (4.21)$$

Proposition 4.2.10. *Consider the problem defined in (4.16) and (4.17) and fix $v \in \mathbb{R}^+$. Suppose the two threshold strategy is optimal. Then the optimal stopping rule τ_v^* takes the form $\tau_v^* = \inf\{t \geq 0 : X_t \notin \mathcal{C}_2\}$ with $\mathcal{C}_2 = (0, \hat{x}_v) \cup (\underline{x}_v, \bar{x}_v)$, where $\underline{x}_v < 1 < \bar{x}_v$ satisfy the following system of equations:*

$$\frac{K^\gamma v (\bar{x}_v^{\gamma-\alpha} - \underline{x}_v^{\gamma-\alpha}) + (\bar{x}_v - 1)^\gamma \bar{x}_v^{-\alpha} + (1 - \underline{x}_v)^\gamma \underline{x}_v^{-\alpha}}{\bar{x}_v^{\beta-\alpha} - \underline{x}_v^{\beta-\alpha}} \quad (4.22)$$

$$= \frac{1}{\beta - \alpha} \left[(\gamma - \alpha) K^\gamma v \underline{x}_v^{\gamma-\beta} + \gamma \lambda (1 - \underline{x}_v)^{\gamma-1} \underline{x}_v^{1-\beta} + \alpha \lambda \bar{x}_v^{-\beta} (1 - \underline{x}_v)^\gamma \right] \quad (4.23)$$

$$= \frac{1}{\beta - \alpha} \left[(\gamma - \alpha) K^\gamma v \bar{x}_v^{\gamma-\beta} + \gamma (\bar{x}_v - 1)^{\gamma-1} \bar{x}_v^{1-\beta} - \alpha \bar{x}_v^{-\beta} (\bar{x}_v - 1)^\gamma \right] \quad (4.24)$$

whereas \hat{x}_v satisfies:

$$K^\gamma (\beta - \gamma) v \hat{x}_v^\gamma = \lambda (1 - \hat{x}_v)^{\gamma-1} ((\gamma - \beta) \hat{x}_v + \beta).$$

Furthermore:

$$\mathcal{H}(v, x) = \begin{cases} K^\gamma v x^\gamma + (x - 1)^\gamma & \text{for } x > \bar{x}_v \\ A^{(2)} x^\beta + B^{(2)} x^\alpha & \text{for } \underline{x}_v \leq x \leq \bar{x}_v \\ K^\gamma v x^\gamma - \lambda(1 - x)^\gamma & \text{for } \hat{x}_v < x < \underline{x}_v \\ (K^\gamma v \hat{x}_v^{\gamma-\beta} - \lambda(1 - \hat{x}_v)^\gamma \hat{x}_v^{-\beta}) x^\beta & \text{for } x \leq \hat{x}_v \end{cases} \quad (4.25)$$

with

$$A_v^{(2)} = \left[\frac{K^\gamma v (\bar{x}_v^{\gamma-\alpha} - \underline{x}_v^{\gamma-\alpha}) + (\bar{x}_v - 1)^\gamma \bar{x}_v^{-\alpha} + \lambda(1 - \underline{x}_v)^\gamma \underline{x}_v^{-\alpha}}{\bar{x}_v^{\beta-\alpha} - \underline{x}_v^{\beta-\alpha}} \right]$$

$$B_v^{(2)} = \left[\frac{K^\gamma v \underline{x}_v^\gamma - \lambda(1 - \underline{x}_v)^\gamma}{\underline{x}_v^\alpha} - \left(\frac{\underline{x}_v^{\beta-\alpha} (K^\gamma v (\bar{x}_v^{\gamma-\alpha} - \underline{x}_v^{\gamma-\alpha}) + (\bar{x}_v - 1)^\gamma \bar{x}_v^{-\alpha} + \lambda(1 - \underline{x}_v)^\gamma \underline{x}_v^{-\alpha})}{\bar{x}_v^{\beta-\alpha} - \underline{x}_v^{\beta-\alpha}} \right) \right]$$

Given that the variable v satisfies $v = \mathcal{H}(v, 1)$, for the case outlined in Proposition 4.2.10, using (4.25), it follows that v satisfies $v = A_v^{(2)} + B_v^{(2)}$.

Shortly we will discuss how the above characterisations can be used to determine the optimality of the two strategies; that is, under which parameter regimes would the agent optimally choose one strategy over the other. Before moving towards this step, recall that in (4.13) we substituted the underlying process of the problem from the wealth process W_t to the asset price process X_t . We hence note that the optimal stopping rules arising from the assumptions of Propositions 4.2.8 and 4.2.10 can be re-written in terms of the wealth process by using (4.2). Since by definition $R_u = w$ over $[0, \tau_v^*)$ and since the game essentially restarts at every liquidation point, we observe that under both scenarios τ_v^* can be re-written as $\tau_v^* = \inf\{t \geq 0 : \frac{W_t}{R_t} \notin \mathcal{C}\}$ where $\mathcal{C} = (0, \bar{x}_v)$ or $\mathcal{C} = (0, \hat{x}_v) \cup (\underline{x}_v, \bar{x}_v)$ depending on which of the two strategies described above is optimal. Thus the agent stops and derives realisation utility only when the ratio of their wealth to current reference level exits the continuation region \mathcal{C} .

Given this characterisation of the optimal stopping rule τ_v^* , the homogeneity property allows us to also derive the solution of the original problem in (4.7). Recall

that:

$$\begin{aligned}
Z(w, r) &= \sup_{\tau} \mathbb{E}_{w, r} \left[e^{-\rho\tau} \left(U(W_{\tau-}, R_{\tau-}) + Z(KW_{\tau-}, KW_{\tau-}) \right) \mathbb{I}_{\{\tau < \infty\}} \right] \\
&= \sup_{\tau} \mathbb{E}_{w, r} \left[e^{-\rho\tau} \left(R_{\tau-}^{\gamma} U\left(\frac{W_{\tau-}}{R_{\tau-}}, 1\right) + (KW_{\tau-})^{\gamma} Z(1, 1) \right) \mathbb{I}_{\{\tau < \infty\}} \right] \\
&= \sup_{\tau} r^{\gamma} \mathbb{E}_{w, r} \left[e^{-\rho\tau} \left(U\left(\frac{W_{\tau-}}{R_{\tau-}}, 1\right) + \left(K \frac{W_{\tau-}}{R_{\tau-}}\right)^{\gamma} Z(1, 1) \right) \mathbb{I}_{\{\tau < \infty\}} \right] \quad (4.26)
\end{aligned}$$

Given that R_t is constant between liquidation times, and noting that $Z(1, 1) = V(1) = v$, we can re-write (4.26) as:

$$\begin{aligned}
Z(w, r) &= \sup_{\tau} r^{\gamma} \mathbb{E} \left[e^{-\rho\tau} \left(U\left(\frac{W_{\tau-}}{r}, 1\right) + \left(K \frac{W_{\tau-}}{r}\right)^{\gamma} V(1) \right) \mathbb{I}_{\{\tau < \infty\}} \mid \frac{W_0}{R_0} = \frac{w}{r} \right] \\
&= \sup_{\tau} r^{\gamma} \mathbb{E} \left[e^{-\rho\tau} \left(U\left(X_{\tau-}, 1\right) + \left(K X_{\tau-}\right)^{\gamma} V(1) \right) \mathbb{I}_{\{\tau < \infty\}} \mid X_0 = \frac{w}{r} \right] \quad (4.27)
\end{aligned}$$

The solution of this problem is a special case to the problem described in Propositions 4.2.8 and 4.2.10 above and the solution of (4.26) is outlined in Proposition 4.2.11 below for completeness.

Proposition 4.2.11. *Consider the problem defined in (4.27) above. The optimal stopping time is given by $\tau_v^* = \inf\{t \geq 0 : \frac{W_t}{R_t} \notin \mathcal{C}\}$ where $\mathcal{C} = (0, \bar{x}_v)$ if the one threshold strategy is optimal or $\mathcal{C} = (\underline{x}_v, \bar{x}_v)$ otherwise. The characterisation of the selling thresholds $\underline{x}_v, \bar{x}_v$ is as given in Proposition 4.2.8 for the one threshold case and Proposition 4.2.10 for the two threshold case. If the one threshold strategy is optimal, $Z(w, r)$ is given by:*

$$Z(w, r) = \begin{cases} r^{\gamma} (K^{\gamma} v (\frac{w}{r})^{\gamma} + ((\frac{w}{r}) - 1)^{\gamma}) & \text{for } \frac{w}{r} \geq \bar{x}_v \\ r^{\gamma} (K^{\gamma} v \bar{x}_v^{\gamma-\beta} + (\bar{x}_v - 1)^{\gamma} \bar{x}_v^{-\beta}) (\frac{w}{r})^{\beta} & \text{for } \frac{w}{r} < \bar{x}_v \end{cases} \quad (4.28)$$

where $v = V(1)$ is specified in (4.21). Otherwise,

$$Z(w, r) = \begin{cases} r^{\gamma} (K^{\gamma} v (\frac{w}{r})^{\gamma} + ((\frac{w}{r}) - 1)^{\gamma}) & \text{for } \bar{x}_v \leq \frac{w}{r} \\ r^{\gamma} A_v^{(2)} (\frac{w}{r})^{\beta} + r^{\gamma} B_v^{(2)} (\frac{w}{r})^{\alpha} & \text{for } \underline{x}_v < \frac{w}{r} < \bar{x}_v \\ r^{\gamma} (K^{\gamma} v (\frac{w}{r})^{\gamma} - \lambda(1 - (\frac{w}{r}))^{\gamma}) & \text{for } \hat{x}_v \leq \frac{w}{r} \leq \underline{x}_v \\ r^{\gamma} (K^{\gamma} v \hat{x}_v^{\gamma-\beta} - \lambda(1 - \hat{x}_v)^{\gamma} \hat{x}_v^{-\beta}) (\frac{w}{r})^{\beta} & \text{for } \frac{w}{r} < \hat{x}_v \end{cases} \quad (4.29)$$

where the constants $A_v^{(2)}$ and $B_v^{(2)}$ are as given in Proposition 4.2.10 and $v = A_v^{(2)} +$

$B_v^{(2)}$.

From (4.28) and (4.29), we note that the value function $Z(w, r)$ is always positive and hence it is always optimal for the agent to invest in the risky asset at time 0 and subsequently to re-invest at all liquidation times. This is due to the fact that the agent's marginal utility is infinite at the reference level r and so it is always advantageous for the agent to enter and re-enter the game. In contrast, in Barberis and Xiong [2012], the authors first assume that the value function is positive and then exhibit a range of parameter values for which this holds. This is because under their framework, it is not always optimal for the agent to enter (or re-enter) the game, which is due to the fact that utility defined by a piece-wise linear function provides finite marginal utility at the reference level.

Having completely characterised the two possible optimal solutions arising for this particular problem, it is now possible to determine conditions on the underlying parameters which distinguish between the two solutions. In fact by using the characterisations of the value function under each of the two cases described in Propositions 4.2.8 and 4.2.10, we show that there exists a critical value for the loss aversion parameter λ which differentiates between the two solutions and determines which of the two strategies is optimal. This result is outlined in Proposition 4.2.12 below and the idea behind it is borrowed from Proposition 1 in Ingersoll and Jin [2013] which describes a similar condition for their specification.

Proposition 4.2.12. *The problem described in (4.27) has a two threshold strategy if and only if λ satisfies $\lambda < \lambda^*$ where:*

$$\lambda^* = \frac{(\bar{x}_* - 1)^{\gamma-1} \bar{x}_*^{-\beta}}{(1 - \underline{x}_*)^{\gamma-1} \underline{x}_*^{-\beta}} \quad (4.30)$$

where the lower and upper thresholds $0 < \underline{x}_* < 1 < \bar{x}_*$ solve:

$$(\gamma - \beta)x + \beta = \gamma K^\gamma x^{\gamma-\beta} \quad (4.31)$$

The proof is given in Appendix C.2. Note that since the variables $\gamma \in (0, 1)$, $\beta > \gamma$ and $K \in (0, 1]$ are fixed, 4.31 has two solutions in $(0, \infty)$. This is because the LHS is linear in x and the RHS is decreasing and convex in x .

Proposition 4.2.12 describes how the agent's choice of strategy varies with the parameters determining their preference structure; specifically λ and γ , and the value of β which is determined by the underlying market dynamics. As discussed within the proof of Proposition 4.2.12, this result follows from the observation that when $\lambda = \lambda^*$, the value function determined by the one-threshold strategy described

in (4.20) is equivalent to the value function determined by the two-threshold strategy described in (4.25). Geometrically this coincides with the idea that when $\lambda = \lambda^*$, the tangent line determining the smallest non-negative concave majorant of $g_v(y)$ passes through the point $(0, 0)$ and touches $g_v(y)$ tangentially at two points $(\underline{y}, g_v(\underline{y}))$ and $(\bar{y}, g_v(\bar{y}))$ with $0 < \underline{y} < 1 < \bar{y}$. An example of such a case is depicted in Figure 4.1 below.

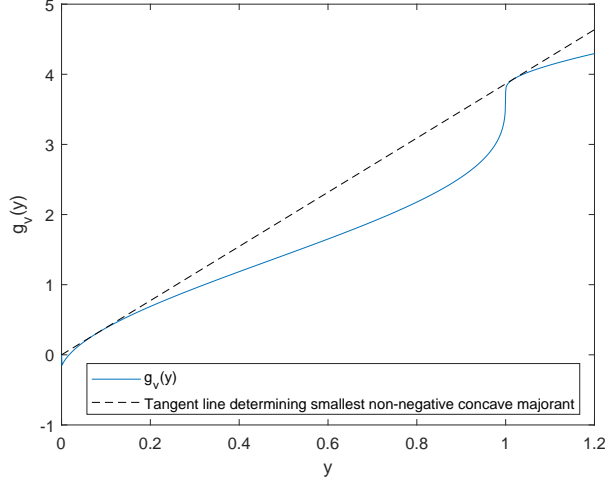


Figure 4.1: The transformed reward function $g_v(y)$ and it's tangent determining the smallest non-negative concave majorant. (Parameter values: $\alpha = -1.5$, $\beta = 5$, $K = 0.9$, $\gamma = 0.3$, $\lambda = 3.9025$)

The definition of the threshold λ^* described in (4.30) or equivalently (C.25) offer some insight on how the agent's strategy will change with the underlying parameters. In fact, as γ increases, given that $\underline{x}_* < 1 < \bar{x}_*$, we see from (C.25) that λ^* will decrease towards 0. By definition the agent's utility function $U(\cdot)$ becomes steeper close to 0 on both the gains side and the losses side as γ decreases to 0. This implies that the agent's risk aversion on both the losses side and the gains side increases as γ decreases and thus the agent is forced to realise their losses earlier.

Since we require $\lambda > 1$ for the problem to be economically feasible, this means that for large enough γ close to β , we expect that the two-threshold strategy will not be optimal. This effect is clearly visible in Figure 4.2 where as γ increases towards β , the set of values of λ for which the agent adopts a two threshold strategy shrinks rapidly. Note that if β is sufficiently close to γ this implies that the agent's growth rate for realisation utility $\gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$ described in Proposition 4.2.2 is sufficiently close to the discount rate ρ . Under such scenarios, the agent is more likely to wait since the effect of discounting is felt less.

In Figure 4.2 we also see that as λ changes, whilst the upper boundary is

relatively stable close to 1, the lower threshold decreases until $\lambda = \lambda^*$, at which point the lower threshold \underline{x}_* jumps downwards to 0. This discontinuity at λ^* can also be observed in the arguments used in the proof of Proposition 4.2.12. The existence of the critical value λ^* relies on the fact that the value function for the two-threshold strategy decreases in λ whereas the value function of the one threshold strategy does not depend on λ . Furthermore for $\lambda = \lambda^*$, the two value functions are equal and we can still find a lower threshold \underline{x}_* which solves (4.31), implying that $\underline{x}_* > 0$. But for $\lambda > \lambda^*$ the one-threshold strategy is optimal implying the discontinuity of \underline{x}_* at λ^* .

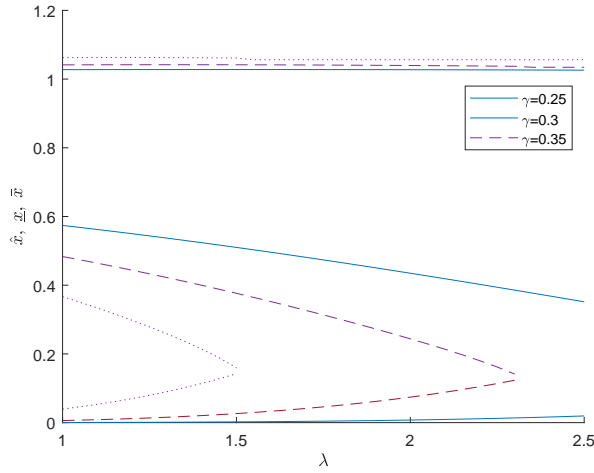


Figure 4.2: Plot illustrating how the optimal strategy changes as λ and γ are varied. (Parameter values: $\alpha = -1.667$, $\beta = 0.667$ and $K=0.9$.)

Figure 4.2 captures an important difference to the results obtained by Ingersoll and Jin [2013] with preferences described by $U_{IJ}(w, r)$ as defined in (4.6). From Figure 2 in Ingersoll and Jin [2013] (Page 732), one notices that under their framework the agent is more likely to sell at a loss as the parameters α_G and α_L increase; where α_G and α_L play a very similar role to γ in our model. Thus under their framework as α_G and α_L decrease and the S-shape becomes more pronounced, the loss threshold decreases. However this is contrary to what one expects when dealing with S-shaped preferences since a more pronounced S-shape means that the agent is more loss-averse. As shown in Figure 4.2, this relationship is captured by our model, since the lower threshold increases as γ decreases.

Proposition 4.2.12 also captures the special case considered in Barberis and Xiong [2012], when $\gamma = 1$, giving a piece-wise linear utility function. In fact plugging $\gamma = 1$ in (4.30) and (4.31), we get $\lambda^* = \frac{x^\beta}{x^\beta}$, and since $\beta > 1$, it is never optimal for the agent to liquidate at a loss since $\lambda^* < 1$.

It is worth noting that in Figure 4.2 we also depict the deep-loss threshold described in Propositions 4.2.10 and 4.2.11. The analytical characterisation of this threshold as given in these Proposition is essential for completeness of the results obtained. However it is worth mentioning that this threshold offers little insight as it is relatively very close to 0 in value. In fact while a similar threshold exists for the problem in Ingersoll and Jin [2013], they completely omit it from their analysis.

To conclude this section, by applying the methodology for optimal stopping in Dayanik and Karatzas [2003], we have solved a liquidation problem for the Cumulative Prospect Theory S-shaped Utility function, first introduced in Tversky and Kahneman [1992]. Our model contains the model of Barberis and Xiong [2012] as a special case. While comparing our solution to a similar problem derived in Ingersoll and Jin [2013], we have also shown that even when considering the classical Cumulative Prospect Theory utility in the realisation utility model devised in Barberis and Xiong [2012], it might be optimal for the agent to sell at a loss. In the next section we will provide a different new framework under which it is again optimal for the agent to sell at a loss.

4.3 Realisation Utility - An Alternative Model

In this section we extend the problem described in Section 4.2 and consider a new utility specification inspired by the solution from the previous section which will allow us to again obtain solutions which include both an upper (profitable) and a lower (loss) selling boundary. Consider again the set-up described in Section 4.2 whilst considering the utility function defined by:

$$\tilde{U}(w, r) = \begin{cases} -\lambda(r - w)^{\gamma_1} \left(\frac{w}{r}\right)^{-\eta_1}, & \text{for } w \leq r \\ (w - r)^{\gamma_2} \left(\frac{w}{r}\right)^{-\eta_2}, & \text{for } w > r \end{cases} \quad (4.32)$$

with $0 < \gamma_1, \gamma_2 < 1$, $\eta_1, \eta_2 \geq 0$, $\lambda \geq 1$, $\eta_2 < \gamma_2$ and $U_w(r-, r) = U_w(r+, r) = \infty$. Again the parameters γ_1 and γ_2 capture relative risk aversion over losses and gains respectively and λ captures loss aversion.

The key difference from the S-shaped utility function defined in (4.5) in Section 4.2 is the application of the multiplicative factors $\left(\frac{w}{r}\right)^{-\eta_1}$ and $\left(\frac{w}{r}\right)^{-\eta_2}$ on the gains side and losses side respectively.

Note that all utility models considered thus far in the context of realisation utility for a setting inspired by the work of Barberis and Xiong [2012] resulted in optimal strategies which depend on the value of the agent's proportion of wealth to reference level. This means that the agent values the proportion $\frac{w}{r}$ in these kind of set-ups, thus inspiring us to include it as part of the agent's preference characterisation. Note that the inclusion of the scaling factors $\left(\frac{w}{r}\right)^{-\eta_1}$ and $\left(\frac{w}{r}\right)^{-\eta_2}$ on the losses and gains sides respectively, take the form of scaling factors to the Cumulative Prospect Theory specification considered in Section 4.2. These factors however play a very different role from the scaling factors considered in the specification by Ingersoll and Jin [2013] as they alter the shape of the utility function, particularly on the losses side. This difference is due to the fact the scaling factors they consider depend solely on the agent's reference level which is constant between liquidations. This has the effect of decreasing the agent's loss aversion between liquidations when compared to the classical Prospect Theory utility considered in Section 4.2, thus making loss taking more probable.

The effects of the factors $\left(\frac{w}{r}\right)^{-\eta_1}$ and $\left(\frac{w}{r}\right)^{-\eta_2}$ in our model are two-fold and can be observed in Figure 4.3 below. Firstly, whilst $\tilde{U}(w, r)$ is still concave on the gains side, larger values of $\eta_2 \in (0, 1)$ contribute to a faster decrease in marginal utility over the gains side. On the losses side, for $\eta_1 \in (0, 1)$, the utility is not bounded below any more and the function $\tilde{U}(w, r)$ decreases to $-\infty$ the closer we get to 0, with the gradient getting steeper with larger values of η_1 . This means

that the agent is highly penalised the closer they get to 0 or equivalently from experiencing very big losses.

It is worth noting in fact, that this assumption in essence captures the interplay between two general ideas in utility theory:

1. The use of reference dependent preferences for realisation utility as proposed by Barberis and Xiong [2012].
2. The idea that the agent experiences infinite marginal utility close to zero.

In fact, the main difference from the Cumulative Prospect Theory utility function defined in (4.5) is that under the usual definition of reference dependent preferences, the agent experiences decreasing marginal utility the closer they get to losing everything, which is not the case any more in the model described in (4.32).

In developing a solution similar to the one discussed in Section 4.2, we again impose the assumption that the parameters γ_1, γ_2 in (4.32) above satisfy $\gamma_1 = \gamma_2 = \gamma$. This implies that the utility function defined in (4.32) is again homogeneous in r of degree γ . We also impose the assumption $\eta_1 = \eta_2 = \eta > 0$, with $\eta < \gamma$; which ensures that U is monotonically increasing. Whilst the assumption $\gamma_1 = \gamma_2 = \gamma$ is necessary for the discussion that follows, the second assumption; $\eta_1 = \eta_2 = \eta$ does not affect the general structure of the solution of the problem.

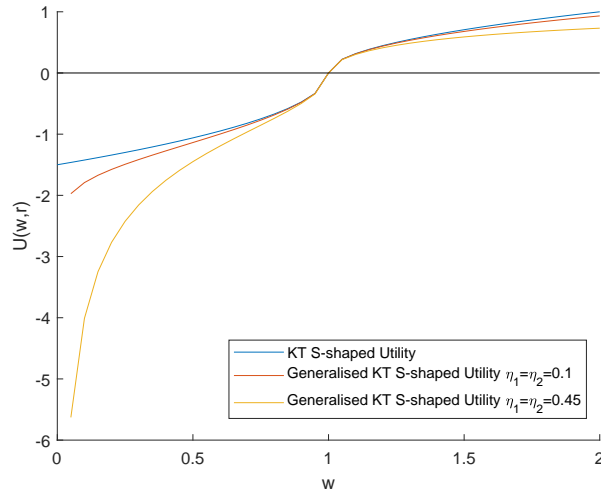


Figure 4.3: Comparison of the KT-Utility defined in (4.5) and the generalised KT-Utility defined in (4.32). (Parameter values: $\gamma_1 = \gamma_2 = 0.5$ and $\lambda = 1.5$.)

We again study the problem of an agent who has a position in a risky asset who wishes to liquidate their position whilst optimising their expected realisation

utility. The agent then reinvests their proceeds in the same asset and the game essentially restarts. This leads us to characterise the agent's problem as:

$$\bar{Z}(w, r) = \sup_{0 \leq \tau_1 \leq \tau_2 \leq \dots} \mathbb{E}_{w, r} \left[\sum_{n=1}^{\infty} e^{-\rho \tau_n} \tilde{U}(W_{\tau_n-}, R_{\tau_n-}) \mathbb{I}_{\{\tau_n < \infty\}} \right] \quad (4.33)$$

where W_t is defined in (4.2), R_t is defined in (4.3), the underlying risky-asset price process X_t defined in (4.1), $U(w, r)$ defined in (4.32) and $\rho > 0$. The following Proposition implies that the same well-posedness conditions to those imposed for the problem discussed in Section 4.2 follow for this problem:

Proposition 4.3.1. $\bar{Z}(w, r) < \infty \iff \rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$ (or equivalently $\gamma \leq \beta$).

A proof is given in Appendix C.4. In the remaining part of this Section we will hence assume that the condition $\rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$ or equivalently $\beta \geq \gamma$ is satisfied. Notice that it follows that the value function of the problem described in (4.33) is again homogeneous in r of degree γ ; which follows from the homogeneity of the underlying reward function. Furthermore the same argument developed in the proof of Proposition 4.2.5 for the Dynamic Programming Principle would still hold. Thus we state without proof the following Lemma and Proposition:

Lemma 4.3.2. $\bar{Z}(w, r)$ is homogeneous in r of degree γ .

