Constrained Optimal Stopping Games

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.
Parts of this thesis have been previously published by the author in the following:

- G. Liang and H. Sun. Dynkin games with Poisson random intervention times. *SIAM Journal on Control and Optimization*, 57(4): 2962–2991, 2019.

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Research was performed in collaboration during the development of this thesis, but does not form part of the thesis:

- Chapter 2.5, Chapter 2.6

- Chapter 3.3

- Chapter 4.5, Chapter 4.6

- Chapter 5.4, Chapter 5.5, Chapter 5.6

Abstract

In this thesis, we consider four optimal stopping problems with stopping constraints. Chapter 2 introduces a new class of Dynkin games, where the two players are allowed to make their stopping decisions at a sequence of exogenous Poisson arrival times. The value function and the associated optimal stopping strategy are characterized by the solution of a backward stochastic differential equation. Furthermore, the chapter applies the model to study the optimal conversion and calling strategies of convertible bonds, and their asymptotics when the Poisson intensity goes to infinity. Chapter 3 generalizes the work in Chapter 2 from the risk-neutral criteria and common signal times for both players to the risk-sensitive criteria and two heterogeneous signal times. Chapter 4 considers a two-player zero-sum optimal switching games with stopping constraints. We prove the chain of inequalities involving the four values of the game, and the values of both the static and dynamic games exist in the case when the running and terminal rewards are separated. Chapter 5 studies a mixed stochastic control and constrained optimal stopping problem which models rollover debt decisions in an incomplete market. In addition to the rollover decisions, the creditor can also choose a control strategy to trade in risky assets correlated with the fundamental assets. In the case of exponential utility, we prove the complete characterization and obtain the exponential indifference bond price and its associated optimal mixed strategy.

Chapter 1

Introduction

Optimal stopping plays a classical and very important role in the field of financial mathematics, due to its various applications in finance and economics. The set-ups in the majority of previous works are either in continuous time where stopping times take any value in a certain time interval, or in discrete time where stopping times only take values in a pre-specified time grid. However, both set-ups have their own limitations: in the former formulation, no restriction is imposed on the class of admissible stopping times, which seems sometimes unrealistic; in the latter setting, to the best knowledge of the author, it seems impossible for us to obtain a closed form solution for optimal stopping in the case of discrete time, which is sometimes valuable for subsequent analysis.

To overcome the aforementioned limitations in both models, in this thesis, we consider a hybrid of continuous and discrete times and investigate some *constrained optimal stopping problems*, where the player(s) is (are) allowed to stop at a sequence(s) of random times generated by an exogenous Poisson process(es) serving as a signal process(es), which can be regarded as exogenous constraints on the players' ability to stop. On the one hand, the constraints may represent the liquidity effects, indicating the times at which the underlying stochastic processes are available to stop. On the other hand, the constraints can also be seen as information constraints. The player(s) is (are) allowed to make stopping decisions at all times, but is (are) only able to observe the underlying stochastic processes at Poisson arrival times.

This kind of constrained optimal stopping problems was first studied by Dupuis and Wang [25], where they used it to model perpetual American options exercised at exogenous Poisson arrival times. See also Lempa [55], Menaldi and Robin [61] and Hobson and Zeng [39] for further extensions of this type of optimal stopping models. From a different perspective, Liang [57] made a connection between constrained optimal stopping problems with penalized backward stochastic differential equations (BSDEs). The corresponding optimal switching (impulse control) models were studied by Liang and Wei [59], and by Menaldi and Robin [62] [63] with more general signal times and state spaces. All of the aforementioned references are concerned with single-player optimisation problems. To the best of our knowledge, multi-player optimal stopping problems with this type of constraints on stopping times, the topics under which Chapter 2-4 fall, have not been studied before. Chapter 5 studies single-player mixed stochastic control and optimal stopping problems with stopping constraints.

In this chapter, we first review constrained *single-player* optimal stopping problems to fix ideas, and then conclude the chapter with a more precise outline of the remainder of the thesis.

1.1 Constrained single-player optimal stopping problems

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $\{T_i\}_{i\geq 0}$ be the arrival times of an independent Poisson process with intensity λ and minimal augmented filtration $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$. Denote the smallest filtration generated by $\mathbb{F}$ and $\mathbb{H}$ as $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Without loss of generality, we also assume that $T_0 = 0$ and $T_\infty = \infty$. Let T be a fixed finite horizon representing the terminal time of the game, and $M : \Omega \mapsto \mathbb{N}$ be an integer-valued random variable such that T_M is the next Poisson arrival time following T, i.e. $M(\omega) := \sum_{i\geq 1} i \mathbb{1}_{\{T_{i-1}(\omega)\leq T(\omega)<T_i(\omega)\}}$. Let $f : \Omega \times [0, T] \to \mathbb{R}$ be an $\mathbb{F}$-progressively measurable process, $S : \Omega \times [0, T] \to \mathbb{R}$ be a continuous $\mathbb{F}$-progressively measurable process, and $\xi : \Omega \to \mathbb{R}$ be an $\mathcal{F}_T$-measurable random variable (all of them satisfying suitable integrability conditions).

Consider the following *constrained single-player optimal stopping problem*

$$v^\lambda = \sup_{\tau \in \mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^{\tau\wedge T} f_s\, ds + S_\tau \mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau\geq T\}}\right]$$

where the control set of the player is defined by

$$\mathcal{R}_{T_i}(\lambda) = \{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega) = T_N(\omega) \text{ where } i \leq N \leq M(\omega)\} \tag{1.1.1}$$

for any integer $i \geq 0$. Liang [57] proved that

$$v^\lambda = V_0^\lambda,$$

and the optimal stopping time is given by

$$\tau^{\lambda,*} = \inf\{T_N \geq T_1 : V_{T_N}^\lambda \leq S_{T_N}\} \wedge T_M,$$

where V^λ is the first component of the solution to the following BSDE

$$V_t^\lambda = \xi + \int_t^T \left[f_s + \lambda \left(S_s - V_s^\lambda \right)^+ \right] ds - \int_t^T Z_s^\lambda \, dW_s \tag{1.1.2}$$

with x^+ denoting the positive part of any real number x, i.e. $x^+ = \max\{x, 0\}$.

Compared to standard single-player optimal stopping problems, there are two new features of the above constrained single-player optimal stopping problems. First, there is a control constraint in the sense that only stopping at Poisson arrival times is allowed. Second, the player is not allowed to stop at the initial starting time. Instead, the player is only allowed to stop from the first Poisson time onwards.

A natural question for this formulation is: how does it differ from a *standard single-player optimal stopping problem*? Indeed, a connection can be established between the constrained problem and the standard problem, where the set-up is the same as above, except the control set is replaced with $\mathcal{R}_0$, which denotes the set of $\mathbb{F}$-stopping times valued in $[0, T]$:

$$v = \sup_{\tau \in \mathcal{R}_0} \mathbb{E} \left[\int_0^{\tau \wedge T} f_s \, ds + S_\tau \mathbb{1}_{\{\tau < T\}} + \xi \mathbb{1}_{\{\tau \geq T\}} \right].$$

El-Karoui et al [28] proved that $v = V_0$ and the optimal stopping time is given by $\tau^* = \inf\{t \geq 0 : V_t = S_t\} \wedge T$, where V is the first component of the solution to the following reflected BSDE with the reflecting barrier S:

$$V_t = \xi + \int_t^T f_s \, ds + \int_t^T dK_s - \int_t^T Z_s \, dW_s \tag{1.1.3}$$

for $t \in [0, T]$, under the constraints (i) $V_t \geq S_t$ for $0 \leq t \leq T$; (ii) $\int_0^T (V_t - S_t) \, dK_t = 0$.

An interesting observation is that BSDE (1.1.2) is often used to construct the solution of the reflected BSDE (1.1.3). Intuitively, when V^λ falls below S, there will be a penalty $\lambda \left(S_s - V_s^\lambda \right)$ incurred, so BSDE (1.1.2) is also refereed to as the penalized equation. Under suitable integrability conditions, El-Karoui et al [28] proved that

$$\lim_{\lambda \uparrow \infty} \mathbb{E} \left[\sup_{t \in [0,T]} |V_t^\lambda - V_t|^2 + \int_0^T |Z_t^\lambda - Z_t|^2 \, dt + \sup_{t \in [0,T]} |K_t^\lambda - K_t|^2 \right] = 0$$

where $K_t^\lambda = \int_0^t \lambda \left(S_s - V_s^\lambda \right)^+ ds$, and thus, the values of the two single-player optimal stopping problems are related by

$$v = \lim_{\lambda \uparrow \infty} v^\lambda.$$

1.2 Outline of the thesis

Dynkin games, as a generalization of single-player optimal stopping problems, are two-player zero-sum game on stopping times, where two players determine their optimal stopping times as their strategies. In *Chapter 2*, we consider a new class of Dynkin games, which we call as *constrained Dynkin games*, where the two players are allowed to make their stopping decisions σ and τ at a sequence of exogenous Poisson arrival times, in order to minimize/maximize the expected value of some payoff function $R(\sigma, \tau)$, i.e.

$$\begin{aligned}\overline{v}^{\lambda} &= \inf_{\sigma \in \mathcal{R}_{T_1}(\lambda)} \sup_{\tau \in \mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[R(\sigma, \tau)\right], \\ \underline{v}^{\lambda} &= \sup_{\tau \in \mathcal{R}_{T_1}(\lambda)} \inf_{\sigma \in \mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[R(\sigma, \tau)\right],\end{aligned}$$

where the control set $\mathcal{R}_{T_1}(\lambda)$ is in (1.1.1). The two value functions $\overline{v}^{\lambda}$ and $\underline{v}^{\lambda}$ are called the upper and lower value of the game, where the names are justified by the following inequality

$$\overline{v}^{\lambda} \geq \underline{v}^{\lambda},$$

because, on the upper (resp. lower) value, the maximizing (resp. minimizing) player is given an advantage by being allowed to look at the minmizing (resp. maximizing) player's stopping strategy before choosing his/her own. We prove the value of the constrained Dynkin game exists (i.e. $\overline{v}^{\lambda} = \underline{v}^{\lambda}$) and characterize the value function and the associated optimal stopping strategy by the solution of a penalized BSDE, which is widely used to approximate the solution of a reflected BSDE with double obstacles and the corresponding continuous time Dynkin game.

We also apply the constrained Dynkin game to study the optimal conversion and calling strategies of convertible bonds. On the one hand, the bondholder decides whether to keep the bond to collect coupons or to convert it to the firm's stocks in order to maximize the bond value. On the other hand, the issuing firm has the right to call the bond, and presumably acts to maximize the equity value of the firm by minimizing the bond value. This creates a two-player, zero-sum Dynkin game.

Chapter 3 generalizes the above model of constrained Dynkin games in two aspects: First, it takes into consideration of both players' attitudes towards risks by replacing the linear expectation $\mathbb{E}[\cdot]$ with the nonlinear expectation

$$\tilde{\mathbb{E}}\left[\cdot\right] := g^{-1}(\mathbb{E}\left[g(\cdot)\right])$$

for some strictly increasing function g as a risk-sensitive function. To the best of our knowledge, the study of risk-sensitive Dynkin games is still lacking, no matter with or without constraints on stopping time strategies. The current chapter offers a first step to understand risk-sensitive Dynkin games. Second, one limitation of the constrained Dynkin game model in Chapter 2 is that both players face the same stopping constraint, which seems sometimes unrealistic. In this chapter, the control constraints for both players are modelled to be different in the sense that they are allowed to stop at two heterogeneous sequences of Poisson arrival times. Due to the introduction of constraints on stopping times and risk-sensitive criteria, we call this new class of Dynkin games as *constrained risk-sensitive Dynkin games.* We prove the value of the constrained risk-sensitive Dynkin game exists and characterize the value function and the associated optimal stopping strategy by the solution of a BSDE. Furthermore, the chapter establishes a connection of constrained risk-sensitive Dynkin games with a class of stochastic differential games via Krylov's randomized stopping technique.

Optimal switching is a generalisation of optimal stopping, where one or more agents determine their optimal sequence of times to switch a system's operational modes, with various applications in economics and finance. In *Chapter 4*, we consider a new type of two-player zero-sum optimal switching games, which we call as *constrained optimal switching games*, where two players are only allowed to switch at two heterogeneous exogenous sequences of Poisson arrival times, in order to minimise/maximise some payoff function. The payoff function also includes the switching payments to the othe player when they make their switching decisions. Similar to the Dynkin game setting in the previous two chapters, to complete the description of the game, it is necessary to specify the information available to each player.

The lower (resp. upper) static value of the game is defined by letting the maximizing (resp. minimizing) player make the decisions first, followed by the minimizing (resp. maximizing) player making the decisions. These two values are called static, because one of the players is able to know the decisions of the other player for the entire duration of the game, where some information about the future is being revealed to the player given advantage. This motivates us to define the dynamic version of the game by introducing the notion of non-anticipating strategies, where one of the players is still given advantage but the information about the other player's decisions are revealed in a dynamic way. The main result of this chapter is the chain of inequalities involving the above four values of the game. We prove the values of both the static and dynamic games exist in the case when the running and terminal rewards are separated. At the end of the chapter, we apply the constrained optimal switching games to study the duopolistic competition in resource extraction, and give a complete description of the structure of switching regions.

Chapter 5 extends the theory of constrained optimal stopping problems in a different direction: we study a *mixed stochastic control and constrained optimal stopping problem* which models a risk-averse creditor's decisions, in an incomplete market, over whether to roll over or to withdraw the funding at a sequence of rollover dates, which are modelled by a sequence of Poisson arrival times. In addition to the rollover decisions, the creditor can also choose a control strategy to trade in risky assets correlated with the fundamental assets. The first main result of this chapter is the verification theorem characterizing the value function of the problem and its associated optimal mixed strategy in terms of the solution of a penalized partial differential equation (PDE). In the case of exponential utility, we prove the complete characterization and obtain the exponential indifference bond price and its associated optimal mixed strategy. Furthermore, we investigate the impacts of parameter values on the bond price and conduct some numerical experiments to examine the shapes of the stopping and continuation regions of the problem in an incomplete market. An interesting observation is that both regions are swapped over for different parameter values.

Chapter 2

Dynkin Games with Poisson Random Intervention Times

2.1 Introduction

Dynkin games are games on stopping times, where two players determine their optimal stopping times as their strategies. The game was first introduced by Dynkin [26], and later generalized by Neveu [68] in 1970s. In this game, two players observe two stochastic processes, say L and U, and their aims are to maximize/minimize the expected value of the payoff

$$R(\sigma, \tau) = L_\tau \mathbb{1}_{\{\tau \le \sigma\}} + U_\sigma \mathbb{1}_{\{\sigma < \tau\}}$$

over stopping times τ and σ, respectively. In a discrete-time setting, under the assumption that $U \ge L$, Neveu proved the existence of the game value and its associated optimal strategy.

Since then, there has been a considerable development of Dynkin games. The corresponding continuous time models were developed, among others, by Bismut [8], Alario-Nazaret et al [1], Lepeltier and Maingueneau [56] and Morimoto [65]. In order to relax the condition $U \ge L$, Yasuda [80] proposed to extend the class of strategies to randomized stopping times, and proved that the game value exists under merely an integrability condition. Rosemberg et al [71], Touzi and Vielle [78] and Laraki and Solan [52] further extended his work in this direction. If the two players in the game are with asymmetric payoffs, then it gives rise to a nonzero-sum Dynkin game. See, for example, Hamadene and Zhang [33] and more recently De Angelis et al [19] with more references therein. A robust version of Dynkin games can be found in Bayraktar and Yao [5] if the players are ambiguous about their probability model.

The set-ups in all the aforementioned works are either in continuous time where stopping times take any value in a certain time interval, or in discrete time where stopping times only take values in a pre-specified time grid. In this chapter, we consider a hybrid of continuous and discrete times, and introduce a new type of Dynkin games, where both players are allowed to stop at a sequence of random times generated by an exogenous Poisson process serving as a signal process. We call such a Dynkin game a *constrained Dynkin game*.

The underlying Poisson process can be regarded as an exogenous constraint on the players' abilities to stop, so it may represent liquidity effects, indicating the times both players are allowed to stop the game freely and at no other time can they exit the game. In the case of hedging some derivative on a thinly-traded asset, the adjustments to the hedge is only allowed when someone in the market is willing to buy or sell the asset. To simplify the problem, we model such a liquidation shock (someone is prepared to buy or sell the asset in the above example) as the arrival times of an exogenous Poisson process, and players only make their decisions when such a shock arrives. Moreover, the Poisson process can also be seen as an information constraint. The players are allowed to make their stopping decisions at all times, but they are only able to observe the underlying stochastic processes at Poisson times.

Our main result is Theorem 2.2.4, which characterizes the value of the constrained Dynkin game and its associated optimal stopping strategy in terms of the solution of a penalized BSDE. The latter is widely used to approximate the solution of a reflected BSDE with double obstacles and the corresponding continuous time Dynkin game. The main idea to solve the constrained Dynkin game is to introduce a family of auxiliary games (see (2.3.13)-(2.3.14)), for which standard dynamic programming principle holds. Furthermore, following from the convergence of penalized BSDE to reflected BSDE (see, for example, [16]) and the penalized BSDE characterization (2.2.6) of the constrained Dynkin game, we also make a connection with standard Dynkin games in continuous time. That is, the value of the constrained Dynkin game will converge to the value of its continuous time counterpart when the Poisson intensity goes to infinity.

We then apply the constrained Dynkin game to study convertible bonds. In a convertible bond, the bondholder decides whether to keep the bond to collect coupons or to convert it to the firm's stocks. She will choose a conversion strategy to maximize the bond value. On the other hand, the issuing firm has the right to call the bond, and presumably acts to maximize the equity value of the firm by minimizing the bond value. This creates a two-person, zero-sum Dynkin game.

Traditionally, convertible bond models often assume that both the bond holder and the firm are allowed to stopped at any stopping time adapted to the firm's fundamental (such as its stock prices). In reality, there may exist some liquidation constraint as an external shock, and both players only make their decisions when such a shock arrives. We model such a liquidation shock as the arrival times of an exogenous Poisson process, and thus the convertible bond model falls into the framework of constrained Dynkin games. A similar idea has first appeared in the modeling of debt run problems (see [58]), which can be formulated as optimal stopping problems with Poisson arrival times.

Furthermore, in a Markovian setting, we derive explicitly the optimal stopping strategies for both the bondholder and the firm. We show that the optimal stopping rules of the two players depend on the relationship between the coupon rate c, dividend rate q, interest rate r and surrender price K. For the firm, its optimal stopping strategy depends on the relationship between c, $\frac{r+\lambda}{q+\lambda}qK$ and rK. If $c \geq rK$, it is optimal for the firm to call the bond back as soon as possible; if $c \leq \frac{r+\lambda}{q+\lambda}qK$, the firm will postpone the calling time of the bond as late as possible; if $\frac{r+\lambda}{q+\lambda}qK < c < rK$, the firm's calling strategy is determined by an optimal calling boundary, which is obtained by solving a free boundary problem. In contrast, the investor's optimal stopping strategy depends on the relationship between c and $\frac{r+\lambda}{q+\lambda}qK$. If $c > \frac{r+\lambda}{q+\lambda}qK$, the investor will delay her conversion time as late as possible; if $c \leq \frac{r+\lambda}{q+\lambda}qK$, her conversion strategy is determined by an optimal conversion boundary.

Turning to the literature, the optimal stopping problem with constraints on the stopping times was introduced by Dupuis and Wang [25], when they used it to model perpetual American options exercised at exogenous Poisson arrival times. See also Lempa [55] and Menaldi and Robin [61] for further extensions of this type of optimal stopping problems. On the other hand, Liang [57] made a connection between such kind of optimal stopping problems with penalized BSDE. The corresponding optimal switching (impulse control) problems were studied by Liang and Wei [59] and more recently by Menaldi and Robin [62] with more general signal times and state spaces.

The study of convertible bonds dated back to Brennan and Schwartz [11] and Ingersoll [44]. However, it was Sirbu et al [73] who first analyzed the optimal strategy of perpetual convertible bonds (see also Sirbu and Shreve [74] for the finite horizon counterpart). They reduced the problem from a Dynkin game to an optimal stopping problem, and discussed when call precedes conversion and vice versa. Several more realistic features of convertible bonds have been taken into account since then. For example, Bielecki et al [7] considered the problem of the decomposition of a convertible bond into bond component and option component. Crepey and Rahal [15] studied the convertible bond with call protection, which is typically path dependent. Chen et al [13] considered the tax benefit and bankruptcy cost for convertible bonds. For a complete literature review, we refer to the aforementioned papers with references therein.

The chapter is organized as follows. Section 2.2 contains the problem formulation and main result, with its proof provided in Section 2.3. In Section

2.4, we establish a connection with standard Dynkin games. In Section 2.5, we apply the constrained Dynkin game to study the convertible bonds in a Markovian setting, and derive the explicit optimal stopping strategies and the corresponding free boundaries under various situations. Section 2.6 carries out an asymptotic analysis of the game values and the free boundaries when the Poisson intensity goes to infinity.

2.2 Constrained Dynkin games

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $\{T_i\}_{i\geq 0}$ be the arrival times of an independent Poisson process with intensity λ and minimal augmented filtration $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$. Denote the smallest filtration generated by $\mathbb{F}$ and $\mathbb{H}$ as $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Without loss of generality, we also assume that $T_0 = 0$ and $T_\infty = \infty$.

Let T be a finite $\mathbb{F}$-stopping time representing the random terminal time of the game, and $M : \Omega \mapsto \mathbb{N}$ be an integer-valued random variable such that T_M is the next Poisson arrival time following T, i.e.

$$M(\omega) := \sum_{i\geq 1} i \mathbb{1}_{\{T_{i-1}(\omega)\leq T(\omega)<T_i(\omega)\}}.$$

For any integer $i \geq 0$, let us define the control set of both players as follows

$$\mathcal{R}_{T_i}(\lambda) = \{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega) = T_N(\omega) \text{ where } i \leq N \leq M(\omega)\}$$

where the subscript T_i in $\mathcal{R}_{T_i}(\lambda)$ represents the smallest stopping time that is allowed to choose, and λ represents the intensity of the underlying Poisson process.

Consider the following *constrained Dynkin game*, where two players choose their respective stopping times $\sigma, \tau \in \mathcal{R}_{T_1}(\lambda)$ in order to minimize/maximize the expected value of the payoff

$$\begin{aligned} R(\sigma, \tau) = & \int_0^{\sigma\wedge\tau\wedge T} e^{-rs} f_s \, ds \\ & + e^{-rT}\xi \mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + e^{-r\tau} L_\tau \mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} + e^{-r\sigma} U_\sigma \mathbb{1}_{\{\sigma<T,\sigma<\tau\}} \end{aligned} \quad (2.2.1)$$

where $r > 0$ is the discount rate, and f, as a real-valued $\mathbb{F}$-progressively measurable process, is the running payoff. The terminal payoff is U if σ happens firstly, L if τ happens firstly or σ and τ happen simultaneously, and ξ otherwise, where L and U are two real-valued $\mathbb{F}$-progressively measurable processes, and ξ is a real-valued $\mathcal{F}_T$-measurable random variable.

Let us define the upper and lower values of the constrained Dynkin game

$$\overline{v}^{\lambda} = \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right], \tag{2.2.2}$$

$$\underline{v}^{\lambda} = \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right]. \tag{2.2.3}$$

The game (2.2.2)-(2.2.3) is said to have value v^{λ} if $v^{\lambda} = \overline{v}^{\lambda} = \underline{v}^{\lambda}$. It is standard to show that if there exists a saddle point $(\sigma^*,\tau^*) \in \mathcal{R}_{T_1}(\lambda)\times\mathcal{R}_{T_1}(\lambda)$ such that $\mathbb{E}\left[R(\sigma^*,\tau)\right] \leq \mathbb{E}\left[R(\sigma^*,\tau^*)\right] \leq \mathbb{E}\left[R(\sigma,\tau^*)\right]$ for every $(\sigma,\tau) \in \mathcal{R}_{T_1}(\lambda)\times\mathcal{R}_{T_1}(\lambda)$, then the value of this game exists and equals $v^{\lambda} = \mathbb{E}\left[R(\sigma^*,\tau^*)\right].$

There are two new features of the above constrained Dynkin game. First, there is a control constraint in the sense that only stopping at Poisson arrival times is allowed. Second, the players are not allowed to stop at the initial starting time. Instead, they are only allowed to stop from the first Poisson time onwards.

We also consider an auxiliary game related to the above constrained Dyknin game by replacing the control set in (2.2.2)-(2.2.3) with $\mathcal{R}_{T_0}(\lambda)$, so the players are also allowed to stop at the initial starting time. That is

$$\overline{\hat{v}}^{\lambda} = \inf_{\sigma\in\mathcal{R}_{T_0}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_0}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right], \tag{2.2.4}$$

$$\underline{\hat{v}}^{\lambda} = \sup_{\tau\in\mathcal{R}_{T_0}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_0}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right]. \tag{2.2.5}$$

Note that the difference between (2.2.4)-(2.2.5) and (2.2.2)-(2.2.3) is that the former is allowed to stop at the initial starting time $T_0 = 0$, while the latter not. In other words, the players in (2.2.4)-(2.2.5) first make their stopping decisions and then move forward, while in (2.2.2)-(2.2.3) they first move forward and then make their decisions. We shall show that if the game (2.2.2)-(2.2.3) has value v^{λ}, then the value of (2.2.4)-(2.2.5) also exists and is given by $\hat{v}^{\lambda} = \min\{U_0, \max\{v^{\lambda}, L_0\}\}$, so the key is to solve the game (2.2.2)-(2.2.3).

2.2.1 Main result of this chapter

To solve the above constrained Dynkin games, we introduce the following BSDE with a random terminal time T:

$$V^{\lambda}_{t\wedge T} = \xi + \int_{t\wedge T}^{T} \left[f_s + \lambda\left(L_s - V^{\lambda}_s\right)^{+} - \lambda\left(V^{\lambda}_s - U_s\right)^{+} - rV^{\lambda}_s \right] ds - \int_{t\wedge T}^{T} Z^{\lambda}_s \, dW_s \tag{2.2.6}$$

for $t \geq 0$. And also we set $V_t \equiv \xi$ for $t \geq T$. Note that the above BSDE (2.2.6) is often used to construct the solution of a reflected BSDE with two reflecting barriers L and U (cf. (2.4.3)). Intuitively, when V^λ falls below L (or goes above U), there will be a penalty $\lambda(L - V^\lambda)$ (or $\lambda(V^\lambda - U)$) incurred, so BSDE (2.2.6) is also refereed to as the penalized equation.

For later use, let us introduce the following spaces: for any given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$,

- $\mathbb{L}^{2,n}_\alpha$: $\mathcal{F}_T$-measurable random variables $\xi : \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[e^{2\alpha T}||\xi||^2\right] < \infty$,
- $\mathbb{H}^{2,n}_\alpha$: $\mathbb{F}$-progressively measurable processes $\varphi : [0,T] \times \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[\int_0^T e^{2\alpha s}||\varphi_s||^2\, ds\right] < \infty$,
- $\mathbb{S}^{2,n}_\alpha$: $\mathbb{F}$-progressively measurable processes $\varphi : [0,T] \times \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[\sup_{s\in[0,T]} e^{2\alpha s}||\varphi_s||^2\right] < \infty$,

where $||\cdot||$ is the Euclidian norm and we denote $\mathbb{L}^{2,n}_0$, $\mathbb{H}^{2,n}_0$ and $\mathbb{S}^{2,n}_0$ by $\mathbb{L}^{2,n}$, $\mathbb{H}^{2,n}$ and $\mathbb{S}^{2,n}$ for the ease of notation.

Remark 2.2.1 *For the convenience of the reader, we give a short introduction of BSDEs, and refer to the lecture notes [9] by Bouchard for the general theory and its details. These equations were first introduced by Pardoux and Peng in their pioneering work [69]. On a Brownian filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$*, given a* $\mathbb{R}$*-valued random variable* $\xi \in \mathbb{L}^{2,1}$ *and* $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ *(called the driver of the BSDE), a solution to the BSDE*

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s \tag{2.2.7}$$

is a pair of $\mathcal{F}_t$*-adapted processes* (Y, Z)*, typically in* $\mathbb{S}^{2,1} \times \mathbb{H}^{2,d}$*, such that (2.2.7) holds. As opposed to forward SDEs, we prescribe its terminal condition* $Y_T = \xi$ *rather than its initial condition* $Y_0 \in \mathbb{S}^{2,1} \times \mathbb{H}^{2,d}$*.*

The key idea of the Z *component of the solution is to ensure that the process* Y *is adapted. For simplicity, let us consider the case* $g \equiv 0$*. Then, a solution* (Y, Z) *to BSDE (2.2.7) must satisfy*

$$Y_t = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s\, dW_t,$$

where Z *is uniquely given by the martingale representation theorem*

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_s\, dW_s.$$

Assumption 2.2.2 *For $t \in [0, T]$, $L_t \leq U_t$, a.s, and moreover, (i) when T is an unbounded stopping time, the running payoff f and the terminal payoffs L, U and ξ are all bounded; (ii) when T is a bounded stopping time, the running payoff $f \in \mathbb{H}^{2,1}$ and the terminal payoffs $L \in \mathbb{S}^{2,1}$, $U \in \mathbb{S}^{2,1}$ and $\xi \in \mathbb{L}^{2,1}$.*

The assumption $L \leq U$ is crucial to the existence of the game value. On the other hand, the conditions (i) and (ii) are to guarantee the existence and uniqueness of the solution to BSDE (2.2.6), which will in turn be used to construct the game value and its associated optimal stopping strategy.

Proposition 2.2.3 *Suppose that Assumption 2.2.2 holds. Then, there exists a unique solution (V^λ, Z^λ) to BSDE (2.2.6). Moreover, (i) when T is an unbounded stopping time, V^λ is continuous and bounded, and Z^λ belongs to $\mathbb{H}^{2,d}_{-r}$; (ii) when T is a bounded stopping time, the solution belongs to $\mathbb{S}^{2,1} \times \mathbb{H}^{2,d}$.*

The proof essentially follows from Theorem 3.3 in [12] (when T is unbounded) and Theorem 4.1 in [69] (when T is bounded) , so we omit its proof and refer to [12] and [69] for the details. We are now in a position to state the main result of this chapter.

Theorem 2.2.4 *Suppose that Assumption 2.2.2 holds. Let (V^λ, Z^λ) be the unique solution to BSDE (2.2.6). Then, the value of the constrained Dynkin game (2.2.2)-(2.2.3) exists and is given by $v^\lambda = \overline{v}^\lambda = \underline{v}^\lambda = V_0^\lambda$. The corresponding optimal stopping strategy is given by*

$$\begin{cases} \sigma^*_{T_1} = \inf\{T_N \geq T_1 : V^\lambda_{T_N} \geq U_{T_N}\} \wedge T_M; \\ \tau^*_{T_1} = \inf\{T_N \geq T_1 : V^\lambda_{T_N} \leq L_{T_N}\} \wedge T_M. \end{cases} \tag{2.2.8}$$

Moreover, the value of the Dynkin game (2.2.4)-(2.2.5) also exists and is given by $\hat{v}^\lambda = \min\{U_0, \max\{v^\lambda, L_0\}\}$, with the associated optimal stopping strategy $\sigma^*_{T_0}$ and $\tau^*_{T_0}$.

2.2.2 Examples

Theorem 2.2.4 solves a wide class of problems in a unified manner, covering from Markovian to non-Markovian situations and from fixed to random finite horizons. In the one-dimensional homogenous Markovian setting, there usually exists a threshold strategy. For this, we will discuss a specific convertible-bond example in Section 2.5. In the rest of the section, we list several path-dependent examples, which are difficult to dealt with under Markovian framework (at least it needs a case-by-case study) but covered by Theorem 2.2.4.

(i) *Path-dependent payoffs L and U.* Let T be fixed so it is a constant stopping time and S be a one-dimensional positive diffusion process adapted to $\mathbb{F}$. For $\delta > 0$, consider an Israeli option written on S with maturity T, where the holder may exercise to get a normal claim but the writer is punished by an amount δS for annulling the contract early (see [49]). The payoffs L and U may take the form $L_t = \max\{m, S_t^*\}$ and $U_t = \max\{m, S_t^*\} + \delta S_t$ for $m > S_0$ and $S_t^* = \sup_{0\le u\le t} S_u$. This is so called Israeli Russian option. For $L_t = \int_0^t S_u du$ and $U_t = \int_0^t S_u du + \delta S_t$, it is called Israeli integral option (see [4]). Under mild integrability assumption on S as in Assumption 2.2.2, Theorem 2.2.4 shows that the values of both Israeli options exist and the associated optimal strategies can be characterized via the solution to (2.2.6).

(ii) *Path-dependent stopping time T.* Stopping times are widely used in insurance as indicators of a variety of risks. Let S be a one-dimensional positive diffusion process adapted to $\mathbb{F}$. We may consider the following stopping times as the terminal time of the game: drawdown stopping time $T = \inf\{t \ge 0 : S_t^* - S_t \ge m\}$ for $m \ge 0$; occupation stopping time $T = \inf\{t \ge m : \int_0^t 1_{\{S_u\in A\}} du \ge m\}$ for $A \subset \mathbb{R}_+$. Note that unlike the standard first-passage-time (see θ^λ in Section 2.5), both types of path-dependent stopping times need tailor-made analysis under Markovian framework, but can be covered by Theorem 2.2.4 in a unified manner.

2.3 Proof of Theorem 2.2.4

We first give an equivalent formulation of the constrained Dynkin game (2.2.2)-(2.2.3). Given the arrival time T_i, define pre-T_i σ-field

$$\mathcal{G}_{T_i} = \left\{ A \in \bigvee_{s\ge 0} \mathcal{G}_s : A \cap \{T_i \le s\} \in \mathcal{G}_s \text{ for } s \ge 0 \right\}$$

and $\tilde{\mathbb{G}} = \{\mathcal{G}_{T_i}\}_{i\ge 0}$. Let us define the following discounted processes

$$\tilde{L}_t = e^{-rt} L_t + \int_0^t e^{-rs} f_s \, ds, \tag{2.3.1}$$

$$\tilde{U}_t = e^{-rt} U_t + \int_0^t e^{-rs} f_s \, ds, \tag{2.3.2}$$

$$\tilde{\xi} = e^{-rT} \xi + \int_0^T e^{-rs} f_s \, ds, \tag{2.3.3}$$

and thus, the upper and lower values of the constrained Dynkin game (2.2.2)-(2.2.3) can be rewritten in the following form

$$\overline{v}^\lambda = \overline{q}^\lambda := \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)\right], \tag{2.3.4}$$

$$\underline{v}^\lambda = \underline{q}^\lambda := \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)\right], \tag{2.3.5}$$

where the modified payoff function is given by

$$\tilde{R}(\sigma,\tau) = \tilde{\xi}\mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + \tilde{L}_\tau\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} + \tilde{U}_\sigma\mathbb{1}_{\{\sigma<T,\sigma<\tau\}}. \tag{2.3.6}$$

Thus, to prove Theorem 2.2.4, it is equivalent to show that

$$Q_0^\lambda = q^\lambda = \overline{q}^\lambda = \underline{q}^\lambda,$$

and the optimal stopping strategy is given by

$$\begin{cases} \sigma_{T_1}^* = \inf\{T_N \geq T_1 : Q_{T_N}^\lambda \geq \tilde{U}_{T_N}\} \wedge T_M, \\ \tau_{T_1}^* = \inf\{T_N \geq T_1 : Q_{T_N}^\lambda \leq \tilde{L}_{T_N}\} \wedge T_M, \end{cases} \tag{2.3.7}$$

where Q^λ is given by

$$Q_t^\lambda = e^{-rt\wedge T}V_t^\lambda + \int_0^{t\wedge T} e^{-rs}f_s\,ds, \tag{2.3.8}$$

with V_t^λ being the first component of the solution to BSDE (2.2.6). Note that, for $t \geq T$,

$$Q_t^\lambda = e^{-rT}\xi + \int_0^T e^{-rs}f_s\,ds = \tilde{\xi}. \tag{2.3.9}$$

To prove the above assertions, we start with the following lemma.

Lemma 2.3.1 *Suppose that Assumption 2.2.2 holds. Then, for any* $1 \leq n \leq M+1$, $Q_{T_{n-1}}^\lambda$, *which is given by (2.3.8), is the unique solution of the recursive equation*

$$Q_{T_{n-1}}^\lambda = \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_n\geq T\}} + \min\{\tilde{U}_{T_n}, \max\{Q_{T_n}^\lambda, \tilde{L}_{T_n}\}\}\mathbb{1}_{\{T_n<T\}}\middle|\,\mathcal{G}_{T_{n-1}}\right]. \tag{2.3.10}$$

Proof. It is obvious (2.3.10) holds for $n = M+1$. In the following, we only focus on the case when $1 \leq n \leq M$. Applying Itô's formula to $\alpha_t Q_t^\lambda$, where $\alpha_t = e^{-\lambda t}$, we obtain that

$$\alpha_{t\wedge T}Q_{t\wedge T}^\lambda = \alpha_T\tilde{\xi} + \int_{t\wedge T}^T \alpha_s\lambda F_s(Q_s^\lambda)\,ds - \int_{t\wedge T}^T \alpha_s\tilde{Z}_s^\lambda\,dW_s,$$

for $t \geq 0$, where

$$F_s(Q_s^\lambda) := Q_s^\lambda + (\tilde{L}_s - Q_s^\lambda)^+ - (Q_s^\lambda - \tilde{U}_s)^+ = \min\{\tilde{U}_s, \max\{Q_s^\lambda, \tilde{L}_s\}\}$$

under the condition $L \leq U$ (so $\tilde{L} \leq \tilde{U}$). Consequently, for $1 \leq n \leq M$,

$$\begin{aligned} Q^\lambda_{T_{n-1}} &= \frac{\alpha_T}{\alpha_{T_{n-1}}}\tilde{\xi} + \int_{T_{n-1}}^T \frac{\alpha_s}{\alpha_{T_{n-1}}}\lambda F_s(Q^\lambda_s)\,ds - \int_{T_{n-1}}^T \frac{\alpha_s}{\alpha_{T_{n-1}}}\tilde{Z}^\lambda_s\,dW_s \\ &= \mathbb{E}\left[e^{-\lambda(T-T_{n-1})}\tilde{\xi} + \int_{T_{n-1}}^T e^{-\lambda(s-T_{n-1})}\lambda F_s(Q^\lambda_s)\,ds \middle| \mathcal{G}_{T_{n-1}}\right], \end{aligned}$$

where the second equality holds by taking the conditional expectation with respect to $\mathcal{G}_{T_{n-1}}$.

On the other hand, we use the conditional density $\lambda e^{-\lambda(x-T_{n-1})}\,dx$ of T_n to calculate the right-hand-side of (2.3.10):

$$\begin{aligned} &\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_n \geq T\}} + \min\{\tilde{U}_{T_n}, \max\{Q^\lambda_{T_n}, \tilde{L}_{T_n}\}\}\mathbb{1}_{\{T_n<T\}} \middle| \mathcal{G}_{T_{n-1}}\right] \\ =&\mathbb{E}\left[e^{-\lambda(T-T_{n-1})}\tilde{\xi} + \int_{T_{n-1}}^T \lambda e^{-\lambda(s-T_{n-1})}\min\{\tilde{U}_s, \max\{Q^\lambda_s, \tilde{L}_s\}\}\,ds \middle| \mathcal{G}_{T_{n-1}}\right], \end{aligned}$$

which proves (2.3.10) holds.

Since $Q^\lambda_{T_{n-1}}$ in (2.3.10) is solved recursively for $n = M+1, M, \cdots, 1$ and T_M is a finite $\mathbb{G}$-stopping time, it is obvious that the backward recursive equation (2.3.10) admits a unique solution, $Q^\lambda_{T_{n-1}}$, which is given by (2.3.8), is then the unique solution of the recursive equation (2.3.10) for $1 \leq n \leq M+1$. ■

As a direct consequence of Lemma 2.3.1, $\hat{Q}^\lambda$, which is defined by

$$\hat{Q}^\lambda_t := \tilde{\xi}\mathbb{1}_{\{t\geq T\}} + \min\{\tilde{U}, \max\{Q^\lambda, \tilde{L}\}\}\mathbb{1}_{\{t<T\}}, \tag{2.3.11}$$

where Q^λ is given by (2.3.8), satisfies the following recursive equation: For $1 \leq n \leq M+1$,

$$\hat{Q}^\lambda_{T_{n-1}} = \tilde{\xi}\mathbb{1}_{\{T_{n-1}\geq T\}} + \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\hat{Q}^\lambda_{T_n}\middle|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\}\mathbb{1}_{\{T_{n-1}<T\}} \tag{2.3.12}$$

which also admits a unique solution since we can calculate its solution backwards in a recursive way for $n = M+1, M, \cdots, 1$.

We will show that $\hat{Q}^\lambda_{T_{n-1}}$ is actually the value of an auxiliary constrained Dynkin game starting from T_{n-1}, whose upper and lower values are defined as

$$\overline{\hat{q}}^\lambda_{T_{n-1}} = \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_{n-1}}(\lambda)} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_{n-1}}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_{n-1}}\right], \tag{2.3.13}$$

$$\underline{\hat{q}}^\lambda_{T_{n-1}} = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_{n-1}}(\lambda)} \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_{n-1}}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_{n-1}}\right], \tag{2.3.14}$$

where $\tilde{R}(\sigma,\tau)$ is given by (2.3.6). The auxiliary constrained Dynkin game (2.3.13)-(2.3.14) is said to have value $\hat{q}^\lambda_{T_{n-1}}$ if $\hat{q}^\lambda_{T_{n-1}} = \overline{\hat{q}}^\lambda_{T_{n-1}} = \underline{\hat{q}}^\lambda_{T_{n-1}}$, and

$(\hat{\sigma}^*_{n-1}, \hat{\tau}^*_{n-1}) \in \mathcal{R}_{T_{n-1}}(\lambda) \times \mathcal{R}_{T_{n-1}}(\lambda)$ is called an optimal stopping strategy of the game if

$$\mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_{n-1}, \tau)|\mathcal{G}_{T_{n-1}}\right] \leq \mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_{n-1}, \hat{\tau}^*_{n-1})|\mathcal{G}_{T_{n-1}}\right] \leq \mathbb{E}\left[\tilde{R}(\sigma, \hat{\tau}^*_{n-1})|\mathcal{G}_{T_{n-1}}\right]$$

for every $(\sigma, \tau) \in \mathcal{R}_{T_{n-1}}(\lambda) \times \mathcal{R}_{T_{n-1}}(\lambda)$.

When $n = 1$, (2.3.13)-(2.3.14) corresponds to the auxiliary constrained Dynkin game (2.2.4)-(2.2.5). The difference between the auxiliary game and the original game is that the players first make their stopping decisions and then move forward in the former game, while in the latter game they first move forward and then make their decisions.

Lemma 2.3.2 *Suppose that Assumption 2.2.2 holds. Then, for any $1 \leq n \leq M+1$, the value of the auxiliary constrained Dynkin game starting from T_{n-1} (2.3.13)-(2.3.14) exists. Its value $\hat{q}^\lambda_{T_{n-1}}$ satisfies the recursive equation (2.3.12), namely,*

$$\hat{q}^\lambda_{T_{n-1}} = \tilde{\xi}\mathbb{1}_{\{T_{n-1}\geq T\}} + \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\hat{q}^\lambda_{T_n}\middle| \mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\}\mathbb{1}_{\{T_{n-1}<T\}}.$$

Hence, $\hat{q}^\lambda_{T_{n-1}} = \hat{Q}^\lambda_{T_{n-1}}$, where the latter is given by (2.3.11). The optimal stopping strategy of the auxiliary constrained Dynkin game (2.3.13)-(2.3.14) is given by

$$\begin{cases} \hat{\sigma}^*_{n-1} = \inf\{T_N \geq T_{n-1} : \hat{q}^\lambda_{T_N} = \tilde{U}_{T_N}\} \wedge T_M; \\ \hat{\tau}^*_{n-1} = \inf\{T_N \geq T_{n-1} : \hat{q}^\lambda_{T_N} = \tilde{L}_{T_N}\} \wedge T_M. \end{cases} \tag{2.3.15}$$

Proof. *Step 1.* We first show that, for $1 \leq n \leq M$, we have

$$\bar{\hat{q}}^\lambda_{T_{n-1}} = \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\bar{\hat{q}}^\lambda_{T_n}|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\}. \tag{2.3.16}$$

Indeed, for $1 \leq n \leq M$, taking conditional expectation on $\mathcal{G}_{T_n}$ yields that

$$\begin{aligned} \bar{\hat{q}}^\lambda_{T_{n-1}} = &\operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_{n-1}}(\lambda)} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_{n-1}}(\lambda)} \mathbb{E}\Big[\tilde{\xi}\mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + \tilde{L}_\tau\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} \\ &+ \tilde{U}_\sigma\mathbb{1}_{\{\sigma<T,\sigma<\tau\}}|\mathcal{G}_{T_{n-1}}\Big] \\ =&\operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_{n-1}}(\lambda)} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_{n-1}}(\lambda)} \mathbb{E}\Big[\tilde{L}_{T_{n-1}}\mathbb{1}_{\{T_{n-1}=\tau\leq\sigma\}} + \tilde{U}_{T_{n-1}}\mathbb{1}_{\{T_{n-1}=\sigma<\tau\}} \\ &+ \mathbb{1}_{\{\sigma\wedge\tau\geq T_n\}}\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + \tilde{L}_\tau\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} + \tilde{U}_\sigma\mathbb{1}_{\{\sigma<T,\sigma<\tau\}}|\mathcal{G}_{T_n}\right]|\mathcal{G}_{T_{n-1}}\Big] \\ =&\min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\bar{\hat{q}}^\lambda_{T_n}|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\} \end{aligned}$$

where the last equality holds since the operations $\operatorname{ess\,inf}_{\sigma\in\mathcal{R}_{T_n}(\lambda)} \operatorname{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)}$ and $\mathbb{E}\left[\cdot|\mathcal{G}_{T_{n-1}}\right]$ are interchangeable, which will be proved in the next step.

This proves (2.3.16) holds, and thus

$$\overline{\hat{q}}^{\lambda}_{T_{n-1}} = \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_n \geq T\}} + \overline{\hat{q}}^{\lambda}_{T_n}\mathbb{1}_{\{T_n<T\}}|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\}$$

since $\overline{\hat{q}}^{\lambda}_{T_M} = \tilde{\xi}$, which follows from the definition of the upper value (2.3.13) of the auxiliary constrained Dynkin game.

It is also obvious that $\overline{\hat{q}}^{\lambda}_{T_{n-1}}$ satisfies (2.3.12) for $n = M+1$, and hence, it satisfies the recursive equation (2.3.12) for $1 \leq n \leq M+1$. Symmetrically, we can obtain that $\underline{\hat{q}}^{\lambda}_{T_{n-1}}$ also satisfies the recursive equation (2.3.12). Since (2.3.12) admits a unique solution, it is clear that $\overline{\hat{q}}^{\lambda}_{T_{n-1}} = \underline{\hat{q}}^{\lambda}_{T_{n-1}} = \hat{q}^{\lambda}_{T_{n-1}} = \hat{Q}^{\lambda}_{T_{n-1}}$, where the latter is given by (2.3.11).

Step 2. In this step, we show the operations $\operatorname{ess\,inf}_{\sigma\in\mathcal{R}_{T_n}(\lambda)} \operatorname{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)}$ and $\mathbb{E}\left[\cdot|\mathcal{G}_{T_{n-1}}\right]$ are interchangeable, i.e. (2.3.19) below holds. To this end, for any $1 \leq n \leq M$ and $\sigma \in \mathcal{R}_{T_n}(\lambda)$, we note that the family

$$\left(\mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_n}\right], \tau \in \mathcal{R}_{T_n}(\lambda)\right) \tag{2.3.17}$$

is an increasing directed set. Indeed, if we choose arbitrary $\tau_1, \tau_2 \in \mathcal{R}_{T_n}(\lambda)$ and let $X_j := \mathbb{E}\left[\tilde{R}(\sigma,\tau_j)|\mathcal{G}_{T_n}\right]$, for $j = 1,2$. Then, there exists $\tau_0 \in \mathcal{R}_{T_n}(\lambda)$, which is given by $\tau_0 = \tau_1\mathbb{1}_{\{X_1\geq X_2\}} + \tau_2\mathbb{1}_{\{X_1<X_2\}}$, such that $\mathbb{E}\left[\tilde{R}(\sigma,\tau_0)|\mathcal{G}_{T_n}\right] \geq \max\{X_1, X_2\}$. Likewise, we have, for any $1 \leq n \leq M$, the family

$$\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_n}\right], \sigma \in \mathcal{R}_{T_n}(\lambda)\right) \tag{2.3.18}$$

is a decreasing directed set. Under Assumption 2.2.2, it is obvious that both (2.3.17) and (2.3.18) are uniformly integrable, and therefore, by Proposition VI-1-1 of Neveu [68], we obtain

$$\begin{aligned}
\mathbb{E}\left[\overline{\hat{q}}^{\lambda}_{T_n}\middle|\mathcal{G}_{T_{n-1}}\right] &= \mathbb{E}\left[\operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_n}(\lambda)} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_n}\right]\middle|\mathcal{G}_{T_{n-1}}\right] \\
&= \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_n}(\lambda)} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_n}\right]\middle|\mathcal{G}_{T_{n-1}}\right] \\
&= \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_n}(\lambda)} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)} \mathbb{E}\left[\mathbb{E}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{T_n}\right]\middle|\mathcal{G}_{T_{n-1}}\right] \\
&= \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{T_n}(\lambda)} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_n}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma,\tau)\middle|\mathcal{G}_{T_{n-1}}\right].
\end{aligned} \tag{2.3.19}$$

Step 3. In this step, we prove that $(\hat{\sigma}^*_{n-1}, \hat{\tau}^*_{n-1})$, which is given by (2.3.15), is indeed the optimal stopping strategy for the auxiliary Dynkin game starting from T_{n-1} (2.3.13)-(2.3.14), i.e. for every $(\sigma,\tau) \in \mathcal{R}_{T_{n-1}}(\lambda) \times \mathcal{R}_{T_{n-1}}(\lambda)$, we have

$$\mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_{n-1},\tau)|\mathcal{G}_{T_{n-1}}\right] \le \mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_{n-1},\hat{\tau}^*_{n-1})|\mathcal{G}_{T_{n-1}}\right] \le \mathbb{E}\left[\tilde{R}(\sigma,\hat{\tau}^*_{n-1})|\mathcal{G}_{T_{n-1}}\right].$$

To this end, we claim the following results hold

(i) $\left(\hat{q}^{\lambda}_{T_m\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\right)_{m\ge n-1}$ is a $\tilde{\mathbb{G}}$-martingale;

(ii) $\left(\hat{q}^{\lambda}_{T_m\wedge\hat{\sigma}^*_{n-1}\wedge\tau}\right)_{m\ge n-1}$ is a $\tilde{\mathbb{G}}$-supermartingale for any $\tau\in\mathcal{R}_{T_{n-1}}(\lambda)$;

(iii) $\left(\hat{q}^{\lambda}_{T_m\wedge\sigma\wedge\hat{\tau}^*_{n-1}}\right)_{m\ge n-1}$ is a $\tilde{\mathbb{G}}$-submartingale for any $\sigma\in\mathcal{R}_{T_{n-1}}(\lambda)$.

If the martingale property (i) holds, then, for $1\le n\le M$,

$$\begin{aligned}\hat{q}^{\lambda}_{T_{n-1}} &= \hat{q}^{\lambda}_{T_{n-1}\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\\ &= \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\ge T\}} + \hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}<T\}}\middle|\mathcal{G}_{T_{n-1}}\right]\\ &= \mathbb{E}\Big[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\ge T\}} + \tilde{L}_{\hat{\tau}^*_{n-1}}\mathbb{1}_{\{\hat{\tau}^*_{n-1}<T,\hat{\tau}^*_{n-1}\le\hat{\sigma}^*_{n-1}\}}\\ &\quad + \tilde{U}_{\hat{\sigma}^*_{n-1}}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}<T,\hat{\sigma}^*_{n-1}<\hat{\tau}^*_{n-1}\}}|\mathcal{G}_{T_{n-1}}\Big]\\ &= \mathbb{E}\Big[\tilde{R}(\hat{\sigma}^*_{n-1},\hat{\tau}^*_{n-1})|\mathcal{G}_{T_{n-1}}\Big]\end{aligned}\tag{2.3.20}$$

where the second last equality follows from the definition (2.3.15) of $(\hat{\sigma}^*_{n-1},\hat{\tau}^*_{n-1})$.

Using the similar arguments, if the supermartingale property (ii) and the submartingale property (iii) hold, then we can have, for any $\tau\in\mathcal{R}_{T_{n-1}}(\lambda)$,

$$\begin{aligned}\hat{q}^{\lambda}_{T_{n-1}} &\ge \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\ge T\}} + \hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\tau}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau<T\}}\middle|\mathcal{G}_{T_{n-1}}\right]\\ &\ge \mathbb{E}\Big[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\ge T\}} + \tilde{L}_{\tau}\mathbb{1}_{\{\tau<T,\tau\le\hat{\sigma}^*_{n-1}\}} + \tilde{U}_{\hat{\sigma}^*_{n-1}}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}<T,\hat{\sigma}^*_{n-1}<\tau\}}|\mathcal{G}_{T_{n-1}}\Big]\\ &= \mathbb{E}\Big[\tilde{R}(\hat{\sigma}^*_{n-1},\tau)|\mathcal{G}_{T_{n-1}}\Big],\end{aligned}\tag{2.3.21}$$

and, for any $\sigma\in\mathcal{R}_{T_{n-1}}(\lambda)$,

$$\hat{q}^{\lambda}_{T_{n-1}} \le \mathbb{E}\Big[\tilde{R}(\sigma,\hat{\tau}^*_{n-1})|\mathcal{G}_{T_{n-1}}\Big].\tag{2.3.22}$$

As a direct consequence of (2.3.20)-(2.3.22), we can obtain $(\hat{\sigma}^*_{n-1},\hat{\tau}^*_{n-1})$, which is given by (2.3.15), is indeed an optimal stopping strategy of the auxiliary constrained Dynkin game (2.3.13)-(2.3.14).

