Extremal density for sparse minors and subdivisions*

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Abstract. We prove an asymptotically tight bound on the extremal density guaranteeing subdivisions of bounded-degree bipartite graphs with a mild separability condition. As corollaries, we answer several questions of Reed and Wood. Among others, $(1 + o(1))t^2$ average degree is sufficient to force the $t \times t$ grid as a topological minor; $(3/2 + o(1))t$ average degree forces *every* t-vertex planar graph as a minor, furthermore, surprisingly, the value is the same for t-vertex graphs embeddable on any fixed surface; average degree $(2 + o(1))t$ forces *every* t-vertex graph in *any* nontrivial minor-closed family as a minor. All these constants are best possible.

Keywords: graph minors, subdivisions, extremal function, average degree, sparse graphs

1 Introduction

Classical extremal graph theory studies sufficient conditions forcing the appearance of substructures. A seminal result of this type is the Erdős–Stone–Simonovits theorem [5, 4], determining the asymptotics of the average degree needed for subgraph containment. We are interested here in the analogous problem of average degree conditions forcing H as a minor. A graph H is a *minor* of G, denoted by $G \succ H$, if it can be obtained from G by vertex deletions, edge deletions and contractions.

The study of this problem has a long history. An initial motivation was Hadwiger's conjecture that every graph of chromatic number t has K_t as a minor, which is a far-reaching generalisation of the four-colour theorem. Since every graph of chromatic number k contains a subgraph of average degree at least $k-1$, a natural angle of attack is to find bounds on the average degree which will

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guarantee a K_t-minor. The first upper bounds for general t were given by Mader [13, 14]. In celebrated work of Kostochka [10] and, independently, Thomason [20], it was improved to the best possible bound $\Theta(t\sqrt{\log t})$, Thomason subsequently determining the optimal constant [21].

For a general graph H, denote

$$d_{\succ}(H) := \inf\{c : \ d(G) \geq c \Rightarrow G \succ H\}.$$

Myers and Thomason [16] determined this function when H is polynomially dense, showing that again $d_{\succ}(H) = \Theta(|H|\sqrt{\log|H|})$ and determining the optimal constant in terms of H. However, for sparse graphs their results only give $d_{\succ}(H) = o(|H|\sqrt{\log|H|})$, similar to the way that the Erdős–Stone–Simonovits theorem gives a degenerate bound for bipartite subgraphs, and so it is natural to ask for stronger bounds in this regime.

Reed and Wood [17] gave improved bounds for sparser graphs, and in particular showed that if H has bounded average degree then $d_{\succ}(H) = \Theta(|H|)$. They asked several interesting questions about the precise asymptotics in this regime. Among sparse graphs, grids play a central role in graph minor theory, and Reed and Wood raised the question of determining $d_{\succ}(\mathsf{G}_{t,t})$, where $\mathsf{G}_{t,t}$ is the $t \times t$ grid. That is, what is the minimum $\beta > 0$ such that every graph with average degree at least βt^2 contains $\mathsf{G}_{t,t}$ as a minor. Trivially $\beta \geq 1$ in order for the graph to have enough vertices, and their results give a bound of $\beta \leq 6.929$.

This question provides the motivating example for our results. However, we shall focus on a special class of minors: subdivisions or topological minors. A *subdivision* of H is a graph obtained from subdividing edges of H to pairwise internally disjoint paths. The name of topological minor comes from its key role in topological graph theory. A cornerstone result in this area is Kuratowski's theorem from 1930 that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. Again it is natural to ask what average degree will force K_t as a topological minor, and we define analogously

$$d_{\mathsf{T}}(H) := \inf\{c : \ d(G) \geq c \Rightarrow G \ \text{ contains } \ H \ \text{ as a topological minor}\}.$$

Clearly, for any H, $d_{\succ}(H) \leq d_{\mathsf{T}}(H)$. However, there can be a considerable gap between the two quantities; Komlós and Szemerédi [9] and, independently, Bollobás and Thomason [2] showed that $d_{\mathsf{T}}(K_t) = \Theta(t^2)$, meaning that clique topological minors are much harder to guarantee than clique minors. Furthermore, the optimal constant is still unknown in this case, and in general much less is known for bounds on average degree guaranteeing sparse graphs as topological minors.

1.1 Main result

Our main result offers the asymptotics of the average degree needed to force subdivisions of a natural class of sparse bipartite graphs, showing that a necessary bound is already sufficient. It reads as follows.

