Nonlinearity

PAPER • OPEN ACCESS

Using periodic boundary conditions to approximate the Navier–Stokes equations on $\mathbb{R}^3$ and the transfer of regularity

To cite this article: James C Robinson 2021 *Nonlinearity* **34** 7683

OPEN ACCESS
IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **34** (2021) 7683–7704 https://doi.org/10.1088/1361-6544/ac2673

Using periodic boundary conditions to approximate the Navier–Stokes equations on $\mathbb{R}^3$ and the transfer of regularity

James C Robinson

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

E-mail: j.c.robinson@warwick.ac.uk

Received 23 November 2020, revised 26 July 2021
Accepted for publication 14 September 2021
Published 1 October 2021

Abstract

This paper considers solutions u_α of the three-dimensional Navier–Stokes equations on the periodic domains $Q_\alpha := (-\alpha, \alpha)^3$ as the domain size $\alpha \to \infty$, and compares them to solutions of the same equations on the whole space. For compactly-supported initial data $u_\alpha^0 \in H^1(Q_\alpha)$, an appropriate extension of u_α converges to a solution u of the equations on $\mathbb{R}^3$, strongly in $L^r(0, T; H^1(\mathbb{R}^3))$, $r \in [1, \infty)$. The same also holds when u_α^0 is the velocity corresponding to a fixed, compactly-supported vorticity. A consequence is that if an initial compactly-supported velocity $u_0 \in H^1(\mathbb{R}^3)$ or an initial compactly-supported vorticity $\omega_0 \in H^1(\mathbb{R}^3)$ gives rise to a smooth solution on $[0, T^*]$ for the equations posed on $\mathbb{R}^3$, a smooth solution will also exist on $[0, T^*]$ for the same initial data for the periodic problem posed on Q_α for α sufficiently large; this illustrates a 'transfer of regularity' from the whole space to the periodic case.

Keywords: Navier–Stokes equations, expanding domains, strong convergence
Mathematics Subject Classification numbers: 35Q30.

1. Introduction

The aim of this paper is to compare solutions of the Navier–Stokes equations

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0, \tag{1.1}$$

posed on 'large' periodic domains $Q_\alpha := (-\alpha, \alpha)^3$ and on the whole space $\mathbb{R}^3$. One would expect, when the initial velocity is sufficiently localised, that the solutions on a 'large enough'

Recommended by Dr Sandra Cerrai.

domain should mimic those on $\mathbb{R}^3$, and this approach is the basis of many numerical experiments. Indeed, discussions with Robert Kerr about his numerical investigations (Kerr 2018) of the trefoil configurations of vorticity from the experiments of Scheeler *et al* (2014) were the original motivation for this paper, which gives a rigorous justification of this intuition.

Section 3 contains an analysis of the velocity fields that arise from such compactly-supported vorticities. The results there both provide a natural family of initial data to consider on the domains Q_α, and also serve to illustrate of some of the arguments that follow in a relatively simple setting.

It is shown that given a fixed compactly-supported vorticity $\omega \in H^1(\mathbb{R}^3)$, the corresponding velocities u_α on Q_α have extensions to $\mathbb{R}^3$, $\tilde{u}_\alpha$, that converge strongly in $H^1(\mathbb{R}^3)$ to the velocity on $\mathbb{R}^3$ reconstructed from ω using the Biot–Savart Law. Obtaining strong convergence in $H^1(\mathbb{R}^3)$ requires uniform bounds on the 'tails'

$$\int_{x\in Q_\alpha:\ |x|\geqslant R} |\nabla u_\alpha|^2,$$

a technique also employed later for solutions of the Navier–Stokes equations, and which goes back at least to Leray (1934).

After recalling some basic existence results for weak and strong solutions of the Navier–Stokes equations in section 4, it is shown that a subsequence of weak solutions on Q_α (solutions bounded in L^2 that satisfy the energy inequality) will converge to a weak solution on $\mathbb{R}^3$, given weak convergence of the initial data in $L^2(\mathbb{R}^3)$. This result is due to Heywood (1988), who used it as a way of proving the existence of weak solutions on the whole space.

The main result of the paper concerns the convergence of strong solutions (i.e. solutions that remain bounded in H^1) given convergence of the initial data in $H^1(\mathbb{R}^3)$; due to uniqueness of the limiting solution this convergence now occurs without the need to extract a subsequence. By bounding the 'tails' of $|u_\alpha|^2$ at infinity it is shown that $\tilde{u}_\alpha$ converges to u strongly in $L^p(0, T; L^2(\mathbb{R}^3))$ for all $p \in [1, \infty)$, and then, via interpolation of the H^1 norm between L^2 and H^2, the boundedness of u_α in $L^2(0, T; H^2(\mathbb{R}^3))$ shows that $\tilde{u}_\alpha$ converges strongly to u in $L^r(0, T; H^1(\mathbb{R}^3))$, $r \in [1, \infty)$.

Finally, using this strong convergence, comes what is perhaps the most striking result of the paper: if $u^0 \in H^1(\mathbb{R}^3)$ with compact support (or $\omega_0 \in H^1(\mathbb{R}^3)$ with compact support) gives rise to a strong solution on $[0, T^*]$ and $u^0_\alpha \in H^1(Q_\alpha)$ converges to u^0 in $H^1(\mathbb{R}^3)$, then for large enough α the equations on Q_α with initial data u^0_α give rise to a unique strong solution on the same interval, and $\tilde{u}_\alpha \to u$ as $\alpha \to \infty$ in $L^r(0, T; H^1(\mathbb{R}^3))$, $r \in [1, \infty)$. This shows that the existence of a regular solution on the whole space implies the existence of a regular solution on a large enough periodic domain.

The relationship between the existence of smooth solutions for the equations in various settings (periodic boundary conditions, Schwartz solutions on $\mathbb{R}^3$, homogeneous and inhomogeneous problems) has also been considered, from a different point of view, by Tao (2013).

There are other 'transfer of regularity' results for the Navier–Stokes equations in different contexts. Constantin (1986) showed that if $u_0 \in H^{s+2}$, $s \geqslant 3$, gives rise to a solution in $L^\infty(0, T^*; H^{s+2})$ of the Euler equations, then for the Navier–Stokes equations with dissipative term $-\nu\Delta u$, one can take ν sufficiently small to ensure that the same initial condition produces an H^s-bounded solution of the Navier–Stokes equations on $[0, T^*]$. A variant of this approach in Chernyshenko *et al* (2007) shows that if u_0 gives rise to a regular solution of the Navier–Stokes equations on $[0, T^*]$ then a sufficiently 'good' numerical scheme will have a similarly smooth solution that will also exist on $[0, T^*]$. Other results that 'transfer regularity' start with two-dimensional flows: Raugel and Sell (1993) considered the problem posed on thin three-dimensional domains, and Gallagher (1997) considered flows with initial data that are 'close to two dimensional'.

There is, of course, another way to view solving the equations on Q_α, $\alpha \geqslant \alpha_0$, with fixed initial data u_0 of compact support. Here, rather than keeping u_0 fixed and increasing α, one could keep the domain fixed and rescale u_0: taking $\alpha_0 = 1$ for simplicity, the problem on Q_α becomes a problem posed on Ω_1 by setting

$$u_0^\alpha(x) = \alpha u_0(\alpha x).$$

A solution $(u(x,t), p(x,t))$ on Q_α becomes the rescaled solution

$$(\alpha u(\alpha x, \alpha^2 t), \alpha^2 p(\alpha x, \alpha^2 t))$$

on Q_1. However, if the solution on Q_α exists for $t \in [0, T]$, then the rescaled solution on Q_1 exists only for $t \in [0, T/\alpha^2]$. It follows that such a rescaling is not a useful tool for considering the behaviour of solutions as $\alpha \to \infty$ in the sense proposed here. Nevertheless, related scaling ideas are used here to check that various inequalities hold with constants independent of the domain parameter α.

2. Preliminaries

The expression $L^2(Q_\alpha)$ denotes the space of functions that are 2α-periodic in every direction, with

$$\int_{Q_\alpha} |u|^2 < \infty,$$

where $Q_\alpha = (-\alpha, \alpha)^3$. Throughout the paper, a dot over a space denotes that the functions have zero average: so, for example, $\dot{L}^2(Q_\alpha)$ denotes the subset of $L^2(Q_\alpha)$ consisting of those functions that also satisfy the condition

$$\int_{Q_\alpha} u = 0. \tag{2.1}$$

The notation $\langle f, g\rangle_{L^2(Q_\alpha)} = \int_{Q_\alpha} f(x)g(x)\,\mathrm{d}x$ is used for the inner product in $L^2(Q_\alpha)$.

The space of 2α-periodic functions with weak derivatives up to order s in $L^2(Q_\alpha)$, again satisfying (2.1), is denoted by $\dot{H}^s(Q_\alpha)$. Due to the zero-average condition, the $\dot{H}^s(Q_\alpha)$ norm defined by setting

$$\|u\|_{\dot{H}^s(Q_\alpha)} := \left(\sum_{|\gamma|=s} \|\partial^\gamma u\|^2_{L^2(Q_\alpha)} \right)^{1/2}$$

is equivalent to the full $H^s(Q_\alpha)$ norm. Indeed, for all $r \geqslant s \geqslant 0$ the generalised Poincaré inequality

$$\|u\|_{\dot{H}^s(Q_\alpha)} \leqslant C_{r,s}\alpha^{r-s}\|u\|_{\dot{H}^r(Q_\alpha)}, \quad u \in \dot{H}^r(Q_\alpha),$$

holds, from which the equivalence follows.

Note also for later use that if $\Delta u \in L^2(Q_\alpha)$ then $u \in H^2(Q_\alpha)$ with

$$\sum_{i,j=1}^{3} \|\partial_i\partial_j u\|^2_{H^2(Q_\alpha)} \leqslant 9\|\Delta u\|^2_{L^2(Q_\alpha)},$$

since for any $f \in C^\infty(Q_1)$ with $f = \sum_{k\in\mathbb{Z}^3} \hat{f}_k \mathrm{e}^{\mathrm{i}k\cdot x}$

$$\|\partial_i\partial_j f\|^2_{L^2(Q_1)} = \sum_{k\in\mathbb{Z}^3} |k_ik_j|^2|\hat{f}_k|^2 \leqslant \sum_{k\in\mathbb{Z}^3} |k|^4|\hat{f}_k|^2 = \|\Delta f\|^2_{L^2(Q_1)}. \tag{2.2}$$

The notation $\dot{C}^\infty(Q_\alpha)$ denotes the space of all C^∞ 2α-periodic functions satisfying the same zero average condition, and $\dot{C}^\infty_\sigma(Q_\alpha)$ the space of all smooth divergence-free functions in $\dot{C}^\infty(Q_\alpha)$. The space $\dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$ is the space of all smooth, compactly-supported, divergence-free functions defined on $\mathbb{R}^3$, with zero integral over $\mathbb{R}^3$. The space $\dot{L}^p_\sigma(Q_\alpha)$ is the completion of $\dot{C}^\infty_\sigma(Q_\alpha)$ in $L^p(Q_\alpha)$; similarly $\dot{L}^p_\sigma(\mathbb{R}^3)$ is the completion of $\dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$. Throughout, the σ subscript indicates that the functions are divergence free.

