Dynamic Joint Decision-Making Problems under Uncertainty in Retail Supply Chains

by

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Declarations

I declare that the work in this thesis has been carried out by me and no part of this thesis was previously presented for another degree or diploma of this or another institution. The information from the literature has been duly acknowledged and a list of references is also provided. Chapter 2 is adopted from the paper titled "Dynamic Production-Pricing Strategies for Multi-Generation Products under Uncertainty" co-authored with Nalan Gulpinar and Nursen Aydin. This paper was published in the *International Journal of Production Economics*, Volume 230, December 2020. Chapter 3 is adopted from the working paper titled "Dynamic Joint Ordering-Pricing Strategies for a Deteriorating Perishable Product under Uncertainty" co-authored with Nalan Gulpinar and Nursen Aydin.

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Abstract

Managing flow of a product smoothly within a retail supply chain is a challenging task due to various complexities arising at different levels in which a retailer needs to simultaneously make several (operational and/or strategic) decisions. The different decisions such as inventory, production, ordering and pricing of the product are inter-related and collectively impact on the performance of the firm interacting with different parties in the supply chain. The inter-connectivity of multiple decisions (so-called joint decision-making process) increases the complexity for management of the retail supply chain. Moreover, stochastic and dynamic nature of the retail supply chain as well as its underlying network and product characteristics add further complexities into the joint decision-making process. It is crucial to adopt coordinated and combined decision-making approaches to manage retail supply chains. In this thesis, we develop joint decision-making policies to enhance the operations management of retail supply chains under uncertainty. In particular, we are concerned with three different joint decision-making problems: i) production and pricing of a multi-generation product line, ii) ordering and markdown policies for a perishable product, and iii) ordering and inventory allocation strategies for a dual-channel supply chain. In the first problem, the firm releases a new version of a product periodically while its older versions continue to sell in the market whereas the retailer deals with a perishable product of fixed and short age in the second problem. While perishability of the product is analysed in view of demand variation in the first two problems, we consider a non-perishable product for the final problem about the dual-channel supply chain under both demand and supply uncertainties. In the dual-channel supply chain network, the firm procures the product from a regular and/or emergency supplier and distributes it through multiple channels. Stochastic dynamic programming is used to model the underlying decision-making problems of the retailer that aims to maximize the expected profit over a planning horizon. The stochastic dynamic models suffer from the curse of dimensionality because of the increasing sizes of state and action spaces. Thus, solving these problems is computationally intractable by using traditional solution approaches. We propose alternative novel approaches to solve these complex problems efficiently. Computational experiments are designed to illustrate performance of the joint decision-making models and the proposed solution approaches. In addition, we derive managerial insights to show the significance of dynamic joint decision-making process. The numerical results indicate that jointly taking different operational decisions outperform single decisions made in isolation. Moreover, they highlight further benefits of joint decision-making in efficiently tackling uncertainties and improving the overall performance of retail supply chains.

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Chapter 1

Introduction

This chapter briefly introduces the background retail supply chain management as well as its challenges and complexities. We also discuss the relevance of dynamic and joint decision-making in dealing with challenges of retail supply chains. Then, the objectives of the thesis is presented and the three research topics are described along with their contributions. Since Stochastic Dynamic Programming is widely used to address the dynamic and joint decisions, a short review of the methodology and its solutions approaches is also provided. Finally, the structure of the thesis is presented.

1.1 Retail Supply Chain Management

Retail supply chain management is a widely studied area which continues to receive significant attention in academia and practice. It involves planning of retail operations with the aim of ensuring a smooth flow of a product along the supply chain. However, there are many challenges in establishing the steady flow of product in the supply chain. The retail sector is a fast-moving industry due to continuous and accelerating advances in technology and innovation. Development in technology further enhances the prominence of e-commerce. E-Commerce has been successfully paving numerous avenues to serve customers (Gaffney 2017). At present e-commerce allows customers to purchase products or services from any vendor without any restrictions. The needs of customers are changing faster than ever (Pearson 2012, McKinsey 2019). In 2014, iPhone 6 sold better than Apple's expectations while in the following year the demand for the next generation iPhone 6S fell short of Apple's forecasts. They also had to slash production by 20 million units of their flagship product iPhone X in 2018 due to the decrease in global demand levels. (Kubota et al. 2019). In a rapidly changing market, the prediction of customer choices and demand is a challenging task which affects the entire supply chain. When Apple struggled with its demand forecast, a major concern was raised by its suppliers who either had an excess of unsold inventories or had to ramp up their productions to meet the uncertainty in demand (Kubota et al. 2019). To improve response to demand uncertainty, businesses have been known to innovate with their sourcing strategies.

The supplier networks of many retail firms have increasingly become global (MacCarthy et al. 2016). Many retailers reach out to suppliers across the globe especially to the ones in Asia because of low production and labour cost. Even though the global suppliers assist in cutting costs, they make the supply chain more susceptible to disruption such as natural disasters, pandemics, accidents, volatile financial and political climate. In the first quarter of 2020 when the Covid pandemic hit China, a large part of it was under a lockdown. The lockdown was the strictest in the regions which are manufacturing hubs of raw materials of auto parts, hi-tech components and steel. As a result, there was a delay in production from China which severely impacted operations of firms across the globe, forcing some of them to even shut down their productions (Yu 2020). The pandemic may be a one-off event but disruptions in a supply chain network are certainly not rare. In 2000, a small fire in a Phillips plant of cellphone chips at New Mexico heavily disrupted the supply chain of Ericsson in Sweden. Even though Philips barely faced a loss of less than 0.6 percent of their annual sales that year, Ericsson faced the real brunt as it reported a significant annual loss of nearly $2.3 billion due to this fire accident. Thus, Ericsson's fallout along with other problems unfortunately led them to retreat from the phone handset market (Sheffi 2007). Hence, a retail supply chain operates in a highly dynamic and uncertain environment.

The dynamic and uncertain retail supply chain is also inherently complex as it has multiple aspects to be taken care of (Pearson 2012). Two important aspects of the retail supply chain are its product characteristics and its underlying network. Product characteristics like age and lifetime of the product highly influence the management of supply chain operations. A product which has a fixed life in terms of its age or selling season is referred as perishable. Retail supply chains consist of many products that are perishable, like food, electronics and fashion products. Electronic and fashion products are also referred perishable as they have a fixed shelf-life. As reported by the Food Market Institute, 50.67% of the total supermarket sales in the US accounted for perishable items (FMI 2019). Meanwhile, the sales of the electronics and fashion industry regularly experience a significant increase as well (Bowers 2019). However, most of the research in supply chain management doesn't give considerable importance to the perishable nature of many products. Both academics, e.g, (Blackburn & Scudder 2009) and practitioners (Webber B 2011) argue that conventional supply chain strategies developed for manufacturing industries sometimes fail when the product has fixed life-time. Even though the major reason behind this failure is age dependency of the product, there are other factors as well. In most perishable industries such as supermarkets, fashion and electronics, inventories are replenishable and both old and new inventories co-exist. Particularly in the electronics sector, the new generation of the product is released periodically while the old generation is still on sale, like mobile phones and computers (Levin et al. 2010, Li et al. 2010). Similarly, the shelves in supermarkets display both fresh and marked down food products. When new and old inventories are present together, it is difficult to predict customer preferences. While offering multiple variants of a product helps retailers to target different segments of customers, it may lead to internal competition between products and cannibalise the sales of old products by the new ones or vice versa. Supermarket and fashion retailers often struggle to decide when and how to conduct a discount or markdown sales as increase in discounts may hamper the sales of the new (or fresh) products. The demand of perishable products is unknown and management of perishables must be done in view of the demand uncertainty.

Traditionally, the underlying network of a supply chain consists of flow of products from wholesalers to retailers and finally to customers. Most retailers have moved from the traditional to non-traditional settings, like omni-channel (MacCarthy et al. 2019) and dual-channel retailing. Omni-channel retailing considers distribution of the product through multiple retail channels to ensure a seamless customer experience (Roberts 2019). Meanwhile, dual-retailing focuses on two distribution channels with the aim of blurring lines between wholesaler and retailer. In the supermarkets and electronic goods sector, prominent names such as Costco, Apple, Samsung, Wallmart's Sam's Club, Price Club have shifted from the traditional wholesaler-retailer setting to dual-retailing set-ups (Morris 2004). In these non-traditional supply chains (so called 'wholesaler's clubs' by (FMI 2019)), the retailers such as Costco, Sam's Club, PriceClub and BJ's have physical stores where they serve the customers directly through their unique wholesale store system and also deal with clients from different sectors like restaurant chains, vendors, caterers, day care centres and small grocery stores (Morris 2004). As another example, Apple successfully plays the dual-role of wholesaler and retailer because they sell their products in Apple stores and also distribute them to third party retailers like Amazon and Walmart. Since the dual-channel network is widely distributed, its operations are severely impacted by disruption at any point of the network. A disruption at the demand or supply side can adversely affect the overall operations. Thus, it is essential to simultaneously consider uncertainty from the demand and supply side of the network while managing it.

