DE GRUYTER Forum Math. 2021; aop

Research Article

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Gross–Prasad periods for reducible representations

https://doi.org/10.1515/forum-2021-0089
Received April 18, 2021; revised July 21, 2021

Abstract: We study $\mathrm{GL}_2(F)$-invariant periods on representations of $\mathrm{GL}_2(A)$, where F is a non-archimedean local field and A/F a product of field extensions of total degree 3. For irreducible representations, a theorem of Prasad shows that the space of such periods has dimension $\leqslant 1$, and is non-zero when a certain ε-factor condition holds. We give an extension of this result to a certain class of reducible representations (of Whittaker type), extending results of Harris–Scholl when A is the split algebra $F \times F \times F$.

Keywords: Branching laws, Gross–Prasad conjectures

MSC 2010: 22E50

Communicated by: Freydoon Shahidi

1 Introduction

One of the central problems in the theory of smooth representations of reductive groups over non-archimedean local fields is to determine when a representation of a group G admits a linear functional invariant under a closed subgroup H (an H-invariant period).

The Gross–Prasad conjectures [5] give a very precise and elegant description of when such periods exist, for many natural pairs (G, H), in terms of ε-factors. However, the original formulation of these conjectures applies to members of *generic* L-packets for G; and the analogous picture for representations in non-generic L-packets is rather more complex. Although the ε-factor is still well-defined for all such L-packets, the conjecture formulated in [4] only applies when the L-parameters satisfy an additional "relevance" condition, raising the natural question of whether the ε-factors for non-relevant L-packets have any significance in terms of invariant periods.

In this short note, we describe some computations of branching laws in the following simple case: G is $\mathrm{GL}_2(A)$, where A/F is a cubic étale algebra, and H is the subgroup $\mathrm{GL}_2(F)$. Our computations suggest an alternative approach to the theory: rather than studying branching laws for non-generic irreducible representations, we focus on representations which are possibly reducible, but satisfy a certain "Whittaker-type" condition. We show that H-invariant periods on these representations are unique if they exist, and that their existence is governed by ε-factors, extending the results of Prasad [16, 17] for irreducible generic representations, and Harris and Scholl [7] for A the split algebra (in which case the ε-factor is always $+1$). In this optic, the "relevance" condition appears as a criterion for the H-invariant period to factor through the unique irreducible quotient.

The result of the present paper, combined with other recent works such as that of Chan [3] in the case $(G, H) = (\mathrm{GL}_n(F) \times \mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$, would seem to suggest that many "Gross–Prasad-style" branching results should extend to Whittaker-type representations, and we hope to explore this further in future works.

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We conclude with an application to global arithmetic. For π a Hilbert modular form over a real quadratic field, the constructions of [6, 8, 9] give rise to a family of cohomology classes taking values in the 4-dimensional Asai Galois representation associated to π. We show that if π is not of CM type and not a base-change from $\mathbf{Q}$, then these elements all lie in a 1-dimensional subspace. This is the analogue for quadratic Hilbert modular forms of the result proved in [7] for Beilinson's elements attached to the Rankin convolution of two modular forms.

2 Statements

Throughout this paper, F denotes a non-archimedean local field of characteristic 0. If G is a reductive group over F, then a "representation" of $G(F)$ shall mean a smooth linear representation on a complex vector space.

2.1 Epsilon-factors

We choose a non-trivial additive character ψ of F. For Weil–Deligne representations ρ of F, we define ε-factors $\varepsilon(\rho) = \varepsilon(\rho, \psi)$ following Langlands (the "ε_L" convention in [19, Section 3.6]), so that $\varepsilon(\rho)$ is independent of ψ if $\det(\rho) = 1$. We note that

$$\varepsilon(\rho_1 \oplus \rho_2) = \varepsilon(\rho_1)\varepsilon(\rho_2), \quad \varepsilon(\rho)\varepsilon(\rho^\vee) = (\det \rho)(-1),$$

where $\det(\rho)$ is identified with a character of $F^\times$ via class field theory.

We write $\mathrm{sp}(n)$ for the n-dimensional Weil–Deligne representation given by the $(n-1)$-st symmetric power of the Langlands parameter of the Steinberg representation, so that the eigenvalues of the Frobenius element on $\mathrm{sp}(n)$ are $q^{\frac{1-n}{2}}, q^{\frac{3-n}{2}}, \dots, q^{\frac{n-1}{2}}$, where q is the size of the residue field.

2.2 The generic Langlands correspondence for GL_2

The classical local Langlands correspondence for GL_2 is a bijection between irreducible smooth representations of $\mathrm{GL}_2(F)$, and 2-dimensional Frobenius-semisimple representations of the Weil–Deligne group of F.

