A New Maximal Subgroup of E_8 in Characteristic 3

David A. Craven, David I. Stewart and Adam R. Thomas

July 5, 2021

Abstract

We prove the existence and uniqueness up to conjugacy of a new maximal subgroup of the algebraic group of type E_8 in characteristic 3. This has type F_4, and was missing from previous lists of maximal subgroups produced by Seitz and Liebeck–Seitz. We also prove a result about the finite group $H = {}^3D_4(2)$, namely that if H embeds in E_8 (in any characteristic p) and has two composition factors on the adjoint module then $p = 3$ and H lies in a conjugate of this new maximal F_4 subgroup.

1 Introduction

The classification of the maximal subgroups of positive dimension of exceptional algebraic groups [14] is a cornerstone of group theory. In the course of understanding subgroups of the finite groups $E_8(q)$ in [3], the first author ran into a configuration that should not occur according to the tables in [14].

We elicit a previously undiscovered maximal subgroup of type F_4 of the algebraic group E_8 over an algebraically closed field of characteristic 3. This discovery corrects the tables in [14], and the original source [19] on which it depends.

Theorem 1.1. *Let* $\mathbf{G}$ *be a simple algebraic group of type* E_8 *over an algebraically closed field* k *of characteristic* 3*. Then* $\mathbf{G}$ *contains a unique conjugacy class of simple maximal subgroups of type* F_4*.*

If $\mathbf{X}$ *is in this class, then the restriction of the adjoint module* $L(\mathbf{G})$ *to* $\mathbf{X}$ *is isomorphic to* $L_{\mathbf{X}}(1000) \oplus L_{\mathbf{X}}(0010)$*, where the first factor is the adjoint module for* $\mathbf{X}$ *of dimension* 52 *and the second is a simple module of dimension* 196 *for* $\mathbf{X}$*.*

(Here we use the notation $L_{\mathbf{X}}(\lambda)$ to refer to the highest-weight module with highest weight λ for the algebraic group $\mathbf{X}$.)

The classification from [14] states that the maximal subgroups of positive dimension are maximal-rank or parabolic subgroups, or one of a short list of reductive subgroups that exist for all but a few small primes, together with G_2 inside F_4 for $p = 7$. This last case arises from a generic embedding of G_2 in E_6, which falls into F_4 on reduction modulo the prime $p = 7$ only. This new F_4 subgroup of E_8 is therefore the only example of a maximal subgroup that exists for a single prime, whose embedding cannot be explained using generic phenomena.

The error in [19] leading to this new maximal subgroup does not propagate into any other arguments or proofs. The mistake is in calculating the multiplicities of the non-negative $\mathbf{T}$-weights of the highest-weight modules $L_{\mathbf{X}}(0010)$ and $L_{\mathbf{X}}(1000)$ in the proof of (15.4), where $\mathbf{T}$ is a 1-dimensional torus of a hypothesized maximal subgroup $\mathbf{X} = F_4$ of $\mathbf{G} = E_8$. The $\mathbf{T}$-weights on $L(\mathbf{G})$ are determined by a labelling of the Dynkin diagram of $\mathbf{G}$ with 0s and 2s (a *labelled diagram*). The argument rules out the existence of such a maximal subgroup $\mathbf{X}$ with $L(\mathbf{G}) \downarrow X = L_{\mathbf{X}}(0010) + L_{\mathbf{X}}(1000)$ by showing that the $\mathbf{T}$-weights on $L(\mathbf{G})$ do not come from such a labelled diagram. See [19, pp.13–16] for more details about $\mathbf{T}$, its weights and labelled diagrams. The correct multiplicities for non-negative $\mathbf{T}$-weights on $L_{\mathbf{X}}(0010) + L_{\mathbf{X}}(1000)$ are as follows:

$$30, 28, 26^2, 24^3, 22^4, 20^5, 18^7, 16^7, 14^9, 12^{10}, 10^{12}, 8^{12}, 6^{14}, 4^{14}, 2^{15}, 0^{16}.$$

The first author wishes to thank the Royal Society for financial support during the course of this research.

There is a labelled diagram which yields these $\mathbf{T}$-weights, so no contradiction is obtained.

We note that we have not traced all applications of the theorems in [19] and [14], for which there are currently 52 and 46 citations, respectively according to MathSciNet. Many will be unaffected, or require an extra case to be considered and/or included in final results.

The structure of this note is as follows. Throughout, we let $\mathbf{G}$ be a simple algebraic group of type E_8 over an algebraically closed field k of characteristic p and let its Lie algebra be denoted by $L(E_8) = \mathfrak{e}_8$. In Section 2, we provide a proof of Theorem 1.1, showing the existence and uniqueness up to conjugacy of a maximal subgroup $\mathbf{X}$ of type F_4 in $\mathbf{G}$ when $p = 3$. To do this we first prove that there is an $\mathfrak{f}_4$ Lie subalgebra of $\mathfrak{e}_8$ that is normalized by the finite group $H = {}^3D_4(2)$ and go on to show that there must be a positive-dimensional subgroup $\mathbf{X} < \mathbf{G}$ containing H, and moreover that $\mathbf{X}$ must be a maximal subgroup of type F_4. In Section 3, a direct construction of $\mathbf{X} < \mathbf{G}$ is given by providing expressions for the root groups of $\mathbf{X}$ in terms of the root groups of $\mathbf{G}$. This can be used to provide an alternative proof of the existence part of Theorem 1.1. More precisely, we provide a Chevalley basis of the Lie subalgebra $\mathfrak{f}_4$ from above, and exponentiating this provides another alternative proof of the existence part of Theorem 1.1.

In the final section we determine various results providing extra details on this new class of maximal subgroups. For each unipotent class in $\mathbf{X}$ we determine the corresponding unipotent class in $\mathbf{G}$ that contains it, and we do the same for nilpotent orbits of the corresponding Lie algebras. We also consider the maximal connected subgroups of $\mathbf{X}$. The maximal parabolic subgroups of $\mathbf{X}$ will be contained in parabolic subgroups of $\mathbf{G}$ by the Borel–Tits Theorem. There are four classes of reductive maximal connected subgroups of $\mathbf{X}$ when $p = 3$ with types $B_4, A_1C_3, A_1G_2, A_2A_2$. We show that all of these classes are contained in other maximal connected subgroups of $\mathbf{G}$ and we specify such an overgroup. Moreover, we determine that the first three classes are $\mathbf{G}$-irreducible but the last class is not. (A subgroup is $\mathbf{H}$-irreducible for some connected reductive algebraic group $\mathbf{H}$ if it is not contained in any proper parabolic subgroup of $\mathbf{H}$.)