Retailers must dynamically evolve operations to combat the uncertainties and efficiently manage their supply chain (Demirel et al. 2019). There is a need for retail firm to make strategic and operational decisions in view of the dynamic and uncertain environment. The firm is required manage the multiple and different aspects of supply chain comprising of sourcing and distribution of the product. In order to efficiently distribute and source the product under uncertainty, the key decisions are related to inventory, production, ordering and pricing of the product. These decisions are inter-linked with each other. If the firm experiences high demand for their product, they may tackle it by ramping up their production or ordering. On the other hand, when there is low demand, the firm must have the flexibility to reduce production levels and/or discount its prices to avoid losses due excess of unsold inventories. Both the decisions related to ordering and pricing are aimed at tackling demand uncertainty to improve the firm's overall performance. When these decisions are simultaneously taken, there is a higher chance of dealing with the uncertainties in comparison to a scenario where each decision is independently taken. Several practitioners also advocate the need for taking inter-related decisions (Pearson 2012, McKinsey 2019).

The academic literature on retail supply chains also highlights the significance of joint decision-making to efficiently manage inter-linked operations such as inventory management, production, ordering and pricing of a product (Simatupang & Sridharan 2002, Barratt 2004, Min et al. 2005). There are several streams of literature dedicated to joint decision-making problems in retail supply chains such as, combined inventory-pricing decisions (see Elmaghraby & Keskinocak (2003), Yano & Gilbert (2005) and Chen & Simchi-Levi (2012)), fulfillment-distributions decisions (see Agatz et al. (2008) and Zhang et al. (2010)) and marketing-production decisions (see Eliashberg & Steinberg (1993) and Upasani & Uzsoy (2008)). However, the focus of most of the joint decision-making literature lies in analysing the inter-connectivity of these decisions via analytical, computational and statistical studies. On the other hand, there are very few joint decision-making studies which aim to capture the complexities arising intrinsically within different retail set-ups of perishable products and dual-retailing networks. Although there are some studies on joint decision-making related to inventory and pricing management of perishable products (Li et al. 2009, Chen & Sapra 2013, Chen et al. 2014), most of them don't consider existence of new and old variants of the perishable products. In comparison to these studies, we capture the gap of considering a retailer who simultaneously sells new and old variants of the perishable products. In addition, we also incorporate differentiation in selling prices of new and old inventories in our research. This way our models are able to encapsulate several relevant features like internal competition and demand cannibalisation between new and old inventories. Most studies on dual-channel distribution network are focused on investigating coordination among multiple channels (Janakiraman et al. 2015, Wang et al. 2017, Hu, Li, Byon & Lawrence 2015, Zhu et al. 2020). Very limited studies related to dual-channel networks discuss its management in view of demand and supply uncertainties. Meanwhile, research on management of retail firms facing supply uncertainty rarely focus on the structure of their distribution network. Thus, we are concerned with finding joint decision-making strategies for a dual-channel network under both demand and supply uncertainties. In this thesis, we investigate dynamic joint decision-making policies for retail supply chains in view of its inherent features like perishability and dual-channel distribution network.

1.2 Objectives and Contributions of Thesis

In this thesis, we address important issues related to product characteristics and the underlying distribution network while designing dynamic joint decision-making for retail supply chains. In order to incorporate product characteristics like perishability in the joint decision-making process, we introduce novel models and solution techniques that capture features related to customer choices and demand uncertainty. In addition, we develop a joint decision-making model for the dual-channel distribution network while considering the important features like demand-supply uncertainties and dual-sourcing. Efficient solution methodologies are also proposed to solve the joint decision-making problem of dual-channel network. We now provide a summary of the three research problems and highlight the corresponding contributions.

Production and Pricing of Multi-Generation Product Line

Due to rapid advances in technology and design, some retail firms periodically release new generations of electronic products such as mobile phones and computers. In order to increase product variability, firms may wish to develop a multiple-generation product line rather than replace the older versions with new ones. However, when multiple generations are available in the market, different generations compete with each other as well as other products in the market. Firms need to take joint decisions for inventory management and dynamic pricing of multiple generations to tackle impact of uncertain demand and market competition. In this research problem, we present a dynamic joint production-pricing decision model to obtain efficient strategies for a firm selling multiple generations of a product. The contributions of this research are highlighted below,

- The existing literature on multi-product pricing and production problems generally focuses on only two products and assumes a constant product value over time. However, during a new product release, the old generation becomes less attractive due to the new generation's technological improvements or additional features. In contrast to the existing literature, we account for the internal competition among multiple generations by considering dynamic changes in product value.

- The joint production-pricing decision making problem of a firm selling multiple generations of a product under demand uncertainty is formulated as a stochastic dynamic programming model. The consideration of multiple generations in the joint production and pricing model flares up the state space. Moreover, the customer choice probabilities depending on pricing decisions lead to a nonlinear (high degree polynomials) optimisation problem to be solved at each state of the system. Therefore, the underlying dynamic programming model is computationally intractable to solve by a traditional (backward) dynamic programming technique. In order to tackle curse of dimensionality on the state space, we propose an approximation method and a heuristic approach.

- The first solution approach considers a forward dynamic programming algorithm for approximately solving the joint production-pricing problem. The second approach applies for a two-stage heuristic algorithm where the pricing decisions (determined in the first stage of the algorithm) are integrated into the optimisation model at the second stage to determine the optimal production level for each product. In order to improve further computational performance of the solution approaches, we investigate different pricing rules determined from the abridged model and a list of prices derived by theoretical bounds.

- We design numerical experiments to illustrate performance of solution approaches and derive some managerial insights. Our numerical analysis indicates that joint production-pricing strategy performs significantly better than the fixed policies since it considers recent changes while matching demand with production. We also observe that the customer choices play an important role on the performance of the joint decision making process.

Ordering and Markdown Policies for Perishable Product

Perishable products are present in abundance in food industries. The firms often promote the freshness level of perishable products to gain strong position in the market. However, they also face the challenge of the increasing amount of wastage of perishable products due to demand uncertainty. A common strategy to reduce wastage is to lower the selling price of inventories close to expiry. In this research problem, we develop markdown and ordering strategies to sell a perishable product under uncertainty in its demand. The joint markdown and ordering strategies hold managerial significance and are also computationally efficient. We highlight the contributions of this research as follows;

- The demand cannibalization between fresh and markdown inventories is evaluated by the dynamic changes in customer choices. The customer choices are then integrated in a demand model depending the pricing and ordering decisions. The joint ordering-pricing decision-making problem for a general lifetime perishable product is formulated as a stochastic dynamic programming model. The challenge with a perishable inventory management problem is the dimensional expansion of the state space, as a result of continuous tracking of the product age. Due to the complexity of this problem, the existing research largely focuses on inventory systems with a two-period lifetime product. In contrast to the existing literature, we consider a general lifetime perishable product and we tackle the curse of dimensionality on the state space by proposing an exact solution methodology yielding optimal solutions.

- The exact solution methodology is constructed on the basis of the theoretical properties of the dynamic programming model. We prove that the value function of the dynamic programming model is k-concave in inventory levels. We then employ the properties of k-concavity to design an algorithm providing the optimal ordering and pricing policy. Numerical experiments are conducted to illustrate the performance of the solution algorithm and to gain managerial insights on the ordering and pricing strategies

- In the numerical study, the joint ordering-markdown policies are compared with various fixed markdown policies followed by practitioners.Our results highlight benefit of following joint decision-making policies over fixed markdown policies. The flexibility in conducting a markdown sale is also investigated in our experiments wherein we compare different flexible strategies motivated from practice. Our finding emphasise the importance of focusing on the age of markdown product more than its price. Moreover, the joint ordering-markdown policies are examined in different customer segments.

Ordering and Delivery Strategies for Dual-Channel Network

In a dual-channel network a firm distributes its products through different channels mainly comprising of its own stores as well as third-party retailers. Since the network of dual-channel firm is extensively and widely spread across different channels, a disruption at any point of the network impacts the overall operations of the firm. Thus, during the management of the dual-channel firm, it is essential to consider disruption at both demand and supply side of the network. To mitigate the disruption from demand or supply side, the firm diversifies its sourcing strategy by ordering from a set of emergency suppliers. In this research, we investigate the joint ordering and delivery strategies of the dual-channel firm under disruption. The contribution of the research are provided as below,

- Existing studies mainly consider either demand, supply uncertainties or both independently in the dual-channel supply chain. However, we find joint ordering and delivery policies to simultaneously tackle both demand and supply uncertainties in contrast to the existing literature. The joint decision-making problem of the dual-channel network under demand and supply uncertainties is formulated as a stochastic dynamic programming problem.
- The consideration of dual-retail channels and multiple decisions flares up the state and action space. Thus, the underlying dynamic programming model is computationally difficult to solve by the standard solution methodology. We proposed two decomposition methods which are tailorly designed for the given dual-channel network. In the first approach, we decompose the model by each channel. To protect the inter-connectivity between multiple channels, we introduce an opportunity cost parameter by considering each channel's effect on the network. The second approach of decomposition is adopted from the practice (RGIS 2013, ASP 2019, Oracle 2019). Due to advances of technology in inventory-tracking, the firm is able to receive information about inventory levels at its owned stores as well as third party retailers. This information assists the firm in improving their decision-making. By using the information, the original model is reformulated such that the firm can track inventories at all channels as well as its central echelon. This model is then decomposed by each inventory-tracking point and solved via two-stage decision making process to obtain ordering and delivery decisions. We conduct a computational study to illustrate the performance of the various solutions methodologies with the standard backward dynamic programming.
- Our approach is compared with a threshold-based policy designed through practitioner and academic reports. An extensive numerical study is also conducted to highlight different features of the joint ordering-delivery model and derive managerial insights. Our results emphasise the importance of collectively considering both demand and supply uncertainties. Moreover, the benefit of joint

ordering and delivery policies are also analysed with varying levels of ordering and penalty costs. Our findings suggest that joint decision-making policy not only does well in terms of profit but it also ensures minimum loss in demand.