Theorem 1. *For given $\varepsilon > 0$ and $\Delta \in \mathbb{N}$, there exist α_0 and d_0 satisfying the following for all $0 < \alpha < \alpha_0$ and $d \geq d_0$. If H is an α-separable bipartite graph with at most $(1-\varepsilon)d$ vertices and $\Delta(H) \leq \Delta$, and G is a graph with average degree at least d, then G contains a subdivision of H.*

Here a graph H is *α-separable* if there exists a set S of at most $\alpha|H|$ vertices such that every component of $H - S$ has at most $\alpha|H|$ vertices. Graphs in many well-known classes are $o(1)$-separable. For example, large graphs in any nontrivial minor-closed family are $o(1)$-separable [1, 15].

As an immediate corollary, our main result answers the above question of Reed and Wood in a strong sense by showing that any $\beta > 1$ is sufficient to force the k-dimensional grid $\mathsf{G}^k_{t,\ldots,t}$ not only as a minor but as a topological minor, and so

$$d_{\mathsf{T}}(\mathsf{G}^k_{t,\ldots,t}) = d_{\succ}(\mathsf{G}^k_{t,\ldots,t}) = (1 + o_t(1))t^k.$$

We remark that the optimal constant 1 in Theorem 1 is no longer sufficient if H is not bipartite. Indeed, if e.g. H is the disjoint union of triangles, then the Corrádi–Hajnal theorem [3] implies that $d_{\succ}(H) = \frac{4}{3}|H| - 2$.

2 Applications

Reed and Wood [17] raised several other interesting questions on the average degree needed to force certain sparse graphs as minors. In particular, they asked the following.

- What is the least constant $c > 0$ such that every graph with average degree at least ct contains every planar graph with t vertices as a minor?
- What is the least function g_1 such that every graph with average degree at least $g_1(k) \cdot t$ contains every graph with t vertices and treewidth at most k as a minor?
- What is the least function g_2 such that every graph with average degree at least $g_2(k) \cdot t$ contains every K_k-minor-free graph with t vertices as a minor?

In applying our results to answer these questions, there are two obstacles to overcome. First, the graph classes considered have bounded average degree, but our main result only covers graphs of bounded maximum degree. Secondly, and more significantly, these classes include non-bipartite graphs. Both issues may be overcome by first constructing a suitable graph H' containing the target graph H as a minor, ensuring that H' is bipartite with bounded average degree but still inherits a suitable separability condition from the original target graph. We then find a subdivision of H' in the host graph. In order to ensure H' has bounded degree it cannot necessarily be a subdivision of H, and so this procedure gives H as a minor, but not necessarily a topological minor.

Passing from a bounded average degree H to a bounded degree graph only requires the addition of $o(t)$ vertices, whereas ensuring that H' is bipartite typically changes the constant required, in a way that depends on the precise class of graphs involved. Thus we obtain a range of different constants for different classes; nevertheless, many of these constants are optimal. In the following results we use the notation

$$d_{\succ}(\mathcal{F},t) := \inf\{c:\ d(G) \geq c \Rightarrow G \succ H,\ \forall H \in \mathcal{F} \text{ with } |H| \leq t\}$$

for a graph family $\mathcal{F}$. We answer the first question above in a strong sense, giving the optimal constant and showing that the answer is the same for graphs which may be drawn on any fixed surface.

Theorem 2. *Writing $\mathcal{F}_g$ for the class of graphs with genus at most g, we have $d_{\succ}(\mathcal{F}_g,t) = (3/2 + o(1))\,t$.*

Many other important classes of graphs are naturally closed under taking minors. The seminal graph minor theorem of Robertson and Seymour (proved in a sequence of papers culminating in [18]) shows that every minor-closed family can be characterised by a finite list of minimal forbidden minors. For example, the linklessly-embeddable graphs are defined by a minimal family of seven forbidden minors, including K_6 and the Petersen graph [19]. We can extend the proof of Theorem 2 to minor-closed families more generally; in fact our results also apply to classes of polynomial expansion, which are not necessarily minor-closed. For each $k \in \mathbb{N}$, define $\alpha_k(G) := \max\{|U| : U \subseteq V(G), \chi(G[U]) = k\}$. So $\alpha_1(G)$ is the usual independence number and $\alpha_2(G)$ is the maximum size of the union of two independent sets.

Theorem 3. *Let $\mathcal{F}$ be a nontrivial minor-closed family, or, more generally, a class of polynomial expansion. For each $F \in \mathcal{F}$ with t vertices, we have*

$$2t - 2\alpha(F) - O(1) \leq d_{\succ}(F) \leq 2t - \alpha_2(F) + o(t).$$

Theorem 3 yields the following consequences, for all of which the constants are best possible (note that the last example is not a minor-closed class).