In this paper, we will use the following modification of the correspondence. A representation of $\mathrm{GL}_2(F)$ is said to be *of Whittaker type* if it is either irreducible and generic, or a reducible principal series representation with 1-dimensional quotient. (These are precisely the representations of $\mathrm{GL}_2(F)$ which have well-defined Whittaker models.) The *generic Langlands correspondence* is a bijection between Whittaker-type representations of $\mathrm{GL}_2(F)$ and 2-dimensional Frobenius-semisimple Weil–Deligne representations; it agrees with the classical Langlands correspondence on irreducible generic representations, and maps a reducible Whittaker-type principal series to the classical Langlands parameter of its 1-dimensional quotient.[1]

In particular, the unramified Weil–Deligne representation with Frobenius acting as $\begin{pmatrix} q^{1/2} & \\ & q^{-1/2}\end{pmatrix}$ corresponds to the reducible principal series Σ_F containing the Steinberg representation St_F as subrepresentation and trivial 1-dimensional quotient. (We omit the subscript F if it is clear from context.)

2.3 Statement of the theorem

We now state our main theorem. Let A/F be a separable cubic algebra, so A is a product of field extensions of F of total degree 3. Let ω_A be the quadratic character of $F^\times$ determined by the class of $\mathrm{disc}(A)$ in $F^\times/F^{\times 2}$. We let $G = \mathrm{GL}_2(A)$, and $H = \mathrm{GL}_2(F)$, embedded in G in the obvious way.

1 This correspondence was introduced in [2]; but our conventions differ from [2] by a power of the norm character, in order that our generic Langlands correspondence extend the classical one.

The Langlands dual group of GL_2/A has a natural 8-dimensional *Asai*, or *multiplicative induction*, representation; in the case $A = F^3$ this is simply the tensor product of the defining representations of the factors. We use this representation, and the generic Langlands correspondence for GL_2 above, to define Asai ε-factors $\varepsilon(\mathrm{As}(\Pi))$ for Whittaker-type representations of $\mathrm{GL}_2(A)$.

Finally, we consider Jacquet–Langlands transfers. Let $H' = D^\times$ where D/F is the unique non-split quaternion algebra. Let $G' = (D \otimes_F A)^\times$, and let Π' be the Jacquet–Langlands transfer of Π to G' if this exists, and 0 otherwise.

Remark 2.1. Note that if $A = E \times F$ for E a quadratic field extension, then $D^\times$ is split over E, and hence

$$G' = \mathrm{GL}_2(E) \times D^\times(F).$$

Thus if $\Pi = \pi \boxtimes \sigma$, for π, σ representations of $\mathrm{GL}_2(E)$ and $\mathrm{GL}_2(F)$, respectively, we have $\Pi' = \pi \boxtimes \sigma'$. In particular, $\Pi' \neq 0$ whenever σ' is discrete series (even if π is principal series, possibly reducible).

Main Theorem. *Let Π be a representation of $\mathrm{GL}_2(A)$ of Whittaker type, whose central character is trivial on $F^\times$ (embedded diagonally in $A^\times$). Then we have*

$$\dim \mathrm{Hom}_H(\Pi, \mathbb{1}) = \begin{cases} 1 & \text{if } \varepsilon(\mathrm{As}(\Pi))\omega_A(-1) = 1, \\ 0 & \text{if } \varepsilon(\mathrm{As}(\Pi))\omega_A(-1) = -1, \end{cases}$$

and

$$\dim \mathrm{Hom}_H(\Pi, \mathbb{1}) + \dim \mathrm{Hom}_{H'}(\Pi', \mathbb{1}) = 1.$$

If Π is an irreducible generic representation, then this is the main result of [16] for A the split algebra, and [17] for non-split A (modulo the case of supercuspidal representations of cubic fields, completed in [18]). The new content of the above theorem is that this also holds for reducible Whittaker-type Π.

Remark 2.2. Any such Π can be written as the specialisation at $s = 0$ of an analytic family of Whittaker-type representations $\Pi(s)$ indexed by a complex parameter s, which are irreducible for generic s and all have central character trivial on $F^\times$. For such families, the ε-factors $\varepsilon(\mathrm{As}\,\Pi(s))$ are locally constant as a function of s; hence, given the results of [17, 18] in the irreducible case, our theorem is equivalent to the assertion that $\dim \mathrm{Hom}_H(\Pi(s), \mathbb{1})$ and $\dim \mathrm{Hom}_{H'}(\Pi(s)', \mathbb{1})$ are locally constant in s.