1.3 Review of Joint Decision-Making Approaches

To efficiently manage the flow of a product in a retail supply chain, firms must jointly consider its operational decisions. Since the different types of decisions are inter-linked, the need for a joint decision-making approach is advocated. Moreover, there is a higher chance of tackling with uncertainties when the decisions are taken simultaneously in comparison to be taken independently. However, the inter-linkage between decisions and the uncertain environment significantly enhances the complexity of modelling and solving the problem. Researchers have tried several ways to model and solve the joint decision-making problems. Some studies use statistical methods such as regressions, multivariate statistical analysis and structural equation modelling, to compare joint versus single decision-making approaches. Chaudhuri et al. (2018) and Danese et al. (2013) consider different data sets and use regression methods to analyse the benefit of integrating decision-making process internally in supply chains. Chaudhuri et al. (2018) examine the benefit vs risks of integrating the decision-making process in various ways. Meanwhile, Danese et al. (2013) test the impact of supply chain integration on firm's responsiveness levels. Munir et al. (2020) use covariance-based structural equation modelling to examine the impact of internal integration to manage risk, handle unexpected disruptions and improve performance of the supply chain. Apart from statistical analysis, some studies specifically design behavioural experiments to evaluate the benefit of joint decision-making. Ramachandran et al. (2018) showcase the improvement in overall performance of firms when decisions related to the pricing and quantity of their products are taken in a combined manner. They also discuss how joint decision-making framework captures the inter-dependences between decisions and reduces the uncertainty. A similar study for a two-stage supply chain comprising of a supplier and a retailer is conducted by Davis & Hyndman (2019).

The above-mentioned studies examine primary or secondary data sets to establish the merit and demerits of joint decision-making. However, there are several research areas where data is either not available or is not viable for collection. Lack of data is successfully replaced by simulating specialised environments. Some papers design simulation-based studies to test the efficiency of combined decision-making.

Moreover, simulation is also used to model real-life operational systems. Datta & Christopher (2011) use an agent-based simulation approach to test different levels of integration in decision-making to tackle uncertainties in the supply chain. Meanwhile, Van Der Vorst et al. (2009) design a specific simulation set-up for food supply chains proposing integration towards logistics, sustainability and food quality. Taghikhah et al. (2021) adopt a system dynamics approach to operationalize agro-food supply chain and simulate adaptive behaviour of farmers, food processors, retailers, and customers. They jointly consider operational factors (e.g., price, quantity, and lead time) and behavioural factors (attitude, perceived control, habits) of the supply chain.

The methodology and research mentioned so far focuses on testing and analysing the impact of different types of integration in decision-making. However, we are concerned with developing (or designing) joint decision-making policies applicable in realistic scenarios (or practice). There are various decision-making models with single or multiple objectives satisfying certain restrictions. Optimisation comprises of finding the best decision by evaluating all possibilities. However, to efficiently apply optimisation, the decision-making problem must be mathematically represented as accurately as possible. When a problem is being modelled mathematically, there are several problem characteristics to be considered, like deterministic or uncertain, static or dynamic nature of the problem. If the optimal decisions are obtained in view of known parameters, it is called deterministic whereas in stochastic optimisation, the parameters are uncertain and must be integrated in the mathematical model. There are various techniques to solve optimisation problems, like linear programming, integer programming and non-linear programming. The reader is referred to Winston (2002) and Williams (2013) for a detailed review of deterministic and static optimisation methods.

Decision-making under uncertainty involves finding best decisions in the face of unknown parameters whose ramifications will only be known at a later stage. Techniques, like stochastic, two-stage and robust optimisation, incorporate uncertainty in decision-making process. In order to incorporate uncertainties in a decision-making problem, one might think of replacing the uncertain parameters with average values or point-wise estimates. By using average values, the decision-making problem can then be formulated with deterministic methods. However, Ben-Tal & Nemirovski (2000) show that the solutions obtained with average values not only perform poorly in an uncertain environment but they can become infeasible as well. Thus, for decision-making problems in an uncertain environment, it is essential to incorporate the uncertainties with maximum mathematical accuracy through sophisticated stochastic optimisation methods. Shen et al. (2003), Wang (2006), Hong et al. (2015) and Liu & Li (2021) use stochastic optimisation methods to solve joint decision-making problems in different retail set-ups. The decision-making in retail set-ups considered in this thesis have to be taken sequentially in regular intervals of time. Moreover, there are different kinds of uncertainties present in the three research problems of the thesis. If the firm takes certain decision, they wouldn't know its outcome or the state of the system in the future. Thus, the firm must formulate the problem using stochastic dynamic programming in view of its dynamic and uncertain nature.

Game theory is another approach that is widely used to investigate the coordination among different players of supply chain (see Moharana et al. (2012), Yang et al. (2015), Hu et al. (2018) and Zhu et al. (2020)). However, retail set-ups of this research are considered from the point of view of the firm. We assume there is limited and fixed interaction among multiple decision-makers. This assumption is based on information on contractual agreements among different players. The details of the agreement between the multiple parties of the supply chain are either established or private in nature. When the participation among multiple players have fixed contracts, it leaves little or no scope for negotiations. Thus, game theory is not suitable to model the decision-making problem among players where there the terms of interactions are already well-established. On the other hand, if there is scope of negotiation, then game theory can be a useful modelling approach. However, in the three retail set-ups considered, there is either limited information on the contractual negotiation or the firms have defined fixed contracts.

Stochastic dynamic programming is extensively used to model decision-making problems of supply chain management. In this thesis, SDP is used to model all three joint decision-making problems. This method will be further reviewed in the next section.

1.3.1 Stochastic Dynamic Programming

Stochastic dynamic programming, abbreviated as SDP, is a combination of stochastic and dynamic programming. Thus, SDP is applied to model and solve complex problems comprising of sequential decision-making under uncertainty. SDP is also known as Markov Decision Process (Bellman 1958, Denardo 2012). It involves a a discrete-time stochastic control process where the outcomes of decisions may or may not be known. In general, probability distributions are used to represent uncertain parameters in the dynamic programming equations. The aim of SDP is to find an optimal policy in the presence of uncertainty. We next present a general formulation of SDP model for single and joint decision-making processes.

Single Decision-Making Model: Let us assume that the planning horizon of the decision-making process comprises of $T+1$ discrete time periods. We also define a set of all possible states of the system, denoted as $\mathcal{S}$. A state of the system at the start of every decision-making period is represented as s_t, where $s_t \in \mathcal{S}$. Given the state s_t of the system at time t, a decision, denoted as x_t, must be taken in view of the uncertainty $\tilde{a}_t$. At time t, the decision x_t belongs to a feasibility set represented as $\mathcal{X}_t$. The immediate reward in terms of cost or profit obtained by taking decision x_t in the state of s_t under uncertainty $\tilde{a}_t$ is expressed as $R_t(x_t, s_t, \tilde{a}_t)$. As we move forward in time, the state of the system must be updated. The state of the system is updated to a constant value of s_{t+1} after taking decision x_t if there is no uncertainty involved. However, due to the presence of uncertainty $\tilde{a}_t$ at time t, it is unsure what would be the ending state. Thus, at time t a probability value is assigned to each possible state the system can be in the next decision-making period $t+1$. The state transition probability for every state j is evaluated as a conditional probability of $P(s_{t+1} = j | x_t, s_t, \tilde{a}_t)$ due to the underlying conditions of the current state s_t, decision x_t and uncertainty $\tilde{a}_t$. The objective the dynamic problem is to find the optimal value of decision x_t such that the overall reward over the planning horizon is maximised. The value function at any time t is expressed as the sum of reward function and the expected future benefit,

$$V_t(s_t) = \max_{x_t \in \mathcal{X}_t} \quad \mathbb{E}[R_t(x_t, s_t, \tilde{a}_t) + \sum_{j \in \mathcal{S}} P(s_{t+1} = j | x_t, s_t, \tilde{a}_t) V_{t+1}(j)] \tag{1.1}$$

Joint Decision-Making Model: The single decision-making model (1.1) presented above comprises of a single decision x_t. In this thesis, as we consider joint decision-making process, we next present its general formulation. Along with decision x_t, we introduce an additional decision, denoted as y_t. Let the feasible set for joint decisions x_t and y_t be represented as $\mathcal{Y}_t$. In addition, the state of joint decision-making problems usually consists of multiple components. Thus, we consider a multi-dimensional state space represented by a vector $\mathbf{s_t} \in \mathcal{S}$. The updated formulation for the joint decision-making problem is provided as below,

$$V_t(\mathbf{s_t}) = \max_{x_t, y_t \in \mathcal{Y}_t} \quad \mathbb{E}[R_t(x_t, y_t, \mathbf{s_t}, \tilde{a}_t) + \sum_{j \in \mathcal{S}} P(\mathbf{s_{t+1}} = j | x_t, y_t, \mathbf{s_t}, \tilde{a}_t) V_{t+1}(j)] \tag{1.2}$$

While including multiple decisions, their inter-connectivity will also be reflected in the decision-making model. This can enhance the efficiency of the overall system in comparison to single decision-making processes. However, the consideration of joint decision-making will also expand the size and dimension of the state and action space. Thus, we now discuss efficient solution approaches for joint decision-making processes.