Proposition 4.3.3. The following Dynamic Programming Principle holds:

$$\bar{Z}(w, r) = \sup_{\tau} \mathbb{E}_{w, r} \left[e^{-\rho \tau} \left(\tilde{U}(W_{\tau-}, R_{\tau-}) + \bar{Z}(KW_{\tau-}, KW_{\tau-}) \right) \mathbb{I}_{\{\tau < \infty\}} \right]. \quad (4.34)$$

Let $\bar{V}(w) = \bar{Z}(w, w)$. Then applying Lemma 4.3.2, we can show that:

$$\bar{V}(1) = \sup_{\tau} \mathbb{E} \left[e^{-\rho \tau} \left(\tilde{U}(X_{\tau}, 1) + (KX_{\tau})^{\gamma} \bar{V}(1) \right) \mathbb{I}_{\{\tau < \infty\}} | X_0 = 1 \right] \quad (4.35)$$

and by temporarily fixing $\bar{V}(1) = \bar{v}$, we can define the following complimentary optimal stopping problem:

$$\bar{\mathcal{H}}(\bar{v}, x, \tau) = \mathbb{E} \left[e^{-\rho \tau} \left(\tilde{U}(X_{\tau}, 1) + (KX_{\tau})^{\gamma} \bar{v} \right) \mathbb{I}_{\{\tau < \infty\}} | X_0 = x \right] \quad (4.36)$$

$$\bar{\mathcal{H}}(\bar{v}, x) = \sup_{\tau} \bar{\mathcal{H}}(\bar{v}, x, \tau). \quad (4.37)$$

Thus the value function of the problem in (4.35) is the solution of the fixed-point problem $\bar{v} = \bar{\mathcal{H}}(\bar{v}, 1)$. A uniqueness result to this fixed point problem is discussed

in the following Lemma:

Lemma 4.3.4. *Let $\bar{\mathcal{H}}(\bar{v}, x)$ be as defined in (4.36) and (4.37). The optimal stopping time τ^* is a hitting time. Furthermore if a solution to the fixed point problem $\bar{v} = \bar{\mathcal{H}}(\bar{v}, 1)$ exists, then it is unique.*

The proof of this result follows from identical arguments to those discussed in the proof of Lemma 4.2.6. Define the reward function of the problem in (4.36) and (4.37) by $h_{\bar{v}}(x)$:

$$h_{\bar{v}}(x) = K^\gamma \bar{v} x^\gamma + \tilde{U}(x, 1) \quad (4.38)$$

and note $h_{\bar{v}}$ is bounded on every compact subset of $\mathbb{R}^+/\{0\}$. Also note that the underlying one-dimensional diffusion describing the underlying price process is equivalent to the one considered in Section 4.2. Thus the functions $\psi(\cdot)$, $\phi(\cdot)$ and $F(\cdot)$ used to transform the reward function so as to apply the methodology outlined by Dayanik and Karatzas [2003] are defined as in Section 4.2. With this in mind, define the corresponding transformed reward function $g_{\bar{v}}(y) = h_{\bar{v}}(F^{-1}(y))/\phi(F^{-1}(y))$. Thus we have:

$$g_{\bar{v}}(y) = \begin{cases} K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} - \lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{\frac{-\eta-\alpha}{\beta-\alpha}}, & \text{for } y \leq 1 \\ K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma y^{\frac{-\eta-\alpha}{\beta-\alpha}}, & \text{for } y > 1 \end{cases} \quad (4.39)$$

The application of the methodology outlined in Chapter 2 to solve this problem, requires us to first analyse the geometry of the function $g_{\bar{v}}(y)$. A discussion of the structure of $g_{\bar{v}}(y)$ is provided in Appendix C.3.

As discussed in Section C.3.1, the types of solutions arising from this problem can be characterised into two cases; namely:

1. The agent will only stop and sell at a gain;
2. The holding region is disconnected and it consists of a neighbourhood of 0 and a neighbourhood of 1. Thus the agent holds the risky asset only if they start at a very deep loss or if the stock's initial gain or loss are relatively small.

Given that in the specification of the problem in (4.35) the starting value of X always takes the value 1, the threshold characterising the neighbourhood around 0 will not affect the behaviour of the agent whatsoever. In view of this, hereinafter we shall refer to the case with a disconnected continuation region as a "*two threshold strategy*". Similarly we shall refer to the case with just an upper threshold as a "*one threshold strategy*".

In Proposition 4.3.5 we characterise $\bar{V}(1)$ and the corresponding stopping threshold for when the one threshold strategy is optimal. A characterisation for the two threshold strategy follows in Proposition 4.3.6. The proof of Proposition 4.3.5 is outlined in Appendix C.4. The proof of Proposition 4.3.6 is omitted as it relies on very similar arguments.

Proposition 4.3.5. *Let $\bar{v} \in \mathbb{R}^+$ be fixed. When the optimal strategy is a one-threshold strategy, the optimal stopping rule $\tau_{\bar{v}}^*$ of the problem defined in (4.36) and (4.37) takes the form $\tau_{\bar{v}}^* = \inf\{t \geq 0 : X_t \notin (0, \bar{x}_{\bar{v}})\}$ where $\bar{x}_{\bar{v}} > 1$ solves the following non-linear equation:*

$$K^\gamma \bar{v} \bar{x}^{\gamma-\alpha} (\beta - \gamma) = \bar{x}^{-\eta-\alpha} (\bar{x} - 1)^{\gamma-1} (\gamma \bar{x} - (\beta + \eta)(\bar{x} - 1)) \quad (4.40)$$

Furthermore, we have:

$$\bar{\mathcal{H}}(\bar{v}, x) = \begin{cases} K^\gamma \bar{v} x^\gamma + (x - 1)^\gamma x^{-\eta}, & \text{for } x > \bar{x}_{\bar{v}} \\ \frac{K^\gamma \bar{v} \bar{x}^{\gamma-\alpha} + (\bar{x}-1)^\gamma \bar{x}^{-\eta-\alpha}}{\bar{x}^{\beta-\alpha}} x^\beta, & \text{for } x \leq \bar{x}_{\bar{v}} \end{cases}$$

and $\bar{V}(1) = \bar{v}$ as defined in (4.35) solves:

$$\bar{V}(1) = \frac{\bar{x}^{-\eta} (\bar{x} - 1)^\gamma}{(\bar{x}^\beta - K^\gamma \bar{x}^\gamma)} \quad (4.41)$$

Proposition 4.3.6. *Let $\bar{v} \in \mathbb{R}^+$ be fixed. Suppose the optimal strategy is a two-threshold strategy. Then the optimal stopping rule $\tau_{\bar{v}}^*$ of the problem defined in (4.36) and (4.37) takes the form $\tau_{\bar{v}}^* = \inf\{t \geq 0 : X_t \notin A\}$ with $A = (0, \hat{x}_{\bar{v}}) \cup (\underline{x}_{\bar{v}}, \bar{x}_{\bar{v}})$ with $\hat{x}_{\bar{v}} < \underline{x}_{\bar{v}} < 1 < \bar{x}_{\bar{v}}$. The thresholds $\underline{x}_{\bar{v}}$ and $\bar{x}_{\bar{v}}$ solve the following system of non-linear equations:*

$$\begin{aligned} & \frac{K^\gamma \bar{v} (\bar{x}_{\bar{v}}^{\gamma-\alpha} - \underline{x}_{\bar{v}}^{\gamma-\alpha}) + (\bar{x}_{\bar{v}} - 1)^\gamma \bar{x}_{\bar{v}}^{-\eta-\alpha} + \lambda (1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\alpha}}{\bar{x}_{\bar{v}}^{\beta-\alpha} - \underline{x}_{\bar{v}}^{\beta-\alpha}} \\ &= \left(\frac{\gamma - \alpha}{\beta - \alpha} \right) K^\gamma \bar{v} \bar{x}_{\bar{v}}^{\gamma-\beta} + \left(\frac{\gamma}{\beta - \alpha} \right) (\bar{x}_{\bar{v}} - 1)^{\gamma-1} \bar{x}_{\bar{v}}^{1-\eta-\beta} - \left(\frac{\eta + \alpha}{\beta - \alpha} \right) (\bar{x}_{\bar{v}} - 1)^\gamma \bar{x}_{\bar{v}}^{-\eta-\beta} \\ &= \left(\frac{\gamma - \alpha}{\beta - \alpha} \right) K^\gamma \bar{v} \underline{x}_{\bar{v}}^{\gamma-\beta} + \left(\frac{\gamma \lambda}{\beta - \alpha} \right) (1 - \underline{x}_{\bar{v}})^{\gamma-1} \underline{x}_{\bar{v}}^{1-\eta-\beta} + \left(\frac{\eta + \alpha}{\beta - \alpha} \right) \lambda (1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\beta} \end{aligned} \quad (4.42)$$

whereas $\hat{x}_{\bar{v}}$ satisfies:

$$K^\gamma \bar{v} \hat{x}^\gamma (\beta - \gamma) = \lambda \hat{x}^{-\eta} (1 - \hat{x})^{\gamma-1} (\gamma \hat{x} + (\beta + \eta)(1 - \hat{x})) \quad (4.43)$$

Furthermore we have:

$$\bar{\mathcal{H}}(\bar{v}, x) = \begin{cases} K^\gamma \bar{v} x^\gamma + (x-1)^\gamma x^{-\eta}, & \text{for } x > \bar{x}_{\bar{v}} \\ A_{\bar{v}} x^\alpha + B_{\bar{v}} x^\beta, & \text{for } \underline{x}_{\bar{v}} \leq x \leq \bar{x}_{\bar{v}} \\ K^\gamma \bar{v} x^\gamma - \lambda(1-x)^\gamma x^{-\eta}, & \text{for } \hat{x}_{\bar{v}} < x < \underline{x}_{\bar{v}} \\ \frac{K^\gamma \bar{v} \hat{x}_{\bar{v}}^{\gamma-\alpha} - \lambda(1-\hat{x}_{\bar{v}})^\gamma \hat{x}_{\bar{v}}^{-\eta-\alpha}}{\hat{x}_{\bar{v}}^{\beta-\alpha}} x^\beta, & \text{for } x \leq \hat{x}_{\bar{v}} \end{cases}$$

where:

$$\begin{aligned} \bar{v} = & \frac{((\bar{x}_{\bar{v}} - 1)^\gamma \bar{x}_{\bar{v}}^{-\eta-\alpha} + \lambda(1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\alpha})(1 - \underline{x}_{\bar{v}}^{\beta-\alpha})}{(\bar{x}_{\bar{v}}^{\beta-\alpha} - \underline{x}_{\bar{v}}^{\beta-\alpha}) - K^\gamma \bar{x}_{\bar{v}}^{\gamma-\alpha}(1 - \underline{x}_{\bar{v}}^{\beta-\alpha}) - K^\gamma \underline{x}_{\bar{v}}^{\gamma-\alpha}(\bar{x}_{\bar{v}}^{\beta-\alpha} - 1)} \\ & - \frac{\lambda(1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\alpha}(\bar{x}_{\bar{v}}^{\beta-\alpha} - \underline{x}_{\bar{v}}^{\beta-\alpha})}{(\bar{x}_{\bar{v}}^{\beta-\alpha} - \underline{x}_{\bar{v}}^{\beta-\alpha}) - K^\gamma \bar{x}_{\bar{v}}^{\gamma-\alpha}(1 - \underline{x}_{\bar{v}}^{\beta-\alpha}) - K^\gamma \underline{x}_{\bar{v}}^{\gamma-\alpha}(\bar{x}_{\bar{v}}^{\beta-\alpha} - 1)} \end{aligned} \quad (4.44)$$

and the constants A_v and B_v are given by:

$$\begin{aligned} A_v &= \frac{K^\gamma \bar{v}(\bar{x}_{\bar{v}}^{\gamma-\alpha} - \underline{x}_{\bar{v}}^{\gamma-\alpha}) + (\bar{x}_{\bar{v}} - 1)^\gamma \bar{x}_{\bar{v}}^{-\eta-\alpha} + \lambda(1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\alpha}}{\bar{x}_{\bar{v}}^{\beta-\alpha} - \underline{x}_{\bar{v}}^{\beta-\alpha}} \\ B_v &= \frac{(K^\gamma v \underline{x}_{\bar{v}}^{\gamma-\alpha} - \lambda(1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\alpha}) \bar{x}_{\bar{v}}^{\beta-\alpha} - (K^\gamma v \bar{x}_{\bar{v}}^{\gamma-\alpha} + (\bar{x}_{\bar{v}} - 1)^\gamma \bar{x}_{\bar{v}}^{-\eta-\alpha}) \underline{x}_{\bar{v}}^{\beta-\alpha}}{\bar{x}_{\bar{v}}^{\beta-\alpha} - \underline{x}_{\bar{v}}^{\beta-\alpha}} \end{aligned}$$

Having characterized the possible solutions for the problem described in (4.36) and (4.37), we note that the solution of the problem we defined in (4.34) follows directly from the solution of the aforementioned problem. In fact, the homogeneity property of the value function $\bar{Z}(w, r)$ and the fact that R_t is constant between liquidations allows us to write:

$$\bar{Z}(w, r) = \sup_{\tau} r^\gamma \mathbb{E} \left[e^{-\rho\tau} \left(\tilde{U} \left(X_{\tau-}, 1 \right) + (K X_{\tau-})^\gamma \bar{V}(1) \right) \mathbb{I}_{\{\tau < \infty\}} \mid X_0 = \frac{w}{r} \right] \quad (4.45)$$

By comparing (4.45) to (4.36) and (4.37) it is obvious that the optimal strategies for the problem in (4.45) are identical to those described in Propositions 4.3.5 and 4.3.6.

The characterisations of the selling thresholds and the constant \bar{v} for both strategies are essential for us to develop an approach to differentiate between the two cases under different parameter regimes. This is discussed in Section 4.3.1 below. We defer the discussion outlining the key differences between Propositions 4.3.5 and 4.3.6 until after we get the necessary conditions to distinguish when each of the two

solutions is optimal.

4.3.1 Distinguishing between the two strategies

Having characterised the possible solutions arising from this problem, in this section we work on determining when either of the two possible solutions is optimal. In the first part of this section we will use the characterisations of the reward function and the corresponding value function to obtain a condition which determines whether the two threshold strategy is optimal. This is presented in Proposition 4.3.8. For completeness, we then use a similar methodology to that employed in Proposition 4.2.12 of Section 4.2 to obtain another condition on λ which determines which of the two solutions is optimal. We then end the section by showing that these two conditions are equivalent.

In what follows we will refer to a solution for the one threshold strategy as a pair $(\bar{x}, \bar{v}) \in \mathbb{R}^2$ where $\bar{x} > 1$, $\bar{v} > 0$ and the pair is a solution to the system of equations described in (4.40) and (4.41). Furthermore, a solution for the two threshold case is given by $(\hat{x}, \bar{x}_l, \bar{x}_u, \bar{v}) \in \mathbb{R}_+^4$ where $\hat{x} < \bar{x}_l < 1 < \bar{x}_u$ and $\bar{v} > 0$ are a solution to the system of equations described in (4.42), (4.43) and (4.44).

We first note that a solution for the system of equations described in (4.40) and (4.41) can always be found. This is due to the fact that given $\beta \geq \gamma$, $g_{\bar{v}}(y)$ as described in (4.39) is concave, positive and $\lim_{y \downarrow 1} g'_{\bar{v}}(y) = \infty$ for any $y > 1$ and $\bar{v} > 0$. Thus we can always construct a line passing through the points $(0, 0)$ and $(\bar{y}, g_{\bar{v}}(\bar{y}))$ satisfying:

$$\frac{g_{\bar{v}}(\bar{y})}{\bar{y}} = \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}}$$

However it is important to note that this solution is not necessarily the solution we are after as it might not define a majorant of the function $g_{\bar{v}}(y)$, let alone the smallest non-negative concave majorant. An example of this can be seen in Figure 4.4 below.

We also note that a solution to the system of equations describing the two threshold case; that is (4.42), (4.43) and (4.44) does not always exist (for example the case described in Figure C.6a). If however, we can find a real-valued solution to both systems of equations, our aim is to obtain a condition with which to determine which of the two solutions is optimal. Before discussing this problem we discuss briefly the uniqueness of solutions.

Suppose that we solve the system of equations in (4.40) and (4.41) giving (\bar{x}, \bar{v}_1) and also solve (4.42), (4.43) and (4.44) giving $(\hat{x}, \bar{x}_l, \bar{x}_u, \bar{v}_2)$. Coupled with both \bar{v}_1 and \bar{v}_2 we have the corresponding reward functions $g_{\bar{v}_1}(y)$ and $g_{\bar{v}_2}(y)$ re-

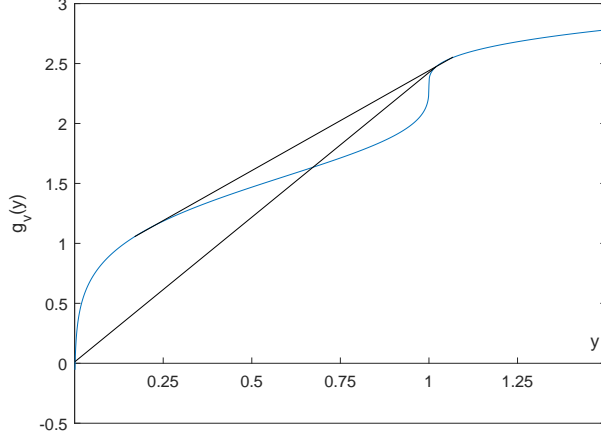


Figure 4.4: Plot of the transformed reward function $g_v(y)$ outlining that the solution obtained from (4.40) and (4.41) is not always optimal. (Parameter values: $\gamma = 0.32$, $\alpha = -0.2$, $\beta = 7.3$, $\eta = 0.3$, $\lambda = 1.6$, $v=2.4$ and $K=0.9$.)

spectively. Note that \bar{v}_1 and \bar{v}_2 are in general not equal and hence the functions $g_{\bar{v}_1}(y)$ and $g_{\bar{v}_2}(y)$ are different. Is it possible that the solution (\bar{x}, \bar{v}_1) defines the smallest non-negative concave majorant with respect to $g_{\bar{v}_1}(y)$ while $(\hat{x}, \bar{x}_l, \bar{x}_u, \bar{v}_2)$ also defines another smallest non-negative concave majorant with respect to $g_{\bar{v}_2}(y)$? This question is closely tied to the idea that the problem in (4.35) has a unique solution. We address this question in Lemma 4.3.7 below and show that the answer is no.

Lemma 4.3.7. *Consider the optimal stopping problem described in (4.36) and (4.37). Let $z_1 = (\bar{x}, \bar{v}_1)$ be a solution to (4.40) and (4.41). If it exists, also let $z_2 = (\hat{x}, \bar{x}_l, \bar{x}_u, \bar{v}_2)$ be the solution corresponding to the system described in (4.42), (4.43) and (4.44). Consider the transformed gain functions $g_{\bar{v}_1}$ and $g_{\bar{v}_2}$ corresponding with each of these sets of solutions respectively. Then only one of z_1 and z_2 defines a non-negative concave majorant with respect to $g_{\bar{v}_1}$ or $g_{\bar{v}_2}$ respectively.*

A proof is provided in Appendix C.4. Lemma 4.3.7 above guarantees that if both systems of equations have a solution, then only one of them defines a non-negative concave majorant with respect to $g_v(y)$.

In solving the system of equations describing the two threshold strategies numerically we notice that more often than not, the solution for \hat{x} is numerically indistinguishable from 0. Note however that the problem defined in (4.35) imposes the requirement $X_0 = 1$ and the parameter \bar{v} satisfies $\bar{v} = \bar{\mathcal{H}}(\bar{v}, 1)$ and hence \bar{v} is uniquely determined by the thresholds characterising the neighbourhood of 1. Thus for the problem in (4.35), the threshold \hat{x} is insignificant. It however is an essential

part of the solution for the problem in (4.34), since w and r are allowed to take any value in \mathbb{R} .

In Proposition 4.3.8 below, we provide a first result to distinguish between the two strategies in Propositions 4.3.5 and 4.3.6.

Proposition 4.3.8. *Consider the pair of systems of equations described in (4.40), (4.41) and (4.42), (4.44) above. The following cases arise:*

1. *If the system described in (4.42) and (4.44) has no real-valued, positive solution then the one threshold strategy described by (4.40) and (4.41) is optimal.*
2. *If both systems of equations allow for a solution to be found, then the two threshold strategy is optimal if and only if*

$$\frac{F(\bar{x}_u)}{F(\bar{x}_l)} \geq \frac{g_v(F(\bar{x}_u))}{g_v(F(\bar{x}_l))} \quad (4.46)$$

A proof is provided in Appendix C.4. The above result allows us to numerically characterise the optimal behaviour of the agent by adopting the condition described in (4.46) as part of our numerical procedure. This condition is fairly simple to check since all parameters are either known a priori or are found when solving the aforementioned systems of equations. Note that when we have equality in the condition in (4.46), this implies that the concave majorant defined by the two-threshold strategy is identical to that defined by the one-threshold strategy. Thus at equality the value functions of the two strategies are equal. This case arises only when $\hat{x}_v = \underline{x}_v$ where \hat{x}_v and \underline{x}_v are as defined in Proposition 4.3.6 and hence the *two threshold* strategy applies at equality.

By using an argument similar to the one used in Proposition 4.2.12 in Section 4.2 we can obtain a similar condition for the loss aversion parameter λ which determines which of the potential solutions is optimal.

Proposition 4.3.9. *The problem described in (4.33) has a two threshold strategy if and only if λ satisfies $\lambda < \lambda^*$ where:*

$$\lambda^* = \frac{(\bar{x}_* - 1)^{\gamma-1} \underline{x}_*^{\eta+\gamma} ((\beta + \eta - \gamma)\bar{x}_* - (\beta + \eta))}{(1 - \underline{x}_*)^{\gamma-1} \bar{x}_*^{\eta+\gamma} ((\beta + \eta - \gamma)\underline{x}_* - (\beta + \eta))} \quad (4.47)$$

where $0 < \underline{x}_* < 1 < \bar{x}_*$ solve:

$$(\beta + \eta - \gamma)x - (\beta + \eta) = K^\gamma x^{\gamma-\beta} (\eta(x - 1) - \gamma) \quad (4.48)$$

A proof is provided in Appendix C.4. In Propositions 4.3.8 and 4.3.9 we have given two alternative conditions determining when the two threshold strategy is optimal. Before discussing the results obtained in this section in more detail, we prove in Proposition 4.3.10 that these two conditions are in fact equivalent. The proof is relegated to Appendix C.4.

Proposition 4.3.10. *The conditions described in Propositions 4.3.8 and 4.3.9 are identical.*

4.3.2 Discussion of Results

Having fully characterized the solution of the problem in (4.36) and (4.37), we are now able to comment on how the agent's behaviour changes with the underlying parameters.

Proposition 4.3.9 distinguishes between the two possible solutions on the basis of the parameter λ and hence offers further insight on how the behaviour of the agent changes with changes in the underlying parameters. As seen in Figure 4.5a below, when the parameter γ is decreased, the loss threshold increases significantly. This is because the underlying utility function becomes steeper close to the reference level with lower values of γ .

The lower threshold also marginally increases as η decreases. This is due to the fact as seen in Figure 4.3, higher values of η push the utility function U lower on the losses side, thus making taking a loss less attractive for the agent. This relation between η and the lower threshold \underline{x} is visible in Figure 4.5b. Figures 4.5a and 4.5b also clearly show that the loss threshold decreases with an increase in λ . This follows from the definition of λ as the agent's loss aversion parameter. In fact the lower (loss) threshold decreases as λ increases to λ^* at which point the lower threshold jumps downwards to 0. Note further from Figures 4.5a and 4.5b that the critical value λ^* defined in (4.47) decreases with an increase in both γ and η , for the same reasons mentioned above.

The upper (profitable) threshold on the other hand is relatively unchanged under changes of both γ and λ . The reason why the upper threshold is always very close to 1 in these kind of problems is mainly due to the concavity of the utility function on the gains side and the fact that the underlying utility function has infinite marginal utility at 1 which decreases significantly fast.

Figures 4.6a and 4.6b depict how the selling thresholds change as the expected rate of return μ changes. The lower threshold decreases with an increase in μ . The upper-threshold increases marginally as μ increases over $(-1, 0)$. However

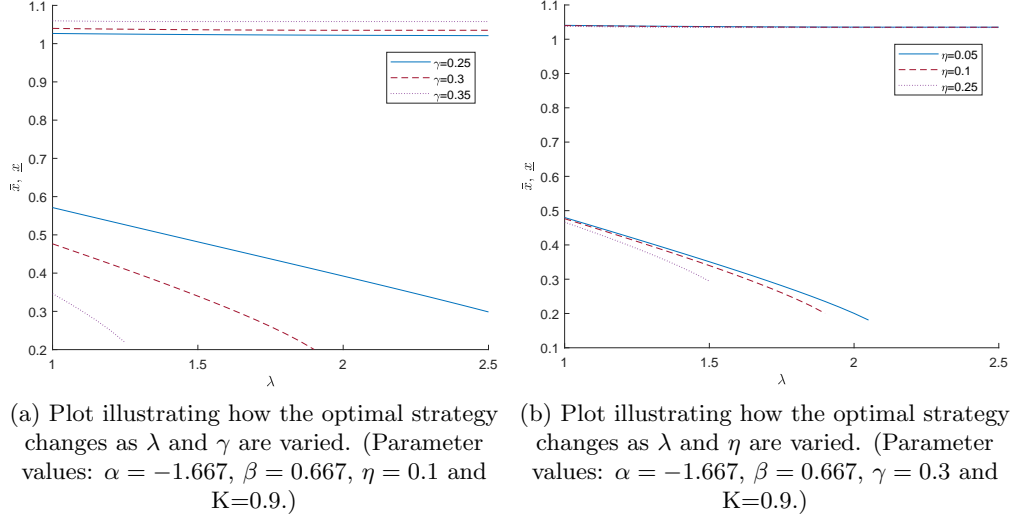


Figure 4.5: Plots describing how the optimal strategy varies with different parameters

as μ increases on the positive side, the upper threshold increases drastically towards $+\infty$. This is because as μ increases, β decreases towards γ . As shown in the proof of Proposition 4.3.1, the case $\beta < \gamma$ implies that it is never optimal for the agent to sell. This is due to the fact that under this case we have $\rho < \gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2$, and thus the effect of the discount term ρ is overpowered by the growth of the agent's realisation utility pushing them towards the case when it is never optimal to sell.

Whilst in Figure 4.6a it might seem that it is always possible for the agent to realise losses when $-\alpha > \eta$, Figure 4.6b indicates another case when this is clearly not true.

Note that while the condition obtained in Proposition 4.3.9 describes a critical value λ^* for the loss aversion λ , this condition could theoretically be re-written to give critical values for all the other parameters in this model. As an example, the critical value of μ and how it changes as we vary λ can clearly be observed in Figure 4.6a.