Note that $\dot{C}^\infty_\sigma(Q_\alpha)$ is dense in $\dot{H}^1_\sigma(Q_\alpha)$ and $\dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$ is dense in $H^1_\sigma(\mathbb{R}^3)$. The second of these two is less obvious, so the proof is given here.

Lemma 2.1. *$\dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$ is dense in $H^1_\sigma(\mathbb{R}^3)$.*

Proof. The density of $C^\infty_{c,\sigma}(\mathbb{R}^3)$ in $H^1_\sigma(\mathbb{R}^3)$ is due to Heywood (1976): so given any $u \in H^1_\sigma(\mathbb{R}^3)$ and $\varepsilon > 0$, there exists $\phi \in C^\infty_{c,\sigma}(\mathbb{R}^3)$ such that $\|u - \phi\|_{H^1(\mathbb{R}^3)} < \varepsilon/2$.

Set $M = \int_{\mathbb{R}^3} \phi(x)\mathrm{d}x$ and choose any $\psi \in C^\infty_{c,\sigma}(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} \psi(x)\mathrm{d}x = 1$. Setting $\psi_M(x) := MR^{-3}\psi(x/R)$ yields a $\psi_M \in C^\infty_{c,\sigma}(\mathbb{R}^3)$ with

$$\int_{\mathbb{R}^3} \psi_M = M, \qquad \int_{\mathbb{R}^3} |\psi_M|^2 = \frac{M^2}{R^3}, \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla\psi_M|^2 = \frac{M^2}{R^5}.$$

Now choose R sufficiently large that $\|\psi_M\|^2_{H^1} = M^2R^{-3} + M^2R^{-5} < \varepsilon^2/4$; setting $\tilde{u} = \phi - \psi_M$ gives $\tilde{u} \in \dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$ with $\|u - \tilde{u}\|_{H^1(\mathbb{R}^3)} < \varepsilon$. □

At various points it is important that the constants in inequalities valid on Q_α do not depend on α, i.e. on the size of the domain. To ensure this, inequalities are shown on Q_1 and then rescaled: given a function f_α defined on Q_α, the rescaled function $f(x) = f_\alpha(\alpha x)$ is defined on Q_1. The L^p norms of derivatives of order k then scale according to

$$\|\partial^\gamma f_\alpha\|_{L^p(Q_\alpha)} = \alpha^{(3/p)-k}\|\partial^\gamma f\|_{L^p(Q_1)}, \quad \text{where } |\gamma| = k. \tag{2.3}$$

3. Convergence of velocities corresponding to compactly-supported vorticity

3.1. Reconstruction of u from ω

One of the issues for the convergence results considered here is to identify a class of initial data that is 'localised' in a reasonable way. One possible choice (although theorem 6.2 is more general) is to take a compactly supported vorticity ω and to consider the corresponding velocity fields obtained by 'inverting' the curl operator on the corresponding domain. This amounts to solving the equations

$$\mathrm{curl}\, u = \omega, \qquad \nabla\cdot u = 0; \tag{3.1}$$

by taking the curl of both equations and using the vector identity

$$\operatorname{curl}\operatorname{curl} u = \nabla(\nabla\cdot u) - \Delta u = -\Delta u$$

it follows that

$$-\Delta u = \operatorname{curl}\omega \quad\Rightarrow\quad u = (-\Delta)^{-1}\operatorname{curl}\omega;$$

the weak form of this system is: given $\omega \in \dot{L}^2_\sigma(\Omega)$,

$$\text{find } u \in \dot{H}^1_\sigma(\Omega) \quad \text{s.t.} \quad \langle \nabla u, \nabla\phi\rangle_{L^2(\Omega)} = \langle \omega, \operatorname{curl}\phi\rangle_{L^2(\Omega)} \quad \forall\ \phi \in \dot{H}^1_\sigma(\Omega), \tag{3.2}$$

for $\Omega = Q_\alpha$, and replacing $\dot{H}^1_\sigma$ with H^1_σ (i.e. relaxing the zero average condition) on $\mathbb{R}^3$. Note the integration by parts in the right-hand side from $\langle \operatorname{curl}\omega, \phi\rangle$, which allows for $\omega \in L^2$ and not only $\omega \in H^1$.

On the whole space, an expression for u can be obtained using the fundamental solution of the Laplacian and an integration by parts, namely the Biot–Savart law

$$u = \operatorname{curl}^{-1}\omega := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)\mathrm{d}y; \tag{3.3}$$

for $\omega \in L^{6/5}(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ this is the unique solution in $H^1_\sigma(\mathbb{R}^3)$ of (3.2).

[In the case of $\mathbb{R}^2$ modified versions of the equivalent to the Biot–Savart law are available that do not require decay of ω and u at infinity, see Serfati (1995) and Ambrose *et al* (2015), for example. For bounded domains see Enciso *et al* (2018), for example.]

On periodic domains, while $u_\alpha = \operatorname{curl}_\alpha^{-1}\omega$ can be written explicitly in terms of the Fourier expansion it will be more useful here to observe that u_α is still the solution of the equation $-\Delta u_\alpha = \operatorname{curl}\omega$.

On the periodic domain Q_1, if $\int_{Q_1} g = 0$, then the equation $-\Delta u = g$, $\int_{Q_1} u = 0$, has a solution given in the form

$$u(x) = \int_{Q_1} K_Q(x,y)g(y)\,\mathrm{d}y, \quad \text{with} \quad K_Q(x,y) = \frac{1}{|x-y|}\phi(x-y) + S(x,y),$$

where ϕ and S are smooth and $\phi(z) = 1$ for $|z| < 1/10$ and $\phi(z) = 0$ for $|z| > 1/4$, see theorem C.5 in Robinson *et al* (2016), for example. Then, when ω has compact support in Q_1,

$$\begin{aligned} u(x) &= \int_{Q_1}\left[\frac{1}{|x-y|}\phi(x-y) + S(x,y)\right][\operatorname{curl}\ \omega](y)\mathrm{d}y \\ &= \int_{Q_1}\frac{1}{|x-y|}\phi(x-y)[\operatorname{curl}\ \omega](y)\,\mathrm{d}y + \int_{Q_1} S(x,y)[\operatorname{curl}\ \omega](y)\,\mathrm{d}y \\ &= \int_{Q_1}\operatorname{curl}_y\left(\frac{1}{|x-y|}\phi(x-y)\right)\omega(y)\,\mathrm{d}y + \int_{Q_1}[\operatorname{curl}_y S](x,y)\omega(y)\,\mathrm{d}y \\ &= -\int_{Q_1}\phi(x-y)\frac{x-y}{|x-y|^3}\times\omega(y)\,\mathrm{d}y + \int_{Q_1}\frac{1}{|x-y|}\nabla\phi(x-y)\times\omega(y)\,\mathrm{d}y \\ &\quad + \int_{Q_1}[\operatorname{curl}_y S](x,y)\omega(y)\,\mathrm{d}y. \end{aligned} \tag{3.4}$$

3.2. Bounds on u from bounds on ω

The following result is extremely useful; it is valid on Q_α for every α and on $\mathbb{R}^3$. While a similar inequality could be obtained using the Calderón–Zygmund theorem and (3.3), equality follows here from a much simpler argument (see equation (1.4.20) in Doering and Gibbon 1995).

Lemma 3.1. *If $u \in H^1_\sigma$ and $\omega = \operatorname{curl} u \in L^2$ then $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$.*

Proof. Assume first that u is smooth and $\omega \in L^2$. Then, since $\omega_i = \epsilon_{ijk}\partial_j u_k$ and $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$,

$$\begin{aligned}\int |\omega|^2 &= \int \epsilon_{ijk}(\partial_j u_k)\epsilon_{ilm}(\partial_l u_m)\\ &= \int [\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}](\partial_j u_k)(\partial_l u_m)\\ &= \int (\partial_j u_k)(\partial_j u_k) - (\partial_j u_k)(\partial_k u_j) = \int \sum_{j,k} |\partial_j u_k|^2,\end{aligned}$$

integrating by parts twice in the final term and using the fact that u is divergence free. Now if $u \in H^1$, $\omega \in L^2$ and mollifying u produces a smooth u_ε with $\nabla \times u_\varepsilon \in L^2$; the same argument shows that since $\omega_\varepsilon \to \omega$, $\partial_i(u_\varepsilon)_j \to \partial_i u_j$ for every i, j, yielding the same equality for these more general u. □

The Biot–Savart law and Young's inequality provide L^q estimates on u given L^p bounds on ω.

Lemma 3.2. *Suppose that $\omega \in L^p_\sigma(\mathbb{R}^3)$ for some $p \in (1, 3)$. Then, for*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3},$$

$u = \operatorname{curl}^{-1}\omega \in L^q_\sigma(\mathbb{R}^3)$ with

$$\|u\|_{L^q(\mathbb{R}^3)} \leqslant C_p\|\omega\|_{L^p(\mathbb{R}^3)}. \tag{3.5}$$

The same estimate also holds when $\omega \in \dot{L}^p_\sigma(Q_\alpha)$: $u_\alpha = \operatorname{curl}_\alpha^{-1}\omega \in \dot{L}^q_\sigma(Q_\alpha)$ with

$$\|u_\alpha\|_{L^q(Q_\alpha)} \leqslant C_p\|\omega\|_{L^p(Q_\alpha)}, \tag{3.6}$$

where C_p is independent of α.

Proof. On the whole space u is given by (3.3). So u is given by the convolution of ω with a kernel of order $|x|^{-2}$; in three dimensions this belongs to the weak Lebesgue space $L^{3/2,\infty}$, and (3.5) follows using the weak-Lebesgue space version of Young's inequality,

$$\|f \star g\|_{L^q} \leqslant C_{p,q,r}\|f\|_{L^{r,\infty}}\|g\|_{L^p}, \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}, \quad 1 < p, q, r < \infty.$$

For the same bound on Q_1, consider the expression in (3.4),

$$\begin{aligned}u(x) = &-\int_{Q_1} \phi(x-y)\frac{x-y}{|x-y|^3} \times \omega(y)\mathrm{d}y + \int_{Q_1} \frac{1}{|x-y|}\nabla\phi(x-y) \times \omega(y)\mathrm{d}y\\ &+ \int_{Q_1} [\operatorname{curl}_y S](x,y)\omega(y)\,\mathrm{d}y.\end{aligned}$$

The kernel in the first term is once again in $L^{3/2,\infty}(Q_1)$ and the kernel in the second term is in $L^{3/2}(Q_1)$; these two terms are thus bounded in $L^q(Q_1)$ using Young's inequality. For the final term $u_3(x)$, Minkowski's inequality yields

$$\|u_3\|_{L^q(Q_1)} \leqslant \int_{Q_1} \|\mathrm{curl}_y S(\cdot, y)\|_{L^q(Q_1)} |\omega(y)| \mathrm{d}y.$$

Noting that S is smooth and that only $x, y \in Q_1$ are relevant, $\|\mathrm{curl}_y S_\alpha(\cdot, y)\|_{L^q(Q_1)} \leqslant M$ and hence

$$\|u_3\|_{L^q(Q_1)} \leqslant M \int_{Q_1} |\omega(y)| \mathrm{d}y \leqslant M\|\omega\|_{L^1(Q_1)} \leqslant M_p \|\omega\|_{L^p(Q_1)},$$

using Hölder's inequality and the fact that Q_1 is bounded.