1.3.2 Solution Approaches

In this section, we present the solution methods for dynamic joint decision-making models. Since it is difficult to obtain closed form solutions of the dynamic models, we discuss different approximate techniques as well.

Backward Recursion Method: It involves a process of solving the decision-making problem backwards in time. The idea is to start from the end point of the problem, find the best sequence of decisions while travelling back to the starting point. While starting the recursion at the final point, all possible outcomes (or states) are evaluated in face of any decision (or action) taken in that time. The same process will be followed in all times till the starting point. Thus, at each decision-making stage, all possible points of the state and action space are evaluated to yield optimal results. The initial state and boundary condition must be clearly specified to obtain an optimal solution using backward recursion

In the joint decision-making problems, there is a significant increase in the dimension of state as well as the action space due to the presence of multiple decisions. Moreover, large size of practical retail problems further enhances the size of state and action space. When the state and action space is high-dimensional and large, it is computationally intractable to evaluate every point in it. It is very challenging to use backward recursion to find optimal solutions. This is also referred as the curse of dimensionality (Bellman 1966).

Approximate Solution Approaches: To deal with the curse of dimensionality, several approximate solution methods have been proposed in the literature like, linear programming approximation, forward dynamic programming and decomposition methods. In LP approximation, the SDP is reformulated as a LP model where the value functions at all possible states are considered decisions variables. Due to the large size of state space, the LP model also becomes computationally challenging to solve. There is research on finding efficient approximations to resolve the issue of large size LP models (Trick & Zin 1993, Schuurmans & Patrascu 2002). Apart from LP approximation, forward dynamic programming & decomposition methods are used to approximate solutions of SDP models. We will now provide a brief overview of them.

Forward Dynamic Programming (FDP): It is a simulation based algorithmic framework and solves the underlying dynamic programming problem by stepping forward in time as opposed to the backward recursion. In addition, FDP also differs from BDP by not evaluting every point in the state and action and being computationally efficient. In addition, FDP moves forward by selecting a sample path for the uncertain parameters. Sample paths are generated Monte Carlo simulation for a number of simulation iterations. In each iteration at the different states of the random path, the value function is updated using regression models. We present a general form of the FDP approach for joint-decision model (1.2). Let $\hat{V}_t(\mathbf{s_t})$ and $\bar{V}_t(\mathbf{s_t})$ denote the optimized and approximated value functions for state $\mathbf{s_t}$ at time t. The iterative updating of the value function is displayed as below,

$$\hat{V}_t(\mathbf{s_t}) = \max_{x_t, y_t \in \mathcal{X}_t} \quad \mathbb{E}[R_t(x_t, y_t, \mathbf{s_t}, \tilde{a}_t) + \sum_{j \in \mathcal{S}} P(\mathbf{s_{t+1}} = j | x_t, y_t, \mathbf{s_t}, \tilde{a}_t) \bar{V}_{t+1}(j)] \tag{1.3}$$

Powell & Topaloglu (2003) and Powell (2007) provide detailed information on FDP.

Decomposition Method: In the decomposition approach, the SDP model is disintegrated with the aim of reducing the dimension of the state and/or action space in smaller-sized model. The decomposed models with the contracted state and/or action space are then solved independently by standard backward recursion. The information from the decomposed model are integrated in a final step when the model is solved forward in time to avoid visiting all possible values of state and action space. We present a general form decomposition approach for joint decision model (1.2). Let the number of decomposition for the model (1.2) be denoted by n. In addition, the compressed state space of the decomposed model is denoted as $\mathbf{s_t^{'}}$ at any time t. The decomposed model i is provided below, where $i = 1, \cdots, n$,

$$v_{it}(\mathbf{s_t^{'}}) = \max_{x_t, y_t \in \mathcal{X}_t} \quad \mathbb{E}[R_t(x_t, \mathbf{s_t^{'}}, \tilde{a}_t) + \sum_{j \in \mathcal{S}} P(\mathbf{s_{t+1}^{'}} = j | x_t, \mathbf{s_t^{'}}, \tilde{a}_t) v_{i,t+1}(j)] \tag{1.4}$$

where boundary condition is $v_{i,T+1}(\mathbf{s_{T+1}^{'}}) = 0$. The information from the decomposed models is tied up together in an approximate value function denoted as $\hat{V}_t(.)$ and expressed as follows,

$$\hat{V}_t(\mathbf{s_t}) = \max_{x_t, y_t \in \mathcal{X}_t} \quad \mathbb{E}[R_t(x_t, y_t, \mathbf{s_t}, \tilde{a}_t) + \sum_{j \in \mathcal{S}} P(\mathbf{s_{t+1}} = j | x_t, y_t, \mathbf{s_t}, \tilde{a}_t) \sum_{i \in \mathcal{D}} v_{i,t+1}(j)] \tag{1.5}$$

The above model is solved in a forward manner and the value of future expected profit is obtained using model (1.5). The decomposition approach avoids searching all the state and action space by solving forward in time while using the information from the decomposed model. For further information, the interested reader is referred to Archibald et al. (1999) and Kunnumkal & Topaloglu (2010). This approach is used in revenue management problems of airline and hotel management (Erdelyi & Topaloglu 2010, Aydin & Birbil 2018).

In this thesis, we consider backward recursion method as a benchmark policy to compare it with the proposed solution approaches in Chapters 3 and 4. In particular, we employ the technique of FDP and decompositions to solve high-dimensional stochastic dynamic models.

1.4 The Structure of Thesis

The thesis comprises of five chapters. Chapter 1 provides a brief introduction and overview of the thesis. It also covers a review of stochastic dynamic programming models and its solution approaches. The dynamic joint decision-making problems under uncertainty of three retail set-ups are discussed in the next three chapters. In the three chapters, we first introduce the motivation and significance of studying the research problem. The most relevant research is also highlighted in literature reviews for each of the research problem. Then, the joint decision-making model, its features and efficient solution approaches are provided in detail. We also discuss the results of numerical experiments highlighting the features of the model and solution approaches. In particular, Chapter 2 focuses on developing production and pricing strategies for a multi-generation product line while Chapter 3 investigates ordering and markdown strategies to tackle demand uncertainty for a deteriorating perishable product. Then, Chapter 4 proposes ordering and delivery policies for a dual-channel network under disruption. Finally, Chapter 5 concludes the thesis by summarizing the research, its key findings and the direction of future research.

Chapter 2

Production and Pricing of Multi-Generation Product Line

Due to rapid advances in technology and design, some retail firms periodically release new generations of electronic products such as mobile phones and computers. In order to increase product variety, a retailer may wish to develop a multiple-generation product line rather than replace the older versions with new ones. However, when multiple generations are available in the market, different generations compete with each other as well as other products in the market. Firms need to take joint decisions for inventory management and dynamic pricing of multiple generations to tackle the impact of uncertain demand and market competition. In this research question, we present a dynamic joint production-pricing decision model to obtain efficient strategies for a firm selling multiple generations of a product.

This chapter describes a multi-generation product line along with its challenges (Bhatia et al. 2020). Then, we highlight the need for joint decision-making to tackle the challenges. A detailed literature review of inventory and pricing management relevant to this research problem is provided. The stochastic dynamic programming formulation along with the customer choice model of joint production-pricing problem is then presented. The two proposed solution methodologies are explained in detail and their respective pseudo-codes are also provided. The computational experiments testing the performance of solution methodologies as well as the joint production-pricing policies are presented. Finally, the concluding remarks are provided.

2.1 Multi-Generation Product Line

In the technology industry of consumer electronics, firms gain competitive advantage by periodically releasing a new-generation product every 12-15 months while keeping multiple generations in the market (Weyhrich 2018). Having a multi-generation product line is more profitable for a firm than selling a single model due to the increase in product variability (Kilicay-Ergin et al. 2015). Many leading firms have developed multi-generation product lines to target different customer segments. For instance, Apple has simultaneously offered four generations of iPhone (namely iPhone 8, XR, XS and 11) since 2017. Similarly, Samsung has had five generations of Galaxy S (Apple 2020, Samsung 2020).