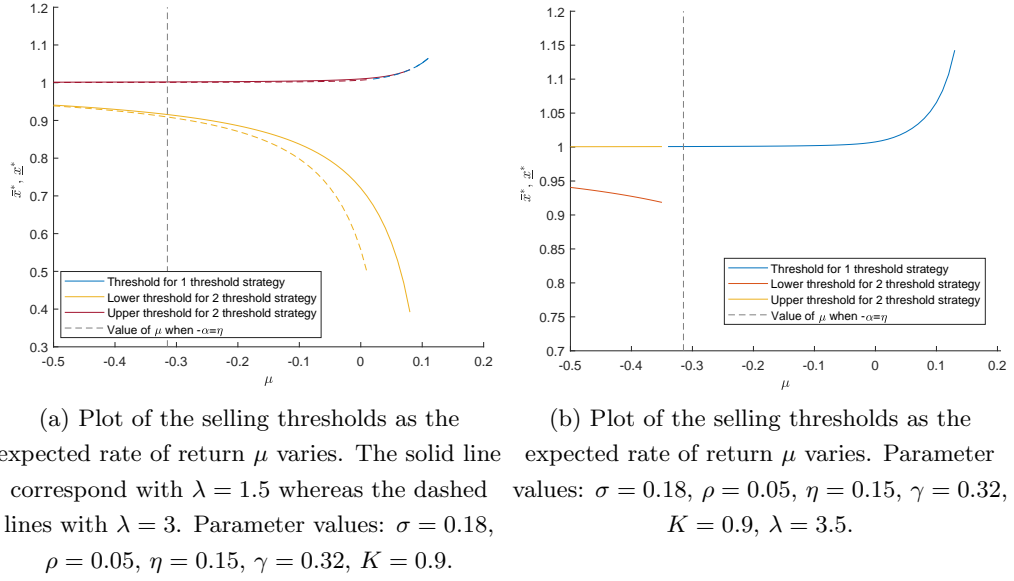


Figure 4.6: Plots describing how the optimal strategy varies with the parameter μ .

The behaviour of the selling thresholds in relation to changes in the volatility parameter σ depends heavily on the values of the drift parameter μ and the discount factor ρ . This change in behaviour can be clearly observed in Figures 4.7a and 4.7b below. It is mainly due to the fact that as we vary σ , the parameter β changes very differently depending on the values of the other parameters. In Figure 4.7a we see that when the agent observes a negative expected return μ , a higher value of σ will make them wait longer before realising a loss. This is because as σ increases, even though μ is negative, a higher value of σ is more likely to push the price upwards towards the upper selling threshold, where they can sell at a profit.

On the other hand if μ is big enough such that the condition $\beta > \gamma$ is still satisfied (Figure 4.7b), the behaviour of the agent relative to the parameter σ appears to be somewhat inverted when compared to the aforementioned case depicted in Figure 4.7a. For small enough σ the agent employs a one-threshold strategy under this case. This is because they expect the price process to drift strongly upwards. As σ increases the interplay between the three parameters σ , μ and ρ can be observed. While the agent expects the price process to drift upwards, higher values of σ increase the chance of experiencing deep losses in the shorter term. This together with the discount factor ρ drive the agent to cut their losses rather than waiting when σ increases, which is why we have a two threshold strategy as σ increases. The same reasoning applies as to why the lower threshold strategy increases as σ increases.

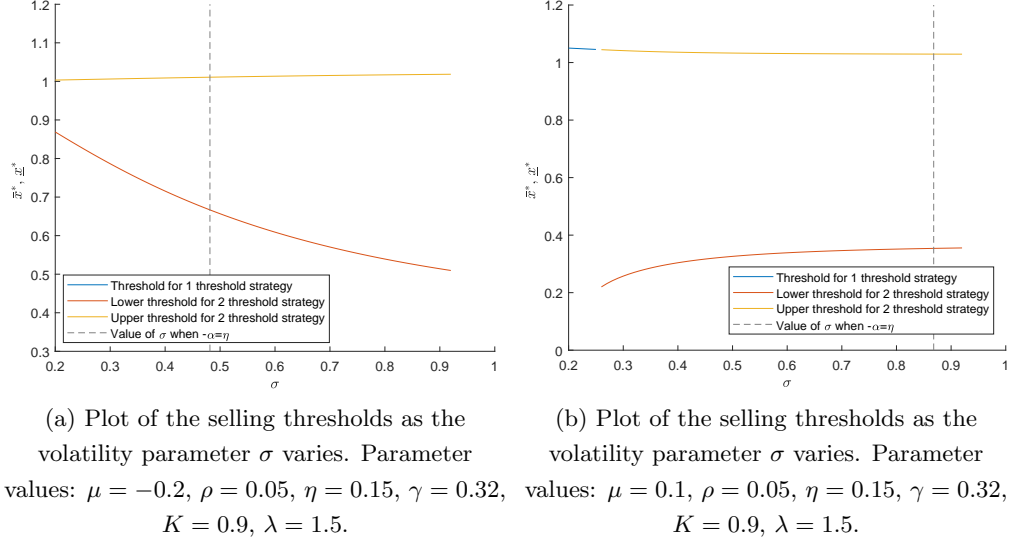


Figure 4.7: Plots describing how the optimal strategy varies with the parameter σ

4.4 Concluding Remarks

In this chapter, we have formulated two optimal stopping problems which portray an agent who derives realisation utility when selling an asset. We build on the work primarily done by Barberis and Xiong [2012]. In Section 4.2 we extend upon the model described in Ingersoll and Jin [2013] and by adopting a different methodology we specialise for an agent whose utility function is given by the standard Cumulative Prospect Theory S-shaped utility defined in Tversky and Kahneman [1992]. By using the methodology outlined in Chapter 2, we show that under this framework the agent can adopt two different strategies and we distinguish between the two.

The solution of the problem described in Section 4.2 then motivates us to extend this problem and consider a new preference structure. The utility function we consider is again reference dependent and it includes additional multiplicative factors which depend on the ratio of current wealth to reference level. This problem is again approached by adopting the methodology described in Dayanik and Karatzas [2003]. We show that again the agent can adopt two different threshold strategies. In Section 4.3.1 we provide two equivalent conditions with which to distinguish which and when each strategy is optimal.

To date, all models considered in literature in relation to realisation utility have a common assumption. This is that after each liquidation, the agent has to re-invest *all* their wealth in the risky asset. A natural extension to this model is to

extend the framework to a portfolio optimisation problem in the spirit of Merton [1969]. This problem takes the form of an impulse control problem and it is currently a work in progress.

Chapter 5

Regret Theory in a Dynamic Setting

5.1 The Regret Problem - An Introduction

The original aim of this chapter was to present our study of models in the literature wherein Regret Theory is applied in a dynamic setting and then branch out and extend upon them in two principal directions. Firstly, our aim was to extend these models; primarily the one described in Strack and Viefers [2015] from a discrete time setting into a continuous time framework. Whilst in various other areas of behavioural finance this extension to continuous time is very prevalent in the literature, there is little to no research done in this context within Regret Theory. Secondly, our aim was to propose and solve for another problem which incorporates regret-rejoice functions $R(\cdot)$ which are truer to the original formulation of Loomes and Sugden [1982]; as discussed in Section 5.1.1.

The main contribution described in this chapter is the re-formulation and extension of the dynamic model presented in Strack and Viefers [2015]. Their model presents a discrete-time optimal liquidation problem for an agent whose preferences incorporate a regret term. The model and some of its underlying assumptions are summarised in Section 5.1.2. Subsequently in Section 5.2, a new dynamic model for Regret Theory in continuous time is presented and solved, extending upon the work of Strack and Viefers [2015]. A brief discussion of some directions for extensions of our work is included in the final part of this chapter.

5.1.1 A Brief Overview of the Theory

Although Expected Utility Theory is still widely used to describe preferences made by individuals under uncertainty it is well accepted that this theory fails to capture various behavioural anomalies which individuals seem to possess when presented with risk. This was already well documented prior to the introduction of theories like Prospect Theory and Regret Theory but as argued by Bleichrodt and Wakker [2015], it was widely accepted that irrational behaviour was actually too chaotic and noisy to model appropriately.

The introduction of Regret Theory as a cornerstone theory in Behavioural Economics came in three independent papers in 1982, each approaching the idea of regret from a slightly different angle. Bell [1982]; one of the three aforementioned papers, focused on discussing how regret could be incorporated into a utility model as an additional attribute to the already existing model. Fishburn [1982] discusses in detail how the Theory can be formulated rigorously in a mathematical setting whereas Loomes and Sugden [1982] enforced the theory through an empirical approach.

In their paper, Bell [1982] theorises that most of the anomalies to Expected Utility Theory stem from the desire of an individual to circumvent the possibility that in the future they will seem to have made a bad decision even if at the time the decision was the '*best*' given all the information available. This argument is used by Bell [1982] to support this proposal that utility derived from the result of a decision should incorporate both monetary satisfaction and minimal prospective regret. The mathematical model capturing this idea shall be briefly discussed below.

The Formulation of Regret Theory

Suppose our Sample Space Ω is the set $\{\omega_1, \dots, \omega_n\}$ where each ω_i represents some state of the world, and we consider a probability measure \mathbb{P} s.t. $\mathbb{P}(\omega_i) = p_i \in (0, 1]$. An individual faces the problem of choosing from a set of actions $\{A_1, \dots, A_m\}$, and associated with each action A_i we have a random variable X_i whose range (x_{i1}, \dots, x_{in}) represents the consequences of action A_i over each state¹.

Loomes and Sugden [1982] assume that associated with every individual is a "choice-less utility function" $C(\cdot)$. Given $x \in \mathbb{R}$, $C(x)$ represents the amount of utility an individual derives from some consequence x if experienced without actually choosing it. This implies that C takes the form of a Bernoullian utility function and hence restricted over \mathbb{R}^+ , C is concave and strictly increasing.

¹In our discussion we restrict x_{ij} to represent increases or decreases in wealth relative to some level but this can be taken in a more general economical context as well.

Assuming only two actions A_1, A_2 are available if the individual chooses action A_1 given the j^{th} state of the world occurs, an individual will experience regret if $x_{1j} < x_{2j}$ and rejoice if $x_{1j} > x_{2j}$. Hence writing c_{ij} for $C(x_{ij})$, Loomes and Sugden [1982] propose modified utility m_{ij}^k to be defined by:

$$m_{ij}^k = M(c_{ij}, c_{kj}) \quad (5.1)$$

representing the utility in state j of an individual who initially chose Action A_i over A_k (The value $m_{ij}^k - c_{ij}$ represents the additional or deduction of utility due to regret/rejoice). From the above discussion a few assumptions can be made on m_{ij}^k , mainly that if $c_{ij} = c_{kj}$ then $m_{ij}^k = c_{ij}$ and also that:

$$\frac{\delta m_{ij}^k}{\delta c_{kj}} \leq 0 \quad \text{and} \quad \frac{\delta m_{ij}^k}{\delta c_{ij}} \geq 0 \quad (5.2)$$

In view of the formulation in (5.1) Loomes and Sugden [1982] propose two assumptions. Firstly that preferences between actions are made on maximizing expected modified utility over all possible actions and also that the degree of one's regret depends on the difference of the choice less utility under the realized action and the choice-less utility of another unrealized action. This leads to the the following formulation:

$$m_{ij}^k = c_{ij} + R(c_{ij} - c_{kj}) \quad (5.3)$$

where R is referred to as the "*regret-rejoice function*". It clearly follows that R is null at 0 and strictly increasing. Given that preferences between actions are established by maximal expected modified utility, then a weak preference of action A_i over A_k is established if and only if:

$$\sum_{j=1}^n p_j Q(c_{ij} - c_{ik}) \geq 0 \quad (5.4)$$

where $Q(\cdot)$ is given by:

$$Q(\xi) = \xi + R(\xi) - R(-\xi) \quad (5.5)$$

Loomes and Sugden [1982] also give the following three alternative assumptions to characterise the function $Q(\cdot)$:

1. $Q(\cdot)$ is linear ($\forall \xi \in \mathbb{R} \quad R''(\xi) = R''(-\xi)$). This implies that individuals will be maximizing expected choice-less utility.

2. $Q(\cdot)$ is concave ($\forall \xi \in \mathbb{R} \quad R''(\xi) \leq R''(-\xi)$).
3. $Q(\cdot)$ is convex ($\forall \xi \in \mathbb{R} \quad R''(\xi) \geq R''(-\xi)$).

While they argue that the choice of Q depend on the underlying assumptions related to human psychology, their experimental results were consistent with the third assumption.

Some Properties of Regret Theory

An important property which contributed to the popularity of Regret Theory is that under this theory, the assumption of preference transitivity is dismissed. Although this was initially considered as highly unorthodox, through the dismissal of this assumption, the theory manages to capture the notion of preference reversal. This phenomenon in economics is a behavioural anomaly occurring when the individual's preferences for objects changes when these preferences are evaluated either separately or jointly.

As discussed in both Bell [1982] and Loomes and Sugden [1982], another concept distinguishing this theory from others is the notion of Dominance (or the dismissal of the Equivalence Axiom)². This manages to provide a perspective in choosing between two equally attractive prospects when viewed from a behavioural perspective. Suppose for an example that we are on the eve of an election between A and B and you own a portfolio which you expect to either go up by 5% if A wins or decline by 3% if B wins with equal probability. Suppose that we look for alternative portfolios to invest in and the two available alternatives are:

- Investment 1: If A wins asset appreciates by 6% or depreciates by 2% if B wins.
- Investment 2: If A wins asset depreciates by 2% or appreciates by 6% if B wins.

Although the two investments seem equally attractive, when viewed in comparison to the portfolio we currently own, Investment 1 might seem more desirable relative to the current portfolio. This is because Investment 2 presents a possible perceived loss of 7% if A wins and a 9% increase if B wins (due to regret).

After the introduction of Regret Theory in 1982, as discussed in Bleichrodt and Wakker [2015], there has been a large body of new literature with empirical studies most of which substantiating predictions made under this theory. Some

²This is also referred to in literature as the juxtaposition effect

such studies supporting this theory are Loomes and Sugden [1987], Starmer and Sugden [1989] and Starmer [1992].

To this day, the theory remains very strongly ascribed to when explaining real world behavioural anomalies in a range of topics, in particular those relating to finance. Muermann et al. [2006] for example show that anticipated regret has a strong effect on the amount of stock an investor holds. Muermann and Volkman [2007] on the other hand show that the well documented disposition effect; that is, the reluctance of investors to realise losses and eagerness to realise gains, can also be explained through Regret averseness.

5.1.2 Regret in a Dynamic Setting

Strack and Viefers [2015] propose a discrete time model describing an agent observing a series of pay-offs X_t ; whose dynamics are described by a multiplicative binomial random walk; that is, given $X_0 = x > 0$ then:

$$X_{t+1} = \begin{cases} hX_t & \text{with probability } p \\ \frac{1}{h}X_t & \text{with probability } 1 - p \end{cases} \quad (5.6)$$

where $h > 1$ and $p \in (\frac{1}{2}, 1)$. At each time $t \in \mathbb{Z}^+$ the agent decides whether to continue observing or stop and receive the pay-off $X_t - K$ where $K > 0$ is some fixed reference level. Further to the above dynamics, Strack and Viefers [2015] assume that the game can also come to an end at any time point with some fixed probability $1 - \delta \in (0, 1)$ which in turn gives a null pay-off. Denote this random termination time of the game by T . The aim of the agent is to choose a stopping strategy which maximises their expected *modified utility*. Utility in this context does not solely capture the idea of consumption but also incorporates an additional term capturing the agent's regret which depends on the value of the ex-post maximum level reached by the price process. These ideas are captured in Strack and Viefers [2015]'s work through a linear preference function which is very similar to the one proposed by Loomes and Sugden [1982] described in (5.3). Let the maximum process of X_t be denoted by $S_t = \max_{u \leq t} X_u \vee s$ where $S_0 = s \geq x$. Strack and Viefers [2015] assume that the relationship explaining the penalty from regret when the agent decides to sell at time $\tau > 0$ at the price level X_τ is given by:

$$G = \mathbb{I}_{\{\tau \leq T\}} (u(S_\tau - K) - u(X_\tau - K))$$

where $u(\cdot)$ is a concave, increasing utility function which also plays the same

role as the *choice-less utility function* described in Section 5.1.1. In view of this, the total utility derived from selling at time τ is then captured through a weighted sum of choice-less utility and regret-penalty; that is,

$$(1 - \lambda)\mathbb{I}_{\{\tau < T\}}(u(X_\tau - K)) - \lambda G = \mathbb{I}_{\{\tau < T\}}(u(X_\tau - K) - \lambda u(S_\tau - K)) \quad (5.7)$$

where $\lambda \in [0, 1)$ represents the intensity of regret. It is natural for the agent's objective to be defined as maximizing their expected modified utility, giving the following value function $V(x, s)$:

$$V(x, s) = \sup_{\tau > 0} \mathbb{E}_{x,s} [\mathbb{I}_{\{\tau < T\}}(u(X_\tau - K) - \lambda u(S_\tau - K))]. \quad (5.8)$$

where $\mathbb{E}_{x,s}[\cdot]$ is shorthand for $\mathbb{E}[\cdot | X_0 = x, S_0 = s]$. An important remark to make here is that although this formulation resembles the formulation originally defined by Loomes and Sugden [1982], this does not capture the notion of rejoice. Furthermore under (5.7) if the agent stops exactly when a new maximum is reached, they will still incur a *penalty* when compared to pure choice-less utility, through the scaling factor $(1 - \lambda)$.

Strack and Viefers [2015] show that the optimal stopping rule τ^* for the above problem is decomposed into three parts, depending on the initial value of the maximum process S_t . They show that there exist constants $c, C \in \mathbb{R}$, $0 \leq c \leq C$ which determine the optimal stopping time τ^* as follows:

$$\tau^* = \begin{cases} \inf\{t > 0 : X_t = c\} & \text{if } S_0 < c \\ \inf\{t > 0 : X_t = S_t\} & \text{if } S_0 \in [c, C) \\ \inf\{t > 0 : X_t = C\} & \text{if } S_0 \geq C \end{cases} \quad (5.9)$$

Note that the existence of the level c under this formulation can be justified by the fact that under this set-up, if the agent never stops then they will have null returns which is still higher than stopping X_t at a level lower than the fixed reference level K . The upper level C is justified by the fact that under this problem the agent will always stop at or before the stopping level obtained under the assumptions of the complementary Expected Utility formulation without regret penalisation.

5.2 The Regret Problem in Continuous Time

In the remainder of this chapter we provide a description of our work on the extension of the model summarised in Section 5.1.2 to a continuous time framework. The first

part of this section will provide a general framework on which the corresponding problem can be appropriately defined. A candidate stopping time is then proposed, inspired from the solution in Strack and Viefers [2015]. The main arguments and proofs to demonstrate the optimality of the proposed stopping time are outlined in the remainder of this chapter.

5.2.1 General Framework

Consider a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ supporting a Brownian Motion $W = \{W_t; t \geq 0\}$ and let $X = \{X_t; t \geq 0\}$ be geometric Brownian Motion characterised by the stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad (5.10)$$

where the parameters $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ are constant and $X_0 = x$. Let $f(\cdot)$ be the corresponding scale function defining the analogous local martingale $Y_t = f(X_t)$ where $f(\cdot)$ is defined by:

$$f(z) = \begin{cases} z^\eta & \eta > 0 \\ -(z^\eta) & \eta < 0 \\ \ln(z) & \eta = 0 \end{cases} \quad (5.11)$$

with $\eta = 1 - \frac{2\mu}{\sigma^2}$. Consider also the corresponding maximum process of X_t denoted by $(S_t^X)_{t \geq 0}$ and defined by:

$$S_t^X = \max_{u \leq t} X_u \vee s \quad \text{with} \quad S_0^X = s. \quad (5.12)$$

where we assume $X_0 = x < s$. The maximum process S_t^Y corresponding to the local martingale Y_t is defined analogously by $S_t^Y = \max_{u \leq t} Y_u \vee f(s)$.

Consider an agent whose choice-less utility function is given by a monotonically increasing concave function $u \in \mathcal{C}^2(\mathbb{R}^+)$. They wish to liquidate an asset whose price dynamics follow (5.10) and at each time point they have to decide whether to continue and forego selling to a later time or sell at the current price. Their utility upon liquidation depends on two factors. Firstly, the agent derives choice-less utility from consuming the returns from the liquidation. Secondly, the agent admits a penalisation due to regret which depends on the ex-post optimal strategy; that is not having sold at the ex-post maximum of the price trajectory. In the same spirit of Strack and Viefers [2015], this regret term is defined by the difference in util-

ity derived from the agent's strategy and the ex-post optimal strategy. Assuming that the agent's final reward is given by a weighted sum of these two factors, the underlying optimisation problem can be formulated by:

$$\begin{aligned} V(x, s) &= \sup_{\tau > 0} \mathbb{E}_{x,s}[(1 - \kappa)u(X_\tau) - \kappa(u(S_\tau^X) - u(X_\tau))] \\ &= \sup_{\tau > 0} \mathbb{E}_{x,s}[u(X_\tau) - \kappa u(S_\tau^X)] \end{aligned} \quad (5.13)$$

where $\kappa \in [0, 1]$ describes the investor's intensity of the penalisation admitted due to regret when compared with the complimentary EU-agent and has a similar role as the parameter λ in Strack and Viefers [2015]'s discrete time model. Note that in this model we do not include the fixed reference level $K > 0$ used by Strack and Viefers [2015] as in (5.8). However this can easily be included as part of this problem, which would require the definition of the utility function $u(\cdot)$ to be extended over $[-K, \infty)$. It is also worth noting that the model in (5.13) assumes an infinite horizon framework under which no random termination of the game is considered.

Note that while the reward function in (5.13) does not capture the notion of rejoice, we can still recover the general form of (5.3) by scaling the choice-less utility function $u(\cdot)$. In fact by denoting $(1 - \kappa)u(\cdot)$ by $\bar{u}(\cdot)$, we have:

$$u(x) - \kappa u(s) = \bar{u}(x) + \frac{\kappa}{(1 - \kappa)}(\bar{u}(x) - \bar{u}(s))$$

giving $R(x) = \frac{\kappa}{1 - \kappa}x$ if choice-less utility is measured by $\bar{u}(\cdot)$. Furthermore the reward function in (5.13) also captures the ideas of the conditions in (5.2). In fact the total reward received by the agent increases as the underlying's price increases; that is, the agent's utility increases by considering a more favourable outcome. Secondly the agent's total utility decreases as the value of S_t increases; that is that the agent admits a bigger penalisation the further away they sell from the best observed price. It is then obvious that the second point only applies when $X_t < S_t$.

For the remainder of this chapter we assume that the agent admits an exponential *choice-less* utility function:

$$u(x) = \frac{1 - \exp(-\gamma x)}{\gamma} \quad (5.14)$$

where $\gamma > 0$ denotes the agent's level of absolute risk aversion. Later on as part of this chapter, a discussion on possible generalisations of $u(\cdot)$ will also follow. In what follows we provide our solution to the problem described in (5.13). A candidate optimal stopping rule is first proposed; inspired from the work of Strack and Viefers

[2015]. The value function corresponding to this stopping rule is derived first. A verification theorem showing that this is indeed the optimal stopping rule then follows. This result heavily depends on two key characteristics of $V(x, s)$; firstly that the process $V(X_t, S_t)$ defines a continuous super-martingale and that the stopped process $V(X_{t \wedge \tau^*}, S_{t \wedge \tau^*})$ is in fact a continuous martingale with:

$$\mathbb{E}_{x,s} \left[\lim_{t \rightarrow \infty} V(X_{t \wedge \tau^*}, S_{t \wedge \tau^*}) \right] = V(x, s).$$

5.2.2 Candidate Stopping Rule and the Characterisation of the Value Function

Consider the stopping problem defined in (5.13) and let $H_a = \inf\{t \geq 0 : X_t \geq a\} = \inf\{t \geq 0 : Y_t \geq f(a)\}$ ³ be the first hitting time for X_t of the level $a \in \mathbb{R}^+$. We propose the candidate stopping rule $\tilde{\tau}$ by:

$$\tilde{\tau} = H_{s \vee b} \wedge H_B \tag{5.15}$$

where $b, B \in \mathbb{R}^+$ are some constants satisfying $0 \leq b < B < \infty$ and $s > 0$ is the starting value of the maximum process S as defined in (5.12). Thus the stopping rule in (5.15) can be decomposed into three cases depending on the initial value of $S_0 = s$:

- If $s \leq b$ then stop the first time X_t hits b ,
- If $s \in (b, B)$ then stop the first time X_t hits s ,
- If $s \geq B$ then stop as soon as $X_t \geq B$ is satisfied.

A representation of $\tilde{\tau}$ is presented in Figure 5.1 where the stopping region is described over the domain of $(X_t, S_t)_{t \geq 0}$. Note that by definition, for all $t > 0$, $X_t \leq S_t$ *a.s.* and hence the pair can only take values in $\{(x, s) \in \mathbb{R}^2 : 0 \leq x \leq s\}$ ⁴.

As we will now describe, this stopping rule will only be optimal under certain values of η . This follows from the fact that under certain *extreme* parameter cases, it is optimal for the agent to never stop (if for example $\mu \gg 0$) or to stop immediately (if for example $\mu \ll 0$).

Consider a complimentary liquidation problem for an expected utility agent:

$$\hat{V}(x) = \sup_{\tau} \mathbb{E}_x[u(X_{\tau})]. \tag{5.16}$$

³This equality follows from the fact that $f(\cdot)$ is strictly increasing over $(0, \infty)$.

⁴The way to think about this representation of (X, S) is that as X_t diffuses, given $X_t < S_t$ is satisfied, the change in value of X_t is the represented as a vertical move upwards or downwards.

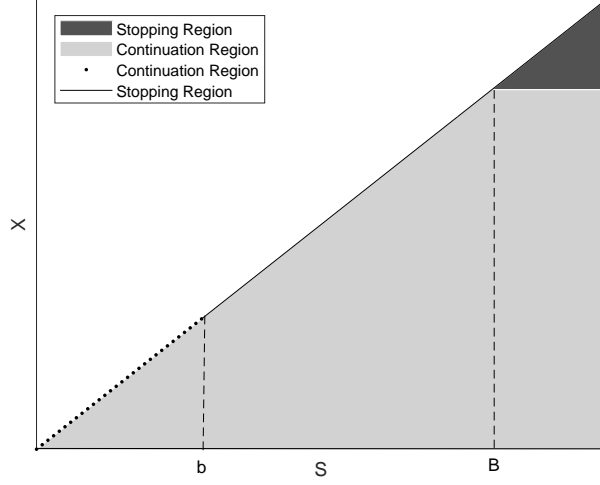


Figure 5.1: Proposed Stopping Rule

Note that the regret agent would always optimally stop either before or with the EU agent as they have less incentive to continue and thus the value function $V(\cdot)$ defined in (5.13) is majorised everywhere by $\hat{V}(\cdot)$. The following result uses the solution of the problem in (5.16) to determine the degenerate solutions of (5.13)⁵.

Proposition 5.2.1. *Consider the optimal stopping problem described in (5.13). If $\eta \geq 1$ then the optimal stopping time is given by $\tau^* = 0$. Furthermore if $\eta \leq 0$ then it is optimal for the agent to never stop; that is, $\tau^* = \infty$.*

The proof of Proposition 5.2.1 is relegated to Appendix D.

