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A UNIQUE PERFECT POWER DECAGONAL NUMBER

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Abstract

Let $\mathcal{P}_s(n)$ denote the nth s-gonal number. We consider the equation

$$\mathcal{P}_s(n) = y^m$$

for integers n, s, y and m. All solutions to this equation are known for $m > 2$ and $s \in \{3, 5, 6, 8, 20\}$. We consider the case $s = 10$, that of decagonal numbers. Using a descent argument and the modular method, we prove that the only decagonal number greater than 1 expressible as a perfect mth power with $m > 1$ is $\mathcal{P}_{10}(3) = 3^3$.

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1. Introduction

The nth s-gonal number, with $s \geq 3$, which we denote by $\mathcal{P}_s(n)$, is given by the formula

$$\mathcal{P}_s(n) = \frac{(s-2)n^2 - (s-4)n}{2}.$$

Polygonal numbers have been studied since antiquity [6, pages 1–39] and relations between different polygonal numbers and perfect powers have received much attention (see, for example, [7] and the references cited therein). Kim *et al.* [7, Theorem 1.2] found all solutions to the equation $\mathcal{P}_s(n) = y^m$ when $m > 2$ and $s \in \{3, 5, 6, 8, 20\}$ for integers n and y. We extend this result (for $m > 1$) to the case $s = 10$, that of decagonal numbers.

THEOREM 1.1. *All solutions to the equation*

$$\mathcal{P}_{10}(n) = y^m, \quad n, y, m \in \mathbb{Z}, \quad m > 1 \tag{1.1}$$

satisfy $n = y = 0, n = |y| = 1$ *or* $n = y = m = 3$.

In particular, the only decagonal number greater than 1 expressible as a perfect m*th power with* $m > 1$ *is* $\mathcal{P}_{10}(3) = 3^3$.

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We will prove Theorem 1.1 by carrying out a descent argument to obtain various ternary Diophantine equations, to which one may associate Frey elliptic curves. The difficulty in solving the equation $\mathcal{P}_s(n) = y^m$ for a fixed value of s is due to the existence of the trivial solution $n = y = 1$ (for any value of m). We note that adapting our method of proof also works for the cases $s \in \{3, 5, 6, 8, 20\}$ mentioned above, but will not extend to any other values of s (see Remark 3.1).

2. Descent and small values of m

We note that it will be enough to prove Theorem 1.1 in the case $m = p$, prime. We write (1.1) as

$$n(4n - 3) = y^p, \quad n, y \in \mathbb{Z}, \quad p \text{ prime} \tag{2.1}$$

and suppose that $n, y \in \mathbb{Z}$ satisfy this equation with $n \neq 0$.

Case 1: $3 \nmid n$. If $3 \nmid n$, then n and $4n - 3$ are coprime, so there exist coprime integers a and b such that

$$n = a^p \quad \text{and} \quad 4n - 3 = b^p.$$

It follows that

$$4a^p - b^p = 3. \tag{2.2}$$

If $p = 2$, we see that $(2a - b)(2a + b) = 3$, so that $a = b = \pm 1$ and so $n = |y| = 1$. If $p = 3$ or $p = 5$, then using the Thue equation solver in `Magma` [5], we also find that $a = b = 1$.

Case 2: $3 \parallel n$. Suppose that $3 \parallel n$ (that is, $\mathrm{ord}_3(n) = 1$). Then, after dividing (2.1) by $3^{\mathrm{ord}_3(y)p}$, we see that there exist coprime integers t and u with $3 \nmid t$ such that

$$n = 3t^p \quad \text{and} \quad 4n - 3 = 3^{p-1}u^p.$$

Then

$$4t^p - 3^{p-2}u^p = 1. \tag{2.3}$$

If $p = 2$, we have $(2t - u)(2t + u) = 1$, which has no solutions. If $p = 3$, then we have $4t^3 - 3u^3 = 1$ and, using the Thue equation solver in `Magma` [5], we verify that $u = t = 1$ is the only solution to this equation. This gives $n = y = 3$. If $p = 5$, `Magma`'s Thue equation solver shows that there are no solutions.

Case 3: $3^2 \mid n$. If $3^2 \mid n$, then $3 \parallel 4n - 3$ and, arguing as in Case 2, there exist coprime integers v and w with $3 \nmid w$ such that

$$n = 3^{p-1}v^p \quad \text{and} \quad 4n - 3 = 3w^p.$$

So,

$$4 \cdot 3^{p-2}v^p - w^p = 1. \tag{2.4}$$

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If $p = 2$, then as in Case 2 we obtain no solutions. If $p = 3$ or $p = 5$, then we use `Magma`'s Thue equation solver to verify that there are no solutions with $v \neq 0$.