Step 4. It remains to prove the martingale property (i), the supermartingale property (ii) and the submartingale property (iii) in Step 3. Indeed, for

$m \geq n-1$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\hat{q}^{\lambda}_{T_{m+1}\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\,\middle|\,\mathcal{G}_{T_m}\right]\\
&=\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_{m+1}\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T\}}+\hat{q}^{\lambda}_{T_{m+1}\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\mathbb{1}_{\{T_{m+1}\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}<T\}}\,\middle|\,\mathcal{G}_{T_m}\right]\\
&=\mathbb{E}\Bigg[\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\leq T_m\}}\left(\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T\}}+\hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}<T\}}\right)\\
&\quad+\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T_{m+1}\}}\left(\tilde{\xi}\mathbb{1}_{\{T_{m+1}\geq T\}}+\hat{q}^{\lambda}_{T_{m+1}}\mathbb{1}_{\{T_{m+1}<T\}}\right)|\mathcal{G}_{T_m}\Bigg]\\
&=\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\leq T_m\}}\left(\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T\}}+\hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}<T\}}\right)\\
&\quad+\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T_{m+1}\}}\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_{m+1}\geq T\}}+\hat{q}^{\lambda}_{T_{m+1}}\mathbb{1}_{\{T_{m+1}<T\}}|\mathcal{G}_{T_m}\right].
\end{aligned}
$$

It follows from the definition (2.3.15) of $(\hat{\sigma}^*_{n-1},\hat{\tau}^*_{n-1})$ that, conditional on the set $\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T_{m+1}\}$, we have

$$
\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_{m+1}\geq T\}}+\hat{q}^{\lambda}_{T_{m+1}}\mathbb{1}_{\{T_{m+1}<T\}}|\mathcal{G}_{T_m}\right]=\tilde{\xi}\mathbb{1}_{\{T_m\geq T\}}+\hat{q}^{\lambda}_{T_m}\mathbb{1}_{\{T_m<T\}},
$$

and thus

$$
\begin{aligned}
&\mathbb{E}\left[\hat{q}^{\lambda}_{T_{m+1}\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\,\middle|\,\mathcal{G}_{T_m}\right]\\
&=\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\leq T_m\}}\left(\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T\}}+\hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}<T\}}\right)\\
&\quad+\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}\geq T_{m+1}\}}\left(\tilde{\xi}\mathbb{1}_{\{T_m\geq T\}}+\hat{q}^{\lambda}_{T_m}\mathbb{1}_{\{T_m<T\}}\right)\\
&=\hat{q}^{\lambda}_{T_m\wedge\hat{\sigma}^*_{n-1}\wedge\hat{\tau}^*_{n-1}},
\end{aligned}
$$

so the martingale property (i) has been proved.

To prove the supermartingale property (ii), for any $\tau\in\mathcal{R}_{T_{n-1}}(\lambda)$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\hat{q}^{\lambda}_{T_{m+1}\wedge\hat{\sigma}^*_{n-1}\wedge\tau}\,\middle|\,\mathcal{G}_{T_m}\right]\\
&=\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\leq T_m\}}\left(\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\geq T\}}+\hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\tau}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau<T\}}\right)\\
&\quad+\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\geq T_{m+1}\}}\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_{m+1}\geq T\}}+\hat{q}^{\lambda}_{T_{m+1}}\mathbb{1}_{\{T_{m+1}<T\}}|\mathcal{G}_{T_m}\right]\\
&\leq\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\leq T_m\}}\left(\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\geq T\}}+\hat{q}^{\lambda}_{\hat{\sigma}^*_{n-1}\wedge\tau}\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau<T\}}\right)\\
&\quad+\mathbb{1}_{\{\hat{\sigma}^*_{n-1}\wedge\tau\geq T_{m+1}\}}\left(\tilde{\xi}\mathbb{1}_{\{T_m\geq T\}}+\hat{q}^{\lambda}_{T_m}\mathbb{1}_{\{T_m<T\}}\right)\\
&=\hat{q}^{\lambda}_{T_m\wedge\hat{\sigma}^*_{n-1}\wedge\tau}
\end{aligned}
$$

where the inequality follows from the fact that, conditional on the set $\{\hat{\sigma}^*_{n-1}\wedge$

$\tau \geq T_{m+1}\} \cap \{T_m < T\}$,

$$\begin{aligned}
\hat{q}^{\lambda}_{T_m} &= \max\left\{\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_{m+1}\geq T\}} + \hat{q}^{\lambda}_{T_{m+1}}\mathbb{1}_{\{T_{m+1}<T\}}\middle|\mathcal{G}_{T_m}\right], \tilde{L}_{T_m}\right\} \\
&\geq \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_{m+1}\geq T\}} + \hat{q}^{\lambda}_{T_{m+1}}\mathbb{1}_{\{T_{m+1}<T\}}\middle|\mathcal{G}_{T_m}\right].
\end{aligned}$$

This proves the supermartingale property (ii). Likewise, the submartingale property (iii) can be proved in a similar way, and the proof of this lemma is thus completed. ■

We are now in a position to prove Theorem 2.2.4. By Lemma 2.3.1 and Lemma 2.3.2, we have

$$\begin{aligned}
Q^{\lambda}_0 &= \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_1\geq T\}} + \hat{Q}^{\lambda}_{T_1}\mathbb{1}_{\{T_1<T\}}\right] \\
&= \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_1\geq T\}} + \hat{q}^{\lambda}_{T_1}\mathbb{1}_{\{T_1<T\}}\right] \\
&= \mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_1\geq T\}} + \mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_1, \hat{\tau}^*_1)|\mathcal{G}_{T_1}\right]\mathbb{1}_{\{T_1<T\}}\right] \\
&= \mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_1, \hat{\tau}^*_1)\right] \qquad (2.3.23)
\end{aligned}$$

where $(\hat{\sigma}^*_1, \hat{\tau}^*_1)$ in (2.3.15) is the optimal stopping strategy of the auxiliary constrained Dynkin game starting from T_1. Similarly, we can obtain that, for any $\tau \in \mathcal{R}_{T_1}(\lambda)$,

$$Q^{\lambda}_0 \geq \mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_1, \tau)\right], \qquad (2.3.24)$$

and, for any $\sigma \in \mathcal{R}_{T_1}(\lambda)$,

$$Q^{\lambda}_0 \leq \mathbb{E}\left[\tilde{R}(\sigma, \hat{\tau}^*_1)\right]. \qquad (2.3.25)$$

It follows from (2.3.24) and (2.3.25) that

$$Q^{\lambda}_0 \geq \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{R}(\hat{\sigma}^*_1, \tau)\right] \geq \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma, \tau)\right] = \overline{q}^{\lambda},$$

and

$$Q^{\lambda}_0 \leq \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma, \hat{\tau}^*_1)\right] \leq \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{R}(\sigma, \tau)\right] = \underline{q}^{\lambda}.$$

It is obvious that $\overline{q}^{\lambda} \geq \underline{q}^{\lambda}$, and thus $Q^{\lambda}_0 = q^{\lambda} = \overline{q}^{\lambda} = \underline{q}^{\lambda}$. As a direct consequence of (2.3.23)-(2.3.25), we can obtain $(\hat{\sigma}^*_1, \hat{\tau}^*_1)$ in (2.3.15) is indeed an optimal stopping strategy of the constrained Dynkin game starting (2.3.4)-(2.3.5).

We conclude the proof by proving $(\hat{\sigma}^*_1, \hat{\tau}^*_1)$ is actually $(\sigma^*_{T_1}, \tau^*_{T_1})$ in (2.3.7).

Indeed,

$$\begin{aligned}\hat{\sigma}_1^* &= \inf\{T_N \geq T_1 : \hat{Q}^\lambda_{T_N} = \tilde{U}_{T_N}\} \wedge T_M \\ &= \inf\{T_N \geq T_1 : Q^\lambda_{T_N} \geq \tilde{U}_{T_N}\} \wedge T_M = \sigma^*_{T_1},\end{aligned}$$

and, similarly, $\hat{\tau}_1^* = \tau^*_{T_1}$.

2.4 Connection with standard Dynkin games

In this section, we make the connection between constrained Dynkin games and standard Dynkin games. Let T be a fixed finite horizon. We show that, when $\lambda \to \infty$, the value v^λ of the constrained Dynkin game converges to the value of the standard Dynkin game.

The set-up of a standard Dynkin game is the same as in Section 2.2 except that the control set of the player is replaced with $\mathcal{R}_0$, which denotes the set of $\mathbb{F}$-stopping times valued in $[0, T]$. Define the corresponding upper and lower values of the standard Dynkin game as

$$\overline{v} = \inf_{\sigma \in \mathcal{R}_0} \sup_{\tau \in \mathcal{R}_0} \mathbb{E}\left[R(\sigma, \tau)\right], \tag{2.4.1}$$

$$\underline{v} = \sup_{\tau \in \mathcal{R}_0} \inf_{\sigma \in \mathcal{R}_0} \mathbb{E}\left[R(\sigma, \tau)\right] \tag{2.4.2}$$

where the payoff function $R(\sigma, \tau)$ is given by (2.2.1). This standard game is said to have value v if $v = \overline{v} = \underline{v}$, and $(\sigma^*, \tau^*) \in \mathcal{R}_0 \times \mathcal{R}_0$ is called a saddle point of the game if $\mathbb{E}\left[R(\sigma^*, \tau)\right] \leq \mathbb{E}\left[R(\sigma^*, \tau^*)\right] \leq \mathbb{E}\left[R(\sigma, \tau^*)\right]$ for every $(\sigma, \tau) \in \mathcal{R}_0 \times \mathcal{R}_0$.

Proposition 2.4.1 *Suppose that Assumption 2.2.2 holds and, moreover, both L and U are continuous and satisfy $L_T \leq \xi \leq U_T$. Then, the value v of the Dynkin game (2.4.1)-(2.4.2) exists and, moreover, $\lim_{\lambda \uparrow \infty} v^\lambda = v$.*

Proof. To solve the Dynkin game (2.4.1)-(2.4.2), we introduce the following reflected BSDE defined on a finite horizon $[0, T]$:

$$V_t = \xi + \int_t^T (f_s - rV_s)ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s\, dW_s \tag{2.4.3}$$

for $t \in [0, T]$, under the constraints (i) $L_t \leq V_t \leq U_t$, for $0 \leq t \leq T$; (ii) $\int_0^T (V_t - L_t)\, dK_t^+ = \int_0^T (U_t - V_t)\, dK_t^- = 0$. By a solution to the reflected BSDE (2.4.3), we mean a triplet of $\mathbb{F}$-progressively measurable processes (V, Z, K), where $K := K^+ - K^-$ with K^+ and K^- being increasing processes starting from $K_0^+ = K_0^- = 0$.

It follows from Cvitanic and Karatzas [16] that (2.4.3) is well-posed and admits a unique solution. [16] shows that the value of the Dynkin game (2.4.1)-(2.4.2) exists and is given by the solution of the reflected BSDE (2.4.3), i.e. $v = \overline{v} = \underline{v} = V_0$.

To prove the second assertion, we note that BSDE (2.2.6) can be regarded as a sequence of penalized BSDEs for (2.4.3), where the local time processes K^+ and K^- are approximated by

$$K_t^{\lambda,+} := \int_0^t \lambda \left(L_s - V_s^\lambda\right)^+ ds; \quad K_t^{\lambda,-} := \int_0^t \lambda \left(V_s^\lambda - U_s\right)^+ ds,$$

with $K^\lambda := K^{\lambda,+} - K^{\lambda,-}$. Since $\lim_{\lambda\uparrow\infty} \mathbb{E}[\sup_{t\in[0,T]} |V_t^\lambda - V_t|^2] = 0$ (see, for example, [16]), the second assertion follows immediately. ■

2.5 Application to convertible bonds with random intervention times

In this section, using the constrained Dynkin game introduced in Section 2.2, we study convertible bonds for which both players are only allowed to stop at a sequence of random intervention times.

Traditionally, convertible bond models often assume that both the bond holder and the issuing firm are allowed to stopped at any stopping time adapted to the firm's fundamental (such as its stock prices). In reality, there may exist some liquidation constraint as an external shock, and both players only make their decisions when such a shock arrives. We model such a liquidation shock as the arrival times of an exogenous Poisson process. A similar idea has first appeared in the modeling of debt run problems (see [58]), which can be formulated as optimal stopping problems with Poisson arrival times.

Assumption 2.5.1 *Let $d = 1$. The firm's stock price S^s, under the risk-neutral probability measure $\mathbb{P}$, follows*

$$S_t^s = s + \int_0^t (r-q) S_u^s \, du + \int_0^t \sigma S_u^s \, dW_u, \tag{2.5.1}$$

with $S_0^s = s > 0$, where the constants r, q, σ represent the risk-free interest rate, the dividend rate and the volatility of the stock, satisfying the parameter condition:

$$r \geq \frac{1}{2}\sigma^2 + q. \tag{2.5.2}$$

Consider an investor purchasing a share of convertible bond, issued by a firm as a perpetuity with a constant coupon rate c, at initial time $t = 0$. By holding the convertible bond, the investor will continuously receive the coupon rate c from the firm until the contract is terminated. The investor has the right to convert her bond to the firm's stocks, while the firm has the right to call the bond and force the bondholder to surrender her bond to the firm at a sequence of Poisson arrival times $\{T_n\}_{n\geq 1}$ with a constant intensity $\lambda > 0$.

In this section, we further assume an automatic conversion is triggered as soon as the firm's stock reaches a set price. The firm will force a conversion of the convertible bond to stocks at the first Poisson arrival time after the stock price exceeds the set price $\bar{s} := K/\gamma$. This additional term, on the one hand, is motivated by real world financial contracts; on the other hand, it is critical in ensuring the dominating condition is satisfied when applying the constrained Dynkin game introduced in Section 2.2.

For later use, we define the first hitting time by an $\mathbb{F}$-stopping time

$$\theta := \inf\{u \geq 0 : S_u^s \geq \bar{s}\},$$

and the first Poisson arrival time following θ by a $\mathbb{G}$-stopping time

$$T_M := \inf\{T_N \geq \theta : N \geq 1\}.$$

Under the parameter condition (2.5.2), it is standard to prove θ is finite (see, for example, Section 3.6 in [72]). In summary, there are three situations that the contract might be terminated:

(i) if the firm calls the bond at some $\mathbb{G}$-stopping time σ firstly, the bondholder will receive a pre-specified surrender price K at time σ;

(ii) if the investor chooses to convert her bond at some $\mathbb{G}$-stopping time τ firstly or both players choose to stop the contract simultaneously, the bondholder will obtain γS_τ at time τ from converting her bond with a pre-specified conversion rate $\gamma \in (0, 1)$;

(iii) if neither the firm nor the investor stops the contract before θ, an automatic conversion is triggered and the bondholder will obtain γS_{T_M} at time T_M from converting her bond with a pre-specified conversion rate $\gamma \in (0, 1)$.

From a perspective of the investor, the expectation of the discounted payoff at initial time $t = 0$ then equals, for $\sigma, \tau \in \mathcal{R}_{T_1}(\lambda)$,

$$\begin{aligned}
\mathbb{E}[P^\lambda(s; \sigma, \tau)] &= \mathbb{E}\bigg[\int_0^{\sigma\wedge\tau\wedge T_M} e^{-ru} c\, du + e^{-rT_M}\gamma S_{T_M}\mathbb{1}_{\{\tau\wedge\sigma\geq\theta\}} + e^{-r\tau}\gamma S_\tau^s \mathbb{1}_{\{\tau<\theta,\tau\leq\sigma\}} \\
&\quad + e^{-r\sigma} K \mathbb{1}_{\{\sigma<\theta,\sigma<\tau\}}\bigg] \\
&= \mathbb{E}\bigg[\int_0^{\sigma\wedge\tau\wedge\theta} e^{-ru} c\, du + e^{-r\theta} L^\lambda(S_\theta^s)\mathbb{1}_{\{\tau\wedge\sigma\geq\theta\}} + e^{-r\tau}\gamma S_\tau^s \mathbb{1}_{\{\tau<\theta,\tau\leq\sigma\}} \\
&\quad + e^{-r\sigma} K \mathbb{1}_{\{\sigma<\theta,\sigma<\tau\}}\bigg],
\end{aligned} \tag{2.5.3}$$

where $L^\lambda(s) := \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s$.

Remark 2.5.2 *For the convenience of the reader, we show the second equality above holds. It follows from the tower property of conditional expectation that*

$$\mathbb{E}[P^\lambda(s;\sigma,\tau)] = \mathbb{E}\Bigg[\int_0^{\sigma\wedge\tau\wedge\theta} e^{-ru}c\,du + \mathbb{E}[\int_\theta^{T_M} e^{-ru}c\,du + e^{-rT_M}\gamma S_{T_M})|\mathcal{F}_\theta]\mathbb{1}_{\{\tau\wedge\sigma\geq\theta\}} \\ + e^{-r\tau}\gamma S_\tau^s \mathbb{1}_{\{\tau<\theta,\tau\leq\sigma\}} + e^{-r\sigma}K\mathbb{1}_{\{\sigma<\theta,\sigma<\tau\}}\Bigg].$$

Using the conditional density $\lambda e^{-\lambda(x-\theta)}\,dx$ *of* T_M*, we can further simplify the conditional expectation as follows*

$$\begin{aligned}
&\mathbb{E}[\int_\theta^{T_M} e^{-ru}c\,du + e^{-rT_M}\gamma S_{T_M})|\mathcal{F}_\theta] \\
=\ &\mathbb{E}[\int_\theta^\infty \lambda e^{-\lambda(m-\theta)}(\int_\theta^m e^{-ru}c\,du + e^{-rm}\gamma S_m^s)\,dm|\mathcal{F}_\theta] \\
=\ &e^{-r\theta}L^\lambda(S_\theta^s).
\end{aligned}$$

The investor will choose $\tau \in \mathcal{R}_{T_1}(\lambda)$ to maximize the bond value, while the firm will choose $\sigma \in \mathcal{R}_{T_1}(\lambda)$ to maximize the equity value of the firm by minimizing the bond value. This leads to a constrained Dynkin game as introduced in Section 2.2. The upper value and lower value of this *constrained convertible bond* are

$$\overline{v}^\lambda(s) = \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[P^\lambda(s;\sigma,\tau)\right], \tag{2.5.4}$$

$$\underline{v}^\lambda(s) = \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[P^\lambda(s;\sigma,\tau)\right]. \tag{2.5.5}$$

By applying Theorem 2.2.4, we can obtain the existence of the value of the convertible bond. In particular, for $s \in (0,\bar{s})$, we characterize the value of the convertible bond and the corresponding optimal stopping strategy via the solution of ordinary differential equations (ODEs) and the associated free boundaries, respectively.

Proposition 2.5.3 *Suppose that Assumption 2.5.1 holds. Define the infinitesimal generator* $\mathcal{L}_0 = \frac{1}{2}\sigma^2 s^2\partial_{ss}^2 + (r-q)s\partial_s - r$*. Then, the value of the constrained convertible bond, denoted as* $v^\lambda(s)$*, exists: For* $s \in [\bar{s},\infty)$*,* $v^\lambda(s) = L^\lambda(s)$*; For* $s \in (0,\bar{s})$*,* $v^\lambda(s)$ *is the unique solution to the following ODE*

$$-\mathcal{L}_0 v^\lambda = c + \lambda(\gamma s - v^\lambda)^+ - \lambda(v^\lambda - K)^+ \tag{2.5.6}$$

with the boundary condition $v^\lambda(\bar{s}) = L^\lambda(\bar{s})$.

Proof. For $s \in [\bar{s}, \infty)$, it is easy to check $v^\lambda(s) = L^\lambda(s)$. Indeed,

$$
\begin{aligned}
\overline{v}^\lambda(s) = \underline{v}^\lambda(s) &= \mathbb{E}\left[\int_0^{T_1} e^{-ru} c\, du + e^{-rT_1} \gamma S^s_{T_1}\right] \\
&= \mathbb{E}\left[\int_0^\infty \lambda e^{-\lambda m} \left(\int_0^m e^{-ru} c\, du + e^{-rm} \gamma S^s_m\right) dm\right] \\
&= \int_0^\infty \lambda e^{-\lambda m} \int_0^m e^{-ru} c\, du\, dm + \lambda\gamma \mathbb{E}\left[\int_0^\infty e^{-(r+\lambda)m} S^s_m\, dm\right] \\
&= \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s = L^\lambda(s).
\end{aligned}
$$

For $s \in (0, \bar{s})$, we apply Theorem 2.2.4 with $T = \theta$, $L_t = \gamma S^s_t$, $U_t = K$, $f_t = c$ and $\xi = L^\lambda(S^s_\theta)$ to (2.5.4)-(2.5.5), and obtain the convertible bond value is $v^\lambda(s) = V^{\lambda,s}_0$, where $V^{\lambda,s}$ is the first component of the solution to the penalized BSDE

$$
\begin{aligned}
V^{\lambda,s}_{t\wedge\theta} = L^\lambda(S^s_\theta) + \int_{t\wedge\theta}^\theta \left[c + \lambda\left(\gamma S^s_u - V^{\lambda,s}_u\right)^+ - \lambda\left(V^{\lambda,s}_u - K\right)^+ - rV^{\lambda,s}_u\right] du \\
- \int_{t\wedge\theta}^\theta Z^{\lambda,s}_u\, dW_u, \quad (2.5.7)
\end{aligned}
$$

for $t \geq 0$, and moreover, the optimal stopping strategy is

$$
\begin{cases}
\sigma^{*,\lambda} = \inf\{T_N \geq T_1 : V^{\lambda,s}_{T_N} \geq K\} \wedge T_M; \\
\tau^{*,\lambda} = \inf\{T_N \geq T_1 : V^{\lambda,s}_{T_N} \leq \gamma S^s_{T_N}\} \wedge T_M.
\end{cases} \quad (2.5.8)
$$

By the Markov property of S, we can have $V^{\lambda,s}_t = v^\lambda(S^s_t)$, where v^λ solves ODE (2.5.6) (the connection between BSDE and ODE is quite standard in the BSDE literature, and thus we refer to Section 4 of [12] and Section 5 of [40] for rigorous proofs). ■

Remark 2.5.4 *In this Remark, we study the boundary condition at $s = 0+$. Firstly, note that this game can be reduced to a one-player optimal stopping problem in the case of $s = 0+$, considering it is never optimal for the investor (maximizer) to convert until T_M because the stock is always worthless, i.e.*

$$
\begin{aligned}
v^\lambda(0+) &= \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^\sigma e^{-ru} c\, du + e^{-r\sigma} K \mathbb{1}_{\{\sigma<\infty\}}\right] \\
&= \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \left[\frac{c}{r}(1 - e^{-r\sigma}) + e^{-r\sigma} K\right].
\end{aligned}
$$

If $c \geq rK$, we can have

$$
v^\lambda(0+) \geq \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \left[K(1 - e^{-r\sigma}) + e^{-r\sigma} K\right] = K,
$$

and similarly, $v^\lambda(0+) < K$ if $c < rK$. Intuitively, when $s = 0+$, the optimal strategy of the firm would depend on the coupon rate c: the firm would prefer calling the bond back in the case of a high coupon rate $c \geq rK$, while would prefer postponing the calling with a low coupon rate $c < rK$.

As a direct consequence of Proposition 2.5.3 and Remark 2.5.4, we simplify ODE (2.5.6) by breaking down the discussion into three situations, which is motivated by the results from the maximum principle.

Corollary 2.5.5 *Suppose that Assumption 2.5.1 holds. Then, for $s \in (0, \bar{s})$, $v^\lambda(s)$ is the unique solution to the following ODEs:*

(i) If $\frac{r+\lambda}{q+\lambda} qK < c < rK$, then $v^\lambda > \gamma s$, and

$$- \mathcal{L}_0 v^\lambda = c - \lambda(v^\lambda - K)^+ \tag{2.5.9}$$

with the boundary condition $v^\lambda(\bar{s}) = L^\lambda(\bar{s})$.

(ii) If $c \geq rK$, then $v^\lambda \geq K > \gamma s$, and

$$- \mathcal{L}_0 v^\lambda = c - \lambda(v^\lambda - K) \tag{2.5.10}$$

with the boundary condition $v^\lambda(\bar{s}) = L^\lambda(\bar{s})$.

(iii) If $c \leq \frac{r+\lambda}{q+\lambda} qK$, then $v^\lambda < K$, and

$$- \mathcal{L}_0 v^\lambda = c + \lambda(\gamma s - v^\lambda)^+ \tag{2.5.11}$$

with the boundary condition $v^\lambda(\bar{s}) = L^\lambda(\bar{s})$.

Proof. To prove the first statement, it is sufficient to prove $v^\lambda > \gamma s$. We first show that γs is a subsolution of (2.5.9) if $\frac{r+\lambda}{q+\lambda} qK < c < rK$, i.e.

$$-\mathcal{L}_0(\gamma s) - c + \lambda(\gamma s - K)^+ = q\gamma s - c < (1 - \frac{r+\lambda}{q+\lambda}) qK \leq 0.$$

Together with the boundary conditions $v^\lambda(0) = \frac{c}{r} > 0$ and $v^\lambda(\bar{s}) = L^\lambda(\bar{s}) = \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda} K > K$, it follows from the maximum principle that $v^\lambda > \gamma s$. The second and third statements can be obtained by the similar arguments. ■

Thanks to the above results, *we focus our analysis to the domain $s \in (0, \bar{s})$ in the rest of this section.* We characterize the optimal stopping strategy of the constrained convertible bond via its associated free boundaries.

2.5.1 Case I: $\frac{r+\lambda}{q+\lambda} qK < c < rK$

It follows from Corollary 2.5.5 that $v^\lambda > \gamma s$ if $\frac{r+\lambda}{q+\lambda} qK < c < rK$. As a direct sequence of (2.5.8), we can conclude the optimal conversion strategy for the investor is $\tau^{*,\lambda} = T_M$, i.e. it is never optimal for the investor to convert until

T_M. Instead, the investor's optimal strategy is to keep the convertible bond to receive its coupons up to T_M.

Furthermore, following from (2.5.9), $v^\lambda = v^{1,\lambda}$ solves the following ODE

$$\begin{cases} -\mathcal{L}_0 v^\lambda - c + \lambda(v^\lambda - K)^+ &= 0, \text{ for } 0 < s < \bar{s}; \\ v^\lambda(\bar{s}) &= L^\lambda(\bar{s}). \end{cases} \tag{2.5.12}$$

Since $v^\lambda(0+) < K$, $v^\lambda(\bar{s}) = \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}K > K$ and v^λ is increasing in s, there must exist $x^{1,\lambda} \in (0, \bar{s})$ such that

$$x^{1,\lambda} = \inf\{s \in (0, \bar{s}) : v^{1,\lambda}(s) \geq K\}, \tag{2.5.13}$$

where by definition it is obvious that $v^{1,\lambda} < K$ for $s \in (0, x^{1,\lambda})$ and $v^{1,\lambda} \geq K$ for $s \in (x^{1,\lambda}, \bar{s})$, and by the continuity of $v^{1,\lambda}$ that $v^{1,\lambda}(x^{1,\lambda}) = K$. In turn, (2.5.12) is equivalent to the following free boundary problem

$$-\mathcal{L}_0 v^\lambda - c = 0, \text{ for } 0 < s < x^{1,\lambda}; \tag{2.5.14}$$

$$-\mathcal{L}_0 v^\lambda - c + \lambda(v^\lambda - K) = 0, \text{ for } x^{1,\lambda} < s < \bar{s}; \tag{2.5.15}$$

$$v^\lambda(\bar{s}) = L^\lambda(\bar{s}); \tag{2.5.16}$$

$$v^\lambda(x^{1,\lambda}-) = K; \tag{2.5.17}$$

$$v^\lambda(x^{1,\lambda}+) = K; \tag{2.5.18}$$

$$\left(v^\lambda\right)'(x^{1,\lambda}-) = \left(v^\lambda\right)'(x^{1,\lambda}+). \tag{2.5.19}$$

The general solution of (2.5.14) has the form $v^{1,\lambda}(s) = A_+ s^{\alpha^+} + A_- s^{\alpha^-} + \frac{c}{r}$ for $0 < s \leq x^{1,\lambda}$, and the general solution of (2.5.15) has the form $v^{1,\lambda} = B_+ s^{\beta^+} + B_- s^{\beta^-} + \frac{c+\lambda K}{r+\lambda}$ for $x^{1,\lambda} \leq s < \bar{s}$, where

$$\alpha^\pm = \frac{-(r - q - \frac{\sigma^2}{2}) \pm \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2}, \tag{2.5.20}$$

$$\beta^\pm = \frac{-(r - q - \frac{\sigma^2}{2}) \pm \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2(r+\lambda)\sigma^2}}{\sigma^2}. \tag{2.5.21}$$

The boundary condition at $s = 0+$ and $\alpha^- < 0$ impliy that

$$v^{1,\lambda}(s) = \begin{cases} A^{1,\lambda} s^\alpha + \frac{c}{r} & \text{if } s \in (0, x^{1,\lambda}] \\ B_+^{1,\lambda} s^{\beta^+} + B_-^{1,\lambda} s^{\beta^-} + \frac{c+\lambda K}{r+\lambda} & \text{if } s \in [x^{1,\lambda}, \bar{s}) \end{cases} \tag{2.5.22}$$

where $\alpha = \alpha^+$ in (2.5.20) and four unknowns $(A^{1,\lambda}, B_+^{1,\lambda}, B_-^{1,\lambda}, x^{1,\lambda})$ are to be determined. Using the continuity (2.5.17)-(2.5.18) and the smooth pasting (2.5.19) across $x^{1,\lambda}$, and the boundary condition (2.5.16) at $s = \bar{s}$, we obtain that $x^{1,\lambda} \in (0, \bar{s})$ is the unique solution to the following algebraic equation

$$C_1 x^{\beta^+} + C_2 x^{\beta^+ - \beta^-} + C_3 = 0,$$

with

$$\begin{cases} C_1 &= (\frac{1}{q+\lambda} - \frac{1}{r+\lambda})\lambda K; \\ C_2 &= -\frac{rK-c}{\beta_+ - \beta_-}(\frac{\beta_+}{r+\lambda} - \frac{\alpha}{r})(\bar{s})^{\beta_-}; \\ C_3 &= -\frac{rK-c}{\beta_+ - \beta_-}(\frac{\alpha}{r} - \frac{\beta_-}{r+\lambda})(\bar{s})^{\beta_+}, \end{cases} \tag{2.5.23}$$

and the coefficients are determined by

$$\begin{cases} A^{1,\lambda} &= \frac{rK-c}{r}(x^{1,\lambda})^{-\alpha}; \\ B_+^{1,\lambda} &= \frac{rK-c}{\beta_+ - \beta_-}(\frac{\alpha}{r} - \frac{\beta_-}{r+\lambda})(x^{1,\lambda})^{-\beta_+}; \\ B_-^{1,\lambda} &= \frac{rK-c}{\beta_+ - \beta_-}(\frac{\beta_+}{r+\lambda} - \frac{\alpha}{r})(x^{1,\lambda})^{-\beta_-}. \end{cases}$$

Note that the existence and uniqueness of the solution to the algebraic equation above can be verified by the following results: $f(x) = C_1 x^{\beta^+} + C_2 x^{\beta^+ - \beta^-} + C_3$ is increasing in x with $f(0) < 0$ and $f(\bar{s}) > 0$.

The optimal calling time for the firm is therefore given as

$$\sigma^{*,\lambda} = \inf\{T_N \geq T_1 : S^s_{T_N} \geq x^{1,\lambda}\} \wedge T_M.$$

In Figure 2.1, we plot the value function $v^{1,\lambda}(s)$. On the one hand, it always lies above the lower obstacle γs, which implies it is never optimal for the investor to convert in the region $s \in (0, \bar{s})$. On the other hand, the bond price crosses the upper obstacle K at $x^{1,\lambda} \in (0, \bar{s})$, which thus can be regarded as the optimal calling boundary for the firm: the firm should call the bond back at the first Poisson time when the stock price exceeds $x^{1,\lambda}$.

2.5.2 Case II: $c \geq rK$

It follows from Corollary 2.5.5 that $v^\lambda \geq K > \gamma s$ if $c \geq rK$. As a direct sequence of (2.5.8), we can conclude the optimal conversion strategy for the investor is $\tau^{*,\lambda} = T_M$, i.e. it is never optimal for the investor to convert until T_M. On the other hand, since the coupon rate c is too high, the firm would prefer to convert as soon as possible to stop paying the bond coupons, i.e. $\sigma^{*,\lambda} = T_1$.

Following from (2.5.10), we further calculate the convertible bond value $v^\lambda = v^{2,\lambda}$ by solving the following ODE explicitly

$$\begin{cases} -\mathcal{L}_0 v^\lambda - c + \lambda(v^\lambda - K) &= 0, \text{ for } 0 < s < \bar{s}; \\ v^\lambda(\bar{s}) &= L^\lambda(\bar{s}). \end{cases} \tag{2.5.24}$$

The general solution of (2.5.24) has the form $v^{2,\lambda} = B_+ s^{\beta^+} + B_- s^{\beta^-} + \frac{c+\lambda K}{r+\lambda}$

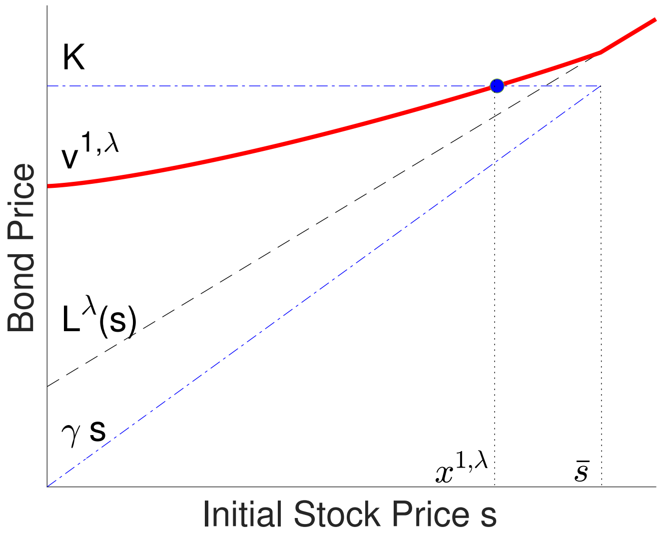


Figure 2.1: Illustrations of the value function in Case I: $\frac{r+\lambda}{q+\lambda}qK < c < rK$

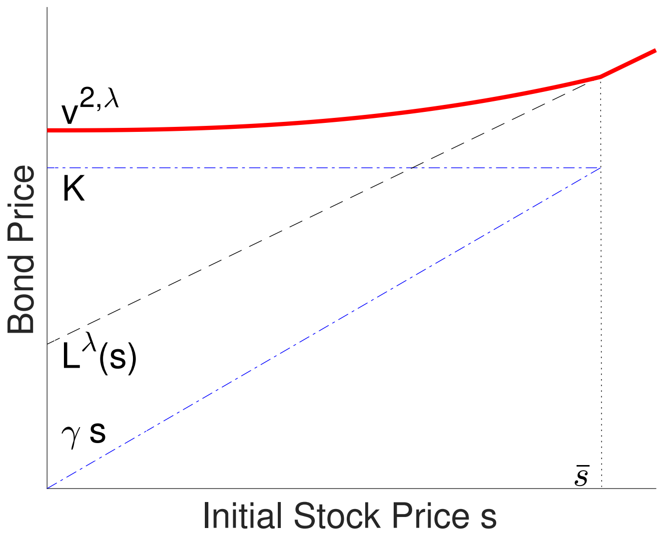


Figure 2.2: Illustrations of the value function in Case II: $c \geq rK$

for $0 < s < \bar{s}$, where $\beta^{\pm}$ are given by (2.5.21). Since $\beta^- < 0$, we obtain $B_- = 0$ by the boundary condition at $s = 0+$. The boundary condition at $s = \bar{s}$ gives

$$v^{2,\lambda}(s) = B^{2,\lambda} s^{\beta} + \frac{c + \lambda K}{r + \lambda}, \tag{2.5.25}$$

where $\beta = \beta^+$ in (2.5.21) and $B^{2,\lambda} = (\frac{1}{q+\lambda} - \frac{1}{r+\lambda})\lambda K(\bar{s})^{-\beta}$.

In Figure 2.2, we plot the value function $v^{2,\lambda}(s)$. The bond price always lies above both the upper obstacle K and the lower obstacle γs, which implies that it is always optimal for the firm to call the bond back at the first Poisson arrival time, and never optimal for the investor to convert in the region $s \in (0, \bar{s})$.

2.5.3 Case III: $c \leq \frac{r+\lambda}{q+\lambda} qK$

It follows from Corollary 2.5.5 that $v^{\lambda} < K$ if $c \leq \frac{r+\lambda}{q+\lambda} qK$. As a direct sequence of (2.5.8), we can conclude the optimal calling strategy for the firm is $\sigma^{*,\lambda} = T_M$, i.e. it is never optimal for the firm to call back the bond until T_M. Instead, the firm's optimal strategy is to postpone the calling up to T_M in light of the low coupon rate. Furthermore, following from (2.5.11), $v^{\lambda} = v^{3,\lambda}$ solves

$$\begin{cases} -\mathcal{L}_0 v^{\lambda} - c - \lambda(\gamma s - v^{\lambda})^+ &= 0, \text{ for } 0 < s < \bar{s}; \\ v^{\lambda}(\bar{s}) &= L^{\lambda}(\bar{s}). \end{cases} \tag{2.5.26}$$

Next, we solve (2.5.26) explicitly. Since $v^{\lambda} > \gamma s$ at $s = 0+$ and $v^{\lambda} \leq \gamma s$ at $s = \bar{s}$, there must exist $x^{3,\lambda} \in (0, \bar{s}]$ such that

$$x^{3,\lambda} = \inf\left\{ s \in (0, \bar{s}] : v^{3,\lambda}(s) \leq \gamma s \right\}. \tag{2.5.27}$$

By definition it is obvious $v^{3,\lambda} > \gamma s$ for $s \in (0, x^{3,\lambda})$, and by the continuity of $v^{3,\lambda}$ that $v^{3,\lambda}(x^{3,\lambda}) = \gamma x^{3,\lambda}$. Let us at the moment assume that $v^{3,\lambda} \leq \gamma s$ for $s \in (x^{*,\lambda}, \bar{s}]$, which will be verified later on. If this condition holds, (2.5.26) is equivalent to the following free boundary problem

$$-\mathcal{L}_0 v^{\lambda} - c = 0, \text{ for } 0 < s < x^{3,\lambda}; \tag{2.5.28}$$
$$-\mathcal{L}_0 v^{\lambda} - c + \lambda(v^{\lambda} - \gamma s) = 0, \text{ for } x^{3,\lambda} < s < \bar{s}; \tag{2.5.29}$$
$$v^{\lambda}(\bar{s}) = L^{\lambda}(\bar{s}); \tag{2.5.30}$$
$$v^{\lambda}(x^{3,\lambda}-) = \gamma x^{3,\lambda}; \tag{2.5.31}$$
$$v^{\lambda}(x^{3,\lambda}+) = \gamma x^{3,\lambda}; \tag{2.5.32}$$
$$\left(v^{\lambda}\right)'(x^{3,\lambda}-) = \left(v^{3,\lambda}\right)'(x^{3,\lambda}+). \tag{2.5.33}$$

We first observe that, with the boundary condition at $s = 0+$, ODEs (2.5.28)-

(2.5.29) imply

$$v^{3,\lambda}(s)=\begin{cases} A^{3,\lambda}s^{\alpha}+\frac{c}{r}, & \text{if } s\in(0,x^{3,\lambda});\\ B_{+}^{3,\lambda}s^{\beta^{+}}+B_{-}^{3,\lambda}s^{\beta^{-}}+\frac{c}{r+\lambda}+\frac{\lambda}{q+\lambda}\gamma s, & \text{if } s\in(x^{3,\lambda},\bar{s}),\end{cases} \tag{2.5.34}$$

where $\alpha=\alpha^{+}$ in (2.5.20), $\beta^{\pm}$ in (2.5.21), and four unknowns $(A^{3,\lambda},B_{+}^{3,\lambda},B_{-}^{3,\lambda},x^{3,\lambda})$ are to be determined. Using the continuity (2.5.31)-(2.5.32) and the smooth pasting (2.5.33) across $x^{3,\lambda}$, and the boundary condition (2.5.30) at $s=\bar{s}$, we obtain that $x^{3,\lambda}\in(0,\bar{s}]$ is the (unique) solution to the following algebraic equation

$$D_1x^{\beta^{+}-\beta^{-}+1}+D_2x^{\beta^{+}-\beta^{-}}+D_3x+D_4=0, \tag{2.5.35}$$

with

$$\begin{cases} D_1=(\frac{q\beta^{+}+\lambda}{q+\lambda}-\alpha)\gamma;\\ D_2=-c(\frac{\beta^{+}}{r+\lambda}-\frac{\alpha}{r});\\ D_3=(\alpha-\frac{q\beta^{-}+\lambda}{q+\lambda})(\bar{s})^{\beta^{+}-\beta^{-}}\gamma;\\ D_4=-c(\frac{\alpha}{r}-\frac{\beta^{-}}{r+\lambda})(\bar{s})^{\beta^{+}-\beta^{-}},\end{cases} \tag{2.5.36}$$

and the coefficients are determined by

$$\begin{cases} A^{3,\lambda} &=& (\gamma x^{3,\lambda}-\frac{c}{r})(x^{3,\lambda})^{-\alpha};\\ B_{+}^{3,\lambda} &=& \frac{1}{\beta^{+}-\beta^{-}}(\alpha-\frac{\beta^{-}q+\lambda}{q+\lambda})\gamma(x^{3,\lambda})^{1-\beta^{+}}-\frac{c}{\beta^{+}-\beta^{-}}(\frac{\alpha}{r}-\frac{\beta^{-}}{r+\lambda})(x^{3,\lambda})^{-\beta^{+}};\\ B_{-}^{3,\lambda} &=& \frac{1}{\beta^{+}-\beta^{-}}(\frac{\beta^{+}q+\lambda}{q+\lambda}-\alpha)\gamma(x^{3,\lambda})^{1-\beta^{-}}-\frac{c}{\beta^{+}-\beta^{-}}(\frac{\beta^{+}}{r+\lambda}-\frac{\alpha}{r})(x^{3,\lambda})^{-\beta^{-}}.\end{cases} \tag{2.5.37}$$

It remains to verify the condition $v^{3,\lambda}\leq\gamma s$ for $s\in(x^{3,\lambda},\bar{s}]$. Indeed, since $A^{3,\lambda}>0$, $\alpha>1$, $B_{+}^{3,\lambda}<0$, $\beta^{+}>1$ and $B_{-}^{3,\lambda}>0$, $\beta^{-}<0$, it is clear that $v^{3,\lambda}$ is convex in the interval $(0,x^{3,\lambda})$ and concave in the interval $(x^{3,\lambda},\bar{s}]$. Moreover, $\left(v^{3,\lambda}\right)'(x^{3,\lambda})<\gamma$, which verifies the required condition.

The optimal conversion time for the investor is therefore given by

$$\tau^{*,\lambda}=\inf\{T_N\geq T_1: S_{T_N}^{s}\geq x^{3,\lambda}\}\wedge T_M.$$

In Figure 2.3, we plot the value function $v^{3,\lambda}(s)$. On the one hand, it always lies below the the upper obstacle K, which implies it is never optimal for the firm to call the bond back in the region $s\in(0,\bar{s})$. On the other hand, the bond price crosses the lower obstacle γs at $x^{3,\lambda}\in(0,\bar{s})$, which thus can be regarded as the optimal conversion boundary for the bondholder: the investor should convert the bond to stocks at the first Poisson time when the stock price falls below $x^{3,\lambda}$.

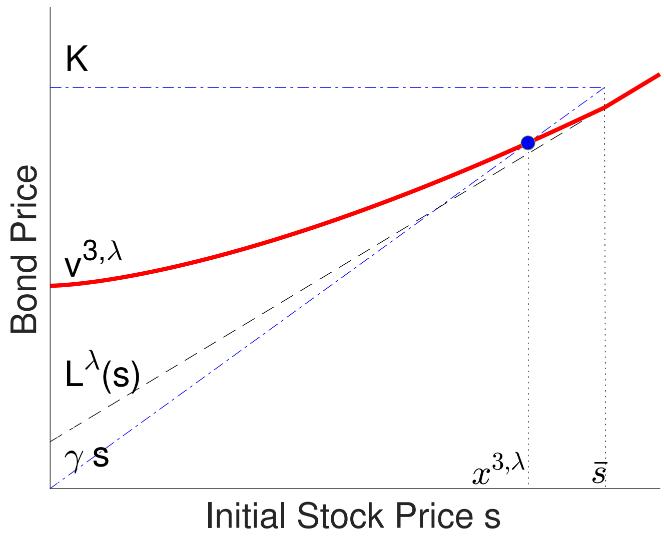


Figure 2.3: Illustrations of the value function in Case III: $c \leq \frac{r+\lambda}{q+\lambda} qK$

2.6 Asymptotics as $\lambda \to \infty$

We study the asymptotic behavior of the convertible bond price and its associated free boundaries when the Poisson intensity $\lambda \to \infty$. Intuitively, they will converge to their continuous time counterparts. We prove this intuition in this section.

2.6.1 Review of standard convertible bonds

The setting is the same as in Section 2.5 except that

- both the investor and the firm choose their respective optimal stopping strategies as $\mathbb{F}$-stopping times taking values in $[0, \theta]$, and
- an automatic conversion is triggered as soon as the firm's stock reaches a set price $\bar{s}$, i.e. the firm will force a conversion of the convertible bond to stocks at θ rather than T_M (the first Poisson arrival time following θ).

Then, the upper and lower values of the standard convertible bond are defined by

$$\overline{v} = \inf_{\sigma \in \mathcal{T}_{0,\theta}} \sup_{\tau \in \mathcal{T}_{0,\theta}} \mathbb{E}\left[P(s; \sigma, \tau)\right], \tag{2.6.1}$$

$$\underline{v} = \sup_{\tau \in \mathcal{T}_{0,\theta}} \inf_{\sigma \in \mathcal{T}_{0,\theta}} \mathbb{E}\left[P(s; \sigma, \tau)\right], \tag{2.6.2}$$

where the expectation of the discount payoff

$$\mathbb{E}[P(s;\sigma,\tau)] = \lim_{\lambda\to\infty} \mathbb{E}[P^\lambda(s;\sigma,\tau)] = \mathbb{E}\bigg[\int_0^{\sigma\wedge\tau\wedge\theta} e^{-ru} c\,du + e^{-r\theta}\gamma S_\theta^s \mathbb{1}_{\{\tau\wedge\sigma\geq\theta\}} + e^{-r\tau}\gamma S_\tau^s \mathbb{1}_{\{\tau<\theta,\tau\leq\sigma\}} + e^{-r\sigma} K \mathbb{1}_{\{\sigma<\theta,\sigma<\tau\}}\bigg]$$

since $L^\lambda(s) \to \gamma s$ as $\lambda \to \infty$, and the control set $\mathcal{T}_{0,\theta}$ is defined by

$$\mathcal{T}_{0,\theta} = \{\mathbb{F}\text{-stopping time } \tau \text{ for } 0 \leq \tau \leq \theta\}.$$

We say this game has value v if $v = \overline{v} = \underline{v}$, and has a saddle point $(\sigma^*, \tau^*) \in \mathcal{T}_{0,\theta} \times \mathcal{T}_{0,\theta}$ if $\mathbb{E}\left[P(s;\sigma^*,\tau)\right] \leq \mathbb{E}\left[P(s;\sigma^*,\tau^*)\right] \leq \mathbb{E}\left[P(s;\sigma,\tau^*)\right]$ for every $(\sigma,\tau) \in \mathcal{T}_{0,\theta} \times \mathcal{T}_{0,\theta}$.

The proof of the following result follows along the similar arguments in [79] and is thus omitted. We refer to [79] for its further details.

Proposition 2.6.1 *Suppose that Assumption 2.5.1 holds. Then, the value of the standard convertible bond, denoted by $v(s)$, exists: For $s \in [\bar{s}, \infty)$, $v(s) = \gamma s$; For $s \in (0, \bar{s})$,*

(i) Case I: $qK < c < rK$,

$$v^1(s) = A^1 s^\alpha + \frac{c}{r} \tag{2.6.3}$$

with $\alpha = \alpha^+$ in (2.5.20) and $A^1 = \frac{rK-c}{r}(\bar{s})^{-\alpha}$, where the optimal stopping strategy is given by $\sigma^* = \tau^* = \theta$.

(ii) Case II: $c \geq rK$,

$$v^2(s) = K, \tag{2.6.4}$$

where the optimal stopping strategy is given by $\sigma^* = 0$ and $\tau^* = \theta$.

(iii) Case III: $c \leq qK$,

$$v^3(s) = \begin{cases} A^3 s^\alpha + \frac{c}{r}, & \text{if } s \in (0, x^3); \\ \gamma s, & \text{if } s \in [x^3, \bar{s}), \end{cases} \tag{2.6.5}$$

with $\alpha = \alpha^+$ in (2.5.20), $A^3 = \left(\gamma x^3 - \frac{c}{r}\right)(x^3)^{-\alpha}$, and the optimal conversion boundary $x^3 = \min(\frac{\alpha}{\alpha-1}\frac{c}{\gamma r}, \bar{s})$, where the optimal stopping strategy is given by $\sigma^* = \theta$ and $\tau^* = \inf\{t \geq 0 : S_t^s \geq x^3\}$.

2.6.2 Asymptotics

We conclude the chapter by studying, when $\lambda \to \infty$, (i) the convergence of the constrained convertible bond price v^λ to its continuous-time counterpart v; (ii) the convergence of the optimal conversion/calling boundaries for the constrained convertible bond to its continuous-time counterparts.

In the case when $c \geq rK$, it is easy to check that $B^{2,\lambda} \to 0$ and $\frac{c+\lambda K}{r+\lambda} \to K$ by using the explicit form (2.5.25). As a consequence, we can have $v^{2,\lambda}(s) \to v^2(s)$. Hence, we only need to establish the convergence results for Case I and Case III. To this end, we first prove the limits of $x^{1,\lambda}$ in (2.5.13) and $x^{3,\lambda}$ in (2.5.27) exist as λ goes to infinity.

Proposition 2.6.2 *Suppose that Assumption 2.5.1 holds. Then, both $x^{1,\lambda}$ in (2.5.13) and $x^{3,\lambda}$ in (2.5.27) have limits, denoted by x^1_∞ and x^3_∞ respectively, as λ goes to infinity.*

Proof. Since $x^{1,\lambda}$ and $x^{3,\lambda}$ are bounded by $\bar{s}$, in order to prove their limits exist, it is sufficient to prove they are increasing in λ. By the definition of $x^{3,\lambda}$ in (2.5.27) and the explicit form of $v^{3,\lambda}$ in (2.5.34), it is sufficient to prove $v^{3,\lambda}$ is increasing in λ.

Recall that $v^{3,\lambda}$ is the solution to the ODE (2.5.26) for $s \in (0, \bar{s})$. Let us suppose $\lambda_1 < \lambda_2$, define the set $\mathcal{N} = \left\{ s \in (0, \bar{s}] : v^{3,\lambda_1}(s) > v^{3,\lambda_2}(s) \right\}$, and suppose that $\mathcal{N} \neq \emptyset$. Then on $\mathcal{N}$, we have

$$\begin{cases} -\mathcal{L}_0 v^{3,\lambda_1} = c + \lambda_1(\gamma s - v^{3,\lambda_1})^+; \\ -\mathcal{L}_0 v^{3,\lambda_2} = c + \lambda_2(\gamma s - v^{3,\lambda_2})^+, \end{cases}$$

which implies

$$\begin{cases} -\mathcal{L}_0(v^{3,\lambda_1} - v^{3,\lambda_2}) = \lambda_1(\gamma s - v^{3,\lambda_1})^+ - \lambda_2(\gamma s - v^{3,\lambda_2})^+ \leq 0; & s \in \mathcal{N} \\ v^{3,\lambda_1} - v^{3,\lambda_2} = 0, & s \in \partial\mathcal{N}. \end{cases}$$

It follows from the maximum principle that $v^{3,\lambda_1} \leq v^{3,\lambda_2}$ on $\mathcal{N}$, which is in contradiction with the definition of $\mathcal{N}$.

Similarly, we can prove $v^{1,\lambda}$ is decreasing in λ, and therefore, by the definition of $x^{1,\lambda}$ in (2.5.13) and the explicit form of $v^{1,\lambda}$ in (2.5.22), we can see $x^{1,\lambda}$ in (2.5.13) is also increasing in λ. ■

We are now in a position to establish the convergence results of $x^{1,\lambda}$ for Case I and $x^{3,\lambda}$ for Case III. As a direct consequence, the convergence of $v^{1,\lambda}$ to v^1 and $v^{3,\lambda}$ to v^3 follows immediately.

Asymptotics: Case I

In this subsection, it is sufficient to prove the unique solution $x^{1,\lambda} \in (0, \bar{s})$ to the following algebraic equation

$$C_1 x^{\beta^+} + C_2 x^{\beta^+ - \beta^-} + C_3 = 0,$$

where C_1, C_2, C_3 are given by (2.5.23), converges to $\bar{s}$ as $\lambda \to \infty$. By letting $y^{1,\lambda} = \frac{\bar{s}}{x^{1,\lambda}} \in (1, \infty)$, we can see $y^{1,\lambda}$ solves the following algebraic equation

$$(\frac{\beta_+}{r+\lambda} - \frac{\alpha}{r})y^{\beta^-} + (\frac{\alpha}{r} - \frac{\beta_-}{r+\lambda})y^{\beta_+} = \frac{r-q}{rK-c}\frac{(\beta_+ - \beta_-)\lambda}{(q+\lambda)(r+\lambda)}K.$$

Sending $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty}\left((\frac{\beta_+}{r+\lambda} - \frac{\alpha}{r})y^{\beta^-} + (\frac{\alpha}{r} - \frac{\beta_-}{r+\lambda})y^{\beta_+}\right) = 0,$$

which forces $y^{1,\lambda} \to 1$. Indeed, if we assume $\lim_{\lambda\to\infty} y^{1,\lambda} > 1$, then the left-hand-side of the above equation would go to infinity, which provides the desired contradiction.

Asymptotics: Case III

In this subsection, it is sufficient to prove the unique solution $x^{3,\lambda} \in (0, \bar{s}]$ to the following algebraic equation

$$D_1 x^{\beta^+ - \beta^- + 1} + D_2 x^{\beta^+ - \beta^-} + D_3 x + D_4 = 0,$$

where D_1, D_2, D_3, D_4 are given by (2.5.36), converges to x^3 as $\lambda \to \infty$. By letting $y^{3,\lambda} = \frac{\bar{s}}{x^{3,\lambda}} \in [1, \infty)$, we can see $y^{1,\lambda}$ solves the following algebraic equation

$$(y^{\beta^+ - \beta^-} - 1)\left[(\alpha - \frac{\beta^- q + \lambda}{q+\lambda})\frac{K}{y} - c(\frac{\alpha}{r} - \frac{\beta^-}{r+\lambda})\right] = (\beta^+ - \beta^-)\left(\frac{c}{r+\lambda} - \frac{qK}{(q+\lambda)y}\right).$$

Sending $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty}\left(\underbrace{(y^{\beta^+ - \beta^-} - 1)}_{I^\lambda}\underbrace{\left[(\alpha - \frac{\beta^- q + \lambda}{q+\lambda})\frac{K}{y} - c(\frac{\alpha}{r} - \frac{\beta^-}{r+\lambda})\right]}_{II^\lambda}\right) = 0,$$

which implies I^λ and/or II^λ has the limit 0.