In establishing the existence of $\mathbf{X}$, we prove the following extra result, of use in the project to classify maximal subgroups of the finite exceptional groups of Lie type.

Proposition 1.2. *Let H be the group ${}^3D_4(2)$, let p be a prime, and suppose that H embeds in the algebraic group E_8 in characteristic p. If the composition factors of the action of H on the adjoint module $L(E_8)$ have dimensions 52 and 196, then $p = 3$ and H is contained in a maximal subgroup $\mathbf{X}$ of type F_4; furthermore, H and $\mathbf{X}$ stabilize the same subspaces of $L(E_8)$.*

2 From the Thompson group to F_4

Let p be an odd prime and let k be an algebraically closed field of characteristic p. From the end of Lemma 2.1 onwards we will assume that $p = 3$.

One path to a construction of the F_4 subgroup of E_8 starts with the Thompson group (acting irreducibly on $L(E_8)$ [20]), which contains a copy of $H \cong {}^3D_4(2)$, acting on $L(E_8)$ with composition factors of dimensions 52 and 196 (using the trace information in [2, p.176] and [6, p.251]). In fact, we will show that every H-invariant alternating bilinear form on the 248-dimensional module is invariant under a suitable copy of $F_4 \leq \mathrm{GL}_{248}(k)$, where k is algebraically closed and of characteristic 3.

We cannot quite show this without a computer. Splitting $L(E_8)$ up as the sum of 52 and 196 fragments the space of alternating forms into six components. For five of these six we can show that the H-invariant maps are F_4-invariant, but for the sixth we cannot do so without a computer. With a computer we can check that this sixth component is at least $F_4(9)$-invariant, and thus every subgroup H of $\mathbf{G}$ is contained in a copy of $F_4(9)$. But $F_4(9)$ contains elements of order 6562, and thus there is an F_4 subgroup of $\mathbf{G}$ containing it, via [12, Proposition 2] and Lemma 2.4 below.

We then show that this F_4 subgroup is unique up to $\mathbf{G}$-conjugacy, obtaining as a by-product that H is unique up to $\mathbf{G}$-conjugacy.

Trace of x^3	Trace of x	Trace of x^3	Trace of x
77	$56\lambda_1 + \lambda_2 + 134$	-4	$56\lambda_1 + 28\lambda_2 + 56$
	$26\lambda_1 + 25\lambda_2 + 53$		$35\lambda_1 + 28\lambda_2 + 38$
	$20\lambda_1 + \lambda_2 + 35$		$26\lambda_1 + 16\lambda_2 + 20$
	$8\lambda_1 + 7\lambda_2 + 8$		$11\lambda_1 + 10\lambda_2 + 20$
	$2\lambda_1 + \lambda_2 + 8$		$8\lambda_1 + \lambda_2 + 11$
	$2\lambda_1 + \lambda_2 - 1$		$2\lambda_1 + \lambda_2 + 2$
5	$54\lambda_1 + 27\lambda_2 + 80$	24	$64\lambda_1 + 14\lambda_2 + 92$
	$45\lambda_1 + 27\lambda_2 + 38$		$43\lambda_1 + 32\lambda_2 + 56$
	$24\lambda_1 + 21\lambda_2 + 32$		$34\lambda_1 + 20\lambda_2 + 29$
	$18\lambda_1 + 9\lambda_2 + 17$		$16\lambda_1 + 8\lambda_2 + 29$
	$9\lambda_1 + 9\lambda_2 + 11$		$13\lambda_1 + 20\lambda_2 + 29$
	$6\lambda_1 + 3\lambda_2 + 5$		$13\lambda_1 + 8\lambda_2 + 11$
	29		$4\lambda_1 + 5\lambda_2 + 11$
	8		$4\lambda_1 + 2\lambda_2 + 2$
	2		$\lambda_1 + 2\lambda_2 + 2$
	-1		

Table 2.1: List of traces of semisimple elements x of order 9 in E_8 on $L(E_8)$. Here, ω_9 is a primitive 9th root of unity, $\lambda_1 = \omega_9 + \omega_9^{-1}$ and $\lambda_2 = \omega_9^2 + \omega_9^{-2}$. Traces are given up to algebraic conjugacy.

We start with a copy J of the Thompson sporadic simple group. This has a 248-dimensional self-dual simple module M over $\mathbb{C}$ (it is a minimal faithful representation), and it remains simple upon reduction modulo all primes. From [2, p.176], we see that there are elements of order 9 with Brauer character value 5 on M. The only integers that are the traces of semisimple elements of order 9 in $\mathbf{G}$ (on the adjoint module) are -1, 2, 8 and 29. (These were computed using the algorithm in [15, Appendix], see Table 2.1.) Thus these elements cannot be semisimple, and in particular, $p \mid 9$. Thus we see that if J embeds in the algebraic group $\mathbf{G}$ in any characteristic p, then $p = 3$. It is a famous result [20] that J does indeed embed in $E_8(3)$, and is unique up to conjugacy.

It is well known that J contains a subgroup H isomorphic to ${}^3D_4(2)$. From [2, p.90] and [6, pp.251–253], we see that in characteristic not 2, the restriction of M to H is the direct sum of a 52-dimensional simple module M_1 and a 196-dimensional simple module M_2 (the sum is direct since M is self-dual). However, all elements of order 9 in H act on $M_1 \oplus M_2$ with trace 2, so we cannot use the previous method to show that H cannot embed in $\mathbf{G}$ in characteristic $p \neq 2, 3$ acting on $L(\mathbf{G})$ as $M_1 \oplus M_2$.

Lemma 2.1. *Let p be an odd prime, let H denote the group ${}^3D_4(2)$, and suppose that H embeds in the algebraic group $\mathbf{G}$ over an algebraically closed field k of characteristic p, acting on the adjoint module with composition factors of dimensions 52 and 196. Then the 52-dimensional submodule carries the structure of a Lie algebra of type F_4, and in addition $p = 3$. Furthermore, such an $\mathfrak{f}_4$-subalgebra of $\mathfrak{e}_8$ does exist for $p = 3$.*

Proof. Let M_1 denote the 52-dimensional kH-submodule of the adjoint module for $\mathbf{G}$ and M_2 the 196-dimensional submodule. Note that $|H| = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$, so either $p \nmid |H|$ and we are essentially in characteristic 0, or $p = 7, 13$, or $p = 3$.