2.4 Relation to results of Mœglin–Waldspurger

Note that [14, Proposition in Section 1.3] gives a formula for branching multiplicities for certain parabolically-induced representations of special orthogonal groups $\mathrm{SO}(d) \times \mathrm{SO}(d')$ (with $d - d'$ odd), expressing these in terms of multiplicities for irreducible tempered representations of smaller special orthogonal groups. These results are applied in [14, Proposition in Section 1.3] to prove the Gross–Prasad conjecture for irreducible representations in non-tempered generic L-packets (by reduction to the tempered case); but the results are also valid for reducible representations.

Since the split form of SO(3) is PGL(2), and SO(4) is closely related to PGL(2) × PGL(2), one can derive many cases of our Main Theorem from their result applied to various forms of SO(3) × SO(4). In fact, if $A = F^3$ or $A = E \times F$ for E quadratic, we can obtain in this way all cases of the Main Theorem not already covered by Prasad's results.

However, the case when A is a cubic field extension does not appear to fit into the framework of [14, Proposition in Section 1.3]; and the proof given in [14] is rather indirect, particularly in the case when the SO(3) representation is reducible, in which case their argument requires a delicate switch back and forth between representations of SO(3) × SO(4) and SO(4) × SO(5). So we hope that the alternative, more direct approach given here will be of interest.

3 Split triple products

We first put $A = F \times F \times F$.

Theorem 3.1 (Prasad, Harris–Scholl). *Let π_1, π_2, π_3 be representations of $\mathrm{GL}_2(F)$ of Whittaker type, with central characters ω_i such that $\omega_1\omega_2\omega_3 = 1$. Then we have*

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{1}) = \begin{cases} 1 & \text{if } \varepsilon(\pi_1 \times \pi_2 \times \pi_3) = +1, \\ 0 & \text{if } \varepsilon(\pi_1 \times \pi_2 \times \pi_3) = -1, \end{cases}$$

and

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{1}) + \dim \mathrm{Hom}_{D^\times(F)}(\pi_1' \otimes \pi_2' \otimes \pi_3', \mathbb{1}) = 1.$$

If the π_i are all irreducible, then the above is the main result of [16]. If one or more of the π_i is isomorphic to a twist of Σ_F, then the ε-factor is automatically $+1$, and $\pi_1' \otimes \pi_2' \otimes \pi_3'$ is the zero representation. So all that remains to be shown is that in this case we have $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{1}) = 1$. This is established in [7, Propositions 1.5, 1.6 and 1.7], except for one specific case, which is when all three of the π_i are twists of Σ by characters.

In this case, by twisting we may assume $\pi_2 = \pi_3 = \Sigma$ and $\pi_1 = \Sigma \otimes \eta$, where η is a character of $F^\times$ with $\eta^2 = 1$. The case $\eta = 1$ is covered by [7, Proposition 1.7], so we assume η is a non-trivial quadratic character. In this case $\mathrm{Hom}_H(\eta \otimes \Sigma_F \otimes \Sigma_F, \mathbb{1}) = \mathrm{Hom}_H(\Sigma_F, \Sigma_F^\vee \otimes \eta) = 0$, so $\mathrm{Hom}_H(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{1})$ injects into $\mathrm{Hom}_H(\eta \mathrm{St}_F \otimes \Sigma_F \otimes \Sigma_F, \mathbb{1})$, which has dimension 1 by [7, Proposition 1.6]. Thus $\mathrm{Hom}_H(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{1})$ has dimension $\leqslant 1$. Since one can easily write down a non-zero element of this space using the Rankin–Selberg zeta integral, we conclude that its dimension is 1 as required.

4 Quadratic fields

We now suppose $A = E \times F$ with E/F quadratic, so $\Pi = \pi \boxtimes \sigma$ for Whittaker-type representations π of $\mathrm{GL}_2(E)$ and σ of $\mathrm{GL}_2(F)$ such that $\omega_\pi|_{F^\times} \cdot \omega_\sigma = 1$. Since the case of π, σ irreducible is proved in [17], it suffices to consider the following cases:

(a) π is irreducible and $\sigma = \Sigma_F$,
(b) σ is irreducible and $\pi = \Sigma_E$,
(c) $\pi = \Sigma_E$ and $\sigma = \Sigma_F \otimes \eta$, where η is a quadratic character.

In cases (a) and (c), we always have $\varepsilon(\mathrm{As}(\pi) \times \sigma)\varepsilon_{E/F}(-1) = 1$, and $\sigma' = \{0\}$, so the Main Theorem amounts to the assertion that $\dim \mathrm{Hom}_H(\pi \boxtimes \sigma, \mathbb{1}) = 1$. In case (b), both signs can occur.