- If $c < \frac{\alpha-1}{\alpha} rK$, we have

$$\lim_{\lambda\to\infty} y^{3,\lambda} = \frac{\bar{s}}{x^3_\infty} \geq \frac{\bar{s}}{x^3} = \frac{\alpha-1}{\alpha}\frac{rK}{c} > 1$$

  which thus forces $\lim_{\lambda\to\infty} II^\lambda = 0$, i.e. $\lim_{\lambda\to\infty} y^{3,\lambda} = \frac{\alpha-1}{\alpha}\frac{rK}{c}$.

- If $c \geq \frac{\alpha-1}{\alpha} rK$, we have

$$\lim_{\lambda\to\infty} II^\lambda = (\alpha-1)\frac{K}{\lim_{\lambda\to\infty} y^{3,\lambda}} - \alpha\frac{c}{r} \leq (\alpha-1)(\frac{K}{\lim_{\lambda\to\infty} y^{3,\lambda}} - K) \leq 0,$$

where either $\lim_{\lambda \to \infty} II^\lambda < 0$ (which implies $\lim_{\lambda \to \infty} I^\lambda = 0$) or $\lim_{\lambda \to \infty} II^\lambda = 0$ gives us $\lim_{\lambda \to \infty} y^{3,\lambda} = 1$.

Chapter 3

Risk-Sensitive Dynkin Games with Heterogeneous Poisson Random intervention Times

3.1 Introduction

Risk-sensitive criteria constitute a genuinely interesting class of performance criteria in optimization problems, in which the linear expectation $\mathbb{E}[\cdot]$ is replaced by the nonlinear expectation

$$\tilde{\mathbb{E}}\left[\cdot\right] := g^{-1}(\mathbb{E}\left[g(\cdot)\right]),$$

for some strictly increasing function g as a risk-sensitive function. The corresponding risk-sensitive control has been developed to reflect an optimizer's attitudes to risks. In particular, the risk-sensitive function g is chosen to model the optimizer's attitudes towards risks (e.g. strict concavity of g reflects risk-aversion of maximization players or risk-seeking of minimization players).

In this chapter, we are interested in *Dynkin games with risk-sensitive criteria*, by taking into account of both players' attitudes to risks. Namely, the two players aim to minimize/maximize some payoff functional $R(\sigma,\tau)$ under the nonlinear expectation $\tilde{\mathbb{E}}[\cdot]$:

$$J(\sigma,\tau) = \tilde{\mathbb{E}}[R\left(\sigma,\tau\right)] = g^{-1}\left(\mathbb{E}\left[g\left(R\left(\sigma,\tau\right)\right)\right]\right),$$

where σ and τ are the stopping times to be chosen by the respective minimization/maximization players. It is called risk-sensitive because[1]

$$J(\sigma,\tau) \approx \mathbb{E}[R(\sigma,\tau)] - \frac{1}{2} l_g \left(\mathbb{E}[R(\sigma,\tau)]\right) \mathrm{Var}[R(\sigma,\tau)],$$

where $l_g(x) = -\frac{g''(x)}{g'(x)}$ is the Arrow-Pratt function of absolute risk aversion. The case $g(x) = x$ corresponds to a risk-neutral attitude of both players since $l_g(x) = 0$. For the case of an exponential utility $g(x) = -e^{-\gamma x}$ with $\gamma > 0$, $l_g(x) = \gamma$ is constant and the risk-sensitivity is only expressed through the risk-sensitivity parameter γ.

The stopping time strategies of the two players are restricted to two independent sequences of Poisson arrival times as the exogenous constraints on the players' abilities to stop. The constraints may represent liquidity effects, indicating the times at which the underlying stochastic processes are available to stop. Applications of such a liquidity model can be found in [58] for bank runs and Chapter 2 for convertible bonds. The constraints can also be seen as information constraints. The players are allowed to make their stopping decisions at all times, but they are only able to observe the underlying stochastic processes at Poisson arrival times. See [25] and [55] for applications to perpetual American options. Due to the introduction of constraints on stopping times and risk-sensitive criteria, we call the Dynkin games considered in this chapter the *constrained risk-sensitive Dynkin games*.

We generalize Chapter 2 on *constrained Dynkin games* in two aspects: First, it takes into consideration of both players' attitudes towards risks via the risk-sensitive function g; Second, there are control constraints for both players and, moreover, the constraints are different in the sense that they are allowed to stop at two heterogeneous sequences of Poisson arrival times. Consequently, since the two players' stopping time strategies are chosen from two different sequences of signal times, the usual condition of the upper obstacle U dominating the lower one L is not required. In Chapter 2, the risk-sensitive function $g(x) = x$ and both players stop at a single sequence of signal times (so $U \geq L$ is assumed therein).

New challenges arise from the above generalizations. Since the two players stop at two different sequences of Poisson arrival times, the first step to solve the constrained risk-sensitive Dynkin game is merging the two Poisson sequences

[1] Here we assume enough regularity of g. We first derive a second-order Taylor approximation of $g(R)$ at $\mathbb{E}[R]$, and taking the expectation for both sides gives us

$$\mathbb{E}[g(R)] \approx g(\mathbb{E}[R]) + \frac{1}{2} g''(\mathbb{E}[R]) \,\mathrm{Var}[R].$$

A first-order Taylor approximation of $g^{-1}(\mathbb{E}[g(R)])$ at $g(\mathbb{E}[R])$ yields

$$g^{-1}(\mathbb{E}[g(R)]) \approx \mathbb{E}[R] + \frac{1}{g'(\mathbb{E}[R])}[\mathbb{E}[g(R)] - g(\mathbb{E}[R])] \approx \mathbb{E}[R] - \frac{1}{2} l_g(\mathbb{E}[R]) \,\mathrm{Var}[R].$$

together while still keeping track of their order. This is crucial when we consider a family of constrained risk-sensitive Dynkin games (3.3.5)-(3.3.6) starting from different signal times in order to apply the dynamic programming principle. Note that the starting times of the games (3.3.5)-(3.3.6) may not be the respective player's own Poisson signal times; instead they could be from the counterparty's signal times. To deal with the nonlinear expectation $\tilde{\mathbb{E}}$ arising from the risk-sensitive function g, we introduce a new transformation resulting in the auxiliary payoff processes (3.2.7)-(3.2.9), which enable us to rewrite the payoff functional under the linear expectation $\mathbb{E}$ instead of the nonlinear expectation $\tilde{\mathbb{E}}$. For a special case of exponential risk-sensitive function g (see Section 3.5.2), the representation formula (3.2.10) of the game value is closely related to Cole-Hopf transformation in the BSDE literature, which is widely used to linearize a class of BSDEs with quadratic growth (see [50]). Our representation formula (3.2.10) can be regarded as a stochastic control version of Cole-Hopf transformation.

We also make a connection of constrained risk-sensitive Dynkin games with a class of stochastic differential games via Krylov's randomized stopping technique (see [51]). It is established in [51] that standard optimal stopping problems (without constraints on stopping times) admit stochastic control representation, which can be further solved via the so-called normalized Bellman equations. The stochastic control representation of the corresponding constrained optimal stopping problems has been established in [57] (see Section 4 therein). In a constrained stopping game setting as considered in the current chapter, it is natural to expect that a stochastic differential game representation should hold accordingly. Indeed, we show that the two players in the stochastic differential game choose their respective running controls and discount rates with binary values 0 or the Poisson intensity λ^i, and the optimal control is the Poisson intensity λ^i whenever the value of the game falls below the lower obstacle process/goes above the upper obstacle process.

Turing to the literature of Dynkin games, there has been a considerable development since the seminal works of Dynkin [26] and Neveu [68]. The continuous time models were developed, among others, by Bismut [8], Alario-Nazaret et al [1], Lepeltier and Maingueneau [56] and Morimoto [65]. In order to relax the dominating condition $U \geq L$ in those papers, Yasuda [80] proposed the strategies of randomized stopping times, and proved that the game value exists under merely an integrability condition. Rosemberg et al [71], Touzi and Vielle [78] and Laraki and Solan [52] further extended his work in this direction. The non-Markovian case was addressed in Cvitanic and Karatzas [16] for a fixed horizon and Hamadene et al [31] for an infinite horizon using the theory of reflected BSDEs. If the two players in the game are with asymmetric payoffs/information, then it gives arise to a nonzero-sum Dynkin game. See, for example, Hamadene and Zhang [33], De Angelis et al [19] and, more recently, De Angelis and Ekstrom [18] with more references therein. A robust version of Dynkin games can be found in Bayraktar and Yao [5] if the players are ambiguous about their probability model.

On the other hand, the risk-sensitive optimal stopping problems have been studied by Nagai [67], Bäuerle and Rieder [3], Bäuerle and Popp [2] and, more recently, Jelito et al [45]. For the risk-sensitive zero-sum and nonzero-sum stochastic differential games, we refer to El-Karoui and Hamadène [27]. To the best of our knowledge, the study of risk-sensitive Dynkin games is still lacking, no matter with or without constraints on stopping time strategies. The current chapter offers a first step to understand risk-sensitive Dynkin games, in particular with constraints on the stopping time strategies.

The constrained optimal stopping problems was first studied by Dupuis and Wang [25], where they used it to model perpetual American options exercised at exogenous Poisson arrival times. See also Lempa [55], Menaldi and Robin [61] and Hobson and Zeng [39] for further extensions of this type of optimal stopping models. From a different perspective, Liang [57] made a connection between such kind of optimal stopping problems with penalized BSDEs. The corresponding optimal switching (impulse control) models were studied by Liang and Wei [59], and by Menaldi and Robin [62] [63] with more general signal times and state spaces. More recently, Liang and Sun (Chapter 2) introduced the corresponding constrained Dynkin games (with the risk-sensitive function $g(x) = x$), where both players were allowed to stop at a sequence of random times generated by a single exogenous Poisson process serving as a signal process.

The chapter is organized as follows. Section 3.2 contains the problem formulation and main result, with its proof provided in Section 3.3. In Section 3.4, we establish its connection with a class of stochastic differential games, and in Section 3.5 we further provide two examples. Finally, Section 3.6 concludes the chapter.

3.2 Constrained risk-sensitive Dynkin games

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The probability space also supports two independent sequences of Poisson arrival times $T^{(1)} = \{T_n^{(1)}\}_{n\geq 0}$ and $T^{(2)} = \{T_n^{(2)}\}_{n\geq 0}$ with their respective intensities $\lambda^{(1)}$ and $\lambda^{(2)}$ and the minimal augmented filtration $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$, satisfying $T_0^{(1)} = T_0^{(2)} = 0$ and $T_\infty^{(1)} = T_\infty^{(2)} = \infty$. Denote the smallest filtration generated by $\mathbb{F}$ and $\mathbb{H}$ as $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and write $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$.

Let T be a finite $\mathbb{F}$-stopping time representing the (random) terminal time of the game. For each player $i \in \{1,2\}$, let us define a random variable $M_i : \Omega \mapsto \mathbb{N}$ such that T_{M_i} is the next arrival time in the Poisson sequence $T^{(i)}$ following T, i.e. $M_i(\omega) := \sum_{n\geq 1} n\mathbb{1}_{\{T^i_{n-1}(\omega)\leq T(\omega)<T^i_n(\omega)\}}$.

For any integer $n \geq 0$, we define the control set for each player $i \in \{1,2\}$ as

$$\mathcal{R}^{(i)}_n = \{\mathbb{G}\text{-stopping time } \sigma \text{ for } \sigma(\omega) = T^{(i)}_N(\omega) \text{ where } n \leq N \leq M_i(\omega)\}, \tag{3.2.1}$$

so the player i chooses from the Poisson arrival times $T^{(i)}$ with intensity $\lambda^{(i)}$, and $T^{(i)}_n$ is the smallest stopping time allowed.

Consider a *constrained risk-sensitive Dynkin game*, where the two players choose their respective stopping times $\sigma \in \mathcal{R}^{(1)}_1$ and $\tau \in \mathcal{R}^{(2)}_1$ in order to minimize/maximize the expected cost functional

$$J(\sigma,\tau) = \tilde{\mathbb{E}}\left[R(\sigma,\tau)\right], \tag{3.2.2}$$

where the nonlinear expectation $\tilde{\mathbb{E}} : \mathbb{R} \to \mathbb{R}$ is defined via the risk-sensitive function g, i.e.

$$\tilde{\mathbb{E}}\left[\cdot\right] := g^{-1}\left(\mathbb{E}\left[g\left(\cdot\right)\right]\right). \tag{3.2.3}$$

The discounted payoff functional $R(\sigma,\tau)$ in (3.2.2) is defined by

$$\begin{aligned} R(\sigma,\tau) = \int_0^{\sigma\wedge\tau\wedge T} e^{-rs} f_s\, ds + e^{-rT}\xi\mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + e^{-r\tau}L_\tau\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} \\ + e^{-r\sigma}U_\sigma\mathbb{1}_{\{\sigma<T,\sigma<\tau\}}, \end{aligned} \tag{3.2.4}$$

where $r > 0$ is the discount rate, and f, as a real-valued $\mathbb{F}$-progressively measurable process, is the running payoff. The terminal payoff is U if σ happens firstly, L if τ happens firstly or σ and τ happen simultaneously, and ξ otherwise, where L and U are two real-valued $\mathbb{F}$-progressively measurable processes, and ξ is a real-valued $\mathcal{F}_T$-measurable random variable.

Let us define the upper and lower values of the constrained risk-sensitve Dynkin game

$$\overline{v}^{\boldsymbol{\lambda}} = \inf_{\sigma\in\mathcal{R}^{(1)}_1}\sup_{\tau\in\mathcal{R}^{(2)}_1} J(\sigma,\tau), \text{ and } \underline{v}^{\boldsymbol{\lambda}} = \sup_{\tau\in\mathcal{R}^{(2)}_1}\inf_{\sigma\in\mathcal{R}^{(1)}_1} J(\sigma,\tau). \tag{3.2.5}$$

The game (3.2.5) is said to have value $v^{\boldsymbol{\lambda}}$ if $v^{\boldsymbol{\lambda}} = \overline{v}^{\boldsymbol{\lambda}} = \underline{v}^{\boldsymbol{\lambda}}$, and a saddle point $(\sigma^*,\tau^*) \in \mathcal{R}^{(1)}_1 \times \mathcal{R}^{(2)}_1$ is called an optimal stopping strategy of the game if

$$J(\sigma^*,\tau) \leq J(\sigma^*,\tau^*) \leq J(\sigma,\tau^*),$$

for every $(\sigma,\tau) \in \mathcal{R}^{(1)}_1 \times \mathcal{R}^{(2)}_1$.

Compared with the constrained Dynkin game introduced in Chapter 2, there are two new features of the game (3.2.5): First, it takes into consideration of the both players' attitudes towards risks via the risk-sensitive function g; Second, there are control constraints for both players and, moreover, the constraints are different in the sense that they are allowed to stop at two heterogeneous sequences of Poisson arrival times. As a consequence, since the two players' stopping time strategies are chosen from two different control sets, the usual dominating condition $U \geq L$ is not required. In Chapter 2, the risk-sensitive function $g(x) = x$ and both players stop at a single sequence of Poisson arrival times (so $U \geq L$ is a critical assumption therein).

3.2.1 Main result of this chapter

To solve the above constrained risk-sensitive Dynkin game, we introduce the characterizing BSDE on a random horizon $[0, T]$:

$$\overline{Q}^{\boldsymbol{\lambda}}_{t\wedge T} = \overline{\xi} + \int_{t\wedge T}^{T} \left[-\lambda^{(1)} \left(\overline{Q}^{\boldsymbol{\lambda}}_s - \overline{U}_s \right)^+ + \lambda^{(2)} \left(\overline{L}_s - \overline{Q}^{\boldsymbol{\lambda}}_s \right)^+ - r\overline{Q}^{\boldsymbol{\lambda}}_s \right] ds - \int_{t\wedge T}^{T} \overline{Z}^{\boldsymbol{\lambda}}_s \, dW_s, \quad (3.2.6)$$

for $t \geq 0$, where the *auxiliary payoff processes* $\overline{L}$, $\overline{U}$ and $\overline{\xi}$ are given by

$$\overline{L}_t = e^{rt} g(e^{-rt} L_t + \int_0^t e^{-ru} f_u \, du), \quad (3.2.7)$$

$$\overline{U}_t = e^{rt} g(e^{-rt} U_t + \int_0^t e^{-ru} f_u \, du), \quad (3.2.8)$$

$$\overline{\xi} = e^{rT} g(e^{-rT} \xi + \int_0^T e^{-ru} f_u \, du), \quad (3.2.9)$$

respectively. And also we set $\overline{Q}^{\boldsymbol{\lambda}}_t \equiv \overline{\xi}$ for $t \geq T$. Moreover, we introduce the following spaces: for any given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$,

- $\mathbb{L}^{2,n}_{\alpha}$: $\mathcal{F}_T$-measurable random variables $\xi : \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[e^{2\alpha T} ||\xi||^2\right] < \infty$,
- $\mathbb{H}^{2,n}_{\alpha}$: $\mathbb{F}$-progressively measurable processes $\varphi : [0, T] \times \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[\int_0^T e^{2\alpha s} ||\varphi_s||^2 \, ds\right] < \infty$,
- $\mathbb{S}^{2,n}_{\alpha}$: $\mathbb{F}$-progressively measurable processes $\varphi : [0, T] \times \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[\sup_{s\in[0,T]} e^{2\alpha s} ||\varphi_s||^2\right] < \infty$,

where $|| \cdot ||$ is the Euclidian norm and we denote $\mathbb{L}^{2,n}_0$, $\mathbb{H}^{2,n}_0$ and $\mathbb{S}^{2,n}_0$ by $\mathbb{L}^{2,n}$, $\mathbb{H}^{2,n}$ and $\mathbb{S}^{2,n}$ for the ease of notation.

We impose the following assumptions on the risk-sensitive function g, the running payoff f and the terminal payoffs L, U and ξ in terms of the auxiliary payoffs $\overline{L}$, $\overline{U}$ and $\overline{\xi}$.

Assumption 3.2.1 *The deterministic risk-sensitive function $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing and, moreover, (i) when T is an unbounded stopping time, $\overline{L}$, $\overline{U}$ and $\overline{\xi}$ are all bounded; (ii) when T is a bounded stopping time, $\overline{L} \in \mathbb{S}^{2,1}$, $\overline{U} \in \mathbb{S}^{2,1}$ and $\overline{\xi} \in \mathbb{L}^{2,1}$, where $\overline{L}$, $\overline{U}$ and $\overline{\xi}$ are given by (3.2.7), (3.2.8) and (3.2.9), respectively.*

On the one hand, since the two players' control sets are different, the usual dominating condition $U \geq L$ is not required. On the other hand, the conditions (i) and (ii) in Assumption 3.2.1 guarantee the existence and uniqueness of the solution to BSDE (3.2.6), which will in turn be used to construct the game value and its associated optimal stopping strategy. Under Assumption 3.2.1, the solvability of BSDE (3.2.6) follows from Theorem 3.3 in [12] (when T is unbounded) and Theorem 4.1 in [69] (when T is bounded), and thus we omit the proof of the following proposition and refer to [12] and [69] for the details.

Proposition 3.2.2 *Suppose that Assumption 3.2.1 holds. Then, there exists a unique solution $(\overline{Q}^{\boldsymbol{\lambda}}, \overline{Z}^{\boldsymbol{\lambda}})$ to BSDE (3.2.6). Moreover, (i) when T is an unbounded stopping time, $\overline{Q}^{\boldsymbol{\lambda}}$ is continuous and bounded, and $\overline{Z}^{\boldsymbol{\lambda}}$ belongs to $\mathbb{H}^{2,d}_{-r}$; (ii) when T is a bounded stopping time, the solution pair $(\overline{Q}^{\boldsymbol{\lambda}}, \overline{Z}^{\boldsymbol{\lambda}})$ belong to $\mathbb{S}^{2,1} \times \mathbb{H}^{2,d}$.*

We are now in a position to state the main result of this chapter.

Theorem 3.2.3 *Suppose that Assumption 3.2.1 holds. Let $(\overline{Q}^{\boldsymbol{\lambda}}, \overline{Z}^{\boldsymbol{\lambda}})$ be the unique solution to BSDE (3.2.6), and define the value process*

$$Q^{\boldsymbol{\lambda}}_t = e^{r(t\wedge T)} g^{-1}(e^{-r(t\wedge T)} \overline{Q}^{\boldsymbol{\lambda}}_t) - \int_0^{t\wedge T} e^{-r(u-t\wedge T)} f_u \, du, \tag{3.2.10}$$

for $t \geq 0$. Then, the value of the constrained risk-sensitive Dynkin game (3.2.5) exists and is given by

$$v^{\boldsymbol{\lambda}} = \overline{v}^{\boldsymbol{\lambda}} = \underline{v}^{\boldsymbol{\lambda}} = Q^{\boldsymbol{\lambda}}_0.$$

Moreover, the optimal stopping strategy of the game is given by

$$\begin{cases} \sigma^* = \inf\{T^{(1)}_N \geq T^{(1)}_1 : Q^{\boldsymbol{\lambda}}_{T^{(1)}_N} \geq U_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1}; \\ \tau^* = \inf\{T^{(2)}_N \geq T^{(2)}_1 : Q^{\boldsymbol{\lambda}}_{T^{(2)}_N} \leq L_{T^{(2)}_N}\} \wedge T^{(2)}_{M_2}. \end{cases}$$

Remark 3.2.4 *For a special case of exponential risk-sensitive function g (see Section 3.5.2), the representation formula (3.2.10) is closely related to*

Cole-Hopf transformation in the BSDE literature, which is widely used to linearize a class of BSDEs with quadratic growth (see [50]). Our representation formula (3.2.10) can be regarded as a stochastic control version of Cole-Hopf transformation.

3.3 Proof of Theorem 3.2.3

Since the two players stop at two different sequences of Poisson arrival times, the first step to prove Theorem 3.2.3 is merging the two Poisson sequences together while still keeping track of their order. To this end, for each $T^{(1)}$ and $T^{(2)}$, we construct an increasing sequence of $\mathbb{G}$-stopping times $\theta = (\theta_k)_{k\geq 0}$ as follows:

$$
\begin{aligned}
&\theta_0 = T_0^{(1)} = T_0^{(2)} = 0,\\
&\theta_1 = \min\left(T_1^{(1)}, T_1^{(2)}\right),\\
&\theta_2 = \min\left(T_1^{(1)}\mathbb{1}_{\{T_1^{(1)}>\theta_1\}} + T_2^{(1)}\mathbb{1}_{\{T_1^{(1)}\leq\theta_1\}}, T_1^{(2)}\mathbb{1}_{\{T_1^{(2)}>\theta_1\}} + T_2^{(2)}\mathbb{1}_{\{T_1^{(2)}\leq\theta_1\}}\right),\\
&\theta_3 = \min\left(T_1^{(1)}\mathbb{1}_{\{T_1^{(1)}>\theta_2\}} + T_3^{(1)}\mathbb{1}_{\{T_1^{(1)}\leq\theta_2\}}, T_2^{(1)}\mathbb{1}_{\{T_2^{(1)}>\theta_2\}} + T_3^{(1)}\mathbb{1}_{\{T_2^{(1)}\leq\theta_2\}},\right.\\
&\qquad\qquad \left. T_1^{(2)}\mathbb{1}_{\{T_1^{(2)}>\theta_2\}} + T_3^{(2)}\mathbb{1}_{\{T_1^{(2)}\leq\theta_2\}}, T_2^{(2)}\mathbb{1}_{\{T_2^{(2)}>\theta_2\}} + T_3^{(2)}\mathbb{1}_{\{T_2^{(2)}\leq\theta_2\}}\right),\\
&\quad\cdots,\\
&\theta_k = \min\Big(\\
&\quad T_1^{(1)}\mathbb{1}_{\{T_1^{(1)}>\theta_{k-1}\}} + T_k^{(1)}\mathbb{1}_{\{T_1^{(1)}\leq\theta_{k-1}\}}, \cdots, T_{k-1}^{(1)}\mathbb{1}_{\{T_{k-1}^{(1)}>\theta_{k-1}\}} + T_k^{(1)}\mathbb{1}_{\{T_{k-1}^{(1)}\leq\theta_{k-1}\}},\\
&\quad T_1^{(2)}\mathbb{1}_{\{T_1^{(2)}>\theta_{k-1}\}} + T_k^{(2)}\mathbb{1}_{\{T_1^{(2)}\leq\theta_{k-1}\}}, \cdots, T_{k-1}^{(2)}\mathbb{1}_{\{T_{k-1}^{(2)}>\theta_{k-1}\}} + T_k^{(2)}\mathbb{1}_{\{T_{k-1}^{(2)}\leq\theta_{k-1}\}}\Big),\\
&\quad\cdots.
\end{aligned}
$$

In Figure 3.1, we illustrate the construction of the merged sequence θ, where the top and the middle line are a realization of $T^{(1)}$ and $T^{(2)}$, and the bottom line is the merged sequence θ. Intuitively, given any $\mathbb{G}$-stopping time θ_{k-1}, $k \geq 1$, (to be used as the starting times for a family of constrained Dynkin games (3.3.5)-(3.3.6) below), we find the first arrival time of each Poisson sequence following θ_{k-1}, say $T_{k_1}^{(1)}$ and $T_{k_2}^{(2)}$ for some $k_1, k_2 \geq 0$, and then define $\theta_k = \min\{T_{k_1}^{(1)}, T_{k_2}^{(2)}\}$. Moreover, given the stopping time θ_k, we define pre-θ_k σ-field:

$$
\mathcal{G}_{\theta_k} = \left\{ A \in \bigvee_{s\geq 0} \mathcal{G}_s : A \cap \{\theta_k \leq s\} \in \mathcal{G}_s \text{ for } s \geq 0 \right\},
$$

and $\tilde{\mathbb{G}} = \{\mathcal{G}_{\theta_k}\}_{k\geq 0}$.

Next, we tackle the nonlinear expectation $\tilde{\mathbb{E}}$ associated with the risk-

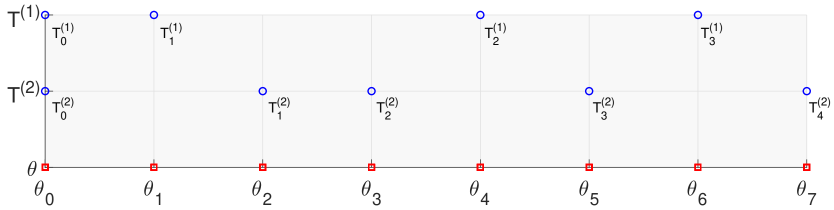


Figure 3.1: An illustration of a merged Poisson arrival sequence θ.

sensitive function g. To this end, introduce the *discounted processes*

$$\tilde{L}_t = e^{-rt} L_t + \int_0^t e^{-ru} f_u \, du, \tag{3.3.1}$$

$$\tilde{U}_t = e^{-rt} U_t + \int_0^t e^{-ru} f_u \, du, \tag{3.3.2}$$

$$\tilde{\xi} = e^{-rT} \xi + \int_0^T e^{-ru} f_u \, du, \tag{3.3.3}$$

and rewrite the discounted payoff functional $R(\sigma, \tau)$ as

$$\tilde{R}(\sigma, \tau) = \tilde{\xi} \mathbb{1}_{\{\sigma \wedge \tau \geq T\}} + \tilde{L}_\tau \mathbb{1}_{\{\tau < T, \tau \leq \sigma\}} + \tilde{U}_\sigma \mathbb{1}_{\{\sigma < T, \sigma < \tau\}} = R(\sigma, \tau). \tag{3.3.4}$$

In turn, consider a *family of constrained risk-sensitive Dynkin games* starting from θ_{k-1}, for $k \geq 1$, whose upper and lower values are defined by

$$\overline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \operatorname*{ess\,inf}_{\sigma \in \tilde{\mathcal{R}}^{(1)}_{\theta_k}} \operatorname*{ess\,sup}_{\tau \in \tilde{\mathcal{R}}^{(2)}_{\theta_k}} \tilde{\mathbb{E}} \left[\tilde{R}(\sigma, \tau) \, | \mathcal{G}_{\theta_{k-1}} \right], \tag{3.3.5}$$

$$\underline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \operatorname*{ess\,sup}_{\tau \in \tilde{\mathcal{R}}^{(2)}_{\theta_k}} \operatorname*{ess\,inf}_{\sigma \in \tilde{\mathcal{R}}^{(1)}_{\theta_k}} \tilde{\mathbb{E}} \left[\tilde{R}(\sigma, \tau) \, | \mathcal{G}_{\theta_{k-1}} \right], \tag{3.3.6}$$

where

$$\tilde{\mathcal{R}}^{(i)}_{\theta_k} = \Big\{ \mathbb{G}\text{-stopping time } \sigma \text{ for } \sigma(\omega) = T^{(i)}_N(\omega) \text{ where } T^{(i)}_N(\omega) \geq \theta_k \text{ and } N \leq M_i(\omega) \Big\}. \tag{3.3.7}$$

Remark 3.3.1 *Note that in the above definition of control set $\tilde{\mathcal{R}}^{(i)}_{\theta_k}$, θ_k is not necessary from the Poisson sequence $T^{(i)}$, so $\tilde{\mathcal{R}}^{(i)}_{\theta_k}$ is in general different from $\mathcal{R}^{(i)}_k$ in (3.2.1). However, they do coincide when $k = 1$: $\tilde{\mathcal{R}}^{(i)}_{\theta_1} = \mathcal{R}^{(i)}_1$.*

On the other hand, thanks to the introduction of the discounted processes $\tilde{L}$, $\tilde{U}$ and $\tilde{\xi}$ in (3.3.1)-(3.3.3), the payoff functional in (3.3.4) can be divided into three disjoint sets and the risk-sensitive function g can be applied to each of them separately. Thus, we can rewrite the payoff in (3.3.5)-(3.3.6) under *the linear expectation* $\mathbb{E}$ *of the auxiliary payoff processes* $\overline{L}$, $\overline{U}$ *and* $\overline{\xi}$ *as*

$$\tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{\theta_{k-1}}\right] = g^{-1}\Big(\mathbb{E}[e^{-rT}\overline{\xi}\mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + e^{-r\tau}\overline{L}_\tau\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} + e^{-r\sigma}\overline{U}_\sigma\mathbb{1}_{\{\sigma<T,\sigma<\tau\}}|\mathcal{G}_{\theta_{k-1}}]\Big).$$

This motivates us to introduce the Cole-Hopf representation formula (3.2.10).

The constrained risk-sensitive Dynkin game (3.3.5)-(3.3.6) is said to have value $q^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ if $q^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \overline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \underline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$, and $(\sigma^*_k,\tau^*_k)\in\tilde{\mathcal{R}}^{(1)}_{\theta_k}\times\tilde{\mathcal{R}}^{(2)}_{\theta_k}$ is called an optimal stopping strategy of the game if

$$\tilde{\mathbb{E}}\left[\tilde{R}(\sigma^*_k,\tau)|\mathcal{G}_{\theta_{k-1}}\right] \leq \tilde{\mathbb{E}}\left[\tilde{R}(\sigma^*_k,\tau^*_k)|\mathcal{G}_{\theta_{k-1}}\right] \leq \tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau^*_k)|\mathcal{G}_{\theta_{k-1}}\right],$$

for every $(\sigma,\tau)\in\tilde{\mathcal{R}}^{(1)}_{\theta_k}\times\tilde{\mathcal{R}}^{(2)}_{\theta_k}$. In particular, when $k=1$, (3.3.5)-(3.3.6) corresponds to the original constrained Dynkin game (3.2.5). Thus, to prove Theorem 3.2.3, it is sufficient to show that

$$q^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \overline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \underline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}},$$

and the optimal stopping strategy is given by

$$\begin{cases} \sigma^*_k = \inf\{T^{(1)}_N \geq \theta_k : \tilde{Q}^{\boldsymbol{\lambda}}_{T^{(1)}_N} \geq \tilde{U}_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1}, \\ \tau^*_k = \inf\{T^{(2)}_N \geq \theta_k : \tilde{Q}^{\boldsymbol{\lambda}}_{T^{(2)}_N} \leq \tilde{L}_{T^{(2)}_N}\} \wedge T^{(2)}_{M_2}, \end{cases} \tag{3.3.8}$$

where $\tilde{Q}^{\boldsymbol{\lambda}}$ is given by

$$\tilde{Q}^{\boldsymbol{\lambda}}_t = g^{-1}(e^{-r(t\wedge T)}\overline{Q}^{\boldsymbol{\lambda}}_t), \tag{3.3.9}$$

with $\overline{Q}^{\boldsymbol{\lambda}}$ being the first component of the solution to BSDE (3.2.6). In turn, the value process $Q^{\boldsymbol{\lambda}}$ in (3.2.10) is given via the discounted process $\tilde{Q}^{\boldsymbol{\lambda}}$ via the relationship

$$Q^{\boldsymbol{\lambda}}_t = e^{r(t\wedge T)}\tilde{Q}^{\boldsymbol{\lambda}}_t - \int_0^{t\wedge T} e^{-r(u-t\wedge T)}f_u du. \tag{3.3.10}$$

Note that, for $t\geq T$,

$$Q^{\boldsymbol{\lambda}}_t = e^{rT}g^{-1}(e^{-rT}\overline{\xi}) - \int_0^T e^{-r(u-T)}f_u du = \xi.$$

Remark 3.3.2 *For the reader's convenience, we recall the notations that have been introduced thus far. For the payoff processes* $h = L, U, \xi$*, we have defined the discounted processes* $\tilde{h}_t = e^{-rt}h_t + \int_0^t e^{-ru}f_u\,du$*, and auxiliary payoff processes* $\overline{h}_t = e^{rt}g(\tilde{h}_t)$*. In terms of the value process* $Q^{\boldsymbol{\lambda}}$*, likewise we have* $\tilde{Q}^{\boldsymbol{\lambda}}_t = e^{-rt}Q^{\boldsymbol{\lambda}}_t + \int_0^t e^{-ru}f_u\,du$*, and* $\overline{Q}^{\boldsymbol{\lambda}}_t = e^{rt}g(\tilde{Q}^{\boldsymbol{\lambda}}_t)$*, for* $t\in[0,T]$*.*

To prove the above assertions (and therefore Theorem 3.2.3), we start with the following lemma.

Lemma 3.3.3 *Suppose that Assumption 3.2.1 holds. Then,* $\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ *given in (3.3.9) satisfies the recursive equation*

$$\begin{aligned}\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \tilde{\mathbb{E}}\Big[&\tilde{\xi}\mathbb{1}_{\{\theta_k \geq T\}} \\ &+ \left(\min\{\tilde{U}_{\theta_k}, \tilde{Q}^{\boldsymbol{\lambda}}_{\theta_k}\}\mathbb{1}_{\{\theta_k \in T^{(1)}\}} + \max\{\tilde{L}_{\theta_k}, \tilde{Q}^{\boldsymbol{\lambda}}_{\theta_k}\}\mathbb{1}_{\{\theta_k \in T^{(2)}\}}\right)\mathbb{1}_{\{\theta_k < T\}}|\mathcal{G}_{\theta_{k-1}}\Big],\end{aligned} \tag{3.3.11}$$

for $k \geq 1$.

Proof. It is equivalent to prove that

$$\begin{aligned}g(\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}) = \mathbb{E}\Big[&g(\tilde{\xi})\mathbb{1}_{\{\theta_k \geq T\}} \\ &+(\min\{g(\tilde{U}_{\theta_k}), g(\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_k})\}\mathbb{1}_{\{\theta_k \in T^{(1)}\}}+\max\{g(\tilde{L}_{\theta_k}), g(\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_k})\}\mathbb{1}_{\{\theta_k \in T^{(2)}\}})\mathbb{1}_{\{\theta_k < T\}}|\mathcal{G}_{\theta_{k-1}}\Big],\end{aligned} \tag{3.3.12}$$

where $g(\tilde{\xi}) = e^{-rT}\overline{\xi}$, $g(\tilde{L}_t) = e^{-rt}\overline{L}_t$ and $g(\tilde{U}_t) = e^{-rt}\overline{U}_t$. For k such that $\theta_{k-1} > T$, it follows from (3.3.9) that $g(\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}) = g(\tilde{\xi})$, and thus (3.3.12) holds. In the rest of the proof, we only focus on the cases where $\theta_{k-1} \leq T$.

By applying Itô's formula to $\alpha_t g(\tilde{Q}^{\boldsymbol{\lambda}}_t)$, where $\alpha_t = e^{-(\lambda^{(1)}+\lambda^{(2)})t}$, we can obtain that

$$\begin{aligned}\alpha_{t\wedge T}g(\tilde{Q}^{\boldsymbol{\lambda}}_{t\wedge T}) &= \alpha_T g(\tilde{\xi}) + \int_{t\wedge T}^{T}\alpha_s\Big[(\lambda^{(1)}+\lambda^{(2)})g(\tilde{Q}^{\boldsymbol{\lambda}}_s) - \lambda^{(1)}\left(g(\tilde{Q}^{\boldsymbol{\lambda}}_s) - g(\tilde{U}_s)\right)^+ \\ &\qquad + \lambda^{(2)}\left(g(\tilde{L}_s) - g(\tilde{Q}^{\boldsymbol{\lambda}}_s)\right)^+\Big]\,ds - \int_{t\wedge T}^{T}\alpha_s e^{-rs}\overline{Z}^{\boldsymbol{\lambda}}_s\,dW_s \\ &= \alpha_T g(\tilde{\xi}) + \int_{t\wedge T}^{T}\alpha_s\Big[\lambda^{(1)}\min\left\{g(\tilde{U}_s), g(\tilde{Q}^{\boldsymbol{\lambda}}_s)\right\} \\ &\qquad + \lambda^{(2)}\max\left\{g(\tilde{L}_s), g(\tilde{Q}^{\boldsymbol{\lambda}}_s)\right\}\Big]\,ds - \int_{t\wedge T}^{T}\alpha_s e^{-rs}\overline{Z}^{\boldsymbol{\lambda}}_s\,dW_s,\end{aligned}$$

for $t \geq 0$. By choosing $t = \theta_{k-1}$ and taking the conditional expectation with respect to $\mathcal{G}_{\theta_{k-1}}$, we further have

$$\begin{aligned}g(\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}) = \mathbb{E}\Big[&e^{-(\lambda^{(1)}+\lambda^{(2)})(T-\theta_{k-1})}g(\tilde{\xi}) + \int_{\theta_{k-1}}^{T}e^{-(\lambda^{(1)}+\lambda^{(2)})(s-\theta_{k-1})} \\ &\left(\lambda^{(1)}\min\left\{g(\tilde{U}_s), g(\tilde{Q}^{\boldsymbol{\lambda}}_s)\right\} + \lambda^{(2)}\max\left\{g(\tilde{L}_s), g(\tilde{Q}^{\boldsymbol{\lambda}}_s)\right\}\right)\,ds|\mathcal{G}_{\theta_{k-1}}\Big],\end{aligned} \tag{3.3.13}$$

for any $k \geq 1$.

On the other hand, by defining $\tilde{T}_t^{(i)}$ as the first arrival time in $T^{(i)}$ following any fixed time t, i.e. $\tilde{T}_t^{(i)} = \inf\{T_N^{(i)} \geq T_1^{(i)} : T_N^{(i)} > t\}$, we can rewrite the right-hand-side of (3.3.12) as

$$\mathbb{E}\left[g(\tilde{\xi})\mathbb{1}_{\{\tilde{T}^{(1)}_{\theta_{k-1}}\wedge\tilde{T}^{(2)}_{\theta_{k-1}}\geq T\}} + \min\left\{g(\tilde{U}_{\tilde{T}^{(1)}_{\theta_{k-1}}}), g(\tilde{Q}^{\boldsymbol{\lambda}}_{\tilde{T}^{(1)}_{\theta_{k-1}}})\right\}\mathbb{1}_{\{\tilde{T}^{(1)}_{\theta_{k-1}}<T,\tilde{T}^{(1)}_{\theta_{k-1}}<\tilde{T}^{(2)}_{\theta_{k-1}}\}}\right.$$
$$\left. + \max\left\{g(\tilde{L}_{\tilde{T}^{(2)}_{\theta_{k-1}}}), g(\tilde{Q}^{\boldsymbol{\lambda}}_{\tilde{T}^{(2)}_{\theta_{k-1}}})\right\}\mathbb{1}_{\{\tilde{T}^{(2)}_{\theta_{k-1}}<T,\tilde{T}^{(2)}_{\theta_{k-1}}\leq\tilde{T}^{(1)}_{\theta_{k-1}}\}}\middle|\mathcal{G}_{\theta_{k-1}}\right]. \tag{3.3.14}$$

Indeed, applying the joint probability density function of $(\tilde{T}^{(1)}_{\theta_{k-1}}, \tilde{T}^{(2)}_{\theta_{k-1}})$ conditional on $\mathcal{G}_{\theta_{k-1}}$,

$$p_{\theta_{k-1}}(S,U) = \lambda^{(1)}e^{-\lambda^{(1)}(S-\theta_{k-1})}\lambda^{(2)}e^{-\lambda^{(2)}(U-\theta_{k-1})},$$

yields that

$$\begin{aligned}
&\mathbb{E}\left[g(\tilde{\xi})\mathbb{1}_{\{\tilde{T}^{(1)}_{\theta_{k-1}}\wedge\tilde{T}^{(2)}_{\theta_{k-1}}\geq T\}}\middle|\mathcal{G}_{\theta_{k-1}}\right]\\
&= \mathbb{E}\left[g(\tilde{\xi})\iint_{S\wedge U\geq T} p_{\theta_{k-1}}(S,U)\,dS\,dU\middle|\mathcal{G}_{\theta_{k-1}}\right]\\
&= \mathbb{E}\Bigg[g(\tilde{\xi})\underbrace{\iint_{U\geq S\geq T}\lambda^{(1)}e^{-\lambda^{(1)}(S-\theta_{k-1})}\lambda^{(2)}e^{-\lambda^{(2)}(U-\theta_{k-1})}\,dS\,dU}_{\text{(I)}}\Bigg|\mathcal{G}_{\theta_{k-1}}\Bigg]\\
&\quad + \mathbb{E}\Bigg[g(\tilde{\xi})\underbrace{\iint_{S\geq U\geq T}\lambda^{(1)}e^{-\lambda^{(1)}(S-\theta_{k-1})}\lambda^{(2)}e^{-\lambda^{(2)}(U-\theta_{k-1})}\,dS\,dU}_{\text{(II)}}\Bigg|\mathcal{G}_{\theta_{k-1}}\Bigg],
\end{aligned}$$

where the first integral

$$\begin{aligned}
\text{(I)} &= \lambda^{(1)}\int_T^\infty e^{-\lambda^{(1)}(S-\theta_{k-1})}\left(\int_S^\infty \lambda^{(2)}e^{-\lambda^{(2)}(U-\theta_{k-1})}\,dU\right)dS\\
&= \frac{\lambda^{(1)}}{\lambda^{(1)}+\lambda^{(2)}}e^{-(\lambda^{(1)}+\lambda^{(2)})(T-\theta_{k-1})},
\end{aligned}$$

and, similarly, the second integral

$$\text{(II)} = \frac{\lambda^{(2)}}{\lambda^{(1)}+\lambda^{(2)}}e^{-(\lambda^{(1)}+\lambda^{(2)})(T-\theta_{k-1})}.$$

In turn, we obtain

$$\mathbb{E}\left[g(\tilde{\xi})\mathbb{1}_{\{\tilde{T}^{(1)}_{\theta_{k-1}}\wedge\tilde{T}^{(2)}_{\theta_{k-1}}\geq T\}}\middle|\mathcal{G}_{\theta_{k-1}}\right] = \mathbb{E}\left[e^{-(\lambda^{(1)}+\lambda^{(2)})(T-\theta_{k-1})}g(\tilde{\xi})\middle|\mathcal{G}_{\theta_{k-1}}\right]. \tag{3.3.15}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E}\left[\min\left\{g(\tilde{U}_{\tilde{T}^{(1)}_{\theta_{k-1}}}), g(\tilde{Q}^{\boldsymbol{\lambda}}_{\tilde{T}^{(1)}_{\theta_{k-1}}})\right\}\mathbb{1}_{\{\tilde{T}^{(1)}_{\theta_{k-1}}<T, \tilde{T}^{(1)}_{\theta_{k-1}}<\tilde{T}^{(2)}_{\theta_{k-1}}\}}\middle|\mathcal{G}_{\theta_{k-1}}\right] \\
&= \mathbb{E}\left[\iint_{\theta_{k-1}<S<T, S<U}\min\left\{g(\tilde{U}_S), g(\tilde{Q}^{\boldsymbol{\lambda}}_S)\right\}p_{\theta_{k-1}}(S,U)\,dS\,dU\middle|\mathcal{G}_{\theta_{k-1}}\right] \\
&= \mathbb{E}\left[\int_{\theta_{k-1}}^{T}\lambda^{(1)}e^{-(\lambda^{(1)}+\lambda^{(2)})(S-\theta_{k-1})}\min\left\{g(\tilde{U}_S), g(\tilde{Q}^{\boldsymbol{\lambda}}_S)\right\}\,dS\middle|\mathcal{G}_{\theta_{k-1}}\right],
\end{aligned} \tag{3.3.16}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\max\left\{g(\tilde{L}_{\tilde{T}^{(2)}_{\theta_{k-1}}}), g(\tilde{Q}^{\boldsymbol{\lambda}}_{\tilde{T}^{(2)}_{\theta_{k-1}}})\right\}\mathbb{1}_{\{\tilde{T}^{(2)}_{\theta_{k-1}}<T, \tilde{T}^{(2)}_{\theta_{k-1}}\leq\tilde{T}^{(1)}_{\theta_{k-1}}\}}\middle|\mathcal{G}_{\theta_{k-1}}\right] \\
&= \mathbb{E}\left[\int_{\theta_{k-1}}^{T}\lambda^{(2)}e^{-(\lambda^{(1)}+\lambda^{(2)})(U-\theta_{k-1})}\max\left\{g(\tilde{L}_U), g(\tilde{Q}^{\boldsymbol{\lambda}}_U)\right\}\,dU\middle|\mathcal{G}_{\theta_{k-1}}\right].
\end{aligned} \tag{3.3.17}$$

It follows from (3.3.13), (3.3.15), (3.3.16) and (3.3.17) that (3.3.12) holds for any $k \geq 1$. Hence, $Q^{\boldsymbol{\lambda}}_{\theta_{k-1}}$, which is given by (3.3.9), satisfies the recursive equation (3.3.11), for $k \geq 1$. ■

As a direct consequence of Lemma 3.3.3, we deduce that $\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ defined by

$$\begin{aligned}
&\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} := \tilde{\xi}\mathbb{1}_{\{\theta_{k-1}\geq T\}} \\
&+\left(\min\{\tilde{U}_{\theta_{k-1}}, \tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}\}\mathbb{1}_{\{\theta_{k-1}\in T^{(1)}\}}+\max\{\tilde{L}_{\theta_{k-1}}, \tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}\}\mathbb{1}_{\{\theta_{k-1}\in T^{(2)}\}}\right)\mathbb{1}_{\{\theta_{k-1}<T\}},
\end{aligned} \tag{3.3.18}$$

where $\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ is given by (3.3.9), satisfies the recursive equation

$$\begin{aligned}
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \tilde{\xi}\mathbb{1}_{\{\theta_{k-1}\geq T\}} + \Bigg(&\min\{\tilde{U}_{\theta_{k-1}}, \tilde{\mathbb{E}}[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_k}|\mathcal{G}_{\theta_{k-1}}]\}\mathbb{1}_{\{\theta_{k-1}\in T^{(1)}\}} \\
&+\max\{\tilde{L}_{\theta_{k-1}}, \tilde{\mathbb{E}}[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_k}|\mathcal{G}_{\theta_{k-1}}]\}\mathbb{1}_{\{\theta_{k-1}\in T^{(2)}\}}\Bigg)\mathbb{1}_{\{\theta_{k-1}<T\}},
\end{aligned} \tag{3.3.19}$$

for $k \geq 1$.

We will show that $\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ in (3.3.18) is actually the unique solution of the recursive equation (3.3.19). The uniqueness is proved by showing that $\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ is the value of an *auxiliary constrained risk-sensitive Dynkin game* starting from

θ_{k-1}, whose upper and lower values are defined by

$$\bar{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}} \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}} \tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{\theta_{k-1}}\right], \tag{3.3.20}$$

$$\underline{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}} \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}} \tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{\theta_{k-1}}\right], \tag{3.3.21}$$

where the payoff functional $\tilde{R}(\sigma,\tau)$ is given by (3.3.4) and the control set $\tilde{\mathcal{R}}^{(i)}_{\theta_{k-1}}$ is given by (3.3.7).

The auxiliary game (3.3.20)-(3.3.21) is said to have value $\hat{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ if $\hat{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \bar{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \underline{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$, and $(\hat{\sigma}^*_{k-1},\hat{\tau}^*_{k-1}) \in \tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}} \times \tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}$ is called an optimal stopping strategy of the game (3.3.20)-(3.3.21) if

$$\tilde{\mathbb{E}}\left[\tilde{R}(\sigma^*_{k-1},\tau)|\mathcal{G}_{\theta_{k-1}}\right] \le \tilde{\mathbb{E}}\left[\tilde{R}(\sigma^*_{k-1},\tau^*_{k-1})|\mathcal{G}_{\theta_{k-1}}\right] \le \tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau^*_{k-1})|\mathcal{G}_{\theta_{k-1}}\right],$$

for every $(\sigma,\tau) \in \tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}} \times \tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}$.

The difference between (3.3.20)-(3.3.21) and (3.3.5)-(3.3.6) is that the players first make their stopping decisions and then move forward in the former game, while in the latter game they first move forward and then make their decisions.

Lemma 3.3.4 *Suppose that Assumption 3.2.1 holds. Then, for any $k \ge 1$, the value of the auxiliary constrained risk-sensitive Dynkin game (3.3.20)-(3.3.21) starting from θ_{k-1} exists. Its value $\hat{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ is the unique solution of the recursive equation (3.3.19). Hence, $\hat{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$, where the latter is given by (3.3.18). The optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game (3.3.20)-(3.3.21) is given by*

$$\begin{cases} \hat{\sigma}^*_{k-1} = \inf\{T^{(1)}_N \ge \theta_{k-1} : \hat{Q}^{\boldsymbol{\lambda}}_{T^{(1)}_N} = \tilde{U}_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1}; \\ \hat{\tau}^*_{k-1} = \inf\{T^{(2)}_N \ge \theta_{k-1} : \hat{Q}^{\boldsymbol{\lambda}}_{T^{(2)}_N} = \tilde{L}_{T^{(2)}_N}\} \wedge T^{(2)}_{M_2}. \end{cases} \tag{3.3.22}$$

Proof. *Step 1.* Let $\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ be a solution of the recursive equation (3.3.19) for $k \ge 1$. We claim the following martingale properties hold:
(i) $\left(\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m\wedge\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}\right)_{m\ge k-1}$ is a $\tilde{\mathbb{G}}$-martingale under the nonlinear expectation $\tilde{\mathbb{E}}$;
(ii) $\left(\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m\wedge\hat{\sigma}^*_{k-1}\wedge\tau}\right)_{m\ge k-1}$ is a $\tilde{\mathbb{G}}$-supermartingale under $\tilde{\mathbb{E}}$, for any $\tau \in \tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}$;
(iii) $\left(\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m\wedge\sigma\wedge\hat{\tau}^*_{k-1}}\right)_{m\ge k-1}$ is a $\tilde{\mathbb{G}}$-submartingale under $\tilde{\mathbb{E}}$, for any $\sigma \in \tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}$.

If the martingale property (i) holds, then, for $k \ge 1$,

$$\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}\wedge\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}} = \tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}|\mathcal{G}_{\theta_{k-1}}\right],$$

and the definition of $(\hat{\sigma}^*_{k-1}, \hat{\tau}^*_{k-1})$ in (3.3.22) further yields that

$$
\begin{aligned}
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} &= \tilde{\mathbb{E}}\Big[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\geq T\}} + \hat{Q}^{\boldsymbol{\lambda}}_{\hat{\tau}^*_{k-1}}\mathbb{1}_{\{\hat{\tau}^*_{k-1}<T, \hat{\tau}^*_{k-1}\leq\hat{\sigma}^*_{k-1}\}} \\
&\qquad + \hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}<T, \hat{\sigma}^*_{k-1}<\hat{\tau}^*_{k-1}\}}|\mathcal{G}_{\theta_{k-1}}\Big] \\
&= \tilde{\mathbb{E}}\Big[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\geq T\}} + \tilde{L}_{\hat{\tau}^*_{k-1}}\mathbb{1}_{\{\hat{\tau}^*_{k-1}<T, \hat{\tau}^*_{k-1}\leq\hat{\sigma}^*_{k-1}\}} \\
&\qquad + \tilde{U}_{\hat{\sigma}^*_{k-1}}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}<T, \hat{\sigma}^*_{k-1}<\hat{\tau}^*_{k-1}\}}|\mathcal{G}_{\theta_{k-1}}\Big] \\
&= \tilde{\mathbb{E}}\Big[\tilde{R}(\hat{\sigma}^*_{k-1}, \hat{\tau}^*_{k-1})|\mathcal{G}_{\theta_{k-1}}\Big]. \qquad (3.3.23)
\end{aligned}
$$

Using the similar arguments, if the supermartingale property (ii) and the submartingale property (iii) hold, then we have, for any $\tau \in \tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}$,

$$
\begin{aligned}
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} &\geq \tilde{\mathbb{E}}\Big[\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\tau}|\mathcal{G}_{\theta_{k-1}}\Big] \\
&= \tilde{\mathbb{E}}\Big[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\tau\geq T\}} + \hat{Q}^{\boldsymbol{\lambda}}_{\tau}\mathbb{1}_{\{\tau<T, \tau\leq\hat{\sigma}^*_{k-1}\}} + \hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}<T, \hat{\sigma}^*_{k-1}<\tau\}}|\mathcal{G}_{\theta_{k-1}}\Big] \\
&\geq \tilde{\mathbb{E}}\Big[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\tau\geq T\}} + \tilde{L}_{\tau}\mathbb{1}_{\{\tau<T, \tau\leq\hat{\sigma}^*_{k-1}\}} + \tilde{U}_{\hat{\sigma}^*_{k-1}}\mathbb{1}_{\{\hat{\sigma}^*_{k-1}<T, \hat{\sigma}^*_{k-1}<\tau\}}|\mathcal{G}_{\theta_{k-1}}\Big] \\
&= \tilde{\mathbb{E}}\Big[\tilde{R}(\hat{\sigma}^*_{k-1}, \tau)|\mathcal{G}_{\theta_{k-1}}\Big], \qquad (3.3.24)
\end{aligned}
$$

and, for any $\sigma \in \tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}$,

$$
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \leq \tilde{\mathbb{E}}\Big[\tilde{R}(\sigma, \hat{\tau}^*_{k-1})|\mathcal{G}_{\theta_{k-1}}\Big]. \qquad (3.3.25)
$$

It follows from (3.3.24) and (3.3.25) that

$$
\begin{aligned}
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} &\geq \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}} \tilde{\mathbb{E}}\Big[\tilde{R}(\hat{\sigma}^*_{k-1}, \tau)|\mathcal{G}_{\theta_{k-1}}\Big] \\
&\geq \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}} \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}} \tilde{\mathbb{E}}\Big[\tilde{R}(\sigma, \tau)|\mathcal{G}_{\theta_{k-1}}\Big] = \bar{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}},
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} &\leq \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}} \tilde{\mathbb{E}}\Big[\tilde{R}(\sigma, \hat{\tau}^*_{k-1})|\mathcal{G}_{\theta_{k-1}}\Big] \\
&\leq \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}} \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_{k-1}}} \tilde{\mathbb{E}}\Big[\tilde{R}(\sigma, \tau)|\mathcal{G}_{\theta_{k-1}}\Big] = \underline{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}}.
\end{aligned}
$$

It is clear that $\bar{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \geq \underline{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$, and therefore the value of the auxiliary constrained risk-sensitive Dynkin game (3.3.20)-(3.3.21) exists, i.e.

$$\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \hat{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \bar{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \underline{\hat{q}}^{\boldsymbol{\lambda}}_{\theta_{k-1}}.$$

This also implies the recursive equation (3.3.19) admits a unique solution. Furthermore, since $\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ given by (3.3.18) satisfies the recursive equation (3.3.19), it is actually the unique solution of (3.3.19). As a direct consequence of (3.3.23)-(3.3.25), we can obtain that $(\hat{\sigma}^*_{k-1}, \hat{\tau}^*_{k-1})$, which is given by (3.3.22), is indeed an optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game (3.3.20)-(3.3.21).