From [4, Table 1.1], there is a unique conjugacy class of subgroups H in F_4 for any odd characteristic p, acting irreducibly on the minimal and adjoint modules. In particular, we see from [4, Section 4.3.4] that $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1)$ is 1-dimensional for all odd primes p. From an ordinary character calculation, we see that

$$\Lambda^2(\chi_{52}) = \chi_{52} + \chi_{1274},$$

where χ_i is the irreducible character of H of degree i. For $p = 7, 13$, the reduction modulo p of χ_{1274} is irreducible (see [6, pp.252–253]), hence the reduction modulo p is irreducible modulo p for all $p > 3$ (as $p = 7, 13$ are the only primes greater than 3 dividing $|H|$). Thus for $p > 3$, if H embeds in $\mathbf{G}$ with the claimed composition factors then the 52-dimensional summand is a Lie subalgebra of $L(\mathbf{G})$.

On the other hand, if $p = 3$ then χ_{1274} has Brauer character constituents of degrees 52 and 1222 from [6, p.251]. Since $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k$, we see that the exterior square is uniserial, with layers of dimensions 52, 1222 and 52. Moreover,

$$\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) = 0.$$

Thus again M_1 forms an H-invariant subalgebra. Thus for all odd primes p, M_1 is an H-invariant Lie subalgebra of $L(\mathbf{G})$ of dimension 52.

Moreover, M_1 must be non-abelian for all p, since $\mathfrak{e}_8$ contains no abelian subspace of dimension 52 by [5, Proposition 2.3]. Furthermore, as M_1 is irreducible for H, the restriction of the Lie bracket to M_1 furnishes it with the structure of a semisimple Lie algebra. Since $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k$, there is at most one isomorphism class of such, but as the algebraic k-group F_4 does contain a subgroup isomorphic to H, acting as χ_{52} on its adjoint module, it follows that $M_1 \cong \mathfrak{f}_4$. In particular, this means that the $\mathfrak{f}_4$ Lie algebra must have a simple module of dimension 196, as it acts H-equivariantly on $L(\mathbf{G})$.

Such a simple module must be restricted: if not the p-closure L_p of the image L of $\mathfrak{f}_4$ in $\mathfrak{e}_8$ will contain a non-trivial centre [22, 2.5.8(2)]; but L has no 1-dimensional submodules on $\mathfrak{e}_8$. By Curtis's theorem, M_2 arises by differentiation of a restricted representation for the algebraic group F_4, whence $p = 3$, from the tables in [16].

The embeddings of ${}^3D_4(2)$ and the $\mathfrak{f}_4$-subalgebra do exist for $p = 3$ via the Thompson group, as seen above. □

For the rest of this paper we therefore assume that $p = 3$.

We will prove that the $\mathfrak{f}_4$-subalgebra is the Lie algebra of an F_4 algebraic subgroup of $\mathbf{G}$. To do so, we will actually prove that every H-invariant alternating product on the kH-module $M = M_1 \oplus M_2$ for $p = 3$ is also F_4-invariant, for the unique $F_4 \leq \mathrm{GL}_{52}(k)$ containing H. To do so, we need to understand the space

$$\mathrm{Hom}_{kH}(\Lambda^2(M), M)$$

of alternating products on M. Using $M = M_1 \oplus M_2$, and the formula

$$\Lambda^2(A \oplus B) \cong \Lambda^2(A) \oplus \Lambda^2(B) \oplus A \otimes B,$$

we split the space of products up into six components. The next result gives the dimensions of these components.

Proposition 2.2. *We have*

$$\begin{aligned} \mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) &= k, & \mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) &= 0, \\ \mathrm{Hom}_{kH}(\Lambda^2(M_2), M_1) &= k, & \mathrm{Hom}_{kH}(\Lambda^2(M_2), M_2) &= k, \\ \mathrm{Hom}_{kH}(M_1 \otimes M_2, M_1) &= 0, & \mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2) &= k. \end{aligned}$$

Proof. One may use a computer to check these with ease. Some may be checked easily by hand as well, using the ordinary character table and the 3-decomposition matrix for H. For example, using those two tables, $M_1 \otimes M_2$ does not possess a composition factor M_1, and thus

$$\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_1) = \mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) = 0.$$

The statement that $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k$ appears in [4, Section 4.3.4] (where it is proved by computer). □

At least the existence, if not the uniqueness, of two of the three remaining non-zero maps is clear from the fact that H embeds in $E_8(3)$ with representation $M_1 \oplus M_2$. If $\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2) = 0$ then M_1 would be an ideal of the Lie algebra (as $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) = 0$), which is not possible. A character calculation shows that $S^2(M_2)$ does not have a composition factor M_1, so

$$\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_1) = \mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2).$$

It is only $\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_2)$ that cannot easily be seen. Indeed, this space will cause us a problem later on.

We now prove that five of the six Hom-spaces extend to the algebraic group $\mathbf{X} = F_4$, with only $\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_2)$ missing. If one is happy to use a computer for all of this, one simply checks that all H-invariant maps are $F_4(3)$- and even $F_4(9)$-invariant, and thus one does not need to prove the next proposition.

Proposition 2.3. *Let $\mathbf{X}$ be an algebraic k-group of type F_4. We have*

$$\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(1000)), L(1000)) = k,$$

$$\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(1000)), L(0010)) = \mathrm{Hom}_{\mathbf{X}}(L(1000) \otimes L(0010), L(1000)) = 0, \text{ and}$$
$$\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(0010)), L(1000)) = \mathrm{Hom}_{\mathbf{X}}(L(1000) \otimes L(0010), L(0010)) = k.$$

Proof. Note that each of these spaces must have dimension at most the dimension of the corresponding space for H. This yields the two 0-dimensional spaces, and that the other spaces have dimension at most 1. The first statement holds because $\mathbf{X}$ is an algebraic group and its adjoint module is $L(1000)$, thus the space is non-zero.

For the last statement, since $S^2(M_2)$ has no composition factor isomorphic to M_1, certainly $S^2(L(0010))$ has no composition factor isomorphic to $L(1000)$. Thus the two Hom-spaces are isomorphic, so it remains to find a non-zero map in the latter space.