Despite the profitability of a multi-generation product line, it creates various challenges at strategic and operational levels. In comparison to the older generation products, the new-generation is always assumed to have innovative and improved features, which have never been exposed to the market yet. Therefore, there is a high uncertainty in customers' initial response towards a new product. Apart from the innovative features, price as another important factor affects customers' behaviour to distinguish among multiple generations. It is expected that price of the older generations drops with the introduction of a new product. Although a new release attracts customers, the older generation products' sales might still increase due to price drop. For instance, when iPhone 7's release date was announced, Apple cut the price of iPhone 6 by $100 and iPhone 6 attained the largest market share in the US (Smith 2016, Munbodh 2016). The surge in sales of old generations is another strong motivation behind the firm maintaining a multi-generation product line. On the other hand, customers' reaction to the older generations, particularly in the presence of new release is unpredictable, as well. In fact, customers may attempt to "get more for less" by delaying purchase in hope of price reductions of the older generations followed by a new release (Levin et al. 2010). Customer anticipation related to the prices and innovative features of different generations may lead to *internal competition* across the multi-generation product line and internal competition cannibalises sales of existing products by the newest generation or vice versa. Thus, internal competition between the old- and new-generation products often creates conflicting benefits for the firm (Ferguson & Koenigsberg 2007, Li et al. 2010).

A forward-looking approach has been often used in practice to predict future business conditions and determine optimal strategies balancing supply and demand uncertainties (Li et al. 2010). This approach involves critical decision-making problems such as production planning and pricing that affect the success of a multi-generation product line. During a new release, a firm needs to determine the prices for both new and old-generations, as well as the production and inventory plan for all generations. The new-generation is usually priced higher than the previous generations based on the additional features and upgrades. On the other hand, the price reduction decision for old generations may be affected by unsold inventory of the current product line. Apart from pricing decisions, the unsold inventory levels also impact firms' production plan for multiple generations. Thus, production and pricing strategies cannot be developed in isolation. Joint decision-making models are necessary to successfully manage multi-generation product lines. Although many academics (Davis 1993, Chen & Simchi-Levi 2004, Talluri & Ryzin 2004, Karaesmen et al. 2011) and practitioners (Webber et al. 2011) advocate the importance of joint pricing-inventory techniques as essential tools to mitigate demand uncertainty for fixed-age products, this research area has not yet received enough attention. Unpredictable customers' response during a new product release does not only cause demand uncertainty for all generations, but also creates internal competition among available generations in the market.

In this chapter, we are concerned with dynamic joint production-pricing strategies for a multi-generation product line by considering demand uncertainty and internal competition. Our contribution in this research is two-fold:

- We formulate the joint production-pricing decision making problem of a firm selling multiple generations of a product under demand uncertainty as a stochastic dynamic programming model. The existing literature on multi-product pricing and production problems, as we review in the next section, generally focuses on only two products and assumes a constant product value over time. However, during a new product release, the old generation becomes less attractive due to the new generation's technological improvements or additional features. In contrast to the existing literature, we account for the internal competition among multiple generations by evaluating the dynamic changes in customer choice model. The consideration of multiple generations in the joint-production and pricing model largely expands the state space. Moreover, the customer choice probabilities depending on pricing decisions lead to a nonlinear (high degree polynomials) optimisation problem to be solved at each state of the system. Therefore, the underlying dynamic programming model is computationally intractable to solve by a traditional (backward) dynamic programming technique. This requires efficient approximation method.
- In order to tackle curse of dimensionality on the state space, we propose an approximation method and a heuristic approach. The first approach considers a forward dynamic programming algorithm for approximately solving the

joint production-pricing problem. At each iteration of the algorithm, for a given customer arrival path, the decision-making model is solved to determine joint production-pricing strategy. The second approach applies a two-stage heuristic algorithm that adopts the idea of partial planning introduced by Chan et al. (2006). The pricing decisions (determined in the first stage of the algorithm) are integrated into the optimisation model at the second stage to determine the optimal production level for each product. In order to improve further computational performance of the solution approaches, we investigate different pricing rules determined from the abridged model and a list of prices derived by theoretical bounds. We design numerical experiments to illustrate performance of solution approaches and derive some managerial insights. In our numerical experiments, we analyse the benefits of selling multiple generations of a product on firm profit. We also quantify the benefits of dynamic joint production-pricing decisions as opposed to using fixed policies based on either production or pricing. Our analysis indicates that joint production-pricing strategy performs significantly better than fixed policies since it considers recent changes while matching demand with production. We also observe that the customer choices play an important role on the performance of the joint decision making process.

The remaining part of the chapter is organized as follows. Section 2.2 focuses on the literature review by providing details of existing studies relevant to our research. The stochastic dynamic programming formulation of the joint inventory-pricing problem is presented in Section 2.3. The solution methodology and computational results are explained in Sections 2.4 and 2.5, respectively. The concluding remarks are provided in Section 2.6.

2.2 Review on Inventory and Pricing Management

The research on dynamic inventory and pricing management problems has attracted significant attention over the years. The recent surveys for these problems are provided by Chen & Simchi-Levi (2012) and Janssen et al. (2016). In this study, we consider a firm producing a multi-generation product such as mobile phones and laptops. Although electronic products do not have a short shelf lifetime like perishable products have, older versions of a multi-generation product are generally discontinued from the market after some time due to technological developments. In this respect, a multi-generation product can be considered as a perishable product. Therefore, we focus our review on joint inventory-pricing management research for both non-perishable and perishable products.

Within the joint inventory-pricing management research on non-perishable products, initial studies consider sale of only one product and assume that the seller has convex production and holding costs, and unlimited production capacity (Thowsen 1975, Federgruen & Heching 1999). By using the properties of the model, they show that the base-stock policy (place an order when the inventory level drops below the base-stock level) is optimal for such problems and the optimal price is a decreasing function of the starting inventory. Similarly, Chen & Simchi-Levi (2004) extend the work of Federgruen & Heching (1999) by considering the fixed setup cost for ordering. The optimal policies of joint ordering and pricing problem for a multi-period problem are derived by assuming an additive demand model. Chao et al. (2012) focus on the model proposed by Chen & Simchi-Levi (2004) and investigate the pricing and inventory control policies under the limited production capacity. Chan et al. (2006) propose partial-planning strategies for pricing and inventory replenishment problem by considering capacity constraints. Under a partial planning strategy, the seller separates the pricing and production decisions and decides either pricing or production schedule at the beginning of the planning horizon. The remaining decision (pricing or production schedule) is made by considering demand uncertainty. They proposed several heuristics based on the proposed dynamic programming model to solve these partial-planning problems.

A few researchers have addressed the joint inventory and pricing management problem for multiple non-perishable products. Gilbert (2000) develops a solution method for the joint decision-making model with deterministic demand. The proposed demand model does not consider the cross-price effect between the non-perishable products. Zhu & Thonemann (2009) extend this case and focus on a two-product model in which demand for each non-perishable product depends linearly on the prices of both products. They show that the optimal inventory policy is similar to the base-stock policy for the one-product problem. Song & Xue (2007) consider a more general demand setting for substitutable multiple products. They formulate the problem as a dynamic program and develop a solution algorithm by exploiting the special problem structure. Yan et al. (2017) study joint production and pricing policies for a firm selling new and remanufactured products by considering possible product returns. They assume that the firm either adopts make-to-order or make-to-stock strategy for the new product and under this set-up, they show that the base-stock type production policy is optimal for the make-to-stock strategy for additive demand model. Elmaghraby & Keskinocak (2003) and Chen & Simchi-Levi (2012) provide a review for joint inventory-pricing management of non-perishable product.

Products with short and fixed lifetimes are known as perishable products, such as vegetables, dairy products and medicines. Perishability is also observed in many high-tech products such as laptops, mobile phones, digital cameras due to the rapid obsolescence in a fast moving market (Ferguson & Koenigsberg 2007). Within the research on joint inventory-pricing management for perishable products, there are various studies exclusively developed for food or healthcare products (Li et al. 2009, Chen & Sapra 2013, Chen et al. 2014, Chintapalli 2015, Herbon 2017). We only discuss the studies which are relevant to management of a multiple generation product line. Ferguson & Koenigsberg (2007) consider a firm selling a food product with exactly two-period life cycle. In the first time period, procurement and pricing decisions of a fresh product is made in the presence of demand uncertainty. At the beginning of second time period, the decisions are concerned with how much leftover inventory from first period (old inventory) to carry over, how much fresh products to procure (new inventory), and what prices to charge for new and old inventories. They assume demand to be known in the second time period. They use a customer utility model to obtain demand functions of new and old products based on price and features of the product using a subgame perfect equilibrium in the second time period. A perfect equilibrium is achievable in a deterministic setting. Sainathan (2013) extends the joint inventory-pricing decision model introduced by Ferguson & Koenigsberg (2007)'s to the case of a firm experiencing uncertain demand. He uses a linear utility customer choice model to derive optimal pricing and replenishment policies for a product with two-period shelf life. He assumes that old and new perishable products compete with each other in the market under demand uncertainty that is incorporated through the process of dynamic demand substitution; i.e. demand for an old food item is replaced by a new one. Chintapalli (2015) works on the joint inventory-pricing decision model for a firm experiencing substitutable demand for a n-period shelf-life food product. However, demand for multiple generation products, like electronics, is not substitutable since price difference among its various versions is higher than the old and new food items due to improved features. In other words, the demand for a customer for a certain generation cannot be substituted by its successive or predecessive generation.