Theorem 4.1 (a). *Let π be an irreducible generic representation of $\mathrm{GL}_2(E)$ such that $\omega_\pi|_{F^\times} = 1$. Then we have $\dim \mathrm{Hom}_H(\pi \boxtimes \Sigma_F, \mathbb{1}) = 1$.*

Remark 4.2. Note that the case when E/F is unramified, and π is unramified and tempered, is part of [6, Theorem 4.1.1]. However, the proof of this statement given in [6] has a minor error which means the argument does not work when π is the normalised induction of the trivial character of B_E. So the argument below fixes this small gap.

Proof. We first observe that $\mathrm{Hom}_H(\pi \boxtimes \Sigma_F, \mathbb{1})$ is non-zero. Since π is generic, it has a Whittaker model $\mathcal{W}(\pi)$ with respect to any non-trivial additive character of E. We may suppose that this additive character is trivial on F, so that we may define the Asai zeta-integral

$$Z(W, \Phi, s) = \int_{N_H \backslash H} W(h)\Phi((0, 1)h)|\det h|^s \, \mathrm{d}h,$$

for $W \in \mathcal{W}(\pi)$ and $\Phi \in \mathcal{S}(F^2)$ (the space of Schwartz functions on F). Here N_H is the upper-triangular unipotent subgroup of H.

It is well known that this integral converges for $\mathbb{R}(s) \gg 0$ and has meromorphic continuation to the whole complex plane; and the values of $Z(-,-,s)$ span a non-zero fractional ideal of $\mathbf{C}[q^s, q^{-s}]$, generated by an L-factor independent of Φ and W, which is the Asai L-factor $L(\mathrm{As}(\pi), s)$. Thus the map

$$(W, \Phi) \mapsto \lim_{s\to 0} \frac{Z(W, \Phi, s)}{L(\mathrm{As}(\pi), s)} \tag{†}$$

defines a non-zero, H-invariant bilinear form $W(\pi) \otimes \mathcal{S}(F^2) \to \mathbf{C}$. Since the maximal quotient of $\mathcal{S}(F^2)$ on which $F^\times$ acts trivially is isomorphic to Σ_F (see for example [10, Proposition 3.3 (b)]), this shows that $\mathrm{Hom}_H(\pi \boxtimes \Sigma_F, \mathbb{1}) \neq 0$ as claimed.

So, to prove Theorem 4.1 (a), it suffices to show that $\dim \mathrm{Hom}_H(\pi \boxtimes \Sigma_F, \mathbb{1}) \leqslant 1$. As π has unitary central character, it is either a discrete-series representation, in which case it is automatically tempered, or an irreducible principal series, which may or may not be tempered. We shall consider these cases separately.

Note that [1, Theorem 1.1] states that if π is an irreducible tempered representation of $\mathrm{GL}_2(E)$, then we have $\dim \mathrm{Hom}_{M(F)}(\pi, \mathbb{1}) = 1$, where $M(F) = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}$ is the mirabolic subgroup of $\mathrm{GL}_2(F)$. If we assume $\omega_\pi|_{F^\times} = 1$, then since $F^\times \cdot M(F) = B(F)$ is the Borel subgroup of $\mathrm{GL}_2(F)$, we have

$$\mathrm{Hom}_{M(F)}(\pi, \mathbb{1}) = \mathrm{Hom}_{B(F)}(\pi, \mathbb{1}) = \mathrm{Hom}_H(\pi, \mathrm{Ind}_{B(F)}^H(\mathbb{1})).$$

As $\mathrm{Ind}_{B(F)}^H(\mathbb{1}) = \Sigma_F^\vee$, this proves Theorem 4.1 (a) for tempered π.

We now consider the principal-series case. For α, β smooth characters of $E^\times$, we write $I_E(\alpha, \beta)$ for the normalised induction to $\mathrm{GL}_2(E)$ of the character $\alpha \boxtimes \beta$ of $B(E)$. Note that this representation is tempered if and only if α and β are unitary. We suppose $\alpha/\beta \neq |\cdot|_E^{\pm 1}$ and $\alpha\beta|_{F^\times} = 1$. Then we have the following results:

- $\mathrm{Hom}_H(\pi \boxtimes \mathrm{St}_F, \mathbb{1})$ is zero if $\alpha\beta^c = 1$, and 1-dimensional otherwise, where β^c denotes the character $x \mapsto \beta(x^c)$. See [17, Remark 4.1.1].
- $\mathrm{Hom}_H(\pi \boxtimes \mathbb{1}, \mathbb{1})$ is 1-dimensional if $\alpha\beta^c = 1$, or if $\alpha|_{F^\times} = \beta|_{F^\times} = 1$; otherwise it is 0. See [13, Theorem 5.2].