Using the Brauer character table of H [6, p.251] (or preferably, a computer), the composition factors of the kH-module $M_1 \otimes M_2$ are of dimensions

$$25, 196, 196, 441, 1963, 2457, 2457, 2457.$$

The highest-weight module $L(1010)$, which must appear as a composition factor in $L(1000) \otimes L(0010)$, has dimension 7371 (see [16, Appendix A.50]), and must restrict to kH to be the sum of the three (non-isomorphic) modules of dimension 2457, as no other combination of dimensions works. The rest of the composition factors, in total, have dimension 2821, so there must be an $\mathbf{X}$-composition factor of dimension between 1963 and 2821. Consulting [16, Appendix A.50], we find exactly one such module: $L(0011)$ of dimension $2404 = 1963 + 441$. The remaining kH-modules, 25, 196 and 196, must be the other composition factors for $\mathbf{X}$, because $\mathbf{X}$ has no simple modules of dimension $25 + 196$, $196 + 196$, or $25 + 196 + 196$.

Thus the composition factors of $L(1000) \otimes L(0010)$ have dimensions 25, 196, 196, 2404 and 7371. Since $L(0010)$ is the unique module to appear more than once, and the tensor product is self-dual, $L(0010)$ must be a submodule, and the maps in $\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2)$ extend to $\mathbf{X}$. ☐

The last remaining Hom-space to check is $\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(0010)), L(0010))$. This seems difficult to do by hand, and we resort to a computer. There are two ways to proceed. The first is to prove that there is an $F_4(3)$-invariant map in the space (this takes a couple of minutes), and thus the group H is contained in a copy of $F_4(3)$ in $\mathbf{G}$. We then apply [3, Proposition 6.8], which states that $F_4(3)$ is contained in a positive-dimensional subgroup stabilizing the same subspaces of $L(\mathbf{G})$, which are M_1 and M_2. This must be a copy of F_4 (as it stabilizes an $\mathfrak{f}_4$-Lie subalgebra), and we are done. Alternatively, we prove the same statement for $F_4(9)$ (which contains elements of order $9^4 + 1 = 6562$, this takes about half an hour on one of the first author's computers) and then apply [12, Proposition 2], which yields the same positive-dimensional subgroup.

We must also show that the subgroup $\mathbf{X}$ is actually F_4. This is easy, and we can do it quite generally.

Lemma 2.4. *Let $\mathbf{G}$ be of type E_8 over k, and let $\mathbf{X}$ be a closed, positive-dimensional subgroup of $\mathbf{G}$. If $\mathbf{X}$ acts on $L(\mathbf{G})$ with composition factors of dimensions 52 and 196 then $\mathbf{X}$ is simple of type F_4. Furthermore, $\mathbf{X}$ is maximal in $\mathbf{G}$.*

Proof. Since $\mathbf{X}$ has no trivial composition factor on $L(\mathbf{G})$ it cannot lie in a parabolic subgroup, hence must be reductive. It also cannot centralize any semisimple element, hence must be semisimple. Since $L(\mathbf{X})$ is a submodule of $L(\mathbf{G})$, it has dimension either 52 or 196 and is simple. There is no simple algebraic group of rank at most 8 and dimension 196, so $\mathbf{X}$ has dimension 52, and must therefore be F_4. Since F_4 has no outer automorphisms, $N_{\mathbf{G}}(\mathbf{X}) = \mathbf{X}$ (as $\mathbf{X}$ has trivial centralizer). Since any closed, positive-dimensional proper subgroup of $\mathbf{G}$ cannot act irreducibly on $L(\mathbf{G})$, any overgroup of $\mathbf{X}$ also acts with composition factors 52 and 196, hence is F_4 by the above proof. Thus $\mathbf{X}$ is maximal, as claimed. □

Thus we obtain the following.

Proposition 2.5. *Let $p = 3$ and let H be a subgroup ${}^3D_4(2)$ of $\mathbf{G}$, acting on $L(\mathbf{G})$ with composition factors of dimensions 52 and 196. Then H is contained in a positive-dimensional subgroup of type F_4, stabilizing exactly the same subspaces of $L(\mathbf{G})$ that are stabilized by H.*

It suffices to ascertain the uniqueness up to conjugacy of the subgroup of type F_4, and as a by-product we also obtain uniqueness of H up to conjugacy.

Let t be an involution in $\mathbf{X} = F_4$ with centralizer $\mathbf{B}$ of type B_4. The trace of t on $L(\mathbf{G})$ is -8 (see [13, Table 4]), and so the centralizer of t in $\mathbf{G}$ is D_8, which acts with composition factors $L(\lambda_2)$ and $L(\lambda_7)$, of dimensions 120 and 128 respectively. The restriction of the $k\mathbf{X}$-module $L_{\mathbf{X}}(1000)$ to $\mathbf{B}$ is $L_{\mathbf{B}}(0001) \oplus L_{\mathbf{B}}(0100)$, with dimensions 16 and 36 respectively. The restriction of $L_{\mathbf{X}}(0010)$ to $\mathbf{B}$ is $L_{\mathbf{B}}(0010) \oplus L_{\mathbf{B}}(1001)$, of dimensions 84 and 112 respectively. (This can be checked using weights or quickly on a computer for $F_4(3)$.) From [23, Table 60] we see that $\mathbf{B}$ is subgroup $E_8(\#45)$ and is unique up to conjugacy. But clearly there is a unique way to assemble the numbers 16, 36, 84 and 112 to make 52 and 196. Thus given any subgroup of type B_4 there exists at most one F_4 containing it, which must stabilize the submodule $L_{\mathbf{B}}(0001) \oplus L_{\mathbf{B}}(0100)$. Thus we obtain the result that $\mathbf{X}$ is unique up to conjugacy.

This completes the proof of uniqueness of $\mathbf{X}$, and thus Theorem 1.1 is proved.

3 An explicit construction of the maximal subgroup F_4

Recall that k is algebraically closed and of characteristic 3. For their application to future explicit computations, we give expressions for the root elements $x_{\pm\beta_i}(t)$ for $t \in k$ and β_i a root of the maximal subgroup F_4 as products of root elements of $\mathbf{G}$; see [1, §4.4] for notation. We also provide the elements $h_{\gamma_i}(t)$ for γ_i a simple root of F_4 and $t \in k^*$, written in terms of the $h_{\alpha_i}(t)$ elements of $\mathbf{G}$. As the factors of $x_{\beta_i}(t) = \prod_j x_{\alpha_j}(c_j t)$ commute we get easily that $x_{-\beta_i}(t) = \prod_j x_{-\alpha_j}(c_j t)$. For this reason we only list the root group elements for positive roots. Note that since the coefficients c_j that we exhibit are elements of GF(3), it follows that the subgroup $\mathbf{Y}$ we produce is defined over GF(3).