These three upper bounds combine to yield (3.6) on Q_1. The fact that the same inequality holds with a constant independent of α follows since both norms in (3.6) behave the same way under the rescaling $x \mapsto \alpha x$, see (2.3). □

3.3. Extension of functions from Q_α to $\mathbb{R}^3$

Given $\omega \in \dot{L}^2_\sigma(\mathbb{R}^3)$ with support contained in Q_{α_0}, lemma 3.2 gives a family $\{u_\alpha\}_{\alpha \geqslant \alpha_0}$ of velocity fields defined on Q_α ($\alpha \geqslant \alpha_0$). In order to be able to take a meaningful limit on the whole of $\mathbb{R}^3$, each u_α will be extended to the whole of $\mathbb{R}^3$ in such a way that the support of $\tilde{u}_\alpha$ is contained in a domain only slightly larger than Q_α.

Given $u_\alpha \in L^2(Q_\alpha)$, denote by $\tilde{u}_\alpha$ the extension of u_α to all of $\mathbb{R}^3$ defined by setting

$$\tilde{u}_\alpha(x) = \psi_\alpha(x) u_\alpha^{\mathrm{p}}(x),$$

where $u_\alpha^{\mathrm{p}}(x)$ is the periodic extension of u_α to $\mathbb{R}^3$ and $\psi_\alpha \in C_c^\infty(\mathbb{R}^3)$ with $0 \leqslant \psi_\alpha \leqslant 1$,

$$\psi_\alpha(x) = \begin{cases} 1 & x \in (-\alpha, \alpha)^3 \\ 0 & x \notin (-(\alpha+1), \alpha+1)^3, \end{cases}$$

$|\nabla \psi_\alpha| \leqslant M_1$, and $|\nabla^2 \psi_\alpha| \leqslant M_2$, uniformly in α.

Bounds on u_α immediately translate to bounds on $\tilde{u}_\alpha$: in particular, for $\alpha \geqslant 1$,

$$\|\tilde{u}_\alpha\|_{L^2(\mathbb{R}^3)} \leqslant e_1 \|u\|_{L^2(Q_\alpha)}, \qquad \|\nabla \tilde{u}_\alpha\|_{L^2(\mathbb{R}^3)} \leqslant e_2 \|u\|_{H^1(Q_\alpha)},$$

and

$$\|\tilde{u}_\alpha\|_{H^2(\mathbb{R}^3)} \leqslant e_3 \|u\|_{H^2(Q_\alpha)}$$

[for explicit values of these constants, one can take $e_1 = 27$, $e_2 = \max(26M_1, 27)$, and $e_3 = \max(27M_2, 52M_1, 27)$].

Later a similar extension will be used for time-dependent functions $u_\alpha(x, t)$; in this case

$$\tilde{u}_\alpha(x, t) := \psi_\alpha(x) u_\alpha^{\mathrm{p}}(x, t),$$

with the cut-off function ψ_α being independent of t. This means, in particular, that

$$\partial_t \tilde{u}_\alpha(x, t) = \psi_\alpha(x) [\partial_t u_\alpha]^{\mathrm{p}}(x, t),$$

so that bounds on $\partial_t \tilde{u}_\alpha$ can be deduced from bounds on $\partial_t u_\alpha$ as done for $\tilde{u}_\alpha$ above.

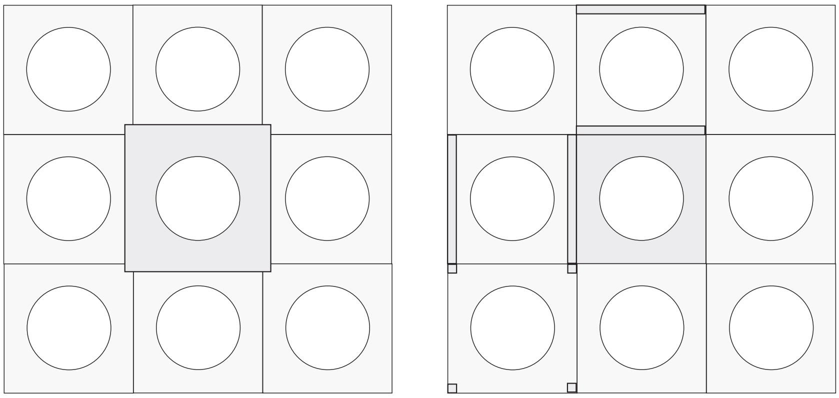

Figure 1. The support of $\tilde{u}_\alpha$ is contained in the large central square in the left-hand figure, and $|\tilde{u}_\alpha| \leqslant |u_\alpha|$ everywhere. Periodised circles of radius R are shown in white. Clearly $\int_{|x|\geqslant R}|\tilde{u}_\alpha|^2 \leqslant 9\int_{x\in Q_\alpha:\ |x|\geqslant R}|u_\alpha|^2$. However, with portions of this darker square moved using periodicity (on the right) this can be improved to $\int_{|x|\geqslant R}|\tilde{u}_\alpha|^2 \leqslant 4\int_{x\in Q_\alpha:\ |x|\geqslant R}|u_\alpha|^2$.

3.4. *Convergence of* $\mathrm{curl}_\alpha^{-1}\,\omega$ *to* $\mathrm{curl}^{-1}\,\omega$ *as* $\alpha \to \infty$

Theorem 3.4 will show that the fields $\tilde{u}_\alpha$ from lemma 3.2 converge to u strongly in $H^1(\mathbb{R}^3)$ whenever $\omega \in H^1(\mathbb{R}^3)$. The following lemma (see Leray 1934, or lemma 6.34 in Ożański and Pooley 2018) can be used to improve the L^2-convergence of $\tilde{u}_\alpha$ to u on compact subsets of $\mathbb{R}^3$ to convergence on the whole of $\mathbb{R}^3$ by bounding the 'tails' of u_α uniformly.

Lemma 3.3. *If* $\{f_\alpha\}_{\alpha\geqslant\alpha_0}, f \in L^2(\mathbb{R}^3)$; $f_\alpha \to f$ *strongly in* $L^2(K)$ *for every compact subset* K *of* $\mathbb{R}^3$*; and for every* $\eta > 0$ *there exist* $R(\eta)$ *and* $\beta(\eta)$ *such that*

$$\int_{|x|\geqslant R} |f_\alpha|^2 < \eta \quad \textit{for all} \ \ \alpha \geqslant \beta, \tag{3.7}$$

then $f_\alpha \to f$ *in* $L^2(\mathbb{R}^3)$.

The argument that follows obtains bounds on the 'tail' of a sequence $u_\alpha \in L^2(Q_\alpha)$; in order to apply lemma 3.3 the corresponding bounds on $\tilde{u}_\alpha$ will be needed. Therefore note here that if $u_\alpha \in L^2(Q_\alpha)$ and $R < \alpha - 1$ then

$$\int_{|x|\geqslant R} |\tilde{u}_\alpha|^2 \,\mathrm{d}x \leqslant 27\int_{x\in Q_\alpha:\ |x|\geqslant R} |u_\alpha|^2 \mathrm{d}x, \tag{3.8}$$

since

$$\bigcup_{\underline{k}\in\mathbb{Z}^3} B(2\alpha\underline{k}, R) \cap \mathrm{supp}(\tilde{u}_\alpha) = B(0, R),$$

i.e. the integral on the left-hand side of (3.8) can at most include the 'tails' from the periodic cells immediately adjacent to Q_α, see figure 1 for an illustration of this in the two-dimensional case, where the corresponding constant is 9. [In 2D this can be improved to 4; following a similar idea the constant in the 3D case can be improved to 10.]

Theorem 3.4. *Suppose that* $\omega \in \dot{L}^2_\sigma(\mathbb{R}^3)$ *has compact support. For every* α *sufficiently large that* $\operatorname{supp}(\omega) \subset Q_\alpha$ *define* $u_\alpha = \operatorname{curl}_\alpha^{-1}\omega$*. Then*

$$\|u_\alpha\|_{L^2} \leqslant C\|\omega\|_{L^{6/5}}, \qquad \|\nabla u_\alpha\|_{L^2} = \|\omega\|_{L^2}, \tag{3.9}$$

$\tilde{u}_\alpha \rightharpoonup$ *curl*$^{-1}\,\omega$ *weakly in* $H^1(\mathbb{R}^3)$ *and* $\tilde{u}_\alpha \to$ *curl*$^{-1}\,\omega$ *strongly in* $L^2(K)$ *for every compact subset* K *of* $\mathbb{R}^3$.

If in addition $\omega \in H^1(\mathbb{R}^3)$ *then* $u_\alpha \in H^2(\mathbb{R}^3)$*,* $\tilde{u}_\alpha \rightharpoonup$ *curl*$^{-1}\,\omega$ *weakly in* $H^2(\mathbb{R}^3)$*, and* $\tilde{u}_\alpha \to$ *curl*$^{-1}\,\omega$ *strongly in* $H^1(\mathbb{R}^3)$.

Proof. If $\omega \in L^2$ then the uniform estimates for u_α in (3.9) follow from lemmas 3.1 and 3.2. Now extend each u_α to a function $\tilde{u}_\alpha$ defined on all of $\mathbb{R}^3$ as outlined above, and in this way obtain a set of functions with $\tilde{u}_\alpha$ uniformly bounded (with respect to α) in $H^1(\mathbb{R}^3)$. Since $H^1(\mathbb{R}^3)$ is reflexive, it follows from reflexive weak sequential compactness that there exists an element $v \in H^1(\mathbb{R}^3)$ such that $\tilde{u}_{\alpha_j} \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$, which in turn implies the strong convergence in $L^2(K)$ for every compact subset K of $\mathbb{R}^3$.

It remains to show that $v = u := \operatorname{curl}^{-1}\omega$ and that the convergence takes place as $\alpha \to \infty$ and not just for a subsequence.