Remark 5.2.2. *Note that by adopting the methodology outlined in Dayanik and Karatzas [2003] it is very straightforward to show that for $\eta \in (0, 1)$ the expected utility agent maximising (5.16) adopts a reservation level $B > 0$ and stops the first time the price process X_t is at or above B .*

In order to restrict our analysis to cases when the solutions are non degenerate, the following assumption is imposed:

Remark 5.2.3. *Assume $\eta \in (0, 1)$.*

Given the candidate stopping time $\tilde{\tau}$ defined in (5.15), the value function \tilde{V} corresponding to $\tilde{\tau}$ is derived in the following result. Given the structure of $\tilde{\tau}$ discussed above, the proof of the following result is split into three parts considering the following sub-regions of \mathbb{R}^2 respectively: $\{(x, s) \in \mathbb{R}^2 : 0 < x \leq s < b\}$, $\{(x, s) \in \mathbb{R}^2 : 0 < x \leq s \text{ \& } s \in (b, B)\}$ and $\{(x, s) \in \mathbb{R}^2 : 0 < x \leq s \text{ \& } s \geq B\}$.

⁵Degenerate solutions are solutions which satisfy $\tau^* = 0$ or $\tau^* = \infty$ almost surely.

Proposition 5.2.4. *Consider the optimal stopping problem described in (5.13). The value function $\tilde{V}(x, s)$ corresponding with the stopping time $\tilde{\tau}$ defined in (5.15) is:*

$$\begin{aligned} \tilde{V}(x, s) = & \mathbb{I}_{\{s < b\}} \left[\frac{u(b)}{f(b)} f(x)(1 - \kappa) - \kappa \left(u(s) \left(1 - \frac{f(x)}{f(s)} \right) + \int_{f(s)}^{f(b)} u(f^{-1}(\omega)) \frac{f(x)}{\omega^2} d\omega \right) \right] \\ & + \mathbb{I}_{\{s \geq B\}} \left[\left(f(x) \frac{u(B)}{f(B)} - \kappa u(s) \right) \mathbb{I}_{\{x \leq B\}} + \left(u(x) - \kappa u(s) \right) \mathbb{I}_{\{x > B\}} \right] \\ & + \mathbb{I}_{\{b \leq s < B\}} \left[u(s) \frac{f(x)}{f(s)} - \kappa u(s) \right]. \end{aligned} \quad (5.17)$$

The proof of Proposition 5.2.4 is relegated to Appendix D. Note that in the above Proposition an assumption is made on the differentiability of \tilde{V} with respect to x and s on the regions $\{s = b\}$, $\{s = B\}$, $\{x = b\}$ and $\{x = B\}$. In determining the optimality of $\tilde{\tau}$, two functions used throughout the rest of the proofs are $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $g(x) = u(f^{-1}(x))$ and $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by:

$$h(x) = \frac{g(x)}{xg'(x)}. \quad (5.18)$$

It can be easily shown that under exponential utility, $h(\cdot)$ is monotonically increasing.

Having obtained an analytical characterisation of $\tilde{V}(x, s)$, it is essential to note that the constants b and B constituting the proposed stopping time $\tilde{\tau}$ are still arbitrary. Following the derivation of $\tilde{V}(x, s)$ it is now possible to determine what values these constants must take for $\tilde{\tau}$ to be considered as a candidate optimal stopping time.

Corollary 5.2.5. *The constant $\tilde{B} = f(B)$ satisfies:*

$$h(\tilde{B}) = 1. \quad (5.19)$$

If $\eta \geq (1 - \kappa)$ then $\tilde{b} = f(b) = 0$. Otherwise \tilde{b} satisfies:

$$h(\tilde{b}) = 1 - \kappa. \quad (5.20)$$

The proof of Corollary 5.2.5 is relegated to Appendix D. Note that the value of reservation price B for the Expected Utility problem (discussed in Remark 5.2.2) matches the level B satisfying (5.19) under the strategy $\tilde{\tau}$. Given that the Regret-agent will by definition always stop at or before the EU agent, it makes sense that

the two agents stop at the same reservation level B when $s \geq B$ since under this scenario the maximum process will remain constant at $s = S_0$ throughout.

For $s < B$, one expects that the behaviour of an agent adopting the strategy $\tilde{\tau}$ of (5.15) will be different from that of the EU-agent, since the maximum process S_t , which directly affects the agent's preference structure, will change value before X_t reaches the Expected Utility reservation price B . Corollary 5.2.5 states that as κ increases, the optimal behaviour of the Regret-agent differs more than that of the EU-agent as the lower reservation level b decreases to 0.

Furthermore, Corollary 5.2.5 also captures another key intuition of how the behaviour of the agent is affected by the underlying price dynamics of X_t . In fact if the ratio $\frac{2\mu}{\sigma^2}$ is small enough that it satisfies $\frac{2\mu}{\sigma^2} \leq \kappa$ (or equivalently $\eta \geq 1 - \kappa$), then the lower reservation level b is equal to 0 and whenever $s < B$, the agent will always stop as soon as X_t reaches the current value of S_t . Note that when $X_t = S_t$, at $t + \delta t$ the agent's utility will increase if X_t moves upwards realising a new maximum or they will instantaneously experience a decrease in utility if the price drops again below S_t . As $\frac{2\mu}{\sigma^2}$ decreases the probability that the agent experiences an instantaneous decrease in X_t increases, which is why for small enough $\frac{2\mu}{\sigma^2}$, the Regret-agent will never choose to continue after X_t reaches the maximum S_t .

From Corollary 5.2.5 it is also clear how the Expected Utility problem in (5.16) is a special case of the regret problem in (5.13). In fact $\kappa = 0$ gives $b = B$ whereas the value of B is unaffected by κ , giving the results discussed above.

5.2.3 Verification of Optimality

Given the characterisation of the value function $\tilde{V}(x, s)$ described in Proposition 5.2.4, this can now be used to determine whether the corresponding stopping time $\tilde{\tau}$ is optimal. In order to prove the main result; that is, Theorem 5.2.9, a few results characterising $\tilde{V}(x, s)$ are first discussed below.

Proposition 5.2.6. *Consider the value function $\tilde{V}(x, s)$ defined in (5.17). The process $\tilde{V}(X_t, S_t^X)$; with X_t and S_t as defined in Section 5.2.1, defines a continuous super-martingale.*

The proof of Proposition 5.2.6 is relegated to Appendix D. In the proof of the above Proposition we note that for $s \in [b, B]$, the condition $\tilde{V}_s(s, x) \leq 0$ is satisfied over the region $\delta\mathcal{C} = \{(x, s) : x = s\}$ as expected. The idea behind this condition is that since it is optimal to stop over this region then $\tilde{V}(x, s)$ would decrease if X_t were allowed to diffuse further.

The next proposition outlines another important property of $\tilde{V}(x, s)$; that is, that corresponding stopped process $\tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}})$ is a uniformly integrable continuous martingale.

Proposition 5.2.7. *Consider the stopped process $\tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}})$, where $\tilde{V}(x, s)$ is as defined in (5.17). This defines a uniformly integrable martingale and thus,*

$$\mathbb{E}_{x,s} \left[\lim_{t \rightarrow \infty} \tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}}) \right] = \lim_{t \rightarrow \infty} \mathbb{E}_{x,s} \left[\tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}}) \right] = \tilde{V}(x, s)$$

The proof of Proposition 5.2.7 is relegated to Appendix D. Lastly, before proving the main result of this section stated in Theorem 5.2.9 the following lemma captures another important property of the value function $\tilde{V}(x, s)$. The proofs of Lemma 5.2.8 and Theorem 5.2.9 are outlined in Appendix D:

Lemma 5.2.8. *For every $(x, s) \in \{(x, s) \in \mathbb{R}^2 : 0 < x \leq s\}$, the value function $\tilde{V}(x, s)$ as characterised in (5.17) majorises the reward function $u(x) - \kappa u(s)$; that is*

$$\tilde{V}(x, s) \geq u(x) - \kappa u(s). \quad (5.21)$$

Theorem 5.2.9. *The proposed stopping time $\tilde{\tau}$ is the optimal stopping time for the problem defined in (5.13); that is,*

$$\mathbb{E}_{x,s} [u(X_{\tilde{\tau}}) - \kappa u(S_{\tilde{\tau}})] = \sup_{\tau} \mathbb{E}_{x,s} [u(X_{\tau}) - \kappa u(S_{\tau})]$$

5.2.4 Concluding Remarks

This chapter provides a first study of modelling Regret preferences in a dynamic, continuous-time setting. The role of regret is captured through a penalisation to utility admitted by the Regret agent when compared with the complimentary EU-agent and is highlighted by showing how it alters their optimal liquidation strategy. In fact we have shown that the strategy depends on two price thresholds $0 \leq b \leq B$ and the value of the maximum process S_t relative to these constants. The higher threshold B is equivalent to that obtained under the Expected Utility Problem, whereas the threshold b depends on the agent's intensity of regret κ and the underlying price dynamics.

Several direct and indirect extensions of this work were attempted throughout our study of the above which unfortunately had to be set aside for the time being mainly due to time restrictions. Firstly an attempt was made in trying to generalise the result to any standard strictly increasing, concave utility function. Most of the results presented here would follow for a general $u(\cdot)$ satisfying:

- $u(0) = 0$,
- $u'(0) < \infty$,
- $u(x)$ and $u'(x)x$ are bounded $\forall x \in \mathbb{R}^+$.

However it is worth noting that the proof of Lemma 5.2.8 directly utilises the definition of $u(\cdot)$ in (5.14) and more work would be required to generalise the proof of this result.

Another line of work which was pursued in relation to Regret Theory was to try and define alternative reward functions to the one used in (5.13) which are closer to the general definition of regret-rejoice functions introduced by Loomes and Sugden [1982] (see (5.3)). In line with this, one of the problems we considered was:

$$V(x, s) = \sup_{\tau} \mathbb{E}_{x,s} [u(X_t) - \kappa(u(S_t) - u(X_t))^\alpha]$$

where $\kappa \in [0, 1)$ and $\alpha > 0$. A considerable amount of time was spent in trying to characterise the optimal stopping time of this problem, mainly through Excursion Theory and by solution methods similar to the one presented in Egami and Oryu [2017], however this line of work had to be set aside due to time restrictions.

Appendices

Appendix A

Appendix for Chapter 2

Proof of Corollary 2.3.10. Suppose $h : (a, b) \rightarrow \mathbb{R}$ is continuous, and $l_a = l_b = 0$. Then from (2.11) we know that $\tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}$, where

$$\Gamma := \{x \in (a, b) : V(x) = h(x)\} = F^{-1}(\hat{\Gamma})$$

and

$$\hat{\Gamma} = \left\{ y \in (F(a), F(b)) : W(y) = \frac{h(F^{-1}(y))}{\phi(F^{-1}(y))} \right\}.$$

Proposition 2.3.8 gives us that $W(y) : [0, \infty) \rightarrow \mathbb{R}$ is the smallest non-negative concave majorant of

$$H(y) = \begin{cases} \frac{h(F^{-1}(y))}{\phi(F^{-1}(y))}, & \text{if } y > 0 \\ l_a & \text{if } y = 0. \end{cases}$$

Note that the continuity of $h(\cdot)$ implies that $H(\cdot)$ is also a continuous function. This implies that $W(\cdot)$ is constructed through chords expanding between points over the graph of H . In other words, we can find $0 \leq b_1 \leq b_2 \leq \dots \leq b_n \leq \infty$ such that $\hat{\Gamma} = [b_1, b_2] \cup [b_3, b_4] \cup \dots [b_{n-1}, b_n]$ if $b_n < \infty$ or $\hat{\Gamma} = [b_1, b_2] \cup [b_3, b_4] \cup \dots [b_{n-1}, \infty)$ if $b_n = \infty$. Since F is bijective then Γ also admits a similar characterisation. Since $\tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}$, it follows that τ^* is a hitting time of X .

□

Appendix B

Appendix for Chapter 3

B.1 Additional Results

Lemma B.1.1. *Let τ_n be an $\{\mathcal{F}_t\}$ measurable stopping time and f a monotonic-increasing continuous-function satisfying $f(0) = 0$. Then the family of \mathcal{F}_{τ_n} measurable random variables $\Gamma = \{Z^\alpha : \alpha \in \mathcal{I}\}$ as defined in the Proof of Proposition 3.3.1 has the lattice property.*

Proof. Let $\alpha, \xi \in \mathcal{I}$, where $\alpha = (\alpha_{n-1}, \dots, \alpha_1)$ and $\xi = (\xi_{n-1}, \dots, \xi_1)$ satisfying:

$$\tau_n \leq \alpha_{n-1} \leq \dots \leq \alpha_1 \quad \text{and} \quad \tau_n \leq \xi_{n-1} \leq \dots \leq \xi_1$$

respectively. Furthermore define:

$$Z^\alpha = \mathbb{E}\left[e^{-\rho(\alpha_1 - \tau_n)} f\left(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\alpha_i}\right) \middle| \mathcal{F}_{\tau_n}\right]$$

$$Z^\xi = \mathbb{E}\left[e^{-\rho(\xi_1 - \tau_n)} f\left(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\xi_i}\right) \middle| \mathcal{F}_{\tau_n}\right]$$

Consider $v = (v_{n-1}, \dots, v_1) \in \mathcal{I}$, defined by¹:

$$v_i = \alpha_i \mathbb{I}_{\{Z^\alpha \geq Z^\xi\}} + \xi_i \mathbb{I}_{\{Z^\alpha < Z^\xi\}}$$

¹The fact each v_i is a stopping time follows from the fact that Z^α and Z^ξ are \mathcal{F}_{τ_n} measurable.

Defining Z^v analogously to Z^α and Z^ξ it follows that:

$$\begin{aligned}
Z^v &= \mathbb{E} \left[e^{-\rho(\alpha_1 - \tau_n)} f(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\alpha_i}) \mathbb{I}_{\{Z^\alpha \geq Z^\xi\}} \right. \\
&\quad \left. + e^{-\rho(\xi_1 - \tau_n)} f(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\xi_i}) \mathbb{I}_{\{Z^\alpha < Z^\xi\}} | \mathcal{F}_{\tau_n} \right] \\
&\geq \mathbb{E} \left[e^{-\rho(\alpha_1 - \tau_n)} f(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\alpha_i}) | \mathcal{F}_{\tau_n} \right] \\
&= Z^\alpha
\end{aligned}$$

and similarly $Z^v \geq Z^\xi$.

□

B.2 Proofs of Results in Chapter 3

Proof of Proposition 3.3.3. The problem in Proposition 3.3.3 can be expressed as the following optimal stopping problem:

$$V_1(y) = \sup_{\tau_1} \mathbb{E}_y [e^{-\rho\tau_1} U(Y_{\tau_1} - y_R)] \quad (\text{B.1})$$

and thus the corresponding reward function is given by $h(y) = U(y - h_R)$ where $U(\cdot)$ is as defined in (3.3). From Proposition 2.3.6 we first need to check whether the limits $l_{-\infty}^1$ and l_{∞}^1 are finite. We have:

$$l_{-\infty}^1 = \lim_{y \downarrow -\infty} \frac{h^+(y)}{\phi(y)} = 0 \quad \text{and} \quad l_{\infty}^1 = \lim_{y \uparrow \infty} \frac{h^+(y)}{\psi(y)} = 0$$

and hence the problem is always well-defined. Now from Proposition 2.3.8, the solution involves finding the smallest positive concave majorant of

$$g_1(\theta) = \begin{cases} \theta^{-\frac{\beta_2}{\beta}} \phi_1 [1 - A\theta^{-\frac{\gamma_1}{\beta}}] & \text{for } \theta \geq \theta_R \\ \theta^{-\frac{\beta_2}{\beta}} \phi_2 [B\theta^{\frac{\gamma_2}{\beta}} - 1] & \text{for } \theta < \theta_R \end{cases}$$

where the constants A, B and θ_R are given by $A = \exp(\gamma_1 y_R)$, $B = \exp(-\gamma_2 y_R)$ and $\theta_R = \exp(\beta y_R)$ respectively and $\theta = F(y)$.

By differentiating $g_1(\cdot)$ twice, we note that g_1 is negative and convex for $\theta \in (0, \theta_R)$ and that g_1 is non-negative and increasing over $[\theta_R, \infty)$. We also note that for $\theta > \theta_R$, the behaviour of $\frac{d^2 g_1}{d\theta^2}(\theta)$ can take two general forms. It is either

always negative; when $-\beta_2 - \beta_1 - \gamma_1 \leq 0$, or, is non-negative for $\theta \in (\theta_R, \infty) \cap \bar{B}(\theta_R, \epsilon)$ for some $\epsilon > 0$ then changes to negative for $\theta \in (\theta_R + \epsilon, \infty)$; which is the case when $-\beta_2 - \beta_1 - \gamma_1 > 0$.

The two cases both imply that under all parameter combinations, there exists $\hat{\theta} \in [\theta_R, \infty)$ such that $g_1(\theta)$ is concave over $(\hat{\theta}, \infty)$. Thus the smallest concave majorant of $g_1(\theta)$ can always be characterised as follows: Find the point $\bar{\theta}_1 > \theta_R$ satisfying

$$\frac{g_1(\bar{\theta}_1)}{\bar{\theta}_1} = \frac{dg}{d\theta}(\theta)|_{\theta=\bar{\theta}_1}$$

The smallest non-negative concave majorant $\bar{g}_1(\theta)$ of $g_1(\theta)$ is then given by the straight line joining $(0, 0)$ and $(\bar{\theta}_1, g_1(\bar{\theta}_1))$ for $\theta \leq \bar{\theta}_1$ and $g_1(\theta)$ for $\theta \geq \bar{\theta}_1$. The above condition implies that the transformed selling threshold $\bar{\theta}_1$ satisfies:

$$\bar{\theta}_1 = \left(\frac{A(\gamma_1 + \beta_1)}{\beta_1} \right)^{\frac{\beta}{\gamma_1}} \quad (\text{B.2})$$

and

$$\bar{g}_1(\theta) = \begin{cases} \theta^{-\frac{\beta_2}{\beta}} \phi_1 [1 - A\theta^{-\frac{\gamma_1}{\beta}}] & \text{for } \theta \geq \bar{\theta}_1 \\ \bar{\theta}_1^{-\frac{\beta_1}{\beta}} \frac{\phi_1 \gamma_1}{\gamma_1 + \beta_1} \theta & \text{for } \theta < \bar{\theta}_1 \end{cases}$$

Thus from Proposition 2.3.8, since $y = F^{-1}(\theta) = \frac{1}{\beta} \ln(\theta)$, we get:

$$\bar{y}_1 = y_R + \frac{1}{\gamma_1} \ln \left(\frac{\gamma_1 + \beta_1}{\beta_1} \right)$$

and

$$V_1(y) = \begin{cases} \phi_1 \left(1 - \exp(-\gamma_1(y - y_R)) \right) & \text{for } y \geq \bar{y}_1 \\ \left[\frac{\phi_1 \gamma_1}{\gamma_1 + \phi_1} \exp(-\beta_1 \bar{y}_1) \right] \exp(\beta_1 y) & \text{for } y < \bar{y}_1 \end{cases}$$

The optimality of the resulting optimal stopping time follows from Theorem 2.3.9. \square

Proof of Proposition 3.3.4. By applying Proposition 3.3.1 and by using the transformation described in (3.13), the multiple optimal stopping problem described in the statement of this Proposition can be expressed as follows:

$$\begin{aligned} V_2(y, x) &= \sup_{\tau_1 \geq \tau_2} \mathbb{E}_y [e^{-\rho \tau_1} U \left(x + \sum_{i=1}^2 (Y_{\tau_i} - y_R) \right)] \\ &= \sup_{\tau_2} \mathbb{E}_y [e^{-\rho \tau_2} V_1(Y_{\tau_2}, x + Y_{\tau_2} - y_R)] \end{aligned} \quad (\text{B.3})$$

where:

$$V_1(y, x) = \sup_{\tau_1} \mathbb{E}_y [e^{-\rho\tau_1} U(x + Y_{\tau_1} - y_R)] \quad (\text{B.4})$$

By applying the same methodology applied in the proof of Proposition 3.3.3 above, we obtain:

$$\bar{y}_1(x) = y_R - x + \frac{1}{\gamma_1} \ln \left(\frac{\gamma_1 + \beta_1}{\beta_1} \right)$$

and

$$V_1(y, x) = \begin{cases} \phi_1 \left(1 - \exp(-\gamma_1(y + x - y_R)) \right) & \text{for } y \geq \bar{y}_1(x) \\ K \exp(\beta_1(y + x - y_R)) & \text{for } y < \bar{y}_1(x) \end{cases}$$

where:

$$K = \left(\frac{\phi_1 \gamma_1}{\gamma_1 + \beta_1} \right) \left(\frac{\gamma_1 + \beta_1}{\beta_1} \right)^{-\frac{\beta_1}{\gamma_1}}$$

Hence the optimal stopping problem in (B.3) has a reward function $h_2(y, x) = V_1(y, x + y - y_R)$ characterised as follows:

$$h_2(y, x) = \begin{cases} \phi_1 \left(1 - \exp(-\gamma_1(2y + x - 2y_R)) \right) & \text{for } y \geq \hat{y}_R(x) \\ K \exp(\beta_1(2y + x - 2y_R)) & \text{for } y < \hat{y}_R(x) \end{cases}$$

with:

$$\hat{y}_R(x) = y_R - \frac{x}{2} + \frac{1}{2\gamma_1} \ln \left(\frac{\gamma_1 + \beta_1}{\beta_1} \right)$$

From Proposition 2.3.6 we again check whether the problem is well defined by checking whether the limits $l_{-\infty}^2$ and l_{∞}^2 are finite. We have:

$$l_{-\infty}^2 = \lim_{y \downarrow -\infty} \frac{h_2^+(y, x)}{\phi(x)} = 0 \quad \text{and} \quad l_{\infty}^2 = \lim_{y \uparrow \infty} \frac{h_2^+(y, x)}{\psi(x)} = 0$$

and thus a solution always exists. By Proposition 2.3.8, the solution involves finding the smallest non-negative concave majorant of

$$g_2(\theta, x) = \begin{cases} \theta^{-\frac{\beta_2}{\beta}} \phi_1 [1 - C\theta^{-\frac{2\gamma_1}{\beta}}] & \text{for } \theta \geq \hat{\theta}_R^2(x) \\ D\theta^{\frac{2\beta_1 - \beta_2}{\beta}} & \text{for } \theta < \hat{\theta}_R^2(x) \end{cases}$$

where the constants C, D and $\hat{\theta}_R^2(x)$ are given by $C = \exp(\gamma_1(2y_R - x))$, $D = K \exp(-\beta_1(2y_R - x))$ and $\hat{\theta}_R^2(x) = \exp(\beta \hat{y}_R(x))$ respectively.

By differentiating $g_2(\theta, x)$ twice w.r.t θ we note that $g_2(\theta, x)$ is convex over $(0, \hat{\theta}_R^2(x))$ and increasing over $[\hat{\theta}_R^2(x), \infty)$. A similar analysis of the second derivative

of $g_2(\theta, x)$ w.r.t θ shows that if $\gamma_1 \geq \frac{-\beta_2 - \beta_1}{2}$ then, $g_2(\theta, x)$ is concave over $[\hat{\theta}_R^2(x), \infty)$. On the other hand if $\gamma_1 < \frac{-\beta_2 - \beta_1}{2}$, there exists $\tilde{\theta}(x)$ such that $g_2(\theta, x)$ is convex over $[\hat{\theta}_R^2(x), \tilde{\theta}(x))$ and concave over $(\tilde{\theta}(x), \infty)$. Thus the smallest concave majorant $\bar{g}_2(\theta, x)$ of $g_2(\theta, x)$ can be characterised in the same way as described in the proof of Proposition 3.3.3, giving:

$$\bar{g}_2(\theta, x) = \begin{cases} \theta^{-\frac{\beta_2}{\beta}} \phi_1 [1 - \exp(\gamma_1(2y_R - x)) \theta^{-\frac{2\gamma_1}{\beta}}] & \text{for } \theta \geq \bar{\theta}_2(x) \\ \bar{\theta}_2^{-\frac{\beta_1}{\beta}} \frac{2\phi_1\gamma_1}{2\gamma_1 + \beta_1} \theta & \text{for } \theta < \bar{\theta}_2(x) \end{cases}$$

with

$$\bar{\theta}_2(x) = \left(\frac{C(\beta_1 + 2\gamma_1)}{\beta_1} \right)^{\frac{\beta}{2\gamma_1}}$$

Thus from Proposition 2.3.8, since $\theta = F^{-1}(y) = \frac{1}{\beta} \ln(\theta)$ and $V_2(y) = V_2(y, 0) = \phi(y) \bar{g}_2(F(y), 0)$ with $\bar{y}_2 = F^{-1}(\bar{\theta}_2(0))$, we get:

$$\bar{y}_2 = y_R + \frac{1}{2\gamma_1} \ln \left(\frac{2\gamma_1 + \beta_1}{\beta_1} \right)$$

and

$$V_2(y) = \begin{cases} \phi_1 \left(1 - \exp(-2\gamma_1(y - y_R)) \right) & \text{for } y \geq \bar{y}_2 \\ \left[\frac{2\phi_1\gamma_1}{2\gamma_1 + \beta_1} \exp(-\beta_1 \bar{y}_2) \right] \exp(\beta_1 y) & \text{for } y < \bar{y}_2 \end{cases}$$

giving the result. The optimality of the resulting optimal stopping times follows from Theorem 2.3.9. □

Appendix C

Appendix for Chapter 4

C.1 Characterising $g_v(y)$

The implementation of the methodology proposed by Dayanik and Karatzas [2003] to solve general optimal stopping problems - summarised in Chapter 2, requires an understanding of the geometry of the underlying transformed reward function in order to ultimately solve the underlying problem. This enables one to characterise how the corresponding non-negative concave majorant can be obtained, thus characterising the stopping and continuation regions of the corresponding problem.

This Appendix will serve as a discussion of the underlying geometry of the transformed reward function $g_v(y)$ resulting from the model described in Section 4.2. This in turn will provide an understanding of what type of solutions to expect under this problem. The transformed reward function $g_v(y)$ under this setting is given by:

$$g_v(y) = \begin{cases} K^\gamma v y^{\frac{\gamma-\alpha}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y \geq 1 \\ K^\gamma v y^{\frac{\gamma-\alpha}{\beta-\alpha}} - \lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y < 1 \end{cases} \quad (\text{C.1})$$

For $y \geq 1$, by taking derivatives w.r.t. y it is directly deducible that $g_v(y)$ is increasing in y . Apart from providing an initial characterisation of $g_v(y)$, the following result indirectly also proves that provided $\beta \geq \gamma$ the problem discussed in Section 4.2 will always admit a solution - in line with Proposition 4.2.2 discussed in the same Section.