Proposition 3.1. *The elements $x_{\pm\beta_i}(t) = \prod_j x_{\alpha_j}(c_j t)$ for $t \in k$ generate a maximal subgroup $\mathbf{X}$ of type F_4 in $\mathbf{G}$. They are the root elements of F_4 with respect to the maximal torus generated by $h_{\gamma_i}(t) \prod_j x_{\alpha_j}(d_j t)$ for $t \in k^*$. Furthermore, the elements $e_{\beta_i}, h_{\gamma_i}$ form a Chevalley basis for $\mathfrak{h} = \mathrm{Lie}(F_4)$ where $e_{\beta_i} = \sum c_j e_{\alpha_j}$, $h_{\gamma_i} = \sum d_j e_{\alpha_j}$.*

$x_{1000}(t) = x_{00010000}(t)x_{00000100}(t)$

$x_{0100}(t) = x_{00100000}(-t)x_{00000010}(t)$

$x_{0010}(t) = x_{10000000}(-t)x_{00000001}(t)x_{00011000}(-t)x_{00001100}(-t)$

$x_{0001}(t) = x_{11110000}(t)x_{01121000}(-t)x_{01111100}(-t)x_{01011110}(t)$

$x_{1100}(t) = x_{00110000}(-t)x_{00000110}(-t)$

$x_{0110}(t) = x_{10100000}(t)x_{00000011}(-t)x_{00111000}(-t)x_{00001110}(-t)$

$x_{0011}(t) = x_{11121000}(t)x_{11111100}(t)x_{01011111}(t)x_{01122100}(-t)$

$x_{1110}(t) = x_{10110000}(t)x_{00000111}(t)x_{00111100}(-t)x_{00011110}(t)$

$x_{0120}(t) = x_{10111000}(-t)x_{00001111}(-t)$

$$x_{0111}(t) = x_{11221000}(t)x_{11111110}(t)x_{01111111}(t)x_{01122110}(-t)$$
$$x_{1120}(t) = x_{10111100}(-t)x_{00011111}(t)$$
$$x_{1111}(t) = x_{11221100}(t)x_{11121110}(-t)x_{01121111}(-t)x_{01122210}(t)$$
$$x_{0121}(t) = x_{11111111}(-t)x_{11222100}(-t)x_{11122110}(t)x_{01122111}(t)$$
$$x_{1220}(t) = x_{10111110}(t)x_{00111111}(-t)$$
$$x_{1121}(t) = x_{11121111}(t)x_{11232100}(t)x_{11122210}(-t)x_{01122211}(-t)$$
$$x_{0122}(t) = x_{12232111}(-t)x_{12233210}(t)$$
$$x_{1221}(t) = x_{11221111}(-t)x_{11232110}(-t)x_{11222210}(t)x_{01122221}(-t)$$
$$x_{1122}(t) = x_{12232211}(t)x_{12243210}(-t)$$
$$x_{1231}(t) = x_{11232111}(t)x_{11222211}(-t)x_{11122221}(-t)x_{11233210}(t)$$
$$x_{1222}(t) = x_{12232221}(t)x_{12343210}(t)$$
$$x_{1232}(t) = x_{22343210}(t)x_{12343211}(t)x_{12243221}(t)x_{12233321}(-t)$$
$$x_{1242}(t) = x_{22343211}(t)x_{12244321}(t)$$
$$x_{1342}(t) = x_{22343221}(-t)x_{12344321}(t)$$
$$x_{2342}(t) = x_{22343321}(t)x_{12354321}(-t)$$
$$h_{1000}(t) = h_{00010000}(t)h_{00000100}(t)$$
$$h_{0100}(t) = h_{00100000}(t)h_{00000010}(t)$$
$$h_{0010}(t) = h_{10000000}(t)h_{00010000}(t)h_{00001000}(t^2)h_{00000100}(t)h_{00000001}(t)$$
$$h_{0001}(t) = h_{1000000}(t)h_{01000000}(t^4)h_{00100000}(t^3)h_{00010000}(t^5)h_{00001000}(t^3)h_{00000100}(t^2)h_{00000010}(t)$$

Proof. It can be checked by hand that these elements satisfy Steinberg's relations (see [1, Theorem 12.1.1]) for a suitable choice of constants for Chevalley's commutator relations and that the corresponding elements of $\mathfrak{e}_8$ form a Chevalley basis for Lie($\mathbf{X}$), a Lie algebra of type F_4. Another check, which theoretically could be done by hand, shows that $\mathbf{X}$ acts on Lie($\mathbf{G}$) as $L_{\mathbf{X}}(1000) \oplus L_{\mathbf{X}}(0010)$, and so by Lemma 2.4 we see that $\mathbf{X}$ is a maximal subgroup of $\mathbf{G}$. $\square$

Remark 3.2. *The proof of Proposition 3.1 provides another proof for the existence of a maximal subgroup of type F_4 in $\mathbf{G}$, which is at least in theory computer-free.*

Remark 3.3. *We note that the* 24*-dimensional unipotent subgroup generated by the positive root elements of F_4 is contained in the* 120*-dimensional unipotent subgroup generated by the positive roots of $\mathbf{G}$. The presentation alluded to in Proposition 3.1 has the slightly unfortunate property that the constants in Chevalley's commutator relations for this subgroup F_4 are not the same as those used in Magma, which are a somewhat standard choice. However, one can if one wishes rectify this by choosing a different basis: let $\tilde{x}_{\alpha_1}(t) = x_{-0100}(t)$, $\tilde{x}_{\alpha_2}(t) = x_{-1242}(t)$, $\tilde{x}_{\alpha_3}(t) = x_{1232}(t)$, $\tilde{x}_{\alpha_4}(t) = x_{-0001}(t)$. Then we generate the same maximal subgroup F_4 but this time the constants in the commutator relations do agree with those in Magma.*

We provide a brief explanation on how we found the generators and Chevalley basis in Proposition 3.1. We start with the 248-dimensional module M with summands of dimensions 52 and 196 for $H \cong {}^3D_4(2)$ (defined at the start of Section 2). We use the space $\mathrm{Hom}_{kH}(\Lambda^2(M), M)$, as used in [5] (described there as the 'Lie product method'), to construct an explicit H-invariant Lie product on M that turns M into a copy of $\mathfrak{e}_8$. This gives us explicit structure constants. The module M splits as the sum of 52- and 196-dimensional H-stable submodules. As explained in Lemma 2.1, the first of these subspaces is forced to be a subalgebra $\mathfrak{h}$ of $\mathfrak{e}_8$ isomorphic to $\mathfrak{f}_4$ and so Magma could write down a basis for it in terms of a basis of $\mathfrak{e}_8$. However, this process left us with basis elements for $\mathfrak{h}$ with around 120 non-zero coefficients in terms of a basis of $\mathfrak{e}_8$.