To this end, take $\varphi \in \dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$. Then, since $\tilde{u}_{\alpha_j} = u_{\alpha_j}$ on Q_{α_j}, once $\operatorname{supp}(\varphi) \subset Q_{\alpha_j}$ we have

$$\begin{aligned}\langle \nabla \tilde{u}_{\alpha_j}, \nabla\varphi\rangle_{L^2(\mathbb{R}^3)} = \langle \nabla u_{\alpha_j}, \nabla\varphi\rangle_{L^2(Q_{\alpha_j})} &= \langle \omega, \operatorname{curl}\varphi\rangle_{L^2(\Omega_{\alpha_j})} \\ &= \langle \omega, \operatorname{curl}\varphi\rangle_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Since $\nabla\tilde{u}_{\alpha_j} \rightharpoonup \nabla u$ weakly in $L^2(\mathbb{R}^3)$, for each fixed φ it follows that

$$\langle \nabla u, \nabla\varphi\rangle_{L^2(\mathbb{R}^3)} = \langle \omega, \operatorname{curl}\varphi\rangle_{L^2(\mathbb{R}^3)}$$

for every $\varphi \in \dot{C}^\infty_{c,\sigma}(\mathbb{R}^3)$; the equality then holds for every $\varphi \in \dot{H}^1_\sigma(\mathbb{R}^3)$ by density (see lemma 2.1). Since $u \in \dot{H}^1_\sigma(\mathbb{R}^3)$ it follows that u is the unique H^1 solution of $-\Delta u = \operatorname{curl}\omega$, which is precisely $\operatorname{curl}^{-1}\omega$. This also shows that the limit of any convergent subsequence must be the same, and it follows that $u_\alpha \to u$ as claimed in the statement of the theorem.

If in addition $\omega \in H^1(\mathbb{R}^3)$ then standard elliptic regularity results (see Evans 2010, for example) gives uniform estimates on $\tilde{u}_\alpha$ in $H^2(\mathbb{R}^3)$, since then

$$\|\Delta u_\alpha\|_{L^2(Q_\alpha)} = \|\operatorname{curl}\omega\|_{L^2(Q_\alpha)}$$

and this yields a bound on the other second derivatives, see (2.2). The weak convergence in $H^2(\mathbb{R}^3)$ now follows since H^2 is reflexive, which implies the strong convergence in $H^1(K)$ for every compact subset K of $\mathbb{R}^3$.

To improve this to strong convergence in $H^1(\mathbb{R}^3)$, take $\phi = u_\alpha\varrho_\alpha$ as the test function in

$$\langle \nabla u_\alpha, \nabla\phi\rangle = \langle \operatorname{curl}\omega, \phi\rangle$$

(cf (3.2)), where ϱ_α is the restriction of

$$\varrho = \begin{cases} 0 & |x| < r \\ \dfrac{|x| - r}{R - r} & r \leqslant |x| \leqslant R \\ 1 & |x| > R, \end{cases} \tag{3.10}$$

to Q_α, where we take $0 < r < R < \alpha$; note that

$$|\nabla \varrho_\alpha| = \begin{cases} 0 & |x| < r \\ \dfrac{1}{R-r} & r < |x| < R \\ 0 & |x| > R. \end{cases}$$

Therefore

$$\int_{Q_\alpha} |\nabla u_\alpha|^2 \varrho_\alpha = -\int_{Q_\alpha} (\nabla u_\alpha) \cdot (\nabla \varrho_\alpha) u_\alpha + \int_{Q_\alpha} (\operatorname{curl} \omega) u_\alpha \varrho_\alpha,$$

and taking r sufficiently large that $\operatorname{supp}(\omega) \subset B(0,r)$ yields

$$\begin{aligned} \int_{x \in Q_\alpha :\ |x| \geqslant R} |\nabla u_\alpha|^2 &\leqslant \frac{1}{R-r} \|\nabla u_\alpha\|_{L^2(Q_\alpha)} \|u_\alpha\|_{L^2(Q_\alpha)} \\ &\leqslant \frac{K}{R-r} \|\omega\|_{L^{6/5}} \|\omega\|_{L^2}. \end{aligned}$$

Lemma 3.3 now guarantees that $\nabla \tilde{u}_\alpha \to \nabla u$ in $L^2(\mathbb{R}^3)$.

It remains to show that $\tilde{u}_\alpha \to u$ in $L^2(\mathbb{R}^3)$. First, since in 3D the Sobolev embedding $\|f\|_{L^6(\mathbb{R}^3)} \leqslant C \|\nabla f\|_{L^2(\mathbb{R}^3)}$ holds for $f \in H^1(\mathbb{R}^3)$, it follows that

$$\|\tilde{u}_\alpha - u\|_{L^6(\mathbb{R}^3)} \leqslant C \|\nabla \tilde{u}_\alpha - \nabla u\|_{L^2(\mathbb{R}^3)},$$

and so $\tilde{u}_\alpha \to u$ in $L^6(\mathbb{R}^3)$. Now, since $\omega \in L^{24/23}(Q_\alpha)$, lemma 3.2 implies that

$$\|u_\alpha\|_{L^{8/5}(Q_\alpha)} \leqslant K \|\omega\|_{L^{24/23}(Q_\alpha)},$$

a bound that holds uniformly in α and yields a similar uniform bound on $\tilde{u}_\alpha$ in $L^{8/5}(\mathbb{R}^3)$. Finally, the Lebesgue interpolation

$$\|\tilde{u}_\alpha - u\|_{L^2(\mathbb{R}^3)} \leqslant \|\tilde{u}_\alpha - u\|_{L^{8/5}(\mathbb{R}^3)}^{8/11} \|\tilde{u}_\alpha - u\|_{L^6(\mathbb{R}^3)}^{3/11}$$

guarantees that $\tilde{u}_\alpha \to u$ in $L^2(\mathbb{R}^3)$.

Combining the convergence of $\tilde{u}_\alpha \to u$ and $\nabla \tilde{u}_\alpha \to \nabla u$ in $L^2(\mathbb{R}^3)$ shows that $\tilde{u}_\alpha \to u$ in $H^1(\mathbb{R}^3)$ as claimed. □

4. Weak and strong solutions of the Navier–Stokes equations

For $\Omega = Q_\alpha$ or $\mathbb{R}^3$, denote by $\mathcal{D}_\sigma(\Omega)$ the space of all test functions on $\Omega \times [0,\infty)$ given by

$$\mathcal{D}_\sigma(\Omega) = \{\phi \in C_c^\infty(\Omega \times [0,\infty)) :\ \nabla \cdot \phi(t) = 0 \text{ for all } t \in [0,\infty)\}.$$

Definition 4.1. A function u is a weak solution of the Navier–Stokes equations corresponding to the initial condition $u_0 \in \dot{L}^2_\sigma(\Omega)$ if

$$u \in L^\infty(0,T;\dot{L}^2_\sigma(\Omega)) \cap L^2(0,T;H^1(\Omega)) \quad \text{for every } T > 0$$

and

$$\int_0^\infty -\langle u, \partial_t \phi \rangle + \int_0^\infty \langle \nabla u, \nabla \phi \rangle + \int_0^\infty \langle (u \cdot \nabla) u, \phi \rangle = \langle u_0, \phi(0) \rangle$$

for all test functions $\phi \in \mathcal{D}_\sigma(\Omega)$.

The following theorem combines the basic existence result for weak solutions (Leray 1934, Hopf 1951) with the property that at least one solution exists that satisfies the strong energy inequality (Leray 1934, Ladyzhenskaya 1969): see theorems 4.4, 4.6, 4.10, and 14.4 in Robinson *et al* (2016).

Theorem 4.2. *For every initial condition $u_0 \in \dot{L}^2_\sigma(\Omega)$ there exists at least one global-in-time weak solution u of the Navier–Stokes equations on Ω that satisfies the strong energy inequality*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_s^t \|\nabla u\|^2_{L^2(\Omega)} \leqslant \frac{1}{2}\|u(s)\|^2_{L^2(\Omega)} \quad \text{for all } \ t > s \tag{4.1}$$

for almost all times $s \in [0,\infty)$, including $s = 0$. [These are known as Leray–Hopf weak solutions.]

Note that it follows from this definition that any weak solution u has a weak time derivative $\partial_t u$ with

$$\partial_t u \in L^{4/3}(0,T;H^{-1}_\sigma(\Omega)) \quad \text{for every } \ T > 0,$$

where $H^{-1}_\sigma(\Omega)$ is the dual space of $\dot{H}^1_\sigma(\Omega)$, with

$$\|\partial_t u\|_{L^{4/3}(0,T;H^{-1}_\sigma(\Omega))} \leqslant c\int_0^T \|\nabla u\|^2\|u\|^{2/3} + T^{1/3}\left(\int_0^T \|\nabla u\|^2\right)^{2/3}, \tag{4.2}$$

with c independent of α; see lemma 3.7 in Robinson *et al* (2016).

Key to later results in this paper is the notion of a strong solution.

Definition 4.3. A function u is a strong solution on $[0,T]$ of the Navier–Stokes equations corresponding to the initial condition $u_0 \in \dot{H}^1_\sigma(\Omega)$ if it is a weak solution and in addition[1]

$$u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

The following theorem on the existence of strong solutions is again valid on Q_α and $\mathbb{R}^3$; the constant c is the same for all these domains. The result as stated combines theorems 6.4, 6.8, 6.15, and 7.5 in Robinson *et al* (2016).

Theorem 4.4. *Any initial condition $u_0 \in \dot{H}^1_\sigma(\Omega)$ gives rise to a unique strong solution of the Navier–Stokes equations at least on the time interval $[0,T]$, where $T = c\|\nabla u_0\|^{-4}_{L^2(\Omega)}$. For such solutions the equation*

$$\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0$$

is satisfied as an equality in $L^2(0,T;L^2(\Omega))$, and in fact u is smooth in space–time on $\Omega \times (0,T]$.

5. Convergence of weak solutions

Convergence of weak solutions as $\alpha \to \infty$ is relatively straightforward; indeed, a similar method has been used by Heywood (1988); (see also theorem 4.10 in Robinson *et al* 2016)

[1] This assumption is in fact sufficient to ensure that $u \in C([0,T];H^1)$.

to prove the existence of weak solutions on the whole space, although with that aim it is probably more natural to consider the equations with Dirichlet boundary conditions on the domains $B(0,\alpha)$, which can easily be extended by zero to all of $\mathbb{R}^3$.

Proposition 5.1. *Suppose that $u_\alpha^0 \in \dot{L}^2_\sigma(Q_\alpha)$ with $\tilde{u}_\alpha^0 \rightharpoonup u^0$ in $L^2(\mathbb{R}^3)$. Let u_α be weak solutions of the equations on Q_α with initial conditions u_α^0 that satisfy the energy inequality*

$$\frac{1}{2}\|u_\alpha(t)\|^2_{L^2(Q_\alpha)} + \int_0^t \|\nabla u_\alpha(s)\|^2_{L^2(Q_\alpha)}\,\mathrm{d}s \leqslant \frac{1}{2}\|u_\alpha^0\|^2_{L^2(Q_\alpha)} \tag{5.1}$$

for almost every $t > 0$. Then there exists a weak solution u of the equations on $\mathbb{R}^3$, and a subsequence u_{α_j} such that, for every $T > 0$, $\tilde{u}_{\alpha_j}$ converges to u weakly in $L^2(0,T;H^1)$ and strongly in $L^2(0,T;L^2(K))$ for every compact subset K of $\mathbb{R}^3$.