We conclude that exactly one of $\mathrm{Hom}_H(\pi \boxtimes \mathrm{St}_F, \mathbb{1})$ and $\mathrm{Hom}_H(\pi \boxtimes \mathbb{1}, \mathbb{1})$ is non-zero (and Theorem 4.1 (a) therefore follows), *unless* π is of the form $I_E(\alpha, \beta)$ with $\alpha|_{F^\times} = \beta|_{F^\times} = 1$ and $\alpha\beta^c \neq 1$. However, in this exceptional case α and β are unitary, and thus π is tempered, so Theorem 4.1 (a) has already been established for π above. This completes the proof of Theorem 4.1 (a). □

Remark 4.3. It follows, in particular, that for a generic irreducible representation π of $\mathrm{GL}_2(E)$, we have $\mathrm{Hom}_H(\pi, \mathbb{1}) \neq 0$ (i.e. π is "F-distinguished") if and only if the zeta-integral (†) factors through the 1-dimensional quotient of Σ_F, and thus vanishes on all Φ with $\Phi(0,0) = 0$; that is, $s = 0$ is an *exceptional pole* of the Asai L-factor. This is the $n = 2$ case of a theorem due to Matringe [12, Theorem 3.1] applying to $\mathrm{GL}_n(E)$-representations. See [10] for analogous results and conjectures regarding poles of zeta-integrals for GSp_4 and $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

For case (b) of the main theorem, we need the following lemma:

Lemma 4.4. *Let σ be an irreducible generic representation of* $\mathrm{GL}_2(F)$ *with* $\omega_\sigma = 1$. *Then*

$$\varepsilon(\mathrm{As}(\Sigma_E) \times \sigma) = \varepsilon(\sigma)\varepsilon(\sigma \times \omega_{E/F}).$$

Moreover, if $\sigma \neq \mathrm{St}_F$, *then we have*

$$\varepsilon(\sigma)\varepsilon(\sigma \times \omega_{E/F}) = \varepsilon(\mathrm{As}(\mathrm{St}_E) \times \sigma),$$

while for $\sigma = \mathrm{St}_F$ *we have*

$$\varepsilon(\mathrm{As}(\mathrm{St}_E) \times \mathrm{St}_F)\omega_{E/F}(-1) = 1 \quad \textit{and} \quad \varepsilon(\mathrm{As}(\Sigma_E) \times \mathrm{St}_F)\omega_{E/F}(-1) = -1.$$

Proof. If σ is not a twist of Steinberg, then its Weil–Deligne representation has trivial monodromy action, so we compute that

$$\varepsilon(\mathrm{As}(\mathrm{St}_E) \times \sigma) = \varepsilon((\mathrm{sp}(3) \oplus \omega_{E/F}) \times \sigma) = \varepsilon(\sigma \times \omega_{E/F})\varepsilon(\sigma)^3 \det(-\mathrm{Frob} : \rho_\sigma^{I_F})^2.$$

Since σ has trivial central character, $\varepsilon(\sigma) = \pm 1$. If σ is supercuspidal we are done, since in this case $\rho_\sigma^{I_F} = 0$. If σ is principal series, then $\rho_\sigma^{I_F}$ must be either 0, or all of ρ_σ, since ρ_σ has determinant 1. Thus $\det(-\mathrm{Frob} : \rho_\sigma^{I_F}) = 1$, so $\varepsilon(\mathrm{As}(\mathrm{St}_E) \times \sigma) = \varepsilon(\sigma)\varepsilon(\sigma \times \omega_{E/F})$, proving the claim in this case. The case when σ is a twist of the Steinberg by a non-trivial (necessarily quadratic) character can be computed similarly. □

Theorem 4.1 (b). *Let σ be an irreducible generic representation of $\mathrm{GL}_2(F)$ with $\omega_\sigma = 1$. Then:*
(i) *If $\varepsilon(\sigma)\varepsilon(\sigma \times \omega_{E/F}) = \omega_{E/F}(-1)$, then $\dim \mathrm{Hom}_H(\Sigma_E \boxtimes \sigma, \mathbb{1}) = 1$ and $\mathrm{Hom}_{H'}(\Sigma_E \boxtimes \sigma', \mathbb{1}) = 0$.*
(ii) *If $\varepsilon(\sigma)\varepsilon(\sigma \times \omega_{E/F}) = -\omega_{E/F}(-1)$, then $\mathrm{Hom}_H(\Sigma_E \boxtimes \sigma, \mathbb{1}) = 0$ and $\dim \mathrm{Hom}_{H'}(\Sigma_E \boxtimes \sigma', \mathbb{1}) = 1$.*

Proof. We first consider the situation for H'. This case is relatively simple, since H' is compact modulo centre, and hence the functor of H'-invariants is exact on the category of H'-representations trivial on $F^\times$. So we have

$$\dim \mathrm{Hom}_{H'}(\Sigma_E \otimes \sigma', \mathbb{1}) = \dim \mathrm{Hom}_{H'}(\sigma', \mathbb{1}) + \dim \mathrm{Hom}_{H'}(\mathrm{St}_E \otimes \sigma', \mathbb{1}).$$

Using Prasad's results for $\mathrm{Hom}_{H'}(\mathrm{St}_E \otimes \sigma', \mathbb{1})$ and the preceding lemma, we see that $\dim \mathrm{Hom}_{H'}(\Sigma_E \otimes \sigma', \mathbb{1})$ has dimension 1 if $\varepsilon(\sigma)\varepsilon(\sigma \times \omega_{E/F}) = -\omega_{E/F}(-1)$ and is zero otherwise, as required.