We found that four of the basis vectors for $\mathfrak{h}$ were toral (meaning $x^{[p]} = x$ [22], which implies x is semisimple) and commuted with each other, thus spanning a maximal toral subalgebra $\mathfrak{t}$. We then searched for a $\mathbf{G}$-conjugate of $\mathfrak{h}$ such that the corresponding conjugate of $\mathfrak{t}$ was contained in the standard toral subalgebra of $\mathfrak{e}_8$. To do this we used the inbuilt `InnerAutomorphism` function in Magma to construct the automorphisms of $\mathfrak{e}_8$ corresponding to $x_\gamma(\pm 1)$ for all roots γ in the root system of $\mathbf{G}$, yielding 480 possible conjugations.

Our strategy was to implement a naive hill climb for the first basis element t_1 of $\mathfrak{t}$. Indeed, we searched through all 480 possible conjugating elements and selected the one that yielded the largest number of zero coefficients when expressing t_1^g in terms of the basis of $\mathfrak{e}_8$. We remembered the elements we used at each step. This meant that when we could no longer increase the number of zero coefficients we could trace our steps back and take the next best conjugating element and continue the process. This lead to a significant increase in the number of zero coefficients but nowhere near the 240 we needed.

We then slightly upgraded our hill climb algorithm to include using a random conjugating element at fixed intervals. Every 100 steps we chose a random conjugating element and used this, regardless of what it did to the number of zero coefficients. This method was not optimized; it could be that a better choice would have been every five steps, or 500 steps. But this hill climb was enough for us; it quickly led us to a conjugating element g_0 which took t_1 into the standard toral subalgebra of $\mathfrak{e}_8$. At this point, the remaining three basis elements t_2, t_3 and t_4 were not sent by g_0 to something in the standard toral subalgebra of $\mathfrak{e}_8$, but they had significantly fewer non-zero coefficients. We then repeated the algorithm looking to increase the total number of zero coefficients in $t_1^g, \ldots, t_4^g$ and this quickly converged, yielding a conjugating element g_1 which sent $t_1, \ldots, t_4$ to the four toral elements corresponding to the generators of the maximal torus given in Proposition 3.1. From the toral subalgebra of $\mathfrak{h}^{g_1}$ it was then routine to take a Cartan decomposition and find the corresponding root elements.

It turned out that the root elements of $\mathfrak{h}$ were expressed as a sum of commuting root elements of $\mathfrak{e}_8$; in fact long root elements of $\mathfrak{h}$ were of type $2A_1$ in $\mathfrak{e}_8$, whereas short root elements were of type $4A_1$. For pairwise commuting root elements $e_{\alpha_1}, \ldots, e_{\alpha_k}$ of $\mathfrak{e}_8$, the operators $\operatorname{ad} e_{\alpha_i}$ pairwise commute and so one has

$$\exp(t_1 \operatorname{ad} e_{\alpha_1} + \cdots + t_m \operatorname{ad} e_{\alpha_m}) = \exp(t_1 e_{\alpha_1}) \ldots \exp(t_m e_{\alpha_m}).$$

Thus if $e_{\beta_i} = \sum t_j e_{\alpha_j} \in \mathfrak{h}$ is of this form, then evidently the left-hand side of the displayed equation normalizes $\mathfrak{h}$ in the group $\mathrm{GL}_{248}(k)$, and the right-hand side belongs to $E_8(k)$, hence for $t \in k$, the elements $x_{\beta_i}(t)$ generate a connected smooth subgroup $\mathbf{Y}$ of E_8 for which $\mathfrak{h} \subseteq \mathrm{Lie}(\mathbf{Y})$. As the restriction to $\mathfrak{h}$ of the adjoint module $\mathfrak{e}_8$ is the direct sum of $\mathfrak{h}$ and a simple module, it is maximal, which forces $\mathrm{Lie}(\mathbf{Y}) = \mathfrak{h}$. The only connected smooth affine k-group whose Lie algebra is a simple Lie algebra of type F_4 is a group of type F_4 itself and using the maximality of $\mathfrak{f}_4$ we conclude that $\mathbf{Y}$ must be a maximal connected subgroup. This yields yet another proof of the existence a maximal subgroup of type F_4 in $\mathbf{G}$.

4 Consequences

As before, k is algebraically closed and of characteristic 3. We extend the results of [10] to this new maximal subgroup, determining which unipotent classes of $\mathbf{G}$ meet the new maximal subgroup $\mathbf{X}$ of type F_4 non-trivially.

Proposition 4.1. *If u is a unipotent element of $\mathbf{X} < \mathbf{G}$, then the class of u in $\mathbf{X}$ and $\mathbf{G}$ is given in Table 4.1.*

Proof. The proof is a fast computer check. Randomly generate elements u of orders 3, 9 and 27 in $F_4(3)$ (given by the presentation alluded to in Lemma 3.1) until we hit each class. (The class to which u belongs can be deduced from [8, Tables 3 and 4].) The Jordan blocks of the action of u on the sum of $L_{\mathbf{X}}(1000)$ and $L_{\mathbf{X}}(0010)$ are trivial to compute then. From [8] we obtain the class in $\mathbf{G}$ to which u belongs.

However, note that there is an error in [8], due to an error in [18], which leads to a single class having the wrong Jordan blocks in characteristic 3. This is corrected in [9], and it concerns exactly the class $E_8(b_6)$ in the table. It has Jordan blocks $9^{26}, 7, 3^2, 1$ on $L(E_8)$, not $9^{25}, 8^2, 2^2, 1^3$ as stated in [8]. With this

Class in F_4	Class in E_8
A_1	$2A_1$
$\tilde{A}_1$	$4A_1$
$A_1 + \tilde{A}_1$	$A_2 + 2A_1$
A_2	$2A_2$
$\tilde{A}_2$	$2A_2$
$A_2 + \tilde{A}_1$	$2A_2 + A_1$
$\tilde{A}_2 + A_1$	$2A_2 + 2A_1$
B_2	$2A_3$
$C_3(a_1)$	$A_4 + 2A_1$
$F_4(a_3)$	$A_4 + A_2$
B_3	A_6
C_3	$D_6(a_1)$
$F_4(a_2)$	$D_5 + A_2$
$F_4(a_1)$	$E_8(b_6)$
F_4	$E_8(b_4)$

Table 4.1: Fusion of unipotent classes of the maximal subgroup $\mathbf{X} = F_4$ into $\mathbf{G} = E_8$. (Horizontal lines separate elements of different orders.)