Proof. Since $\tilde{u}_\alpha^0$ is a weakly-convergent sequence it must be bounded in $L^2(\mathbb{R}^3)$; so u_α^0 is uniformly bounded in $L^2(Q_\alpha)$, and it is immediate from the energy inequality (5.1) that u_α is uniformly bounded (with respect to α) in $L^\infty(0,T;L^2(Q_\alpha))$ and $L^2(0,T;H^1(Q_\alpha))$. The inequality (4.2) also provides uniform bounds on the time derivative $\partial_t u_\alpha$ in $L^{4/3}(0,T;H^{-1}_\sigma(Q_\alpha))$.

These uniform bounds on u_α become uniform bounds on the extended functions $\tilde{u}_\alpha$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ and $L^2(0,T;L^2(\mathbb{R}^3))$, so there exists an element $u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ and a subsequence $\tilde{u}_{\alpha_j}$ that converges to u weakly-$*$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ and for which

$$\nabla\tilde{u}_{\alpha_j} \rightharpoonup \nabla u \quad \text{in} \quad L^2(0,T;L^2(\mathbb{R}^3)).$$

However, it is not necessarily the case that $\partial_t\tilde{u}_\alpha$ is uniformly bounded in $L^{4/3}(0,T;H^{-1}_\sigma(\mathbb{R}^3))$, since there is no reason why the restriction of a 'test function' $\phi \in \dot{H}^1_\sigma(\mathbb{R}^3)$ to Q_α should respect the periodic boundary conditions or integrate to zero, i.e. be an element of $\dot{H}^1_\sigma(Q_\alpha)$. To obtain strong convergence in $L^2(0,T;L^2(K))$ for compact subsets K of $\mathbb{R}^3$, instead observe that for each $R > 0$, once $\alpha > R$

$$(\partial_t\tilde{u}_\alpha)|_{B(0,R)} = (\partial_t u_\alpha)|_{B(0,R)},$$

and that if $\alpha > 3R$ then any $\phi \in H^1_{0,\sigma}(B(0,R)) := H^1_0(B(0,R)) \cap L^2_\sigma(B(0,R))$ can be extended to an element $\hat{\phi} \in \dot{H}^1_\sigma(Q_\alpha)$ with

$$\|\hat{\phi}\|_{H^1(Q_\alpha)} = 2\|\phi\|_{H^1(B(0,R))},$$

by setting

$$\hat{\phi}(x) = \begin{cases} \phi(x) & x \in B(0,R) \\ -\phi(x) & x \in B((2R,0,0),R) \\ 0 & \text{otherwise;} \end{cases}$$

the part of the extension where $\hat{\phi}(x) = -\phi(x)$ ensures that $\int_{Q_\alpha}\hat{\phi} = 0$. It follows that once $\alpha > 3R$,

$$\|\partial_t\tilde{u}_\alpha\|_{H^{-1}_{0,\sigma}(B(0,R))} \leqslant 2\|\partial_t u_\alpha\|_{H^{-1}_\sigma(Q_\alpha)}.$$

It is also clear that

$$\|\tilde{u}_\alpha\|_{H^1(B(0,R))} \leqslant \|u_\alpha\|_{H^1(Q_\alpha)},$$

so $\tilde{u}_\alpha$ is uniformly bounded in $L^2(0,T;H^1_\sigma(Q_\alpha))$. An application of the Aubin–Lions compactness theorem (see Simon 1987) now yields a subsequence that converges strongly in $L^2(0,T;L^2(B(0,R)))$ for every $R>0$, and hence in $L^2(0,T;L^2(K))$ for every compact subset K of $\mathbb{R}^3$.

It remains only to show that u is a solution of the equations on the whole space.

To do this, take any test function $\phi\in\mathcal{D}_\sigma(\mathbb{R}^3)$ and let M and T be large enough that the support of ϕ is contained in $Q_M\times[0,T)$. Then for all $\alpha\geqslant M$ it follows from definition 4.1, since $\tilde{u}_\alpha=u_\alpha$ on Q_α, that

$$-\int_0^\infty\langle\tilde{u}_{\alpha_j},\partial_t\phi\rangle+\int_0^\infty\langle\nabla\tilde{u}_{\alpha_j},\nabla\phi\rangle+\int_0^\infty\langle(\tilde{u}_{\alpha_j}\cdot\nabla)\tilde{u}_{\alpha_j},\phi\rangle=\langle\tilde{u}^0_{\alpha_j},\phi(0)\rangle.$$

Passing to the limit as $j\to\infty$ (using the weak convergence of gradients, the strong convergence in $L^2(0,T;L^2(\Omega_M))$, and the fact that $\tilde{u}^0_{\alpha_j}\rightharpoonup u^0$) shows that u is a weak solution of the equations on $\mathbb{R}^3$ with initial condition u^0, as required. □

Note that the above proof does not show that the solution u on $\mathbb{R}^3$ satisfies the energy inequality; this is why the limiting procedure here is not the ideal way to generate solutions of the equations on $\mathbb{R}^3$.

6. Convergence of strong solutions

The main result of this paper, theorem 6.2, will show that given a suitably convergent family of initial data $u^0_\alpha\in H^1(Q_\alpha)$, the 'solutions' $\tilde{u}_\alpha$ converge strongly to u in $L^2(0,T;H^1(\mathbb{R}^3))$.

6.1. Uniform inequalities

Key to obtaining uniform estimates for strong solutions on expanding domains are the following inequalities.

Lemma 6.1 (Uniform inequalities). *There exist constants C_A and C_6, which do not depend on α, such that*

$$\|u\|_{L^\infty(Q_\alpha)}\leqslant C_A\|\nabla u\|^{1/2}_{L^2(Q_\alpha)}\|\Delta u\|^{1/2}_{L^2(Q_\alpha)}\quad \textit{for all}\ \ u\in\dot{H}^2(Q_\alpha),\tag{6.1}$$

and

$$\|u\|_{L^6(Q_\alpha)}\leqslant C_6\|\nabla u\|_{L^2(Q_\alpha)},\quad \textit{for all}\ \ u\in\dot{H}^1(Q_\alpha).\tag{6.2}$$

If $-\Delta p=\nabla\cdot[(u\cdot\nabla)u]$ *with* $\int_{Q_\alpha}p=\int_{Q_\alpha}u=0$ *then*

$$\|p\|_{L^2(Q_\alpha)}\leqslant C_Z\|u\|^2_{L^4(Q_\alpha)}.\tag{6.3}$$

where C_Z is independent of α.

Proof. The validity of the estimate (6.1) for a fixed value of α is standard, and follows by splitting the Fourier series expansion of u into 'low modes' and 'high modes' (see exercise 1.10 in Robinson *et al* (2016), for example): so, taking $\alpha=1$, for all $v\in H^2(\Omega_1)$

$$\|v\|_{L^\infty(\Omega_1)}\leqslant C_A\|\nabla v\|^{1/2}_{L^2(\Omega_1)}\|\Delta v\|^{1/2}_{L^2(\Omega_1)}.$$

The rescalings in (2.3) now show that this inequality is valid with the same constant on Q_α.

Inequality (6.2) in the case $\alpha = 1$ is a consequence of the embedding $H^1(Q_1) \subset L^6(Q_1)$ valid for three-dimensional domains, and the Poincaré inequality $\|u\|_{L^2(Q_1)} \leqslant C_P\|\nabla u\|_{L^2(Q_1)}$ which holds when $\int_{Q_1} u = 0$. A similar rescaling argument shows that the same constant works for every α.

Finally, on Q_1, the estimate (6.3) follows using the Calderón–Zygmund theorem,

$$\|p\|_{L^2(Q_1)} \leqslant C_Z\|u\|^2_{L^4(Q_1)} \tag{6.4}$$

(see appendix B in Robinson *et al* (2016) for example). To see that the constant is uniform in α, given $(\tilde{p}, \tilde{u})$ that satisfy the equations on Q_α, define (p, u) on Q_1 by setting $p(x) = \alpha^2\tilde{p}(\alpha x)$ and $u(x) = \alpha\tilde{u}(\alpha x)$. Then

$$[-\Delta p](x) = -\alpha^4(\Delta\tilde{p})(\alpha x) \quad \text{and} \quad \nabla\cdot[(u\cdot\nabla)u](x) = \alpha^4[(\tilde{u}\cdot\nabla)\tilde{u}](\alpha x),$$

so $-\Delta p = \nabla\cdot[(u\cdot\nabla)u]$, whence (p, u) satisfy (6.4). Now observe that $\|p\|_{L^2(Q_1)} = \alpha^{1/2}\|\tilde{p}\|_{L^2(Q_\alpha)}$ and $\|u\|_{L^4(Q_1)} = \alpha^{1/4}\|\tilde{u}\|_{L^2(Q_\alpha)}$ to obtain (6.3). □

6.2. Convergence in $L^2(0,T;H^1(\mathbb{R}^3))$ when $u_\alpha \in H^1(Q_\alpha)$

For initial $u^0_\alpha \in \dot{L}^2_\sigma(Q_\alpha)\cap H^1(Q_\alpha)$, such that $\tilde{u}^0_\alpha \to u^0$ in $H^1(\mathbb{R}^3)$, the following theorem shows that the corresponding strong solutions converge in $L^2(0,T;H^1(\mathbb{R}^3))$. One particular example of such a family is provided by theorem 3.4: take a fixed compactly-supported vorticity, and set $u^0_\alpha = \mathrm{curl}^{-1}_\alpha\,\omega$ and $u^0 = \mathrm{curl}^{-1}\,\omega$. Alternatively, simply take a compactly-supported initial condition $u^0 \in H^1_\sigma(\mathbb{R}^3)$ and let $u^0_\alpha = u^0|_{Q_\alpha}$ once α is sufficiently large.

There is a uniform time for which the existence of a smooth solution u_α (on Q_α) and u (on $\mathbb{R}^3$) can be guaranteed, starting with this initial condition. The following theorem shows that the extended solutions $\tilde{u}_\alpha$ must converge to u. That there is weak convergence [as in proposition 5.1] is fairly standard and follows directly from uniform bounds on u_α; that the convergence is strong in $L^2(0,T;H^1(\mathbb{R}^3))$ is more surprising, and requires a more careful analysis. This strong convergence is crucial for the 'transference of regularity' result that follows in section 7.

Theorem 6.2. *Suppose that $u_0 \in H^1_\sigma(\mathbb{R}^3)$, $u^0_\alpha \in \dot{H}^1_\sigma(Q_\alpha)$, and $\tilde{u}^0_\alpha \to u^0$ in $H^1(\mathbb{R}^3)$, with $\|u^0_\alpha\|^2_{H^1(Q_\alpha)} \leqslant M$ for all $\alpha \geqslant \alpha_0$. [For the definition of the extension $\tilde{u}^0_\alpha$ see section 3.3.]*

Set $T = 2/[9C_A^4M^2]$, where C_A is the constant from (6.1). Denote by u_α the strong solution of the Navier–Stokes equations on Q_α with initial data u^0_α, and by u the solution on $\mathbb{R}^3$ with initial data u^0; all of these solutions exist on $[0,T]$. Then for all $1 \leqslant s < 2$

$$\tilde{u}_\alpha \to u \quad \text{in} \quad L^r(0,T;H^{1+s}(\mathbb{R}^3)), \qquad r \in [1, 2/(s-1)); \tag{6.5}$$

in particular, $\tilde{u}_\alpha \to u$ strongly in $L^r(0,T;H^1(\mathbb{R}^3))$ for all $r \in [1,\infty)$.