Step 2. It remains to prove the martingale property (i), the supermartingale property (ii) and the submartingale property (iii) in Step 1.

Indeed, for $m \geq k-1$, we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}\wedge\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}\middle|\mathcal{G}_{\theta_m}\right]\\
&=\tilde{\mathbb{E}}\left[\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\leq\theta_m\}}\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}+\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\geq\theta_{m+1}\}}\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}}|\mathcal{G}_{\theta_m}\right]\\
&=\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\leq\theta_m\}}\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}+\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\geq\theta_{m+1}\}}\tilde{\mathbb{E}}[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}}|\mathcal{G}_{\theta_m}]\\
&=\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\leq\theta_m\}}\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}+\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}\geq\theta_{m+1}\}}\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m}\\
&=\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m\wedge\hat{\sigma}^*_{k-1}\wedge\hat{\tau}^*_{k-1}}
\end{aligned}$$

where the second last equality follows from the definition (3.3.22) of $(\hat{\sigma}^*_{k-1}, \hat{\tau}^*_{k-1})$, and thus the martingale property (i) has been proved.

To prove the supermartingale property (ii), for any $\tau \in \tilde{\mathcal{R}}^{(2)}_{\theta_{k-1}}$, we have

$$\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}\wedge\hat{\sigma}^*_{k-1}\wedge\tau}\middle|\mathcal{G}_{\theta_m}\right]=\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\tau\leq\theta_m\}}\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\tau}+\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\tau\geq\theta_{m+1}\}}\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}}|\mathcal{G}_{\theta_m}\right].$$

Conditional on the set $\{\hat{\sigma}^*_{k-1} \wedge \tau \geq \theta_{m+1}\} \cap \{\theta_m < T\}$, we have

$$\begin{aligned}
\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m}&=\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}}\middle|\mathcal{G}_{\theta_m}\right]\mathbb{1}_{\{\theta_m\in T^{(1)}\}}+\max\left\{\tilde{L}_{\theta_m},\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}}\middle|\mathcal{G}_{\theta_m}\right]\right\}\mathbb{1}_{\{\theta_m\in T^{(2)}\}}\\
&\geq\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}}\middle|\mathcal{G}_{\theta_m}\right],
\end{aligned}$$

and thus

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\hat{Q}^{\boldsymbol{\lambda}}_{\theta_{m+1}\wedge\hat{\sigma}^*_{k-1}\wedge\tau}\middle|\mathcal{G}_{\theta_m}\right]\\
&\leq\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\tau\leq\theta_m\}}\hat{Q}^{\boldsymbol{\lambda}}_{\hat{\sigma}^*_{k-1}\wedge\tau}+\mathbb{1}_{\{\hat{\sigma}^*_{k-1}\wedge\tau\geq\theta_{m+1}\}}\left(\tilde{\xi}\mathbb{1}_{\{\theta_m\geq T\}}+\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m}\mathbb{1}_{\{\theta_m<T\}}\right)\\
&=\hat{Q}^{\boldsymbol{\lambda}}_{\theta_m\wedge\hat{\sigma}^*_{k-1}\wedge\tau},
\end{aligned}$$

which proves the supermartingale property (ii). Likewise, the submartingale property (iii) can be proved in a similar way, and the proof of this lemma is thus completed. ■

We are now in a position to prove Theorem 3.2.3. Let $\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ be a solution of the recursive equation (3.3.11), and in turn,

$$\begin{aligned}\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} &= \tilde{\mathbb{E}}\left[\tilde{\xi}\mathbb{1}_{\{\theta_k\geq T\}} + \hat{Q}^{\boldsymbol{\lambda}}_{\theta_k}\mathbb{1}_{\{\theta_k<T\}}\middle|\mathcal{G}_{\theta_{k-1}}\right] \\ &= \tilde{\mathbb{E}}\left[\tilde{\xi}\mathbb{1}_{\{\theta_k\geq T\}} + \tilde{\mathbb{E}}\left[\tilde{R}(\hat{\sigma}^*_k,\hat{\tau}^*_k)|\mathcal{G}_{\theta_k}\right]\mathbb{1}_{\{\theta_k<T\}}\middle|\mathcal{G}_{\theta_{k-1}}\right] \\ &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[\tilde{\xi}\mathbb{1}_{\{\theta_k\geq T\}} + \tilde{R}(\hat{\sigma}^*_k,\hat{\tau}^*_k)\mathbb{1}_{\{\theta_k<T\}}|\mathcal{G}_{\theta_k}\right]\middle|\mathcal{G}_{\theta_{k-1}}\right] \\ &= \tilde{\mathbb{E}}\Big[\tilde{\xi}\left(\mathbb{1}_{\{\theta_k\geq T\}} + \mathbb{1}_{\{\hat{\sigma}^*_k\wedge\hat{\tau}^*_k\geq T\}}\mathbb{1}_{\{\theta_k<T\}}\right) + \tilde{L}_{\hat{\tau}^*_k}\mathbb{1}_{\{\hat{\tau}^*_k<T,\hat{\tau}^*_k\leq\hat{\sigma}^*_k\}}\mathbb{1}_{\{\theta_k<T\}} \\ &\quad + \tilde{U}_{\hat{\sigma}^*_k}\mathbb{1}_{\{\hat{\sigma}^*_k<T,\hat{\sigma}^*_k<\hat{\tau}^*_k\}}\mathbb{1}_{\{\theta_k<T\}}|\mathcal{G}_{\theta_{k-1}}\Big].\end{aligned}$$

Using the relationship $\{\theta_k \geq T\} \subseteq \{\hat{\sigma}^*_k \wedge \hat{\tau}^*_k \geq T\}$, $\{\hat{\tau}^*_k < T, \hat{\tau}^*_k \leq \hat{\sigma}^*_k\} \subseteq \{\theta_k < T\}$ and $\{\hat{\sigma}^*_k < T, \hat{\sigma}^*_k < \hat{\tau}^*_k\} \subseteq \{\theta_k < T\}$, we can further obtain that

$$\begin{aligned}\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} &= \tilde{\mathbb{E}}\left[\tilde{\xi}\mathbb{1}_{\{\hat{\sigma}^*_k\wedge\hat{\tau}^*_k\geq T\}} + \tilde{L}_{\hat{\tau}^*_k}\mathbb{1}_{\{\hat{\tau}^*_k<T,\hat{\tau}^*_k\leq\hat{\sigma}^*_k\}} + \tilde{U}_{\hat{\sigma}^*_k}\mathbb{1}_{\{\hat{\sigma}^*_k<T,\hat{\sigma}^*_k<\hat{\tau}^*_k\}}|\mathcal{G}_{\theta_{k-1}}\right] \\ &= \tilde{\mathbb{E}}\left[\tilde{R}(\hat{\sigma}^*_k,\hat{\tau}^*_k)|\mathcal{G}_{\theta_{k-1}}\right],\end{aligned} \tag{3.3.26}$$

where $(\hat{\sigma}^*_k, \hat{\tau}^*_k)$ is the optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game starting from θ_k given in (3.3.22). Similarly, we can obtain that, for any $\tau \in \tilde{\mathcal{R}}^{(2)}_{\theta_k}$,

$$\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \geq \tilde{\mathbb{E}}\left[\tilde{R}(\hat{\sigma}^*_k,\tau)|\mathcal{G}_{\theta_{k-1}}\right], \tag{3.3.27}$$

and, for any $\sigma \in \tilde{\mathcal{R}}^{(1)}_{\theta_k}$,

$$\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \leq \tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\hat{\tau}^*_k)|\mathcal{G}_{\theta_{k-1}}\right]. \tag{3.3.28}$$

It follows from (3.3.27) and (3.3.28) that

$$\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \geq \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_k}}\tilde{\mathbb{E}}\left[\tilde{R}(\hat{\sigma}^*_k,\tau)|\mathcal{G}_{\theta_{k-1}}\right] \geq \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_k}}\operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_k}}\tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{\theta_{k-1}}\right] = \overline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}},$$

and

$$\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \leq \operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_k}}\tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\hat{\tau}^*_k)|\mathcal{G}_{\theta_{k-1}}\right] \leq \operatorname*{ess\,sup}_{\tau\in\tilde{\mathcal{R}}^{(2)}_{\theta_k}}\operatorname*{ess\,inf}_{\sigma\in\tilde{\mathcal{R}}^{(1)}_{\theta_k}}\tilde{\mathbb{E}}\left[\tilde{R}(\sigma,\tau)|\mathcal{G}_{\theta_{k-1}}\right] = \underline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}.$$

It is clear that $\overline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} \geq \underline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$, and therefore the value of the constrained risk-sensitive Dynkin game starting from θ_{k-1} (3.3.5)-(3.3.6) exists, i.e.

$$\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = q^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \overline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}} = \underline{q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}.$$

This also implies the recursive equation (3.3.11) admits a unique solution. Furthermore, since $\tilde{Q}^{\boldsymbol{\lambda}}_{\theta_{k-1}}$ given by (3.3.9) satisfies the recursive equation (3.3.11), it is actually the unique solution of (3.3.11). As a direct consequence of (3.3.26)-(3.3.28), we can obtain that $(\hat{\sigma}^*_k, \hat{\tau}^*_k)$, which is given by (3.3.22), is indeed an optimal stopping strategy of the constrained risk-sensitive Dynkin game (3.3.5)-(3.3.6).

We conclude the proof by proving $(\hat{\sigma}^*_k, \hat{\tau}^*_k)$ are actually (σ^*_k, τ^*_k) in (3.3.8). Indeed,

$$\begin{aligned}
\hat{\sigma}^*_k &= \inf\{T^{(1)}_N \geq \theta_k : \hat{Q}^{\boldsymbol{\lambda}}_{T^{(1)}_N} = \tilde{U}_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1} \\
&= \inf\{T^{(1)}_N \geq \theta_k : \tilde{Q}^{\boldsymbol{\lambda}}_{T^{(1)}_N} \geq \tilde{U}_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1} = \sigma^*_k,
\end{aligned}$$

and, similarly, $\hat{\tau}^*_k = \tau^*_k$.

3.4 Connection with stochastic differential games via randomized stopping

In this section, we connect constrained risk-sensitive Dynkin games with a class of stochastic differential games via randomized stopping first introduced by Krylov (see [51]). In particular, we generalize the optimal control representation of constrained optimal stopping problems in [57] (see Section 4 therein).

Let us introduce the basic idea of randomized stopping in a two-player setting as follows. Consider a nonnegative control process $(a_t)_{t\geq 0}$ (resp. $(b_t)_{t\geq 0}$), and let Player I (resp. II) stop with probability

$$\mathbb{P}(\tau \leq t + \Delta | \tau > t) = 1 - e^{-\int_t^{t+\Delta} a_s\, ds} \approx a_t \Delta$$

(resp.

$$\mathbb{P}(\tau \leq t + \Delta | \tau > t) = 1 - e^{-\int_t^{t+\Delta} b_s\, ds} \approx b_t \Delta)$$

in an infinitesimal interval $(t, t+\Delta)$. Then the probability that Player I (resp. II) does not stop before time t is

$$e^{-\int_0^t a_u\, du} \quad \left(\text{resp. } e^{-\int_0^t b_u\, du}\right),$$

and the probability that both players do not stop before time t and Player I (resp. II) does stop in the infinitesimal interval $(t, t+\Delta)$ is

$$e^{-\int_0^t (a_u+b_u)\,du} a_t \Delta \quad \left(\text{resp. } e^{-\int_0^t (a_u+b_u)\,du} b_t \Delta\right).$$

Recall that T is a finite $\mathbb{F}$-stopping time representing the (random) terminal time of the game, and $r > 0$ represents the discount rate. The discounted payoff is assumed to be $e^{-rt}\overline{U}_t$ if Player I stops firstly at time $t < T$, $e^{-rt}\overline{L}_t$ if Player II stops firstly at time $t < T$, and $e^{-rT}\overline{\xi}$ if neither players stop in the time interval $[0, T]$, where the auxiliary payoff processes $\overline{U}$, $\overline{L}$ and $\overline{\xi}$ are given in (3.2.8), (3.2.7), and (3.2.9), respectively. Thus, the discounted payoff functional associated with the control processes a and b is given by

$$J(a,b) = \int_0^T e^{-\int_0^t (a_u+b_u+r)\,du} \left(a_t\overline{U}_t + b_t\overline{L}_t\right)\, dt + e^{-\int_0^T (a_u+b_u+r)\,du}\overline{\xi},$$

or in terms of the original processes L, U and ξ,

$$\begin{aligned}J(a,b) = &\int_0^T e^{-\int_0^t (a_u+b_u)\,du}\Bigg[a_t g(e^{-rt}U_t + \int_0^t e^{-ru} f_u du) \\ &+ b_t g(e^{-rt}L_t + \int_0^t e^{-ru} f_u du)\Bigg] + e^{-\int_0^T (a_u+b_u)\,du} g(e^{-rT}\xi + \int_0^T e^{-ru} f_u du).\end{aligned}$$

We define the control set $\mathcal{A}(\lambda^{(1)})$ (resp. $\mathcal{B}(\lambda^{(2)})$) for Player I (resp. II) as

$$\mathcal{A}(\lambda^{(1)}) = \{\mathbb{F}\text{-adapted process } (a_t)_{t\geq 0} : a_t = 0 \text{ or } \lambda^{(1)}\}$$

(resp.

$$\mathcal{B}(\lambda^{(2)}) = \{\mathbb{F}\text{-adapted process } (b_t)_{t\geq 0} : b_t = 0 \text{ or } \lambda^{(2)}\}),$$

and the upper and lower values of the stochastic differential game as

$$\overline{v}^{\boldsymbol{\lambda},SDG} = \inf_{a\in\mathcal{A}(\lambda^{(1)})} \sup_{b\in\mathcal{B}(\lambda^{(2)})} g^{-1}\left(\mathbb{E}[J(a,b)]\right), \tag{3.4.1}$$

$$\underline{v}^{\boldsymbol{\lambda},SDG} = \sup_{b\in\mathcal{B}(\lambda^{(2)})} \inf_{a\in\mathcal{A}(\lambda^{(1)})} g^{-1}\left(\mathbb{E}[J(a,b)]\right), \tag{3.4.2}$$

where g^{-1} is the inverse function of the risk-sensitive function g. The game (3.4.1)-(3.4.2) is said to have value $v^{\boldsymbol{\lambda},SDG}$ if $v^{\boldsymbol{\lambda},SDG} = \overline{v}^{\boldsymbol{\lambda},SDG} = \underline{v}^{\boldsymbol{\lambda},SDG}$, and $(a^*, b^*) \in \mathcal{A}(\lambda^{(1)}) \times \mathcal{B}(\lambda^{(2)})$ is said to be an optimal pair of controls if $v^{\boldsymbol{\lambda},SDG} = g^{-1}\left(\mathbb{E}[J(a^*, b^*)]\right)$.

We are now in a position to present the main result of this section.

Proposition 3.4.1 *Suppose that Assumption 3.2.1 holds. Let $(\overline{Q}^{\boldsymbol{\lambda}}, \overline{Z}^{\boldsymbol{\lambda}})$ be the unique solution to BSDE (3.2.6). Then, the value of the stochastic differential game (3.4.1)-(3.4.2) exists and equals the value $v^{\boldsymbol{\lambda}}$ of the constrained risk-*

sensitive Dynkin game (3.2.5), i.e.

$$v^{\boldsymbol{\lambda},SDG} = \overline{v}^{\boldsymbol{\lambda},SDG} = \underline{v}^{\boldsymbol{\lambda},SDG} = v^{\boldsymbol{\lambda}} = g^{-1}\left(\overline{Q}_0^{\boldsymbol{\lambda}}\right). \tag{3.4.3}$$

Moreover, the optimal pair of controls is given by

$$a_t^* = \lambda^{(1)} \mathbb{1}_{\{\overline{Q}_t^{\boldsymbol{\lambda}} \geq \overline{U}_t\}}, \quad b_t^* = \lambda^{(2)} \mathbb{1}_{\{\overline{Q}_t^{\boldsymbol{\lambda}} \leq \overline{L}_t\}} \tag{3.4.4}$$

for $t \geq 0$.

Proof. Following the similar arguments to the proof of Lemma 3.3.3, it can be shown that, for any pair of controls $(a, b) \in \mathcal{A}(\lambda^{(1)}) \times \mathcal{B}(\lambda^{(2)})$, $\mathbb{E}[J(a,b)] = V_0^{\boldsymbol{\lambda}}(a,b)$, where the latter is the first component of the unique solution to the following BSDE with a random terminal time T:

$$\begin{aligned} V_{t\wedge T}^{\boldsymbol{\lambda}}(a,b) = \overline{\xi} + \int_{t\wedge T}^{T} \left[a_u(\overline{U}_u - V_u^{\boldsymbol{\lambda}}(a,b)) + b_u(\overline{L}_u - V_u^{\boldsymbol{\lambda}}(a,b)) - rV_u^{\boldsymbol{\lambda}}(a,b) \right] du \\ - \int_{t\wedge T}^{T} Z_u^{\boldsymbol{\lambda}}(a,b)\, dW_u, \end{aligned}$$

for $t \geq 0$. On the other hand, recall that $\overline{Q}^{\boldsymbol{\lambda}}$ is the first component of the solution to BSDE (3.2.6):

$$\begin{aligned} \overline{Q}_{t\wedge T}^{\boldsymbol{\lambda}} = \overline{\xi} + \int_{t\wedge T}^{T} \left[-\lambda^{(1)} \left(\overline{Q}_u^{\boldsymbol{\lambda}} - \overline{U}_u \right)^+ + \lambda^{(2)} \left(\overline{L}_u - \overline{Q}_u^{\boldsymbol{\lambda}} \right)^+ - r\overline{Q}_u^{\boldsymbol{\lambda}} \right] du \\ - \int_{t\wedge T}^{T} \overline{Z}_u^{\boldsymbol{\lambda}}\, dW_u, \end{aligned}$$

for $t \geq 0$. By letting $b_t^* = \lambda^{(2)} \mathbb{1}_{\{\overline{Q}_t^{\boldsymbol{\lambda}} \leq \overline{L}_t\}}$, we obtain the inequality

$$-\lambda^{(1)}(\overline{Q}_u^{\boldsymbol{\lambda}} - \overline{U}_u)^+ + \lambda^{(2)}(\overline{L}_u - \overline{Q}_u^{\boldsymbol{\lambda}})^+ - r\overline{Q}_u^{\boldsymbol{\lambda}} \leq a_u(\overline{U}_u - \overline{Q}_u^{\boldsymbol{\lambda}}) + b_u^*(\overline{L}_u - \overline{Q}_u^{\boldsymbol{\lambda}}) - r\overline{Q}_u^{\boldsymbol{\lambda}}$$

holds for any control $a \in \mathcal{A}(\lambda^{(1)})$, and thus, the BSDE comparison result (see Corollary 4.4.2 in [17]) yields that

$$\overline{Q}_{t\wedge T}^{\boldsymbol{\lambda}} \leq V_{t\wedge T}^{\boldsymbol{\lambda}}(a, b^*), \tag{3.4.5}$$

for $t \geq 0$ and any control $a \in \mathcal{A}(\lambda^{(1)})$. Similarly, by letting $a_t^* = \lambda^{(1)} \mathbb{1}_{\{\overline{Q}_t^{\boldsymbol{\lambda}} \geq \overline{U}_t\}}$, we obtain

$$\overline{Q}_{t\wedge T}^{\boldsymbol{\lambda}} \geq V_{t\wedge T}^{\boldsymbol{\lambda}}(a^*, b), \tag{3.4.6}$$

for $t \geq 0$ and any control $b \in \mathcal{B}(\lambda^{(2)})$, and by letting $a_t^* = \lambda^{(1)} \mathbb{1}_{\{\overline{Q}_t^{\boldsymbol{\lambda}} \geq \overline{U}_t\}}$ and

$b_t^* = \lambda^{(2)} \mathbb{1}_{\{\overline{Q}_t^{\boldsymbol{\lambda}} \leq \overline{L}_t\}}$, we obtain the equality

$$\overline{Q}_{t\wedge T}^{\boldsymbol{\lambda}} = V_{t\wedge T}^{\boldsymbol{\lambda}}(a^*, b^*). \tag{3.4.7}$$

It follows from (3.4.5) that

$$\begin{aligned}
g^{-1}\left(\overline{Q}_0^{\boldsymbol{\lambda}}\right) &\leq \inf_{a\in\mathcal{A}(\lambda^{(1)})} g^{-1}\left(V_0^{\boldsymbol{\lambda}}(a, b^*)\right) \\
&= \inf_{a\in\mathcal{A}(\lambda^{(1)})} g^{-1}\left(\mathbb{E}[J(a, b^*)]\right) \\
&\leq \sup_{b\in\mathcal{B}(\lambda^{(2)})} \inf_{a\in\mathcal{A}(\lambda^{(1)})} g^{-1}\left(\mathbb{E}[J(a, b)]\right) \\
&= \underline{v}^{\boldsymbol{\lambda},SDG}.
\end{aligned}$$

Likewise, (3.4.6) yields that $g^{-1}\left(\overline{Q}_0^{\boldsymbol{\lambda}}\right) \geq \overline{v}^{\boldsymbol{\lambda},SDG}$. Hence, it follows from $\overline{v}^{\boldsymbol{\lambda},SDG} \geq \underline{v}^{\boldsymbol{\lambda},SDG}$ that (3.4.3) holds. As a direct consequence of (3.4.5)-(3.4.7), we can obtain (a^*, b^*) in (3.4.4) is an optimal pair of controls. ■

3.5 Examples

3.5.1 Example I: Constrained risk-neutral Dynkin games

As the first example, we take the risk-sensitive function to be $g(x) = x$. This means both players are risk neutral and, therefore, the corresponding games are called *constrained risk-neutral Dynkin games*. In this case, the cost functional in (3.2.2) is evaluated under the linear expectation $\mathbb{E}$:

$$\tilde{\mathbb{E}}\left[R(\sigma, \tau)\right] = \mathbb{E}\left[R(\sigma, \tau)\right]$$

with the payoff functional $R(\sigma, \tau)$ given by (3.2.4). Hence, the upper and lower values of the constrained risk-neutral Dynkin game are defined as

$$\overline{v}^{\boldsymbol{\lambda},RN} = \inf_{\sigma\in\mathcal{R}_1^{(1)}} \sup_{\tau\in\mathcal{R}_1^{(2)}} \mathbb{E}[R(\sigma, \tau)], \text{ and } \underline{v}^{\boldsymbol{\lambda},RN} = \sup_{\tau\in\mathcal{R}_1^{(2)}} \inf_{\sigma\in\mathcal{R}_1^{(1)}} \mathbb{E}[R(\sigma, \tau)]. \tag{3.5.1}$$

The game (3.5.1) is said to have value $v^{\boldsymbol{\lambda},RN}$ if $v^{\boldsymbol{\lambda},RN} = \overline{v}^{\boldsymbol{\lambda},RN} = \underline{v}^{\boldsymbol{\lambda},RN}$, and $(\sigma^{*,RN}, \tau^{*,RN}) \in \mathcal{R}_1^{(1)} \times \mathcal{R}_1^{(2)}$ is called an optimal stopping strategy of the game if

$$\mathbb{E}\left[R(\sigma^{*,RN}, \tau)\right] \leq \mathbb{E}\left[R(\sigma^{*,RN}, \tau^{*,RN})\right] \leq \mathbb{E}\left[R(\sigma, \tau^{*,RN})\right]$$

for every $(\sigma, \tau) \in \mathcal{R}_1^{(1)} \times \mathcal{R}_1^{(2)}$.

Recall

$$Q_t^{\boldsymbol{\lambda}} = \overline{Q}_t^{\boldsymbol{\lambda}} - \int_0^{t\wedge T} e^{-r(u-t\wedge T)} f_u \, du$$

in (3.2.10), where $(\overline{Q}^{\boldsymbol{\lambda}}, \overline{Z}^{\boldsymbol{\lambda}})$ is the unique solution to the characterizing BSDE

(3.2.6). Thus, we deduce the so-called penalized BSDE with double obstacles on a random horizon $[0, T]$ (see [16] for the case of a fixed terminal time T),

$$Q^{\boldsymbol{\lambda}}_{t\wedge T} = \xi + \int_{t\wedge T}^{T} \left[f_s - \lambda^{(1)} \left(Q^{\boldsymbol{\lambda}}_s - U_s \right)^+ + \lambda^{(2)} \left(L_s - Q^{\boldsymbol{\lambda}}_s \right)^+ - rQ^{\boldsymbol{\lambda}}_s \right] ds \\ - \int_{t\wedge T}^{T} \overline{Z}^{\boldsymbol{\lambda}}_s \, dW_s, \quad (3.5.2)$$

and $Q^{\boldsymbol{\lambda}}_t = \overline{\xi} - \int_0^T e^{-r(u-T)} f_u \, du = \xi$ for $t \geq T$.

Assumption 3.5.1 *The risk-sensitive function $g(x) = x$. Moreover, (i) when T is an unbounded stopping time, f, L, U and ξ are all bounded; (ii) when T is a bounded stopping time, $f \in \mathbb{H}^{2,1}$, $L \in \mathbb{S}^{2,1}$, $U \in \mathbb{S}^{2,1}$ and $\xi \in \mathbb{L}^{2,1}$.*

Note that the above assumption implies Assumption 3.2.1 and, therefore, it follows from Theorem 3.2.3 that BSDE (3.5.2) admits a unique solution $(Q^{\boldsymbol{\lambda}}, \overline{Z}^{\boldsymbol{\lambda}})$. Moreover, the value of the constrained risk-neutral Dynkin game (3.5.1) exists and is given by

$$v^{\boldsymbol{\lambda},RN} = \overline{v}^{\boldsymbol{\lambda},RN} = \underline{v}^{\boldsymbol{\lambda},RN} = Q^{\boldsymbol{\lambda}}_0.$$

The optimal stopping strategy is given by

$$\begin{cases} \sigma^{*,RN} = \inf\{T^{(1)}_N \geq T^{(1)}_1 : Q^{\boldsymbol{\lambda}}_{T^{(1)}_N} \geq U_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1}; \\ \tau^{*,RN} = \inf\{T^{(2)}_N \geq T^{(2)}_1 : Q^{\boldsymbol{\lambda}}_{T^{(2)}_N} \leq L_{T^{(2)}_N}\} \wedge T^{(2)}_{M_2}. \end{cases}$$

Remark 3.5.2 *The special case $g(x) = x$ generalizes the results obtained in [57] and Chapter 2. To be more specific, when $\lambda^{(1)} = 0$ (resp. $\lambda^{(2)} = 0$), Player I (resp. II) is with a zero intensity control set and is never allowed to stop, so the value of the constrained risk-neutral Dynkin game (3.5.1) equals the value of the one-player optimal stopping problem with Poisson intervention times introduced in [57]. On the other hand, when the two intensities coincide, i.e. $\lambda^{(1)} = \lambda^{(2)}$, the value of the constrained risk-neutral Dynkin game (3.5.1) equals the value of the Dynkin game with Poisson intervention times introduced in Chapter 2 (i.e. the game (2.2.2)-(2.2.3) introduced in Section 2.2).*

3.5.2 Example II: Constrained Dynkin games with exponential utility

The second example for the risk-sensitive function g is an exponential utility: $g(x) = -e^{-\gamma x}$ for $\gamma > 0$. In this case, the cost functional in (3.2.2) becomes

$$\tilde{\mathbb{E}}\left[R(\sigma, \tau)\right] = -\frac{1}{\gamma} \ln \mathbb{E}\left[\exp(-\gamma R(\sigma, \tau))\right]$$

with the payoff functional $R(\sigma,\tau)$ given by (3.2.4). Hence, the upper and lower values of the constrained risk-sensitive Dynkin game are defined as

$$\begin{aligned}
\overline{v}^{\boldsymbol{\lambda},EU} &= \inf_{\sigma\in\mathcal{R}_1^{(1)}} \sup_{\tau\in\mathcal{R}_1^{(2)}} -\frac{1}{\gamma}\ln\mathbb{E}[\exp(-\gamma R(\sigma,\tau))], && (3.5.3)\\
\underline{v}^{\boldsymbol{\lambda},EU} &= \sup_{\tau\in\mathcal{R}_1^{(2)}} \inf_{\sigma\in\mathcal{R}_1^{(1)}} -\frac{1}{\gamma}\ln\mathbb{E}[\exp(-\gamma R(\sigma,\tau))]. && (3.5.4)
\end{aligned}$$

The game (3.5.3)-(3.5.4) is said to have value $v^{\boldsymbol{\lambda},EU}$ if $v^{\boldsymbol{\lambda},EU}=\overline{v}^{\boldsymbol{\lambda},EU}=\underline{v}^{\boldsymbol{\lambda},EU}$, and $(\sigma^{*,EU},\tau^{*,EU})\in\mathcal{R}_1^{(1)}\times\mathcal{R}_1^{(2)}$ is called an optimal stopping strategy of the game if

$$\tilde{\mathbb{E}}\left[R(\sigma^{*,EU},\tau)\right]\le\tilde{\mathbb{E}}\left[R(\sigma^{*,EU},\tau^{*,EU})\right]\le\tilde{\mathbb{E}}\left[R(\sigma,\tau^{*,EU})\right]$$

for every $(\sigma,\tau)\in\mathcal{R}_1^{(1)}\times\mathcal{R}_1^{(2)}$.

Recall

$$Q_t^{\boldsymbol{\lambda}} = -\frac{1}{\gamma}e^{r(t\wedge T)}\ln(-e^{-r(t\wedge T)}\overline{Q}_t^{\boldsymbol{\lambda}}) - \int_0^{t\wedge T} e^{-r(u-t\wedge T)}f_u\,du \qquad (3.5.5)$$

in (3.2.10), where $(\overline{Q}^{\boldsymbol{\lambda}},\overline{Z}^{\boldsymbol{\lambda}})$ is the unique solution to the characterizing BSDE (3.2.6). By applying Itô's lemma to $Q_t^{\boldsymbol{\lambda}}$ in (3.5.5), we can deduce the following BSDE with quadratic growth on a random horizon $[0,T]$ (see [50] for the case of a fixed maturity T):

$$\begin{aligned}
Q_{t\wedge T}^{\boldsymbol{\lambda}} = \xi + \int_{t\wedge T}^{T}\Bigg[& f_u - \frac{\lambda^{(1)}}{\gamma}e^{ru}(e^{\gamma(e^{-ru}Q_u^{\boldsymbol{\lambda}}-e^{-ru}U_u)}-1)^+ \\
&+\frac{\lambda^{(2)}}{\gamma}e^{ru}(1-e^{\gamma(e^{-ru}Q_u^{\boldsymbol{\lambda}}-e^{-ru}L_u)})^+ - rQ_u^{\boldsymbol{\lambda}} - \frac{\gamma}{2}e^{-ru}||Z_u^{\boldsymbol{\lambda}}||^2\Bigg]du \\
&-\int_{t\wedge T}^{T} Z_u^{\boldsymbol{\lambda}}\,dW_u, \qquad (3.5.6)
\end{aligned}$$

for $t\ge 0$, where $Z_u^{\boldsymbol{\lambda}}=-e^{ru}\overline{Z}_u^{\boldsymbol{\lambda}}/(\gamma\overline{Q}_u^{\boldsymbol{\lambda}}), u\in[0,T]$, and

$$Q_t^{\boldsymbol{\lambda}} = -\frac{1}{\gamma}e^{rT}\ln(-e^{-rT}\overline{\xi}) - \int_0^{T} e^{-r(u-T)}f_u\,du = \xi$$

for $t\ge T$. Note that $\overline{Q}_t^{\boldsymbol{\lambda}}<0$ for $t\in[0,T]$, and therefore $Z_t^{\boldsymbol{\lambda}}$ is well-defined. Indeed, $\overline{L},\overline{U}<0$ by the construction of the risk-sensitive function $g(x)=-e^{-\gamma x}$, which implies the driver of BSDE (3.2.6) satisfying

$$-\lambda^{(1)}\left(0-\overline{U}_t\right)^+ + \lambda^{(2)}\left(\overline{L}_t-0\right)^+ - r\cdot 0\le 0.$$

for $t\in[0,T]$. Given the terminal condition $\overline{\xi}<0$ by construction, it follows from standard comparison results that $\overline{Q}_t^{\boldsymbol{\lambda}}<0$ holds for $t\in[0,T]$.

Assumption 3.5.3 *The risk-sensitive function $g(x) = -e^{-\gamma x}$ for $\gamma > 0$, and f, L, U and ξ are all bounded.*

Note that the above assumption implies Assumption 3.2.1 and, therefore, it follows from Theorem 3.2.3 that BSDE (3.5.6) admits a unique solution $(Q^{\boldsymbol{\lambda}}, Z^{\boldsymbol{\lambda}})$. Moreover, the value of the constrained risk-sensitive Dynkin game (3.5.3)-(3.5.4) exists and is given by

$$v^{\boldsymbol{\lambda},EU} = \overline{v}^{\boldsymbol{\lambda},EU} = \underline{v}^{\boldsymbol{\lambda},EU} = Q_0^{\boldsymbol{\lambda}}.$$

The optimal stopping strategy is given by

$$\begin{cases} \sigma^{*,EU} = \inf\{T_N^{(1)} \geq T_1^{(1)} : Q^{\boldsymbol{\lambda}}_{T_N^{(1)}} \geq U_{T_N^{(1)}}\} \wedge T_{M_1}^{(1)}; \\ \tau^{*,EU} = \inf\{T_N^{(2)} \geq T_1^{(2)} : Q^{\boldsymbol{\lambda}}_{T_N^{(2)}} \leq L_{T_N^{(2)}}\} \wedge T_{M_2}^{(2)}. \end{cases}$$

3.6 Conclusion and future work

In this chapter, we have solved a new class of Dynkin games with a general risk-sensitive criterion function g and two heterogenous Poisson arrival times as the permitted stopping time strategies for the two players. Moreover, we have made a connection with a class of stochastic differential games via the so-called randomized stopping technique.

The approach and the results herein may be extended in various directions. First, one may consider stochastic intensity models, an undoubtedly important case since the two players' signal times may affect each other's intensities. For example, for $i \in \{1, 2\}$, if the player i's first signal time $T_1^{(i)}$ occurs, it will have an impact (either positive or negative) on the other player's intensity:

$$\lambda_t^{(1)} = \lambda^{(1)} + \overline{\lambda}^{(1)} \mathbb{1}_{\{T_1^{(2)} \leq t\}}, \quad \lambda_t^{(2)} = \lambda^{(2)} + \overline{\lambda}^{(2)} \mathbb{1}_{\{T_1^{(1)} \leq t\}},$$

for some constants $\lambda^{(i)}, \overline{\lambda}^{(i)}$ such that the process $(\lambda_t^{(i)})_{t\geq 0}$ is always nonnegative. However, various nontrivial technical difficulties arise. In particular, the resulting characterizing BSDEs will become a family of recursive equations, whose solvability is far from clear yet.

Second, one may consider that the two players have different attitudes towards risks and are associated with different information sets. For example, one player is risk-neutral with $g^{(1)}(x) = x$ and the other has an exponential utility with $g^{(2)}(x) = -e^{-\gamma x}$. This leads to heterogenous payoff functionals and, therefore a nonzero-sum constrained Dynkin game arises. The corresponding characterizing equations will become a BSDE system. Both extensions will be left for the future research.

Chapter 4

Optimal Switching Games with Poisson Random Intervention Times

4.1 Introduction

Optimal switching is a generalisation of optimal stopping, where one or more agents determine their optimal sequence of times to switch a system's operational modes. Optimal switching has various applications in economics and finance, in particular for real options (see, for example, Brekke and Øksendal [10] and Duckworth and Zervos [24]). In the literature, there are mainly two ways to solve optimal switching problems: either an analytical one using PDE (see, for example, Bensoussan and Lions [6] and Tang and Yong [76]) and a probabilistic one using martingale approach and BSDE (see, for example, Djehiche et al [23] and Hu and Tang [42]).

Unlike single-player optimisation problems, which all of the aforementioned references are concerned with, multiple-player optimal switching games did not attract much interest in the literature. Two-player zero-sum optimal switching games with strictly positive switching costs for deterministic systems were first considered and solved by Yong ([81] in a finite horizon case and [82] in an infinite horizon case) using a viscosity solution approach. Tang and Hou [77] formulated and solved a similar game in a stochastic setting. Hu and Tang [43] proved the existence of the solution to a system of reflected BSDEs with interconnected obstacles, which were claimed to be associated to two-player zero-sum optimal switching games. In the case when the running and terminal rewards are separated (see (4.5.4)), Djehiche et al [22] made a connection between the game value and the system of reflected BSDEs with interconnected obstacles. To the best of our knowledge, the existence of the value of a two-player zero-sum optimal switching game, without additional conditions on the running and terminal rewards (for example, (4.5.4)), is still an open question.

In this chapter, we consider a new type of two-player zero-sum optimal switching games, where two players are only allowed to switch at two heterogeneous exogenous sequences of Poisson arrival times. We call this kind of optimal switching games as *constrained optimal switching games*.

Our main result is Theorem 4.2.5, providing the chain of inequalities (4.2.8) involving the lower and upper static values (4.2.4), the lower and upper dynamic values (4.2.5) of constrained optimal switching games and the solution of a BSDE system, where the latter can be regarded as a "penalized version" of a system of reflected BSDEs with interconnected obstacles. The basic idea comes from the Dynkin game representation for one dimensional penalized BSDEs (see Section 4.4.2). Under some additional conditions on the running and terminal rewards, as imposed in [22], we show the value of constrained optimal switching games exists and equals the solution of the above BSDE system, and establish its connection with constrained single-player optimal switching problems.

Finally at the end of this chapter, we study the duopolistic competition in resource extraction when the resource price follows a one-dimensional geometric Brownian motion. Both producers are allowed to either open a field for producing at most a given amount of resource or close down a field at Poisson arrival times, aiming to maximise/minimise the difference of their expected profits. This creates a constrained optimal switching game, whose structure of switching regions can be fully described. The problem we solve is closely related with the classical model imposed by Brekke and Øksendal [10].

The chapter is organized as follows. Section 4.2 contains the problem formulation and main result on the chain of inequalities involving the game values and the solution of a BSDE system, whose solvability is proved in Section 4.3. Section 4.4 provides the proof of the main result. In Section 4.5, under some additional conditions on the running and terminal rewards, we show the game has a value and establish a connection with constrained single-player optimal switching problems. In Section 4.6, we apply the constrained optimal switching games to study the duopolistic competition in resource extraction, and give a complete description of the structure of switching regions.

4.2 Constrained optimal switching games

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $T^{(1)} = \{T_n^{(1)}\}_{n\geq 0}$ and $T^{(2)} = \{T_n^{(2)}\}_{n\geq 0}$ be two sequences of the arrival times of independent Poisson processes with intensities $\lambda^{(1)} > 0$ and $\lambda^{(2)} > 0$, and minimal augmented filtration $\mathbb{H}^{\boldsymbol{\lambda}} = \{\mathcal{H}_t^{\boldsymbol{\lambda}}\}_{t\geq 0}$ given the parameter pair $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$. Denote the smallest filtrations generated by $\mathbb{F}$ and $\mathbb{H}^{\boldsymbol{\lambda}}$ as $\mathbb{G}^{\boldsymbol{\lambda}} = \{\mathcal{G}_t^{\boldsymbol{\lambda}}\}_{t\geq 0}$, i.e. $\mathcal{G}_t^{\boldsymbol{\lambda}} = \mathcal{F}_t \vee \mathcal{H}_t^{\boldsymbol{\lambda}}$. Without loss of generality, we follow the convention that $T_0^{(1)} = T_0^{(2)} = 0$ and $T_\infty^{(1)} = T_\infty^{(2)} = \infty$. Moreover, for each player $k \in \{1, 2\}$, given the stopping time $T_n^{(k)}$, define pre-$T_n^{(k)}$ σ-field:

$$\mathcal{G}^{\boldsymbol{\lambda}}_{T_n^{(k)}} = \left\{ A \in \bigvee_{s\geq 0} \mathcal{G}_s : A \cap \{T_n^{(k)} \leq s\} \in \mathcal{G}_s \text{ for } s \geq 0 \right\}$$

for $n \geq 0$. Let $T \in [0, +\infty]$ be a fixed horizon, representing the terminal time of the game.

Let $\Lambda^k := \{1, \cdots, m_k\}$ denote the set of switching modes for each player $k \in \{1, 2\}$, and $\Lambda = \Lambda^1 \times \Lambda^2$ with cardinality $|\Lambda| = m = m_1 \times m_2$. Given m_1 (resp. m_2) switching modes, player I (resp. II) starts in mode $i \in \Lambda^1$ at time 0, and makes his (resp. her) switching decisions sequentially at a sequence of Poisson arrival times $T^{(1)}$ (resp. $T^{(2)}$) until the terminal time T of the game.

Let us define individual admissible switching controls for both players as follows.

Definition 4.2.1 *An admissible switching control for Player I (resp. II) is defined to be a pair of sequences $\alpha = (\sigma_n, a_n)_{n\geq 0}$ (resp. $\beta = (\tau_n, b_n)_{n\geq 0}$) such that*

1. *for all $n \geq 0$, $\sigma_n \in T^{(1)}$ such that $\sigma_n < \sigma_{n+1}$ $\mathbb{P}$-a.s. (resp. $\tau_n \in T^{(2)}$ with $\tau_n < \tau_{n+1}$ $\mathbb{P}$-a.s.),*
2. *for all $n \geq 0$, a_n is a $\mathcal{G}^{\boldsymbol{\lambda}}_{\sigma_n}$-measurable Λ^1-valued random variable (resp. b_n is a $\mathcal{G}^{\boldsymbol{\lambda}}_{\tau_n}$-measurable Λ^2-valued random variable),*
3. *for all $n \geq 1$, on $\{\sigma_n < T\}$ we have $a_n \neq a_{n-1}$ while on $\{\sigma_n \geq T\}$ we have $a_n = a_{n-1}$ (resp. on $\{\tau_n < T\}$ we have $b_n \neq b_{n-1}$ while on $\{\tau_n \geq T\}$ we have $b_n = b_{n-1}$).*

Let $\boldsymbol{A}$ (resp. $\boldsymbol{B}$) denote the set of controls for player I (resp. II), and $\boldsymbol{A}^i$ (resp. $\boldsymbol{B}^j$) denote the set of controls $\alpha \in \boldsymbol{A}$ (resp. $\beta \in \boldsymbol{B}$) satisfying $\sigma_0 = 0$ and $a_0 = i \in \Lambda^1$ (resp. $\tau_0 = 0$ and $b_0 = j \in \Lambda^2$).

His (resp. her) switching decision at any time $t \geq 0$ can be represented as

$$a(t) = a_0 \mathbb{1}_{\{\sigma_0\}}(t) + \sum_{n=1}^{\infty} a_{n-1} \mathbb{1}_{(\sigma_{n-1}, \sigma_n]}(t) \tag{4.2.1}$$

(resp.

$$b(t) = b_0 \mathbb{1}_{\{\tau_0\}}(t) + \sum_{n=1}^{\infty} b_{n-1} \mathbb{1}_{(\tau_{n-1}, \tau_n]}(t)). \tag{4.2.2}$$

4.2.1 Static version

Let $(i,j) \in \Lambda$ be the initial state and consider the following *static version of constrained optimal switching game*, where player I and II choose their respective admissible switching controls $\alpha \in \boldsymbol{A}^i$ and $\beta \in \boldsymbol{B}^j$ in order to minimise/maximise the following payoff

$$J(\alpha,\beta) = \mathbb{E}\bigg[\int_0^T e^{-rt} f_t^{a(t),b(t)}\,dt - \sum_{n=1}^{\infty}\Big(e^{-r\tau_n} l_{\tau_n}^{b_{n-1},b_n} - e^{-r\sigma_n} k_{\sigma_n}^{a_{n-1},a_n}\Big) \\ + e^{-rT} h^{a(T),b(T)} \mathbb{1}_{\{T<\infty\}}\bigg] \quad (4.2.3)$$

where $r > 0$ is the discount rate. For $(i,j) \in \Lambda$, $f^{i,j}$ defines a running reward paid by player I to player II and $h^{i,j}$ defines a terminal reward paid by player I to player II, when the active modes of both players are i and j respectively. For $i_1, i_2 \in \Lambda^1$, k^{i_1,i_2} defines a payment from player I to player II when the former switches from mode i_1 to i_2. For $j_1, j_2 \in \Lambda^2$, l^{j_1,j_2} defines a payment from player II to player I when the former switches from mode j_1 to j_2.

For the initial state $(i,j) \in \Lambda$, let us define the lower and upper values for the static game as follows

$$\underline{v}^{i,j} = \sup_{\beta\in\boldsymbol{B}^j}\inf_{\alpha\in\boldsymbol{A}^i} J(\alpha,\beta) \text{ and } \overline{v}^{i,j} = \inf_{\alpha\in\boldsymbol{A}^i}\sup_{\beta\in\boldsymbol{B}^j} J(\alpha,\beta), \quad (4.2.4)$$

and the static game (4.2.4) is said to have a value $v^{i,j}$ if $v^{i,j} = \underline{v}^{i,j} = \overline{v}^{i,j}$. We call $\underline{v}^{i,j}$ and $\overline{v}^{i,j}$ as the *lower static value* and *upper static value*, respectively. Clearly, on $\underline{v}^{i,j}$ (resp. $\overline{v}^{i,j}$), player I (resp. II), the minimizer (resp. the maximizer), is given an advantage over player II (resp. I) because he is able to look at the other player's control before choosing his own.

4.2.2 Dynamic version

Both $\underline{v}^{i,j}$ and $\overline{v}^{i,j}$ are called static because on each of them, one of the players is given advantage by being allowed to know the control of the other player for the *entire* duration of the game, i.e. $[0,T]$. In static games, clearly some information about the future is being revealed to the player with advantage. This motivates us to introduce a *dynamic version of constrained optimal switching game*, where no information about the future is revealed. One of the players will be given a limited advantage in the sense that the information about the other player's control is revealed in a dynamic way, as time goes by. This is achieved by introducing non-anticipating strategies for both players.

Before that, we first define equivalent admissible controls. Let $0 \le t \le s \le T$, two controls $\alpha^1, \alpha^2 \in \boldsymbol{A}$ (resp. $\beta^1, \beta^2 \in \boldsymbol{B}$) with $\alpha^1 = (\sigma_n^1, a_n^1)_{n\ge 0}$ and

$\alpha^2 = (\sigma_n^2, a_n^2)_{n\geq 0}$ (resp. $\beta^1 = (\tau_n^1, b_n^1)_{n\geq 0}$ and $\beta^2 = (\tau_n^2, b_n^2)_{n\geq 0}$) are said to be equivalent, denoting this by $\alpha^1 \equiv \alpha^2$ (resp. $\beta^1 \equiv \beta^2$), on $[t,s]$ if we have

$$a_0^1 \mathbb{1}_{\{\sigma_0^1\}}(u) + \sum_{n=1}^{\infty} a_{n-1}^1 \mathbb{1}_{(\sigma_{n-1}^1, \sigma_n^1]}(u) = a_0^2 \mathbb{1}_{\{\sigma_0^2\}}(u) + \sum_{n=1}^{\infty} a_{n-1}^2 \mathbb{1}_{(\sigma_{n-1}^2, \sigma_n^2]}(u)$$

(resp.

$$b_0^1 \mathbb{1}_{\{\tau_0^1\}}(u) + \sum_{n=1}^{\infty} b_{n-1}^1 \mathbb{1}_{(\tau_{n-1}^1, \tau_n^1]}(u) = b_0^2 \mathbb{1}_{\{\tau_0^2\}}(u) + \sum_{n=1}^{\infty} b_{n-1}^2 \mathbb{1}_{(\tau_{n-1}^2, \tau_n^2]}(u))$$

for $u \in [t,s]$ $\mathbb{P}$-a.s..

Definition 4.2.2 *A non-anticipative strategy for player I (resp. II) is a mapping $\overline{\alpha} : \boldsymbol{B} \to \boldsymbol{A}$ (resp. $\overline{\beta} : \boldsymbol{A} \to \boldsymbol{B}$) such that if $\beta^1 \equiv \beta^2$ (resp. $\alpha^1 \equiv \alpha^2$) on $[t,s]$, we have $\overline{\alpha}(\beta^1) \equiv \overline{\alpha}(\beta^2)$ (resp. $\overline{\beta}(\alpha^1) \equiv \overline{\beta}(\alpha^2)$) on $[t,s]$. Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the set of non-anticipative strategies for player I (resp. II), and $\mathcal{A}^i$ (resp. $\mathcal{B}^j$) denote the set of non-anticipative strategies $\overline{\alpha} \in \mathcal{A}$ (resp. $\overline{\beta} \in \mathcal{B}$) satisfying $\sigma_0 = 0$ and $a_0 = i \in \Lambda^1$ (resp. $\tau_0 = 0$ and $b_0 = j \in \Lambda^2$).*

For the initial state $(i,j) \in \Lambda$, let us define the lower and upper values for the dynamic game as follows

$$\underline{V}^{i,j} = \inf_{\overline{\alpha} \in \mathcal{A}^i} \sup_{\beta \in \boldsymbol{B}^j} J(\overline{\alpha}(\beta), \beta) \text{ and } \overline{V}^{i,j} = \sup_{\overline{\beta} \in \mathcal{B}^j} \inf_{\alpha \in \boldsymbol{A}^i} J(\alpha, \overline{\beta}(\alpha)) \tag{4.2.5}$$

and the dynamic game (4.2.5) is said to have a value $V^{i,j}$ if $V^{i,j} = \underline{V}^{i,j} = \overline{V}^{i,j}$. We call $\underline{V}^{i,j}$ and $\overline{V}^{i,j}$ as the *lower dynamic value* and *upper dynamic value*, respectively.

4.2.3 Main result of this chapter

To solve the above constrained optimal switching games (4.2.4) and (4.2.5), we introduce the following BSDE system defined on $[0,T]$:

$$\begin{aligned} Y_t^{i,j} = h^{i,j} \mathbb{1}_{\{T<\infty\}} + \int_t^T \Big[f_s^{i,j} + \lambda^{(2)} \left(L_s^{i,j} - Y_s^{i,j} \right)^+ - \lambda^{(1)} \left(Y_s^{i,j} - U_s^{i,j} \right)^+ \\ - r Y_s^{i,j} \Big] ds - \int_t^T Z_s^{i,j} \, dW_s \end{aligned} \tag{4.2.6}$$

for $t \in [0,T]$ and $(i,j) \in \Lambda$, where

$$L_s^{i,j} := \max_{j' \neq j} \{ Y_s^{i,j'} - l_s^{j,j'} \} \text{ and } U_s^{i,j} := \min_{i' \neq i} \{ Y_s^{i',j} + k_s^{i,i'} \}. \tag{4.2.7}$$

For later use, let us introduce the following spaces: for any given $\alpha \in \mathbb{R}$ and $n, \bar{n} \in \mathbb{N}$,

- $\mathbb{L}_\alpha^{2,n}$: $\mathcal{F}_T$-measurable random variables $\xi : \Omega \mapsto \mathbb{R}^n$ with $\mathbb{E}\left[e^{2\alpha T}||\xi||^2\right] < \infty$,
- $\mathbb{H}_\alpha^{2,n\times\bar{n}}$: $\mathbb{F}$-progressively measurable processes $\varphi : [0,T] \times \Omega \mapsto \mathbb{R}^{n\times\bar{n}}$ with
$$\mathbb{E}\left[\int_0^T e^{2\alpha s}||\varphi_s||^2\,ds\right] < \infty,$$
- $\mathbb{S}_\alpha^{2,n}$: $\mathbb{F}$-progressively measurable processes $\varphi : [0,T] \times \Omega \mapsto \mathbb{R}^n$ with
$$\mathbb{E}\left[\sup_{s\in[0,T]} e^{2\alpha s}||\varphi_s||^2\right] < \infty,$$

where $||\cdot||$ is the Euclidian norm and we denote $\mathbb{L}_0^{2,n}$, $\mathbb{H}_0^{2,n\times\bar{n}}$ and $\mathbb{S}_0^{2,n}$ by $\mathbb{L}^{2,n}$, $\mathbb{H}^{2,n\times\bar{n}}$ and $\mathbb{S}^{2,n}$ for the ease of notation.

In order to solve (4.2.6), we impose the following assumptions on the running reward function f, the terminal reward function h and the switching cost functions k and l.

Assumption 4.2.3 *1. For $i \in \Lambda^1$ and $j \in \Lambda^2$, $k_t^{i,i} = 0$ and $l_t^{j,j} = 0$ $\mathbb{P}$-a.s. for $t \geq 0$.*

2. For $(i,j) \in \Lambda$, $i_1, i_2 \in \Lambda^1$ and $j_1, j_2 \in \Lambda^2$, (i) when $T = \infty$, $f^{i,j}, k^{i_1,i_2}$ and l^{j_1,j_2} are all bounded; (ii) when $T < \infty$, $f^{i,j} \in \mathbb{H}^{2,1}$, $h^{i,j} \in \mathbb{L}^{2,1}$, $k^{i_1,i_2} \in \mathbb{S}^{2,1}$ and $l^{j_1,j_2} \in \mathbb{S}^{2,1}$.

The last condition is to guarantee the existence and uniqueness of the solution to BSDE system (4.2.6), which will be used to construct the game values and associated optimal switching strategies.

Proposition 4.2.4 *Suppose that Assumption 4.2.3 holds. Then there exists a unique solution (Y, Z) to BSDE system (4.2.6). Moreover, (i) when $T = \infty$, Y is bounded and $Z \in \mathbb{H}_{-r}^{2,m\times d}$; (ii) when $T < \infty$, $(Y, Z) \in \mathbb{S}^{2,m} \times \mathbb{H}^{2,m\times d}$.*

Under Assumption 4.2.3, the solvability of BSDE system (4.2.6) on a finite horizon essentially follows from Theorem 3.3 in [69], and thus we omit its proof and refer to [69] for the details. We only provide the proof of Proposition 4.2.4 on an infinite horizon in the next section.