In some high-tech industries, the firm's aim to release a new product is to eventually replace the older versions. The transition from the current product to a new one does not occur instantaneously but rather involves a period of time, referred as the product transition or product rollover (Li et al. 2010). A firm can either completely replace the old generation by the new one or continue to sell multiple generations until the sales of the old generations diminishes. These strategies are known as the single and the dual product rollover, respectively (Corey Billington & Tang 1998, Feryal Erhun & Hopman 2007). Several studies compare the benefits of both strategies by considering the internal competition between two generations (Lim & Tang 2006, Arslan et al. 2009, Zhou et al. 2015). Liang et al. (2014) extend the earlier work on rollover strategies by analysing the interaction between rollover strategy and strategic waiting behaviour. They formulate a two-period problem in which a firm releases a new generation of the product in each period. By analysing the customers' optimal purchase decisions, they conclude that optimal rollover strategy significantly depends on the new product's innovation and the number of strategic customers in the market. In a related work, Liu et al. (2018) compare product rollover strategies when customers are allowed to trade-in the older version of the product with the new one. They propose a two-period dynamic game model and analyse the value of trade-in policy for rollover strategies. Li et al. (2010) focus on an inventory management problem with no replenishment during the product transition from an old generation to a new one. They develop a dynamic model to find inventory levels for two generations of products where the release date of the new generation is assumed to be unknown. Li & Graves (2012) present a dynamic pricing model for the product transition stage in which two generations are sold simultaneously with no product replenishment. All of the above models address the transition between exactly two generations of a product line. However, in various cases like mobile phones, computers and e-tablets, the motive of a new release is not to replace the older versions, rather develop a product line of multiple generations. In fact, in some cases, a new release boosts the sales of its predecessors, due to their reduced prices (Munbodh 2016). Moreover, simultaneously selling multiple generations is reported to be profitable for the firm (Kilicay-Ergin et al. 2015). Akçay et al. (2010) consider the case of a firm simultaneously selling multiple perishable products over a finite time. They derive optimal pricing policies by introducing a linear random utility framework to model consumer choices in a differentiated assortment of products. Thus, their focus is a pricing problem where no replenishment opportunities are present during the planning horizon.

Table 2.1: Classification of relevant research papers

Research papers	Product type	Age of product	Internal competition	New release	Decisions	
					Replenishment	Pricing
Ferguson & Koenigsberg (2007)	Food	2	✓		✓	✓
Li et al. (2010)	Electronic	2	✓		✓	
Akçay et al. (2010)	Electronic	n	✓			✓
Li & Graves (2012)	Electronic	2		✓	✓	✓
Sainathan (2013)	Food	2	✓		✓	✓
Liang et al. (2014)	Electronic	2		✓		
Chintapalli (2015)	Food	n			✓	✓
Liu et al. (2018)	Electronic	2	✓			
Our research	Electronic	n	✓	✓	✓	✓

In this chapter, we develop a joint inventory-pricing model for a firm selling multiple generations simultaneously in the market using stochastic dynamic program. We primarily investigate the release of newer generations in the presence of older ones. The most relevant studies using stochastic dynamic programming to formulate the underlying pricing or/and inventory management problems for food and electronic products are summarized in Table 2.1. In particular, research for electronic products focuses on how the product can be rolled over from its old version to the new one. In this thesis, we are concerned with transition between exactly two generations. We develop joint inventory-pricing policies for a multi-generation product line. This model differs from the joint production and pricing models introduced by Ferguson & Koenigsberg (2007) and Sainathan (2013) in terms of problem set-up and demand function. At every decision stage, orders for all generations in market are placed, in contrast to Ferguson & Koenigsberg (2007) and Sainathan (2013). Moreover, we develop a customer choice model based on the models proposed by Caplin & Nalebuff (1991) and Petrin & Train (2003) to account for both uncertain demand and internal competition among multiple generations.

The curse of dimensionality is often experienced while obtaining the solution for the dynamic joint decision-making problem of multiple products. Therefore, most of the literature discussed above limits the joint production-pricing model to a simplified two product case. The intertwined structure of the multiple inventory levels with their production and pricing decisions results in a complex and non-decomposable decision problem, which is computationally intractable by the classical dynamic programming solution method (backward dynamic programming). Thus, forward dynamic programming (FDP) has emerged as a successful methodology to tackle the curse of dimensionality by reducing the state space calculations. Although FDP has been widely used in various areas, the literature on FDP for solving the joint inventory-pricing problems is limited. For instance, Çimen & Kirkbride (2017) consider the FDP approach for the flexible production-inventory problem for multiple products at multiple locations. Coşgun et al. (2013) use FDP to solve the problem of markdown optimization between substitutable products in retail chains.

In this research, the curse of dimensionality is encountered for solving the underlying dynamic programming model for management of a multiple generation product line. Thus, we propose two approximation methods to solve the joint production-pricing problem. The first approximation method is a two-stage heuristic based on the idea of partial planning model introduced by Chan et al. (2006). In the first stage, we solve an abridged adaption of the original problem as a dynamic programming model to determine the pricing policy. In the second stage, this pricing information is then used to obtain the joint production pricing policies. Our second approximation method is based on the forward dynamic programming approach to approximately solve the joint production-pricing model. The numerical experiments (presented in Section 2.5) show that the two-stage heuristic in conjunction with different pricing rules produces policies with higher expected profits while FDP is accounted as more efficient than other approaches.

2.3 The Dynamic Production-Pricing Model

Consider a firm (such as Apple and IBM) that designs and develops various generations of innovative products (such as electronics, mobile phones and computers). The firm releases a new generation of the product whilst the older versions still continue to sell in the market. The new generation of the product uses innovative technologies and/or involves improved features in comparison to previous models. It is difficult to predict customers' reaction toward the latest technological developments of the new generation of the product as well as the older generations in presence of a new release. Thus, demand for all generations of the product is assumed to be uncertain. The firm tackles demand uncertainty and internal competition among multi-generations of the product by a forward-looking planning of joint decisions on production level and their prices. We formulate the firm's joint production-pricing problem using a finite discrete-time stochastic dynamic programming model. In this section, we first present the underlying production-pricing problem and then formulate the problem as a dynamic program under a customer choice model. Before that, we introduce notation used for the problem formulation.

We interchangeably use *models*, *versions* and *generations* of the underlying (electronics) product in this chapter. The maximum function $(a)^+ = \max\{a, 0\}$ takes value of a if and only if $a > 0$; otherwise, it is zero. On the other hand, the minimum function $\min\{b, c\}$ for $b, c \geq 0$ takes value of b if $b \leq c$; otherwise, it is equal to c. Both "innovation level" and "quality of product" that basically represent "level of desirability" for the product are interchangeably used in this chapter. As the innovation level (quality) of a product increases, the level of desirability increases. Even though the level of desirability $\alpha_{k,t}$ for version k of a product is the same for different classes of customers in terms of price and quality sensitive, their utilities at time t will be different because of customers quality preferences.

Notation: We use tilde ($\tilde{*}$) to denote randomness; e.g., $\tilde{y}$ denotes random variable y. Boldface is used to denote vectors; for example, $\mathbf{a} \in \mathbf{R}^n$ is a n-dimensional vector.

In particular, we denote a vector of ones by $\mathbf{1} = [1, \cdots, 1]$ in appropriate dimension and "$\mathbf{x} \cdot \mathbf{y}$" displays a scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$. A description of the notation used in the chapter is in the Table A1.