For the group H, the situation is a little more complicated: since σ is generic, we have $\mathrm{Hom}_H(\sigma, \mathbb{1})$ is zero, and hence there is an exact sequence

$$0 \to \mathrm{Hom}_H(\Sigma_E \otimes \sigma, \mathbb{1}) \to \mathrm{Hom}_H(\mathrm{St}_E \otimes \sigma, \mathbb{1}) \to \mathrm{Ext}^1_{\mathrm{PGL}_2(F)}(\sigma, \mathbb{1}).$$

Claim. *The group $\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}(\sigma, \mathbb{1})$ is 1-dimensional if $\sigma = \mathrm{St}_F$, and zero otherwise.*

Proof of Claim. If σ is supercuspidal, then the result is immediate, since σ is projective in the category of $\mathrm{PGL}_2(F)$-representations. The remaining cases can be handled directly using Frobenius reciprocity, or alternatively, one can appeal to Schneider–Stuhler duality (as reformulated in [15, Theorem 2]) to show that the Ext group is dual to $\mathrm{Hom}_H(\mathbb{1}, D(\sigma))$ where D is the Aubert–Zelevinsky involution, which sends St_F to the trivial representation. □

This gives the desired formula for $\dim \mathrm{Hom}_H(\Sigma_E \otimes \sigma, \mathbb{1})$ in all cases except when $\sigma = \mathrm{St}_F$, in which case we must show that the non-trivial H-invariant period of $\mathrm{St}_E \otimes \mathrm{St}_F$ does not lift to $\Sigma_E \otimes \mathrm{St}_F$. This can be done directly: we can compute $\Sigma_E|_{\mathrm{GL}_2(F)}$ via Mackey theory, using the two orbits of H on $\mathbf{P}^1(E)$ to obtain the exact sequence

$$0 \to \mathrm{cInd}_{E^\times}^{\mathrm{GL}_2(F)}(\mathbb{1}) \to \Sigma_E \to I_F(|\cdot|_F, |\cdot|_F^{-1}) \to 0.$$

The latter representation is irreducible and has no homomorphisms to St_F; and we saw in the proof of Theorem 4.1 (a) that

$$\mathrm{Hom}_H(\mathrm{cInd}_{E^\times}^{H}(\mathbb{1}) \otimes \mathrm{St}_F, \mathbb{1}) = \mathrm{Hom}_{E^\times}(\mathrm{St}_F, \mathbb{1}) = 0.$$

This shows that $\mathrm{Hom}_H(\Sigma_E \boxtimes \mathrm{St}_F, \mathbb{1}) = 0$, completing the proof. □

Remark 4.5. We are grateful to the anonymous referee for pointing out the significance of the vanishing of $\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}(\sigma, \mathbb{1})$; the original version of this paper used a different and rather more complicated argument.

Theorem 4.1 (c). *Let η be a quadratic character of $F^\times$ (possibly trivial). Then we have*

$$\dim \mathrm{Hom}_H(\Sigma_E \boxtimes \Sigma_F, \eta) = 1.$$

Proof. The computation of the ε-factor is immediate; and by a zeta-integral argument as before, we can show that $\mathrm{Hom}_H(\Sigma_E \boxtimes \Sigma_F, \eta) \neq 0$ (since the representation Σ_E, despite being reducible, has a well-defined Whittaker model). So it suffices to show that the hom-space has dimension $\leqslant 1$.

If η is not the trivial character, then

$$\mathrm{Hom}_H(\mathbb{1} \boxtimes \Sigma_F, \eta) = 0,$$

so the desired Hom-space injects into $\mathrm{Hom}_H(\mathrm{St}_E \boxtimes \Sigma_F, \eta)$, which is 1-dimensional by Theorem 4.1 (a).

If η is trivial, then we have seen above that $\mathrm{Hom}_H(\Sigma_E \boxtimes \mathrm{St}_F, \mathbb{1})$ is zero. So

$$\mathrm{Hom}_H(\Sigma_E \boxtimes \Sigma_F, \mathbb{1}) = \mathrm{Hom}_H(\Sigma_E, \mathbb{1}).$$

From the Mackey decomposition of $\Sigma_E|_{\mathrm{GL}_2(F)}$ above, one sees easily that this space is 1-dimensional. □

5 Cubic fields

We briefly discuss the case where A is a cubic extension of F.