We are now in a position to present the main result of this chapter.

Theorem 4.2.5 *Suppose that Assumption 4.2.3 holds. Let (Y, Z) be the unique solution to BSDE system (4.2.6). For every initial state $(i,j) \in \Lambda$, the following chain of inequalities hold:*

$$\overline{v}^{i,j} \geq \overline{V}^{i,j} \geq Y_0^{i,j} \geq \underline{V}^{i,j} \geq \underline{v}^{i,j}, \tag{4.2.8}$$

and moreover, there exists a pair of controls $(\alpha^*, \beta^*) \in \boldsymbol{A}^i \times \boldsymbol{B}^j$ *and non-anticipative strategies* $\overline{\alpha^*} \in \mathcal{A}^i$ *and* $\overline{\beta^*} \in \mathcal{B}^j$ *such that*

$$Y_0^{i,j} = J(\alpha^*, \beta^*) = \sup_{\beta \in \boldsymbol{B}^j} J(\overline{\alpha^*}(\beta), \beta) = \inf_{\alpha \in \boldsymbol{A}^i} J(\alpha, \overline{\beta^*}(\alpha)).$$

Remark 4.2.6 *Note that the admissible switching controls* α^*, β^* *and non-anticipative strategies* $\overline{\alpha^*}, \overline{\beta^*}$ *are related by* $\alpha^* = \overline{\alpha^*}(\beta^*)$ *and* $\beta^* = \overline{\beta^*}(\alpha^*)$. *When Player II uses the non-anticipative strategy* $\overline{\beta^*}$, *then* α^* *by Player I gives the minimum possible value for the upper dynamic game over all controls* α. *Symmetrically, when Player I uses the non-anticipative strategy* $\overline{\alpha^*}$, *then* β^* *by Player II gives the maximum possible value for the lower dynamic game over all controls* β. *In Section 4.5, we will show that, under some additional conditions on the running and terminal rewards,* (α^*, β^*) *is indeed a Nash equilibrium in this constrained optimal switching game.*

4.3 Proof of Proposition 4.2.4 on an infinite horizon

The proof is based on the multidimensional comparison result, which was first established by Hu and Peng [41]. In this section, we use a slightly different but more general version, which was provided by Hu et al [40]. We omit the proof of the following lemma and refer to [40] for the details.

Lemma 4.3.1 *Let deterministic terminal time* $T > 0$ *be fixed. Consider a system of BSDE*$(\xi^{i,j}, G^{i,j})$ *with the terminal data* $\xi^{i,j}$ *and the driver* $G^{i,j}$:

$$Y_t^{i,j} = \xi^{i,j} + \int_t^T [G_s^{i,j}(Y_s^{i,j}, Y_s^{-i,-j})]\, ds - \int_t^T Z_s^{i,j}\, dW_s,$$

where

$$Y_s^{-i,-j} := (Y_s^{1,1}, \cdots, Y_s^{1,m_2}, Y_s^{2,1}, \cdots, Y_s^{2,m_2}, \cdots, Y_s^{i,j-1}, Y_s^{i,j+1}, \cdots, Y_s^{m_1,m_2}).$$

Let $(\overline{Y}^{i,j}, \overline{Z}^{i,j})$ *be the solution of another system of BSDE*$(\overline{\xi}^{i,j}, \overline{G}^{i,j})$. *Suppose that*

1. $\xi^{i,j}, \overline{\xi}^{i,j} \in \mathbb{L}^{2,1}$ *and satisfying* $\xi^{i,j} \leq \overline{\xi}^{i,j}$ *for* $(i,j) \in \Lambda$;

2. $G_s^{i,j}(y^{i,j}, y^{-i,-j})$ *is Lipschitz continuous in* $y = (y^{i,j}, y^{-i,-j})$ *for* $(i,j) \in \Lambda$, *and nondecreasing in* $y^{i',j'}$ *for* $(i',j') \in \Lambda \backslash \{(i,j)\}$;

3. *the following inequality holds:* $G_s^{i,j}(\overline{Y}_s^{i,j}, \overline{Y}_s^{-i,-j}) \leq \overline{G}_s^{i,j}(\overline{Y}_s^{i,j}, \overline{Y}_s^{-i,-j})$ *for* $(i,j) \in \Lambda$.

Then, $Y_t^{i,j} \leq \overline{Y}_t^{i,j}$ *for* $t \in [0,T]$ *and* $(i,j) \in \Lambda$.

The idea of the proof of Proposition 4.2.4 is as follows. We construct a solution to BSDE system (4.2.6) by following an approximation procedure. For fixed $m \geq 1$ and $t \in [0, m]$, we consider the following finite horizon BSDE system

$$Y_t^{i,j,(m)} = \int_t^m \left[f_s^{i,j} + \lambda^{(2)} \left(L_s^{i,j,(m)} - Y_s^{i,j,(m)} \right)^+ - \lambda^{(1)} \left(Y_s^{i,j,(m)} - U_s^{i,j,(m)} \right)^+ \right. \\ \left. - rY_s^{i,j,(m)} \right] ds - \int_t^m Z_s^{i,j,(m)} \, dW_s \quad (4.3.1)$$

for $t \in [0, m]$, and $Y_t^{i,j,(m)} = Z_t^{i,j,(m)} \equiv 0$ for $t > m$, where

$$L_s^{i,j,(m)} := \max_{j' \neq j} \{ Y_s^{i,j',(m)} - l_s^{j,j'} \} \text{ and } U_s^{i,j,(m)} := \min_{i' \neq i} \{ Y_s^{i',j,(m)} + k_s^{i,i'} \}.$$

Note that (4.3.1) is a finite horizon BSDE system with Lipschitz continuous driver, so it admits a unique solution $(Y^{i,j,(m)}, Z^{i,j,(m)})_{(i,j) \in \Lambda}$. We will show the pair of processes $(Y^{i,j,(m)}, Z^{i,j,(m)})_{m \geq 1}$ form a Cauchy sequence in an appropriate space, whose limit provides a solution to the infinite horizon BSDE system (4.2.6). Moreover, the uniqueness of the solution follows from the comparison result in Lemma 4.3.1.

We are now in a position to prove Proposition 4.2.4 in the infinite horizon case.

Step 1. A priori estimate: We show that $(Y^{i,j,(m)})_{(i,j) \in \Lambda}$, which is the first component of the solution to the finite horizon BSDE system (4.3.1), has the following estimate

$$|Y^{i,j,(m)}| \leq K_y,$$

where the constant K_y is independent of m. Since $f^{i,j}, k^{i_1,i_2}$ and l^{j_1,j_2} are all bounded, then there exist the constants K_f, K_k, K_l such that $|f^{i,j}| \leq K_f$, $|k^{i_1,i_2}| \leq K_k$ and $|l^{j_1,j_2}| \leq K_l$. Let

$$G_s^{i,j}(y^{i,j}, y^{-i,-j}) := f_s^{i,j} + \lambda^{(2)} \left(\max_{j' \neq j} \{ y^{i,j'} - l_s^{j,j'} \} - y^{i,j} \right)^+ \\ - \lambda^{(1)} \left(y^{i,j} - \min_{i' \neq i} \{ y^{i',j} + k_s^{i,i'} \} \right)^+ - ry^{i,j}.$$

Note that $G_s^{i,j}(y^{i,j}, y^{-i,-j})$ is Lipschitz continuous and nondecreasing in $y^{i',j'}$ for $(i', j') \neq (i, j)$. Moreover,

$$G_s^{i,j}(\overline{Y}_s, \overline{Y}_s^{-i,-j}) = f_s^{i,j} + \lambda^{(2)} \left(\max_{j' \neq j} \{ -l_s^{j,j'} \} \right)^+ - \lambda^{(1)} \left(- \min_{i' \neq i} \{ k_s^{i,i'} \} \right)^+ - r\overline{Y}_s,$$

where $\overline{Y}^{-i,-j} := (\underbrace{\overline{Y}, \cdots, \overline{Y}}_{m-1})$ and $\overline{Y}$ solves the ODE

$$\overline{Y}_t = \int_t^m \left[K_f + \lambda^{(2)} K_l + \lambda^{(1)} K_k - r\overline{Y}_s\right] ds.$$

Then it follows from Lemma 4.3.1 that

$$Y_t^{i,j,(m)} \le \overline{Y}_t \le K_y$$

for $t \in [0, m]$ amd $(i, j) \in \Lambda$, where $K_y := \frac{1}{r}\left[K_f + \lambda^{(2)} K_l + \lambda^{(1)} K_k\right]$. Likewise, we can prove $Y_t^{i,j,(m)} \ge -K_y$, and thus $|Y_t^{i,j,(m)}| \le K_y$.

Step 2. Existence: We first prove that $(Y^{i,j,(m)})_{m\ge 1}$ is a Cauchy sequence. For $m \ge n \ge 1$ and $t \in [0, m]$, let

$$\delta Y_t^{i,j,(m,n)} := Y_t^{i,j,(m)} - Y_t^{i,j,(n)} \text{ and } \delta Z_t^{i,j,(m,n)} := Z_t^{i,j,(m)} - Z_t^{i,j,(n)}.$$

Then we have $(\delta Y_t^{i,j,(m,n)}, \delta Z_t^{i,j,(m,n)})$ is the unique solution to the following BSDE system

$$\delta Y_t^{i,j,(m,n)} = \int_t^m G_s^{i,j}(\delta Y_t^{i,j,(m,n)}, \delta Y_t^{-i,-j,(m,n)})\, ds - \int_t^m \delta Z_t^{i,j,(m,n)}\, dW_s \tag{4.3.2}$$

where

$$\begin{aligned} G_s^{i,j}(y^{i,j}, y^{-i,-j}) := f_s^{i,j} \mathbb{1}_{\{s\ge n\}} + \lambda^{(2)} &\Bigg[(\max_{j'\ne j}\{y^{i,j'} - y^{i,j} + Y_s^{i,j',(n)} - l_s^{j,j'}\} - Y_s^{i,j,(n)})^+ \\ &- (L_s^{i,j,(n)} - Y_s^{i,j,(n)})^+\Bigg] - \lambda^{(1)}\Bigg[(Y_s^{i,j,(n)} - \min_{i'\ne i}\{y^{i',j} - y^{i,j} \\ &+ Y_s^{i',j,(n)} + k_s^{i,i'}\})^+ - (Y_s^{i,j,(n)} - U_s^{i,j,(n)})^+\Bigg] - ry^{i,j} \end{aligned}$$

for $y = (y^{i,j}, y^{-i,-j}) \in \mathbb{R}^m$ with $(i, j) \in \Lambda$. Moreover, $G_s^{i,j}(\overline{Y}_s, \overline{Y}_s^{-i,-j}) = f_s^{i,j} \mathbb{1}_{\{s\ge n\}} - r\overline{Y}_s$, where $\overline{Y}$ solves the ODE

$$\overline{Y}_t = \int_t^m \left[K_f \mathbb{1}_{\{s\ge n\}} - r\overline{Y}_s\right] ds.$$

It follows from Lemma 4.3.1 that

$$\begin{aligned} \delta Y_t^{i,j,(m,n)} \le \overline{Y}_t &= K_f \int_t^m e^{-r(s-t)} \mathbb{1}_{\{s\ge n\}}\, ds \\ &\le K_f \int_n^m e^{-r(s-t)}\, ds = \frac{K_f}{r} e^{rt}(e^{-rn} - e^{-rm}) \end{aligned}$$

for $t \in [0, m]$ amd $(i, j) \in \Lambda$. Similarly, we can prove $\delta Y_t^{i,j,(m,n)} \ge -\frac{K_f}{r} e^{rt}(e^{-rn} -$

e^{-rm}), so

$$|\delta Y_t^{i,j,(m,n)}| \le \frac{K_f}{r} e^{rt}(e^{-rn} - e^{-rm}).$$

Sending $m, n \to \infty$, we can obtain that, for any $T > 0$, $\sup_{t\in[0,T]} |\delta Y_t^{i,j,(m,n)}| \to 0$, and therefore there exists a limit process $Y^{i,j}$ such that $Y_t^{i,j,(m)} \to Y_t^{i,j}$ with $|Y_t^{i,j}| \le K_y$.

We next prove that $(Z^{i,j,(m)})_{m\ge1}$ is also a Cauchy sequence. By applying Ito's formula to $e^{-2rt}|\delta Y_t^{i,j,(m,n)}|^2$ and using (4.3.2), we can obtain that

$$\begin{aligned}
&\int_0^m e^{-2rs}|\delta Z_s^{i,j,(m,n)}|^2\, ds \\
= &-|\delta Y_0^{i,j,(m,n)}|^2 + \int_0^m 2e^{-2rs}\delta Y_s^{i,j,(m,n)}\left(f_s^{i,j}\mathbb{1}_{s\ge n} + \tilde{G}_s^{i,j,(m,n)}\right) ds \\
&- \int_0^m 2e^{-rs}\delta Y_s^{i,j,(m,n)}\delta Z_s^{i,j,(m,n)}\, dW_s, \qquad (4.3.3)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{G}_s^{i,j,(m,n)} := \lambda^{(2)}&\left((L_s^{i,j,(m)} - Y_s^{i,j,(m)})^+ - (L_s^{i,j,(n)} - Y_s^{i,j,(n)})^+\right) \\
&- \lambda^{(1)}\left((Y_s^{i,j,(m)} - U_s^{i,j,(m)})^+ - (Y_s^{i,j,(n)} - U_s^{i,j,(n)})^+\right).
\end{aligned}$$

Taking expectation on both sides of (4.3.3) and using the boundedness result of $Y^{i,j,(m)}$ yields

$$\mathbb{E}\left[\int_0^m e^{-2rs}|\delta Z_s^{i,j,(m,n)}|^2\, ds\right] \le C_z\mathbb{E}\left[\int_0^m e^{-2rs}\delta Y_s^{i,j,(m,n)}\, ds\right]$$

where the constant $C_z = 2\left[K_f + \lambda^{(2)}(2K_y + K_l) + \lambda^{(1)}(2K_y + K_k)\right]$. It follows from the dominated convergence theorem that $\delta Z^{i,j,(m,n)} \to 0$ in $\mathbb{H}^{2,d}_{-r}$, and therefore there exists a limit process $Z^{i,j}$ such that $Z^{i,j,(m)} \to Z^{i,j}$ in $\mathbb{H}^{2,d}_{-r}$.

It is standard to check the pair of limit processes $(Y^{i,j}, Z^{i,j})_{(i,j)\in\Lambda}$ satisfy the inifinite horizon BSDE system (4.2.6) (see, for example, Section 5 of [12]).
Step 3. Uniqueness: Suppose $(Y^{i,j}, Z^{i,j})_{(i,j)\in\Lambda}$ and $(\overline{Y}^{i,j}, \overline{Z}^{i,j})_{(i,j)\in\Lambda}$ are two solutions to (4.2.6). For $t \ge 0$, we define

$$\delta Y_t^{i,j} := e^{-rt}(Y_t^{i,j} - \overline{Y}_t^{i,j}) \text{ and } \delta Z_t^{i,j} := e^{-rt}(Z_t^{i,j} - \overline{Z}_t^{i,j}).$$

For $T \ge t$, we have $(\delta Y_t^{i,j}, \delta Z_t^{i,j})_{t\in[0,T]}$ is the solution to the following BSDE system:

$$\delta Y_t^{i,j} = \delta Y_T^{i,j} + \int_t^T G_s^{i,j}(\delta Y_s^{i,j}, \delta Y_s^{-i,-j})\, ds - \int_t^T \delta Z_s^{i,j}\, dW_s \qquad (4.3.4)$$

where

$$G_s^{i,j}(y^{i,j}, y^{-i,-j}) := e^{-rs}\Big[\lambda^{(2)}(\max_{j'\neq j}\{e^{rs}(y^{i,j'} - y^{i,j}) + \overline{Y}_s^{i,j'} - l_s^{j,j'}\} - \overline{Y}_s^{i,j})^+ \\ - \lambda^{(2)}(\overline{L}_s^{i,j} - \overline{Y}_s^{i,j})^+ - \lambda^{(1)}(\overline{Y}_s^{i,j} - \min_{i'\neq i}\{e^{rs}(y^{i',j} - y^{i,j}) \\ + \overline{Y}_s^{i',j} + k_s^{i,i'}\})^+ + \lambda^{(1)}(\overline{Y}_s^{i,j} - \overline{U}_s^{i,j})^+\Big]$$

with

$$\overline{L}_s^{i,j} := \max_{j'\neq j}\{\overline{Y}_s^{i,j'} - l_s^{j,j'}\} \text{ and } \overline{U}_s^{i,j} := \min_{i'\neq i}\{\overline{Y}_s^{i',j} + k_s^{i,i'}\}.$$

It is easy to see that $G_s^{i,j}(y^{i,j}, y^{-i,-j})$ is Lipschitz continuous and therefore $(\delta Y^{i,j}, \delta Z^{i,j})$ is the unique solution to (4.3.4).

Let $\overline{Y}_t = 2K_y e^{-rT}$. Since $|\delta Y_T^{i,j}| \leq \overline{Y}_T$ and $G_s^{i,j}(\overline{Y}_s, \overline{Y}_s^{-i,-j}) = 0$, it follows from Lemma 4.3.1 that $|\delta Y_t^{i,j}| \leq \overline{Y}_T$. By sending $T \to \infty$, we can obtain $\delta Y_t^{i,j} = 0$, and therefore $\delta Z_t^{i,j} = 0$.

4.4 Proof of Theorem 4.2.5

4.4.1 Coupling of controls

In order to prove Theorem 4.2.5, we first give an equivalent formulation of the constrained optimal switching games (4.2.4) and (4.2.5), by defining the coupling of two controls $\alpha \in \boldsymbol{A}$ and $\beta \in \boldsymbol{B}$, where the latter is inspired by [32].

Definition 4.4.1 *Given admissible switching controls $\alpha \in \boldsymbol{A}$ and $\beta \in \boldsymbol{B}$, define the coupling $\gamma(\alpha, \beta) = (\theta_n, c_n)_{n\geq 0}$, where $\{\theta_n\}_{n\geq 0}$ is the merged arrival sequence such that*

$$\theta_n = \sigma_{r_n} \wedge \tau_{s_n}$$

with $r_0 = s_0 = 0$, $r_1 = s_1 = 1$ and for $n \geq 2$,

$$r_n = r_{n-1} + \mathbb{1}_{\{\sigma_{r_{n-1}} < \tau_{s_{n-1}}\}}, \tag{4.4.1}$$

$$s_n = s_{n-1} + \mathbb{1}_{\{\sigma_{r_{n-1}} \geq \tau_{s_{n-1}}\}}, \tag{4.4.2}$$

and c_n is a $\mathcal{G}_{\theta_n}^{\boldsymbol{\lambda}}$-measurable Λ-valued random variable such that $c_0 = (a_0, b_0)$ and for $n \geq 1$,

$$c_n = (c_n^{(1)}, c_n^{(2)}) = \begin{cases} (a_{r_n}, c_{n-1}^{(2)}), & \text{if } \sigma_{r_n} < \tau_{s_n}, \sigma_{r_n} < T, \\ (c_{n-1}^{(1)}, b_{s_n}), & \text{if } \tau_{s_n} \leq \sigma_{r_n}, \tau_{s_n} < T, \\ c_{n-1}, & \text{if } \sigma_{r_n} \wedge \tau_{s_n} \geq T. \end{cases}$$

Let $\boldsymbol{C}$ denote the set of couplings $\gamma(\alpha, \beta)$ for both players, and $\boldsymbol{C}^{i,j}$ denote the set of couplings $\gamma(\alpha, \beta) \in \boldsymbol{C}$ satisfying $\theta_0 = 0$ and $c_0 = (i, j) \in \Lambda$.

The coupling switching decision at any time $t \geq 0$ can be represented as

$$c(t) = (c^{(1)}(t), c^{(2)}(t)) = c_0 \mathbb{1}_{\{\theta_0\}}(t) + \sum_{n=1}^{\infty} c_{n-1} \mathbb{1}_{(\theta_{n-1},\theta_n]}(t), \tag{4.4.3}$$

and thus, we can rewrite the payoff (4.2.3) as

$$\begin{aligned} J(\alpha,\beta) = J(\gamma(\alpha,\beta)) = \mathbb{E}\bigg[\int_0^T e^{-rt} f_t^{c(t)} \, dt - \sum_{n=1}^{\infty} e^{-r\theta_n} \left(l_{\theta_n}^{c_{n-1}^{(1)}, c_n^{(1)}} - k_{\theta_n}^{c_{n-1}^{(2)}, c_n^{(2)}} \right) \\ + e^{-rT} h^{c(T)} \mathbb{1}_{\{T<\infty\}} \bigg]. \end{aligned} \tag{4.4.4}$$

4.4.2 Constrained Dynkin game representation

The proof of Theorem 4.2.5 crucially depends on the constrained Dynkin game representation for BSDE system (4.2.6). The new feature of this kind of Dynkin game is that there are control constraints for both players, in the sense that both players are only allowed to stop at two heterogeneous exogenous Poisson arrival times $T^{(1)}$ and $T^{(2)}$.

For each player $k \in \{1,2\}$, define a random variable $M_k : \Omega \mapsto \overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ such that $M_k := \sum_{n=1}^{\infty} n \mathbb{1}_{\{T_{n-1}^{(k)} \leq T < T_n^{(k)}\}}$ if $T < \infty$ and $M_k := \infty$ if $T = \infty$. For every $T^{(1)}$ and $T^{(2)}$, we can construct a merged Poisson arrival times $\delta = (\delta_n)_{n\geq 0}$ such that

$$\delta_n = T_{r_n}^{(1)} \wedge T_{s_n}^{(2)}$$

with $r_0 = s_0 = 0$, $r_1 = s_1 = 1$ and for $n \geq 2$, r_n and s_n are given by

$$\begin{aligned} r_n &= r_{n-1} + \mathbb{1}_{\{T_{r_{n-1}}^{(1)} < T_{s_{n-1}}^{(2)}\}}, \\ s_n &= s_{n-1} + \mathbb{1}_{\{T_{r_{n-1}}^{(1)} \geq T_{s_{n-1}}^{(2)}\}}, \end{aligned}$$

and define a random variable $M : \Omega \mapsto \overline{\mathbb{N}}$ such that $M := \sum_{n=1}^{\infty} n \mathbb{1}_{\{\delta_{n-1} \leq T < \delta_n\}}$ if $T < \infty$ and $M := \infty$ if $T = \infty$.

Remark 4.4.2 *The merged Poisson arrival times $\delta = (\delta_n)_{n\geq 0}$ constructed above is exactly the increasing sequence of stopping times $\theta = (\theta_k)_{k\geq 0}$ constructed in Section 3.3. The idea is nothing but to merge the two Poisson sequences together while still keeping track of their order, but here we use a different definition in the spirit of Definition 4.4.1.*

For any integer $n \geq 0$, let us define the control set for both players:

$$\mathcal{R}^{(k)}_{\delta_n} = \Big\{ \mathbb{G}\text{-stopping time } \sigma \text{ for } \sigma(\omega) = T^{(k)}_N(\omega) \text{ where } T^{(k)}_N(\omega) \geq \delta_n \text{ and } N \leq M_k \Big\}$$

for $k \in \{1, 2\}$. In the following constrained Dynkin game, two players choose their respective stopping times $\sigma \in \mathcal{R}^{(1)}_{\delta_n}$ and $\tau \in \mathcal{R}^{(2)}_{\delta_n}$ in order to minimize/maximize the cost functional

$$\tilde{J}^{i,j}_{\delta_{n-1}}(\sigma, \tau) = \mathbb{E}\Big[\int_{\delta_{n-1}}^{\sigma\wedge\tau\wedge T} e^{-rs} f^{i,j}_s \, ds + e^{-r\tau} L^{i,j}_\tau \mathbb{1}_{\{\tau<T, \tau\leq\sigma\}} + e^{-r\sigma} U^{i,j}_\sigma \mathbb{1}_{\{\sigma<T,\sigma<\tau\}} + e^{-rT} h^{i,j} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\sigma\wedge\tau\geq T\}} | \mathcal{G}^{\boldsymbol{\lambda}}_{\delta_{n-1}} \Big]$$

for $(i, j) \in \Lambda$ and $1 \leq n \leq M$, where $f^{i,j}, h^{i,j}, L^{i,j}, U^{i,j}$ are the data in BSDE system (4.2.6).

For $(i, j) \in \Lambda$ and $1 \leq n \leq M$, let us define the lower and upper values of this auxiliary constrained Dynkin game with starting time δ_{n-1} and starting mode (i, j):

$$\underline{y}^{i,j}_{\delta_{n-1}} = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}^{(2)}_{\delta_n}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}^{(1)}_{\delta_n}} \tilde{J}^{i,j}_{\delta_{n-1}}(\sigma, \tau) \text{ and } \overline{y}^{i,j}_{\delta_{n-1}} = \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}^{(1)}_{\delta_n}} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}^{(2)}_{\delta_n}} \tilde{J}^{i,j}_{\delta_{n-1}}(\sigma, \tau), \tag{4.4.5}$$

where the constrained Dynkin game (4.4.5) is said to have value $y^{i,j}_{\delta_{n-1}}$ if $y^{i,j}_{\delta_{n-1}} = \underline{y}^{i,j}_{\delta_{n-1}} = \overline{y}^{i,j}_{\delta_{n-1}}$ a.s., and it is standard to show that if there exists a saddle point $(\sigma^{i,j}_{\delta_{n-1}}, \tau^{i,j}_{\delta_{n-1}}) \in \mathcal{R}^{(1)}_{\delta_n} \times \mathcal{R}^{(2)}_{\delta_n}$ such that

$$\tilde{J}^{i,j}_{\delta_{n-1}}(\sigma^{i,j}_{\delta_{n-1}}, \tau) \leq \tilde{J}^{i,j}_{\delta_{n-1}}(\sigma^{i,j}_{\delta_{n-1}}, \tau^{i,j}_{\delta_{n-1}}) \leq \tilde{J}^{i,j}_{\delta_{n-1}}(\sigma, \tau^{i,j}_{\delta_{n-1}}) \text{ a.s.} \tag{4.4.6}$$

for every $(\sigma, \tau) \in \mathcal{R}^{(1)}_{\delta_n} \times \mathcal{R}^{(2)}_{\delta_n}$, then the value of the Dynkin game (4.4.5) exists and equals

$$y^{i,j}_{\delta_{n-1}} = \tilde{J}_{\delta_{n-1}}(\sigma^{i,j}_{\delta_{n-1}}, \tau^{i,j}_{\delta_{n-1}}) \quad a.s..$$

Remark 4.4.3 *Although $\sigma^{i,j}_{\delta_{n-1}}$ and $\tau^{i,j}_{\delta_{n-1}}$ belong to $\mathcal{R}^{(1)}_{\delta_n}$ and $\mathcal{R}^{(2)}_{\delta_n}$ respectively, the subscripts δ_{n-1} in the optimal stopping strategies represent the starting time of the constrained Dynkin game.*

Proposition 4.4.4 *Suppose that Assumption 4.2.3 holds. Let (Y, Z) be the unique solution to BSDE system (4.2.6). For $(i, j) \in \Lambda$ and $1 \leq n \leq M$, the value of the constrained Dynkin game (4.4.5) with starting time δ_{n-1} and starting mode (i, j) exists and is given by*

$$y^{i,j}_{\delta_{n-1}} = \underline{y}^{i,j}_{\delta_{n-1}} = \overline{y}^{i,j}_{\delta_{n-1}} = \tilde{Y}^{i,j}_{\delta_{n-1}} \ a.s.,$$

and moreover, the optimal stopping strategy of the game is given by

$$\begin{cases} \sigma^{i,j}_{\delta_{n-1}} = \inf\{T^{(1)}_N \geq \delta_n : \tilde{Y}^{i,j}_{T^{(1)}_N} \geq \tilde{U}^{i,j}_{T^{(1)}_N}\} \wedge T^{(1)}_{M_1}, \\ \tau^{i,j}_{\delta_{n-1}} = \inf\{T^{(2)}_N \geq \delta_n : \tilde{Y}^{i,j}_{T^{(2)}_N} \leq \tilde{L}^{i,j}_{T^{(2)}_N}\} \wedge T^{(2)}_{M_2}, \end{cases} \tag{4.4.7}$$

where $\tilde{Y}^{i,j}_t := e^{-rt}Y^{i,j}_t$, $\tilde{L}^{i,j}_t := e^{-rt}L^{i,j}_t$ *and* $\tilde{U}^{i,j}_t := e^{-rt}U^{i,j}_t$ *for* $t \in [0,T]$.

This proposition, when T is finite, essentially follows from Section 3.5.1. For completeness and readers' convenience, we provide the proof of Proposition 4.4.4 on an infinite horizon in the Appendix (see Section 4.A).

4.4.3 Main part of the proof

For the ease of notation, let us define the following discounted processes

$$\tilde{f}^{i,j}_t = e^{-rs}f^{i,j}_t, \quad \tilde{l}^{i,j}_t = e^{-rs}l^{i,j}_t, \quad \tilde{k}^{i,j}_t = e^{-rs}k^{i,j}_t,$$

for $t \in [0,T]$, and $\tilde{h}^{i,j} = e^{-rT}h^{i,j}$, for $(i,j) \in \Lambda$. Then, the orginal payoff (4.4.4) can be rewritten as

$$J(\gamma(\alpha,\beta)) = \mathbb{E}\left[\int_0^T \tilde{f}^{c(t)}_t\,dt - \sum_{n=1}^{\infty}\left(\tilde{l}^{c^{(1)}_{n-1},c^{(1)}_n}_{\theta_n} - \tilde{k}^{c^{(2)}_{n-1},c^{(2)}_n}_{\theta_n}\right) + \tilde{h}^{c(T)}\mathbb{1}_{\{T<\infty\}}\right], \tag{4.4.8}$$

and we are now in a position to prove Theorem 4.2.5.

Step 1. Let us construct a sequence $(\theta_n, c_n)_{n\geq 0}$ such that $\theta_0 = 0$, $c_0 = (i,j) \in \Lambda$ and for $n \geq 1$,

$$\theta_n = \sigma^{c_{n-1}}_{\theta_{n-1}} \wedge \tau^{c_{n-1}}_{\theta_{n-1}} \tag{4.4.9}$$

where $\sigma^{c_{n-1}}_{\theta_{n-1}}$ and $\tau^{c_{n-1}}_{\theta_{n-1}}$, defined by (4.4.7), are the optimal strategy of the constrained Dynkin game (4.4.5) with starting time θ_{n-1} and starting mode c_{n-1}, and

$$c_n = \begin{cases} \left(\arg\min_{i'\neq c^{(1)}_{n-1}}\{\tilde{Y}^{i',c^{(2)}_{n-1}}_{\theta_n} + \tilde{k}^{c^{(1)}_{n-1},i'}_{\theta_n}\}, c^{(2)}_{n-1}\right), & \text{if } \sigma^{c_{n-1}}_{\theta_{n-1}} < \tau^{c_{n-1}}_{\theta_{n-1}}, \sigma^{c_{n-1}}_{\theta_{n-1}} < T, \\ \left(c^{(1)}_{n-1}, \arg\max_{j'\neq c^{(2)}_{n-1}}\{\tilde{Y}^{c^{(1)}_{n-1},j'}_{\theta_n} - \tilde{l}^{c^{(2)}_{n-1},j'}_{\theta_n}\}\right), & \text{if } \tau^{c_{n-1}}_{\theta_{n-1}} \leq \sigma^{c_{n-1}}_{\theta_{n-1}}, \tau^{c_{n-1}}_{\theta_{n-1}} < T, \\ c_{n-1}, & \text{if } \sigma^{c_{n-1}}_{\theta_{n-1}} \wedge \tau^{c_{n-1}}_{\theta_{n-1}} \geq T, \end{cases} \tag{4.4.10}$$

and define sequences $\alpha^* = (\sigma^*_n, a^*_n)_{n\geq 0}$ and $\beta^* = (\tau^*_n, b^*_n)_{n\geq 0}$ such that $\sigma^*_0 = \tau^*_0 = 0$, $(a^*_0, b^*_0) = (i,j) \in \Lambda$ and for $n \geq 1$,

$$\begin{cases} \sigma^*_n = \inf\{T^{(1)}_N > \sigma^*_{n-1} : c^{(1)}(T^{(1)}_N) \neq a^*_{n-1}\} \wedge T^{(1)}_{M_1}, & a^*_n = c^{(1)}(\sigma^*_n+), \\ \tau^*_n = \inf\{T^{(2)}_N > \tau^*_{n-1} : c^{(2)}(T^{(2)}_N) \neq b^*_{n-1}\} \wedge T^{(2)}_{M_2}, & b^*_n = c^{(2)}(\tau^*_n+), \end{cases} \tag{4.4.11}$$

where $c(\cdot)$ in (4.4.3) is defined by (4.4.9)-(4.4.10). By the construction of

α^*, a_n^* is $\mathcal{G}^{\boldsymbol{\lambda}}_{\sigma_n^*}$-measurable since $\mathbb{G}^{\boldsymbol{\lambda}}$ is right-continuous, on $\{\sigma_n^* \geq T\}$ we have $a_n^* = a_{n-1}^*$, and hence $\alpha^* \in \boldsymbol{A}^i$. Similarly we can verify that $\beta^* \in \boldsymbol{B}^j$, and hence $\gamma(\alpha^*, \beta^*) \in \boldsymbol{C}^{i,j}$.

We now prove the following equality

$$\tilde{Y}_0^{i,j} = y_0^{i,j} = J(\gamma(\alpha^*, \beta^*)). \tag{4.4.12}$$

Indeed, it follows from Proposition 4.4.4 that

$$\begin{aligned} y_0^{i,j} = \mathbb{E}\bigg[\int_0^{\sigma_0^{i,j} \wedge \tau_0^{i,j} \wedge T} \tilde{f}_t^{i,j}\, dt + \tilde{L}^{i,j}_{\tau_0^{i,j}} \mathbb{1}_{\{\tau_0^{i,j} < T, \tau_0^{i,j} \leq \sigma_0^{i,j}\}} + \tilde{U}^{i,j}_{\sigma_0^{i,j}} \mathbb{1}_{\{\sigma_0^{i,j} < T, \sigma_0^{i,j} < \tau_0^{i,j}\}} \\ + \tilde{h}^{i,j} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\sigma_0^{i,j} \wedge \tau_0^{i,j} \geq T\}} \bigg] \end{aligned}$$

where $\sigma_0^{i,j}$ and $\tau_0^{i,j}$ are defined in (4.4.7). By the definition of a_1^*, on $\{\sigma_0^{i,j} < T, \sigma_0^{i,j} < \tau_0^{i,j}\}$, it is obvious that $\theta_1 = \sigma_0^{i,j} \wedge \tau_0^{i,j} = \sigma_0^{i,j}$, and then

$$\tilde{U}^{i,j}_{\sigma_0^{i,j}} = \min_{i' \neq i} \left\{ \tilde{Y}^{i',j}_{\sigma_0^{i,j}} + \tilde{k}^{i,i'}_{\sigma_0^{i,j}} \right\} = \tilde{Y}^{a_1^*,j}_{\theta_1} + \tilde{k}^{i,a_1^*}_{\theta_1}.$$

Similarly, by the definition of b_1^*, on $\{\tau_0^{i,j} < T, \tau_0^{i,j} \leq \sigma_0^{i,j}\}$, we have $\tilde{L}^{i,j}_{\tau_0^{i,j}} = \tilde{Y}^{i,b_1^*}_{\theta_1} - \tilde{l}^{j,b_1^*}_{\theta_1}$. Then,

$$\begin{aligned} y_0^{i,j} = \mathbb{E}\bigg[\int_0^{\theta_1 \wedge T} \tilde{f}_t^{c(t)}\, dt + \tilde{Y}^{c_1}_{\theta_1} \mathbb{1}_{\{\theta_1 < T\}} - \left(\tilde{l}^{c_0^{(2)}, c_1^{(2)}}_{\theta_1} - \tilde{k}^{c_0^{(1)}, c_1^{(1)}}_{\theta_1} \right) \\ + \tilde{h}^{c_0} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\theta_1 \geq T\}} \bigg]. \end{aligned} \tag{4.4.13}$$

On the other hand, we also have, conditional on $\{\theta_1 < T\}$,

$$\begin{aligned} \tilde{Y}^{c_1}_{\theta_1} &= y^{c_1}_{\theta_1} \\ &= \mathbb{E}\bigg[\int_{\theta_1}^{\sigma^{c_1}_{\theta_1} \wedge \tau^{c_1}_{\theta_1} \wedge T} \tilde{f}_t^{c_1}\, dt + \tilde{L}^{c_1}_{\tau^{c_1}_{\theta_1}} \mathbb{1}_{\{\tau^{c_1}_{\theta_1} < T, \tau^{c_1}_{\theta_1} \leq \sigma^{c_1}_{\theta_1}\}} + \tilde{U}^{c_1}_{\sigma^{c_1}_{\theta_1}} \mathbb{1}_{\{\sigma^{c_1}_{\theta_1} < T, \sigma^{c_1}_{\theta_1} < \tau^{c_1}_{\theta_1}\}} \\ &\quad + \tilde{h}^{c_1} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\sigma^{c_1}_{\theta_1} \wedge \tau^{c_1}_{\theta_1} \geq T\}} | \mathcal{G}^{\boldsymbol{\lambda}}_{\theta_1} \bigg] \\ &= \mathbb{E}\bigg[\int_{\theta_1}^{\theta_2 \wedge T} \tilde{f}_t^{c(t)}\, dt + \tilde{Y}^{c_2}_{\theta_2} \mathbb{1}_{\{\theta_2 < T\}} - \left(\tilde{l}^{c_1^{(2)}, c_2^{(2)}}_{\theta_2} - \tilde{k}^{c_1^{(1)}, c_2^{(1)}}_{\theta_2} \right) \\ &\quad + \tilde{h}^{c_1} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\theta_2 \geq T\}} | \mathcal{G}^{\boldsymbol{\lambda}}_{\theta_1} \bigg]. \end{aligned} \tag{4.4.14}$$

By plugging (4.4.14) into (4.4.13), we can obtain

$$
\begin{aligned}
y_0^{i,j} = \mathbb{E}\Big[& \int_0^{\theta_2 \wedge T} \tilde{f}_t^{c(t)}\, dt + \tilde{Y}_{\theta_2}^{c_2} \mathbb{1}_{\{\theta_2 < T\}} - \sum_{n=1}^{2} \left(\tilde{l}_{\theta_n}^{c_{n-1}^{(2)}, c_n^{(2)}} - \tilde{k}_{\theta_n}^{c_{n-1}^{(1)}, c_n^{(1)}} \right) \\
& + \sum_{n=1}^{2} \tilde{h}^{c_{n-1}} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\theta_{n-1} < T \le \theta_n\}} \Big].
\end{aligned}
$$

We can repeat the above procedure M times, since $\theta_M \geq \delta_M > T$, we have

$$
\begin{aligned}
y_0^{i,j} &= \mathbb{E}\Big[\int_0^{\theta_M \wedge T} \tilde{f}_t^{c(t)}\, dt + \tilde{Y}_{\theta_M}^{c_M} \mathbb{1}_{\{\theta_M < T\}} - \sum_{n=1}^{M} \left(\tilde{l}_{\theta_n}^{c_{n-1}^{(2)}, c_n^{(2)}} - \tilde{k}_{\theta_n}^{c_{n-1}^{(1)}, c_n^{(1)}} \right) \\
&\quad + \sum_{n=1}^{M} \tilde{h}^{c_{n-1}} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\theta_{n-1} < T \le \theta_n\}} \Big] \\
&= \mathbb{E}\left[\int_0^{T} \tilde{f}_t^{c(t)}\, dt - \sum_{n=1}^{\infty} \left(\tilde{l}_{\theta_n}^{c_{n-1}^{(2)}, c_n^{(2)}} - \tilde{k}_{\theta_n}^{c_{n-1}^{(1)}, c_n^{(1)}} \right) + \tilde{h}^{c(T)} \mathbb{1}_{\{T<\infty\}} \right] \\
&= J(\gamma(\alpha^*, \beta^*)).
\end{aligned}
$$

Step 2. For any given $\beta = (\tau_n, b_n)_{n\geq 0} \in \boldsymbol{B}^j$, define $\overline{\alpha^*}(\beta) = (\sigma_n^*, a_n^*)_{n\geq 0}$ such that $\sigma_0^* = 0$, $a_0^* = i \in \Lambda^1$ and for $n \geq 1$, $\sigma_n^* = \inf\{T_N^{(1)} > \sigma_{n-1}^* : c^{(1)}(T_N^{(1)}) \neq a_{n-1}^*\} \wedge T_{M_1}^{(1)}$, $a_n^* = c^{(1)}(\sigma_n^*+)$, with $c(\cdot)$ is defined using the sequence $(\theta_n, c_n)_{n\geq 0}$ such that $\theta_0 = 0$, $c_0 = (i,j) \in \Lambda$ and for $n \geq 1$,

$$
\theta_n = \sigma_{\theta_{n-1}}^{c_{n-1}} \wedge \tau_{s_n} \tag{4.4.15}
$$

where $\sigma_{\theta_{n-1}}^{c_{n-1}}$, defined by (4.4.7), is the optimal strategy for Player I of the constrained Dynkin game (4.4.5) with starting time θ_{n-1} and starting mode c_{n-1}, and

$$
c_n = \begin{cases} \left(\arg\min_{i' \neq c_{n-1}^{(1)}} \{ \tilde{Y}_{\theta_n}^{i', c_{n-1}^{(2)}} + \tilde{k}_{\theta_n}^{c_{n-1}^{(1)}, i'} \}, c_{n-1}^{(2)} \right), & \text{if } \sigma_{\theta_{n-1}}^{c_{n-1}} < \tau_{s_n}, \sigma_{\theta_{n-1}}^{c_{n-1}} < T, \\ \left(c_{n-1}^{(1)}, b_{s_n} \right), & \text{if } \tau_{s_n} \leq \sigma_{\theta_{n-1}}^{c_{n-1}}, \tau_{s_n} < T, \\ c_{n-1}, & \text{if } \sigma_{\theta_{n-1}}^{c_{n-1}} \wedge \tau_{s_n} \geq T, \end{cases} \tag{4.4.16}
$$

with the sequence $\{s_n\}_{n\geq 0}$ defined iteratively by $s_0 = 0$, $s_1 = 1$ and for $n \geq 2$,

$$
s_n = s_{n-1} + \mathbb{1}_{\{\tau_{s_{n-1}} \leq \sigma_{\theta_{n-2}}^{c_{n-2}}\}}.
$$

By the construction, it is obvious $\overline{\alpha^*} \in \mathcal{A}^i$.

We now prove the following equality

$$
\tilde{Y}_0^{i,j} = y_0^{i,j} = \sup_{\beta \in \boldsymbol{B}^j} J(\gamma(\overline{\alpha^*}(\beta), \beta)). \tag{4.4.17}
$$

Indeed, it follows from Proposition 4.4.4 and (4.4.6) that

$$
\begin{aligned}
y_0^{i,j} \geq \mathbb{E}\Big[\int_0^{\sigma_0^{i,j}\wedge\tau_1\wedge T} & \tilde{f}_t^{i,j}\, dt + \tilde{L}_{\tau_1}^{i,j} \mathbb{1}_{\{\tau_1 < T, \tau_1 \leq \sigma_0^{i,j}\}} + \tilde{U}_{\sigma_0^{i,j}}^{i,j} \mathbb{1}_{\{\sigma_0^{i,j} < T, \sigma_0^{i,j} < \tau_1\}} \\
& + \tilde{h}^{i,j} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\sigma_0^{i,j}\wedge\tau_1 \geq T\}} \Big].
\end{aligned}
$$

By the definition of a_1^* and $\tilde{L}_{\tau_1}^{i,j}$, we obtain

$$
\begin{aligned}
y_0^{i,j} \geq \mathbb{E}\Big[\int_0^{\theta_1\wedge T} & \tilde{f}_t^{c(t)}\, dt + \tilde{Y}_{\theta_1}^{c_1} \mathbb{1}_{\{\theta_1 < T\}} - \left(\tilde{l}_{\theta_1}^{c_0^{(2)}, c_1^{(2)}} - \tilde{k}_{\theta_1}^{c_0^{(1)}, c_1^{(1)}} \right) \\
& + \tilde{h}^{c_0} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\theta_1 \geq T\}} \Big].
\end{aligned}
$$

We can repeat the above procedure M times, since $\theta_M \geq \delta_M > T$, we obtain

$$
\begin{aligned}
y_0^{i,j} &\geq \mathbb{E}\Big[\int_0^{\theta_M\wedge T} \tilde{f}_t^{c(t)}\, dt + \tilde{Y}_{\theta_M}^{c_M} \mathbb{1}_{\{\theta_M < T\}} - \sum_{n=1}^{M} \left(\tilde{l}_{\theta_n}^{c_{n-1}^{(2)}, c_n^{(2)}} - \tilde{k}_{\theta_n}^{c_{n-1}^{(1)}, c_n^{(1)}} \right) \\
&\qquad + \sum_{n=1}^{M} \tilde{h}^{c_{n-1}} \mathbb{1}_{\{T<\infty\}} \mathbb{1}_{\{\theta_{n-1} < T \leq \theta_n\}} \Big] \\
&= \mathbb{E}\Big[\int_0^{T} \tilde{f}_t^{c(t)}\, dt - \sum_{n=1}^{\infty} \left(\tilde{l}_{\theta_n}^{c_{n-1}^{(2)}, c_n^{(2)}} - \tilde{k}_{\theta_n}^{c_{n-1}^{(1)}, c_n^{(1)}} \right) + \tilde{h}^{c(T)} \mathbb{1}_{\{T<\infty\}} \Big] \\
&= J(\gamma(\overline{\alpha^*}(\beta), \beta))
\end{aligned}
$$

for any $\beta \in \boldsymbol{B}^j$. This implies (4.4.17), since

$$
y_0^{i,j} \geq \sup_{\beta\in\boldsymbol{B}^j} J(\gamma(\overline{\alpha^*}(\beta), \beta)) \geq J(\gamma(\overline{\alpha^*}(\beta^*), \beta^*)) = J(\gamma(\alpha^*, \beta^*)) = y_0^{i,j}
$$

where β^* is given in Step 1 (i.e. (4.4.11)) and the last equality follows from (4.4.12).

Similarly, for any given $\alpha = (\sigma_n, a_n)_{n\geq 0} \in \boldsymbol{A}^i$, define $\overline{\beta^*}(\alpha) = (\tau_n^*, b_n^*)_{n\geq 0}$ such that $\tau_0^* = 0$, $b_0^* = j \in \Lambda^2$ and for $n \geq 1$, $\tau_n^* = \inf\{T_N^{(2)} > \tau_{n-1}^* : c^{(2)}(T_N^{(2)}) \neq b_{n-1}^*\} \wedge T_{M_2}^{(2)}$, $b_n^* = c^{(2)}(\tau_n^*+)$, with $c(\cdot)$ is defined using the sequence $(\theta_n, c_n)_{n\geq 0}$ such that $\theta_0 = 0$, $c_0 = (i,j) \in \Lambda$ and for $n \geq 1$, $\theta_n = \sigma_{r_n} \wedge \tau_{\theta_{n-1}}^{c_{n-1}}$, where $\tau_{\theta_{n-1}}^{c_{n-1}}$, defined by (4.4.7), is the optimal strategy for Player II of the constrained Dynkin game (4.4.5) with starting time θ_{n-1} and starting mode c_{n-1}, and

$$
c_n = \begin{cases} \left(a_{r_n}, c_{n-1}^{(2)} \right), & \text{if } \sigma_{r_n} < \tau_{\theta_{n-1}}^{c_{n-1}}, \sigma_{r_n} < T, \\ \left(c_{n-1}^{(1)}, \arg\max_{j' \neq c_{n-1}^{(2)}} \{ \tilde{Y}_{\theta_n}^{c_{n-1}^{(1)}, j'} - \tilde{l}_{\theta_n}^{c_{n-1}^{(2)}, j'} \} \right), & \text{if } \tau_{\theta_{n-1}}^{c_{n-1}} \leq \sigma_{r_n}, \tau_{\theta_{n-1}}^{c_{n-1}} < T, \\ c_{n-1}, & \text{if } \sigma_{r_n} \wedge \tau_{\theta_{n-1}}^{c_{n-1}} \geq T, \end{cases}
$$

with the sequence $\{r_n\}_{n\geq 0}$ defined iteratively by $r_0 = 0$, $r_1 = 1$ and for $n \geq 2$,

$$r_n = r_{n-1} + \mathbb{1}_{\{\sigma_{r_{n-1}} < \tau^{c_{n-2}}_{\theta_{n-2}}\}}.$$

By the construction, $\overline{\beta^*} \in \mathcal{B}^j$, and using the similar arguments, we can have

$$\tilde{Y}_0^{i,j} = y_0^{i,j} = \inf_{\alpha \in \boldsymbol{A}^i} J(\gamma(\alpha, \overline{\beta^*}(\alpha))). \tag{4.4.18}$$

Step 3. Finally, using (4.4.17) and (4.4.18), we can have

$$\begin{aligned}
\overline{v}^{i,j} &= \inf_{\alpha \in \boldsymbol{A}^i} \sup_{\beta \in \boldsymbol{B}^j} J(\gamma(\alpha, \beta)) \\
&\geq \sup_{\overline{\beta} \in \mathcal{B}^j} \inf_{\alpha \in \boldsymbol{A}^i} J(\gamma(\alpha, \overline{\beta}(\alpha))) = \overline{V}^{i,j} \\
&\geq \inf_{\alpha \in \boldsymbol{A}^i} J(\gamma(\alpha, \overline{\beta^*}(\alpha))) = \tilde{Y}_0^{i,j} = \sup_{\beta \in \boldsymbol{B}^j} J(\gamma(\overline{\alpha^*}(\beta), \beta)) \\
&\geq \inf_{\overline{\alpha} \in \mathcal{A}^i} \sup_{\beta \in \boldsymbol{B}^j} J(\gamma(\overline{\alpha}(\beta), \beta)) = \underline{V}^{i,j} \\
&\geq \sup_{\beta \in \boldsymbol{B}^j} \inf_{\alpha \in \boldsymbol{A}^i} J(\gamma(\alpha, \beta)) = \underline{v}^{i,j}
\end{aligned}$$

which completes the proof of Theorem 4.2.5.

4.5 Connection with constrained single-player optimal switching problems

In this section, we will show, under some additional conditions on the running reward $f^{i,j}$ and terminal reward $h^{i,j}$, the values of both the static game (4.2.4) and the dynamic game (4.2.5) exist. The relationship of (4.2.4)-(4.2.5) and the value of a constrained single-player optimal switching problem is also studied.

Constrained single-player optimal switching problems were first introduced in [59]. The setup is the same as in Section 4.2 except that there is only a single player in the problem, Player I (resp. II) chooses his (resp. her) admissible switching controls $\alpha = (\sigma_n, a_n)_{n\geq 0} \in \boldsymbol{A}^i$ (resp. $\beta = (\tau_n, b_n)_{n\geq 0} \in \boldsymbol{B}^j$) in order to maximise the following payoff functional

$$J^{(1)}(\alpha) = \mathbb{E}\left[\int_0^T e^{-rt} f_t^{(1),a(t)}\, dt - \sum_{n=1}^{\infty} e^{-r\sigma_n} k_{\sigma_n}^{a_{n-1},a_n} + e^{-rT} h^{(1),a(T)} \mathbb{1}_{\{T<\infty\}}\right] \tag{4.5.1}$$

(resp.

$$J^{(2)}(\beta) = \mathbb{E}\left[\int_0^T e^{-rt} f_t^{(2),b(t)}\, dt - \sum_{n=1}^{\infty} e^{-r\tau_n} l_{\tau_n}^{b_{n-1},b_n} + e^{-rT} h^{(2),b(T)} \mathbb{1}_{\{T<\infty\}}\right]) \tag{4.5.2}$$

where $a(\cdot)$ (resp. $b(\cdot)$) is given by (4.2.1) (resp. (4.2.2)). For $i \in \Lambda^1$ (resp. $j \in \Lambda^2$), $f^{(1),i}$ and $h^{(1),i}$ (resp. $f^{(2),j}$ and $h^{(2),j}$) define a running reward and a terminal reward received by the player, whose active mode is i (resp. j). For $i_1, i_2 \in \Lambda^1$ (resp. $j_1, j_2 \in \Lambda^2$), k^{i_1,i_2} (resp. l^{j_1,j_2}) defines a payment from the player, who switches the active mode from mode i_1 (resp. j_1) to i_2 (resp. j_2).

Let us define the value of the constrained single-player optimal switching problem for Player I (resp. II), with the initial mode $i \in \Lambda^1$ (resp. $j \in \Lambda^2$), as

$$v^{(1),i} = \sup_{\alpha \in \boldsymbol{A}^i} J^{(1)}(\alpha) \quad \left(\text{resp. } v^{(2),j} = \sup_{\beta \in \boldsymbol{B}^j} J^{(2)}(\beta)\right), \tag{4.5.3}$$

and $\alpha^* \in \boldsymbol{A}^i$ (resp. $\beta^* \in \boldsymbol{B}^j$) attaining the supremum in (4.5.3) is called his (resp. her) optimal switching strategy.

We impose the following additional condition on the running and terminal rewards.

Assumption 4.5.1 *For $(i,j) \in \Lambda$, the $f^{i,j}$ and $h^{i,j}$ are separated with respect to i and j, i.e.*

$$f^{i,j} := f^{(2),j} - f^{(1),i} \text{ and } h^{i,j} := h^{(2),j} - h^{(1),i}, \tag{4.5.4}$$

and moreover, (i) when $T = \infty$, $f^{(1),i}$ and $f^{(2),j}$ are bounded; (ii) when $T < \infty$, $f^{(1),i}, f^{(2),j} \in \mathbb{H}^{2,1}$ and $h^{(1),i}, h^{(2),j} \in \mathbb{L}^{2,1}$.