A1: Description of notation

	Model parameters
T	planning horizon discretised by time periods $t = 0, 1, \cdots, T$
G_t	set of generations available in the market at time t
$\bar{m}$	maximum number of versions available in the market
$\alpha_{k,t}$	level of desirability for version $k \in G_t$ at time t
$\gamma_{k,t}$	choice probability of version $k \in G_t$ at time t
$\mathbf{c}_t$	vector of unit ordering costs $c_{k,t}$ of generation $k \in G_t$ at time t
h_t	unit inventory holding cost from time $t-1$ to t
κ	production capacity
δ	discount factor
$\tilde{\mathbf{d}}_t$	vector of uncertain demand $\tilde{d}_{k,t}$ for generation $k \in G_t$ at time t
η	step size
	State variables and actions
$\mathbf{x}_t$	vector of inventory levels $x_{k,t}$ for generation $k \in G_t$ at time t
$\mathbf{p}_t$	vector of prices $p_{k,t}$ for generation $k \in G_t$ at time t
$\mathbf{q}_t$	vector of production levels $q_{k,t}$ for generation $k \in G_t$ at time t

Dynamic Programming Model: Assume that the firm releases different generations (indexed by r) of the product over a finite planning horizon. The planning horizon is discretized into $t = 0, 1, \cdots, T$ time periods where the operational (such as production and pricing) decisions and tactical decisions (such as release of new generation) are made; in particular, $t = 0$ represents today. We assume that the firm regularly launches new generations of the product and the lead time for procurement is zero so that the products can be received instantaneously. In addition, release time of new generation is predetermined in the model. Figure 2.1 displays a graphical timeline of firm's decision-making process. The release time of generation r is denoted as t_r such that $0 \leq t_r \leq T$. The firm may sell at most $\bar{m} \geq 2$ generations at each time period. We denote a set of different generations of the product available in the market at time t by G_t.

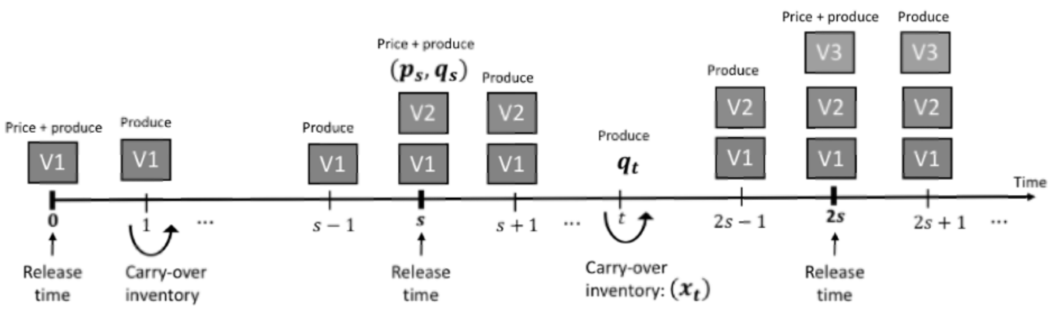

Figure 2.1: Decision-making process in a multi-generation product line

Let $\tilde{\mathbf{d}}_t = \{\tilde{d}_{r,t} : r \in G_t\}$ represent a vector of uncertain demand $\tilde{d}_{r,t}$ of generations $r \in G_t$ at time t. In addition, let $\mathbf{c}_t = \{c_{r,t} : r \in G_t\}$ show a vector of unit production cost $c_{r,t}$ of all generations $r \in G_t$ at time t. The carrying cost h_t for holding one unit of inventory from $t-1$ to t remains the same for all generations.

We assume that the inventory level $x_{r,t}$ of generations $r \in G_t$ (that are currently selling in the market) is reviewed at the beginning of time period t before the production process begins. We define vector $\mathbf{x}_t = \{x_{r,t} : r \in G_t\}$ of the inventory levels of all generations available in the market as a state of the dynamic system at time t. Given a state of the system, an action set consists of decisions made at each i) release time as *production and selling prices of the new generation as well as previously released generations of the product*, and ii) intermediate time period between two consecutive release times in terms of *production of all generations available in the market*. Let $p_{r,t}$ and $q_{r,t}$ denote unit selling price and amount of products to be produced for each generation $r \in G_t$ at time t, respectively. Similarly, we introduce vectors $\mathbf{p}_t = \{p_{r,t} | \ p_{r,t} \geq 0, r \in G_t\}$ and $\mathbf{q}_t = \{q_{r,t} | \ q_{r,t} \geq 0, r \in G_t\}$ corresponding to market prices and amount of production of generations at time t, respectively.

The system dynamics lead to state transition of inventory levels from t to $t+1$. The following balance equations

$$\mathbf{x}_{t+1} = \max\{\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t, 0\}, \text{ for } t = 0, 1, \cdots, T$$

imply that the inventory $\mathbf{x}_{t+1}$ to be carried over from t to $t+1$ is determined by the inventory level $\mathbf{x}_t$, amount of production $\mathbf{q}_t$ and customers' demand $\tilde{\mathbf{d}}_t$ at time t. Note that if $\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t > 0$, then the firm incurs an inventory holding cost h_t per unit to carry unsold inventory to the next period. On the other hand, if demand exceeds the current inventory level, i.e., $\tilde{\mathbf{d}}_t - \mathbf{x}_t + \mathbf{q}_t > 0$, then the firm is unable to fulfill the customers' demand. In this case, the unmet demand is assumed to be lost (not backlogged).

Let κ represent the production capacity of the firm. We ensure that the total number of products to be produced at time t do not exceed the available production capacity: $\mathbf{1}(\mathbf{q}_t) \leq \kappa$.

The firm aims to maximize the expected profit over the planning horizon while maintaining a multiple-generation product line through a joint inventory-pricing decision framework. The expected profit is computed as the expected revenue minus the expected total cost of production and holding. Given the selling price of all generations, the revenue depends on future realisations of the customer demand. When the demand at time t is low (i.e., $\tilde{\mathbf{d}}_t \leq \mathbf{x}_t + \mathbf{q}_t$), the revenue becomes $\mathbf{p}_t \cdot \tilde{\mathbf{d}}_t$. In case of high demand (i.e., $\tilde{\mathbf{d}}_t > \mathbf{x}_t + \mathbf{q}_t$), we compute the revenue as $\mathbf{p}_t \cdot (\mathbf{x}_t + \mathbf{q}_t)$. Thus, we can state the revenue earned at time t as $\mathbf{p}_t \cdot \min\{\mathbf{x}_t + \mathbf{q}_t, \tilde{\mathbf{d}}_t\}$. The total ordering and holding costs are expressed as $\mathbf{c}_t \cdot \mathbf{q}_t$ and $h_t \mathbf{x}_{t+1}$, respectively. The single-period profit $\pi_t(\mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t)$ at time t is obtained as

$$\pi_t(\mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t) = \mathbf{p}_t \cdot \min\{\tilde{\mathbf{d}}_t, \mathbf{x}_t + \mathbf{q}_t\} - h_t \mathbf{x}_{t+1} - \mathbf{c}_t \cdot \mathbf{q}_t.$$

We formulate the joint production-pricing decision making problem as a stochastic dynamic optimisation model that requires different action sets at the release time of a new generation. During the launch of a new generation, along with production decisions, the firm needs to determine the price of the latest generation of the product, based on enhanced features and/or innovations, while adjusting prices of older generations by evaluating the market value of innovative evolution of technologies and predicting customers' willingness to pay for the new and old innovative improvements. Let us assume that a new generation r is to be released at time t (i.e., $t = t_r$). The firm needs to determine how many products to produce for each generation and what market price to assign for all generations $r \in G_t$ in the market. Let $V_t(\mathbf{x}_t)$ denote the value function at time t given a state $\mathbf{x}_t$ of the system. The value function for the dynamic joint production-pricing decision making problem can be written as follows;

$$\begin{aligned} V_t(\mathbf{x}_t) = \quad & \max_{\mathbf{p}_t, \mathbf{q}_t} \; \mathbb{E}\left[\pi_t(\mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t) + \delta V_{t+1}(\mathbf{x}_{t+1})\right] \\ & \text{s.t.} \;\; \mathbf{x}_{t+1} = (\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ \\ & \qquad \mathbf{1} \cdot (\mathbf{q}_t) \leq \kappa, \quad \mathbf{q}_t \geq \mathbf{0} \\ & \qquad \mathbf{p}_t \in \mathcal{F}_t \end{aligned} \tag{2.1}$$

where $\mathcal{F}_t$ is a set of feasible prices of all generations of the product. We will further refine construction of this set in the next section by introducing the customer choice model that takes into account internal competition of different generations of the product and also the customer preferences in terms of quality and price of the products selling in the market. The expectation operator in the value function is always taken over customer demand uncertainty. Note that in this formulation, the ending inventory for the last time period is assumed to be held till the end of the planning horizon and then it is discarded at zero cost. Finally, the boundary condition at the end of planning horizon is $V_{T+1}(\mathbf{x}_{T+1}) = 0$.

The firm can produce any version of the product at intermediate time periods between consecutive release times. The problem formulation (2.1) then becomes the following dynamic programming model where the production decisions are made at any intermediate time periods t' for $t_r < t' < t_{r+1}$ assuming that the unit prices of different versions (G_{t_r}) available in the market at time t' remain the same from t_r till the next launch at t_{r+1}.

$$\begin{aligned} V_{t'}(\mathbf{x}_{t'}|\mathbf{p}_{t_r}) = \quad & \max_{\mathbf{q}_{t'}} \ \mathbb{E}\Big[\mathbf{p}_{t_r} \cdot \min\{\tilde{\mathbf{d}}_{t'}, \mathbf{x}_{t'} + \mathbf{q}_{t'}\} - h_{t'}\mathbf{x}_{t'+1} + \delta V_{t'+1}(\mathbf{x}_{t'+1})\Big] - \mathbf{c}_{t'} \cdot \mathbf{q}_{t'} \\ & \text{s.t.} \ \ \mathbf{x}_{t'+1} = (\mathbf{x}_{t'} + \mathbf{q}_{t'} - \tilde{\mathbf{d}}_{t'})^{+} \\ & \qquad \mathbf{1} \cdot (\mathbf{q}_{t'}) \leq \kappa, \quad \mathbf{q}_{t'} \geq \mathbf{0} \end{aligned} \tag{2.2}$$

Notice that when intermediate time periods between any two consecutive release times are ignored, the problem complexity of the dynamic program in (2.1) can be slightly reduced. Then the dynamic optimisation model (so-called an *'abridged model'*) provides an approximate solution to the initial joint production-pricing problem. The abridged model will be used to develop a two-stage heuristic and it is discussed in Section 2.4.2.