Theorem 5.1. *Let π be a Whittaker-type representation of $\mathrm{GL}_2(E)$. Then the space $\mathrm{Hom}_H(\pi, \mathbb{1})$ has dimension 1 if $\varepsilon(\mathrm{As}(\pi))\omega_A(-1) = 1$ and is zero otherwise.*

Proof. The case of irreducible generic π is proved in [17] assuming π non-supercuspidal, and the supercuspidal case is filled in by [18]. In this case, the only example of a reducible Whittaker-type representation of G is $\Sigma_E \otimes \eta$, where η is a character of $E^\times$; and the central-character condition implies that $\lambda = \eta|_{F^\times}$ must be trivial or quadratic.

The ε-factors $\varepsilon(\mathrm{As}(\mathrm{St}_E) \times \lambda)$ are computed in [17, Section 8]. We find that $\varepsilon(\mathrm{As}(\Sigma_E) \times \lambda)\omega_{E/F}(-1)$ is always $+1$. On the other hand, $\varepsilon(\mathrm{As}(\mathrm{St}_E) \times \lambda)\omega_{E/F}(-1)$ is $+1$ if λ is non-trivial quadratic, and -1 if $\lambda = 1$. So it follows that exactly one of $\mathrm{Hom}_H(\mathbb{1}, \lambda)$ and $\mathrm{Hom}_H(\mathrm{St}_E, \lambda)$ is non-zero, implying that $\dim \mathrm{Hom}_H(\Sigma_E \otimes \eta, \mathbb{1}) \leqslant 1$.

To complete the proof, we must show that when $\lambda \neq 1$, the H-invariant homomorphism $\mathrm{Hom}_H(\mathrm{St}_E, \lambda)$ extends to Σ_E. However, this is clear since the obstruction lies in $\mathrm{Ext}^1_H(\mathbb{1}, \lambda)$, which is zero. □

This completes the proof of the Main Theorem.

6 An application to Euler systems

We now give a global application, a strengthening of some results from [9] and [6] on Euler systems for quadratic Hilbert modular forms. Let $K/\mathbf{Q}$ be a real quadratic field and write $G = \mathrm{Res}_{K/\mathbf{Q}}(\mathrm{GL}_2)$, $H = \mathrm{GL}_{2/\mathbf{Q}} \subset G$; set $G_f = G(\mathbf{A}_f) = \mathrm{GL}_2(\mathbf{A}_{K,f})$ and H_f similarly.

6.1 Adelic representations

Let χ be a finite-order character of $\mathbf{A}_f^\times$ and define a representation of H_f by

$$\mathcal{I}(\chi) = \bigotimes_\ell{}' \, \mathcal{I}_\ell(\chi_\ell),$$

where $\mathcal{I}_\ell(\chi_\ell)$ denotes the representation of H_ℓ given by normalised induction of the character $\chi_\ell|\cdot|^{\frac{1}{2}} \boxtimes |\cdot|^{-\frac{1}{2}}$ of the Borel subgroup. For $\chi = 1$, we let $\mathcal{I}^0(1)$ denote the codimension 1 subrepresentation of $\mathcal{I}(1)$. Exactly as in [7, Section 2], the local results above imply the following branching law for G_f-representations:

Proposition 6.1. *Let π be an irreducible admissible representation of G_f, all of whose local factors are generic, with $\omega_\pi|_{\mathbf{A}_f^\times} = \chi^{-1}$.*

- *We have $\dim \mathrm{Hom}_{H_f}(\pi \otimes \mathcal{I}(\chi), \mathbb{1}) = 1$.*
- *If $\chi = 1$ and there exists some ℓ such that $\mathrm{Hom}_{H_\ell}(\pi_\ell, \mathbb{1}) = 0$, then $\dim \mathrm{Hom}_{H_f}(\pi \otimes \mathcal{I}^0(1), \mathbb{1}) = 1$ and the natural restriction map $\mathrm{Hom}_{H_f}(\pi \otimes \mathcal{I}(1), \mathbb{1}) \to \mathrm{Hom}_{H_f}(\pi \otimes \mathcal{I}^0(1), \mathbb{1})$ is a bijection.*
- *If $\chi = 1$ and $\mathrm{Hom}_{H_\ell}(\pi_\ell, \mathbb{1}) \neq 0$ for all ℓ, then $\dim \mathrm{Hom}_{H_f}(\pi \otimes \mathcal{I}^0(1), \mathbb{1}) = \infty$.*

6.2 Hilbert modular forms

Suppose now that π is (the finite part of) a cuspidal automorphic representation, arising from a Hilbert modular cusp form of parallel weight $k + 2 \geqslant 2$, normalised so that ω_π has finite order.