Class in $\mathfrak{f}_4$	Class in $\mathfrak{e}_8$
A_1	$2A_1$
$\tilde{A}_1$	$4A_1$
$A_1 + \tilde{A}_1$	$A_2 + 2A_1$
A_2	$2A_2$
$\tilde{A}_2$	$A_2 + 3A_1$
$A_2 + \tilde{A}_1$	$2A_2 + A_1$
B_2	$2A_3$
$\tilde{A}_2 + A_1$	$2A_2 + A_1$
$C_3(a_1)$	$A_4 + 2A_1$
$F_4(a_3)$	$A_4 + A_2$
B_3	A_6
C_3	$D_5 + A_1$
$F_4(a_2)$	$E_7(a_4)$
$F_4(a_1)$	$E_6(a_1)$
F_4	E_6

Table 4.2: Fusion of nilpotent classes of maximal $\mathfrak{f}_4$ into $\mathfrak{e}_8$.

correction, the Jordan block structure of u on $L(\mathbf{G})$ determines the unipotent class to which u belongs, and thus we are done. $\square$

For completeness we do the same thing for nilpotent orbits of $\mathfrak{f}_4$.

Proposition 4.2. *If x is a nilpotent element of the maximal $\mathfrak{f}_4$-subalgebra of $\mathfrak{e}_8$, then the class of x in $\mathfrak{f}_4$ and $\mathfrak{e}_8$ is given in Table 4.2.*

Proof. Using the root elements constructed for $\mathfrak{f}_4$ as a subalgebra of $\mathfrak{e}_8$ (as explained in Section 3) we find a set of nilpotent orbit representatives for $\mathfrak{f}_4$ using [21, Appendix]. For each representative x we use Magma to calculate the Jordan block structure for the adjoint action of x on $\mathfrak{e}_8$ and the normalizer of each term of the derived series of $C_{\mathfrak{e}_8}(x)$. Using [7, Proposition 1.5], we then find the $\mathfrak{e}_8$-class of x. $\square$

Before stating the next result we explain some notation. We use $\bar{A}_2$ to denote an A_2 subgroup generated by long root elements.

In $\mathbf{G}$ there are two conjugacy classes of B_4 subgroups contained in D_8 embedded via their spin module $L_{B_4}(0001)$. When p is odd, these are distinguished by whether or not they are contained in a maximal subgroup of type A_8. We denote the class not contained in A_8 by $B_4(\dagger)$.

Proposition 4.3. *If $\mathbf{M}$ is a maximal connected reductive subgroup of $\mathbf{X} < \mathbf{G}$, then $\mathbf{M}$ is conjugate to one of the following four subgroups.*

(i) $\mathbf{M}_1 = B_4(\dagger) < D_8$ *embedded via the spin module* $L_{B_4}(0001)$. *It is* $\mathbf{G}$*-irreducible and denoted* $E_8(\#45)$ *in [23].*

(ii) $\mathbf{M}_2 = A_1C_3 < D_8$ *embedded via* $L_{A_1}(2) \oplus L_{C_3}(010)$. *This subgroup is* $\mathbf{G}$*-irreducible and denoted by* $E_8(\#774)$ *in [23].*

(iii) $\mathbf{M}_3 = A_1G_2 < A_1E_7$ *embedded as follows:* E_7 *has a maximal subgroup of type* A_1G_2 *(when* $p \neq 2$*). Therefore* A_1E_7 *has a maximal subgroup* $A_1A_1G_2$ *and* M_3 *is embedded diagonally in this subgroup. One has to twist the embedding in the first* A_1 *factor by the Frobenius morphism. This subgroup is again* $\mathbf{G}$*-irreducible and denoted by* $E_8(\#967^{\{1,0\}})$ *in [23].*

(iv) $\mathbf{M}_4 = A_2A_2 < \bar{A}_2E_6$ *embedded as follows:* E_6 *has a maximal subgroup* A_2G_2*, and* G_2 *has a maximal subgroup* $\tilde{A}_2$ *generated by short root subgroups of the* G_2 *when* $p = 3$*. Thus* $\bar{A}_2E_6$ *has a subgroup* $H = \bar{A}_2A_2\tilde{A}_2$ *(denoted* $E_8(\#1012)$ *in [23]). The first* A_2 *factor of* $\mathbf{M}_4$ *is the second* A_2 *factor of* H *and the second* A_2 *factor of* $\mathbf{M}_4$ *is diagonally embedded in the first and third factors of* H *(with no twisting by field or graph automorphisms). Moreover,* $\mathbf{M}_4$ *is not* $\mathbf{G}$*-irreducible.*

Proof. By [14, Corollary 2], F_4 has four conjugacy classes of reductive maximal connected subgroups in characteristic 3, which are indeed of types B_4, A_1C_3, A_1G_2 and A_2A_2. The first two maximal subgroups are centralizers of involutions. It follows from the action of F_4 on $\mathrm{Lie}(G)$ that the centralizer in $\mathbf{G}$ of both of these involutions is D_8. Thus B_4 and A_1C_3 are contained in a maximal subgroup of type D_8. By [11, Table 8.1], there are only three $\mathbf{G}$-conjugacy classes of B_4 subgroups in D_8. Calculating the composition factors of the action of $B_4 < F_4$ on $\mathrm{Lie}(G)$ yields that the B_4 subgroup of F_4 is indeed conjugate to $\mathbf{M}_1$, which is $\mathbf{G}$-irreducible by [23, Theorem 1]. Calculating the composition factors of $A_1C_3 < F_4$ on $\mathrm{Lie}(G)$, we find that it has no trivial composition factors. Therefore it must be $\mathbf{G}$-irreducible (by [23, Corollary 3.8]) and we may use the classification of irreducible subgroups determined in [23]. In particular, the composition factors on the Lie algebra of $\mathbf{G}$ are enough to determine conjugacy and it follows that A_1C_3 is conjugate to $\mathbf{M}_2$, as required.