Proposition 4.5.2 *Suppose that Assumption 4.2.3 and 4.5.1 hold. Then, there exists a unique solution $(Y^{(1)}, Z^{(1)})$ to the following BSDE system defined on $[0,T]$:*

$$\begin{aligned} Y_t^{(1),i} = h^{(1),i}\mathbb{1}_{\{T<\infty\}} + \int_t^T \Bigg[f_s^{(1),i} + \lambda^{(1)} \left(\max_{i' \neq i} \{ Y_s^{(1),i'} - k_s^{i,i'} \} - Y_s^{(1),i} \right)^+ \\ - rY_s^{(1),i} \Bigg] ds - \int_t^T Z_s^{(1),i}\, dW_s \end{aligned} \tag{4.5.5}$$

for $t \in [0,T]$ and $i \in \Lambda^1$, and a unique solution $(Y^{(2)}, Z^{(2)})$ to the following BSDE system defined on $[0,T]$:

$$\begin{aligned} Y_t^{(2),j} = h^{(2),j}\mathbb{1}_{\{T<\infty\}} + \int_t^T \Bigg[f_s^{(2),j} + \lambda^{(2)} \left(\max_{j' \neq j} \{ Y_s^{(2),j'} - l_s^{j,j'} \} - Y_s^{(2),j} \right)^+ \\ - rY_s^{(2),j} \Bigg] ds - \int_t^T Z_s^{(2),j}\, dW_s \end{aligned} \tag{4.5.6}$$

for $t \in [0,T]$ and $j \in \Lambda^2$. For every initial state $(i,j) \in \Lambda$, the values of both the static game (4.2.4) and the dynamic game (4.2.5) exist, i.e.

$$\overline{v}^{i,j} = \underline{v}^{i,j} = \overline{V}^{i,j} = \underline{V}^{i,j} = v^{(2),j} - v^{(1),i} = Y_0^{(2),j} - Y_0^{(1),i}, \tag{4.5.7}$$

and the optimal pair of controls $(\alpha^*, \beta^*) \in \boldsymbol{A}^i \times \boldsymbol{B}^j$ of the constained optimal switching games (4.2.4) is the sequences $\alpha^* = (\sigma_n^*, a_n^*)_{n\geq 0}$ such that $\sigma_0^* = 0, a_0^* = i$, and for $n \geq 1$,

$$\left\{ \begin{array}{l} \sigma_n^* = \inf \left\{ T_N^{(1)} > \sigma_{n-1}^* : Y_{T_N^{(1)}}^{(1),a_{n-1}^*} \leq \max_{i' \neq a_{n-1}^*} \left\{ Y_{T_N^{(1)}}^{(1),i'} - k_{T_N^{(1)}}^{a_{n-1}^*,i'} \right\} \right\} \wedge T_{M_1}^{(1)} \\ a_n^* = \arg\max_{i' \neq a_{n-1}^*} \left\{ Y_{\sigma_n^*}^{(1),i'} - k_{\sigma_n^*}^{a_{n-1}^*,i'} \right\} \end{array} \right. ; \tag{4.5.8}$$

and $\beta^* = (\tau_n^*, b_n^*)_{n\geq 0}$ such that $\tau_0^* = 0, b_0^* = j$, and for $n \geq 1$,

$$\left\{ \begin{array}{l} \tau_n^* = \inf \left\{ T_N^{(2)} > \tau_{n-1}^* : Y_{T_N^{(2)}}^{(2),b_{n-1}^*} \leq \max_{j' \neq b_{n-1}^*} \left\{ Y_{T_N^{(2)}}^{(2),j'} - l_{T_N^{(2)}}^{b_{n-1}^*,j'} \right\} \right\} \wedge T_{M_2}^{(2)} \\ b_n^* = \arg\max_{j' \neq b_{n-1}^*} \left\{ Y_{\tau_n^*}^{(2),j'} - l_{\tau_n^*}^{b_{n-1}^*,j'} \right\} \end{array} \right. . \tag{4.5.9}$$

Proof. Since $f^{i,j}$ and $h^{i,j}$ are separated with respect to i and j, for any pair of controls $(\alpha, \beta) \in \boldsymbol{A}^i \times \boldsymbol{B}^j$, we have

$$J(\alpha, \beta) = J^{(2)}(\beta) - J^{(1)}(\alpha)$$

where $J^{(1)}(\alpha)$ and $J^{(2)}(\beta)$ are given by (4.5.1) and (4.5.2). Thus, we have

$$\overline{v}^{i,j} = \underline{v}^{i,j} = \sup_{\beta \in \boldsymbol{B}^j} J^{(2)}(\beta) - \sup_{\alpha \in \boldsymbol{A}^i} J^{(1)}(\alpha) = v^{(2),j} - v^{(1),i},$$

where the last equality follows from BSDE characterization (4.5.5)-(4.5.6) of the constrained optimal switching problems (4.5.3) (see [59] for more details), i.e.

$$Y_0^{(1),i} = v^{(1),i} = J^{(1)}(\alpha^*)$$

with the optimal switching strategy α^* given by (4.5.8), and

$$Y_0^{(2),j} = v^{(2),j} = J^{(2)}(\beta^*)$$

with the optimal switching strategy β^* given by (4.5.9). It remains to prove that (α^*, β^*) is the optimal pair of controls of the constained optimal switching games (4.2.4). Indeed, for any pair of controls $(\alpha, \beta) \in \boldsymbol{A}^i \times \boldsymbol{B}^j$, we have

$$\begin{aligned} J(\alpha^*, \beta) = J^{(2)}(\beta) - J^{(1)}(\alpha^*) &\leq J^{(2)}(\beta^*) - J^{(1)}(\alpha^*) \\ &= J(\alpha^*, \beta^*) \\ &\leq J^{(2)}(\beta^*) - J^{(1)}(\alpha) = J(\alpha, \beta^*). \end{aligned}$$

■

4.6 Application to the duopolistic competition in resource extraction

In this section, we study the duopolistic competition in resource extraction for which both producers are only allowed to make their decisions at Poisson arrival times. The problem is adapted from the classical model imposed by Brekke and Øksendal [10], where they considered the case of a single producer without any switching constraints. Traditionally, duopolistic competition models often assume that both producers are allowed to make decisions at any stopping time. In reality, there may exist some liquidation constraint as an external shock, and both players only make their decisions when such a shock arrives. We model such a liquidation shock as the arrival times of two exogenous Poisson processes.

4.6.1 Problem formulation

There are two large producers in the market. We assume that the price P_t at time t per unit of the resource follows a geometric Brownian motion starting from $P_0 = p \in \mathbb{R}^+$ with constant drift b and constant volatility $\sigma > 0$:

$$dP_t = bP_t\,dt + \sigma P_t\,dW_t.$$

For producer $k \in \{1, 2\}$, let $Q_t^{(k)}$ denote the stock of remaining resources in the field . We assume when the field is open, extraction rate is proportional to the amount of remaining reserves. In other words, $Q^{(k)}$ follows

$$dQ_t^{(k)} = -\eta^{(k)} X_t^{(k)} Q_t^{(k)}\,dt$$

where $\eta^{(k)} > 0$ is a constant, $Q_0^{(k)} = q^{(k)}$ and $X^{(k)}$ is a $\mathbb{G}^{\boldsymbol{\lambda}}$-adapted, finite variation, càglàd process with values in $\{0, 1\}$, i.e.

$$X_t^{(k)} = \begin{cases} 1 & \text{if the field is open at time } t \\ 0 & \text{if the field is closed at time } t. \end{cases}$$

At time 0, we assume $X_0^{(k)} = x^{(k)} \in \{0, 1\}$. The extraction can operate in two modes, open and closed. The transition from one operating mode to the other is immediate and the management is only allowed to make decisions at a sequence of Possion random intervention times $T^{(k)} = \{T_n^{(k)}\}_{n \geq 0}$ with the constant intensity $\lambda^{(k)} > 0$. Let $\boldsymbol{A}^i$ (resp. $\boldsymbol{B}^j$) denote the family of $X^{(1)}$ (resp. $X^{(2)}$) satisfying $x^{(1)} = i$ (resp. $x^{(2)} = j$).

We assume that, when the field is open, there is a constant profit rate constraint $M^{(k)} > 0$ per time unit, which is imposed by the government to avoid the producers obtaining supernormal profits. We also assume there is a constant standby cost $C^{(k)} > 0$ per time unit resulting from a closed field. Then the net profit rate for each producer k is given by

$$f^{(k)}(t, p, q, x) = e^{-rt}\left(\min\{\eta^{(k)} pq, M^{(k)}\}x - C^{(k)}(1 - x)\right)$$

where $r > \max\{0, b\}$ is the discount rate.

We further assume that the constants $K_0^{(k)}, K_1^{(k)} > 0$ represent the positive costs resulting from switching the extraction mode from the open to the closed mode and vice versa.

We model the duopolistic competition between two large producers, whose performance can be measured by the difference of their expected profits, i.e.

$$J(p, q^{(1)}, q^{(2)}, X^{(1)}, X^{(2)}) = J^{(2)}(p, q^{(2)}, X^{(2)}) - J^{(1)}(p, q^{(1)}, X^{(1)}) \qquad (4.6.1)$$

where the individual performance criterion $J^{(k)}(p, q^{(k)}, X^{(k)})$ is given by

$$\begin{aligned}&J^{(k)}(p, q^{(k)}, X^{(k)})\\ &= \mathbb{E}\left[\int_0^\infty f^{(k)}(t, P_t, Q_t^{(k)}, X_t^{(k)})\, dt - \sum_{0\le t} e^{-rt}\left(K_1^{(k)}(\Delta X_t^{(k)})^+ + K_0^{(k)}(\Delta X_t^{(k)})^-\right)\right]\end{aligned}$$

with $\Delta X_t^{(k)} = X_{t+}^{(k)} - X_t^{(k)}$, for each player $k \in \{1, 2\}$.

For the initial state $(i, j) \in \{0, 1\}^2$, producer II will choose $X^{(2)} \in \boldsymbol{B}^j$ to maximize the profit difference (4.6.1), while producer I will choose $X^{(1)} \in \boldsymbol{A}^i$ to minimize the difference. Either in positive or negative direction, the large difference means one large producer dominates the market and wins the game. This leads to an aforementioned constrained optimal switching game with separated running and terminal rewards.

Let us define the upper and lower value of this constrained duopolistic competition as follows

$$\overline{v}^{i,j}(p, q^{(1)}, q^{(2)}) = \inf_{X^{(1)}\in\boldsymbol{A}^i} \sup_{X^{(2)}\in\boldsymbol{B}^j} J(p, q^{(1)}, q^{(2)}, X^{(1)}, X^{(2)}), \qquad (4.6.2)$$

$$\underline{v}^{i,j}(p, q^{(1)}, q^{(2)}) = \sup_{X^{(2)}\in\boldsymbol{B}^j} \inf_{X^{(1)}\in\boldsymbol{A}^i} J(p, q^{(1)}, q^{(2)}, X^{(1)}, X^{(2)}). \qquad (4.6.3)$$

Applying Proposition 4.5.2, we obtain this constrained duopolistic competition has value

$$\overline{v}^{i,j}(p, q^{(1)}, q^{(2)}) = \underline{v}^{i,j}(p, q^{(1)}, q^{(2)}) = v^{i,j}(p, q^{(1)}, q^{(2)}) = Y_0^{(2),j} - Y_0^{(1),i}$$

where $Y^{(1),i}$ (resp. $Y^{(2),j}$) is the first component of the solution to the following infinite horizon BSDE system:

$$
\begin{aligned}
Y_t^{(1),i} = & \int_t^\infty \min\left\{\eta^{(1)} P_s Q_s^{(1),i}, M^{(1)}\right\} i - C^{(1)}(1-i) - rY_s^{(1),i} \\
& + \lambda^{(1)} \left(Y_s^{(1),1-i} - iK_0^{(1)} - (1-i)K_1^{(1)} - Y_s^{(1),i}\right)^+ ds - \int_t^\infty Z_s^{(1),i}\, dW_s
\end{aligned}
\tag{4.6.4}
$$

(resp.

$$
\begin{aligned}
Y_t^{(2),j} = & \int_t^\infty \min\left\{\eta^{(2)} P_s Q_s^{(2),j}, M^{(2)}\right\} j - C^{(2)}(1-j) - rY_s^{(2),j} \\
& + \lambda^{(2)} \left(Y_s^{(2),1-j} - jK_0^{(2)} - (1-j)K_1^{(2)} - Y_s^{(2),j}\right)^+ ds - \int_t^\infty Z_s^{(2),j}\, dW_s)
\end{aligned}
$$

for $t \geq 0$ and $i \in \{0,1\}$ (resp. $j \in \{0,1\}$), with $Q^{(1),i}$ (resp. $Q^{(2),j}$) following

$$
dQ_t^{(1),i} = -\eta^{(1)} i Q_t^{(1),i}\, dt, \quad Q_0^{(1),i} = q^{(1)}
$$

(resp.

$$
dQ_t^{(2),j} = -\eta^{(2)} j Q_t^{(2),j}\, dt, \quad Q_0^{(2),j} = q^{(2)}).
$$

Moreover, the optimal switching strategy $(\alpha^*, \beta^*) \in \boldsymbol{A}^i \times \boldsymbol{B}^j$ of this constrained duopolistic competition (4.6.2)-(4.6.3) is given by the sequence $\alpha^* = (\sigma_n^*, a_n^*)_{n\geq 0}$ such that $\sigma_0^* = 0, a_0^* = i \in \{0,1\}$, and for $n \geq 1$,

$$
\begin{cases}
\sigma_n^* = \inf\left\{T_N^{(1)} > \sigma_{n-1}^* : Y_{T_N^{(1)}}^{(1),a_{n-1}^*} \leq Y_{T_N^{(1)}}^{(1),a_n^*} - a_{n-1}^* K_0^{(1)} - (1-a_{n-1}^*)K_1^{(1)}\right\} \\
a_n^* = 1 - a_{n-1}^*
\end{cases} ;
\tag{4.6.5}
$$

and $\beta^* = (\tau_n^*, b_n^*)_{n\geq 0}$ such that $\tau_0^* = 0, b_0^* = j \in \{0,1\}$, and for $n \geq 1$,

$$
\begin{cases}
\tau_n^* = \inf\left\{T_N^{(2)} > \tau_{n-1}^* : Y_{T_N^{(2)}}^{(2),b_{n-1}^*} \leq Y_{T_N^{(2)}}^{(2),b_n^*} - b_{n-1}^* K_0^{(2)} - (1-b_{n-1}^*)K_1^{(2)}\right\} \\
b_n^* = 1 - b_{n-1}^*
\end{cases} .
\tag{4.6.6}
$$

4.6.2 The structure of switching regions

In the rest of this section, we investigate the structure of switching regions of both players. By the results (4.6.4)-(4.6.6), we can observe both players will make similar optimal switching decisions and their decisions are independent from each other. We only study Player I's switching regions since the other player's counterparts follow immediately.

To ease the notation, we omit the superscripts (1) from now on. By observing P_t and Q_t only enter the performance criterion as P_tQ_t, we define

$$\Gamma_t = P_tQ_t$$

which satisfies the SDE

$$d\Gamma_t = (b - \eta X_t)\Gamma_t \, dt + \sigma\Gamma_t \, dW_t, \quad \Gamma_0 = pq := z.$$

As a direct consequence of the previous subsection, for $i \in \{0, 1\}$, we have

$$v^i(z) = \sup_{X\in \boldsymbol{A}^i} \tilde{J}(z, X) = Y_0^i$$

where the modified cost functional is given by

$$\begin{aligned}\tilde{J}(z, X) = \mathbb{E}\Big[\int_0^\infty e^{-rt} \left(\min\{\eta\Gamma_t, M\}X_t - C(1 - X_t)\right) \, dt \\ - \sum_{0\le t} e^{-rt} \left(K_1(\Delta X_t)^+ + K_0(\Delta X_t)^-\right) \Big]\end{aligned} \tag{4.6.7}$$

and Y^i is the first component of the solution to the following infinite horizon BSDE system:

$$\begin{aligned}Y_t^i = \int_t^\infty \min\left\{\eta\Gamma_s^i, M\right\} i - C(1-i) - rY_s^i + \lambda\left(Y_s^{1-i} - iK_0 - (1-i)K_1 - Y_s^i\right)^+ \, ds \\ - \int_t^\infty Z_s^i \, dW_s\end{aligned} \tag{4.6.8}$$

for $t \ge 0$, with Γ^i following

$$d\Gamma_t^i = (b - \eta i)\Gamma_t^i \, dt + \sigma\Gamma_t^i \, dW_t, \quad \Gamma_0^i = z.$$

By the Markov property of Γ^i, we can have $Y_t^i = v^i(\Gamma_t^i)$, where $v = (v^0, v^1)$ solve the ODE system

$$-\mathcal{L}^0v^0 + rv^0 = \lambda\left(v^1 - K_1 - v^0\right)^+ - C, \tag{4.6.9}$$

$$-\mathcal{L}^1v^1 + rv^1 = \lambda\left(v^0 - K_0 - v^1\right)^+ + \min\{\eta z, M\}, \tag{4.6.10}$$

with the operator $\mathcal{L}^i = \frac{1}{2}\sigma^2z^2\partial_{zz}^2 + (b - \eta i)z\partial_z$ (the connection between BSDE and ODE is quite standard in the BSDE literature, and thus we refer to Section 4 of [12] and Section 5 of [40] for rigorous proofs).

For later use, we define the stopping region ($\mathcal{S}^i$) and the continuation region

$(\mathcal{C}^i)$ as

$$\begin{aligned}\mathcal{S}^i &= \{z \in \mathbb{R}^+ : v^i(z) \leq v^{1-i}(z) - iK_0 - (1-i)K_1\},\\ \mathcal{C}^i &= \{z \in \mathbb{R}^+ : v^i(z) > v^{1-i}(z) - iK_0 - (1-i)K_1\},\end{aligned}$$

for $i \in \{0, 1\}$. Furthermore, we define

$$\underline{z}^i = \inf S^i \in [0, \infty] \text{ and } \overline{z}^i = \sup S^i \in [0, \infty]$$

with the usual convention $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

The main result of this section is the following characterization of the switching regions of the above constrained duopolistic competition problem (4.6.2)-(4.6.3).

Theorem 4.6.1 *Suppose the assumptions in this section hold, and that the value function $v = (v^0, v^1)$ is twice continuously differentiable in z. Then, we have the following structures of the switching regions $\mathcal{S}^0$ and $\mathcal{S}^1$:*

$$\mathcal{S}^0 = \begin{cases} (0, \infty), & \text{if } -rK_1 \geq -C \\ [\underline{z}^0, \infty) \text{ for some } \underline{z}^0 \in (0, \infty), & \text{if } -rK_1 < -C \leq M - rK_1 \\ \emptyset, & \text{if } M - rK_1 < -C \end{cases},$$

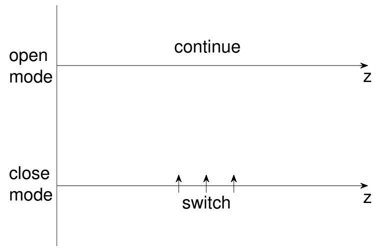
and $\mathcal{S}^1 = \emptyset$.

The economic intuition behind Theorem 4.6.1 is as follows. Firstly, note that the open mode (mode 1) is more favorable than the closed mode (mode 0). Since the switching cost from open mode to closed K_0 is always positive, then one has no interest to switch to closed mode from open. On the other hand, if the switching cost from closed to open K_1 is less than the standby cost resulting from a closed field, i.e. $-rK_1 \geq -C$, the producer would switch to the open mode (higher regime) as soon as possible; if the switching cost can never be compensated by the net running profit, i.e. $M - rK_1 < -C$, the producer would never switch to the open mode; if the net running profit may exceed the loss due to the switching cost in some state, i.e. $-rK_1 < -C \leq M - rK_1$, then one may switch to open mode when the net running profit rate reaches some level at Poisson arrival times. The various structures of the switching regions are demonstrated in Figure 4.1.

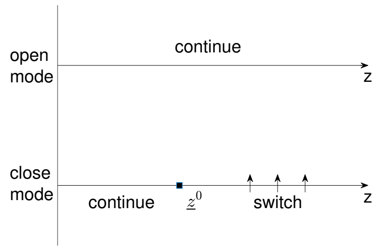
4.6.3 Proof of Theorem 4.6.1

Step 1. We first prove that $\mathcal{S}^1 = \emptyset$. By picking $\overline{X}^1 \equiv 1 \in \boldsymbol{A}^1$, we have

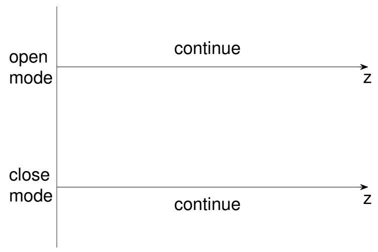
$$v^1(z) \geq \tilde{J}(z, \overline{X}^1) \geq \tilde{J}(z, X)$$



(a) $-rK_1 \geq -C$



(b) $-rK_1 < -C \leq M - rK_1$



(c) $M - rK_1 < -C$

Figure 4.1: The various structures of the switching regions.

where $\tilde{J}$ is given by (4.6.7), for any $X \in \boldsymbol{A}^0$. This implies

$$v^1(z) \geq v^0(z) > v^0(z) - K_0$$

for $\forall z \in (0, \infty)$, i.e. $\mathcal{S}^1 = \emptyset$.

Step 2. In this step, we show the following priori results on v^1:

$$0 \leq \partial_z v^1 \leq \frac{\eta}{r - b + \eta}, \tag{4.6.11}$$

$$\partial^2_{zz} v^1 \leq 0, \tag{4.6.12}$$

and v^1 has the explicit form

$$v^1(z) = \left(A_+ z^{\alpha_+} + \frac{\eta}{r - b + \eta} z \right) \mathbb{1}_{\left\{z \leq \frac{M}{\eta}\right\}} + \left(B_- z^{\alpha_-} + \frac{M}{r} \right) \mathbb{1}_{\left\{z > \frac{M}{\eta}\right\}} \tag{4.6.13}$$

where A_+, B_- are given by (4.6.16) and $\alpha_\pm$ are given by (4.6.15).

As a direct consequence of step 1, we have

$$v^1(z) = \mathbb{E}\left[\int_0^\infty e^{-rt} \min\{\eta \Gamma_t^{1,z}, M\}\, dt \right]$$

where $\Gamma_t^{1,z} = z \exp((b - \eta - \frac{1}{2}\sigma^2)t + \sigma W_t)$ with the initial value $\Gamma_0^{1,z} = z$.

For any $z \geq \bar{z} > 0$, we have $\Gamma_t^{1,z} \geq \Gamma_t^{1,\bar{z}}$ for all $t \geq 0$, and therefore $v^1(z) \geq v^1(\bar{z})$, which proves the first inequality of (4.6.11).

For any $z, \bar{z} \in \mathbb{R}^+$, we have

$$\begin{aligned}
\left|v^1(z) - v^1(\bar{z})\right| &\leq \mathbb{E}\left[\int_0^\infty e^{-rt}\left|\min\{\eta\Gamma_t^{1,z}, M\} - \min\{\eta\Gamma_t^{1,\bar{z}}, M\}\right| dt\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt}\eta\left|\Gamma_t^{1,z} - \Gamma_t^{1,\bar{z}}\right| dt\right] \\
&= \int_0^\infty e^{-(r-b+\eta)t}\eta|z - \bar{z}|\, dt \\
&= \frac{\eta}{r-b+\eta}|z - \bar{z}|,
\end{aligned}$$

which proves the second inequality of (4.6.11).

For any $z, \bar{z} \in \mathbb{R}^+$ and $\kappa \in [0,1]$, we have

$$\begin{aligned}
&\kappa v^1(z) + (1-\kappa)v^1(\bar{z}) \\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\kappa\min\{\eta\Gamma_t^{1,z}, M\} + (1-\kappa)\min\{\eta\Gamma_t^{1,\bar{z}}, M\}\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt}\min\{\eta(\kappa\Gamma_t^{1,z} + (1-\kappa)\Gamma_t^{1,\bar{z}}), M\}\, dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}\min\{\eta\Gamma_t^{1,\kappa z+(1-\kappa)\bar{z}}, M\}\, dt\right] \\
&= v^1(\kappa z + (1-\kappa)\bar{z}),
\end{aligned}$$

which proves (4.6.12).

As another direct consequence of step 1, (4.6.10) is reduced to the following equation

$$-\mathcal{L}^1 v^1 + rv^1 = \min\{\eta z, M\}. \tag{4.6.14}$$

Standard calculations yield that

$$\begin{aligned}
v^1(z) = &\left(A_+ z^{\alpha_+} + A_- z^{\alpha_-} + \frac{\eta}{r-b+\eta}z\right)\mathbb{1}_{\left\{z\leq\frac{M}{\eta}\right\}} \\
&+ \left(B_+ z^{\alpha_+} + B_- z^{\alpha_-} + \frac{M}{r}\right)\mathbb{1}_{\left\{z>\frac{M}{\eta}\right\}}
\end{aligned}$$

where α_+ and α_- are the two roots of the following quadratic equation

$$\frac{1}{2}\alpha^2 + \left(\frac{b-\eta}{\sigma^2} - \frac{1}{2}\right)\alpha - \frac{r}{\sigma^2} = 0$$

i.e.

$$\alpha_\pm = \left(\frac{1}{2} - \frac{b-\eta}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{b-\eta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, \tag{4.6.15}$$

satisfying $\alpha_+ > 1$ and $\alpha_- < 0$. The boundary conditions imply that

$$A_- = B_+ = 0.$$

Indeed, as $z \to 0$, we have $\Gamma_t^{1,z} \to 0$ for any $t \geq 0$, then $v^1(0+) = 0$, and as $z \to \infty$, we have $\Gamma_t^{1,z} \to \infty$ for any $t \geq 0$, then $v^1(\infty) = \frac{M}{r}$.

Moreover, using the continuity of $v^1(z)$ and $\left(v^1\right)'(z)$ across the point $z = \frac{M}{\eta}$, we can determine the two unknowns A_+ and B_-, i.e.

$$A_+ = \frac{(\alpha_- - 1)\frac{M}{r-b+\eta} - \frac{M}{r}\alpha_-}{(\alpha_+ - \alpha_-)\left(\frac{M}{\eta}\right)^{\alpha_+}}, \text{ and } B_- = \frac{(\alpha_+ - 1)\frac{M}{r-b+\eta} - \frac{M}{r}\alpha_+}{(\alpha_+ - \alpha_-)\left(\frac{M}{\eta}\right)^{\alpha_-}}. \quad (4.6.16)$$

Step 3. We next prove $\mathcal{S}^0 = (0, \infty)$ if $-rK_1 \geq -C$.

Define $G_0(z) = v^0(z) - v^1(z) + K_1$, then the switching region

$$\mathcal{S}^0 = \{z \in \mathbb{R}^+ : G_0(z) \leq 0\}.$$

We can then obtain G_0 is the solution to the following ODE:

$$-\mathcal{L}^0 G_0 - \eta z \partial_z v^1 + rG_0 = \lambda(-G_0)^+ - \min\{\eta z, M\} - C + rK_1. \quad (4.6.17)$$

Because the terms on the right-hand-side of (4.6.17) are not continuously differentiable, in order to prove $G_0'(z) \leq 0$, we construct a penalty approximation of (4.6.17). Suppose $G_{0,\epsilon}$ satisfies

$$-\mathcal{L}^0 G_{0,\epsilon} - \eta z \partial_z v^1 + rG_{0,\epsilon} = \lambda(-G_{0,\epsilon})^+ - \pi_\epsilon(\eta z - M) - M - C + rK_1 \quad (4.6.18)$$

where $\pi_\epsilon(z)$ satisfies that $\pi_\epsilon(z) \in C^\infty$, $0 \leq \pi_\epsilon'(z) \leq 1$, $\pi_\epsilon'(0) \geq \frac{\eta}{r-b+\eta}$, $\pi_\epsilon''(z) \leq 0$, $\lim_{\epsilon \to 0+} \pi_\epsilon(z) = \min(z, 0)$, and

$$\pi_\epsilon(z) = \begin{cases} z, & z \leq -\epsilon, \\ \nearrow, & |z| \leq \epsilon, \\ 0, & z \geq \epsilon. \end{cases}$$

Differentiating both sides of (4.6.18) yields that

$$\begin{aligned} -\frac{1}{2}\sigma^2 z^2 \partial_{zz}^2 G_{0,\epsilon}' - (b + \sigma^2) z \partial_z G_{0,\epsilon}' + (r - b + \lambda H(-G_{0,\epsilon})) G_{0,\epsilon}' \\ = \eta\left(\partial_z v^1 - \pi_\epsilon'(\eta z - M) + z\partial_{zz}^2 v^1\right) \end{aligned}$$

where $H(x) = \mathbb{1}_{[0,\infty)}(x)$. Using the priori results (4.6.11)-(4.6.13) on v^1 obtained in Step 1, we can prove that the right-hand-side of the above equation is no greater than 0. Indeed, for $z \leq \frac{M}{\eta}$, we have

$$\eta\left(\partial_z v^1 - \pi_\epsilon'(\eta z - M) + z\partial_{zz}^2 v^1\right) \leq \eta\left(\partial_z v^1 - \pi_\epsilon'(0) + z\partial_{zz}^2 v^1\right) \leq 0 \quad (4.6.19)$$

by (4.6.11)-(4.6.12), and for $z > \frac{M}{\eta}$, we have

$$\eta\left(\partial_z v^1 - \pi'_\epsilon(\eta z - M) + z\partial^2_{zz}v^1\right) \le \eta\left(\partial_z v^1 + z\partial^2_{zz}v^1\right) = \eta B_- z^{\alpha_- -1}\alpha_-^2 \le 0 \tag{4.6.20}$$

since $B_- \le 0$. Thus, it follows from the comparision results (see, for example, Section 4.4 in [70]) that $G'_{0,\epsilon}(z) \le 0$. By letting $\epsilon \to 0$, we have a continuous limit (of a subsequence if necessary) $G'_0(z)$, that is, $G'_{0,\epsilon}(z) \to G'_0(z)$ uniformly in $C(\mathbb{R}^+)$, and then we obtain the required result (see, for example, Proof of Lemma 5 in [14]).

Given the boundary condition

$$\begin{aligned} v^0(0+) = & \max\Big(\mathbb{E}[\int_0^\infty e^{-rt}(-C)\,dt], \\ & \qquad \mathbb{E}[\int_0^{T_1} e^{-rt}(-C)\,dt + \int_{T_1}^\infty e^{-rt}\cdot 0\,dt - e^{-rT_1}K_1]\Big) \\ = & \max\left(-\frac{C}{r}, -\frac{C+\lambda K_1}{\lambda + r}\right) \end{aligned}$$

we can compute that, in the case of $-rK_1 \ge -C$,

$$G_0(0+) = v^0(0+) - v^1(0+) + K_1 = -\frac{C - rK_1}{\lambda + r} \le 0.$$

This proves that $\mathcal{S}^0 = (0, \infty)$ if $-rK_1 \ge -C$.
Step 4. We now prove $\mathcal{S}^0 = [\underline{z}^0, \infty)$ for some $\underline{z}^0 \in (0, \infty)$ if $-rK_1 < -C \le M - rK_1$ and $\mathcal{S}^0 = \emptyset$ if $M - rK_1 < -C$.

Note that $G'_0(z) \le 0$, and for both cases, we have $-rK_1 < -C$, and then

$$G_0(0+) = v^0(0+) - v^1(0+) + K_1 = -\frac{C - rK_1}{r} > 0.$$

In the case of $-rK_1 < -C \le M - rK_1$, we only need to prove that $\underline{z}^0 \ne \infty$. If not, then $G_0(z) > 0$ for $\forall z \in (0, \infty)$, and then (4.6.17) is reduced to

$$-\mathcal{L}^0 G_0 + rG_0 = F_0(z)$$

where $F_0(z) := \eta z \partial_z v^1 - \min\{\eta z, M\} - C + rK_1$. Feynman-Kac formula implies that

$$G_0(z) = \mathbb{E}\left[\int_0^\infty e^{-rt} F_0(\Gamma_t^{0,z})\,dt\right].$$

Using Fatou's lemma and the explicity form (4.6.13) of v^1, we have

$$\begin{aligned} G_0(\infty) &= \limsup_{z\to\infty} \mathbb{E}\left[\int_0^\infty e^{-rt} F_0(\Gamma_t^{0,z})\, dt\right] \\ &\leq \mathbb{E}\left[\int_0^\infty e^{-rt} F_0(\infty)\, dt\right] \\ &= \frac{-M - C + rK_1}{r} \\ &\leq 0, \end{aligned}$$

which provides the desired contradiction.

In the case of $M - rK_1 < -C$, we need to show that $\underline{z}^0 = \infty$. If not, then $0 < \underline{z}^0 < \infty$. Due to the continuity of $G_0(z)$, we have $G_0(\underline{z}^0) = 0$. Then it follows that $G_0(z) \leq 0$ for $z \in (\underline{z}^0, \infty)$, and therefore, (4.6.17) is reduced to

$$-\mathcal{L}^0 G_0 + (r + \lambda) G_0 = F_0(z)$$

on $(\underline{z}^0, \infty)$. If we can show $F_0(z) > 0$, then it follows from the comparison results that $G_0(z) > 0$ on $(\underline{z}^0, \infty)$, which provides the desired contradiction. Indeed, it is straightforward to prove $F_0(z) > 0$ since we have $F_0' \leq 0$ by (4.6.19)-(4.6.20), and $F_0(\infty) = -M - C + rK_1 > 0$.

4.A Proof of Proposition 4.4.4 on an infinite horizon

The proof is adapted from the proof of Section 3.5.1, where a finite horizon problem was considered.

We first give an equivalent formulation of the problem. We define the following processes:

$$\overline{L}_t^{i,j} = e^{-rt} L_t^{i,j} + \int_0^t e^{-ru} f_u^{i,j}\, du, \quad \overline{U}_t^{i,j} = e^{-rt} U_t^{i,j} + \int_0^t e^{-ru} f_u^{i,j}\, du,$$

and the lower and upper values of the revised auxiliary constrained Dynkin game with starting time δ_{n-1} and starting mode (i,j):

$$\underline{q}_{\delta_{n-1}}^{i,j} = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{\delta_n}^{(2)}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{\delta_n}^{(1)}} \mathbb{E}\left[\tilde{R}^{i,j}(\sigma,\tau)\,|\mathcal{G}_{\delta_{n-1}}\right], \tag{4.A.1}$$

$$\overline{q}_{\delta_{n-1}}^{i,j} = \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}_{\delta_n}^{(1)}} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{\delta_n}^{(2)}} \mathbb{E}\left[\tilde{R}^{i,j}(\sigma,\tau)\,|\mathcal{G}_{\delta_{n-1}}\right], \tag{4.A.2}$$

where the revised payoff function is given by

$$\tilde{R}^{i,j}(\sigma,\tau) = \overline{L}_\tau^{i,j} \mathbb{1}_{\{\tau<\infty,\tau\leq\sigma\}} + \overline{U}_\sigma^{i,j} \mathbb{1}_{\{\sigma<\infty,\sigma<\tau\}}.$$

Thus, to prove Proposition 4.4.4 on an infinite horizon, it is equivalent to prove that $q^{i,j}_{\delta_{n-1}} = \overline{q}^{i,j}_{\delta_{n-1}} = \underline{q}^{i,j}_{\delta_{n-1}} = \overline{Y}^{i,j}_{\delta_{n-1}}$, and the optimal stopping strategy of the game is given by

$$\begin{cases} \sigma^{i,j}_{\delta_{n-1}} = \inf\{T^{(1)}_N \geq \delta_n : \overline{Y}^{i,j}_{T^{(1)}_N} \geq \overline{U}^{i,j}_{T^{(1)}_N}\}, \\ \tau^{i,j}_{\delta_{n-1}} = \inf\{T^{(2)}_N \geq \delta_n : \overline{Y}^{i,j}_{T^{(2)}_N} \leq \overline{L}^{i,j}_{T^{(2)}_N}\}, \end{cases} \tag{4.A.3}$$

where $\overline{Y}^{i,j}$ is given by $\overline{Y}^{i,j}_t := e^{-rt}Y^{i,j}_t + \int_0^t e^{-ru} f^{i,j}_u \, du$, for $t \geq 0$.

We are now in a position to prove Proposition 4.4.4.

Step 1. We first show that $\overline{Y}^{i,j}_{\delta_{n-1}}$ satisfies the following recursive equation:

$$\begin{aligned} \overline{Y}^{i,j}_{\delta_{n-1}} = \mathbb{E}\bigg[&\min\{\overline{U}^{i,j}_{\delta_n}, \overline{Y}^{i,j}_{\delta_n}\} \mathbb{1}_{\{\delta_n < \infty, \delta_n \in T^{(1)}\}} \\ &+ \max\{\overline{L}^{i,j}_{\delta_n}, \overline{Y}^{i,j}_{\delta_n}\} \mathbb{1}_{\{\delta_n < \infty, \delta_n \in T^{(2)}\}} | \mathcal{G}_{\delta_{n-1}} \bigg] \end{aligned} \tag{4.A.4}$$

for $n \geq 1$. On the one hand, applying Itô's formula to $\alpha_t \overline{Y}^{i,j}_t$, where $\alpha_t = e^{-(\lambda^{(1)}+\lambda^{(2)})t}$, yields that

$$\begin{aligned} \alpha_t \overline{Y}^{i,j}_t = & \int_t^\infty \alpha_s \left[\lambda^{(1)} \min\{\overline{U}_s, \overline{Y}^{i,j}_s\} + \lambda^{(2)} \max\{\overline{L}_s, \overline{Y}^{i,j}_s\} \right] ds \\ & - \int_t^\infty \alpha_s e^{-rs} Z^{i,j}_s \, dW_s, \end{aligned}$$

and thus, by choosing $t = \delta_{n-1}$ and taking the conditional expectation with respect to $\mathcal{G}_{\delta_{n-1}}$, we further have

$$\begin{aligned} \overline{Y}^{i,j}_{\delta_{n-1}} = \mathbb{E}\bigg[&\int_{\delta_{n-1}}^\infty e^{-(\lambda^{(1)}+\lambda^{(2)})(s-\delta_{n-1})} [\lambda^{(1)} \min\{\overline{U}_s, \overline{Y}^{i,j}_s\} \\ &+ \lambda^{(2)} \max\{\overline{L}_s, \overline{Y}^{i,j}_s\}] \, ds \bigg| \mathcal{G}_{\theta_{k-1}} \bigg] \end{aligned}$$

for any $n \geq 1$. On the other hand, by applying the probability density function of δ_n conditional on $\mathcal{G}_{\delta_{n-1}}$, we can also obtain the right-hand-side of (4.A.4) equals the right-hand-side of the above equation (see the proof of Lemma 3.3.3). This proves that $\overline{Y}^{i,j}_{\delta_{n-1}}$ satisfies the recursive equation (4.A.4) for $n \geq 1$.

Step 2. As a direct consequence, we deduce that $\hat{Y}^{i,j}_{\delta_{n-1}}$ defined by

$$\begin{aligned} \hat{Y}^{i,j}_{\delta_{n-1}} = &\min\{\overline{U}^{i,j}_{\delta_{n-1}}, \overline{Y}^{i,j}_{\delta_{n-1}}\} \mathbb{1}_{\{\delta_{n-1} < \infty, \delta_{n-1} \in T^{(1)}\}} \\ &+ \max\{\overline{L}^{i,j}_{\delta_{n-1}}, \overline{Y}^{i,j}_{\delta_{n-1}}\} \mathbb{1}_{\{\delta_{n-1} < \infty, \delta_{n-1} \in T^{(2)}\}} \end{aligned}$$

satisfies the following recursive equation:

$$
\begin{aligned}
\hat{Y}^{i,j}_{\delta_{n-1}} &= \min\{\overline{U}^{i,j}_{\delta_{n-1}}, \mathbb{E}[\hat{Y}^{i,j}_{\delta_n}|\mathcal{G}_{\delta_{n-1}}]\}\mathbb{1}_{\{\delta_{n-1}<\infty,\delta_{n-1}\in T^{(1)}\}} \\
&\quad + \max\{\overline{L}^{i,j}_{\delta_{n-1}}, \mathbb{E}[\hat{Y}^{i,j}_{\delta_n}|\mathcal{G}_{\delta_{n-1}}]\}\mathbb{1}_{\{\delta_{n-1}<\infty,\delta_{n-1}\in T^{(2)}\}}
\end{aligned} \tag{4.A.5}
$$

for any $n \geq 1$. In this step, we will show that $\hat{Y}^{i,j}_{\delta_{n-1}}$ is actually the value of an auxiliary constrained Dynkin game starting from δ_{n-1}, whose lower and upper values are defined by

$$
\underline{\hat{q}}^{i,j}_{\delta_{n-1}} = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}^{(2)}_{\delta_{n-1}}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}^{(1)}_{\delta_{n-1}}} \mathbb{E}\left[\tilde{R}^{i,j}(\sigma,\tau)\,|\mathcal{G}_{\delta_{n-1}}\right], \tag{4.A.6}
$$

$$
\overline{\hat{q}}^{i,j}_{\delta_{n-1}} = \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}^{(1)}_{\delta_{n-1}}} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}^{(2)}_{\delta_{n-1}}} \mathbb{E}\left[\tilde{R}^{i,j}(\sigma,\tau)\,|\mathcal{G}_{\delta_{n-1}}\right], \tag{4.A.7}
$$

with the optimal stopping strategy of the auxiliary game is given by

$$
\begin{cases}
\hat{\sigma}^{i,j}_{\delta_{n-1}} = \inf\{T^{(1)}_N \geq \delta_{n-1} : \hat{Y}^{i,j}_{T^{(1)}_N} = \overline{U}^{i,j}_{T^{(1)}_N}\}, \\
\hat{\tau}^{i,j}_{\delta_{n-1}} = \inf\{T^{(2)}_N \geq \delta_{n-1} : \hat{Y}^{i,j}_{T^{(2)}_N} = \overline{L}^{i,j}_{T^{(2)}_N}\}.
\end{cases} \tag{4.A.8}
$$

Using the similar arguments as in the proof of Lemma 3.3.4, it is easy to show the following martingale properties hold:

(i) $\left(\hat{Y}^{i,j}_{\delta_m\wedge\hat{\sigma}^{i,j}_{\delta_{n-1}}\wedge\hat{\tau}^{i,j}_{\delta_{n-1}}}\right)_{m\geq n-1}$ is a $\tilde{\mathbb{G}}$-martingale;

(ii) $\left(\hat{Y}^{i,j}_{\delta_m\wedge\hat{\sigma}^{i,j}_{\delta_{n-1}}\wedge\tau}\right)_{m\geq n-1}$ is a $\tilde{\mathbb{G}}$-supermartingale, for any $\tau \in \mathcal{R}^{(2)}_{\delta_{n-1}}$;

(iii) $\left(\hat{Y}^{i,j}_{\delta_m\wedge\sigma\wedge\hat{\tau}^{i,j}_{\delta_{n-1}}}\right)_{m\geq n-1}$ is a $\tilde{\mathbb{G}}$-submartingale, for any $\sigma \in \mathcal{R}^{(1)}_{\delta_{n-1}}$.

It follows from the martingale property (i) that

$$
\begin{aligned}
\hat{Y}^{i,j}_{\delta_{n-1}} &= \hat{Y}^{i,j}_{\delta_{n-1}\wedge\hat{\sigma}^{i,j}_{\delta_{n-1}}\wedge\hat{\tau}^{i,j}_{\delta_{n-1}}} \\
&= \mathbb{E}\left[\hat{Y}^{i,j}_{\hat{\sigma}^{i,j}_{\delta_{n-1}}\wedge\hat{\tau}^{i,j}_{\delta_{n-1}}}|\mathcal{G}_{\delta_{n-1}}\right] \\
&= \mathbb{E}\Bigg[\hat{Y}^{i,j}_{\hat{\tau}^{i,j}_{\delta_{n-1}}}\mathbb{1}_{\{\hat{\tau}^{i,j}_{\delta_{n-1}}<\infty,\hat{\tau}^{i,j}_{\delta_{n-1}}\leq\hat{\sigma}^{i,j}_{\delta_{n-1}}\}} \\
&\qquad + \hat{Y}^{i,j}_{\hat{\sigma}^{i,j}_{\delta_{n-1}}}\mathbb{1}_{\{\hat{\sigma}^{i,j}_{\delta_{n-1}}<\infty,\hat{\sigma}^{i,j}_{\delta_{n-1}}<\hat{\tau}^{i,j}_{\delta_{n-1}}\}}|\mathcal{G}_{\delta_{n-1}}\Bigg],
\end{aligned}
$$

and using (4.A.8), we can obtain

$$\begin{aligned}\hat{Y}^{i,j}_{\delta_{n-1}} &= \mathbb{E}\Big[\overline{L}^{i,j}_{\hat{\tau}^{i,j}_{\delta_{n-1}}}\mathbb{1}_{\{\hat{\tau}^{i,j}_{\delta_{n-1}}<\infty,\hat{\tau}^{i,j}_{\delta_{n-1}}\leq\hat{\sigma}^{i,j}_{\delta_{n-1}}\}}\\ &\quad+\overline{U}^{i,j}_{\hat{\sigma}^{i,j}_{\delta_{n-1}}}\mathbb{1}_{\{\hat{\sigma}^{i,j}_{\delta_{n-1}}<\infty,\hat{\sigma}^{i,j}_{\delta_{n-1}}<\hat{\tau}^{i,j}_{\delta_{n-1}}\}}|\mathcal{G}_{\delta_{n-1}}\Big]\\ &= \mathbb{E}\left[\tilde{R}^{i,j}\left(\hat{\sigma}^{i,j}_{\delta_{n-1}},\hat{\tau}^{i,j}_{\delta_{n-1}}\right)|\mathcal{G}_{\delta_{n-1}}\right].\end{aligned}$$

Likewise, it follows from the supermartingale property (ii) and the submartingale property (iii) that

$$\mathbb{E}\left[\tilde{R}^{i,j}\left(\hat{\sigma}^{i,j}_{\delta_{n-1}},\tau\right)|\mathcal{G}_{\delta_{n-1}}\right]\leq\hat{Y}^{i,j}_{\delta_{n-1}}\leq\mathbb{E}\left[\tilde{R}^{i,j}\left(\sigma,\hat{\tau}^{i,j}_{\delta_{n-1}}\right)|\mathcal{G}_{\delta_{n-1}}\right]$$

for any $(\sigma,\tau)\in\mathcal{R}^{(1)}_{\delta_{n-1}}\times\mathcal{R}^{(2)}_{\delta_{n-1}}$. It then follows that the value of the auxiliary game (4.A.6)-(4.A.7) exists, i.e.

$$\hat{Y}^{i,j}_{\delta_{n-1}}=\hat{q}^{i,j}_{\delta_{n-1}}=\overline{\hat{q}}^{i,j}_{\delta_{n-1}}=\underline{\hat{q}}^{i,j}_{\delta_{n-1}},$$

and $(\hat{\sigma}^{i,j}_{\delta_{n-1}},\hat{\tau}^{i,j}_{\delta_{n-1}})$ in (4.A.8) is indeed an optimal stopping strategy of the auxiliary game (4.A.6)-(4.A.7). The uniqueness of the solution to the recursive equation (4.A.5) is an immediate consequence.

Step 3. Let $\overline{Y}^{i,j}_{\delta_{n-1}}$ be a solution to the recursive equation (4.A.4), and in turn,

$$\begin{aligned}\overline{Y}^{i,j}_{\delta_{n-1}} &= \mathbb{E}\left[\hat{Y}^{i,j}_{\delta_n}|\mathcal{G}_{\delta_{n-1}}\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\tilde{R}^{i,j}\left(\hat{\sigma}^{i,j}_{\delta_n},\hat{\tau}^{i,j}_{\delta_n}\right)|\mathcal{G}_{\delta_n}\right]|\mathcal{G}_{\delta_{n-1}}\right]\\ &= \mathbb{E}\left[\tilde{R}^{i,j}\left(\hat{\sigma}^{i,j}_{\delta_n},\hat{\tau}^{i,j}_{\delta_n}\right)|\mathcal{G}_{\delta_{n-1}}\right],\end{aligned}$$

and similarly,

$$\mathbb{E}\left[\tilde{R}^{i,j}\left(\hat{\sigma}^{i,j}_{\delta_n},\tau\right)|\mathcal{G}_{\delta_{n-1}}\right]\leq\overline{Y}^{i,j}_{\delta_{n-1}}\leq\mathbb{E}\left[\tilde{R}^{i,j}\left(\sigma,\hat{\tau}^{i,j}_{\delta_n}\right)|\mathcal{G}_{\delta_{n-1}}\right]$$

for any $(\sigma,\tau)\in\mathcal{R}^{(1)}_{\delta_n}\times\mathcal{R}^{(2)}_{\delta_n}$. It then follows that the value of the game (4.A.1)-(4.A.2) exists, i.e.

$$\overline{Y}^{i,j}_{\delta_{n-1}}=q^{i,j}_{\delta_{n-1}}=\overline{q}^{i,j}_{\delta_{n-1}}=\underline{q}^{i,j}_{\delta_{n-1}},$$

and $(\hat{\sigma}^{i,j}_{\delta_n},\hat{\tau}^{i,j}_{\delta_n})$ is indeed an optimal stopping strategy of the game (4.A.1)-(4.A.2). It is immediate to verify $(\hat{\sigma}^{i,j}_{\delta_n},\hat{\tau}^{i,j}_{\delta_n})$ are indeed $(\sigma^{i,j}_{\delta_{n-1}},\tau^{i,j}_{\delta_{n-1}})$ in (4.A.3).

Chapter 5

Pricing Rollover Debt in an Incomplete Market

5.1 Introduction

Short-term debt has often been blamed for triggering the financial crisis of 2007-2008. However, several reasons support the use of short-term borrowing in addition to its cost advantage. Diamond and Rajan [20] argue that the direction of causality is the opposite to the one traditionally suggested, and that a ban on short-term financing may cause a more serious crisis. Moreover, He and Xiong [34] argue that, as a disciplinary device for firms, short-term financing can be used to alleviate adverse selection problems and to reduce the cost of auditing firms. In reality, those firms, which finance their long-term fundamental assets by short-term debt contracts, typically spread out their debt expirations over time to reduce liquidity risk. This realistic staggered debt structure motivates the study of rollover debt.

Rollover debt was first introduced by Leland [53] and Leland and Toft [54], with Hilberink and Rogers [38] providing further technical details. The idea is to assume a random duration of debt to reflect the maturity mismatch between the assets and the liabilities sides. Recently, a similar idea has appeared in the modeling of debt run problems to capture this kind of liquidity constraints (see, for example, [34] and [58]), where debt maturities are modelled as arrival times of a Poisson process. Once a creditor lends money to the firm, the debt contract lasts until the arrival of an independent Poisson shock. The creditor can decide, at each rollover date, whether to roll over or to withdraw her funding.

In this chapter, we study this kind of rollover debt in incomplete market environments. Incompleteness comes from the fact that the fundamental assets might not be freely traded or their payoffs might not be perfectly spanned by other assets. The creditor can at best trade in risky assets correlated with the fundamental assets. This provides a hedging opportunity for the creditor, but she still faces the idiosyncratic risk, which is unhedgeable. We use a utility maximization framework in which the creditor chooses her withdrawal time and a hedge position to maximize her expected utility of wealth, which comes from her bond payoff and hedge portfolio. We investigate the creditor's rollover decisions and price rollover debt in an incomplete market.

Turning to the literature, the impact of incomplete markets on investment timing was first considered by Miao and Wang [64] in a model with consumption and portfolio. Henderson [35] examines the impact of market incompleteness and values the option to invest using a model closer to the canonical complete real option model of McDonald and Siegel [60] and Dixit and Pindyck [21]. Although different kinds of real options are considered, our set-up still relates to Henderson's.

The key contribution of this chapter is to provide a rigorous formulation for a class of rollover-decision and rollover-debt-pricing models in an incomplete market, by introducing a new class of *mixed stochastic control and constrained optimal stopping problems* (see (5.2.3)). Standard mixed stochastic control and optimal stopping problems were first studied by Karatzas and Kou [46] to price American contingent claims under constraints. More examples include Karatzas and Sudderth [47], Karatzas and Wang [48], and Henderson and Hobson [37].

Compared to standard mixed stochastic control and optimal stopping problems, the new class of problems has a stopping constraint in the sense that the player is only allowed to stop at a sequence of exogenous Poisson arrival times (in the current set-up, the creditor is only allowed to withdraw her funding at a sequence of rollover dates generated by an exogenous Poisson process). The optimal stopping problem with stopping constraints was introduced by Dupuis and Wang [25] to model perpetual American options exercised at exogenous Poisson arrival times. See also Lempa [55], Liang [57], Liang and Wei [59], and Menaldi and Robin [61]-[62] for further extensions of this type of optimal stopping problems. Recently, Liang and Sun (Chapter 2) introduce the Dynkin games with constaints on both players' stopping times and apply it to model perpetual convertible bonds with liquidation constraints.

Our first main result is Theorem 5.3.1, the verification theorem, characterizing, under some assumptions, the value function of the problem and its associated optimal control / stopping strategy in terms of the solution of a penalized PDE, where the latter is widely used to approximate the solution of a variational inequality (VI).

The second main result is Theorem 5.4.4, which gives the complete characterization in the case of exponential utility. Theorem 5.4.4 also characterizes exponential indifference bond price and its associated optimal control / stopping strategy, both of which are proved to be wealth independent. Thanks to the above results, we further investigate the impact of parameters on the bond price. Furthermore, following from the convergence of penalized PDEs to VIs, we make a connection with mixed stochastic control and optimal stopping problems without stopping constraints (see (5.5.1)). That is, the value function of the problem with stopping constraints will converge to that without stopping constraints when the Poisson intensity goes to infinity. Inspired by the theoretical analysis of exponential indifference bond price in a complete market, we conduct some numerical experiments to examine the shapes of the stopping and continuation regions of the problem in an incomplete market, and observe that both regions are swapped over for different parameter values.

The chapter is organized as follows. Section 5.2 contains the problem formulation. Section 5.3 presents the verification theorem. In Section 5.4, in the case of exponential utility, we verify the assumptions imposed to the verification theorem for the complete characterization and obtain some properties of exponential indifference bond price. In Section 5.5, we establish a connection with standard mixed stochastic control and optimal stopping problems and solve in closed form both the price and optimal stopping strategy of a rollover debt without stopping constraints in a complete market. Section 5.6 concludes the chapter by some numerical results for optimal stopping strategies.

5.2 The model

Let $\mathcal{W} = (W^1, W^2)$ be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, where $\mathcal{F}_t$ is the augmented σ-algebra generated by $\mathcal{W}$. Let $(T_i)_{i\geq 0}$ be the arrival times of an independent Poisson process with intensity λ and minimal augmented filtration $\{\mathcal{H}_t\}_{t\geq 0}$. Denote the smallest filtration generated by $\mathbb{F}$ and $\mathbb{H}$ as $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Without loss of generality, we assume that $T_0 = 0$ and $T_\infty = \infty$. For a finite horizon $T < \infty$ representing the terminal time of the problem, there exists an interger-valued random variable $M < \infty$ such that $T_{M-1} \leq T < T_M$, i.e. $M(\omega) = \sum_{n\geq 1} n \mathbb{1}_{\{T_n(\omega)\leq T < T_n(\omega)\}}$.

Consider a finitely-lived, risk averse creditor, who purchased a rollover bond issued by a firm with fundamental asset value V, at initiation time 0 until maturity T_1. The face value of the debt is K^1. At T_1, the bond matures, and the creditor can decide to whether withdraw her funding receiving Ke^{rT_1}, the sum of the face value and accrued interest payments, or successively roll over until the next rollover date, where the constant r is the spread rate (either positive or negative), representing the yield difference between this bond and

[1] We express all amounts in discounted units or equivalently take the risk-free bond as numeraire.

risk-free bond, and the maturity dates (or rollover dates) are modelled by the aforementioned independent Poisson arrival times $(T_i)_{i\geq 1}$.