Lemma C.1.1. *Given $\beta \geq \gamma$, for $y \geq 1$, there exists some constant $\tilde{y} \geq 1$ such that $g_v(y)$ is concave over $y \in (\tilde{y}, \infty)$.*

Proof. By taking the second derivative of $g_v(y)$ with respect to y it can be shown

that $\frac{d^2}{dy^2}g_v(y) \leq 0$ for $y \geq 1$ if and only if:

$$\begin{aligned} \beta\alpha(y^{\frac{1}{\beta-\alpha}} - 1)^\gamma + \gamma(\gamma - 1)y^{\frac{2}{\beta-\alpha}}(y^{\frac{1}{\beta-\alpha}} - 1)^{\gamma-2} - 2\alpha\gamma y^{\frac{1}{\beta-\alpha}}(y^{\frac{1}{\beta-\alpha}} - 1)^{\gamma-1} \\ + (\alpha - \beta + 1)\gamma y^{\frac{1}{\beta-\alpha}}(y^{\frac{1}{\beta-\alpha}} - 1)^{\gamma-1} \leq 0 \end{aligned}$$

By letting $z = y^{\frac{1}{\beta-\alpha}}$ and dividing throughout by $(y^{\frac{1}{\beta-\alpha}})^{\gamma-2}$ the above inequality translates into the following inequality:

$$(\beta\alpha + \gamma(\gamma - \beta) - \alpha\gamma)z^2 + (\alpha\gamma - 2\beta\alpha - \gamma(1 - \beta))z + \beta\alpha \leq 0 \quad (\text{C.2})$$

Given that we assume $\beta \geq \gamma$, the coefficient of z^2 is negative. Thus the left hand side of (C.2) defines an inverted parabola and thus there exists some constant $\tilde{y} > 1$ such that the (C.2) is satisfied for all $y \geq \tilde{y}$. \square

The implication that $g_v(y)$ always admits a minimal non-negative concave-majorant under the assumptions described in Lemma C.1.1 follows by noting that any finite continuous function defined over a bounded interval (a, b) ($-\infty < a < b < \infty$) admits a minimal non-negative concave majorant. Thus by noting that $g_v(y)$ is strictly positive and finite over $(1, \infty)$ and $g_v(y)$ is concave over (\tilde{y}, ∞) , it follows that a minimal non-negative concave majorant can always be constructed.

Next we discuss the characterisation of $g_v(y)$ for $y < 1$. We first note that $g_v(y) \rightarrow 0$ as $y \downarrow 0$ and $g_v(y) \uparrow K^\gamma v$ as $y \uparrow 1$. The following Lemma provides an overview of some other important characteristics of $g_v(y)$.

Lemma C.1.2. *There exists $\epsilon \in (0, 1)$ such that $\frac{d}{dy}g_v(y) < 0$ over $(0, \epsilon)$. Furthermore $g_v(y)$ has a unique turning point over the interval $(0, 1)$.*

Proof. By definition:

$$\frac{d}{dy}g_v(y) = \frac{1}{\beta - \alpha} \left(K^\gamma v(\gamma - \alpha)y^{\frac{\gamma-\beta}{\beta-\alpha}} + \gamma\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma-1}y^{\frac{1-\beta}{\beta-\alpha}} + \alpha\lambda y^{\frac{-\beta}{\beta-\alpha}}(1 - y^{\frac{1}{\beta-\alpha}})^\gamma \right)$$

and hence result is true if there exists $\epsilon \in (0, 1)$ such that over $(0, \epsilon)$ the following inequality is satisfied:

$$K^\gamma v(\gamma - \alpha)y^{\frac{\gamma-\beta}{\beta-\alpha}} + \gamma\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma-1}y^{\frac{1-\beta}{\beta-\alpha}} \leq -\alpha\lambda y^{\frac{-\beta}{\beta-\alpha}}(1 - y^{\frac{1}{\beta-\alpha}})^\gamma$$

or equivalently:

$$K^\gamma v(\gamma - \alpha)z^\gamma + \gamma\lambda(1 - z)^{\gamma-1}z \leq -\alpha\lambda(1 - z)^\gamma \quad (\text{C.3})$$

for $z \in (0, \epsilon^{\beta-\alpha})$ where (C.3) is obtained by letting $z = y^{\frac{1}{\beta-\alpha}}$ and dividing by $z^{-\beta}$ throughout.

Recall that $\alpha < 0$. Furthermore since $\gamma \in (0, 1)$, as $z \downarrow 0$, $(1 - z)^\gamma \uparrow 1$ whilst $z^\gamma \downarrow 0$ and $(1 - z)^{\gamma-1}z \downarrow 0$. Thus $\exists \epsilon > 0$ such that the inequality in (C.3) is satisfied over $(0, \epsilon^{\beta-\alpha})$. This proves the first part of the statement of this Lemma.

The first order condition implies that $y \in (0, 1)$ is a turning point of $g_v(y)$ if it satisfies:

$$K^\gamma v(\gamma - \alpha)y^{\frac{\gamma}{\beta-\alpha}} + \gamma\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma-1}y^{\frac{1}{\beta-\alpha}} + \alpha\lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma = 0 \quad (\text{C.4})$$

or alternatively for $z = y^{\frac{1}{\beta-\alpha}}$:

$$(\gamma - \alpha)K^\gamma v z^\gamma (1 - z)^{1-\gamma} + \lambda(\gamma - \alpha)z = -\alpha\lambda. \quad (\text{C.5})$$

Given that the left hand side is continuous in z and the value $\lambda(\gamma - \alpha) > -\alpha\lambda$, the intermediate value theorem implies that (C.5) has at least one solution.

Furthermore note that the functions $(\gamma - \alpha)y^{\frac{\gamma}{\beta-\alpha}}$, $\gamma\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma-1}y^{\frac{1}{\beta-\alpha}}$ and $\alpha\lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma$ in (C.4) are all increasing in y over $(0, 1)$ and hence the solution is unique. \square

Equipped with the characteristics obtained in Lemmas C.1.1 and C.1.2 together with the definition of $g_v(y)$ in (C.1), a good exposition of the geometry of the function $g_v(y)$ can now be derived.

Note that the definition of $g_v(y)$ over $(0, 1)$ in (C.1) is the sum of two function components; call them $f_1 : (0, 1) \rightarrow \mathbb{R}$ and $f_2 : (0, 1) \rightarrow \mathbb{R}$ respectively. The function $f_1(y) = K^\gamma v y^{\frac{\gamma-\alpha}{\beta-\alpha}}$ is concave over $(0, 1)$ and increases from $f_1(0) = 0$ to $f_1(1) = K^\gamma v$ over this interval. The function $f_2(y) = -\lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{\frac{-\alpha}{\beta-\alpha}}$ defines a non-positive, U-shaped curve and decreases from $f_2(0) = 0$ to a unique minimum achieved at $y = (\frac{-\alpha}{\gamma-\alpha})^{\beta-\alpha}$ and then increases to $f_2(1) = 0$. This behaviour is clearly observable in Figures C.1a and C.1b below.

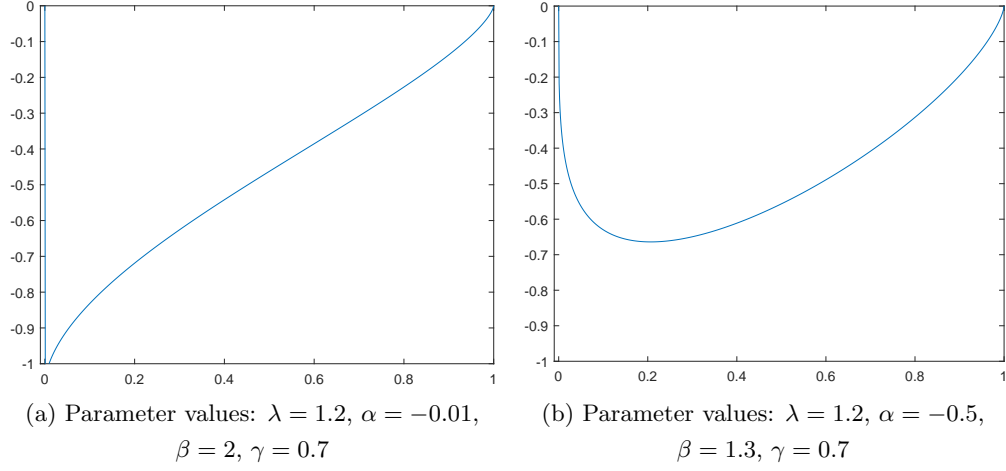


Figure C.1: Plots of the function $f_2(y) = -\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma} y^{\frac{-\alpha}{\beta-\alpha}}$

From Lemma C.1.2 we know that for $y < 1$, the graph of the function $g_v(y)$ decreases from $g_v(0) = 0$ to achieve a unique minimum point after which it increases to $g_v(1) = K^{\gamma}v$.¹ This together with the statement of Lemma C.1.1 and the general form of the functions $f_1(\cdot)$ and $f_2(\cdot)$ described above imply that the geometry of the function $g_v(y)$ takes the general form portrayed in Figures C.2a and C.2b; that is, *an S-shaped like curve with a skewed parabola close to zero*.

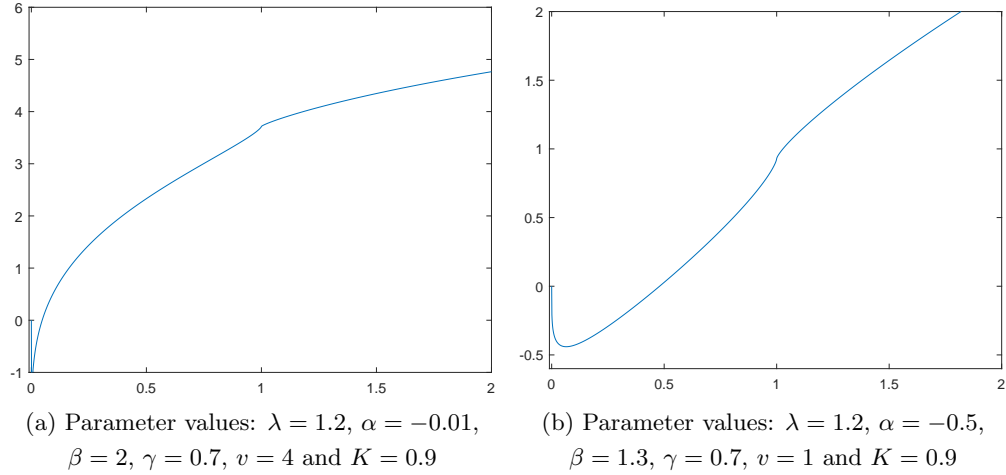


Figure C.2: Plots of the function $g_v(y)$

This general form, and as clearly observable in Figures C.2a and C.2b the

¹By considering the definition of $g_v(y)$ in (C.1) it can also be easily deduced that as the value of the constant $K^{\gamma}v$ increases, the constant ϵ in Lemma C.1.2 decreases.

minimal non-negative concave majorant of $g_v(y)$ can be constructed in one of two ways depending on the case:

- Case 1: The concave majorant is constructed by fitting a chord between the points $(0, 0)$ and $(\hat{y}, g_v(\hat{y}))$ with $\hat{y} \in (0, 1)$ and satisfying:

$$\frac{g_v(\hat{y})}{\hat{y}} = \frac{d}{dy}g_v(y) \Big|_{y=\hat{y}}$$

and another chord between the points $(y_l, g_v(y_l))$ and $(y_u, g_v(y_u))$ satisfying $0 < \hat{y} \leq y_l < 1 < y_u$ and:

$$\frac{g_v(y_u) - g_v(y_l)}{y_u - y_l} = \frac{d}{dy}g_v(y) \Big|_{y=y_l} = \frac{d}{dy}g_v(y) \Big|_{y=y_u}$$

Figure C.2a provides an example where such a construction would be applicable.

- Case 2: The concave majorant is constructed by fitting a chord between the points $(0, 0)$ and $(y_u, g_v(y_u))$ where $y_u > 1$ and:

$$\frac{g_v(y_u)}{y_u} = \frac{d}{dy}g_v(y) \Big|_{y=y_u}$$

An example of this case is provided in Figure C.2b.

C.2 Proofs of Results in Section 4.2

Proof of Proposition 4.2.2. Suppose $\rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma-1)\sigma^2$. We first note that there exists $C \in \mathbb{R}^+$ such that for any $x > 0$, we have $U(x, 1) \leq C(1 + x^\gamma)$. This can be verified easily using the definition of $U(x, 1)$.

Thus for any stopping times $\tau_1 \leq \tau_2 \leq \dots$ we have:

$$\begin{aligned}
& \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho\tau_n} U(W_{\tau_n-}, R_{\tau_n-}) \mathbb{I}_{\{\tau_n < \infty\}} \right] \\
&= \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho\tau_n} (R_{\tau_n-})^{\gamma} U\left(\frac{W_{\tau_n-}}{R_{\tau_n-}}, 1\right) \mathbb{I}_{\{\tau_n < \infty\}} \right] \\
&\leq C \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho\tau_n} (R_{\tau_n-})^{\gamma} \left(1 + \left(\frac{W_{\tau_n-}}{R_{\tau_n-}} \right)^{\gamma} \right) \mathbb{I}_{\{\tau_n < \infty\}} \right] \\
&= C \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho\tau_n} \left((R_{\tau_n-})^{\gamma} + (W_{\tau_n-})^{\gamma} \right) \mathbb{I}_{\{\tau_n < \infty\}} \right] \\
&= \sum_{n=1}^{\infty} C \mathbb{E}_{w,r} \left[e^{-\rho\tau_n} \left((R_{\tau_n-})^{\gamma} + (W_{\tau_n-})^{\gamma} \right) \mathbb{I}_{\{\tau_n < \infty\}} \right] \tag{C.6}
\end{aligned}$$

Since $R_{\tau_n-} = W_{\tau_{n-1}} = KW_{\tau_{n-1}-}$ for $n \geq 2$, we need to consider the case when $n = 1$ separately. Thus for $n = 1$, by the assumption discussed in Remark 4.2.1 we have:

$$\begin{aligned}
& \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} \left((R_{\tau_1-})^{\gamma} + (W_{\tau_1-})^{\gamma} \right) \mathbb{I}_{\{\tau_1 < \infty\}} \right] \\
&= \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} \left(r^{\gamma} + (W_{\tau_1-})^{\gamma} \right) \mathbb{I}_{\{\tau_1 < \infty\}} \right] \\
&\leq r^{\gamma} + \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} (W_{\tau_1-})^{\gamma} \mathbb{I}_{\{\tau_1 < \infty\}} \right] \tag{C.7}
\end{aligned}$$

However since $|W_{t \wedge \tau_1}| \leq c_1$ almost surely, we have:

$$\begin{aligned}
& \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} (W_{\tau_1-})^{\gamma} \mathbb{I}_{\{\tau_1 < \infty\}} \right] \\
&= w^{\gamma} \mathbb{E}_{w,r} \left[\exp \left(\left(-\rho + \gamma\mu - \frac{1}{2}\gamma(1-\gamma)\sigma^2 \right) \tau_{1-} - \frac{1}{2}\gamma^2\sigma^2\tau_{1-} + \gamma\sigma B_{\tau_{1-}} \right) \mathbb{I}_{\{\tau_1 < \infty\}} \right] \\
&\leq w^{\gamma} \mathbb{E}_{w,r} \left[\exp \left(-\frac{1}{2}\gamma^2\sigma^2\tau_{1-} + \gamma\sigma B_{\tau_{1-}} \right) \mathbb{I}_{\{\tau_1 < \infty\}} \right] \tag{C.8}
\end{aligned}$$

$$= w^{\gamma}. \tag{C.9}$$

where (C.8) follows from our initial assumptions and (C.9) follows by the Optional Sampling Theorem. By applying a similar argument to the above, we can show that for $n \geq 2$ we have:

$$\begin{aligned}
& \mathbb{E}_{w,r}[e^{-\rho\tau_n} \left((R_{\tau_n-})^\gamma + (W_{\tau_n-})^\gamma \right) \mathbb{I}_{\{\tau_n < \infty\}}] \\
&= \mathbb{E}_{w,r}[e^{-\rho\tau_n} \left((W_{\tau_{n-1}})^\gamma + (W_{\tau_n-})^\gamma \right) \mathbb{I}_{\{\tau_n < \infty\}}] \\
&= \left(wK^{n-1} \right)^\gamma \mathbb{E}_{w,r}[e^{-\rho\tau_n} \left(\left(\frac{X_{\tau_{n-1}}}{X_0} \right)^\gamma + \left(\frac{X_{\tau_n}}{X_0} \right)^\gamma \right) \mathbb{I}_{\{\tau_n < \infty\}}] \\
&\leq 2 \left(wK^{(n-1)} \right)^\gamma
\end{aligned} \tag{C.10}$$

Amalgamating (C.6), (C.7), (C.9) and (C.10) gives:

$$\begin{aligned}
Z(w, r) &\leq C(r^\gamma + w^\gamma) + \sum_{n=2}^{\infty} 2C \left(wK^{(n-1)} \right)^\gamma \\
&= C(r^\gamma + w^\gamma) + 2Cw^\gamma \left(\frac{K}{1 - K^\gamma} \right) \\
&< \infty
\end{aligned}$$

giving the result.

For the converse argument, suppose $\rho < \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$.

Let $\beta = \sigma^{-2} \left[-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\delta\sigma^2} \right]$. Then it can be easily shown that the conditions $\gamma > \beta$ and $\rho < \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$ are equivalent. Now given $a \in \mathbb{R}^+$, consider the sub-optimal strategy $\tau_1 = H_a$ and $\tau_2 = \tau_3 = \dots = \infty$, where H_a is defined by:

$$H_a = \inf\{t \geq 0 : W_t = a\}$$

The definition of $U(w, r)$ in (4.5) implies that $\exists C_1, C_2 \in \mathbb{R}^+$ such that $U(x, 1) \geq C_1(x^\gamma - C_2)$ for $x \geq 0$. Thus:

$$\begin{aligned}
Z(w, r) &\geq \mathbb{E}_{w,r}[e^{-\rho H_a} U(W_{H_a-}, R_{H_a-}) \mathbb{I}_{\{H_a < \infty\}}] \\
&= \mathbb{E}_{w,r}[e^{-\rho H_a} (R_{H_a-})^\gamma u\left(\frac{W_{H_a-}}{R_{H_a-}}, 1\right) \mathbb{I}_{\{H_a < \infty\}}] \\
&\geq \mathbb{E}_{w,r}[e^{-\rho H_a} (R_{H_a-})^\gamma C_1 \left(\left(\frac{W_{H_a-}}{R_{H_a-}}\right)^\gamma - C_2\right) \mathbb{I}_{\{H_a < \infty\}}] \\
&\geq \mathbb{E}_{w,r}[e^{-\rho H_a} r^\gamma C_1 \left(\left(\frac{a}{r}\right)^\gamma - C_2\right) \mathbb{I}_{\{H_a < \infty\}}]
\end{aligned}$$

From Borodin and Salminen [2012] we know that $\mathbb{E}_{w,r}[e^{-\rho H_a} \mathbb{I}_{\{H_a < \infty\}}] = \left(\frac{w}{a}\right)^\beta$ and hence from above it follows that

$$Z(w, r) \geq w^\beta C_1 \left(a^{\gamma-\beta} - C_2 r^\gamma a^{-\beta}\right) \quad (\text{C.11})$$

But since $\gamma > \beta$ we see that the right hand side is increasing in a and thus under this parameter regime, even if the agent follows a strategy of this form, it is always optimal for the investor to delay selling implying the result. \square

Proof of Proposition 4.2.5. By definition in (4.4), we have:

$$Z(w, r) = \sup_{0 \leq \tau_1 \leq \tau_2 \leq \dots} \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho \tau_n} U(W_{\tau_n-}, R_{\tau_n-}) \mathbb{I}_{\{\tau_n < \infty\}} \right] \quad (\text{C.12})$$

Let $\zeta = (\tau_1, \tau_2, \tau_3, \dots)$ and let $J(w, r, \zeta)$ be defined by:

$$J(w, r, \zeta) = \mathbb{E}_{w,r} \left[\sum_{n=1}^{\infty} e^{-\rho \tau_n} U(W_{\tau_n-}, R_{\tau_n-}) \mathbb{I}_{\{\tau_n < \infty\}} \right] \quad (\text{C.13})$$

Furthermore let $F(w, r, \tau)$ denote the corresponding gain function in (4.7), that is:

$$F(w, r, \tau) = e^{-\rho \tau} \left(U(W_{\tau-}, R_{\tau-}) + Z(W_{\tau}, R_{\tau}) \right) \mathbb{I}_{\{\tau < \infty\}} \quad (\text{C.14})$$

In view of the above, our objective is to prove that:

$$Z(w, r) = \sup_{\tau} \mathbb{E}_{w,r}[F(w, r, \tau)] \quad (\text{C.15})$$

Fix ζ . Then by using the tower-property for conditional expectation in (C.13), we

get:

$$\begin{aligned}
J(w, r, \zeta) &= \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} U(W_{\tau_1-}, R_{\tau_1-}) \mathbb{I}_{\{\tau_1 < \infty\}} + \right. \\
&\quad \left. e^{-\rho\tau_1} \mathbb{E} \left[\sum_{n=2}^{\infty} e^{-\rho(\tau_n - \tau_1)} U(W_{\tau_n-}, R_{\tau_n-}) \mathbb{I}_{\{\tau_n < \infty\}} \middle| \mathcal{F}_{\tau_1} \right] \mathbb{I}_{\{\tau_1 < \infty\}} \right] \\
&= \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} U(W_{\tau_1-}, R_{\tau_1-}) \mathbb{I}_{\{\tau_1 < \infty\}} + \right. \\
&\quad \left. e^{-\rho\tau_1} \mathbb{E} \left[\sum_{n=1}^{\infty} e^{-\rho(\tau_{n+1} - \tau_1)} U(W_{\tau_{n+1}-}, R_{\tau_{n+1}-}) \mathbb{I}_{\{\tau_{n+1} < \infty\}} \middle| \mathcal{F}_{\tau_1} \right] \mathbb{I}_{\{\tau_1 < \infty\}} \right]
\end{aligned}$$

Note that $(B_{\tau_1+s} - B_{\tau_1})_{s \geq 0}$ is a standard Brownian Motion. Hence there exists \mathcal{F}_s -measurable stopping times $0 \leq \tau_1 \leq \bar{\tau}_2 \leq \bar{\tau}_3 \leq \dots$ such that conditional on τ_1 , $((B_{\tau_1+s} - B_{\tau_1})_{s \geq 0}, \tau_2 - \tau_1, \tau_3 - \tau_1, \dots)$ is identically distributed to $((B_s)_{s \geq 0}, \bar{\tau}_2, \bar{\tau}_3, \dots)$. Denote $\bar{\zeta} = (\bar{\tau}_2, \bar{\tau}_3, \dots)$. Then:

$$\begin{aligned}
J(w, r, \zeta) &= \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} U(W_{\tau_1-}, R_{\tau_1-}) \mathbb{I}_{\{\tau_1 < \infty\}} + e^{-\rho\tau_1} J(W_{\tau_1}, W_{\tau_1}, \bar{\zeta}) \mathbb{I}_{\{\tau_1 < \infty\}} \right] \\
&\leq \mathbb{E}_{w,r} \left[e^{-\rho\tau_1} \left(U(W_{\tau_1-}, R_{\tau_1-}) + Z(W_{\tau_1}, W_{\tau_1}) \right) \mathbb{I}_{\{\tau_1 < \infty\}} \right]
\end{aligned}$$

giving:

$$Z(w, r) \leq \sup_{\tau} \mathbb{E}_{w,r} [F(w, r, \tau)] \quad (\text{C.16})$$

Conversely, fix $\epsilon > 0$. We know that $Z(w, r)$ is homogeneous in r of degree γ from Lemma 4.2.4. Thus by definition, there exists an \mathcal{F}_s -stopping time $\tilde{\tau} \geq 0$ such that:

$$\sup_{\tau} \mathbb{E}_{w,r} [F(w, r, \tau)] - \epsilon \leq \mathbb{E}_{w,r} [F(w, r, \tilde{\tau})] \quad (\text{C.17})$$

where we have:

$$F(w, r, \tilde{\tau}) = e^{-\rho\tilde{\tau}} \left(U(W_{\tilde{\tau}-}, R_{\tilde{\tau}-}) + (KW_{\tilde{\tau}-})^{\gamma} Z(1, 1) \right) \mathbb{I}_{\{\tilde{\tau} < \infty\}}$$

Furthermore, there exists $\hat{\zeta} = (\hat{\tau}_2, \hat{\tau}_3, \dots)$ such that $Z(1, 1) - \epsilon \leq J(1, 1, \hat{\zeta})$. Thus:

$$F(w, r, \tilde{\tau}) \leq e^{-\rho\tilde{\tau}} \left(U(W_{\tilde{\tau}-}, R_{\tilde{\tau}-}) + (KW_{\tilde{\tau}-})^{\gamma} (J(1, 1, \hat{\zeta}) + \epsilon) \right) \mathbb{I}_{\{\tilde{\tau} < \infty\}}$$

Note that we can again find stopping times (τ_2, τ_3, \dots) such that conditional on $\tilde{\tau}$, $((B_{\tilde{\tau}+s})_{s \geq 0}, \tau_2 - \tilde{\tau}, \tau_3 - \tilde{\tau}, \dots)$ is identically distributed to $((B_s)_{s \geq 0}, \hat{\tau}_2, \hat{\tau}_3, \dots)$. Let $\zeta = (\tilde{\tau}, \tau_2, \tau_3, \dots)$. Then by again applying the Tower Property conditioned on $\mathcal{F}_{\tilde{\tau}}$,

we have:

$$\mathbb{E}_{w,r}[F(w, r, \tilde{\tau})] \leq \mathbb{E}_{w,r}[J(w, r, \zeta)] + \epsilon \mathbb{E}_w[e^{-\rho \tilde{\tau}}(KW_{\tilde{\tau}-})^\gamma \mathbb{I}_{\{\tilde{\tau} < \infty\}}] \quad (\text{C.18})$$

If we show that $\mathbb{E}[e^{-\rho \tilde{\tau}}(W_{\tilde{\tau}-})^\gamma \mathbb{I}_{\{\tilde{\tau} < \infty\}}] = C < \infty$ for some constant $C \in \mathbb{R}^+$, then from (C.17) and (C.18) we have:

$$\begin{aligned} \sup_{\tau} \mathbb{E}_{w,r}[F(w, r, \tau)] &\leq \sup_{\zeta} \mathbb{E}_{w,r}[J(w, r, \zeta)] + \epsilon(1 + K^\gamma C) \\ &= Z(w, r) + \epsilon(1 + \tilde{C}) \end{aligned}$$

The arbitrariness of $\epsilon > 0$ yields:

$$Z(w, r) \geq \sup_{\tau} \mathbb{E}_{w,r}[F(w, r, \tau)]$$

giving the result. Hence what remains to be shown is that for a finite \mathcal{F}_s -stopping time τ , $\mathbb{E}_w[e^{-\rho \tau}(W_{\tau-})^\gamma \mathbb{I}_{\{\tau < \infty\}}] = C < \infty$ for some constant $C \in \mathbb{R}^+$. But by following a similar argument to that outlined in the proof of Proposition 4.2.2 particularly in (C.9) we have:

$$\mathbb{E}_w[e^{-\rho \tau}(W_{\tau-})^\gamma \mathbb{I}_{\{\tau < \infty\}}] \leq w^\gamma$$

giving the result. □

Proof to Lemma 4.2.6. Firstly note that given $\beta \geq \gamma$, we have $l_0 = 0$ and $l_\infty = 0$, where the limits l_0 and l_∞ are as defined in (2.9). By applying the statement of Corollary 2.3.10, it directly follows that the optimal stopping time τ^* is a hitting time of X_t . Furthermore given that $X_0 = 1$, this implies that there exists $a, b \in \mathbb{R}^+$ satisfying $0 \leq a \leq 1 \leq b$ and $\tau^* = H_a \wedge H_b$.