Proof. Since the solution u_α is smooth on $[0,T]$ it is admissible to take the inner product with u_α in $L^2(Q_\alpha)$ to obtain

$$\frac{1}{2}\|u_\alpha(t)\|^2_{L^2(Q_\alpha)} + \int_0^t \|\nabla u_\alpha(s)\|^2_{L^2(Q_\alpha)}\,\mathrm{d}s \leqslant \frac{1}{2}\|u^\alpha_0\|^2_{L^2(Q_\alpha)} \leqslant \frac{M}{2}. \tag{6.6}$$

This gives bounds on u_α in $L^\infty(0,T;L^2(Q_\alpha))$ and $L^2(0,T;H^1(Q_\alpha))$ that are uniform with respect to α.

Equation (6.6) shows that the solutions u_α satisfy the energy inequality (5.1), so proposition 5.1 already guarantees that a subsequence (at least) converges to a weak solution on $\mathbb{R}^3$ with initial data u^0. However, although u^0 gives rise to a strong solution, weak-strong uniqueness (see theorem 6.10 in Robinson *et al* (2016), for example) cannot be used here, since the limiting solution u from proposition 5.1 does not necessarily satisfy the energy inequality (which is required in the proof of weak-strong uniqueness).

Better convergence of $\tilde{u}_\alpha$ to u can be obtained via bounds on u_α in H^1 and bounds on u_α in H^2. Take the inner product of the equation with $-\Delta u_\alpha$ in $L^2(Q_\alpha)$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_\alpha\|^2_{L^2(Q_\alpha)} + \|\Delta u_\alpha\|^2_{L^2(Q_\alpha)} &= \langle (u_\alpha\cdot\nabla)u_\alpha, \Delta u_\alpha\rangle_{L^2(Q_\alpha)} \\ &\leqslant \|u_\alpha\|_{L^\infty(Q_\alpha)}\|\nabla u_\alpha\|_{L^2(Q_\alpha)}\|\Delta u_\alpha\|_{L^2(Q_\alpha)} \\ &\leqslant C_A\|\nabla u_\alpha\|^{3/2}_{L^2(Q_\alpha)}\|\Delta u_\alpha\|^{3/2}_{L^2(Q_\alpha)},\end{aligned}$$

where the constant C_A does not depend on α (see lemma 6.1). It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_\alpha\|^2_{L^2(Q_\alpha)} + \|\Delta u_\alpha\|^2_{L^2(Q_\alpha)} \leqslant \frac{27}{16}C_A^4\|\nabla u_\alpha\|^6_{L^2(Q_\alpha)}, \tag{6.7}$$

and therefore

$$\|\nabla u_\alpha(t)\|^2_{L^2(Q_\alpha)} \leqslant \frac{\|\nabla u_0^\alpha\|^2_{L^2(Q_\alpha)}}{\sqrt{1-\frac{27}{8}C_A^4 t\|\nabla u_0^\alpha\|^4_{L^2(Q_\alpha)}}}. \tag{6.8}$$

Taking $T = 2/[9C_A^4M^2]$ it follows that

$$\|\nabla u_\alpha(t)\|^2_{L^2(Q_\alpha)} \leqslant 2\|\nabla u_0^\alpha\|^2_{L^2(Q_\alpha)} \leqslant 2M \quad \text{for all} \;\; t\in[0,T]$$

and, integrating (6.7) from 0 to T and using the bound in (6.8), that

$$\int_0^T \|\Delta u_\alpha(t)\|^2_{L^2(Q_\alpha)}\,\mathrm{d}t \leqslant \frac{5M}{2}. \tag{6.9}$$

Therefore u_α is bounded uniformly in $L^\infty(0,T;H^1(Q_\alpha))$ and in $L^2(0,T;H^2(Q_\alpha))$.

To obtain bounds on the time derivative, since the equation

$$\partial_t u_\alpha = \Delta u_\alpha - (u_\alpha\cdot\nabla)u_\alpha - \nabla p_\alpha$$

holds as an equality in $L^2(0,T;L^2(Q_\alpha))$ it follows that

$$\|\partial_t u_\alpha\|_{L^2(Q_\alpha)} \leqslant \|\Delta u_\alpha\|_{L^2(Q_\alpha)} + \|(u_\alpha\cdot\nabla)u_\alpha\|_{L^2(Q_\alpha)} + \|\nabla p_\alpha\|_{L^2(Q_\alpha)}.$$

The Helmholtz decomposition provides a bound on ∇p_α in $L^2(Q_\alpha)$: write

$$L^2(Q_\alpha) = L^2_\sigma(Q_\alpha)\oplus G(Q_\alpha),$$

where

$$G(Q_\alpha) = \{\nabla\psi : \;\psi\in H^1(Q_\alpha)\}.$$

These two spaces are orthogonal: for any $v\in H(Q_\alpha)$ and $\nabla\psi\in G(Q_\alpha)$

$$\langle v,\nabla\psi\rangle_{L^2(Q_\alpha)} = 0.$$

Take any $\phi \in L^2(Q_\alpha)$ and write $\phi = v + \nabla\psi$, where $v \in H(Q_\alpha)$ and $\nabla\psi \in G(Q_\alpha)$. Then

$$\langle \nabla p_\alpha, \phi\rangle = \langle \nabla p_\alpha, \nabla\psi\rangle = \langle \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha\cdot\nabla)u_\alpha, \nabla\psi\rangle = \langle (u_\alpha\cdot\nabla)u_\alpha, \nabla\psi\rangle,$$

since $\nabla\psi$ is orthogonal to any divergence-free function. It follows that

$$\begin{aligned}|\langle \nabla p_\alpha, \phi\rangle| &\leqslant \|(u_\alpha\cdot\nabla)u_\alpha\|_{L^2(Q_\alpha)}\|\nabla\psi\|_{L^2(Q_\alpha)}\\ &\leqslant \|(u_\alpha\cdot\nabla)u_\alpha\|_{L^2(Q_\alpha)}\|\phi\|_{L^2(Q_\alpha)},\end{aligned}$$

which shows that

$$\|\nabla p_\alpha\|_{L^2(Q_\alpha)} \leqslant \|(u_\alpha\cdot\nabla)u_\alpha\|_{L^2(Q_\alpha)}.$$

It follows that

$$\begin{aligned}\|\partial_t u_\alpha\|_{L^2(Q_\alpha)} &\leqslant \|\Delta u_\alpha\|_{L^2(Q_\alpha)} + 2\|(u_\alpha\cdot\nabla)u_\alpha\|_{L^2(Q_\alpha)}\\ &\leqslant \|\Delta u_\alpha\|_{L^2(Q_\alpha)} + 2\|u_\alpha\|_{L^\infty(Q_\alpha)}\|\nabla u_\alpha\|_{L^2(Q_\alpha)},\end{aligned}$$

so $\partial_t u_\alpha$ is bounded uniformly in $L^2(0,T;L^2(Q_\alpha))$.

All these bounds carry over uniformly to the extended functions $\tilde{u}_\alpha$, which are therefore bounded uniformly in $L^\infty(0,T;L^2(\mathbb{R}^3))$ and $L^2(0,T;H^1(\mathbb{R}^3))$, with $\partial_t\tilde{u}_\alpha$ bounded uniformly in $L^2(0,T;L^2(\mathbb{R}^3))$.

It follows — using weak-$*$ sequential compactness, weak sequential compactness in reflexive Banach spaces (see chapter 27 in Robinson 2020, for example), and the Aubin–Lion compactness theorem (see Simon 1987) — that there is a subsequence $\tilde{u}_{\alpha_j}$ that converges to some limit $u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3))$, with

$$\tilde{u}_{\alpha_j} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H^1(\mathbb{R}^3)), \qquad \tilde{u}_{\alpha_j} \rightharpoonup u \text{ in } L^2(0,T;H^2(\mathbb{R}^3)),$$

and $\tilde{u}_{\alpha_j} \to u$ strongly in $L^2(0,T;H^1(K))$ for every compact subset K of $\mathbb{R}^3$.

We know from before that u is at least a weak solution on $[0,T)$: these bounds now show that u has the required regularity to be a strong solution. By the uniqueness of strong solutions (in their own class) it follows that in fact $\tilde{u}_\alpha$ converges to u in all senses above as $\alpha\to\infty$, and not only through the sequence α_j. (See lemma 3.1 in Robinson (2004), for example.)

To obtain strong convergence of $\tilde{u}_\alpha$ to u solutions in $L^r(0,T;H^1(\mathbb{R}^3))$, the idea is first to use lemma 3.3 to prove that $\tilde{u}_\alpha \to u$ in $L^p(0,T;L^2(\mathbb{R}^3))$, $p\in[1,\infty)$, by showing that

$$\int_{x\in Q_\alpha:\ |x|\geqslant R} |u_\alpha(t)|^2 \tag{6.10}$$

can be made small (uniformly for α sufficiently large and $t\in[0,T]$) by taking R large. Towards this, observe that it follows from the assumptions on u_α^0 that for every $\eta>0$ there exists $r=r(\eta)$ and $\beta=\beta(\eta)\geqslant r(\eta)$ such that

$$\int_{x\in Q_\alpha:\ |x|\geqslant r} |u_\alpha^0(x)|^2\,\mathrm{d}x < \eta \quad \text{for every } \ \alpha\geqslant\beta. \tag{6.11}$$

To obtain the bound (6.10) on u_α, take the inner product [in $L^2(Q_\alpha)$] of

$$\partial_t u_\alpha - \Delta u_\alpha + (u_\alpha\cdot\nabla)u_\alpha + \nabla p_\alpha = 0$$

with $\varrho_\alpha u_\alpha$, where ϱ_α is the function defined in (3.10).