For any $i \geq 0$, let us define the creditor's withdrawal control set $\mathcal{R}^{\lambda}_{T_i}$ as

$$\mathcal{R}^{\lambda}_{T_i} = \{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega) = T_N(\omega) \text{ where } i \leq N \leq M\},$$

and the payoff of this bond would be delivered at either the withdrawal time $\tau \in \mathcal{R}^{\lambda}_{T_1}$ if τ happens before the terminal time T, or T otherwise.

Moreover, the cash flow at the time of delivery t is assumed to be in the following form

$$\phi(t, V_t) := \min\{V_t, K_t\}$$

where $K_t = Ke^{rt}$. This means, at the time of delivery t, the creditor will collect K_t if the firm is solvent. However, if the firm fails to repay the creditor, i.e. $V_t < K_t$, it has to prematurely liquidate the fundamental asset at a fire-sale price, which, for simplicity, we assume equals its fundamental value.

The market consists of risk-free bonds and a traded risky asset S, where the latter is correlated with V. We assume the risky asset value S and the fundamental asset value V follow from geometric Brownian motion processes:

$$\frac{dS_t}{S_t} = \sigma_S(\xi\, dt + dW^1_t),$$

and

$$\frac{dV_t}{V_t} = \sigma_V(\eta\, dt + \rho\, dW^1_t + \sqrt{1-\rho^2}\, dW^2_t), \tag{5.2.1}$$

where Sharpe ratios ξ, η and volatilities σ_S, σ_V are constants.

This bond is exposure to the traded or market risk, represented by Brownian motion W^1, and the non-traded idiosyncratic risk, represented by W^2. When correlation is one, the model is complete. However, one faces idiosyncratic risk and incomplete markets if $|\rho| < 1$. The creditor invests in the risky asset S so that she can, to some extent, hedge the market risk, represented by W^1.

Let π denote the holdings in the risky asset S, resulting in X^π, the total wealth invested in the risky asset and risk-free bonds. Then X^π has the following dynamics:

$$dX^\pi_t = \pi_t \frac{dS_t}{S_t} = \pi_t \sigma_S \left(\xi\, dt + dW^1_t\right). \tag{5.2.2}$$

A $\mathbb{R}$-valued $\mathbb{F}$-predictable process $\pi = (\pi_t)_{0\leq t\leq T}$ is called a *self-financing* trading strategy if π satisfies the integrability condition $\mathbb{E}[\int_0^T |\pi_t|^2\, dt] < \infty$. This integrability condition ensures the existence and uniqueness of a strong solution to stochastic differential equation (SDE) (5.2.2).

The creditor chooses her withdrawal time τ and *admissible* self-financing trading strategy π, in order to maximize her expected performance from both investing in the market and receiving the bond payoff. This leads to *the following mixed stochastic control and constrained optimal stopping problem*:

$$Y^{\lambda}(x,v) = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}^{\lambda}_{T_1},\pi\in\mathcal{U}_{0,\tau\wedge T}} \mathbb{E}\left[U\left(\tau\wedge T, X^{\pi}_{\tau\wedge T} + \phi(\tau\wedge T, V_{\tau\wedge T})\right)\middle| X^{\pi}_0 = x, V_0 = v\right], \tag{5.2.3}$$

where $\mathcal{U}$ is the admissible control set which will be defined in Definition 5.2.1, and $U(t,x)$ is a time-dependent deterministic utility function satisfying

- for each $t \geq 0$, the mapping $x \mapsto U(t,x)$ is increasing and strictly concave in $x \in \mathbb{R}$ and
- for $0 \leq t \leq s < \infty$, we have

$$U(t, X^{\pi}_t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{U}_{t,s}} \mathbb{E}\left[U\left(X^{\pi}_s\right)\middle| \mathcal{F}_t\right].$$

Note that, in related literature, this time-dependent utility function U can be regarded as a deterministic case of forward performance processes in [66] and the second condition is also referred to as the horizon-unbiased condition in [36].

A pair $(\tau^*, \pi^*) \in \mathcal{R}^{\lambda}_{T_1} \times \mathcal{U}_{0,\tau^*\wedge T}$ is called an optimal control / stopping strategy for the problem (5.2.3) if

$$Y^{\lambda}(x,v) = \mathbb{E}\left[U\left(\tau^*\wedge T, X^{\pi^*}_{\tau^*\wedge T} + \phi(\tau^*\wedge T, V_{\tau^*\wedge T})\right)\middle| X^{\pi^*}_0 = x, V_0 = v\right].$$

In order to solve the main problem (5.2.3), we choose an admissible set from which we can select the optimal trading strategy. Note that the integrability condition in the following definition guarantees that there is no arbitrage, while the class (D) condition is technical.

Definition 5.2.1 *The set of admissible trading strategies $\mathcal{U}$ consists of all $\mathbb{R}$-valued $\mathbb{F}$-predictable processes π satisfying the integrability condition: $\mathbb{E}[\int_0^T |\pi_t|^2\,dt] < \infty$, and the class (D) condition:*

$$\{U(\nu, X^{\pi}_{\nu}) : \nu \text{ is a stopping time taking values in } [0,T]\}$$

is a uniformly integrable family. We denote by $\mathcal{U}_{a,b}$, the set of admissible strategies over the period $[a,b]$, for $0 \leq a \leq b \leq T$.

We are now ready to provide the definition of the creditor's *indifference bond price with stopping constraints.* The creditor's indifference bond price is defined as the cash amount such that the creditor is indifferent between two positions: (i) optimal investment with the bond, (ii) optimal investment without the bond but instead with extra initial wealth.

Definition 5.2.2 *The creditor's indifference bond price with stopping constraints $P^\lambda(x,v)$ is defined by the equation*

$$Y^\lambda(x,v) = U(0, x + P^\lambda(x,v)), \tag{5.2.4}$$

where Y^λ is given by (5.2.3), provided it exists.

5.3 Verification theorem for Y^λ

Theorem 5.3.1 *Let y be a function in $C^{1,2,2}([0,T) \times \mathbb{R} \times \mathbb{R}^+) \cap C([0,T] \times \mathbb{R} \times \mathbb{R}^+)$, satisfying*

$$\begin{aligned} -\partial_t y(t,x,v) - \sup_{\pi \in \mathbb{R}} \mathcal{L}_1^\pi y(t,x,v) - \lambda \left(U(t, x + \phi(t,v)) - y(t,x,v)\right)^+ = 0, \\ y(T,x,v) = U\left(T, x + \phi(T,v)\right), \end{aligned} \tag{5.3.1}$$

where

$$\mathcal{L}_1^\pi := \frac{1}{2}\sigma_V^2 v^2 \partial_{vv}^2 + \sigma_V \eta v \partial_v + \frac{1}{2}\sigma_S^2 \pi^2 \partial_{xx}^2 + \pi \sigma_S (\xi \partial_x + \rho \sigma_V v \partial_{xv}^2).$$

Suppose that

- *$\{y(\nu, X_\nu^\pi, V_\nu) : \nu$ is a stopping time taking values in $[0,T]\}$ is uniformly integrable for any $\pi \in \mathcal{U}_{0,T}$,*
- *for all $(t,x,v) \in [0,T) \times \mathbb{R} \times \mathbb{R}^+$, there exists a measurable function $\pi^* : [0,T) \times \mathbb{R} \times \mathbb{R}^+ \to \mathcal{U}$ such that*

$$\pi^*(t,x,v) \in \arg\max_{\pi \in \mathbb{R}} \mathcal{L}_1^\pi y(t,x,v), \tag{5.3.2}$$

- *the SDE*

$$dX_t^* = \pi^*(t, X_t^*, V_t)\sigma_S \left(\xi\, dt + dW_t^1\right), \quad X_0^* = x \tag{5.3.3}$$

  admits a unique solution,

- *and the process $(\pi^*(t, X_t^*, V_t))_{0 \le t \le T} \in \mathcal{U}_{0,T}$,* *then* $Y^\lambda(x,v) = y(0,x,v)$ *for* $(x,v) \in \mathbb{R}\times\mathbb{R}^+$, *and* (τ^*,π^*) *is an optimal control / stopping strategy, where*

$$\tau^* = \inf\Big\{T_i \geq T_1 : y(T_i\wedge T, X^*_{T_i\wedge T}, V_{T_i\wedge T}) \\ \leq U(T_i\wedge T, X^*_{T_i\wedge T} + \phi(T_i\wedge T, V_{T_i\wedge T}))\Big\}. \quad (5.3.4)$$

Proof. In this proof, we denote by $\{X_s^{t,x,\pi}, t\leq s\leq T\}$ the solution of (5.2.2) with control process π and initial condition $X_t^{t,x,\pi} = x$, and denote by $\{V_s^{t,v}, t\leq s\leq T\}$ the solution of (5.2.1) with initial condition $V_t^{t,v} = v$.
Step 1. Let $\pi \in \mathcal{U}_{0,T}$ be an arbitrary control process, and define the stopping time

$$\theta_n = (T - n^{-1}) \wedge \inf\left\{s > 0 : |X_s^{0,x,\pi} - x| + |V_s^{0,v} - v| \geq n\right\}.$$

By Itô's formula, we have

$$\begin{aligned} &y(0,x,v) \\ &= e^{-\lambda\theta_n} y(\theta_n, X_{\theta_n}^{0,x,\pi}, V_{\theta_n}^{0,v}) - \int_0^{\theta_n} e^{-\lambda s}(\partial_t + \mathcal{L}_1^{\pi_s} - \lambda) y(s, X_s^{0,x,\pi}, V_s^{0,v})\, ds \\ &\quad - \int_0^{\theta_n} e^{-\lambda s}\pi_s\sigma_S\partial_x y(s, X_s^{0,x,\pi}, V_s^{0,v})\, dW_s^1 \\ &\quad - \int_0^{\theta_n} e^{-\lambda s}\sigma_V V_s^{0,v}\partial_v y(s, X_s^{0,x,\pi}, V_s^{0,v})\, d\tilde{W}_s^2 \end{aligned}$$

where $\tilde{W}^2 := \rho W^1 + \sqrt{1-\rho^2}W^2$ is a Brownian motion. Since both stochastic integrals are martingales on $[0,\theta_n]$, a consequence of the continuity of $\partial_x y$ and $\partial_v y$, we then take expectation on both sides

$$\begin{aligned} &y(0,x,v) \\ &= \mathbb{E}\left[e^{-\lambda\theta_n} y(\theta_n, X_{\theta_n}^{0,x,\pi}, V_{\theta_n}^{0,v}) - \int_0^{\theta_n} e^{-\lambda s}(\partial_t + \mathcal{L}_1^{\pi_s} - \lambda) y(s, X_s^{0,x,\pi}, V_s^{0,v})\, ds\right] \\ &\geq \mathbb{E}\left[e^{-\lambda\theta_n} y(\theta_n, X_{\theta_n}^{0,x,\pi}, V_{\theta_n}^{0,v}) + \int_0^{\theta_n} e^{-\lambda s}\lambda\hat{y}(s, X_s^{0,x,\pi}, V_s^{0,v})\, ds\right] \qquad (5.3.5) \end{aligned}$$

where we define $\hat{y}(t,x,v) := \max\{y(t,x,v), U(t, x+\phi(t,v))\}$. By the uniform integrabilities of $y(\cdot, X_\cdot^{0,x,\pi}, V_\cdot^{0,v})$ and $U(\cdot, X_\cdot^{0,x,\pi} + \phi(\cdot, V_\cdot^{0,v}))$, and then by taking the limit as $n\to\infty$, it follows from the dominated convergence theorem that

$$
\begin{aligned}
& y(0,x,v) \\
& \geq \mathbb{E}\left[e^{-\lambda T} y(T, X_T^{0,x,\pi}, V_T^{0,v}) + \int_0^T e^{-\lambda s} \lambda \hat{y}(s, X_s^{0,x,\pi}, V_s^{0,v})\, ds\right] \\
& = \mathbb{E}\left[U(T, X_T^{0,x,\pi} + \phi(T, V_T^{0,v}))\mathbb{1}_{\{T_1 \geq T\}} + \hat{y}(T_1, X_{T_1}^{0,x,\pi}, V_{T_1}^{0,v})\mathbb{1}_{\{T_1 < T\}}\right] \\
& = \mathbb{E}\left[\hat{y}(T_1 \wedge T, X_{T_1\wedge T}^{0,x,\pi}, V_{T_1\wedge T}^{0,v})\right] \qquad (5.3.6)
\end{aligned}
$$

where the second equality holds by applying the probability density function of T_1. If we can claim the following inequality

$$
\hat{y}(T_1 \wedge T, \overline{x}, \overline{v}) \geq \mathbb{E}\left[U\left(\tau\wedge T, X_{\tau\wedge T}^{T_1\wedge T, \overline{x}, \pi} + \phi(\tau \wedge T, V_{\tau\wedge T}^{T_1\wedge T, \overline{v}})\right)\middle| \mathcal{G}_{T_1\wedge T}\right] \quad (5.3.7)
$$

holds for any $(\tau,\pi) \in \mathcal{R}_{T_1}^{\lambda} \times \mathcal{U}_{T_1\wedge T, \tau\wedge T}$. By plugging (5.3.7) into (5.3.6), we can obtain

$$
y(0,x,v) \geq \mathbb{E}\left[U\left(\tau \wedge T, X_{\tau\wedge T}^{0,x,\pi} + \phi(\tau\wedge T, V_{\tau\wedge T}^{0,v})\right)\right]
$$

for $\tau \in \mathcal{R}_{T_1}^{\lambda}$. Since the above inequality holds for any $(\tau,\pi) \in \mathcal{R}_{T_1}^{\lambda} \times \mathcal{U}_{0,\tau\wedge T}$, this gives

$$
y(0,x,v) \geq Y^{\lambda}(x,v).
$$

It remains to prove (5.3.7) holds. Indeed, using the similar arguments to prove (5.3.6), we have

$$
y(T_1 \wedge T, \overline{x}, \overline{v}) \geq \mathbb{E}\left[\hat{y}(T_2\wedge T, X_{T_2\wedge T}^{T_1\wedge T, \overline{x}, \pi}, V_{T_2\wedge T}^{T_1\wedge T, \overline{v}})\middle| \mathcal{G}_{T_1\wedge T}\right] \quad (5.3.8)
$$

for any $\pi \in \mathcal{U}_{T_1\wedge T, T_2\wedge T}$. The definition of $\hat{y}$ and (5.3.8) give that

$$
\hat{y}(T_1 \wedge T, \overline{x}, \overline{v}) \geq \mathbb{E}\left[\hat{y}(T_2\wedge T, X_{T_2\wedge T}^{T_1\wedge T, \overline{x}, \pi}, V_{T_2\wedge T}^{T_1\wedge T, \overline{v}})\middle| \mathcal{G}_{T_1\wedge T}\right]. \quad (5.3.9)
$$

Likewise, we can have

$$
\hat{y}(T_2 \wedge T, \overline{\overline{x}}, \overline{\overline{v}}) \geq \mathbb{E}\left[\hat{y}(T_3\wedge T, X_{T_3\wedge T}^{T_2\wedge T, \overline{\overline{x}}, \pi}, V_{T_3\wedge T}^{T_2\wedge T, \overline{\overline{v}}})\middle| \mathcal{G}_{T_2\wedge T}\right] \quad (5.3.10)
$$

for any $\pi \in \mathcal{U}_{T_2\wedge T, T_3\wedge T}$. By plugging (5.3.10) into (5.3.9), we have

$$
\hat{y}(T_1 \wedge T, \overline{x}, \overline{v}) \geq \mathbb{E}\left[\hat{y}(T_3\wedge T, X_{T_3\wedge T}^{T_1\wedge T, \overline{x}, \pi}, V_{T_3\wedge T}^{T_1\wedge T, \overline{v}})\middle| \mathcal{G}_{T_1\wedge T}\right] \quad (5.3.11)
$$

for any $\pi \in \mathcal{U}_{T_1\wedge T, T_3\wedge T}$. From (5.3.9) and (5.3.11), it is easy to see the following inequality holds:

$$\begin{aligned}\hat{y}(T_1\wedge T,\overline{x},\overline{v}) &\geq \mathbb{E}\left[\hat{y}(\tau\wedge T, X_{\tau\wedge T}^{T_1\wedge T,\overline{x},\pi}, V_{\tau\wedge T}^{T_1\wedge T,\overline{v}})\Big|\,\mathcal{G}_{T_1\wedge T}\right]\\ &\geq \mathbb{E}\left[U(\tau\wedge T, X_{\tau\wedge T}^{T_1\wedge T,\overline{x},\pi}+\phi(\tau\wedge T, V_{\tau\wedge T}^{T_1\wedge T,\overline{v}}))\Big|\,\mathcal{G}_{T_1\wedge T}\right]\end{aligned}$$

for any $(\tau,\pi)\in\mathcal{R}_{T_2}^{\lambda}\times\mathcal{U}_{T_1\wedge T,\tau\wedge T}$, and hence, together with the definition of $\hat{y}$, (5.3.7) follows.

Step 2. In order to prove the reverse inequality, let us introduce the following auxiliary problem associated with (5.2.3):

$$\begin{aligned}&\hat{Y}^{\lambda}(T_1\wedge T,\overline{x},\overline{v})\\ &=\operatorname*{ess\,sup}_{\tau\in\mathcal{R}_{T_1}^{\lambda},\pi\in\mathcal{U}_{T_1\wedge T,\tau\wedge T}}\mathbb{E}\left[U\left(\tau\wedge T, X_{\tau\wedge T}^{T_1\wedge T,\overline{x},\pi}+\phi(\tau\wedge T, V_{\tau\wedge T}^{T_1\wedge T,\overline{v}})\right)\Big|\,\mathcal{G}_{T_1\wedge T}\right],\end{aligned}\tag{5.3.12}$$

and $(\hat{\tau}^*,\hat{\pi}^*)\in\mathcal{R}_{T_1}^{\lambda}\times\mathcal{U}_{T_1\wedge T,\hat{\tau}^*\wedge T}$ is called an optimal control / stopping strategy if

$$\hat{Y}^{\lambda}(T_1\wedge T,\overline{x},\overline{v})=\mathbb{E}\left[U\left(\hat{\tau}^*\wedge T, X_{\hat{\tau}^*\wedge T}^{T_1\wedge T,\overline{x},\hat{\pi}^*}+\phi(\hat{\tau}^*\wedge T, V_{\hat{\tau}^*\wedge T}^{T_1\wedge T,\overline{v}})\right)\Big|\,\mathcal{G}_{T_1\wedge T}\right].$$

Note that the difference between (5.3.12) and (5.2.3) is that the former is allowed to stop its corresponding initial starting time, while the latter not.

In this step, we aim to show that

$$\hat{y}(T_1\wedge T,\overline{x},\overline{v})=\hat{Y}^{\lambda}(T_1\wedge T,\overline{x},\overline{v}),\tag{5.3.13}$$

and the optimal control / stopping strategy $(\hat{\tau}^*,\hat{\pi}^*)$ is

$$\begin{cases}\hat{\tau}^* = \inf\Big\{T_i\geq T_1:\hat{y}(T_i\wedge T, X_{T_i\wedge T}^{T_1\wedge T,\overline{x},*}, V_{T_i\wedge T}^{T_1\wedge T,\overline{v}})\\ \qquad\qquad = U(T_i\wedge T, X_{T_i\wedge T}^{T_1\wedge T,\overline{x},*}+\phi(T_i\wedge T, V_{T_i\wedge T}^{T_1\wedge T,\overline{v}}))\Big\}\\ \hat{\pi}_t^*=\pi_t^*,\quad T_1\wedge T\leq t<T\end{cases}\tag{5.3.14}$$

where $X^{T_1\wedge T,\overline{x},*}$ is defined in (5.3.3) with initial condition $X_{T_1\wedge T}^{T_1\wedge T,\overline{x},*}=\overline{x}$.

To this end, it is sufficient to prove that

$$\left(\hat{y}(\hat{\tau}^*\wedge T_i\wedge T, X_{\hat{\tau}^*\wedge T_i\wedge T}^{T_1\wedge T,\overline{x},*}, V_{\hat{\tau}^*\wedge T_i\wedge T}^{T_1\wedge T,\overline{v}})\right)_{i\geq 1}\tag{5.3.15}$$

is a uniformly integrable martingale. Indeed, if this is true, then it follows from the optional sampling theorem that

$$\begin{aligned}
\hat{y}(T_1 \wedge T, \overline{x}, \overline{v}) &= \mathbb{E}\left[\hat{y}(\hat{\tau}^* \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{\hat{\tau}^* \wedge T}, V^{T_1 \wedge T, \overline{v}}_{\hat{\tau}^* \wedge T}) \middle| \mathcal{G}_{T_1 \wedge T} \right] \\
&= \mathbb{E}\left[U\left(\hat{\tau}^* \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{\hat{\tau}^* \wedge T} + \phi(\hat{\tau}^* \wedge T, V^{T_1 \wedge T, \overline{v}}_{\hat{\tau}^* \wedge T})\right) \middle| \mathcal{G}_{T_1 \wedge T} \right] \\
&\leq \hat{Y}^{\lambda}(T_1 \wedge T, \overline{x}, \overline{v}),
\end{aligned}$$

which, together with (5.3.7), implies (5.3.13) holds and $(\hat{\tau}^*, \hat{\pi}^*)$ defined in (5.3.14) is the optimal control / stopping strategy to the auxiliary problem (5.3.12).

Now, it remains to prove that (5.3.15) is a uniformly integrable martingale. The uniform integrability property is obvious, and the martingale property can be proved as follows. Using the similar arguments as in step 1, we can observe the control π^* achieves equality at the crucial step (5.3.5), and therefore,

$$y(T_{i-1} \wedge T, \overline{x}, \overline{v}) = \mathbb{E}\left[\hat{y}(T_i \wedge T, X^{T_{i-1} \wedge T, \overline{x}, *}_{T_i \wedge T}, V^{T_{i-1} \wedge T, \overline{v}}_{T_i \wedge T}) \middle| \mathcal{G}_{T_{i-1} \wedge T} \right] \quad (5.3.16)$$

for any $1 \leq i \leq M$. Then we have, for $2 \leq i \leq M$,

$$\begin{aligned}
&\mathbb{E}\left[\hat{y}(\hat{\tau}^* \wedge T_i \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{\hat{\tau}^* \wedge T_i \wedge T}, V^{T_1 \wedge T, \overline{v}}_{\hat{\tau}^* \wedge T_i \wedge T}) \middle| \mathcal{G}_{T_{i-1} \wedge T} \right] \\
&= \hat{y}(\hat{\tau}^*, X^{T_1 \wedge T, \overline{x}, *}_{\hat{\tau}^*}, V^{T_1 \wedge T, \overline{v}}_{\hat{\tau}^*}) \mathbb{1}_{\{\hat{\tau}^* < T_i \wedge T\}} \\
&\quad + \mathbb{E}\left[\hat{y}(T_i \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{T_i \wedge T}, V^{T_1 \wedge T, \overline{v}}_{T_i \wedge T}) \middle| \mathcal{G}_{T_{i-1} \wedge T} \right] \mathbb{1}_{\{\hat{\tau}^* \geq T_i \wedge T\}}. \quad (5.3.17)
\end{aligned}$$

Conditional on the set $\{\hat{\tau}^* \geq T_i \wedge T\}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\hat{y}(T_i \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{T_i \wedge T}, V^{T_1 \wedge T, \overline{v}}_{T_i \wedge T}) \middle| \mathcal{G}_{T_{i-1} \wedge T} \right] \\
&= \mathbb{E}\left[\hat{y}\left(T_i \wedge T, X^{T_{i-1} \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{T_{i-1} \wedge T}, *}_{T_i \wedge T}, V^{T_{i-1} \wedge T, V^{T_1 \wedge T, \overline{v}}_{T_{i-1} \wedge T}}_{T_i \wedge T} \right) \middle| \mathcal{G}_{T_{i-1} \wedge T} \right] \\
&= y(T_{i-1} \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{T_{i-1} \wedge T}, V^{T_1 \wedge T, \overline{v}}_{T_{i-1} \wedge T}) \\
&= \hat{y}(T_{i-1} \wedge T, X^{T_1 \wedge T, \overline{x}, *}_{T_{i-1} \wedge T}, V^{T_1 \wedge T, \overline{v}}_{T_{i-1} \wedge T})
\end{aligned}$$

where the second equality follows from (5.3.16), and the last equality follows from (5.3.14). By plugging the above equation into (5.3.17), we obtain the required martingale property of (5.3.15).

Step 3. Now, we are in a position to prove the main result. Since (5.3.16) and

(5.3.13), we have

$$
\begin{aligned}
& y(0,x,v) \\
&= \mathbb{E}\left[\hat{y}(T_1\wedge T, X^{0,x,*}_{T_1\wedge T}, V^{0,v}_{T_1\wedge T})\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[U\left(\hat{\tau}^*\wedge T, X^{T_1\wedge T, X^{0,x,*}_{T_1\wedge T},*}_{\hat{\tau}^*\wedge T} + \phi(\hat{\tau}^*\wedge T, V^{T_1\wedge T, V^{0,v}_{T_1\wedge T}}_{\hat{\tau}^*\wedge T})\right)\middle|\mathcal{G}_{T_1\wedge T}\right]\right] \\
&= \mathbb{E}\left[U\left(\hat{\tau}^*\wedge T, X^{0,x,*}_{\hat{\tau}^*\wedge T} + \phi(\hat{\tau}^*\wedge T, V^{0,v}_{\hat{\tau}^*\wedge T})\right)\right].
\end{aligned}
$$

This gives $y(0,x,v) = Y^\lambda(x,v)$ and (τ^*, π^*) is the optimal control / stopping strategy. Finally, we conclude the proof by proving $\hat{\tau}^*$ is actually τ^* in (5.3.4). Indeed,

$$
\begin{aligned}
\hat{\tau}^* = & \inf\Big\{T_i \geq T_1 : \hat{y}(T_i\wedge T, X^{T_1\wedge T, X^{0,x,*}_{T_1\wedge T},*}_{T_i\wedge T}, V^{T_1\wedge T, V^{0,v}_{T_1\wedge T}}_{T_i\wedge T}) \\
& \qquad = U(T_i\wedge T, X^{T_1\wedge T, X^{0,x,*}_{T_1\wedge T},*}_{T_i\wedge T} + \phi(T_i\wedge T, V^{T_1\wedge T, V^{0,v}_{T_1\wedge T}}_{T_i\wedge T}))\Big\} \\
= & \inf\Big\{T_i \geq T_1 : \hat{y}(T_i\wedge T, X^{0,x,*}_{T_i\wedge T}, V^{0,v}_{T_i\wedge T}) \\
& \qquad = U(T_i\wedge T, X^{0,x,*}_{T_i\wedge T} + \phi(T_i\wedge T, V^{0,v}_{T_i\wedge T}))\Big\} \\
= & \inf\Big\{T_i \geq T_1 : y(T_i\wedge T, X^{0,x,*}_{T_i\wedge T}, V^{0,v}_{T_i\wedge T}) \\
& \qquad \leq U(T_i\wedge T, X^{0,x,*}_{T_i\wedge T} + \phi(T_i\wedge T, V^{0,v}_{T_i\wedge T}))\Big\} \\
= & \tau^*.
\end{aligned}
$$

■

5.4 Exponential indifference bond price

In the remaining of the chapter, we model the creditor's risk preferences via horizon-unbiased exponential performance utility

$$U(t,x) = -e^{-\gamma x + \frac{1}{2}\xi^2 t}. \tag{5.4.1}$$

where $\gamma > 0$ is the creditor's risk aversion parameter. This horizon-unbiased exponential performance utility has been widely used in the related literature, see, for example, [35] and [14].

Using the analytical properties of exponential utility, we can eliminate wealth-dependence and reduce to a one-spatial-dimensional free boundary problem. The benefit is we can easily obtain the existence of optimal trading strategy, and then verify the assumptions imposed to Theorem 5.3.1 for the complete characterization. Other utilities could be studied, though the solution would be less tractable due to an increase in spatial dimension.

In the case of exponential utility, we present the following verification theorem for indifference bond price $P^\lambda(x,v)$.

Proposition 5.4.1 *Let p be a function in $C^{1,2}([0,T)\times\mathbb{R}^+)\cap C([0,T]\times\mathbb{R}^+)$, satisfying*

$$\begin{aligned} -\partial_t p(t,v) - \mathcal{L}_2 p(t,v) + \lambda\gamma^{-1}\min\{e^{-\gamma(\phi(t,v)-p(t,v))}-1,0\} = 0, \\ p(T,v) = \phi(T,v) \end{aligned} \tag{5.4.2}$$

where

$$\mathcal{L}_2 := \frac{1}{2}\sigma_V^2 v^2\partial_{vv}^2 + \sigma_V(\eta-\rho\xi)v\partial_v - \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2 v^2(\partial_v)^2,$$

and if moreover, both $p(t,v)$ and $v\partial_v p(t,v)$ are uniformly bounded, then PDE (5.3.1) is uniquely solvable in $C^{1,2,2}([0,T)\times\mathbb{R}\times\mathbb{R}^+)\cap C([0,T]\times\mathbb{R}\times\mathbb{R}^+)$, $y(\cdot,X_\cdot^\pi,V_\cdot)$ is uniformly integrable for any $\pi\in\mathcal{U}_{0,T}$, there exists a unique pair of (π^*,X^*) satisfying (5.3.2) and (5.3.3), and $\pi^*\in\mathcal{U}_{0,T}$. As a result, $P^\lambda(x,v)=p(0,v)$ for $v\in\mathbb{R}^+$, i.e. $P^\lambda(x,v)$ is independent of the initial wealth x, and moreover, the control / stopping strategy given by

$$\begin{cases} \tau^* = \inf\{T_i \geq T_1 : p(T_i\wedge T, V_{T_i\wedge T}) \leq \phi(T_i\wedge T, V_{T_i\wedge T})\} \\ \pi_t^* = \xi(\gamma\sigma_S)^{-1} - \rho\sigma_V\sigma_S^{-1}V_t\partial_v p(t,V_t) \end{cases} \tag{5.4.3}$$

is optimal of the mixed stochastic control and constrained optimal stopping problem (5.2.3) with exponential performance utility.

Proof. Let $y(t,x,v)=U(t,x+p(t,v))$, we have $y\in C^{1,2,2}([0,T)\times\mathbb{R}\times\mathbb{R}^+)\cap C([0,T]\times\mathbb{R}\times\mathbb{R}^+)$. By the boundedness of $p(t,v)$ and the definition of the admissible control set, we can have $y(\cdot,X_\cdot^\pi,V_\cdot)$ is uniformly integrable for any $\pi\in\mathcal{U}_{0,T}$. Since $\partial_{xx}^2 y(t,x,v)=\gamma^2 y(t,x,v)<0$, then

$$\begin{aligned} \pi^*(t,x,v) &= -\xi\sigma_S^{-1}\frac{\partial_x y(t,x,v)}{\partial_{xx}^2 y(t,x,v)} - \rho\sigma_V\sigma_S^{-1}v\frac{\partial_{xv}^2 y(t,x,v)}{\partial_{xx}^2 y(t,x,v)} \\ &= \xi(\gamma\sigma_S)^{-1} - \rho\sigma_V\sigma_S^{-1}v\partial_v p(t,v) \end{aligned}$$

which is well defined, attains the maximum of $\mathcal{L}_1^\pi$. Note that π^* is independent of x, we can rewrite $\pi^*(t,x,v)$ in the following form

$$\pi^*(t,v) = \xi(\gamma\sigma_S)^{-1} - \rho\sigma_V\sigma_S^{-1}v\partial_v p(t,v).$$

Then we can define the optimal trading strategy π^* as follows

$$\pi_t^* = \pi^*(t,V_t) = \xi(\gamma\sigma_S)^{-1} - \rho\sigma_V\sigma_S^{-1}V_t\partial_v p(t,V_t)$$

for $t \in [0,T)$. Since the optimal trading strategy π^* is independent of X^*, SDE (5.3.3) admits a unique solution.

We can also verify that π^* is an admissible strategy, i.e. $\pi^* \in \mathcal{U}_{0,T}$. Indeed, π^* is bounded due to the boundedness of $v\partial_v p(t,v)$, and hence satisfies both the integrability condition and the class (D) condition.

As a a result, the conditions of Theorem 5.3.1 are all satisfied, which implies

$$Y^\lambda(x,v) = y(0,x,v) = U(0, x + p(0,v)),$$

and therefore $P^\lambda(x,v) = p(0,v)$ by (5.2.4). Moreover, τ^* can be rewritten as follows

$$\begin{aligned}
\tau^* = \ & \inf\Big\{T_i \geq T_1 : y(T_i \wedge T, X^*_{T_i\wedge T}, V_{T_i\wedge T}) \\
& \qquad\qquad \leq \ U(T_i \wedge T, X^*_{T_i\wedge T} + \phi(T_i \wedge T, V_{T_i\wedge T}))\Big\} \\
= & \inf\Big\{T_i \geq T_1 : U(T_i \wedge T, X^*_{T_i\wedge T} + p(T_i \wedge T, V_{T_i\wedge T})) \\
& \qquad\qquad \leq \ U(T_i \wedge T, X^*_{T_i\wedge T} + \phi(T_i \wedge T, V_{T_i\wedge T}))\Big\} \\
= & \inf\Big\{T_i \geq T_1 : p(T_i \wedge T, V_{T_i\wedge T}) \leq \ \phi(T_i \wedge T, V_{T_i\wedge T})\Big\}.
\end{aligned}$$

■

The above verification proposition only gives the conditional characterization of mixed stochastic control and constrained optimal stopping problem (5.2.3) with exponential performance utility. The complete characterization, however, is based on the solvability of PDE (5.4.2) and the uniform boundedness of $p(t,v)$ and $v\partial_v p(t,v)$, which will be studied in the following lemma.

For the convenience in the analysis, we let $\theta = T - t$, $z = \ln v$, $u(\theta, z) = p(t,v)$ in (5.4.2), then

$$\begin{aligned}
& \mathcal{L}_2 p(t,v) \\
= & \ \frac{1}{2}\sigma_V^2 v^2 \partial^2_{vv} p + \sigma_V(\eta - \rho\xi) v \partial_v p - \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2 v^2 (\partial_v p)^2 \\
= & \ \frac{1}{2}\sigma_V^2 \partial^2_{zz} u + \sigma_V(\eta - \rho\xi - \frac{1}{2}\sigma_V)\partial_z u - \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2 (\partial_z u)^2 := \mathcal{L}u(\theta, z),
\end{aligned}$$

and therefore, $u(\theta, z)$ satisfies

$$\partial_\theta u(\theta,z) - \mathcal{L}u(\theta,z) + \lambda\gamma^{-1}\min\{e^{-\gamma(\phi(T-\theta, e^z) - u(\theta,z))} - 1, 0\} = 0, \qquad u(0,z) = \phi(T, e^z), \quad (5.4.4)$$

on $(0,T] \times \mathbb{R}$.

Remark 5.4.2 *For the convenience of the reader, we recall the definition of viscosity sub-and supersolutions for later use. Consider a non-linear second order degenerate partial differential equation*

$$F(x, u(x), Du(x), D^2u(x)) = 0 \text{ for } x \in \mathcal{O}, \tag{5.4.5}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ and F is a continuous map from $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ satisfying the so-called ellipticity condition:

$$F(x, r, p, A) \leq F(x, r, p, B) \text{ whenever } A \geq B$$

for all $(x, r, p) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$ and $A, B \in \mathcal{S}_d$.

Let $u : \mathcal{O} \to \mathbb{R}$ be a continuous function:

1. *We say that u is a viscosity supersolution of (5.4.5) if*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

for all pairs of $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a minimizer of the difference $(u - \varphi)$ on $\mathcal{O}$.

2. *We say that u is a viscosity subsolution of (5.4.5) if*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$$

for all pairs of $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a maximizer of the difference $(u - \varphi)$ on $\mathcal{O}$.

3. *We say that u is a viscosity solution of (5.4.5) if it is both a viscosity supersolution and subsolution of (5.4.5).*

Lemma 5.4.3 *Suppose that there exists a solution $u(\theta, z) \in C^{1,2}((0, T] \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ to PDE (5.4.4). Then, $u(\theta, z)$ satisfies*

$$0 \leq u(\theta, z) \leq Ke^{r^+T}, \tag{5.4.6}$$

$$0 \leq \partial_z u(\theta, z) \leq Ke^{r^+T}. \tag{5.4.7}$$

Proof. We prove both estimates (5.4.6)-(5.4.7) using the comparison principle. Firstly, the estimate (5.4.6) follows immediately from the observation that $w_1 = 0$ and $w_2 = Ke^{r^+T}$ are the subsolution and supersolution of (5.4.4).

Secondly, the estimate (5.4.7) can be verified by constructing a penalty approximation of (5.4.4). Suppose $u_\epsilon(\theta, z)$ satisfies

$$\partial_\theta u_\epsilon(\theta,z) - \mathcal{L}u_\epsilon(\theta,z) + \lambda\gamma^{-1}\pi_\epsilon(e^{-\gamma(\pi_\epsilon(e^z - K_{T-\theta}) + K_{T-\theta} - u_\epsilon(\theta,z))} - 1) = 0,$$
$$u_\epsilon(0,z) = \pi_\epsilon(e^z - K_T) + K_T, \quad (5.4.8)$$

where $\pi_\epsilon(z)$ satisfies that $\pi_\epsilon(z) \in C^\infty$, $0 \le \pi'_\epsilon(z) \le 1$, $\pi''_\epsilon(z) \le 0$, $\lim_{\epsilon\to 0+}\pi_\epsilon(z) = \min(z, 0)$, and

$$\pi_\epsilon(z) = \begin{cases} z, & z \le -\epsilon, \\ \nearrow, & |z| \le \epsilon, \\ 0, & z \ge \epsilon. \end{cases}$$

If we differentiate (5.4.8) w.r.t. z, then $w(\theta, z) := \partial_z u_\epsilon(\theta, z)$ satisfies

$$\partial_\theta w - \frac{1}{2}\sigma_V^2 \partial_{zz}^2 w - \sigma_V(\eta - \rho\xi - \frac{1}{2}\sigma_V)\partial_z w + \gamma(1-\rho^2)\sigma_V^2 w \partial_z w$$
$$+ \lambda \pi'_\epsilon(e^y - 1)e^y \left(w - \pi'_\epsilon(e^z - K_{T-\theta})e^z\right) = 0, \quad (5.4.9)$$

where $y = -\gamma(\pi_\epsilon(e^z - K_{T-\theta}) + K_{T-\theta} - u_\epsilon(\theta, z))$, with the initial condition $w(0, z) = \pi'_\epsilon(e^z - K_T)e^z$.

Since $w_1 = 0$ and $w_2 = Ke^{r^+T} + \epsilon$ are the subsolution and supersolution of (5.4.9), we can obtain

$$0 \le \partial_z u_\epsilon(\theta, z) \le Ke^{r^+T} + \epsilon$$

by the comparison principle. By letting $\epsilon \to 0$, we have a continuous limit (of a subsequence if necessary) $\partial_z u(\theta, z)$, it follows from Arzela-Ascoli Compactness Criterion that $\partial_z u_\epsilon(\theta, z) \to \partial_z u(\theta, z)$ uniformly in $C((0, T] \times \mathbb{R})$, and thus we obtain the required result (see, for example, Proof of Lemma 5 in [14]). ∎

We are now in a position to present the complete characterization of the value function Y^λ of mixed stochastic control and constrained optimal stopping problem (5.2.3) with exponential performance utility (5.4.1), and exponential indifference bond price P^λ. Applying Theorem 5.3.1, Proposition 5.4.1 and Lemma 5.4.3, we can conclude the following Theorem 5.4.4.

Theorem 5.4.4 *Suppose that there exists a solution $u(\theta, z) \in C^{1,2}((0, T] \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ to PDE (5.4.4). Then, the following statements hold.*

1. *Exponential indifference bond price with stopping constraints $P^\lambda(x, v)$, which is defined in (5.2.4) with exponential performance utility (5.4.1), is independent of the initial wealth x, and $P^\lambda(x, v) = p(0, v)$, where $p(t, v)$ is the unique $C^{1,2}([0, T) \times \mathbb{R}^+) \cap C([0, T] \times \mathbb{R}^+)$ solution to (5.4.2);*

2. *The value function $Y^\lambda(x, v)$ of the mixed stochastic control and constrained optimal stopping problem (5.2.3) with exponential performance utility (5.4.1) has the form $Y^\lambda(x, v) = U(0, x + P^\lambda(x, v))$, and*

$Y^\lambda(x,v) = y(0,x,v)$, *where* $y(t,x,v)$ *is the unique* $C^{1,2,2}([0,T) \times \mathbb{R} \times \mathbb{R}^+) \cap C([0,T] \times \mathbb{R} \times \mathbb{R}^+)$ *solution to (5.3.1); and moreover, the corresponding optimal control / stopping strategy is given by (5.4.3).*

5.4.1 Properties of exponential indifference bond price

In this subsection, we investigate the impact of parameters on exponential indifference bond price as a consequence of the complete characterization Theorem 5.4.4.

Proposition 5.4.5 *Exponential indifference bond price* $P^\lambda(x,v)$ *is increasing w.r.t.* η*,* r *and* λ*, decreasing w.r.t.* γ*, and increasing w.r.t.* ξ *if* $\rho \le 0$ *and decreasing w.r.t.* ξ *if* $\rho \ge 0$*.*

Proof. *Step 1.* Suppose that $\eta_1 > \eta_2$, and $u_{\epsilon(i)}(\theta,z)$ is the solution to the following problem on $(0,T] \times \mathbb{R}$,

$$\partial_\theta u_{\epsilon(i)} - \frac{1}{2}\sigma_V^2 \partial_{zz}^2 u_{\epsilon(i)} - \sigma_V(\eta_i - \rho\xi - \frac{1}{2}\sigma_V)\partial_z u_{\epsilon(i)} + \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(i)})^2 \\ + \lambda\gamma^{-1}\pi_\epsilon(e^{-\gamma(\pi_\epsilon(e^z - K_{T-\theta}) + K_{T-\theta} - u_{\epsilon(i)})} - 1) = 0,$$

with the initial condition $u_{\epsilon(i)}(0,z) = \pi_\epsilon(e^z - K_T) + K_T$. Then $w(\theta,z) := u_{\epsilon(1)}(\theta,z) - u_{\epsilon(2)}(\theta,z)$ satisfies

$$\partial_\theta w - \frac{1}{2}\sigma_V^2 \partial_{zz}^2 w - \sigma_V(\eta_1 - \rho\xi - \frac{1}{2}\sigma_V)\partial_z w + \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(1)} + \partial_z u_{\epsilon(2)})\partial_z w \\ + \lambda\pi'_\epsilon(\cdot)e^{-\gamma\cdot} w = \sigma_V(\eta_1 - \eta_2)\partial_z u_{\epsilon(2)} \ge 0.$$

Combined with the initial condition $w(0,z) = 0$, we have $u_{\epsilon(1)}(\theta,z) \ge u_{\epsilon(2)}(\theta,z)$. By letting $\epsilon \to 0$, we obtain $u(\theta,z)$ is increasing w.r.t. η.

Similarly, we can prove the monotonicities wr.t. λ and ξ.

Step 2. Suppose that $r_1 > r_2$, and $u_{\epsilon(i)}(\theta,z)$ is the solution to the following problem on $(0,T] \times \mathbb{R}$,

$$\partial_\theta u_{\epsilon(i)} - \frac{1}{2}\sigma_V^2 \partial_{zz}^2 u_{\epsilon(i)} - \sigma_V(\eta - \rho\xi - \frac{1}{2}\sigma_V)\partial_z u_{\epsilon(i)} + \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(i)})^2 \\ + \lambda\gamma^{-1}\pi_\epsilon(e^{-\gamma(\pi_\epsilon(e^z - K^i_{T-\theta}) + K^i_{T-\theta} - u_{\epsilon(i)})} - 1) = 0,$$

with the initial condition $u_{\epsilon(i)}(0,z) = \pi_\epsilon(e^z - K^i_T) + K^i_T$, where $K^i_t = Ke^{r_i t}$.

Then $w(\theta,z) := u_{\epsilon(1)}(\theta,z) - u_{\epsilon(2)}(\theta,z)$ satisfies

$$
\begin{aligned}
&\partial_\theta w - \frac{1}{2}\sigma_V^2\partial_{zz}^2 w - \sigma_V(\eta - \rho\xi - \frac{1}{2}\sigma_V)\partial_z w \\
&\quad + \frac{1}{2}\gamma(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(1)} + \partial_z u_{\epsilon(2)})\partial_z w + \lambda\pi_\epsilon'(\cdot)e^{-\gamma\cdot} w \\
&= \lambda\gamma^{-1}[\pi_\epsilon(e^{-\gamma(\pi_\epsilon(e^z - K^2_{T-\theta}) + K^2_{T-\theta} - u_{\epsilon(2)})} - 1) - \pi_\epsilon(e^{-\gamma(\pi_\epsilon(e^z - K^1_{T-\theta}) + K^1_{T-\theta} - u_{\epsilon(2)})} - 1)] \\
&= \lambda\pi_\epsilon'(\cdot)e^{-\gamma\cdot}\left[\pi_\epsilon(e^z - K^1_{T-\theta}) - \pi_\epsilon(e^z - K^2_{T-\theta}) + K^1_{T-\theta} - K^2_{T-\theta}\right] \\
&= \lambda\pi_\epsilon'(\cdot)e^{-\gamma\cdot}(1 - \pi_\epsilon'(\cdot))(K^1_{T-\theta} - K^2_{T-\theta}) \geq 0.
\end{aligned}
$$

Combined with the initial condition

$$
w(0,z) = \pi_\epsilon(e^z - K_T^1) - \pi_\epsilon(e^z - K_T^2) + K_T^1 - K_T^2 = (1 - \pi_\epsilon'(\cdot))(K_T^1 - K_T^2) \geq 0,
$$

we have $u_{\epsilon(1)}(\theta,z) \geq u_{\epsilon(2)}(\theta,z)$. By letting $\epsilon \to 0$, we obtain $u(\theta,z)$ is increasing w.r.t. r.

Step 3. Suppose that $\gamma_1 > \gamma_2$, and $u_{\epsilon(i)}(\theta,z)$ is the solution to the following problem on $(0,T] \times \mathbb{R}$,

$$
\begin{aligned}
\partial_\theta u_{\epsilon(i)} - \frac{1}{2}\sigma_V^2\partial_{zz}^2 u_{\epsilon(i)} - \sigma_V(\eta - \rho\xi - \frac{1}{2}\sigma_V)\partial_z u_{\epsilon(i)} + \frac{1}{2}\gamma_i(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(i)})^2& \\
+ \lambda\gamma_i^{-1}\pi_\epsilon(e^{-\gamma_i(\pi_\epsilon(e^z - K_{T-\theta}) + K_{T-\theta} - u_{\epsilon(i)})} - 1) = 0&,
\end{aligned}
$$

with the initial condition $u_{\epsilon(i)}(0,z) = \pi_\epsilon(e^z - K_T) + K_T$. Then $w(\theta,z) := u_{\epsilon(1)}(\theta,z) - u_{\epsilon(2)}(\theta,z)$ satisfies

$$
\begin{aligned}
&\partial_\theta w - \frac{1}{2}\sigma_V^2\partial_{zz}^2 w - \sigma_V(\eta - \rho\xi - \frac{1}{2}\sigma_V)\partial_z w \\
&\quad + \frac{1}{2}\gamma_1(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(1)} + \partial_z u_{\epsilon(2)})\partial_z w + \lambda\pi_\epsilon'(\cdot)e^{-\gamma_1\cdot} w \\
&= \frac{1}{2}(\gamma_2 - \gamma_1)(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(2)})^2 - \lambda\gamma_1^{-1}\pi_\epsilon(e^{-\gamma_1(\pi_\epsilon(e^z - K_{T-\theta}) + K_{T-\theta} - u_{\epsilon(2)})} - 1) \\
&\quad + \lambda\gamma_2^{-1}\pi_\epsilon(e^{-\gamma_2(\pi_\epsilon(e^z - K_{T-\theta}) + K_{T-\theta} - u_{\epsilon(2)})} - 1) \\
&\leq \frac{1}{2}(\gamma_2 - \gamma_1)(1-\rho^2)\sigma_V^2(\partial_z u_{\epsilon(2)})^2 \leq 0
\end{aligned}
$$

where the second inequality holds since it is easy to prove that the mapping $\gamma \mapsto \gamma^{-1}\pi_\epsilon(e^{-\gamma y} - 1)$ is increasing w.r.t. γ for any $y \in \mathbb{R}$. Combined with the initial condition $w(0,z) = 0$, we have $u_{\epsilon(1)}(\theta,z) \leq u_{\epsilon(2)}(\theta,z)$. By letting $\epsilon \to 0$, we obtain $u(\theta,z)$ is decreasing w.r.t. γ. ■

The intuition behind the impacts with respect to η and ξ is that greater chance of the bond providing better opportunities will impact on its value in the same direction, where "better" is in the sense of a greater payoff in the future. This happens more frequently with the greater Sharpe ratio of V or the lower relative Sharpe ratio of S, where the correlation ρ palys a role in the relativity.

With higher spread rate r, the creditor will benefit from receiving more interest payment from the firm, which therefore makes the bond more attractive. The higher intensity of the signaling Poisson process λ will also impact on the bond value in the same direction, since the higher λ provides more flexibility for the creditor. In the extreme situation of $\lambda \to \infty$, where the creditor is allowed to withdraw her money at any time, the original problem (5.2.3) is reduced to a standard mixed control and stopping problem, where the creditor is not exposed to stopping constraints. This will be further discussed in the next section.

The higher risk aversion γ will impact on the bond value in the opposite direction. Waiting involves facing random fluctuations in V, which can only be partially hedged away by trading in incomplete markets. If the creditor is more risk averse, she will less appreciate this bond with idiosyncratic risk. In the extreme case of one correlation, the market is complete, and there is no idiosyncratic risk, and hence the value of γ will have no impact on the bond value.

However, the general monotonicity with respect to the correlation ρ is uncertain due to the role of the Sharpe ratio of S, i.e. ξ. ρ will impact on the bond price in two ways. Since V can only be partially hedged, the higher $|\rho|$ implies the less idiosyncratic risk she is exposed to, which resultis in the higher bond price. This is the first way. The second way is through the relative Sharpe ratio of S, i.e. $\rho\xi$. As explained earlier, the higher relative Sharpe ratio will impact on the bond price in the opposite direction. These two ways might lead the bond price into opposite direction, and hence we can only obtain the conditional monotonicity, in the case where these two ways have the same direction.

The following proposition gives the conditional monotonicity of exponential indifference bond price with respect to the correlation.

Proposition 5.4.6 *Exponential indifference bond price $P^{\lambda}(x,v)$ is increasing w.r.t. ρ if $\rho \geq 0$ and $\xi \leq 0$, and decreasing w.r.t. ρ if $\rho \leq 0$ and $\xi \geq 0$.*

Proof. Suppose that $\rho_1 > \rho_2 \geq 0$ with $\xi \leq 0$, and $u_{\epsilon(i)}(\theta, z)$ is the solution to the following problem on $(0,T] \times \mathbb{R}$,

$$\partial_\theta u_{\epsilon(i)} - \frac{1}{2}\sigma_V^2 \partial_{zz}^2 u_{\epsilon(i)} - \sigma_V(\eta - \rho_i \xi - \frac{1}{2}\sigma_V)\partial_z u_{\epsilon(i)} + \frac{1}{2}\gamma(1-\rho_i^2)\sigma_V^2(\partial_z u_{\epsilon(i)})^2 \\ + \lambda\gamma^{-1}\pi_\epsilon(e^{-\gamma(\pi_\epsilon(e^z - K) + K - u_{\epsilon(i)})} - 1) = 0,$$

with the initial condition $u_{\epsilon(i)}(0,z) = \pi_\epsilon(e^z - K) + K$. Then $w(\theta, z) :=$

$u_{\epsilon(1)}(\theta,z) - u_{\epsilon(2)}(\theta,z)$ satisfies

$$
\begin{aligned}
&\partial_\theta w - \frac{1}{2}\sigma_V^2 \partial_{zz}^2 w - \sigma_V(\eta - \rho_1\xi - \frac{1}{2}\sigma_V)\partial_z w \\
&+ \frac{1}{2}\gamma(1-\rho_1^2)\sigma_V^2(\partial_z u_{\epsilon(1)} + \partial_z u_{\epsilon(2)})\partial_z w + \lambda\gamma^{-1}\pi_\epsilon'(\cdot)w \\
= &\ \sigma_V(\rho_2 - \rho_1)\xi\partial_z u_{\epsilon(2)} + \frac{1}{2}\gamma(\rho_1^2 - \rho_2^2)\sigma_V^2(\partial_z u_{\epsilon(2)})^2 \geq 0.
\end{aligned}
$$

Combined with the initial condition $w(0,z) = 0$, we have $u_{\epsilon(1)}(\theta,z) \geq u_{\epsilon(2)}(\theta,z)$. By letting $\epsilon \to 0$, we obtain $u(\theta,z)$ is increasing w.r.t. ρ if $\rho \geq 0$ and $\xi \leq 0$. We can prove the second statement using the similar arguments. ■

5.5 Asymptotics as $\lambda \to \infty$

A relevant question for the problem (5.2.3) is the following: what is the cost of having such constraints on the stopping times? In other words, how does this problem differ from standard mixed stochastic control and optimal stopping problem, which can be regarded as the limiting model when $\lambda \to \infty$.

5.5.1 Review of standard mixed control/stopping problems

Let $\mathcal{T}_{[t,T]}$ denote the collection of all $\mathbb{F}$-stopping times with values in $[t,T]$. Define the following *standard mixed stochastic control and optimal stopping problem* corresponding to the mixed stochastic control and constrained optimal stopping problem (5.2.3):

$$
Y(x,v) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}, \pi \in \mathcal{U}_{0,\tau\wedge T}} \mathbb{E}\left[U\left(\tau\wedge T, X_{\tau\wedge T}^\pi + \phi(\tau\wedge T, V_{\tau\wedge T})\right)\right], \tag{5.5.1}
$$

and $(\tau^*, \pi^*) \in \mathcal{T}_{[0,T]} \times \mathcal{U}_{0,\tau^*\wedge T}$ is called an optimal control / stopping strategy for the problem (5.5.1) if

$$
Y(x,v) = \mathbb{E}\left[U\left(\tau^*\wedge T, X_{\tau^*\wedge T}^{\pi^*} + \phi(\tau^*\wedge T, V_{\tau^*\wedge T})\right)\right].
$$

The corresponding creditor's *indifference bond price without stopping constraints* $P(x,v)$ is defined by the following equation

$$
Y(x,v) = U(0, x + P(x,v)) \tag{5.5.2}
$$

where Y is given by (5.5.1).

Similar to Theorem 5.4.4, in the case of exponential performance utility (5.4.1), the following theorem gives the complete characterization of the value function Y and indifference bond price without stopping constraints P. What's more, we also make the connection between the problems with and without stopping constraints.