By applying this characterisation of τ^* in (4.14) and (4.15), $\mathcal{H}(v)$ satisfies the following linear equation in v :

$$\begin{aligned} \mathcal{H}(v) &= \mathcal{H}(0) + \mathbb{E}_1[e^{\rho \tau^*}(KX_{\tau^*})^\gamma]v \\ &= \mathcal{H}(0) + \tilde{C}v \end{aligned} \quad (\text{C.19})$$

for some $\tilde{C} \in \mathbb{R}^+$. This linearity in v of (C.19), gives the uniqueness result for the fixed point problem $\mathcal{H}(v) = v$. □

Proof of Proposition 4.2.8. For $\beta > \gamma$ we have that:

$$l_0 = \limsup_{x \downarrow 0} \frac{h_v^+(x)}{\phi(x)} = 0 \quad \text{and} \quad l_\infty = \limsup_{x \uparrow \infty} \frac{h_v^+(x)}{\psi(x)} = 0$$

where l_0 and l_∞ are as defined in (2.9). Then by Proposition 2.3.8 the solution depends primarily on finding the smallest concave majorant of the transformed gain function $g_v(y) = h_v(F^{-1}(y))/\phi(F^{-1}(y))$ given by:

$$g_v(y) = \begin{cases} \left(K^\gamma v y^{\frac{\gamma}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma \right) y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y \geq 1 \\ \left(K^\gamma v y^{\frac{\gamma}{\beta-\alpha}} - \lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma \right) y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y < 1 \end{cases} \quad (\text{C.20})$$

Denote by $\bar{g}_v(y) : [0, \infty) \rightarrow \mathbb{R}$ the smallest concave majorant of $g_v(y)$. Then from Proposition 2.3.8 we have:

$$\mathcal{H}(v, x) = \begin{cases} \phi(x) \bar{g}_v(F(x)) & \text{for } x > 0 \\ l_0 = 0 & \text{for } x = 0 \end{cases} \quad (\text{C.21})$$

If the one-threshold strategy is optimal then the smallest non-negative concave majorant $\bar{g}_v(y)$ of $g_v(y)$ can be characterised as follows:

1. Find the point $\bar{\theta} > 1$ satisfying:

$$\frac{g_v(\bar{\theta})}{\bar{\theta}} = \left. \frac{dg_v(y)}{dy} \right|_{y=\bar{\theta}} \quad (\text{C.22})$$

2. Then for $y \geq \bar{\theta}$, we have $\bar{g}_v(y) = g_v(y)$ and for $y < \bar{\theta}$, $\bar{g}_v(y)$ is defined by the line passing through the points $(0, 0)$ and $(\bar{\theta}, g_v(\bar{\theta}))$.

This gives:

$$\bar{g}_v(y) = \begin{cases} \left(K^\gamma v y^{\frac{\gamma}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma \right) y^{\frac{-\alpha}{\beta-\alpha}} & \text{for } y \geq \bar{\theta} \\ \frac{\left(K^\gamma v \bar{\theta}^{\frac{\gamma}{\beta-\alpha}} + (\bar{\theta}^{\frac{1}{\beta-\alpha}} - 1)^\gamma \right) \bar{\theta}^{\frac{-\alpha}{\beta-\alpha}}}{\bar{\theta}} y & \text{for } y < \bar{\theta} \end{cases} \quad (\text{C.23})$$

By applying the transformation in (C.21) above, and by noting that $y = x^{\beta-\alpha}$, we get:

$$\mathcal{H}(v, x) = \begin{cases} K^\gamma v x^\gamma + (x - 1)^\gamma & \text{for } x \geq \bar{x}_v \\ (K^\gamma v \bar{x}_v^{\gamma-\beta} + (\bar{x}_v - 1)^\gamma \bar{x}_v^{-\beta}) x^\beta & \text{for } x < \bar{x}_v \end{cases} \quad (\text{C.24})$$

where $\bar{x}_v = F^{-1}(\bar{\theta})$. Next, to characterise $\bar{\theta}$ and hence \bar{x}_v , the condition in (C.22)

gives:

$$\begin{aligned} (K^\gamma v \bar{\theta}^{\frac{\gamma}{\beta-\alpha}} + (\bar{\theta}^{\frac{1}{\beta-\alpha}} - 1)^\gamma) \bar{\theta}^{\frac{-\beta}{\beta-\alpha}} = \\ \frac{\bar{\theta}^{\frac{-\beta}{\beta-\alpha}}}{\beta - \alpha} \left((\gamma - \alpha) K^\gamma v \bar{\theta}^{\frac{\gamma}{\beta-\alpha}} + \gamma (\bar{\theta}^{\frac{-\alpha}{\beta-\alpha}} - 1)^{\gamma-1} \bar{\theta}^{\frac{1}{\beta-\alpha}} - \alpha (\bar{\theta}^{\frac{-\alpha}{\beta-\alpha}} - 1)^\gamma \right) \end{aligned}$$

Rearranging and rewriting in terms of \bar{x}_v we get:

$$K^\gamma v \bar{x}_v = \left(\left(\frac{\beta}{\beta - \gamma} \right) - \bar{x}_v \right) (\bar{x}_v - 1)^{\gamma-1}$$

giving (4.19). \square

Proof of Proposition 4.2.12. From Proposition 4.2.8 we see that the value function $\mathcal{H}(v, x)$ satisfies $\mathcal{H}(v, x) = A_v^{(1)} x^\beta$ when the one threshold strategy is optimal (for some constant $A_v^{(1)} \in \mathbb{R}^+$). Furthermore under this setting, it is clear from the characterisation of $\mathcal{H}(v, x)$ in (4.20), that the value function does not depend on λ when the solution contains one selling threshold. On the other hand for the two threshold strategy, from Proposition 4.2.10, we have $\mathcal{H}(v, x) = A_v^{(2)} x^\beta + B_v^{(2)} x^\alpha$ and $\mathcal{H}(v, x)$ is decreasing in λ . Hence there must exist a unique value of λ at which the two value functions are equal and hence $A_v^{(1)} = A_v^{(2)} = A_v$, $B_v^{(2)} = 0$ and $v = V(1)$ satisfies (4.21). We have:

$$A_v^{(1)} = K^\gamma v \bar{x}_*^{\gamma-\beta} + (\bar{x}_* - 1)^\gamma \bar{x}_*^{-\beta}$$

and

$$A_v^{(2)} = \left[\frac{K^\gamma v (\bar{x}_*^{\gamma-\alpha} - \underline{x}_*^{\gamma-\alpha}) + (\bar{x}_* - 1)^\gamma \bar{x}_*^{-\alpha} + (1 - \underline{x}_*)^\gamma \underline{x}_*^{-\alpha}}{\bar{x}_*^{\beta-\alpha} - \underline{x}_*^{\beta-\alpha}} \right].$$

Equating the two, plugging in v as given in (4.21) and arranging for λ we obtain:

$$\lambda^* = - \frac{(\bar{x}_* - 1)^\gamma (\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma)}{(1 - \underline{x}_*)^\gamma (\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma)} \quad (\text{C.25})$$

Note that $\mathcal{H}(v, x) = h_v(x)$ for $x = \bar{x}_*$ or $x = \underline{x}_*$. Furthermore by definition we have that $v = \mathcal{H}(v, 1) = A_v$ and $\mathcal{H}(v, \bar{x}_*) = A_v \bar{x}_*^\beta$ and $\mathcal{H}(v, \underline{x}_*) = A_v \underline{x}_*^\beta$. Thus we have:

$$\begin{aligned} A_v \bar{x}_*^\beta &= (\bar{x}_* - 1)^\gamma + K^\gamma \bar{x}_*^\gamma A_v \\ A_v \underline{x}_*^\beta &= -\lambda (1 - \underline{x}_*)^\gamma + K^\gamma \underline{x}_*^\gamma A_v \end{aligned}$$

implying:

$$A_v = \frac{(\bar{x}_* - 1)^\gamma}{\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma} = \frac{-\lambda(1 - \underline{x}_*)^\gamma}{\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma} \quad (\text{C.26})$$

By applying the smooth fit principle described in Proposition 2.3.11 we also obtain the following system of equations after re-arranging:

$$\begin{aligned} A_v \beta \bar{x}_*^{\beta-1} &= \gamma(\bar{x}_* - 1)^{\gamma-1} + \gamma K^\gamma \bar{x}_*^{\gamma-1} A_v \\ A_v \beta \underline{x}_*^{\beta-1} &= \gamma\lambda(1 - \underline{x}_*)^{\gamma-1} + \gamma K^\gamma \underline{x}_*^{\gamma-1} A_v \end{aligned}$$

giving:

$$A_v = \frac{\gamma(\bar{x}_* - 1)^{\gamma-1}}{\beta \bar{x}_*^{\beta-1} - \gamma K^\gamma \bar{x}_*^{\gamma-1}} = \frac{\gamma\lambda(1 - \underline{x}_*)^{\gamma-1}}{\beta \underline{x}_*^{\beta-1} - \gamma K^\gamma \underline{x}_*^{\gamma-1}} \quad (\text{C.27})$$

Since the quotients in both (C.26) and (C.27) are equal to A_v we get:

$$\frac{(\bar{x}_* - 1)^\gamma}{\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma} = \frac{\gamma(\bar{x}_* - 1)^{\gamma-1}}{\beta \bar{x}_*^{\beta-1} - \gamma K^\gamma \bar{x}_*^{\gamma-1}} \quad \text{and} \quad \frac{-\lambda(1 - \underline{x}_*)^\gamma}{\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma} = \frac{\gamma\lambda(1 - \underline{x}_*)^{\gamma-1}}{\beta \underline{x}_*^{\beta-1} - \gamma K^\gamma \underline{x}_*^{\gamma-1}} \quad (\text{C.28})$$

and hence both \underline{x}_* and \bar{x}_* satisfy:

$$(\gamma - \beta)x + \beta = \gamma K^\gamma x^{\gamma-\beta} \quad (\text{C.29})$$

giving (4.31). The expressions in (C.28) can be re-arranged to show that \bar{x}_* and \underline{x}_* both satisfy:

$$x^\beta - K^\gamma x^\gamma = \frac{K^\gamma x^\gamma (\beta - \gamma)(x - 1)}{(\gamma - \beta)x + \beta} \quad (\text{C.30})$$

Plugging (C.30) in (C.25) gives

$$\lambda^* = \frac{(\bar{x}_* - 1)^{\gamma-1} \underline{x}_*^\gamma ((\gamma - \beta)\bar{x}_* + \beta)}{(1 - \underline{x}_*)^{\gamma-1} \bar{x}_*^\gamma ((\gamma - \beta)\underline{x}_* + \beta)}$$

and result follows from the relation in (C.29). □

C.3 Characterising $g_{\bar{v}}(y)$

Understanding the geometry of the underlying scaled reward function is an integral step to implement the methodology outlined in Chapter 2. This section will serve as an overview of the general structure of the function $g_{\bar{v}}(y)$ in (4.39) to obtain a similar characterisation to the one obtained in Appendix C.1 for the problem described in Section 4.2. The transformed reward function $g_{\bar{v}}(y)$ for the optimal stopping problem described in Section 4.3 is given by:

$$g_{\bar{v}}(y) = \begin{cases} K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} - \lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{-\frac{\eta+\alpha}{\beta-\alpha}}, & \text{for } y \leq 1 \\ K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma y^{-\frac{\eta+\alpha}{\beta-\alpha}}, & \text{for } y > 1. \end{cases} \quad (\text{C.31})$$

In-line with the assumptions considered throughout Section 4.3, we will here assume that $\beta > \gamma$ is satisfied. The first important distinction which needs to be made when analysing the geometry of $g_{\bar{v}}(y)$ is that it varies depending on where the parameters α and η lie relative to each other. In fact note that given $-\alpha < \eta$, $\lim_{y \downarrow 0} g_{\bar{v}}(y) = -\infty$ whereas when $-\alpha > \eta$ we have $\lim_{y \downarrow 0} g_{\bar{v}}(y) = 0$.

The discussion is split into two parts, outlining the cases when $-\alpha > \eta$ and $-\alpha < \eta$ separately. For the case when $\alpha = -\eta$, a similar argument can also be made.

Case 1: $-\alpha < \eta$

The definition of $g_{\bar{v}}(y)$ over $(0, 1)$ in (C.31) is composed of a sum of two functions; $f_1 : (0, 1) \rightarrow \mathbb{R}$ and $f_2 : (0, 1) \rightarrow \mathbb{R}$ respectively. The function $f_1(y) = K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}}$ is concave over $(0, 1)$ and increases from $f_1(0) = 0$ to $f_1(1) = K^\gamma \bar{v}$. When $-\alpha < \eta$, the function $f_2(y) = -\lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{-\frac{\eta+\alpha}{\beta-\alpha}}$ is increasing, negative and its range over this interval is $(-\infty, 0)$. Furthermore by examining $f_2''(y)$, it follows that $\exists C_{\bar{v}} \in (0, 1)$ such that $f_2(y)$ is concave over $(0, C_{\bar{v}})$. All of these properties of $f_2(\cdot)$ are clearly observable in Figures C.3a and C.3b below.

For $y \geq 1$, $g_{\bar{v}}(y)$ is also composed of a sum of two functions; $f_1 : (0, 1) \rightarrow \mathbb{R}$ and $f_3 : (0, 1) \rightarrow \mathbb{R}$ with $f_3(y) = (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma y^{-\frac{\eta+\alpha}{\beta-\alpha}}$ respectively. Note that $f_3(\cdot)$ is increasing and $\lim_{y \uparrow \infty} f_3(y) \geq \lim_{z \uparrow \infty} \left(\frac{z-1}{z}\right)^\gamma z^{-\alpha} = \infty$. Furthermore $f_3(\cdot) = h(g(\cdot))$ with $h(z) = (z-1)^\gamma z^{-\eta-\alpha}$ and $g(y) = y^{\frac{1}{\beta-\alpha}}$. The second order condition gives that $h(\cdot)$ is concave and since $g(\cdot)$ is either concave or convex over $(1, \infty)$, then $f_3(\cdot)$ is concave over this interval. This implies that $g_{\bar{v}}(y)$ is concave over $(1, \infty)$ as it is the sum of two concave functions.

The properties outlining the general form of $f_1(\cdot)$ and $f_2(\cdot)$ described above

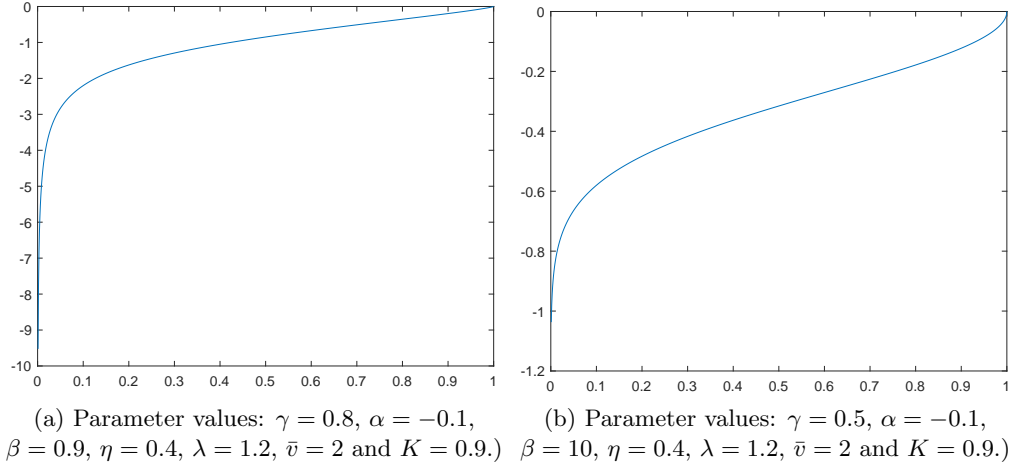


Figure C.3: Plots describing the function $f_2(y) = -\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma} y^{-\frac{\eta+\alpha}{\beta-\alpha}}$.

together with the geometry of $g_{\bar{v}}(y)$ over $(1, \infty)$ implies that over $(0, \infty)$, $g_{\bar{v}}(y)$ takes the general form portrayed in Figures C.4a and C.4b; that is, *an S-shaped like curve which decreases to $-\infty$ close to 0*.

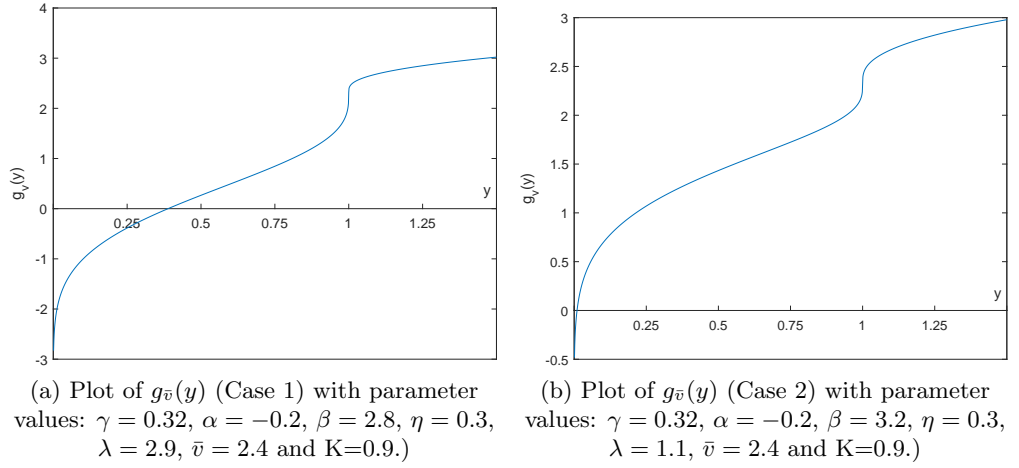


Figure C.4: Plots describing the shape of the function $g_{\bar{v}}(y)$ when $-\alpha < \eta$.

Case 2: $-\alpha > \eta$

For $y \geq 1$, we again have $g_{\bar{v}}(y) = f_1(y) + f_3(y)$, where both $f_1(\cdot)$ and $f_3(\cdot)$ are strictly increasing. Given that $f_1(y)$ is concave over $(1, \infty)$ and $\lim_{y \uparrow \infty} f_3'(y) = 0$, it follows that there exists $\tilde{y} \geq 1$ such that $g_{\bar{v}}(y) = f_1(y) + f_3(y)$ is concave over (\tilde{y}, ∞) .

For $y \leq 1$, $g_{\bar{v}}(y)$ is given by $f_1(y) + f_2(y)$ where $f_1(y)$ and $f_2(y)$ are as defined in the previous case.

Lemma C.3.1. *There exists $\epsilon \in (0, 1)$ such that $\frac{d}{dy}g_{\bar{v}}(y) < 0$ over $(0, \epsilon)$. Furthermore $g_{\bar{v}}(y)$ has a unique turning point over the interval $(0, 1)$.*

Proof. By definition:

$$\frac{d}{dy}g_{\bar{v}}(y) = \frac{1}{\beta - \alpha} \left(K^{\gamma} \bar{v}(\gamma - \alpha) y^{\frac{\gamma - \beta}{\beta - \alpha}} + \gamma \lambda (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma - 1} y^{\frac{1 - \beta - \eta}{\beta - \alpha}} + (\eta + \alpha) \lambda y^{-\frac{\beta + \eta}{\beta - \alpha}} (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma} \right)$$

and hence result is true if there exists $\epsilon \in (0, 1)$ such that over $(0, \epsilon)$ the following inequality is satisfied:

$$K^{\gamma} \bar{v}(\gamma - \alpha) y^{\frac{\gamma - \beta}{\beta - \alpha}} + \gamma \lambda (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma - 1} y^{\frac{1 - \beta - \eta}{\beta - \alpha}} \leq -(\eta + \alpha) \lambda y^{-\frac{\beta + \eta}{\beta - \alpha}} (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma}$$

or equivalently:

$$K^{\gamma} \bar{v}(\gamma - \alpha) z^{\gamma + \eta} + \gamma \lambda (1 - z)^{\gamma - 1} z \leq -(\eta + \alpha) \lambda (1 - z)^{\gamma} \quad (\text{C.32})$$

for $z \in (0, \epsilon^{\beta - \alpha})$ where (C.32) is obtained by letting $z = y^{\frac{1}{\beta - \alpha}}$ and dividing by $z^{-\beta}$ throughout.

Recall that $\alpha < 0$. Furthermore since $\gamma \in (0, 1)$, as $z \downarrow 0$, $(1 - z)^{\gamma} \uparrow 1$ whilst $z^{\gamma + \eta} \downarrow 0$ and $(1 - z)^{\gamma - 1} z \downarrow 0$. Thus $\exists \epsilon > 0$ such that the inequality in (C.32) is satisfied over $(0, \epsilon^{\beta - \alpha})$. This proves the first part of the statement of this Lemma.

The first order condition implies that $y \in (0, 1)$ is a turning point of $g_{\bar{v}}(y)$ if it satisfies:

$$K^{\gamma} \bar{v}(\gamma - \alpha) y^{\frac{\gamma}{\beta - \alpha}} + \gamma \lambda (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma - 1} y^{\frac{1 - \eta}{\beta - \alpha}} + \lambda(\alpha + \eta) (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma} y^{-\frac{\eta}{\beta - \alpha}} = 0 \quad (\text{C.33})$$

or alternatively for $z = y^{\frac{1}{\beta - \alpha}}$:

$$(\gamma - \alpha) K^{\gamma} \bar{v} z^{\gamma + \eta} (1 - z)^{1 - \gamma} + \lambda(\gamma - (\eta + \alpha)) z = -\lambda(\eta + \alpha). \quad (\text{C.34})$$

Given that the left hand side is continuous in z and the inequality $\lambda(\gamma - (\alpha + \eta)) > -\lambda(\alpha + \eta)$ holds, the intermediate value theorem implies that (C.34) has at least one solution.

Furthermore note that the functions $(\gamma - \alpha) y^{\frac{\gamma}{\beta - \alpha}}$, $\gamma \lambda (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma - 1} y^{\frac{1 - \eta}{\beta - \alpha}}$ and $\lambda(\alpha + \eta) (1 - y^{\frac{1}{\beta - \alpha}})^{\gamma} y^{-\frac{\eta}{\beta - \alpha}}$ in (C.33) are all increasing in y over $(0, 1)$ and hence the

solution is unique. \square

The above Lemma together with the characteristics of $g_{\bar{v}}(y)$ over $(0, 1)$ discussed earlier provide a good stepping-stone towards obtaining an overview of the geometry of $g_{\bar{v}}(y)$ when $-\alpha > \eta$. It is worth mentioning here that the proof of Lemma C.3.1 specifically the inequality in (C.32), that as the constant $K^{\gamma\bar{v}}$ increases, the constant $\epsilon > 0$ decreases towards 0.

Over $(0, 1)$, recall that $g_{\bar{v}}(y) = f_1(y) + f_2(y)$ where $f_1(y)$ is concave and increasing over the range specified by $f_1(0) = 0$ and $f_1(1) = K^{\gamma\bar{v}}$. Given the constant α satisfies $-\alpha > \eta$, the function $f_2(y) = -\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma} y^{-\frac{\eta+\alpha}{\beta-\alpha}}$ defines a negative U-shaped curve which decreases from $f_2(0) = 0$ to a unique minimum achieved at $y = \left(\frac{-(\alpha+\eta)}{\gamma-(\alpha+\eta)}\right)^{\beta-\alpha}$ and increases to $f_2(1) = 0$; as outlined in Figures C.5a and C.5b below.

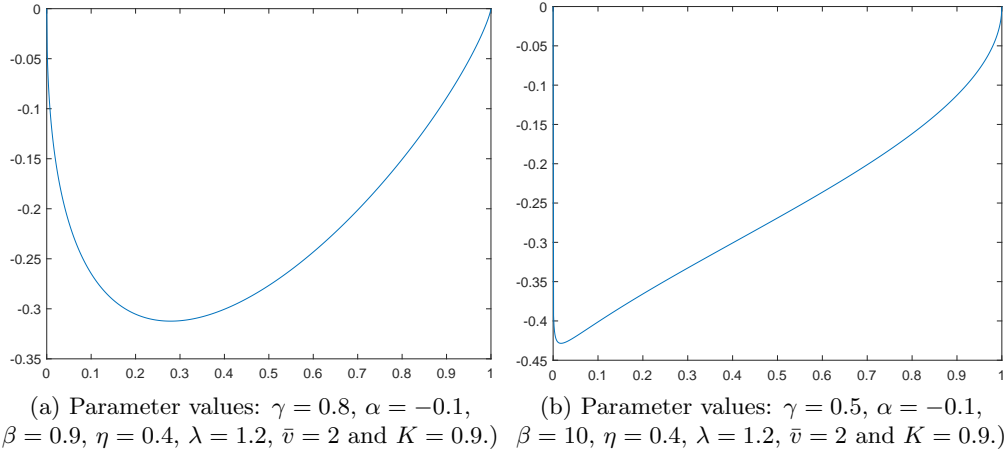


Figure C.5: Plots describing the function $f_2(y) = -\lambda(1 - y^{\frac{1}{\beta-\alpha}})^{\gamma} y^{-\frac{\eta+\alpha}{\beta-\alpha}}$.

In view of the above discussion, for $y < 1$ the graph of $g_{\bar{v}}(y)$ decreases from $g_{\bar{v}}(0) = 0$ to a unique minimum point and then increases to $g_{\bar{v}}(1) = K^{\gamma\bar{v}}$. Furthermore, in view of the general form of $f_1(\cdot)$ and $f_2(\cdot)$ described above we can conclude that the geometry of $g_{\bar{v}}(y)$ when $-\alpha > \eta$ is as portrayed in Figures C.6a and C.6b; that is, *an S-shaped like curve with a skewed parabola close to zero*².