Proposition 6.2. *Suppose π is not a twist of a base-change from $\mathrm{GL}_{2/\mathbf{Q}}$. Then, for any Dirichlet character τ, there exist infinitely many primes ℓ such that $\mathrm{Hom}_{H_\ell}(\pi_\ell \otimes \tau_\ell, \mathbb{1}) = 0$.*

Proof. See [6, Proposition 7.2.5]. □

There is a natural H_f-representation $\mathcal{O}^\times(Y)_{\mathbf{C}}$ of *modular units*, where Y is the infinite-level modular curve (the Shimura variety for GL_2). Note that this representation is smooth, but not admissible. It fits into a long exact sequence

$$0 \to (\mathbf{Q}^{\mathrm{ab}})^\times \otimes \mathbf{C} \to \mathcal{O}^\times(Y)_{\mathbf{C}} \to \mathcal{I}^0(1) \oplus \bigoplus_{\eta \neq 1} \mathcal{I}(\eta) \to 0,$$

with H_f acting on $(\mathbf{Q}^{\mathrm{ab}})^\times$ via the Artin reciprocity map of class field theory, and the sum is over all even Dirichlet characters η.

There is a canonical homomorphism, the *Asai–Flach map*, constructed in [9] (building on several earlier works such as [8]):

$$\mathcal{AF}^{[\pi,k]} : (\pi \otimes \mathcal{O}^\times(Y)_{\mathbf{C}})_{H_f} \to H^1(\mathbf{Q}, V^{\mathrm{As}}(\pi)^*(-k)),$$

where $V^{\mathrm{As}}(\pi)$ is the Asai Galois representation attached to π, and we have fixed an isomorphism $\overline{\mathbf{Q}}_p \cong \mathbf{C}$. The subscript H_f indicates H_f-coinvariants.

Theorem 6.3. *Suppose π is not a twist of a base-change from $\mathbf{Q}$. Then the Asai–Flach map factors through $\pi \otimes \mathcal{I}(\chi)$, and its image is contained in a 1-dimensional subspace of $H^1(\mathbf{Q}, V^{\mathrm{As}}(\pi)^*(-k))$.*

Proof. Using Proposition 6.2, we see that $\mathcal{AF}^{[\pi,k]}$ must vanish on $(\mathbf{Q}^{\mathrm{ab}})^\times \otimes \mathbf{C}$, so it factors through $\pi \otimes \mathcal{I}(\chi)$ if $\chi \neq 1$, or $\pi \otimes \mathcal{I}^0(\chi)$ if $\chi = 1$, where $\chi = (\omega_\pi|_{\mathbf{A}_f^\times})^{-1}$ as above. Using Proposition 6.1, combined with a second application of Proposition 6.2 if ω_π is trivial on $\mathbf{Q}$, the result follows. □

As in the GSp_4 case described in [11, Section 6.6], one can remove the dependency on the test data entirely: using zeta-integrals, we can construct a canonical basis vector $Z_{\mathrm{can}} \in \mathrm{Hom}(\pi_f \otimes \mathcal{I}(\chi), \mathbb{1})$, and define $\mathcal{AF}^{[\pi,k]}_{\mathrm{can}} \in H^1(\mathbf{Q}, V^{\mathrm{As}}(\pi)^*(-k))$ as the unique class such that

$$\mathcal{AF}^{[\pi,k]} = Z_{\mathrm{can}} \cdot \mathcal{AF}^{[\pi,k]}_{\mathrm{can}}.$$

We hope that this perspective may be useful in formulating and proving explicit reciprocity laws in the Asai setting.

Remark 6.4. The constructions of [9] also apply to other twists of $V^{\mathrm{As}}(\pi)$, and to Hilbert modular forms of non-parallel weight; but in these other cases the input data for the Asai–Flach map lies in an irreducible principal series representation of H_f, so the necessary multiplicity-one results are standard. (The delicate cases are those which correspond to near-central values of L-series.)

Acknowledgment: I am grateful to Giada Grossi and Dipendra Prasad for interesting conversations in connection with this paper, and especially to Nadir Matringe for his answer to a question of mine on MathOverflow, which provided the key to Theorem 4.1 (a). I would also like to thank Kei Yuen Chan, for pointing out the relevance of a result of Mœglin–Waldspurger recalled in Section 2.4; and the anonymous referee, for suggesting a much cleaner proof of Theorem 4.1 (b).

Funding: The author was supported by Royal Society University Research Fellowship UF160511.

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