Before stating the main theorem of this section, we introduce the following two Sobolev spaces with certain regularity: for any $p \geq 1$,

(i) $W_p^{1,2}([0,T) \times \mathbb{R}^+)$ is the completion of $C^\infty([0,T) \times \mathbb{R}^+)$ under the norm

$$||u||_{W_p^{1,2}([0,T)\times\mathbb{R}^+)} = \left[\int_{[0,T)\times\mathbb{R}^+} (|u|^p + |\partial_t u|^p + |\partial_v u|^p + |\partial_{vv} u|^p)\, dt\, dv\right]^{1/p}.$$

(ii) $W_p^{1,2,2}([0,T) \times \mathbb{R} \times \mathbb{R}^+)$ is the completion of $C^\infty([0,T) \times \mathbb{R} \times \mathbb{R}^+)$ under the norm

$$\begin{aligned} ||u||_{W_p^{1,2}([0,T)\times\mathbb{R}\times\mathbb{R}^+)} = \bigg[\int_{[0,T)\times\mathbb{R}\times\mathbb{R}^+} & (|u|^p + |\partial_t u|^p + |\partial_x u|^p + |\partial_v u|^p \\ & + |\partial_{xv} u|^p + |\partial_{xx} u|^p + |\partial_{vv} u|^p)\, dt\, dv\, dx\bigg]^{1/p}. \end{aligned}$$

Theorem 5.5.1 *The following statements hold.*

1. *Exponential indifference bond price without stopping constraints $P(x,v)$, which is defined in (5.5.2) with exponential performance utility (5.4.1), is independent of the initial wealth x, and $P(x,v) = p^c(0,v)$, where $p^c(t,v)$ is the unique $W_p^{1,2}([0,T) \times \mathbb{R}^+) \cap C([0,T] \times \mathbb{R}^+)$ solution to the following equation:*

$$\begin{aligned} \min\{-\partial_t p^c(t,v) - \mathcal{L}_2 p^c(t,v), p^c(t,v) - \phi(t,v)\} &= 0, \\ p^c(T,v) &= \phi(T,v); \end{aligned} \tag{5.5.3}$$

2. *The value function $Y(x,v)$ of the mixed stochastic control and optimal stopping problem (5.5.1) with exponential performance utility (5.4.1) has the form $Y(x,v) = U(0, x + P(x,v))$, and $Y(x,v) = y(0,x,v)$, where $y^c(t,x,v)$ is the unique $W_p^{1,2,2}([0,T) \times \mathbb{R} \times \mathbb{R}^+) \cap C([0,T] \times \mathbb{R} \times \mathbb{R}^+)$ solution to the following equation:*

$$\min\left\{-\partial_t y^c(t,x,v) - \sup_{\pi\in\mathbb{R}} \mathcal{L}_1^\pi y^c(t,x,v), y^c(t,x,v) - U(t, x+\phi(t,v))\right\} = 0,$$
$$y^c(T,x,v) = U\left(T, x + \phi(T,v)\right); \tag{5.5.4}$$

and moreover, the corresponding optimal control / stopping strategy is given by

$$\begin{cases} \tau^* = \inf\{t \geq 0 : p^c(t\wedge T, V_{t\wedge T}) = \phi(t \wedge T, V_{t\wedge T})\}, \\ \pi_t^* = \xi(\gamma\sigma_S)^{-1} - \rho\sigma_V\sigma_S^{-1} V_t \partial_v p^c(t, V_t); \end{cases} \tag{5.5.5}$$

3. As λ goes to infinity, we have the following convergence results

$$\begin{aligned} P(x,v) &= \lim_{\lambda\to\infty} P^\lambda(x,v), \\ Y(x,v) &= \lim_{\lambda\to\infty} Y^\lambda(x,v). \end{aligned}$$

Proof. The proof of the first two statements follows along the similar arguments in [14] and is thus omitted. To prove the convergence result for Y, we note that PDE (5.3.1) can be regarded as a sequence of penalized PDEs for (5.5.4) (see, for example, [28]), then the convergence follows. ■

5.5.2 A complete market model

In this subsection, we consider a rollover dedt without stopping constraints in a complete market, where we can solve in closed form both the bond price and optimal stopping strategy. The set-up is the same as in Section 5.5.1 except for $\rho = 1$. For later use, we define the stopping region ($\mathcal{S}$) and the continuation region ($\mathcal{C}$) for this problem as

$$\begin{aligned} \mathcal{S} &:= \{(t,v) \in [0,T) \times \mathbb{R}^+ : p^c(t,v) = \phi(t,v)\} \\ \mathcal{C} &:= \{(t,v) \in [0,T) \times \mathbb{R}^+ : p^c(t,v) > \phi(t,v))\} \end{aligned}$$

where p^c is the solution to (5.5.3), and therefore, the optimal stopping strategy τ^*, given by (5.5.5), has the following representation

$$\tau^* = \inf\{t \geq 0 : (t, V_t) \in \mathcal{S}\} \wedge T.$$

Theorem 5.5.2 *In the complete market where $\rho = 1$,*
(a) *if $r \leq 0$ and $\eta \leq \xi$, then $\mathcal{S} = [0,T) \times \mathbb{R}^+$ and $P(x,v) = \min\{v, K\}$;*
(b) *if $r \leq 0$ and $\eta > \xi$, then $\mathcal{S} = \{(t,v) \in [0,T) \times \mathbb{R}^+ : v \geq K_t\}$ and*

$$P(x,v) = \begin{cases} \overline{p}^c(0,v) & v < K \\ K & v \geq K \end{cases}$$

where $\overline{p}^c$ is the classical solution of

$$\begin{cases} -\partial_t \overline{p}^c - \hat{\mathcal{L}}_2 \overline{p}^c = 0 & \text{on } [0,T) \times (0, K_t) \\ \overline{p}^c(T,v) = v & \text{on } (0, K_T] \\ \overline{p}^c(t, K_t) = K_t & \text{on } [0,T) \end{cases} \tag{5.5.6}$$

with $\hat{\mathcal{L}}_2 = \frac{1}{2}\sigma_V^2 v^2 \partial_{vv}^2 + \sigma_V(\eta - \xi) v \partial_v$.

(c) *if $r > 0$ and $\eta \leq \xi$, then $\mathcal{S} = \{(t, v) \in [0, T) \times \mathbb{R}^+ : v \leq K_t\}$ and*

$$P(x, v) = \begin{cases} v & v \leq K \\ \hat{p}^c(0, v) & v > K \end{cases}$$

where $\hat{p}^c$ is the classical solution of

$$\begin{cases} -\partial_t \hat{p}^c - \hat{\mathcal{L}}_2 \hat{p}^c = 0 & on\ [0, T) \times (K_t, \infty) \\ \hat{p}^c(T, v) = K_T & on\ [K_T, \infty) \\ \hat{p}^c(t, K_t) = K_t & on\ [0, T) \end{cases} . \tag{5.5.7}$$

(d) *if $r > 0$ and $\eta > \xi$, let us define the auxiliary function*

$$\begin{aligned} f(t) = {} & e^{r(T-t)} \left(-\frac{\sigma_V(\eta - \xi) - r - \sigma_V^2/2}{\sigma_V^2/2} \right) \Phi \left(-\frac{\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2}{\sigma_V} \sqrt{T-t} \right) \\ & + \ e^{\sigma_V(\eta-\xi)(T-t)} \left(\frac{\sigma_V(\eta - \xi) - r + \sigma_V^2/2}{\sigma_V^2/2} \right) \Phi \left(-\frac{\sigma_V(\eta - \xi) - r + \frac{1}{2}\sigma_V^2}{\sigma_V} \sqrt{T-t} \right) \\ & - \ \frac{8r}{\sqrt{2\pi}\sigma_V} e^{(r - \frac{(\sigma_V(\eta-\xi) - r - \frac{1}{2}\sigma_V^2)^2}{2\sigma_V^2})(T-t)} \sqrt{T-t} \end{aligned} \tag{5.5.8}$$

where $\Phi(\cdot)$ is the cumulative distribution function for a standard normal distribution,

- *and, moreover, if $f(0) \geq 0$, then $\mathcal{S} = \{(t, v) \in [0, T) \times \mathbb{R}^+ : v = K_t\}$ and $P(x, v) = \tilde{p}^c(0, v)$, where the latter is given by*

$$\tilde{p}^c(t, v) = \begin{cases} \overline{p}^c(t, v) & v \leq K_t \\ \hat{p}^c(t, v) & v \geq K_t \end{cases} ; \tag{5.5.9}$$

- *otherwise, then $\mathcal{S} = \{(t, v) \in [t_*^c, T) \times \mathbb{R}^+ : v = K_t\}$, where*

$$t_*^c := \inf\{t \geq 0 : f(t) \geq 0\} > 0, \tag{5.5.10}$$

and $P(x, v) = \check{p}^c(0, v)$, where $\check{p}^c(t, v) = \tilde{p}^c(t, v)$ in (5.5.9) for $t \in [t_*^c, T]$, and $\check{p}^c$ is the classical solution of

$$\begin{cases} -\partial_t \check{p}^c - \hat{\mathcal{L}}_2 \check{p}^c = 0 & on\ [0, t_*^c) \times (0, \infty) \\ \check{p}^c(t_*^c, v) = \tilde{p}^c(t_*^c, v) & on\ (0, \infty) \end{cases} \tag{5.5.11}$$

for $t \in [0, t_*^c)$.

5.5.3 Proof of Theorem 5.5.2

By Theorem 5.5.1, $P(x,v) = p^c(0,v)$, where p^c is the solution to the following equation:

$$\begin{aligned} \min\Big\{-\partial_t p^c(t,v) - \hat{\mathcal{L}}_2 p^c(t,v), p^c(t,v) - \phi(t,v)\Big\} &= 0, \\ p^c(T,v) &= \phi(T,v). \end{aligned} \tag{5.5.12}$$

Case (a).

If $r \leq 0$ and $\eta \leq \xi$, both $p_1^c = K_t$ and $p_2^c = v$ are the supersolutions of (5.5.12), using the comparison principle, we can obtain that $p^c \leq \phi(t,v)$, and therefore $p^c = \phi(t,v)$. This implies

$$\mathcal{S} = [0,T) \times \mathbb{R}^+,$$

i.e. $\tau^* = 0$; and moreover, $P(x,v) = p^c(0,v) = \min\{v, K\}$.

Case (b).

If $r \leq 0$ and $\eta > \xi$, we prove that $p^c(t,v)$, which is given by

$$p^c(t,v) = \begin{cases} \overline{p}^c(t,v), & \text{on } [0,T) \times (0,K_t) \\ K_t, & \text{on } [0,T) \times [K_t,\infty) \end{cases},$$

is the viscosity solution of (5.5.12), where $\overline{p}^c$ solves (5.5.6). Indeed, since $\overline{p}_1^c = K_t$ and $\overline{p}_2^c = v$ are the supersolution and subsolution of (5.5.6), using the comparison principle, we can obtain that $v < \overline{p}^c \leq K_t$ on $[0,T) \times (0,K_t)$, which implies that

$$\partial_v p^c(t, K_t-) \geq 0. \tag{5.5.13}$$

It is sufficient to prove that p^c is a viscosity supersolution of the equation

$$-\partial_t p^c - \hat{\mathcal{L}}_2 p^c = 0.$$

Since p^c is a classical solution except on the curve $v = K_t$, it is sufficient to focus on the equation on this curve. Thanks to (5.5.13), there are two following cases.

- If $\partial_v p^c(t,K_t-) = 0$, then $\partial_v p^c(t,K_t) = 0$. For all $\varphi \in C^{1,2}$ such that $p^c(t,v) - \varphi(t,v) \geq 0$ and $p^c(t_0,K_{t_0}) - \varphi(t_0,K_{t_0}) = 0$, and therefore we have

$$\partial_v p^c(t_0,K_{t_0}) = \partial_v \varphi(t_0,K_{t_0}) \text{ and } \partial_t p^c(t_0,K_{t_0}) \geq \partial_t \varphi(t_0,K_{t_0}). \tag{5.5.14}$$

We claim the following inequality

$$\partial_{vv}p^c(t_0, K_{t_0}-) - \partial_{vv}\varphi(t_0, K_{t_0}) \geq 0 \tag{5.5.15}$$

holds, and by (5.5.6), (5.5.14) and (5.5.15), we can obtain that

$$-\partial_t\varphi(t_0, K_{t_0}) - \hat{\mathcal{L}}_2\varphi(t_0, K_{t_0}) \geq 0$$

which completes the proof. Indeed, in order to prove (5.5.15), we assume $\partial_{vv}p^c(t_0, K_{t_0}-) - \partial_{vv}\varphi(t_0, K_{t_0}) < 0$, then $\partial_v p^c(t_0, v) - \partial_v\varphi(t_0, v)$ is decreasing in the left neighbourhood $(K_{t_0} - \delta, K_{t_0})$. This implies $\partial_v p^c(t_0, v) - \partial_v\varphi(t_0, v) > 0$ for $v \in [K_{t_0} - \delta, K_{t_0}]$, and thus $p^c(t_0, v) - \varphi(t_0, v)$ is increasing in the left neighbourhood $(K_{t_0} - \delta, K_{t_0})$, which contradicts the assumption that $p^c - \varphi$ attains its minimum at (t_0, K_{t_0}).

- If $\partial_v p^c(t, K_t-) > 0$, the supersolution property is obviously satisfied since there is no smooth function $\varphi \in C^{1,2}$ satisfying the minimum condition.

This implies that

$$\mathcal{S} = \{(t, v) \in [0, T) \times \mathbb{R}^+ : v \geq K_t\},$$

and moreover, we solve PDE (5.5.6) in Appendix 5.A.1 and obtain that

$$P(x, v) = p^c(0, v) = \begin{cases} \overline{p}^c(0, v) & v < K \\ K & v \geq K \end{cases}$$

where $\overline{p}^c$ is given by (5.A.3).

Case (c).

If $r > 0$ and $\eta \leq \xi$, we prove that $p^c(t, v)$, which is given by

$$p^c(t, v) = \begin{cases} v, & \text{on } [0, T) \times (0, K_t] \\ \hat{p}^c(t, v), & \text{on } [0, T) \times (K_t, \infty) \end{cases},$$

is the viscosity solution of (5.5.12), where $\hat{p}^c$ solves (5.5.7). Indeed, since $\hat{p}^c_1 = v$ and $\hat{p}^c_2 = K_t$ are the supersolution and subsolution of (5.5.7), using the comparison principle, we can obtain that $K_t < \hat{p}^c \leq v$ on $[0, T) \times (0, K_t)$, which implies that

$$\partial_v p^c(t, K_t+) \leq 1. \tag{5.5.16}$$

It is sufficient to show p^c is a viscosity supersolution of the equation

$$-\partial_t p^c - \hat{\mathcal{L}}_2 p^c = 0,$$

which can be proved using the similar arguments of Case (b). This implies that

$$\mathcal{S} = \{(t, v) \in [0, T) \times \mathbb{R}^+ : v \le K_t\},$$

and moreover, we solve PDE (5.5.7) in Appendix 5.A.2 and obtain that

$$P(x, v) = p^c(0, v) = \begin{cases} v & v \le K \\ \hat{p}^c(0, v) & v > K \end{cases}$$

where $\hat{p}^c$ is given by (5.A.4).

Case (d).

If $r > 0$ and $\eta > \xi$, we can see $p_1^c = K_t$ is the subsolution of of (5.5.12) on $[0, T) \times (0, K_t)$ and $p_2^c = v$ is the subsolution of (5.5.12) on $[0, T) \times (K_t, \infty)$. Using the comparison principle, we can obtain that $p^c > v$ on $[0, T) \times (0, K_t)$ and $p^c > K_t$ on $[0, T) \times (K_t, \infty)$. This implies that the possible set in $\mathcal{S}$ is the curve $v = K_t$.

The key point to prove whether $\mathcal{S}$ is the whole curve $v = K_t$ (i.e. $\tilde{p}^c$ in (5.5.9) is the solution of (5.5.12)) is the sign of the function f in (5.5.8). Thanks to the explicit form of $\overline{p}^c$ and $\hat{p}^c$, which are given by (5.A.3) and (5.A.4) respectively, standard calculation shows that

$$f(t) = \partial_v \overline{p}^c(t, K_t-) - \partial_v \hat{p}^c(t, K_t+). \tag{5.5.17}$$

In other words, if $f(t) \ge 0$ for all $t \in [0, T]$, $\tilde{p}^c$ is concave at (t, K_t), and then $\tilde{p}^c$ in (5.5.9) is the solution of (5.5.12). To this end, we start with the following lemma on the monotonicity property of the auxiliary function f.

Lemma 5.5.3 *$f(t)$ in (5.5.8) is increasing with $f(T) = 1$.*

Proof. By the transformation

$$\overline{u}(\tau, w) = e^{-r(T-\tau)} \overline{p}^c(T - \tau, K_{T-\tau} e^w) - K, \tag{5.5.18}$$

it follows from (5.5.6) that $\overline{u}(\tau, w)$ satisfies

$$\begin{cases} \partial_\tau \overline{u} - \frac{1}{2}\sigma_V^2 \partial_{ww}^2 \overline{u} - \left(\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2\right) \partial_w \overline{u} - r\overline{u} = rK & \text{on } (0, T] \times (-\infty, 0) \\ \overline{u}(0, w) = K(e^w - 1) & \text{on } (-\infty, 0] \\ \overline{u}(\tau, 0) = 0 & \text{on } (0, T] \end{cases},$$

and it is easy to prove that $\partial_\tau \overline{u} \ge 0$. Thus, for any $0 \le t_1 < t_2 \le T$, by the definition of left derivative and the fact that $\partial_v \overline{p}^c(t, K_t-) = \frac{1}{K}\partial_w \overline{u}^c(T-t, 0-)$,

$$
\begin{aligned}
& \partial_v \overline{p}^c(t_1, K_{t_1}-) - \partial_v \overline{p}^c(t_2, K_{t_2}-) \\
= & \frac{1}{K} \cdot \lim_{\epsilon \to 0-} \frac{1}{\epsilon}\left[(\overline{u}^c(T-t_1, \epsilon) - \overline{u}^c(T-t_1, 0)) - (\overline{u}^c(T-t_2, \epsilon) - \overline{u}^c(T-t_2, 0))\right] \\
= & \frac{1}{K} \cdot \lim_{\epsilon \to 0-} \frac{1}{\epsilon}\left[\overline{u}^c(T-t_1, \epsilon) - \overline{u}^c(T-t_2, \epsilon)\right] \\
\leq & \ 0,
\end{aligned}
$$

which proves $\partial_v \overline{p}^c(t, K_t-)$ is increasing in t. Likewise, we can prove $\partial_v \hat{p}^c(t, K_t+)$ is decreasing in t. Using (5.5.17), we have $f(t)$ is increasing in t, with

$$
f(T) = \partial_v \overline{p}^c(T, K_T-) - \partial_v \hat{p}^c(T, K_T+) = 1 - 0 = 1.
$$

■

As a direct consequence, by the definition (5.5.10) of t_*^c, we have $f(t) \geq 0$ for $t \in [t_*^c, T]$. We are now in a position to prove Case (d).

On the one hand, if $f(0) \geq 0$, it follows that $\partial_{vv}^2 \tilde{p}^c(t, K_t) \leq 0$ for any $t \in [0, T)$, by (5.5.6)-(5.5.7), and thus

$$
-\partial_t \tilde{p}^c - \hat{\mathcal{L}}_2 \tilde{p}^c \geq 0
$$

on $[0, T) \times (0, \infty)$. Moreover, we have $\tilde{p}^c(t, v) \geq \phi(t, v)$, where the equality only holds on the curve $v = K_t$. This gives that $\mathcal{S} = \{(t, v) \in [0, T) \times \mathbb{R}^+ : v = K_t\}$.

On the other hand, if $f(0) < 0$, then $t_*^c \in (0, T)$. It is clear that $\check{p}^c$ is the solution of (5.5.12) on $[t_*^c, T]$. To prove $\check{p}^c$ is the solution of (5.5.12) on $[0, t_*^c)$, we only need to prove that $\check{p}^c \geq \phi(t, v)$. Indeed, note that

$$
\partial_t \overline{p}^c(t_*^c, K_{t_*^c}) = \partial_t \hat{p}^c(t_*^c, K_{t_*^c}) \text{ and } \partial_v \overline{p}^c(t_*^c, K_{t_*^c}-) = \partial_v \hat{p}^c(t_*^c, K_{t_*^c}+),
$$

and thus $\partial_{vv}^2 \overline{p}^c(t_*^c, K_{t_*^c}-) = \partial_{vv}^2 \hat{p}^c(t_*^c, K_{t_*^c}+)$ by (5.5.6)-(5.5.7), which implies $\partial_{vv}^2 \check{p}^c(t_*^c, K_{t_*^c})$ exists. Again, using the transformation

$$
\check{u}(\tau, w) = e^{-r(T-\tau)} \check{p}^c(T-\tau, K_{T-\tau}e^w) - K, \tag{5.5.19}
$$

we can have $\partial_\tau \check{u}^c(\tau, w)$ satisfies

$$
\partial_\tau(\partial_\tau \check{u}^c) - \frac{1}{2}\sigma_V^2 \partial_{ww}^2(\partial_\tau \check{u}^c) - \left(\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2\right)\partial_w(\partial_\tau \check{u}^c) - r(\partial_\tau \check{u}^c) = 0
$$

on $(T - t_*^c, T] \times (-\infty, \infty)$, with the initial condition

$$
\partial_\tau \check{u}^c(T - t_*^c, w) = \partial_\tau \overline{u}^c(T - t_*^c, w) \geq 0,
$$

and it follows from the comparison principle that $\partial_\tau \check{u}^c(\tau, w) > 0$ on $(T - t_*^c, T] \times (-\infty, \infty)$. In turn,

$$\check{u}^c(\tau, w) > \overline{u}^c(T - t_*^c, w)$$

for $(\tau, w) \in (T - t_*^c, T] \times (-\infty, \infty)$, or equivalently,

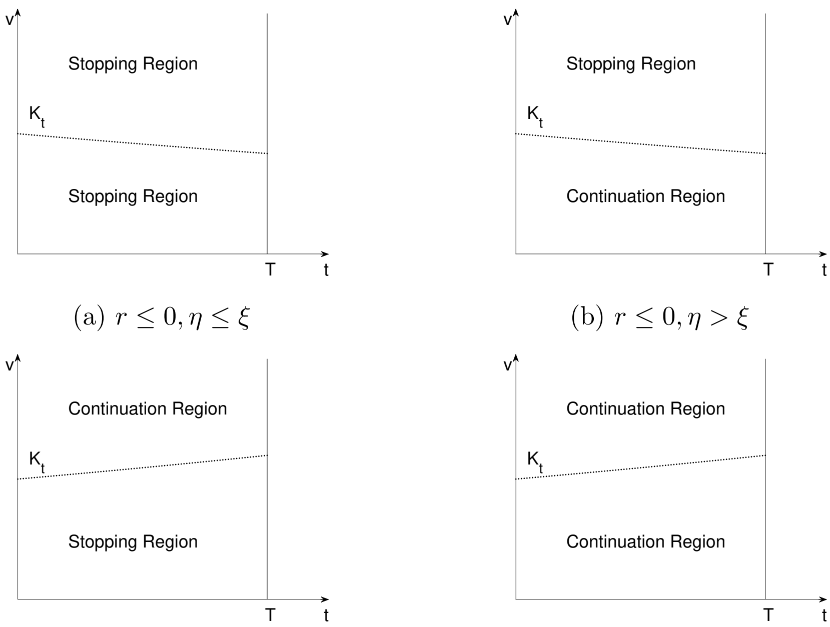
$$\check{p}^c(t, v) > e^{r(t-t_*^c)} \check{p}^c(t_*^c, e^{-r(t-t_*^c)} v) \geq \phi(t, v)$$

for $(t, v) \in [0, t_*^c) \times (0, \infty)$. Also, this gives $\check{p}^c(t, K_t) > K_t$ for $t \in [0, t_*^c)$, i.e.

$$\mathcal{S} = \{(t, v) \in [t_*^c, T) \times \mathbb{R}^+ : v = K_t\}.$$

The proof of Theorem 5.5.2 is now completed.

Different optimal stopping strategies of the rollover debt without stopping constraints in the compelet market, where $\rho = 1$, are depicted in Figure 5.1. The theoretical analysis in Theorem 5.5.2 shows that the form of the stopping and continuation regions (at least excluding the curve $v = K_t$) is determined by the signs of r and $\eta - \xi$. An interesting observation is that both regions are swapped over for different parameter values. The financial intuition behind it will be further discussed in the next section.



(a) $r \leq 0, \eta \leq \xi$ (b) $r \leq 0, \eta > \xi$

(c) $r > 0, \eta \leq \xi$ (d) $r > 0, \eta > \xi$

Figure 5.1: Different optimal stopping strategies of the rollover debt without stopping constraints in a compelet market.

5.6 Numerical results for optimal stopping strategy

In this section, we investigate the optimal stopping strategy of the mixed stochastic control and constrained optimal stopping problem in (5.2.3) with exponential performance utility (5.4.1). Similar to Section 5.5.2, we define the stopping region ($\mathcal{S}^\lambda$) and the continuation region ($\mathcal{C}^\lambda$) for the problem:

$$\mathcal{S}^\lambda := \{(t,v) \in [0,T) \times \mathbb{R}^+ : p(t,v) \le \phi(t,v)\} \tag{5.6.1}$$

$$\mathcal{C}^\lambda := \{(t,v) \in [0,T) \times \mathbb{R}^+ : p(t,v) > \phi(t,v)\} \tag{5.6.2}$$

where p is the solution to (5.4.2), and therefore, the optimal stopping strategy τ^*, given by (5.4.3), has the following representation

$$\tau^* = \inf\left\{T_i \ge T_1 : (T_i, V_{T_i}) \in \mathcal{S}^\lambda\right\} \wedge T_M.$$

Inspired by Theorem 5.5.2, we can expect the form of the stopping and continuation regions depends on parameter values. The following corollary gives some parts of the stopping regions for different parameter values, and this result is consistent with what was obtained in Section 5.5.2.

Corollary 5.6.1 *If $r \le 0$, the following statement holds*

$$\{(t,v) \in [0,T) \times \mathbb{R}^+ : v \ge K_t\} \subset \mathcal{S}^\lambda.$$

If $\eta \le \rho\xi$, the following statement holds

$$\{(t,v) \in [0,T) \times \mathbb{R}^+ : v \le K_t\} \subset \mathcal{S}^\lambda.$$

Proof. In the case of $r \le 0$, we can see $p_1 = K_t$ is the supersolution of (5.4.2). Using the comparison principle, we can obtain that $p \le K_t$. This implies $p \le \phi$ on $\{(t,v) \in [0,T) \times \mathbb{R}^+ : v \ge K_t\}$, and therefore, the first statement holds. Similarly, we can prove the second statement using the fact that $p_2 = v$ is the supersolution of (5.4.2) if $\eta \le \rho\xi$. ■

The financial intuition behind this corollary is as follows. On the one hand, with a nonpositive spread rate (i.e. $r \le 0$), the creditor should withdraw her money as soon as the firm has enough debt-paying ability to cover its debt obligations. On the other hand, when the bond fails to provide better opportunities than the risky asset S to obtain a greater payoff in the future (i.e. $\eta \le \rho\xi$), given the firm's weak debt-paying potential, it is unwise for the creditor to wait for the firm to improve its value, and therefore, her best interest is to withdraw her money as soon as the firm is insolvent.

At the end of this chapter, we conduct some numerical experiments to examine the shapes of $\mathcal{S}^\lambda$ (the stopping region) and $\mathcal{C}^\lambda$ (the continuation region), which are defined by (5.6.1) and (5.6.2) respectively, for different parameter values, where a numerical algorithm to approximate the solution of PDE (5.4.2) is provided in Appendix 5.B.

The structure of $\mathcal{S}^\lambda$ in the case of $r \leq 0$

If $\eta \leq \rho\xi$, as a direct consequence of Corollary 5.6.1, we can obtain

$$\mathcal{S}^\lambda = [0,T) \times \mathbb{R}^+$$

which means the creditor should withdraw her money as soon as possible. As shown in Figure 5.2a, with three simulated firm value paths and three (given) Poisson times T_1=.15, T_2=.5 and T_3=.8, she should choose to withdraw her money at time .15 for all the paths.

However, with higher value of $\eta - \rho\xi$ (other parameters being fixed), the bond starts to provide better opportunities than the risky asset S, then the creditor's best interest is to continue to roll over the debt, even when the firm is currently insolvent, to wait for the firm to improve its value in the future. In Figure 5.2b, we can observe that

$$\{(t,v) \in [0,T) \times \mathbb{R}^+ : v \geq K_t\} \subset \mathcal{S}^\lambda = \{(t,v) \in [0,T) \times \mathbb{R}^+ : v \geq v^*(t)\}$$

where $v^* : [0,T) \to \mathbb{R}^+$ is the optimal withdrawal boundary. With three simulated firm value paths and three (given) Poisson times T_1=.15, T_2=.5 and T_3=.8, she should choose to withdraw her money at time .15, .8 and 1, respectively.

The structure of $\mathcal{S}^\lambda$ in the case of $r > 0$

If $\eta \leq \rho\xi$, Corollary 5.6.1 gives that

$$\{(t,v) \in [0,T) \times \mathbb{R}^+ : v \leq K_t\} \subset \mathcal{S}^\lambda,$$

which implies that the creditor should withdraw her money if the firm is insolvent. In Figure 5.2c, numerically, we can observe

$$\{(t,v) \in [0,T) \times \mathbb{R}^+ : v \leq K_t\} \subset \mathcal{S}^\lambda = \{(t,v) \in [0,T) \times \mathbb{R}^+ : v \leq v^*(t)\}$$

where $v^* : [0,T) \to \mathbb{R}^+$ is the optimal withdrawal boundary. The creditor should choose to continue when the firm value exceeds than the boundary v^*. This makes sense since, with a high firm value, the creditor is always better off by not withdrawing her money and continuing to receive positive interest rate spread. With three simulated firm value paths and three (given) Poisson times

T_1=.15, T_2=.5 and T_3=.8, she should choose to withdraw her money at time .15 for all the paths.

Similar to the case of $r \leq 0$, by increasing the value of $\eta - \rho\xi$ and keeping other parameters constant, we can observe, in Figure 5.2d, there exist two seperate continuation regions and one stopping region, i.e.

$$\{(t,v) \in [0,T)\times\mathbb{R}^+ : v = K_t\} \subset \mathcal{S}^\lambda = \{(t,v) \in [0,T)\times\mathbb{R}^+ : \underline{v}^*(t) \leq v \leq \overline{v}^*(t)\}$$

where $\underline{v}^* : [0,T) \to \mathbb{R}^+$ and $\overline{v}^* : [0,T) \to \mathbb{R}^+$ are the lower and upper optimal withdrawal boundary, respectively. It is in her best interest to continue to roll over the debt when the firm value falls below the lower boundary or above the upper boundary. The creditor will choose to withdraw her money when the firm value is relatively close to the sum of its debt obligations K_t, since the gamma at K_T is minus infinity at T, which means extremly serious loss will happen. With three simulated firm value paths and three (given) Poisson times T_1=.15, T_2=.5 and T_3=.8, she should choose to withdraw her money at time .15, 1 and 1, respectively.

However, if we continue to increase the value of $\eta - \rho\xi$, we can observe, in Figure 5.2e, there are only one continuation region and one stopping region, i.e.

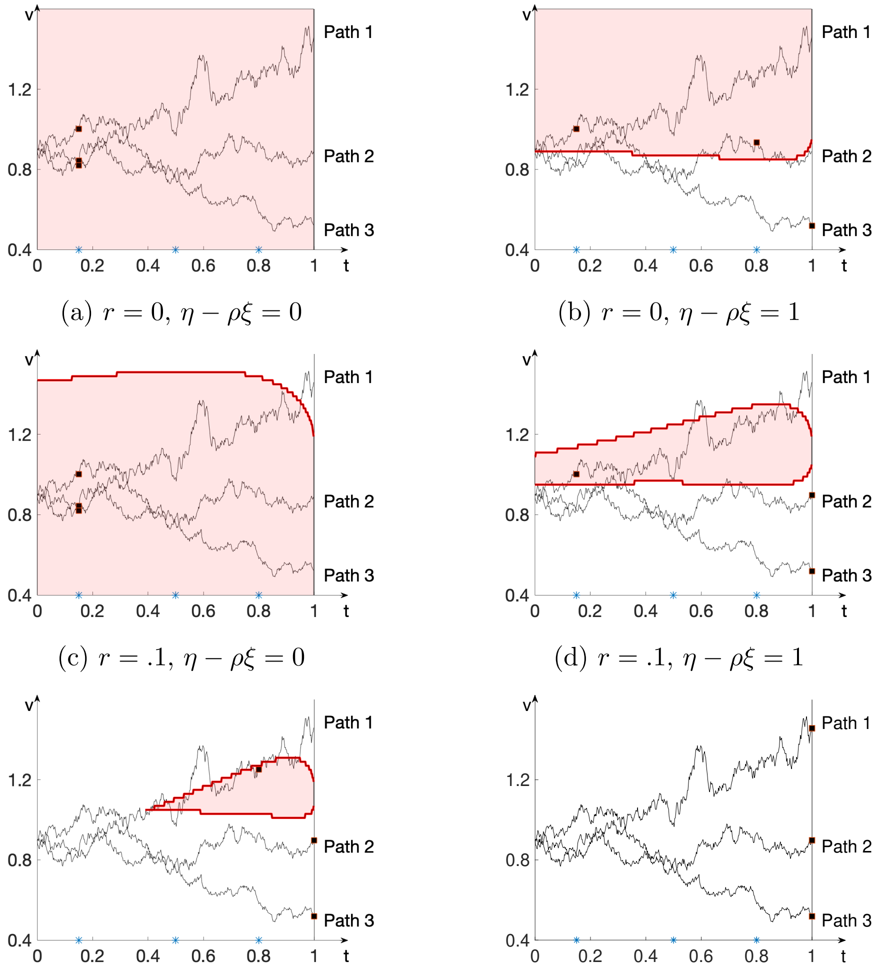
$$\{(t,v) \in [t^*,T)\times\mathbb{R}^+ : v = K_t\} \subset \mathcal{S}^\lambda = \{(t,v) \in [t^*,T)\times\mathbb{R}^+ : \underline{v}^*(t) \leq v \leq \overline{v}^*(t)\}$$

where $t^* \in [0,T)$ is the optimal withdrawal temporal boundary satisfying $\underline{v}^*(t^*) = \overline{v}^*(t^*)$, and $\underline{v}^* : [t^*,T) \to \mathbb{R}^+$ and $\overline{v}^* : [t^*,T) \to \mathbb{R}^+$ are the lower and upper optimal withdrawal boundary, respectively. Up to the optimal withdrawal temporal boundary t^*, she should continue to roll over the debt regardless of the firm value. After t^*, it is in her best interest to withdraw the money when the firm value lies between the lower and upper boundaries. With three simulated firm value paths and three (given) Poisson times T_1=.15, T_2=.5 and T_3=.8, she should choose to withdraw her money at time .8, 1 and 1, respectively.

If we further increase the value of $\eta - \rho\xi$, in Figure 5.2f, we can observe the stopping region disappears and the whole region is the continuation region, i.e.

$$\mathcal{S}^\lambda = \emptyset$$

which means the creditor should continue to roll over the debt regardless of the firm value. With three simulated firm value paths and three (given) Poisson times T_1=.15, T_2=.5 and T_3=.8, she should never choose to withdraw her money until time 1 for all the paths.



(a) $r = 0,\ \eta - \rho\xi = 0$

(b) $r = 0,\ \eta - \rho\xi = 1$

(c) $r = .1,\ \eta - \rho\xi = 0$

(d) $r = .1,\ \eta - \rho\xi = 1$

(e) $r = .1,\ \eta - \rho\xi = 1.5$

(f) $r = .1,\ \eta - \rho\xi = 50$

Figure 5.2: Scenario simulation. These figures show the shapes of the stopping and continuation regions of a rollover debt with different parameter values. Three firm value paths in the model are simulated. The initial firm value is set to be $v = .9$ and other parameter values are σ_S=.2, $\eta = .67$, $\sigma_V = .3$, $\rho = -.5$, $\gamma = 1$, $\lambda = 5$, $K = 1$ and $T = 1$. The red curves and shaded areas describe the optimal withdrawal boundaries and the stopping regions for each cases. Three Poisson times (marked by the asterisks) is given by $T_1 = .15$, $T_2 = .5$ and $T_3 = .8$, and the optimal withdrawal strategies for the creditor are marked by the black squares.

5.A The explicit solutions of PDEs (5.5.6) and (5.5.7)

5.A.1 PDE (5.5.6)

By the transformation

$$w(t,v) = \ln\left(\frac{K_t}{v}\right), \quad \tau(t,v) = T - t, \quad u(\tau, w) = e^{-rt}\overline{p}^c(t,v) - K,$$

the problem (5.5.6) is reduced to

$$\begin{cases} \partial_\tau u - \frac{1}{2}\sigma_V^2 \partial_{ww}^2 u + \left(\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2\right)\partial_w u - ru = rK & \text{on } (0,T] \times (0,\infty) \\ u(0,w) = K(e^{-w} - 1) & \text{on } [0,\infty) \\ u(\tau, 0) = 0 & \text{on } (0,T]. \end{cases}$$

We define $\bar{u}(\tau, w)$ by

$$u(\tau, w) = \bar{u}(\tau, w)e^{\alpha\tau + \beta w}$$

where

$$\alpha = r - \frac{1}{2\sigma_V^2}\left(\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2\right)^2, \quad \beta = \frac{1}{\sigma_V^2}\left(\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2\right), \tag{5.A.1}$$

and thus $\bar{u}(\tau, w)$ satisfies

$$\begin{cases} \partial_\tau \bar{u} - \frac{1}{2}\sigma_V^2 \partial_{ww}^2 \bar{u} = rK & \text{on } (\tau, w) \in (0,T] \times (0,\infty) \\ \bar{u}(0,w) = K\left(e^{-(1+\beta)w} - e^{-\beta w}\right) & \text{on } w \in [0,\infty) \\ \bar{u}(\tau, 0) = 0 & \text{on } \tau \in (0,T]. \end{cases}$$

Using the fundamental solution to heat equations, we can have

$$\begin{aligned} \bar{u}(\tau, w) = &\int_0^\infty \frac{1}{\sqrt{2\sigma_V^2\pi\tau}}\left(e^{-\frac{(w-y)^2}{2\sigma_V^2\tau}} - e^{-\frac{(w+y)^2}{2\sigma_V^2\tau}}\right)\left(Ke^{-(1+\beta)y} - Ke^{-\beta y}\right)dy \\ &+ \int_0^\tau \int_0^\infty \frac{1}{\sqrt{2\sigma_V^2\pi(\tau - s)}}\left(e^{-\frac{(w-y)^2}{2\sigma_V^2(\tau-s)}} - e^{-\frac{(w+y)^2}{2\sigma_V^2(\tau-s)}}\right) rK\, dy\, ds. \end{aligned}$$

Standard calculations yield that

$$\begin{aligned} &\int_0^\infty \frac{1}{\sqrt{2\sigma_V^2\pi\tau}}\left(e^{-\frac{(w-y)^2}{2\sigma_V^2\tau}} - e^{-\frac{(w+y)^2}{2\sigma_V^2\tau}}\right)e^{\delta y}\, dy \\ &= e^{\frac{1}{2}\sigma_V^2\delta^2\tau}\left(e^{\delta w}\Phi\left(\frac{w + \sigma_V^2\delta\tau}{\sigma_V\sqrt{\tau}}\right) - e^{-\delta w}\Phi\left(\frac{-w + \sigma_V^2\delta\tau}{\sigma_V\sqrt{\tau}}\right)\right), \end{aligned} \tag{5.A.2}$$

and then we can obtain that

$$\bar{u}(\tau,w) = e^{(r-\alpha)\tau-\beta w}K\Big[-\Phi\left(\bar{d}_1\right)+e^{2\beta w}\Phi\left(\bar{d}_2\right)-e^{(1+2\beta)(\frac{1}{2}\sigma_V^2\tau+w)}\Phi\left(\bar{d}_3\right)$$
$$+e^{\frac{1}{2}\sigma_V^2(1+2\beta)\tau-w}\Phi\left(\bar{d}_4\right)+re^{-(r-\alpha)\tau+\beta w}\int_0^\tau \Phi\left(\bar{d}_5\right)-\Phi\left(\bar{d}_6\right)\,ds\Big]$$

where $\Phi(\cdot)$ is the cumulative distribution function for a standard normal distribution, and

$$\begin{aligned}
\bar{d}_1 &= \frac{w-\sigma_V^2\beta\tau}{\sigma_V\sqrt{\tau}}\\
\bar{d}_2 &= -d_1-2\sigma_V\beta\sqrt{\tau}\\
\bar{d}_3 &= d_2-\sigma_V\sqrt{\tau}\\
\bar{d}_4 &= d_1-\sigma_V\sqrt{\tau}\\
\bar{d}_5 &= \frac{w}{\sigma_V\sqrt{\tau-s}}\\
\bar{d}_6 &= -d_5.
\end{aligned}$$

After rearranging the equations, we can obtain

$$\begin{aligned}
&\overline{p}^c(t,v)\\
&= K_t - K_T\Phi\left(d_1\right)+e^{r(T-t)}v\left(\frac{v}{K_t}\right)^{-\frac{\sigma_V(\eta-\xi)-r}{\sigma_V^2/2}}\Phi\left(d_2\right)\\
&\quad - K_te^{\sigma_V(\eta-\xi)(T-t)}\left(\frac{v}{K_t}\right)^{-\frac{\sigma_V(\eta-\xi)-r}{\sigma_V^2/2}}\Phi\left(d_3\right)+ve^{\sigma_V(\eta-\xi)(T-t)}\Phi\left(d_4\right)\\
&\quad + rK_Te^{-\frac{(\sigma_V(\eta-\xi)-r-\frac{1}{2}\sigma_V^2)^2}{2\sigma_V^2}(T-t)}\left(\frac{v}{K_t}\right)^{-\frac{\sigma_V(\eta-\xi)-r-\frac{1}{2}\sigma_V^2}{\sigma_V^2}}\int_0^{T-t}\Phi\left(d_5\right)-\Phi\left(d_6\right)\,ds
\end{aligned} \tag{5.A.3}$$

where

$$\begin{aligned}
d_1(t,v) &= \frac{\ln\left(\frac{K_t}{v}\right)-\left(\sigma_V(\eta-\xi)-r-\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_2(t,v) &= \frac{-\ln\left(\frac{K_t}{v}\right)-\left(\sigma_V(\eta-\xi)-r-\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_3(t,v) &= \frac{-\ln\left(\frac{K_t}{v}\right)-\left(\sigma_V(\eta-\xi)-r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_4(t,v) &= \frac{\ln\left(\frac{K_t}{v}\right)-\left(\sigma_V(\eta-\xi)-r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_5(s,t,v) &= \frac{\ln\left(\frac{K_t}{v}\right)}{\sigma_V\sqrt{T-t-s}}\\
d_6(s,t,v) &= \frac{-\ln\left(\frac{K_t}{v}\right)}{\sigma_V\sqrt{T-t-s}}.
\end{aligned}$$

5.A.2 PDE (5.5.7)

By the transformation

$$w(t,v) = \ln\left(\frac{v}{K_t}\right), \quad \tau(t,v) = T - t, \quad u(\tau, w) = e^{-rt}\overline{\overline{p}}^c(t,v) - K,$$

the problem (5.5.7) is reduced to

$$\begin{cases} \partial_\tau u - \frac{1}{2}\sigma_V^2 \partial_{ww}^2 u - \left(\sigma_V(\eta - \xi) - r - \frac{1}{2}\sigma_V^2\right)\partial_w u - ru = rK & \text{on } (0,T] \times (0,\infty) \\ u(0,w) = 0 & \text{on } [0,\infty) \\ u(\tau, 0) = 0 & \text{on } (0,T]. \end{cases}$$

We define $\bar{u}(\tau, w)$ by

$$u(\tau,w) = \bar{u}(\tau,w)e^{\bar{\alpha}\tau + \bar{\beta}w}$$

with $\bar{\alpha} = \alpha$ and $\bar{\beta} = -\beta$, where α and β are given by (5.A.1), and thus $\bar{u}(\tau, w)$ satisfies

$$\begin{cases} \partial_\tau \bar{u} - \frac{1}{2}\sigma_V^2 \partial_{ww}^2 \bar{u} = rK & \text{on } (\tau, w) \in (0,T] \times (0,\infty) \\ \bar{u}(0,w) = 0 & \text{on } w \in [0,\infty) \\ \bar{u}(\tau,0) = 0 & \text{on } \tau \in (0,T]. \end{cases}$$

Using the fundamental solution to heat equations and (5.A.2), we can have

$$\begin{aligned} \bar{u}(\tau, w) &= \int_0^\tau \int_0^\infty \frac{1}{\sqrt{2\sigma_V^2 \pi(\tau - s)}} \left(e^{-\frac{(w-y)^2}{2\sigma_V^2(\tau-s)}} - e^{-\frac{(w+y)^2}{2\sigma_V^2(\tau-s)}} \right) rK \, dy \, ds \\ &= rK \int_0^\tau \Phi\left(\bar{d}_7\right) - \Phi\left(\bar{d}_8\right) \, ds \end{aligned}$$

where $\bar{d}_7 = \frac{w}{\sigma_V \sqrt{\tau - s}}$ and $\bar{d}_8 = -d_7$. After rearranging the equations, we can obtain that

$$\begin{aligned} &\overline{\overline{p}}^c(t,v) \\ &= K_t + rK_T e^{-\frac{\left(\sigma_V(\eta-\xi) - r - \frac{1}{2}\sigma_V^2\right)^2}{2\sigma_V^2}(T-t)} \left(\frac{v}{K_t}\right)^{-\frac{\sigma_V(\eta-\xi)-r-\frac{1}{2}\sigma_V^2}{\sigma_V^2}} \int_0^{T-t} \Phi\left(d_7\right) - \Phi\left(d_8\right) \, ds \end{aligned} \tag{5.A.4}$$

where

$$\begin{aligned} d_7(s,t,v) &= \frac{\ln\left(\frac{v}{K_t}\right)}{\sigma_V\sqrt{T-t-s}} \\ d_8(s,t,v) &= \frac{-\ln\left(\frac{v}{K_t}\right)}{\sigma_V\sqrt{T-t-s}}. \end{aligned}$$

5.B Numerical approximation of the solution to PDE (5.4.2)

We now use finite difference method to numerically solve PDE (5.4.2).

By letting $a := \frac{1}{2}\sigma_V^2$, $b := \sigma_V(\eta - \rho\xi)$ and $c = -\frac{1}{2}\gamma(1-\rho^2)\sigma_V^2$, we can rewrite (5.4.2) as

$$-\partial_t p - av^2\partial_{vv}^2 p - bv\partial_v p - cv^2(\partial_v p)^2 + \lambda\gamma^{-1}\min\{e^{-\gamma(\phi(t,v)-p)} - 1, 0\} = 0 \quad (5.B.1)$$

with the terminal condition $p(T,v) = \phi(T,v)$. In order to solve (5.B.1) numerically, it is necessary to impose the boundary conditions at $v = 0+$ and $v = +\infty$. When $v = 0+$, the payoff at withdrawal time is expected to be negligible, which gives the boundary condition at $v = 0+$:

$$\lim_{v\to 0+} p(t,v) = 0.$$

When $v = +\infty$, the payoff at withdrawal time τ is expected to be K_τ, which is increasing with time if $r \geq 0$ and decreasing with time otherwise. This motivates the creditor to postpone her withdrawal until time T if $r \geq 0$, i.e.

$$\lim_{v\to +\infty} p(t,v) = K_T \quad (5.B.2)$$

in the case of $r \geq 0$. However, it is in her best interest to withdraw her money as soon as possible if $r < 0$, this gives that

$$\lim_{v\to +\infty} p(t,v) = \mathbb{E}[K_{T_i\wedge T} | V_t = v]$$

where T_i is the first arrival Poisson time after time t, i.e. $T_{i-1} \leq t < T_i$. By applying the probability density function of T_i, we can obtain

$$\lim_{v\to +\infty} p(t,v) = K\int_t^\infty \lambda e^{-\lambda(s-t)} e^{rs\wedge T}\, ds = \frac{\lambda K_t - re^{-\lambda(T-t)}K_T}{\lambda - r} \quad (5.B.3)$$

in the case of $r < 0$. In the following, we only consider the algorithm in the case of $r \geq 0$ for simplicity, since the algorithm for $r < 0$ follows immediately by replacing the boundary condition (5.B.2) with (5.B.3).

In order to remove the square term in (5.B.1), we define

$$\tilde{p}(t,v) := e^{\frac{c}{a}p(t,v)} = e^{-\gamma(1-\rho^2)p(t,v)}$$

which satisfies the following equation

$$\frac{\partial \tilde{p}}{\partial t} + av^2\frac{\partial^2 \tilde{p}}{\partial v^2} + bv\frac{\partial \tilde{p}}{\partial v} + \lambda(1-\rho^2)\min\left\{e^{-\gamma\phi(t,v)}\tilde{p}^{1-\frac{1}{1-\rho^2}} - \tilde{p}, 0\right\} = 0,$$

with correponding terminal and boundary conditions.

We choose $\overline{v}$ is a very large constant and $\underline{v}$ is a very small constant such that realization of v outside the region $[\underline{v}, \overline{v}]$ occurs with negligible probability. By defining

$$w = \ln\left(v/\underline{v}\right), \quad \tau = T - t, \quad W(\tau, w) = \tilde{p}(t, v),$$

we can have

$$\frac{\partial W}{\partial \tau} = a\frac{\partial^2 W}{\partial w^2} + (b-a)\frac{\partial W}{\partial w} + \lambda(1-\rho^2)\min\left\{e^{-\gamma\phi(T-\tau,\underline{v}e^w)}W^{1-\frac{1}{1-\rho^2}} - W, 0\right\} \tag{5.B.4}$$

with the initial and boundary conditions

$$\begin{aligned} W(0,w) &= e^{-\gamma(1-\rho^2)\phi(T,\underline{v}e^w)} \\ W(\tau,0) &= 1 \\ W(\tau,\ln(\overline{v}/\underline{v})) &= e^{-\gamma(1-\rho^2)K_T}. \end{aligned}$$

In the following, we derive the implict finite difference equation for PDE (5.B.4). Let $\Delta\tau$ denote the step size between two updates of the value function W in the time dimension, and Δw denote the step size between grid points in the space dimension of W. The range of two variables is taken to be $(\tau, w) \in [0,T]\times[0,\overline{w}]$, where $\overline{w} = \ln\left(\overline{v}/\underline{v}\right)$. At each grid point, we define

$$W_{(i,j)} = W(i\Delta\tau, j\Delta w),$$

for $0 \le i \le \frac{T}{\Delta\tau} := N_\tau$ and $0 \le j \le \frac{\overline{w}}{\Delta w} := N_w$, satisfying the implicit finite difference equation

$$\begin{aligned} &\frac{W_{(i+1,j)} - W_{(i,j)}}{\Delta\tau} \\ &= a\frac{W_{(i+1,j+1)} - 2W_{(i+1,j)} + W_{(i+1,j-1)}}{\Delta w^2} + (b-a)\frac{W_{(i+1,j+1)} - W_{(i+1,j-1)}}{2\Delta w} \\ &\quad + \lambda(1-\rho^2)\min\left\{e^{-\gamma\phi(T-\tau,\underline{v}e^{j\Delta w})}W_{(i+1,j)}^{1-\frac{1}{1-\rho^2}} - W_{(i+1,j)}, 0\right\} \end{aligned}$$

which can be written as the following nonlinear algebraic equation

$$AW_{i+1} - \overline{\xi}\min\left\{\left(\overline{\eta}_{i+1}, W_{i+1}^{1-\frac{1}{1-\rho^2}}\right) - W_{i+1}, 0\right\} = C_i \tag{5.B.5}$$

where the constant $\overline{\xi} := \lambda(1-\rho^2)\Delta\tau$, the vectors of $N_w - 1$ elements

$$\begin{aligned} W_i &:= \left[W_{(i,1)}, W_{(i,2)}, \cdots, W_{(i,N_w-1)}\right]^T \\ C_i &:= W_i - \left[\alpha W_{(i+1,0)}, 0, \ldots, 0, \beta W_{(i+1,N_w)}\right]^T \\ \overline{\eta}_{i+1} &:= \left[e^{-\gamma\phi(T-i\Delta\tau,\underline{v}e^{1\cdot\Delta w})}, \cdots, e^{-\gamma\phi(T-i\Delta\tau,\underline{v}e^{j\Delta w})}, \cdots, e^{-\gamma\phi(T-i\Delta\tau,\underline{v}e^{(N_w-1)\cdot\Delta w})}\right]^T \end{aligned}$$

and A is a $(N_w - 1) \times (N_w - 1)$ tridiagonal matrix

$$A := \begin{pmatrix} \theta & \beta & 0 & 0 \\ \alpha & \theta & \beta & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \theta & \beta \\ 0 & \cdots & \alpha & \theta \end{pmatrix}, \qquad (5.B.6)$$

with $\alpha := -\frac{a\Delta\tau}{\Delta w^2} + \frac{(b-a)\Delta\tau}{2\Delta w}$, $\beta := -\frac{a\Delta\tau}{\Delta w^2} - \frac{(b-a)\Delta\tau}{2\Delta w}$, and $\theta := 1 - \alpha - \beta$. The corresponding initial and boundary conditions of (5.B.5) are

$$\begin{aligned} W_{(0,j)} &= e^{-\gamma(1-\rho^2)\phi(T,\underline{v}e^{j\Delta w})} \\ W_{(i,0)} &= 1 \\ W_{(i,N_w)} &= e^{-\gamma(1-\rho^2)K_T}. \end{aligned}$$

Finally, for $i = 0, 1, \cdots, N_\tau$, we use the standard Newton method to solve (5.B.5) as follows:

Step 1. Set $\tilde{W}_1 = W_i$;

Step 2. For $m = 0, 1, 2, \cdots$, solve $\tilde{W}_{m+1}$ recursively by the following equation

$$A\tilde{W}_m - \overline{\xi}\min\left\{\left(\overline{\eta}_{i+1}, \tilde{W}_m^{1-\frac{1}{1-\rho^2}}\right) - \tilde{W}_m, 0\right\} + B_m(\tilde{W}_{m+1} - \tilde{W}_m) = C_i,$$

which is equivalent to

$$\tilde{W}_{m+1} = \tilde{W}_m + B_m^{-1}\left(C_i - A\tilde{W}_m + \overline{\xi}\min\left\{\left(\overline{\eta}_{i+1}, \tilde{W}_m^{1-\frac{1}{1-\rho^2}}\right) - \tilde{W}_m, 0\right\}\right),$$

until $\sup|\tilde{W}_{m+1} - \tilde{W}_m| < eps$, where

$$\begin{aligned} &B_m \\ &= A - \overline{\xi}\begin{pmatrix} \min(\eta_{i+1,1}\tilde{W}_{(m,1)}^{-\frac{1}{1-\rho^2}} - 1, 0) & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & \min(\eta_{i+1,N_w-1}\tilde{W}_{(m,N_w-1)}^{-\frac{1}{1-\rho^2}} - 1, 0) \end{pmatrix}; \end{aligned}$$

Step 3. Suppose the above loop runs M times, then set $W_{i+1} = \tilde{W}_M$.

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