C.3.1 The Resulting Solution Types

The general form of $g_{\bar{v}}(y)$ arising under both parameter cases discussed provides us with a clear indication of what types of solutions will arise for the optimal stopping

²In Figure C.6b it is not very clear that $g_{\bar{v}}(y)$ is decreasing at 0. However this is only the case since the constant ϵ discussed in Lemma C.3.1 is very close to 0 under this case.

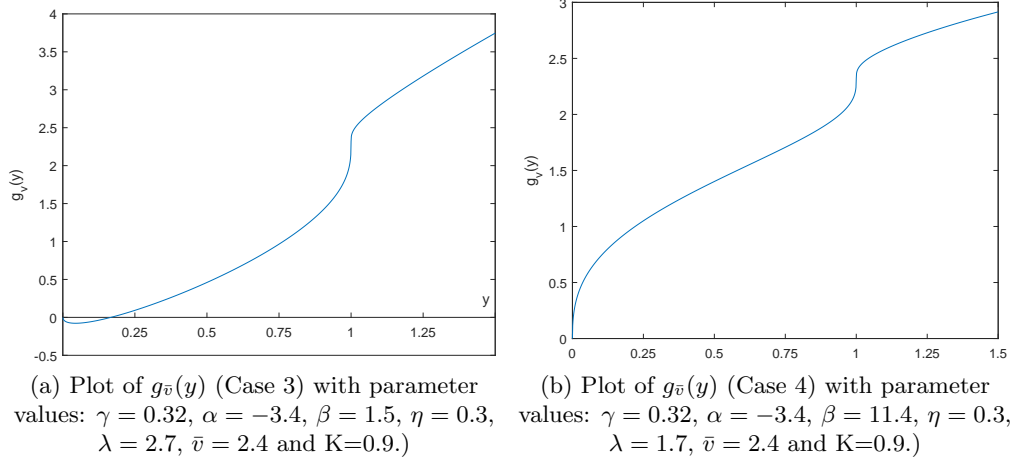


Figure C.6: Plots describing the shape of the function $g_{\bar{v}}(y)$ when $-\alpha > \eta$.

problem being considered. As noted in Chapter 2; specifically Proposition 2.3.8³, the types of solutions arising from a problem of this form are directly related to the construction of a minimal non-negative concave majorant of the transformed reward function $g_{\bar{v}}(y)$. It is worth noting here that the non-negativity assumption implies a different solution type under the cases depicted in Figures C.4a and C.4b.

In view of the above we can conclude that two types of solutions will arise:

1. The agent stops only at a gain (as is the case in Figures C.4a and C.6a)).
2. The continuation region is disconnected and it consists of a neighbourhood of 0 and a neighbourhood of 1.

C.4 Proofs of Results in Section 4.3

Proof of Proposition 4.3.1. Suppose $\rho \geq \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$. We again note that for utility function $U(w, r)$ defined in (4.32), $\exists C \in \mathbb{R}^+$ such that $\tilde{U}(x, 1) \leq C(1 + x^\gamma)$. A similar argument to that used in the proof of Proposition 4.2.2 can be transposed for this case giving $Z(w, r) < \infty$.

For the converse, suppose that $\rho < \gamma\mu + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$ or equivalently, $\beta < \gamma$ where β is as defined in (4.9). Given w and r , let $a \in \mathbb{R}^+$ such that $a > r$ and consider the (sub-optimal) strategy $\tau_1 = H_a$ and $\tau_2 = \tau_3 = \dots = \infty$, where

$$H_a = \inf\{t \geq 0 : W_t = a\} \quad (\text{C.35})$$

³Assume for now that l_0 and l_∞ defined in (2.3.6) are both equal to 0. This will be shown later.

Then we have:

$$\begin{aligned}
\bar{Z}(w, r) &\geq \mathbb{E}_{w,r} [e^{-\rho H_a} \tilde{U}(W_{H_a-}, R_{H_a-}) \mathbb{I}_{\{H_a < \infty\}}] \\
&= r^\gamma \mathbb{E}_{w,r} \left[e^{-\rho H_a} \tilde{U}\left(\frac{W_{H_a-}}{r}, 1\right) \mathbb{I}_{\{H_a < \infty\}} \right] \\
&= r^\gamma \left(\frac{a}{r} - 1\right)^\gamma \left(\frac{a}{r}\right)^{-\eta} \mathbb{E}_{w,r} \left[e^{-\rho H_a} \mathbb{I}_{\{H_a < \infty\}} \right] \\
&= r^\gamma \left(\frac{a}{r} - 1\right)^\gamma \left(\frac{a}{r}\right)^{-\eta} w^\beta a^{\gamma-\beta} \\
&\geq r^\gamma w^\beta \left(\left(\frac{a}{r}\right)^\gamma - 1\right) \left(\frac{a}{r}\right)^{-\eta} a^{\gamma-\beta}
\end{aligned} \tag{C.36}$$

But $(x^\gamma - 1)x^{-\eta}$ is positive and increasing in x for $x \geq 1$. Hence given the initial choice of $a \in \mathbb{R}^+$, in (C.36) we have the product of two positive, increasing functions in a . Thus even if we were to restrict our strategies to hitting of the types defined in (C.35), it would always be optimal to wait and sell at a larger threshold, a thus waiting indefinitely, giving the result. \square

Proof of Proposition 4.3.5. Let $\beta > \gamma$ and consider \bar{l}_0 and \bar{l}_∞ as defined in (2.9). Then we have:

$$\bar{l}_0 = \limsup_{x \downarrow 0} \frac{h_v^+(x)}{\phi(x)} = 0 \quad \text{and} \quad \bar{l}_\infty = \limsup_{x \uparrow \infty} \frac{h_v^+(x)}{\psi(x)} = 0$$

By Proposition 2.3.8, assuming the one-threshold strategy is optimal, the solution follows after finding the smallest concave majorant of the transformed gain function $g_{\bar{v}}(y) = h_{\bar{v}}(F^{-1}(y))/\phi(F^{-1}(y))$ given by:

$$g_{\bar{v}}(y) = \begin{cases} K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} + (y^{\frac{1}{\beta-\alpha}} - 1)^\gamma y^{-\frac{\eta+\alpha}{\beta-\alpha}} & \text{for } y \geq 1 \\ K^\gamma \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} - \lambda(1 - y^{\frac{1}{\beta-\alpha}})^\gamma y^{-\frac{\eta+\alpha}{\beta-\alpha}} & \text{for } y < 1 \end{cases} \tag{C.37}$$

Denote the smallest concave majorant of $g_{\bar{v}}(y)$ by $\bar{g}_{\bar{v}}(y) : [0, \infty) \rightarrow \mathbb{R}$. Then from Proposition 2.3.8 we have:

$$\bar{\mathcal{H}}(\bar{v}, x) = \begin{cases} \phi(x) \bar{g}_{\bar{v}}(F(x)) & \text{for } x > 0 \\ l_0 = 0 & \text{for } x = 0 \end{cases} \tag{C.38}$$

If the one threshold strategy is optimal then the smallest concave majorant $\bar{g}_v(y)$ of $g_v(y)$ is given by a chord connecting the point $(0, 0)$ to a point $(\bar{\theta}, g_{\bar{v}}(\bar{\theta}))$ with $\bar{\theta} > 1$

satisfying:

$$\frac{g_{\bar{v}}(\bar{\theta})}{\bar{\theta}} = \frac{dg_{\bar{v}}(y)}{dy} \Big|_{y=\bar{\theta}} \quad (\text{C.39})$$

Thus $\bar{g}_{\bar{v}}(y)$ is given by:

$$\bar{g}_{\bar{v}}(y) = \begin{cases} K^{\gamma} \bar{v} y^{\frac{\gamma-\alpha}{\beta-\alpha}} + \left(y^{\frac{1}{\beta-\alpha}} - 1\right)^{\gamma} y^{-\frac{\eta+\alpha}{\beta-\alpha}} & \text{for } y \geq \bar{\theta} \\ \frac{K^{\gamma} \bar{v} \bar{\theta}^{\frac{\gamma-\alpha}{\beta-\alpha}} + \left(\bar{\theta}^{\frac{1}{\beta-\alpha}} - 1\right)^{\gamma} \bar{\theta}^{-\frac{\eta+\alpha}{\beta-\alpha}}}{\bar{\theta}} y & \text{for } y < \bar{\theta} \end{cases} \quad (\text{C.40})$$

and by (C.38), since $y = x^{\beta-\alpha}$:

$$\bar{\mathcal{H}}(\bar{v}, x) = \begin{cases} K^{\gamma} \bar{v} x^{\gamma} + (x - 1)^{\gamma} x^{-\eta} & \text{for } x \geq \bar{x}_{\bar{v}} \\ \frac{K^{\gamma} \bar{v} \bar{x}_{\bar{v}}^{\frac{\gamma-\alpha}{\beta-\alpha}} + (\bar{x}_{\bar{v}} - 1)^{\gamma} \bar{x}_{\bar{v}}^{-\frac{\eta+\alpha}{\beta-\alpha}}}{\bar{x}_{\bar{v}}^{\frac{\beta-\alpha}{\beta-\alpha}}} x^{\beta} & \text{for } x < \bar{x}_{\bar{v}} \end{cases} \quad (\text{C.41})$$

where $\bar{x}_{\bar{v}} = F^{-1}(\bar{\theta})$. The condition in (C.39) gives (4.40) and $\bar{v} = \bar{\mathcal{H}}(\bar{v}, 1)$ characterises \bar{v} as in (4.41). □

Proof of Lemma 4.3.7. Consider the optimal stopping problem described in (4.36) and (4.37) and let $z_1 = (\bar{x}, \bar{v}_1)$ be a solution to (4.40) and (4.41). If it exists, also let $z_2 = (\hat{x}, \bar{x}_l, \bar{x}_u, \bar{v})$ be the solution corresponding to the system described in (4.42), (4.43) and (4.44).

Suppose that $\bar{v}_1 \neq \bar{v}_2$. The proof for the case when $\bar{v}_1 = \bar{v}_2 = \bar{v}$ follows directly from the fact that the function $g_{\bar{v}}(y)$ has a unique non-negative concave majorant.

Suppose that both z_1 and z_2 define a non-negative concave majorant with respect to $g_{\bar{v}_1}$ and $g_{\bar{v}_2}$ respectively. The solutions z_1 and z_2 allow us to define the corresponding stopping times $\tau_1 = \{t \geq 0 : X_t \geq \bar{x}\}$ and $\tau_2 = \{t \geq 0 : X_t \notin (0, \hat{x}) \cup (\bar{x}_l, \bar{x}_u)\}$ respectively. Furthermore from (4.35), our assumption implies that \bar{v}_1 and \bar{v}_2 satisfy $\bar{v}_1 = \mathcal{H}(\bar{v}_1, 1, \tau_1)$ and $\bar{v}_2 = \mathcal{H}(\bar{v}_2, 1, \tau_2)$ respectively.

From Proposition 2.3.8, this implies that τ_1 and τ_2 are both optimal stopping times for the problem:

$$\bar{V}(1) = \sup_{\tau} \mathbb{E} \left[e^{-\rho\tau} \left(\tilde{U}(X_{\tau}, 1) + (KX_{\tau})^{\gamma} \bar{V}(1) \right) \mathbb{I}_{\{\tau < \infty\}} \mid X_0 = 1 \right]$$

described in (4.35). But this implies that $\bar{v}_1 = \bar{v}_2$ as by definition τ_1 and τ_2 both achieve the maximal expected reward. This gives the required contradiction. □

Proof of Proposition 4.3.8. The geometry of $g_{\bar{v}}(y)$ implies that one of the types of solutions described in Propositions 4.3.5 and 4.3.6 is optimal. If the system of equations described in (4.42) and (4.44) has no real-valued solution then it follows that the one threshold strategy is optimal, giving (1).

Suppose now that the system of equations described in (4.36) and (4.37) gives a solution of the form $z_1 = (\bar{x}, \bar{v}_1)$, and let $z_2 = (\bar{x}_l, \bar{x}_u, \bar{v}_2)$ be a solution to the system of equations described in (4.42) and (4.44). Furthermore we let $z_1^y = (\bar{y}, \bar{v}_1)$ and $z_2^y = (\bar{y}_l, \bar{y}_u, \bar{v}_2)$ where $\bar{y} = F(\bar{x})$, $\bar{y}_l = F(\bar{x}_l)$ and $\bar{y}_u = F(\bar{x}_u)$. Finally define the constant a by:

$$a = g_{\bar{v}}(\bar{y}_l) - \frac{\bar{y}_l}{\bar{y}_u - \bar{y}_l} (g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)) \quad (\text{C.42})$$

The condition in (4.46) is equivalent to $a \geq 0$

– Case 1 ($-\alpha < \eta$):

Under this case we have $g_{\bar{v}}(y) \downarrow -\infty$ as $y \downarrow 0$. Furthermore we know that $g_{\bar{v}}(y)$ is concave over $(1, \infty)$ and as described in Appendix C.3, there exists a constant $C_{\bar{v}} \in (0, 1)$ such that $g_{\bar{v}}(y)$ is concave over $(0, C_{\bar{v}})$. In view of this, both systems of equations discussed above have a real-valued solution; that is, z_1^y and z_2^y exist and $\bar{y}_l < 1 < \bar{y}_u$ satisfy:

$$\frac{g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)}{\bar{y}_u - \bar{y}_l} = \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_l} = \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_u} \quad (\text{C.43})$$

The constant a defined in (C.42) above corresponds with the intercept of the line passing through the points $(\bar{y}_l, g_{\bar{v}}(\bar{y}_l))$ and $(\bar{y}_u, g_{\bar{v}}(\bar{y}_u))$ with the vertical axis. If $g_{\bar{v}}(\bar{y}_l) \leq 0$, then the two threshold strategy is obviously not optimal as the non-negativity assumption is not satisfied. In fact if $g_{\bar{v}}(\bar{y}_l) \leq 0$, the monotonicity of $g_{\bar{v}}(\cdot)$ gives $a < 0$. Let us hence assume that $g_{\bar{v}}(\bar{y}_l) > 0$.

(\Rightarrow) If the two threshold strategy is optimal, the system of equations in (4.42), (4.43) and (4.44) defines a non-negative concave majorant of $g_{\bar{v}}(y)$. This means that there exists $\hat{y} \leq \bar{y}_l$ satisfying:

$$\frac{g_{\bar{v}}(\hat{y})}{\hat{y}} = \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\hat{y}}$$

Since $C_{\bar{v}} > \bar{y}_l \geq \hat{y}$ and $g_{\bar{v}}(y)$ is concave over $(0, C_{\bar{v}})$ then:

$$\left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\hat{y}} \geq \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_l} = \frac{g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)}{\bar{y}_u - \bar{y}_l}$$

Furthermore the concavity of $g_{\bar{v}}(y)$ also implies that:

$$\frac{g_{\bar{v}}(\bar{y}_l)}{\bar{y}_l} \geq \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_l}$$

and hence

$$\frac{g_{\bar{v}}(\bar{y}_l)}{\bar{y}_l} \geq \frac{g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)}{\bar{y}_u - \bar{y}_l}$$

giving $a \geq 0$.

(\Leftarrow) Suppose now that $a \geq 0$. We show that the two threshold strategy z_2^y defines a non-negative concave majorant of $g_{\bar{v}}(y)$ and thus the uniqueness result in Lemma 4.3.7 implies that the two threshold strategy is optimal. Given that $a \geq 0$ and the definition of z_y^2 , we have:

$$\left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_u} = \frac{g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)}{\bar{y}_u - \bar{y}_l} = \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_l} \quad \text{and} \quad \frac{g_{\bar{v}}(\bar{y}_l)}{\bar{y}_l} - \frac{g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)}{\bar{y}_u - \bar{y}_l} \geq 0$$

Recall that since $-\alpha < \eta$, $g_{\bar{v}}(y)$ is concave over an interval $(0, C_{\bar{v}})$ for some constant $C_{\bar{v}} < 1$. Given $a \geq 0$, we have $g_{\bar{v}}(\bar{y}_l) > 0$ and hence:

$$\frac{g_{\bar{v}}(\bar{y}_l)}{\bar{y}_l} \geq \left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\bar{y}_l}.$$

Also by definition of $g_{\bar{v}}(y)$, there exists $\tilde{y} \leq \bar{y}_l$ such that $g_{\bar{v}}(\tilde{y}) = 0$ and

$$\left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\tilde{y}} \geq 0$$

The concavity of $g_{\bar{v}}(y)$ over $(0, C_{\bar{v}})$ implies that $\frac{dg_{\bar{v}}}{dy}(y)$ is decreasing over this interval. Moreover $\frac{g_{\bar{v}}(y)}{y}$ is continuous over (\tilde{y}, \bar{y}_l) . These two properties imply that there exists $\hat{y} \in (\tilde{y}, \bar{y}_l)$ satisfying:

$$\left. \frac{dg_{\bar{v}}}{dy} \right|_{y=\hat{y}} = \frac{g_{\bar{v}}(\hat{y})}{\hat{y}}$$

and hence $(\hat{y}, \bar{y}_l, \bar{y}_u, \bar{v})$ defines a smallest concave majorant of $g_{\bar{v}}(y)$ which concludes the proof for this case.

– Case 2 ($-\alpha \geq \eta$):

As discussed in Appendix C.3 under this case we have $\lim_{y \downarrow 0} g_{\bar{v}}(y) = 0$. Furthermore, there exists $\epsilon \in (0, 1)$ such that $\frac{d}{dy} g_{\bar{v}}(y) < 0$ over $(0, \epsilon)$ and $g_{\bar{v}}(y)$

has a unique turning point over $(0, 1)$. Suppose that the solutions z_1^y and z_2^y exist and hence $\bar{y}_l < 1 < \bar{y}_u$ satisfy:

$$\frac{g_{\bar{v}}(\bar{y}_u) - g_{\bar{v}}(\bar{y}_l)}{\bar{y}_u - \bar{y}_l} = \frac{dg_{\bar{v}}}{dy} \Big|_{y=\bar{y}_l} = \frac{dg_{\bar{v}}}{dy} \Big|_{y=\bar{y}_u} \quad (\text{C.44})$$

We can again assume that $g_{\bar{v}}(\bar{y}_l) \geq 0$, since otherwise $a < 0$. Using these arguments a similar proof to the one discussed in Case 1 above follows. \square

Proof of Proposition 4.3.9. Proposition 4.3.5 implies that when the one threshold strategy is optimal, the value function $\bar{\mathcal{H}}(\bar{v}, x)$ satisfies $\bar{\mathcal{H}}(\bar{v}, x) = \bar{A}_{\bar{v}}^{(1)} x^\beta$ for some constant $\bar{A}_{\bar{v}}^{(1)} \in \mathbb{R}^+$. Furthermore $\bar{A}_{\bar{v}}^{(1)}$ does not depend on λ , and thus the value function is independent of λ . For the two threshold strategy Proposition 4.3.6 implies that we have $\bar{\mathcal{H}}(\bar{v}, x) = \bar{A}_{\bar{v}}^{(2)} x^\beta + \bar{B}_{\bar{v}}^{(2)} x^\alpha$ for some constants $\bar{A}_{\bar{v}}^{(2)}$, $\bar{B}_{\bar{v}}^{(2)}$ and is decreasing in λ . Hence there must exist a unique value of λ ; call it λ^* , at which the two value functions are equal and hence $\bar{A}_{\bar{v}}^{(1)} = \bar{A}_{\bar{v}}^{(2)} = \bar{A}_{\bar{v}}$, $\bar{B}_{\bar{v}}^{(2)} = 0$ and \bar{v} satisfies (4.41). By definition we have that $\bar{v} = \bar{\mathcal{H}}(\bar{v}, 1) = \bar{A}_{\bar{v}}$ and $\bar{\mathcal{H}}(\bar{v}, \bar{x}_*) = \bar{A}_{\bar{v}} \bar{x}_*^\beta$ and $\bar{\mathcal{H}}(\bar{v}, \underline{x}_*) = \bar{A}_{\bar{v}} \underline{x}_*^\beta$.

$$\bar{A}_{\bar{v}} = \frac{(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta}}{\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma} = \frac{-\lambda(1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta}}{\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma} \quad (\text{C.45})$$

and hence:

$$\lambda^* = -\frac{(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta} (\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma)}{(1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta} (\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma)} \quad (\text{C.46})$$

By applying the smooth fit principle described in Proposition 2.3.11 we also obtain the following system of equations after re-arranging:

$$\begin{aligned} \bar{A}_{\bar{v}} \beta \bar{x}_*^{\beta-1} &= \gamma(\bar{x}_* - 1)^{\gamma-1} \bar{x}_*^{-\eta} - \eta(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta-1} + \gamma K^\gamma \bar{x}_*^{\gamma-1} \bar{A}_{\bar{v}} \\ \bar{A}_{\bar{v}} \beta \underline{x}_*^{\beta-1} &= \gamma \lambda (1 - \underline{x}_*)^{\gamma-1} \underline{x}_*^{-\eta} + \eta \lambda (1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta-1} + \gamma K^\gamma \underline{x}_*^{\gamma-1} \bar{A}_{\bar{v}} \end{aligned}$$

giving:

$$\bar{A}_{\bar{v}} = \frac{\gamma(\bar{x}_* - 1)^{\gamma-1} \bar{x}_*^{-\eta} - \eta(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta-1}}{\beta \bar{x}_*^{\beta-1} - \gamma K^\gamma \bar{x}_*^{\gamma-1}} = \frac{\gamma \lambda (1 - \underline{x}_*)^{\gamma-1} \underline{x}_*^{-\eta} + \eta \lambda (1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta-1}}{\beta \underline{x}_*^{\beta-1} - \gamma K^\gamma \underline{x}_*^{\gamma-1}} \quad (\text{C.47})$$

Since the quotients in both (C.26) and (C.27) are equal to $\bar{A}_{\bar{v}}$ we get:

$$\frac{(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta}}{\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma} = \frac{\gamma(\bar{x}_* - 1)^{\gamma-1} \bar{x}_*^{-\eta} - \eta(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta-1}}{\beta \bar{x}_*^{\beta-1} - \gamma K^\gamma \bar{x}_*^{\gamma-1}} \quad (\text{C.48})$$

and

$$\frac{-(1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta}}{\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma} = \frac{\gamma(1 - \underline{x}_*)^{\gamma-1} \underline{x}_*^{-\eta} + \eta(1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta-1}}{\beta \underline{x}_*^{\beta-1} - \gamma K^\gamma \underline{x}_*^{\gamma-1}} \quad (\text{C.49})$$

and hence after re-arranging (C.48) and (C.49), both \underline{x}_* and \bar{x}_* satisfy:

$$(\beta + \eta - \gamma)x - (\beta + \eta) = K^\gamma x^{\gamma-\beta} (\eta(x-1) - \gamma) \quad (\text{C.50})$$

giving (4.48). The expressions in (C.48) and (C.49) can be re-arranged to show that \bar{x}_* and \underline{x}_* both satisfy:

$$x^\beta - K^\gamma x^\gamma = \frac{K^\gamma x^\gamma (\gamma - \beta)(x - 1)}{(\beta + \eta - \gamma)x - (\beta + \eta)} \quad (\text{C.51})$$

Plugging (C.51) in (C.46) gives

$$\lambda^* = \frac{(\bar{x}_* - 1)^{\gamma-1} \underline{x}_*^{\eta+\gamma} ((\beta + \eta - \gamma)\bar{x}_* - (\beta + \eta))}{(1 - \underline{x}_*)^{\gamma-1} \bar{x}_*^{\eta+\gamma} ((\beta + \eta - \gamma)\underline{x}_* + (\beta + \eta))}$$

□

Proof of Proposition 4.3.10. Proposition 4.3.8 implies that the two threshold strategy is optimal if and only if the upper and lower thresholds $\bar{x}_{\bar{v}}$ and $\underline{x}_{\bar{v}}$ satisfy:

$$\frac{F(\bar{x}_{\bar{v}})}{F(\underline{x}_{\bar{v}})} \geq \frac{g_v(F(\bar{x}_{\bar{v}}))}{g_v(F(\underline{x}_{\bar{v}}))}$$

which yields:

$$\frac{\bar{x}_{\bar{v}}^{\beta-\alpha}}{\underline{x}_{\bar{v}}^{\beta-\alpha}} \geq \frac{K^\gamma \bar{v} \bar{x}_{\bar{v}} + (\bar{x}_{\bar{v}} - 1)^\gamma \bar{x}_{\bar{v}}^{-\eta-\alpha}}{K^\gamma \bar{v} \underline{x}_{\bar{v}} - \lambda(1 - \underline{x}_{\bar{v}})^\gamma \underline{x}_{\bar{v}}^{-\eta-\alpha}} \quad (\text{C.52})$$

Plugging in the value of \bar{v} as given in (4.44) into (C.52) and re-arranging we get:

$$\lambda \leq - \frac{(\bar{x}_* - 1)^\gamma \bar{x}_*^{-\eta} (\underline{x}_*^\beta - K^\gamma \underline{x}_*^\gamma)}{(1 - \underline{x}_*)^\gamma \underline{x}_*^{-\eta} (\bar{x}_*^\beta - K^\gamma \bar{x}_*^\gamma)}$$

But from (C.46), this implies $\lambda \leq \lambda^*$ where λ^* is as given in Proposition 4.3.9. Hence the equivalence.

□

Appendix D

Appendix for Chapter 5

Proof of Proposition 5.2.1. Consider the case when $\eta \geq 1$. We will show that under this case the optimal strategy for the Expected Utility agent is to stop immediately. The result then follows from the fact that the regret agent always stops before or with the EU agent. Consider the problem described in (5.16). We will show that $\hat{V} = u$ implying that the stopping region is equivalent to the whole domain of X . Note that by the super-harmonic characterisation of the value function (See Theorem 2.4 in Peskir and Shiryaev [2006]), \hat{V} is the smallest super-harmonic function dominating $u(\cdot)$. If $u(\cdot)$ is in itself a super-harmonic function of X then $\hat{V} \equiv u$ and we are done. It hence suffices to show that for every stopping time τ and $x \in \mathcal{I}$,

$$\mathbb{E}_x[u(X_\tau)] \leq u(x) \quad (\text{D.1})$$

Let \mathbb{L}_X be the infinitesimal generator of X . Since $u(\cdot)$ is concave and increasing over \mathcal{I} and given $\eta \geq 1$, we have:

$$\mathbb{L}_X u(x) = \frac{1}{2}\sigma^2 x^2 u_{xx}(x) + \mu x u_x(x) \leq 0 \quad (\text{D.2})$$

By Ito's lemma:

$$\begin{aligned} u(X_\tau) &= u(x) + \int_0^\tau u'(X_s) dX_s + \int_0^\tau \frac{1}{2} u''(X_s) d[X]_s \\ &= u(x) + \int_0^\tau u'(X_s) \sigma X_s dW_s + \int_0^\tau \mathbb{L}_X u(X_s) ds \end{aligned}$$

From (D.2) it follows that:

$$\mathbb{E}_x[u(X_\tau)] \leq u(x) + \sigma \mathbb{E}_x \left[\int_0^\tau u'(X_s) X_s dW_s \right] \quad (\text{D.3})$$

The process $H_t = \int_0^t u'(X_s) X_s dW_s$ is a continuous martingale starting at $H_0 = 0$. This follows from the fact that $xu'(x)$ is bounded over $(0, \infty)$ giving $\mathbb{E}_x[[H]_t] < \infty$, $\forall t \in [0, \infty)$. Hence from (D.3), it follows that

$$\mathbb{E}_x[u(X_\tau)] \leq u(x)$$

giving the result when $\eta \geq 1$.