- The class $\mathcal{T}_k$ of treewidth at most k satisfies $d_{\succ}(\mathcal{T}_k,t) = \big(\frac{2k}{k+1} + o(1)\big)t$; in particular, $g_1(k) = 2 - o_k(1)$.
- $g_2(k) = 2 - o_k(1)$.
- For any nontrivial minor closed family $\mathcal{F}$, we have $d_{\succ}(\mathcal{F},t) \leq (2 + o(1))t$.
- The class $\mathcal{L}$ of linklessly embeddable graphs satisfies $d_{\succ}(\mathcal{L},t) = (8/5 + o(1))t$.
- The class $\mathcal{P}_1$ of 1-planar graphs satisfies $d_{\succ}(\mathcal{P}_1,t) = (5/3 + o_t(1))t$.

While for some families we are able to show that the upper and lower bounds from Theorem 3 match, giving the precise constant, in others this is not clear. In particular, for the K_k-minor-free graphs Hadwiger's conjecture would imply matching bounds.

3 Outline of the proof

Our proof utilises both pseudorandomness from Szemerédi's regularity lemma and expansions for sparse graphs. The particular expander that we shall make use of is an extension of the one introduced by Komlós and Szemerédi [8, 9], which has played an important role in some recent developments on sparse graph embedding problems, see e.g. [7, 11, 12].

To prove Theorem 1, we first pass to a robust sublinear expander subgraph without losing much on the average degree. Depending on the density of this expander, we use different approaches. Roughly speaking, when the expander has positive edge density, we will utilise pseudorandomness via the machinery of the graph regularity lemma and the blow-up lemma, and otherwise we exploit its sublinear expansion property. Full proofs may be found in [6].

3.1 Embeddings in dense graphs

The regularity lemma essentially partitions our graph G into a bounded number of parts, in which the bipartite subgraphs induced by most of the pairs of parts behave pseudorandomly. The information of this partition is then stored to a (weighted) fixed-size so-called reduced graph R which inherits the density of G. We seek to embed H in G using the blow-up lemma, which boils down to finding a 'balanced' bounded-degree homomorphic image of H in R. This is where the additional separable assumption on H kicks in, enabling us to cut H into small pieces to offer suitable 'balanced' homomorphic images. If the reduced graph R is not bipartite, the density of R inherited from G is just large enough to guarantee an odd cycle in R long enough to serve as our bounded-degree homomorphic image of H. However, an even cycle of the same length would not be sufficient, since H could be an extremely asymmetric bipartite graph. To overcome this problem, when R is bipartite we make use of a 'sun' structure. This is a bipartite graph consisting of a cycle with some additional leaves, which help in balancing out any asymmetry of H.

3.2 Embeddings in robust expanders with medium density

The robust sublinear expansion underpins all of our constructions of H-subdivisions when the graph G is no longer dense. At a high level, in G, we anchor on some carefully chosen vertices and embed paths between anchors (corresponding to the edge set of H) one at a time. As these paths in the subdivision need to be internally vertex disjoint, to realise this greedy approach we will need to build a path avoiding a certain set of vertices. This set of vertices to avoid contains previous paths that we have already found and often some small set of 'fragile' vertices that we wish to keep free.

To carry out such robust connections, we use the small-diameter property of sublinear expanders. We aim to anchor at vertices with large 'boundary' compared to the total size of all paths needed, that is, being able to access many vertices within short distance. If there are d vertices of sufficiently high degree, we can anchor on them. Assuming this is not the case essentially enables us to view G as if it is a 'relatively regular' graph. We now use a web structure in which each core vertex is connected by a tree to a large 'exterior'. Using the relative regularity of G, together with the fact that it is not too sparse, we can pull out many reasonably large stars and link them up to find webs. We then anchor on their core vertices and connect pairs via the exteriors of the corresponding webs, while being careful to avoid the fragile centre parts of other webs.

3.3 Embeddings in sparse robust expanders

The method of building and connecting webs breaks down if the expander is too sparse, and we need to use other structures in this case.

For the easier problem of finding minors, it suffices to find d large balls and link them up by internally disjoint paths according to the structure of H; contracting each ball gives H as a minor. In order to be able to find the paths, we ensure the balls are sufficiently far apart that any given pair of balls can be expanded to very large size, avoiding all others, and then connect the pairs one by one.

Coming back to embedding H-subdivisions, we shall follow a similar general strategy. However, an immediate obstacle we encounter is that we need to be able to lead a constant number of paths arriving at each ball disjointly to the anchor vertex. In other words, each anchor vertex has to expand even after removing a constant number of disjoint paths starting from itself. Our expansion property is simply too weak for this.

We therefore use a new structure we call a 'nakji'. Each nakji consists of several 'legs', which are balls pairwise far apart, linked to a central well-connected 'head'. This structure is designed precisely to circumvent the above problem by doing everything in reverse order. Basically, instead of looking for anchor vertices that expand robustly, we rather anchor on nakjis and link them via their legs first and then extend the paths from the legs in each nakji's head using connectivity. The remaining task is then to find many nakjis. This is done essentially by linking small subexpanders within G, after removing the few high-degree vertices.

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