Another calculation shows that $A_1G_2 < F_4$ has no trivial composition factors on the Lie algebra of $\mathbf{G}$. We can therefore use the same method as in the previous case to deduce that it is conjugate to $\mathbf{M}_3$.

For $\mathbf{AB} = A_2A_2 < F_4$ we start by considering the first A_2 factor $\mathbf{A}$, which we define to be the A_2 subgroup generated by long root subgroups of F_4. This is the derived subgroup of a long root A_2-Levi subgroup, and is thus a subgroup of $B_4 < F_4$. Since B_4 is conjugate to $\mathbf{M}_1$, we find that $\mathbf{A}$ is contained in a maximal subgroup of type A_8 and acts as $L_{A_2}(10) \oplus L_{A_2}(01) \oplus L_{A_2}(00)^{\oplus 3}$ on the natural 9-dimensional module of A_8. Given this action, it follows that $\mathbf{A}$ is a diagonal subgroup (without field or graph twists) of the derived subgroup of an A_2A_2-Levi subgroup of A_8 and hence of $\mathbf{G}$. We now claim that the connected centralizer of $\mathbf{A}$ in $\mathbf{G}$ is $\bar{A}_2G_2$. Indeed, we use [23, Theorem 1] to see that $\mathbf{C} = \mathbf{A}\bar{A}_2G_2$ is $\mathbf{G}$-irreducible (denoted by $E_8(\#978)$) and the only reductive connected overgroups of $\mathbf{C}$ are the maximal subgroups $\bar{A}_2E_6$ and G_2F_4. Therefore, we must have that $\bar{A}_2G_2 \leq C_{\mathbf{G}}(\mathbf{A})^\circ$. Moreover, this must be an equality of subgroups since $\mathbf{A}C_{\mathbf{G}}(\mathbf{A})^\circ$ is a reductive connected overgroup of $\mathbf{C}$, but clearly cannot be $\bar{A}_2E_6$ or G_2F_4 ($\mathbf{A}$ is not conjugate to $\bar{A}_2$).

From the previous calculations, $\mathbf{AB} < \mathbf{C}$ is a subgroup of $\bar{A}_2E_6$ and $\mathbf{B} < C_{\mathbf{G}}(\mathbf{A})$ is contained in $\bar{A}_2G_2$. It is straightforward to list all A_2 subgroups of $\bar{A}_2G_2$, noting that G_2 has precisely two classes of A_2 subgroups when the characteristic is 3. Computing the composition factors of $\mathbf{B}$ on the Lie algebra of $\mathbf{G}$, by restricting first to F_4 and then $\mathbf{B}$, shows that $\mathbf{B}$ is conjugate to the subgroup claimed and hence $\mathbf{AB}$ is conjugate to $\mathbf{M}_4$. The fact that $\mathbf{M}_4$ is $\mathbf{G}$-reducible follows from [23, Theorem 1]. □

References

[1] Roger W. Carter, *Simple groups of Lie type*, John Wiley & Sons Inc., 1989.

[2] John Conway, Robert Curtis, Simon Norton, Richard Parker, and Robert Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[3] David A. Craven, *On medium-rank Lie primitive and maximal subgroups of exceptional groups of Lie type*, *Mem. Amer. Math. Soc.*, to appear.

[4] ______, *The maximal subgroups of the exceptional groups $F_4(q)$, $E_6(q)$ and ${}^2E_6(q)$ and related almost simple groups*, preprint, 2021. arXiv:2103.04869

[5] ______, *The maximal subgroups of the exceptional groups $E_7(q)$ and related almost simple groups*, preprint, 2021.

[6] Christoph Jansen, Klaus Lux, Richard Parker, and Robert Wilson, *An Atlas of Brauer characters*, Oxford University Press, New York, 1995.

[7] Mikko Korhonen, David Stewart, and Adam Thomas, *Representatives for unipotent classes and nilpotent orbits*, preprint, 2021. arXiv:2105.04347

[8] Ross Lawther, *Jordan block sizes of unipotent elements in exceptional algebraic groups*, Comm. Algebra **23** (1995), 4125–4156.

[9] ______, *Correction to: "Jordan block sizes of unipotent elements in exceptional algebraic groups"*, Comm. Algebra **26** (1998), 2709.

[10] ______, *Unipotent classes in maximal subgroups of exceptional algebraic groups*, J. Algebra **322** (2009), 270–293.

[11] Martin Liebeck and Gary Seitz, *Reductive subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **580** (1996), no. 580, vi+111.

[12] ______, *On the subgroup structure of exceptional groups of Lie type*, Trans. Amer. Math. Soc. **350** (1998), 3409–3482.

[13] ______, *On finite subgroups of exceptional algebraic groups*, J. reine angew. Math. **515** (1999), 25–72.

[14] ______, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Mem. Amer. Math. Soc. **169** (2004).

[15] Alastair Litterick, *Finite simple subgroups of exceptional algebraic groups*, Ph.D. thesis, Imperial College, London, 2013.

[16] Frank Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.

[17] Anatoly Mal'cev, *Commutative subalgebras of semi-simple Lie algebras*, Bull. Acad. Sci. URSS Ser. Math. **9** (1945), 291–300.

[18] Kenzo Mizuno, *The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8*, Tokyo J. Math. **3** (1980), 391–461.

[19] Gary Seitz, *Maximal subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **90** (1991), no. 441, iv+197.

[20] Peter Smith, *A simple subgroup of M? and $E_8(3)$*, Bull. London Math. Soc. **8** (1976), 161–165.

[21] David Stewart, *On the minimal modules for exceptional Lie algebras: Jordan blocks and stabilizers*, LMS J. Comput. Math. **19** (2016), 235–258.

[22] Helmut Strade and Rolf Farnsteiner, *Modular Lie algebras and their representations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 116, Marcel Dekker Inc., New York, 1988.

[23] Adam Thomas, *The irreducible subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. 268 (2021), no. 1307, vi+191.

David A. Craven, School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom

E-mail address: d.a.craven@bham.ac.uk

David I. Stewart, School of Mathematics, Statistics and Physics, Herschel Building, Newcastle University, Newcastle, NE1 7RU, United Kingdom

E-mail address: david.stewart@ncl.ac.uk

Adam R. Thomas, Mathematics Institute, Zeeman Building, University of Warwick, Coventry, CV4 7AL, United Kingdom

E-mail address: Adam.R.Thomas@warwick.ac.uk