Then (cf proof of proposition 14.3 in Robinson *et al* 2016) an integration by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{Q_\alpha}\varrho_\alpha|u_\alpha|^2+\int_{Q_\alpha}\varrho_\alpha|\nabla u_\alpha|^2$$
$$=-\int_{Q_\alpha}(\partial_j u_{\alpha,i})u_{\alpha,i}(\partial_j\varrho_\alpha)+\int_{Q_\alpha}|u_\alpha|^2(u_\alpha\cdot\nabla)\varrho_\alpha+\int_{Q_\alpha}p_\alpha(u_\alpha\cdot\nabla)\varrho_\alpha.$$

Integrating from 0 to t and using the definition of ϱ_α yields

$$\frac{1}{2}\int_{x\in Q_\alpha:\ |x|>R}|u_\alpha(t)|^2\leqslant\frac{1}{2}\int_{x\in Q_\alpha:\ |x|>r}|u_\alpha^0|^2$$
$$+\frac{1}{R-r}\int_0^t\int_{Q_\alpha}|\nabla u_\alpha||u_\alpha|+|u_\alpha|^3+|p_\alpha||u_\alpha|.$$

Since $\|u_\alpha(s)\|_{L^2(Q_\alpha)}\leqslant\|u_\alpha^0\|_{L^2(Q_\alpha)}$ the second term on the right-hand side can be bounded by

$$\frac{1}{R-r}\|u_\alpha^0\|_{L^2(Q_\alpha)}\int_0^t\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}+\|u_\alpha(s)\|_{L^4(Q_\alpha)}^2+\|p_\alpha\|_{L^2(Q_\alpha)}\,\mathrm{d}s.$$

The first term of this integral can be estimated by

$$\int_0^t\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}\,\mathrm{d}s\leqslant t^{1/2}\int_0^t\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^2\,\mathrm{d}s.$$

Using the Calderón–Zygmund estimate $\|p_\alpha\|_{L^2(Q_\alpha)}\leqslant C_Z\|u_\alpha\|_{L^4(Q_\alpha)}^2$ from (6.3) the second and third terms can be combined; then using the Lebesgue interpolation inequality $\|f\|_{L^4}\leqslant\|f\|_{L^2}^{1/4}\|f\|_{L^6}^{3/4}$ and the Sobolev embedding $\|f\|_{L^6(Q_\alpha)}\leqslant C_6\|\nabla f\|_{L^2(Q_\alpha)}$ from (6.2)

$$\int_0^t\|u_\alpha(s)\|_{L^4(Q_\alpha)}^2\,\mathrm{d}s\leqslant\int_0^t\|u_\alpha(s)\|_{L^2(Q_\alpha)}^{1/2}\;\|u_\alpha(s)\|_{L^6(Q_\alpha)}^{3/2}\,\mathrm{d}s$$
$$\leqslant C_6^{3/2}\|u_\alpha^0\|_{L^2(Q_\alpha)}^{1/2}\int_0^t\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^{3/2}\,\mathrm{d}s$$
$$\leqslant C_6^{3/2}\|u_\alpha^0\|_{L^2(Q_\alpha)}^{1/2}t^{1/4}\left(\int_0^t\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^2\,\mathrm{d}s\right)^{3/4}.$$

Therefore, for all $t\in[0,T]$,

$$\frac{1}{2}\int_{x\in Q_\alpha:\ |x|>R}|u_\alpha(t)|^2\leqslant\frac{1}{2}\int_{x\in Q_\alpha:\ |x|>r}|u_\alpha^0|^2+\frac{\|u_\alpha^0\|_{L^2(Q_\alpha)}}{R-r}\left[T^{1/2}\int_0^T\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^2\,\mathrm{d}s\right.$$
$$\left.+\,2C_6^{3/2}\|u_\alpha^0\|_{L^2(Q_\alpha)}^{1/2}T^{1/4}\left(\int_0^T\|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^2\,\mathrm{d}s\right)^{3/4}\right];$$

or

$$\int_{x\in Q_\alpha:\ |x|>R}|u_\alpha(t)|^2\leqslant\int_{x\in Q_\alpha:\ |x|>r}|u_\alpha^0|^2+\frac{\Gamma}{R-r},$$

where Γ can be chosen to be independent of α. Given $\eta > 0$, it follows from (6.11) that there exist β and r such that

$$\int_{x\in Q_\alpha:\ |x|>r} |u_\alpha^0|^2 < \eta/2 \quad \text{for } \alpha \geqslant \beta.$$

Now choose R sufficiently large that $\Gamma/(R-r) < \eta/2$, and then increase β if necessary so that $\beta > R+1$. There therefore exist $R(\eta)$ and $\beta(\eta)$ such that

$$\int_{x\in Q_\alpha:\ |x|>R(\eta)} |u_\alpha(t)|^2 \leqslant \eta \quad \text{for } \alpha \geqslant \beta(\eta),\ \ t\in[0,T],$$

with $\beta(\eta) > R(\eta)+1$, which was (6.10). Finally, it follows from (3.8) that

$$\int_{|x|>R(\eta)} |\tilde{u}_\alpha(t)|^2 \leqslant 27\eta \quad \text{for } \alpha \geqslant \beta(\eta). \tag{6.12}$$

Since $\tilde{u}_\alpha \to u$ in $L^2(0,T;L^2(K))$ for every compact subset K of $\mathbb{R}^3$, it follows that $\tilde{u}_\alpha(t) \to u(t)$ in $L^2(B(0,n))$ for every $n\in\mathbb{N}$ and for almost every $t\in\mathbb{R}$. Given the estimate in (6.12), it now follows from lemma 3.3 that $\tilde{u}_\alpha(t)\to u(t)$ in $L^2(\mathbb{R}^3)$ for almost every t, i.e. $\|\tilde{u}_\alpha(t)-u(t)\|_{L^2(\mathbb{R}^3)} \to 0$ for almost every t. Now observe that

$$\begin{aligned}\|\tilde{u}_\alpha(t)-u(t)\|_{L^2(\mathbb{R}^3)} &\leqslant \|\tilde{u}_\alpha(t)\|_{L^2(\mathbb{R}^3)} + \|u(t)\|_{L^2(\mathbb{R}^3)}\\ &\leqslant 27\|u_\alpha(t)\|_{L^2(Q_\alpha)} + \|u^0\|_{L^2(\mathbb{R}^3)}\\ &\leqslant 27\|u_\alpha^0\|_{L^2(Q_\alpha)} + \|u^0\|_{L^2(\mathbb{R}^3)} \leqslant 28\sqrt{M};\end{aligned}$$

it follows, using the dominated convergence theorem, that $\tilde{u}_\alpha \to u$ in $L^2(0,T;L^2(\mathbb{R}^3))$ (and in fact in $L^p(0,T;L^2(\mathbb{R}^3))$ for every $p\in[1,\infty)$).

The fact that $\tilde{u}_\alpha\to u$ strongly in $L^2(0,T;L^2(\mathbb{R}^3))$ can now be used to improve the convergence of $\tilde{u}_\alpha$ to u from weak in $L^2(0,T;H^1(\mathbb{R}^3))$ to strong in $L^r(0,T;H^1(\mathbb{R}^3))$ for all $r\in[1,\infty)$; rather than having to bound the 'tails' of $\int_{|x|\geqslant R}|\nabla\tilde{u}_\alpha|^2$, all that is required is the additional information that $\tilde{u}_\alpha$ is uniformly bounded in $L^\infty(0,T;H^1(\mathbb{R}^3))$ and in $L^2(0,T;H^2(\mathbb{R}^3))$ (which is guaranteed by (6.9)). Assume that $r\geqslant 2$; given convergence in any such $L^r(0,T;H^1(\mathbb{R}^3))$, convergence with $r\in[1,2)$ follows immediately. Now note that the Sobolev interpolation inequality

$$\|f\|_{H^1(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(\mathbb{R}^3)}^{1/2}\|f\|_{H^2(\mathbb{R}^3)}^{1/2}$$

implies that

$$\begin{aligned}\int_0^T \|\tilde{u}_\alpha-u\|_{H^1(\mathbb{R}^3)}^r\,\mathrm{d}t &\leqslant \|\tilde{u}_\alpha-u\|_{L^\infty(0,T;H^1(\mathbb{R}^3))}^{r-2}\int_0^T\|\tilde{u}_\alpha-u\|_{H^1(\mathbb{R}^3)}^2\,\mathrm{d}t\\ &\leqslant C\|\tilde{u}_\alpha-u\|_{L^\infty(0,T;H^1(\mathbb{R}^3))}^{r-1}\int_0^T\|\tilde{u}_\alpha-u\|_{L^2(\mathbb{R}^3)}\|\tilde{u}_\alpha-u\|_{H^2(\mathbb{R}^3)}\,\mathrm{d}t\\ &\leqslant C\|\tilde{u}_\alpha-u\|_{L^\infty(0,T;H^1(\mathbb{R}^3))}^{r-1}\left(\int_0^T\|\tilde{u}_\alpha-u\|_{L^2(\mathbb{R}^3)}^2\,\mathrm{d}t\right)^{1/2}\\ &\quad\times\left(\int_0^T\|\tilde{u}_\alpha-u\|_{H^2(\mathbb{R}^3)}^2\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

Since $\tilde{u}_\alpha$ (and hence u) are uniformly bounded in $L^\infty(0,T;H^1(\mathbb{R}^3))$ and in $L^2(0,T;H^2(\mathbb{R}^3))$, this implies that $\tilde{u}_\alpha \to u$ in $L^r(0,T;H^1(\mathbb{R}^3))$ as claimed.

To finish the proof, if $s = 1+\theta$ with $\theta \in (0,1)$ and $r \in [1, 2/\theta)$, then

$$\|f\|_{H^{1+\theta}(\mathbb{R}^3)} \leqslant C\|f\|^{1-\theta}_{H^1(\mathbb{R}^3)}\|f\|^{\theta}_{H^2(\mathbb{R}^3)}$$

and so

$$\int_0^T \|\tilde{u}_\alpha - u\|^r_{H^{1+\theta}(\mathbb{R}^3)}\,\mathrm{d}t \leqslant C\int_0^T \|\tilde{u}_\alpha - u\|^{(1-\theta)r}_{H^1(\mathbb{R}^3)}\|\tilde{u}_\alpha - u\|^{\theta r}_{H^2(\mathbb{R}^3)}\,\mathrm{d}t$$
$$\leqslant C\left(\int_0^T \|\tilde{u}_\alpha - u\|^{2r(1-\theta)/(2-r\theta)}_{H^1(\mathbb{R}^3)}\,\mathrm{d}t\right)^{(2-r\theta)/2}\left(\int_0^T \|\tilde{u}_\alpha - u\|^2_{H^2(\mathbb{R}^3)}\,\mathrm{d}t\right)^{r\theta/2}.$$

□

7. 'Transfer of regularity' from the whole space to the periodic case

This final section shows that the existence of a solution on the whole space for a particular choice of initial condition is transferred to the periodic case when α is large enough.

7.1. The transfer of regularity result

The following theorem shows that if u_0 gives rise to a smooth solution on $[0, T^*]$ on the whole space, the corresponding periodic problems will have smooth solutions on the same time interval once the size of the periodic domain is sufficiently large. Note that T^* does not need to be a 'guaranteed local existence time' from the proof of the existence of strong solutions, but could be significantly longer.

The simplest particular cases of the theorem are when $u^0_\alpha \equiv u^0 \in \dot{H}^1_\sigma(\mathbb{R}^3)$ for all α sufficiently large or when $u^0_\alpha = \mathrm{curl}^{-1}_\alpha\,\omega_0$ for some compactly-supported $\omega_0 \in \dot{H}^1_\sigma(\mathbb{R}^3)$.