3. Frey curves and the modular method

To prove Theorem 1.1, we will associate Frey curves to equations (2.2), (2.3) and (2.4) and apply Ribet's level-lowering theorem [8, Theorem 1.1] to obtain a contradiction. We describe this process as *level-lowering* the Frey curve. We have considered the cases $p = 2, 3$ and 5 in Section 2 and so we will assume that $m = p$ is prime with $p \geq 7$.

We note that at this point we could directly apply [3, Theorem 1.2] to conclude that the only solutions to (3.1) are $a = b = 1$, giving $n = 1$, and apply [2, Theorem 1.2] to show that (3.2) and (3.3) have no solutions. The computations for (3.1) are not explicitly carried out in [3], so for the convenience of the reader and to highlight why the case $s = 10$ is somewhat special, we provide some details of the arguments.

Case 1: $3 \nmid n$. We write (2.2) as

$$-b^p + 4a^p = 3 \cdot 1^2, \tag{3.1}$$

which we view as a generalised Fermat equation of signature $(p, p, 2)$. We note that the three terms are integral and coprime.

We suppose that $ab \neq \pm 1$. Following the recipes of [3, pages 26–31], we associate Frey curves to (3.1). We first note that b is odd, since $b^p = 4n - 3$. If $a \equiv 1 \pmod 4$, we set

$$E_1 : Y^2 = X^3 - 3X^2 + 3a^p X.$$

If $a \equiv 3 \pmod 4$, we set

$$E_2 : Y^2 = X^3 + 3X^2 + 3a^p X.$$

If a is even, we set

$$E_3 : Y^2 + XY = X^3 - X^2 + \frac{3a^p}{16}X.$$

We level-lower each Frey curve and find that for $i = 1, 2, 3$, we have $E_i \sim_p f_i$ for f_i a newform at level N_{p_i}, where $N_{p_1} = 36, N_{p_2} = 72$ and $N_{p_3} = 18$. The notation $E \sim_p f$ means that the mod-p Galois representation of E arises from f. There are no newforms at level 18 and so we focus on the curves E_1 and E_2. There is a unique newform, f_1, at level 36, and a unique newform, f_2, at level 72.

The newform f_1 has complex multiplication by the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$. This allows us to apply [3, Proposition 4.6]. Since $2 \nmid ab$ and $3 \nmid ab$, we conclude that $p = 7$ or 13 and that all elliptic curves of conductor $2p$ have positive rank over $\mathbb{Q}(\sqrt{-3})$. However, it is straightforward to check that this is not the case for $p = 7$ and 13. We conclude that $E_1 \nsim_p f_1$.

Let F_2 denote the elliptic curve with Cremona label 72a2 whose isogeny class corresponds to f_2. This elliptic curve has full two-torsion over the rationals and has j-invariant $2^4 \cdot 3^{-2} \cdot 13^3$. We apply [3, Proposition 4.4], which uses an image of inertia argument, to obtain a contradiction in this case too.

REMARK 3.1. The trivial solution $a = b = 1$ (or $n = y = 1$) corresponds to the case $i = 1$ above. The only reason we are able to discard the isomorphism $E_1 \sim_p f_1$ is because the newform f_1 has complex multiplication. The modular method would fail to eliminate the newform f_1 otherwise. For each value of s, we can associate to (1.1) generalised Fermat equations of signature $(p, p, 2)$, $(p, p, 3)$ and (p, p, p). We found we could only obtain newforms with complex multiplication (when considering the case corresponding to the trivial solution) when $s = 3, 6, 8, 10$ or 20. A similar strategy of proof also works for $s = 5$ using the work of Bennett [1, page 3] on equations of the form $(a + 1)x^n - ay^n = 1$ to deal with the trivial solution.

Case 2: $3 \parallel n$. We rewrite (2.3) as

$$4t^p - 3^{p-2}u^p = 1 \cdot 1^3, \tag{3.2}$$

which we view as a generalised Fermat equation of signature $(p, p, 3)$. The three terms are integral and coprime. We suppose that $tu \neq \pm 1$. Using the recipes of [4, pages 1401–1406], we associate to (3.2) the Frey curve

$$E_4 : Y^2 + 3XY - 3^{p-2}u^p\, Y = X^3.$$

We level-lower E_4 and find that $E_4 \sim_p f$, where f is a newform at level 6, an immediate contradiction, as there are no newforms at level 6.

Case 3: $3^2 \mid n$. We rewrite (2.4) as

$$-w^p + 4 \cdot 3^{p-2}v^p = 1 \cdot 1^3, \tag{3.3}$$

which we view as a generalised Fermat equation of signature $(p, p, 3)$. The three terms are integral and coprime. We suppose that $vw \neq \pm 1$. The Frey curve we attach to (3.3) is

$$E_5 : Y^2 + 3XY + 4 \cdot 3^{p-2}v^p\, Y = X^3.$$

We level-lower and find that $E_5 \sim_p f$, where f is a newform at level 6, a contradiction as in Case 2.

This completes the proof of Theorem 1.1.

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