Consider now the case when $\eta \leq 0$. Let $a \in \mathbb{R}^+$ be an arbitrary constant satisfying $a \geq s$ where $s = S_0$. Consider the stopping time $H_a = \inf\{t \geq 0 : X_t \geq a\}$ and let $\nu = \frac{\mu}{\sigma^2} - \frac{1}{2}$. Under H_a the agent attains the value:

$$\begin{aligned} \mathbb{E}_{x,s}[u(X_{H_a}) - \kappa u(S_{H_a})] &= (1 - \kappa)u(a)\mathbb{P}_x\left[\sup_{0 \leq t < \infty} X_t > a\right] \\ &= (1 - \kappa)u(a)\left(\frac{x}{a}\right)^{|\nu| - \nu} \\ &= (1 - \kappa)u(a) \end{aligned}$$

This implies that when $\eta \leq 0$, the expected reward for the agent is increasing in a and thus it is always optimal for the agent to keep delaying stopping. This gives $\tau^* = \infty$ under $\eta \leq 0$. \square

Proof of Proposition 5.2.4. Let $b, B \in \mathbb{R}^+$ be some constants satisfying $0 \leq b < B$ and consider $\hat{\tau}$ as defined in (5.15). Furthermore, define $g(\cdot) = u(f^{-1}(\cdot))$.

Part 1: Consider the case when $X_0 = x$ and $S_0 = s$ satisfy $0 < x \leq s \leq b$. Then by definition $\hat{\tau} = H_b$ almost surely. Hence:

$$\begin{aligned} \tilde{V}(x, s) &= \mathbb{E}_{x,s}\left[u(X_{\hat{\tau}}) - \kappa u(S_{\hat{\tau}})\right] \\ &= \mathbb{E}_{x,s}\left[u(X_{H_b}) - \kappa u(S_{H_b})\right] \\ &= \mathbb{E}_{y,\tilde{s}}\left[g(Y_{\tilde{H}_b}) - \kappa g(S_{\tilde{H}_b}^Y)\right] \\ &= g(\tilde{b})\frac{y}{\tilde{b}} - \kappa \mathbb{E}_{y,\tilde{s}}[g(S_{\tilde{H}_b}^Y)] \end{aligned} \tag{D.4}$$

where we use the fact that the stopped process $Y_{t \wedge H_b}$ is an $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale. The last term in (D.4) above can be determined analytically by partitioning the sample space into three; that is,

- Y_t reaches the level \tilde{b} in finite time,
- Y_t remains below the level \tilde{s} indefinitely ,
- Y_t reaches the level \tilde{s} in finite time but remains below the level \tilde{b} .

Thus the last term in (D.4) can be re-written as follows:

$$\mathbb{E}_{y,\tilde{s}}[g(S_{\tilde{H}_b}^Y)] = \mathbb{E}_{y,\tilde{s}}[g(S_{\tilde{H}_b}^Y)(\mathbb{I}_{\{\tilde{H}_b < \infty\}} + \mathbb{I}_{\{\tilde{H}_s = \infty\}} + \mathbb{I}_{\{\tilde{H}_s < \infty \text{ \& } \tilde{H}_b = \infty\}})] \quad (\text{D.5})$$

$$\begin{aligned} &= g(b)\mathbb{P}_{y,\tilde{s}}[S_{\tilde{H}_b}^Y = \tilde{b}] + g(\tilde{s})\mathbb{P}_{y,\tilde{s}}[\tilde{H}_s = \infty] \\ &\quad + \mathbb{E}_{y,\tilde{s}}[g(S_{\tilde{H}_b}^Y)\mathbb{I}_{\{\tilde{H}_s < \infty \text{ \& } \tilde{H}_b = \infty\}}] \\ &= (g(\tilde{b})\frac{y}{\tilde{b}} + g(\tilde{s})\frac{\tilde{s} - y}{\tilde{s}}) + \int_{\tilde{s}}^{\tilde{b}} g(z)\mathbb{P}_{y,\tilde{s}}(S_{\infty}^Y \in (z, z + dz)) \end{aligned} \quad (\text{D.6})$$

But:

$$\begin{aligned} \int_{\tilde{s}}^{\tilde{b}} g(z)\mathbb{P}_{y,\tilde{s}}(S_{\infty}^Y \in (z, z + dz)) &= \int_{\tilde{s}}^{\tilde{b}} g(z)\mathbb{P}_{y,\tilde{s}}(\{S_{\infty}^Y \in (z, z + dz)\} \cap \{\tilde{H}_z < \infty\}) \\ &= \int_{\tilde{s}}^{\tilde{b}} g(z)\mathbb{P}_{y,\tilde{s}}(S_{\infty}^Y \in (z, z + dz) \mid \tilde{H}_z < \infty)\mathbb{P}_{y,\tilde{s}}(\tilde{H}_z < \infty) \\ &= \int_{\tilde{s}}^{\tilde{b}} g(z)\frac{y}{z}\mathbb{P}_{y,\tilde{s}}(S_{\infty}^Y \in (z, z + dz) \mid \tilde{H}_z < \infty) \\ &= \int_{\tilde{s}}^{\tilde{b}} g(z)\frac{y}{z}\mathbb{P}_z(\tilde{H}_{z+dz} = \infty) \\ &= \int_{\tilde{s}}^{\tilde{b}} g(z)\frac{y}{z}\frac{dz}{z + dz} \\ &\approx \int_{\tilde{s}}^{\tilde{b}} g(z)\frac{y}{z^2} dz \end{aligned} \quad (\text{D.7})$$

From (D.4), (D.6) and (D.7) above, it hence follows that for $x \leq s$ satisfying $0 < x \leq s < b$:

$$\tilde{V}(x, s) = u(\tilde{b})\frac{f(x)}{f(b)}(1 - \kappa) - \kappa \left(u(s) \left(1 - \frac{f(x)}{f(s)} \right) + \int_{f(s)}^{f(b)} u(f^{-1}(\omega)) \frac{f(x)}{\omega^2} d\omega \right) \quad (\text{D.8})$$

$$= g(\tilde{b})\frac{y}{\tilde{b}}(1 - \kappa) - \kappa \left(g(\tilde{s})\frac{\tilde{s} - y}{\tilde{s}} + \int_{\tilde{s}}^{\tilde{b}} g(\omega)\frac{y}{\omega^2} d\omega \right) \quad (\text{D.9})$$

Part 2: Consider now the case when $(x, s) \in \{(x, s) \in \mathbb{R}^2 : 0 < x \leq s \text{ \& } s \geq B\}$. Note that under $\tilde{\tau}$, this region can be further split into a continuation region (i.e. $0 < x < B$) and a stopping region (i.e. $B \leq x \leq s$). Furthermore given the definition of $\tilde{\tau}$ in (5.15), for $s > B$ it follows that $S_{\tilde{\tau}} = S_0 = s$.

Consider first the case $0 < x < B$. Letting $\tilde{B} = f(B)$ it follows that for $s \geq B$, $\tilde{\tau} = H_B = \tilde{H}_{\tilde{B}} = \tilde{\tau}_Y$ almost surely. The corresponding value function can

hence be characterised as follows:

$$\begin{aligned}\tilde{V}(x, s) &= \mathbb{E}_{x,s}[u(X_{\tilde{\tau}}) - \kappa u(s)] \\ &= \mathbb{E}_{y,\tilde{s}}[g(Y_{\tilde{\tau}_Y})] - \kappa g(\tilde{s}) \\ &= g(\tilde{B}) \frac{y}{\tilde{B}} - \kappa g(\tilde{s})\end{aligned}\tag{D.10}$$

$$= u(B) \frac{f(x)}{f(B)} - \kappa u(s)\tag{D.11}$$

since the stopped process $Y_{t \wedge H_B}$ is an $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale.

Given that under $\tilde{\tau}$, the agent stops immediately if $B \leq x \leq s$ it follows that over this region,

$$\tilde{V}(x, s) = u(x) - \kappa u(s)\tag{D.12}$$

Part 3: Finally for $S_0 = s \in (b, B)$ the proposed stopping rule $\tilde{\tau}$ is equivalent to the first hitting time $H_s = \inf_{t \geq 0} \{X_t \geq S_0 = s\}$ implying:

$$\begin{aligned}\tilde{V}(x, s) &= \mathbb{E}_{x,s}[u(X_{H_s}) - \kappa u(S_{H_s})] \\ &= \mathbb{E}_{x,s}[u(X_{H_s}) - \kappa u(s)] \\ &= \mathbb{E}_{y,\tilde{s}}[g(Y_{\tilde{H}_s})] - \kappa g(\tilde{s}) \\ &= g(\tilde{s}) \frac{y}{\tilde{s}} - \kappa g(\tilde{s}) \\ &= u(s) \frac{f(x)}{f(s)} - \kappa u(s)\end{aligned}\tag{D.13}$$

since the stopped process $Y_{t \wedge H_s}$ is an $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale.

The result follows by combining (D.8), (D.11), (D.12) and (D.13). \square

Proof of Corollary 5.2.5. For $\tilde{\tau}$ to be considered as a candidate optimal stopping time, the constants b and B have to be chosen in such a way that they maximise the corresponding value function $\tilde{V}(x, s)$. From (D.10) this implies that $\tilde{B} = f(B)$ must maximise:

$$c_1(\tilde{B}) = g(\tilde{B}) \frac{y}{\tilde{B}} - \kappa g(\tilde{s})$$

giving:

$$\frac{g(\tilde{B})}{g'(\tilde{B})\tilde{B}} = 1.$$

Similarly from (D.9) it follows that the parameter $\tilde{b} = f(b)$ maximises:

$$c_2(\tilde{b}) = g(\tilde{b}) \frac{y}{\tilde{b}} (1 - \kappa) - \kappa \left(g(\tilde{s}) \frac{\tilde{s} - y}{\tilde{s}} + \int_{\tilde{s}}^{\tilde{b}} g(\omega) \frac{y}{\omega^2} d\omega \right)$$

Through the first order condition, the maximum of $c_2(\cdot)$ is achieved at \tilde{b} satisfying:

$$\frac{g(\tilde{b})}{g'(\tilde{b})\tilde{b}} = 1 - \kappa. \quad (\text{D.14})$$

The first order condition in (D.14) does not ensure that $\tilde{b} \geq 0$. Note however that $h(\cdot)$ is monotonically increasing and

$$h(x) = \frac{g(x)}{g'(x)x} = \frac{\eta}{\gamma} x^{-\frac{1}{\eta}} \left(\exp(\gamma y^{\frac{1}{\eta}}) - 1 \right) > (1 - \kappa)$$

for all $x \in \mathbb{R}^+$ if and only if $\eta > (1 - \kappa)$. Thus a constant $\tilde{b} \geq 0$ satisfying (D.14) exists only when $\eta \leq (1 - \kappa)$, with $\tilde{b} = 0$ when $\eta = (1 - \kappa)$ and no positive solution to (D.14) when $\eta > (1 - \kappa)$. However, under the assumption that $\eta > (1 - \kappa)$ it can be easily that $c_2(\cdot)$ is monotonically decreasing over \mathbb{R}^+ and thus $c_2(\cdot)$ achieves its' maximum value over \mathbb{R}^+ at $\tilde{b} = 0$. \square

Proof of Proposition 5.2.6. The continuity of $\tilde{V}(X_t, S_t)$ follows from the definition in (5.17). Applying Ito's Lemma to $\tilde{V}(X_t, S_t)$ gives

$$d\tilde{V}(X_t, S_t) = \tilde{V}_x(X_t, S_t)dX_t + \frac{1}{2}\tilde{V}_{xx}(X_t, S_t)d[X]_t + \tilde{V}_s(X_t, S_t)dS_t$$

where:

$$\begin{aligned} & \tilde{V}_x(X_t, S_t)dX_t + \frac{1}{2}\tilde{V}_{xx}(X_t, S_t)d[X]_t \\ &= \left[\mathbb{I}_{\{S_t < b\}} \left(\frac{u(b)}{f(b)}(1 - \kappa) + \kappa \frac{u(S_t)}{f(S_t)} - \kappa \int_{f(S_t)}^{f(b)} \frac{u(f^{-1}(z))}{z^2} dz \right) \right. \\ & \quad \left. + \mathbb{I}_{\{b \leq S_t \leq B\}} \left(\frac{u(S_t)}{f(S_t)} \right) + \mathbb{I}_{\{X_t \leq B, S_t > B\}} \left(\frac{u(B)}{f(B)} \right) \right] \times \\ & \quad \left(f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X]_t \right) \\ & \quad + \mathbb{I}_{\{X_t > B\}} \left(u'(X_t)dX_t + \frac{1}{2}u''(X_t)d[X]_t \right) \end{aligned} \quad (\text{D.15})$$

But by definition, $\forall x \in \mathcal{I}$ the scale function f satisfies $\mathbb{L}_X f(x) = 0$, implying:

$$\tilde{V}_x(X_t, S_t) dX_t + \frac{1}{2} \tilde{V}_{xx}(X_t, S_t) d[X]_t \quad (\text{D.16})$$

$$\begin{aligned} &= \left[\mathbb{I}_{\{S_t < b\}} \left(\frac{u(b)}{f(b)} (1 - \kappa) + \kappa \frac{u(S_t)}{f(S_t)} - \kappa \int_{f(S_t)}^{f(b)} \frac{u(f^{-1}(z))}{z^2} dz \right) \right. \\ &\quad \left. + \mathbb{I}_{\{b \leq S_t \leq B\}} \left(\frac{u(S_t)}{f(S_t)} \right) + \mathbb{I}_{\{X_t \leq B, S_t > B\}} \left(\frac{u(B)}{f(B)} \right) \right] f'(X_t) \sigma X_t dW_t \\ &\quad + \mathbb{I}_{\{X_t > B\}} (\mathbb{L}_X u(X_t) dt + u'(X_t) \sigma X_t dW_t) \\ &= (G(X_t, S_t) f'(X_t) \sigma X_t \mathbb{I}_{\{X_t \leq B\}}) dW_t \\ &\quad + \mathbb{I}_{\{X_t > B\}} (\mathbb{L}_X u(X_t) dt + u'(X_t) \sigma X_t dW_t) \end{aligned} \quad (\text{D.17})$$

where:

$$\begin{aligned} G(x, s) &= \mathbb{I}_{\{s < b\}} \left(\frac{u(b)}{f(b)} (1 - \kappa) + \kappa \frac{u(s)}{f(s)} - \kappa \int_{f(s)}^{f(b)} \frac{u(f^{-1}(z))}{z^2} dz \right) \\ &\quad + \mathbb{I}_{\{b \leq s \leq B\}} \left(\frac{u(s)}{f(s)} \right) + \mathbb{I}_{\{x \leq B, s > B\}} \left(\frac{u(B)}{f(B)} \right) \end{aligned} \quad (\text{D.18})$$

We note that for all $t \geq 0$, $|G(X_t, S_t)| \leq C_s$ for some constant $C_s \in \mathbb{R}^+$ depending on the initial value $S_0 = s$. Letting $M_t = \int_0^t (G(X_u, S_u) f'(X_u) \sigma X_u \mathbb{I}_{\{X_u \leq B\}}) dW_u$ it hence follows that:

$$\begin{aligned} \mathbb{E}[M_t] &= \mathbb{E} \left[\int_0^t (\eta \sigma G(X_u, S_u) X_u^\eta)^2 \mathbb{I}_{\{X_u \leq B\}} du \right] \\ &\leq \tilde{K}_s t \end{aligned}$$

This gives $\mathbb{E}[M_t] < \infty$, $\forall t \in [0, \infty)$ and hence M_t is a martingale (See Billingsley [2013] page 72 Corollary 3).

Since the region $\{(x, s) \in \mathbb{R}^2 : x \geq B\}$ corresponds with the stopping region of $\tilde{\tau}$, $\mathbb{L}_X u(x) \leq 0$ is satisfied over this region. Moreover, the boundedness of $u'(x)x$ gives the martingale property for $\int_0^t u'(X_s) \sigma X_s dW_s$. The above arguments imply that for all $0 \leq x \leq s$,

$$\mathbb{E}_{x,s} \left[\int_0^t \tilde{V}_x(X_u, S_u) dX_u + \frac{1}{2} \int_0^t \tilde{V}_{xx}(X_u, S_u) d[X]_u \right] \leq 0 \quad (\text{D.19})$$

Moreover we have:

$$\begin{aligned} V_s(X_t, S_t)dS_t = & \left[-\mathbb{I}_{\{S_t < b\}} \left(\kappa u'(S_t) \left(1 - \frac{f(X_t)}{f(S_t)} \right) \right) + \mathbb{I}_{\{S_t > B\}} \left(-\kappa u'(S_t) \right) \right. \\ & \left. + \mathbb{I}_{\{b \leq S_t \leq B\}} \left(u'(S_t) \left(\frac{f(X_t)}{f(S_t)} - \kappa \right) - \frac{u(S_t)}{(f(S_t))^2} f(X_t) f'(S_t) \right) \right] dS_t \end{aligned} \quad (\text{D.20})$$

By definition dS_t terms are naturally zero whenever $X_t < S_t$, implying that the first term in (D.20) above vanishes. Furthermore given that $dS_t = (1/f'(S_t))dS_t^Y$ then the third term in (D.20) above can be re-written as:

$$\mathbb{I}_{\{\tilde{b} \leq S_t^Y \leq \tilde{B}\}} \left[(1 - \kappa) g'(S_t^Y) - \frac{g(S_t^Y)}{S_t^Y} \right] dS_t^Y$$

and given that the function $h(x) = \frac{g(x)}{xg'(x)}$ is increasing and $h(\tilde{b}) \geq (1 - \kappa)$, it follows that this term is always negative. This together with (D.19) give the result. \square

Proof of Proposition 5.2.7. By applying Ito's Lemma to (5.17) and following a similar argument to the one used in the proof of Proposition 5.2.6,

$$\begin{aligned} \tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}}) &= v(x, s) \\ &+ \int_0^{t \wedge \tilde{\tau}} \left[\mathbb{I}_{\{S_u < b\}} \left(\frac{u(b)}{f(b)} (1 - \kappa) + \kappa \frac{u(S_u)}{f(S_u)} - \kappa \int_{f(S_u)}^{f(b)} \frac{u(f^{-1}(z))}{z^2} dz \right) \right. \\ &\quad \left. + \mathbb{I}_{\{b \leq S_u \leq B\}} \left(\frac{u(S_u)}{f(S_u)} \right) + \mathbb{I}_{\{X_u \leq B, S_u > B\}} \left(\frac{u(B)}{f(B)} \right) \right] f'(X_u) \sigma X_u dW_u \\ &= v(x, s) + \sigma \eta \int_0^{t \wedge \tilde{\tau}} G(X_u, S_u) X_u^\eta dW_u \end{aligned}$$

where $G(x, s)$ is as defined in (D.18). Given $|G(X_u, S_u)| \leq C_s$ for every $u > 0$ then for every $t > 0$:

$$\begin{aligned} \mathbb{E}_{x,s} \left[[\tilde{V}(X, S)]_{t \wedge \tau^*} \right] &\leq \tilde{C}_s \mathbb{E}_{x,s} \left[\int_0^{t \wedge \tilde{\tau}} X_u^{2\eta} du \right] \\ &\leq \tilde{C}_s \mathbb{E}_{x,s} \left[\int_0^t X_u^{2\eta} \mathbb{I}_{\{X_u \leq B\}} du \right] \end{aligned}$$

for some constant \tilde{C}_s . This implies that the stopped process $\tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}})$ is a

martingale (See Billingsley [2013] page 72 Corollary 3), since:

$$\mathbb{E}_{x,s} \left[[\tilde{V}(X, S)]_{t \wedge \tilde{\tau}} \right] < \infty \quad \forall t \in [0, \infty)$$

Furthermore from the characterisation of $\tilde{V}(x, s)$ in (5.17) it follows that the stopped process $V(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}})$ is bounded and hence uniformly integrable. This implies:

$$\mathbb{E}_{x,s} \left[\lim_{t \rightarrow \infty} \tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}}) \right] = \lim_{t \rightarrow \infty} \mathbb{E}_{x,s} \left[\tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}}) \right] = \tilde{V}(x, s)$$

□

Proof of Lemma 5.2.8. Recall the characterisation of $\tilde{V}(x, s)$ in (5.17):

$$\begin{aligned} \tilde{V}(x, s) &= \mathbb{I}_{\{s < b\}} \left[\frac{u(b)}{f(b)} f(x)(1 - \kappa) - \kappa \left(u(s) \left(1 - \frac{f(x)}{f(s)} \right) + \int_{f(s)}^{f(b)} u(f^{-1}(\omega)) \frac{f(x)}{\omega^2} d\omega \right) \right] \\ &+ \mathbb{I}_{\{s \geq B\}} \left[\left(f(x) \frac{u(B)}{f(B)} - \kappa u(s) \right) \mathbb{I}_{\{x \leq B\}} + \left(u(x) - \kappa u(s) \right) \mathbb{I}_{\{x > B\}} \right] \\ &+ \mathbb{I}_{\{b \leq s < B\}} \left[u(s) \frac{f(x)}{f(s)} - \kappa u(s) \right] \end{aligned}$$

and let:

$$h(y) = \frac{g(y)}{y g'(y)} = \frac{\eta}{\gamma} y^{-\frac{1}{\eta}} (\exp(\gamma y^{\frac{1}{\eta}}) - 1).$$

The inequality in (5.21) holds trivially over the region $\{(x, s) \in \mathbb{R}^2 : 0 \leq x \leq s, s \geq B \text{ and } x \geq B\}$. Consider now the region $\{(x, s) \in \mathbb{R}^2 : 0 \leq x \leq s, s \geq B \text{ and } x \leq B\}$. The inequality holds over this region if:

$$\frac{f(x)}{u(x)} \geq \frac{f(B)}{u(B)} \quad \text{for } x \leq B$$

This follows from the fact that $\frac{f(x)}{u(x)}$ has a unique minimum x_{\min} , is strictly decreasing over $(0, x_{\min})$ and $x_{\min} = B$. A similar argument can be made to prove the inequality over the region $\{(x, s) \in \mathbb{R}^2 : 0 \leq x \leq s \text{ and } 0 \leq b \leq s \leq B\}$.

Finally consider the region $\mathcal{D} = \{(x, s) \in \mathbb{R}^2 : 0 \leq x \leq s \text{ and } 0 \leq s \leq b\}$. Recall from Corollary 5.2.5 that $b > 0$ if and only if $\eta < (1 - \kappa)$ and hence we can

assume that $\eta < (1 - \kappa)$. The result follows if for all $(x, s) \in \mathcal{D}$,

$$\frac{u(b)}{f(b)}(1 - \kappa) + \kappa \frac{u(s)}{f(s)} - \frac{u(x)}{f(x)} \geq \kappa \int_{f(s)}^{f(b)} \frac{u(f^{-1}(\omega))}{\omega^2} d\omega \quad (\text{D.21})$$

However since $\frac{u(x)}{f(x)}$ is increasing over $(0, B)$, and (D.21) must be satisfied for all x satisfying $0 \leq x \leq s \leq b$ then the inequality in (D.21) holds if:

$$(1 - \kappa) \left(\frac{u(b)}{f(b)} - \frac{u(s)}{f(s)} \right) \geq \kappa \int_{f(s)}^{f(b)} \frac{u(f^{-1}(\omega))}{\omega^2} d\omega \quad (\text{D.22})$$

By applying $y = f^{-1}(\omega)$, the inequality in (D.22) can be re-written as:

$$\left(\frac{u(b)}{f(b)} - \frac{u(s)}{f(s)} \right) \geq \kappa \gamma \int_s^b \exp(-\gamma y) y^{-\eta} dy \quad (\text{D.23})$$

Define:

$$D(s) = \left(\frac{u(b)}{f(b)} - \frac{u(s)}{f(s)} \right) - \kappa \gamma \int_s^b \exp(-\gamma y) y^{-\eta} dy \quad (\text{D.24})$$

Note that $D(b) = 0$ and $\frac{d}{ds}D(s) \leq 0$ which follows easily by noting that $s \leq b$, $h(b) = 1 - \kappa$ and $h(y)$ is increasing. Thus $D(s) \geq 0$ for $s \in (0, b)$ which gives (D.24) and hence the result. \square

Proof of Theorem 5.2.9. Since $\tilde{V}(X_t, S_t)$ defines a continuous super-martingale, given any stopping time τ :

$$\tilde{V}(x, s) \geq \mathbb{E}_{x,s}[\tilde{V}(X_{t \wedge \tau}, S_{t \wedge \tau})]$$

But from the characterisation of $\tilde{V}(x, s)$ in (5.17), it follows that $\tilde{V}(X_{t \wedge \tau}, S_{t \wedge \tau})$ is bounded. Hence by letting $t \rightarrow \infty$ by the Dominated Convergence Theorem and Lemma 5.2.8, we get:

$$\tilde{V}(x, s) \geq \mathbb{E}_{x,s}[V(X_\tau, S_\tau)] \geq \mathbb{E}_{x,s}[u(X_\tau) - \kappa u(S_\tau)] \quad (\text{D.25})$$

This holds for every stopping time τ and thus:

$$\tilde{V}(x, s) \geq \sup_{\tau} \mathbb{E}_{x,s}[u(X_\tau) - \kappa u(S_\tau)] \quad (\text{D.26})$$

On the other hand from Proposition 5.2.7 we have:

$$\begin{aligned}
\tilde{V}(x, s) &= \lim_{t \rightarrow \infty} \mathbb{E}_{x,s} [\tilde{V}(X_{t \wedge \tilde{\tau}}, S_{t \wedge \tilde{\tau}})] \\
&= \mathbb{E}_{x,s} [\tilde{V}(X_{\tilde{\tau}}, S_{\tilde{\tau}})] \\
&= \mathbb{E}_{x,s} [u(X_{\tilde{\tau}}) - \kappa u(S_{\tilde{\tau}})]
\end{aligned}$$

by definition of $\tilde{V}(x, s)$. Thus it follows that:

$$\tilde{V}(x, s) \leq \sup_{\tau} \mathbb{E}_{x,s} [u(X_{\tau}) - \kappa u(S_{\tau})] \quad (\text{D.27})$$

Equations (D.26) and (D.27) give the result. \square

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