Theorem 7.1. *Suppose that $u^0_\alpha \in \dot{H}^1_\sigma(Q_\alpha)$ and $u_0 \in H^1_\sigma(\mathbb{R}^3)$, with $\tilde{u}^0_\alpha \to u^0$ in $H^1(\mathbb{R}^3)$. [For the definition of the extension $\tilde{u}^0_\alpha$ see section 3.3.] Suppose in addition that there exists $T^* > 0$ such that the equations on $\mathbb{R}^3$ with initial condition u^0 admit a solution*

$$u \in L^\infty([0,T^*];H^1(\mathbb{R}^3)) \cap L^2(0,T^*;H^2(\mathbb{R}^3)).$$

Then for α sufficiently large the equations on the periodic domain Q_α with initial data u^0_α have a smooth solution

$$u_\alpha \in L^\infty(0,T^*;H^1(Q_\alpha)) \cap L^2(0,T^*;H^2(Q_\alpha))$$

and $\tilde{u}_\alpha \to u$ in $L^r(0,T^*;H^1)$, $r \in [1,\infty)$, as $\alpha \to \infty$.

Proof. Since $u \in L^\infty([0,T^*];H^1(\mathbb{R}^3))$ there exists $M > 0$ such that

$$\|u(t)\|^2_{H^1(\mathbb{R}^3)} \leqslant M \quad \text{for all } t \in [0,T^*].$$

Theorem 4.4 guarantees that there exists a uniform time τ such that any solution with $u(0) = v_0$, where $\|v_0\|^2_{H^1(\mathbb{R}^3)} \leqslant 2M$, exists at least on the time interval $[0,\tau]$.

Set $N = 2T^*/\tau$ and fix $r \in [1,\infty)$.

Theorem 6.2 ensures that $\tilde{u}_\alpha \to u$ in $L^r(0,T;H^1(\mathbb{R}^3))$ as $\alpha \to \infty$. In particular, $\tilde{u}_\alpha(t) \to u(t)$ in $H^1(\mathbb{R}^3)$ for almost every $t \in (0,\tau)$; choose one such t with $t > \tau/2$ and call this t_1.

Choose α_1 such that $\|\tilde{u}_\alpha(t_1)\|_{H^1(\mathbb{R}^3)} \leqslant 2M$ for all $\alpha \geqslant \alpha_1$. Since

$$\|u_\alpha(t_1)\|_{H^1(Q_\alpha)} \leqslant \|\tilde{u}_\alpha(t_1)\|_{H^1(\mathbb{R}^3)},$$

this bound is enough to ensure that, uniformly for $\alpha \geqslant \alpha_1$, the solutions on Q_α starting from $u_\alpha(t_1)$ exist on the time interval $[t_1, t_1 + \tau] \supset [\tau, 3\tau/2]$.

Since $\tilde{u}_\alpha(t_1) \to u(t_1)$ in $H^1(\mathbb{R}^3)$, theorem 6.2 can again be used to guarantee that as $\alpha \to \infty$ ($\alpha \geqslant \alpha_1$), have $\tilde{u}_\alpha \to u$ in $L^r(t_1, t_1 + \tau; H^1(\mathbb{R}^3))$. Again, the convergence in $H^1(\mathbb{R}^3)$ for almost-every time means that there exists $t_2 \in (t_1, t_1 + \tau)$ with $t_2 > t_1 + \tau/2 > \tau$ such that $\tilde{u}_\alpha(t_2) \to u(t_2)$ in $H^1(\mathbb{R}^3)$; in particular, there exists $\alpha_2 \geqslant \alpha_1$ such that $\|u_\alpha(t_2)\|_{H^1(Q_\alpha)} \leqslant 2M$ for all $\alpha \geqslant \alpha_2$.

Continue in this way, noting that at each step the interval of existence of the solutions on Q_α (for $\alpha \geqslant \alpha_n$) increases by at least $\tau/2$. After N steps the entire interval $[0, T^*]$ has been covered, showing that the solution on Q_α starting at u_α^0 is strong on $[0, T^*]$ for all $\alpha \geqslant \alpha_N$. $\square$

Note that this result does not say that if the equations are regular on $\mathbb{R}^3$—i.e. if *any* smooth (compactly-supported) initial condition gives rise to a smooth solution for all $t > 0$—then they are regular on Q_α for α large enough (which would then imply regularity on Q_α for any α). Indeed, it does not even guarantee that if a particular compactly-supported smooth initial condition u_0 gives rise to a globally-defined solution on $\mathbb{R}^3$ then the equations on Q_α will also have a globally-defined solution for α sufficiently large. Rather, for a fixed (compactly-supported) initial condition, regularity on $\mathbb{R}^3$ on a given finite time interval $[0, T^*]$ carries over to Q_α for α sufficiently large (so α depends on both u_0 and on T^*). (In this way it is reminiscent of the result of Constantin (1986) mentioned in the introduction, which transfers regularity from a smooth Euler solution on $[0, T]$, starting at $u(0) = u_0$, to the Navier–Stokes equations with sufficiently small viscosity $\nu < \nu_0$; in his result, ν_0 depends on both u_0 and T.)

A full 'transfer of regularity' from one problem to another would require (among other ingredients) a convergence result in which the distance between solutions on $\mathbb{R}^3$ and Q_α could be bounded in terms of the H^1 norm of the initial data, which appears to require much more sophisticated methods that the compactness-based arguments employed here. [For results in this direction for the Ginzburg–Landau equation see Mielke (1997) and for the two-dimensional Navier–Stokes equations see Zelik (2013).]

8. Conclusion

Given fixed sufficiently regular initial data with compact support, solutions of the Navier–Stokes equations on expanding periodic domains converge to the corresponding solution on the whole space; and this can to some extent be 'reversed', in that a compactly-supported initial condition that leads to a strong solution on a time interval $[0, T^*]$ (which could be significantly longer than what is guaranteed by standard existence theorems) will give rise to a strong solution on the same time interval on a sufficiently large periodic domain.

It is natural to conjecture that a similar result holds given any choice of smooth, simply-connected, bounded subset Ω of $\mathbb{R}^3$, replacing $(-\alpha, \alpha)^3$ by $\alpha\Omega$ and imposing no-slip (Dirichlet) boundary conditions on the boundary of $\alpha\Omega$. However, the estimates on the pressure required in the proof given here become much more delicate in the case of a bounded domain (see Sohr and von Wahl 1986, for example).

While the results here demonstrate convergence, they give no error estimates; this appears to be a significantly harder problem, but a particularly interesting one if one is to view solving the equations on a periodic domain as a 'numerical approximation' to the solution of the equations on the whole space. Ożański (2021) has recently obtained such error estimates, comparing solutions of the equations on the whole space and on bounded domains with Dirichlet boundary conditions, by finding a way to treat the bounded domain problem as a perturbation of the problem posed on the whole space.

Acknowledgments

Many thanks to Robert Kerr, for interesting discussions that motivated the problem considered here. Thanks also to Wojciech Ożański and José Rodrigo for helpful conversations about some of the issues that needed to be resolved, and for numerous insightful comments on a preliminary version of the paper. In particular, Wojciech pointed out that a minor change in the argument would give convergence in $L^p(0, T; H^1(\mathbb{R}^3))$ for all $1 \leqslant p < \infty$.

References

Ambrose D M, Kelliher J P, Lopes Filho M C and Nussenzveig Lopes H J 2015 Serfati solutions to the 2D Euler equations on exterior domains *J. Differ. Equ.* **259** 4509–60

Chernyshenko S I, Constantin P, Robinson J C and Titi E S 2007 A posterior irregularity of the three-dimensional Navier–Stokes equations from numerical computations *J. Math. Phys.* **48** 065204

Constantin P 1986 Note on loss of regularity for solutions of the 3D incompressible Euler and related equations *Commun. Math. Phys.* **104** 311–26

Doering C R and Gibbon J D 1995 *Applied Analysis of the Navier–Stokes Equations* (Cambridge: Cambridge University Press)

Enciso A, García-Ferrero M Á and Peralta-Salas D 2018 The Biot–Savart operator of a bounded domain *J. Math. Pures Appl.* **119** 85–113

Evans L C 2010 *Partial Differential Equations* (Providence, RI: American Mathematical Society)

Gallagher I 1997 The tridimensional Navier–Stokes equations with almost bidimensional data: stability, uniqueness, and life span *Int. Math. Res. Not.* **1997** 919–35

Heywood J G 1976 On uniqueness questions in the theory of viscous flow *Acta Math.* **136** 61–102

Heywood J G 1988 Epochs of regularity for weak solutions of the Navier–Stokes equations in unbounded domains *Tohoku Math.* **40** 293–313

Hopf E 1951 Über die Aufgangswertaufgave für die hydrodynamischen Grundliechungen *Math. Nachr.* **4** 213–31

Kerr R M 2018 Enstrophy and circulation scaling for Navier–Stokes reconnection *J. Fluid Mech.* **839** R2

Ladyzhenskaya O A 1969 *The Mathematical Theory of Viscous Incompressible Flow* (London: Gordon and Breach)

Leray J 1934 Sur le mouvement d'un liquide visqueux emplissant l'espace *Acta Math.* **63** 193–248

Mielke A 1997 The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors *Nonlinearity* **10** 199–222

Ożański W 2021 Quantitative transfer of regularity of the incompressible Navier–Stokes equations from $\mathbb{R}^3$ to the case of a bounded domain (arXiv:2107.01803)

Ożański W and Pooley B 2018 Leray's fundamental work on the Navier–Stokes equations: a modern review of 'Sur le mouvement d'un liquide visqueux emplissant l'espace' *Partial Differential Equations in Fluid Mechanics (LMS Lecture Notes)* ed C L Fefferman, J L Rodrigo and J C Robinson (Cambridge: Cambridge University Press) pp 113–203

Raugel G and Sell G R 1993 Navier–Stokes equations on thin 3D domains: I. Global attractors and global regularity of solutions *J. Am. Math. Soc.* **6** 503

Robinson J C 2004 A coupled particle-continuum model: well-posedness and the limit of zero radius *Proc. R. Soc.* A **460** 131–1334

Robinson J C 2020 *An Introduction to Functional Analysis* (Cambridge: Cambridge University Press)

Robinson J C, Rodrigo J L and Sadowski W 2016 *The Three-Dimensional Navier–Stokes Equations (Cambridge studies in advanced mathematics)*vol 157) (Cambridge: Cambridge University Press)

Scheeler M W, Kleckner D, Proment D, Kindlmann G L and Irvine W T M 2014 Helicity conservation by flow across scales in reconnecting vortex links and knots *Proc. Natl Acad. Sci.* **111** 15350–5

Serfati P 1995 Solutions C^∞ en temps, n-log Lipschitz bornées en espace et équation d'Euler *C. R. Acad. Sci., Paris* I **320** 555–8 https://gallica.bnf.fr/ark:/12148/bpt6k62036636/f16.item#
Simon J 1987 Compact sets in the space $L^p(0, T; B)$ *Ann. Mat. Pura Appl.* **146** 65–96
Sohr H and von Wahl W 1986 On the regularity of the pressure of weak solutions of Navier–Stokes equations *Arch. Math* **46** 428–39
Tao T 2013 Localisation and compactness properties of the Navier–Stokes global regularity problem *Anal. PDE* **6** 25–107
Zelik S 2013 Infinite energy solutions for damped Navier–Stokes equations in $\mathbb{R}^2$ *J. Math. Fluid Mech.* **15** 717–45