2.3.1 The Customer Choice Model

When the firm releases a new generation of the product, customers anticipate both price at which the new version is released and also potential change in selling prices of the older versions. As reported by Li et al. (2010), the customer's anticipation for prices of multiple versions may cannibalize sales of existing generations by the new one and vice versa. The cannibalization of sales occurs because new and existing generations internally compete to be a preferable choice of customers. A customer choice model examines various factors governing the customer's decision to buy a specific version of the product or leave without a purchase. In general, there are different observable and unobservable factors that determine the customer's choice of buying a particular version (Train 2009). Price, innovation and technological levels of a generation, and the customer's sensitivity toward technology can be listed as examples of observable factors impacting their choices. On the other hand, unobservable factors influencing the customer's predilection toward buying a specific version are described as idiosyncratic and mostly depend on individual's preferences, like personal style and acceptance of innovative technology. In this chapter, we consider only observable factors such as price and quality of generations to analyze the customer's choice. Next a description of our customer choice model follows.

We assume that a customer is rational while making a decision and aims to maximize his/her own utility. Moreover, the latest generation of the product is assumed to be more attractive than the previous models in terms of technological features of multiple versions. Let $U_{k,t}$ denote the utility that a customer achieves from purchasing generation $k \in G_t$ at time t. Following Train (2009), Akçay et al. (2010) and Sainathan (2013), the customer's utility can be expressed by a linear function of the innovation level $\alpha_{k,t}$ and price $p_{k,t}$ of generation k at time t as follows;

$$U_{k,t} = \theta \alpha_{k,t} - p_{k,t}, \quad k \in G_t, \ t = 0, 1, \cdots, T, \tag{2.3}$$

where θ represents the customer's quality sensitivity toward the technology. The innovation level $\alpha_{k,t}$ measures the attractiveness of a generation and is explicitly associated with technological features of multiple versions available in the market. Although the innovation level and the market price of each generation are the same for all customers, the quality sensitivities of customers toward innovative technology may vary. Note that one can also define the utility for an individual customer by using specific sensitivity parameter associated with the individual customer; the reader is referred to Akçay et al. (2010) and Sainathan (2013) for further information on the customer specific choice models.

We assume that parameter θ (representing customer's quality sensitivity) follows a uniform distribution over an interval of $[0, 1]$. The customer prefers the latest generation of the product possessing innovative technologies when $\theta = 1$ that indicates the highest quality sensitivity. On the other end, $\theta = 0$ reflects the least quality sensitivity where the customer prefers not to buy any version of the product. This assumption was also considered in Train (2009), Akcay et al. (2010), and Sainathan (2013). Let us now consider a set $G_t = \{k | \ k = 1, 2, \cdots, m\}$ of $m \leq \bar{m}$ generations available in the market at time t where $k = 1$ represents the earliest generation and $k = m$ is the latest generation of the product. Further assumptions regarding the customer choice model are enlisted below.

- A1: Since the recent generation of the product is always perceived to have a better quality than the older models, the corresponding innovation level of the newest generation is assumed to be higher than those of previous models. Thus, we can construct the following relationship $\alpha_{m,t} \geq \alpha_{m-1,t} \geq, \ldots, \geq \alpha_{1,t}$ among innovation levels of generations available in the market.
- A2: The market prices $p_{m,t}, p_{m-1,t}, \cdots, p_{1,t}$ of generations should also reflect quality difference in terms of innovative technologies and/or improved features employed in development of generations. In order to balance between prices and features of generations, we impose the following conditions:

$$\frac{p_{m,t} - p_{m-1,t}}{\alpha_{m,t} - \alpha_{m-1,t}} \geq \frac{p_{m-1,t} - p_{m-2,t}}{\alpha_{m-1,t} - \alpha_{m-2,t}} \geq, \cdots, \geq \frac{p_{2,t} - p_{1,t}}{\alpha_{2,t} - \alpha_{1,t}} \geq \frac{p_{1,t}}{\alpha_{1,t}}. \tag{2.4}$$

This relationship is based on the quality-aligned prices condition provided by Akçay et al. (2010). This condition basically states that a recently released model of the product would be priced higher than the older models available in the market due to improved or additional features. This relationship allows the firm to charge a larger price for a higher quality model.

It is worthwhile to mention that a set of linear constraints in (2.4) must be included into the joint production-pricing model (2.1) as they construct the feasibility set $\mathcal{F}_t$ for the dynamic pricing problem.

Next, we will describe how to compute the customer choice probability in view of different features of the multi-generation product. As mentioned above, while purchasing a specific generation of the product, customers are assumed to compare its attributes in terms of price and innovation level with its predecessors as well as possible successive generations. Thus, due to internal competition among all versions $k \in G_t$ of the product available in the market, the choice probability $\gamma_{k,t}$ of generation k at time t depends on its own price and innovation level of successive generation $k+1$ as well as predecessor generation $k-1$ for $k = 2, 3, \cdots, m-1$. Since the latest generation $k = m$ has no successor at time t and the oldest generation $k = 1$ has no predecessor, their choice probabilities need to be computed accordingly. The following proposition states the choice probabilities for all generations of the product available in the market.

Proposition 1 *For the given set G_t of currently available generations $k = 1, \cdots, m-1, m$ (in order from the earliest to the latest released versions) at time t, the customer's choice probabilities $\gamma_{k,t}$ are determined as follows;*

$$\gamma_{k,t} = \begin{cases} \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} - \frac{p_{k,t}}{\alpha_{k,t}}, & k = 1, \\ \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} - \frac{p_{k,t}-p_{k-1,t}}{\alpha_{k,t}-\alpha_{k-1,t}}, & k = 2, \cdots, m-1, \\ 1 - \frac{p_{k,t}-p_{k-1,t}}{\alpha_{k,t}-\alpha_{k-1,t}}, & k = m. \end{cases}$$

Proof: The proof is provided in Appendix A. ■

As the latest generation possesses the highest innovative technology, which hasn't been exposed to the market before, the customer's response toward the newest version of the product is unpredictable. Moreover, the release of a new version reduces the prices of the older versions and may increase the customer's willingness to buy. Thus, during a new release, the customer's response toward the older generations is also difficult to predict.

In our model, customer demand for multiple generations is assumed to be uncertain and the underlying random variable follows a probability distribution. At time t, demand of generation k specifically depends on the customer choice probability for generation k and also the number of customer arrivals. We also suppose that customer arrival is uncertain and follows a discrete probability distribution. Let $\lambda_{j,t}$ denote the probability of j customer arrivals at time t such that $\sum_{j=0}^{M} \lambda_{j,t} = 1$ where M represents the maximum number of customers expected to arrive at any time period. The following proposition expresses the probability mass function of demand for multiple versions.

Proposition 2 *Let $f_{k,t}(.)$ denote a probability mass function for generation $k \in G_t$ at time t. Then the probability of having demand for j number of products from generation $k \in G_t$ at time t can be computed as follows:*

$$f_{k,t}(j) = \quad Pr(\tilde{d}_{k,t} = j) = \sum_{i=j}^{M} \binom{i}{j} (\gamma_{k,t})^j (1-\gamma_{k,t})^{i-j} \lambda_{i,t}, \ for\ j = 0, 1, \cdots, M.$$

Proof: The proof is provided in Appendix A. ■

Note that Proposition 2 will be used to define the probability mass function of having demand for certain number of products within the stochastic dynamic programming model. Next, we will focus on approximation methods for solving the dynamic joint production-pricing model. In particular, we introduce a simulation based stochastic dynamic programming method (namely forward dynamic programming) and a two-stage heuristic method.

2.4 Approximation Methods

As in most real-life stochastic dynamic programming applications, finding an optimal policy for the joint production-pricing problem of multi-generation products under uncertainty is computationally expensive due to a large number of states. The traditional dynamic programming algorithm uses the backward recursion principle where the optimal decisions and value functions are calculated iteratively starting from the terminal time and stepping backwards in time. Although this procedure can produce an exact analytical solution, it is affected by the curse of dimensionality since the value function is computed at each state and all possible actions are evaluated and stored in look-up tables. Because, at each decision epoch, enumeration of the entire state and its feasible action spaces becomes computationally expensive.

For the joint production-pricing model, the action and state spaces magnify with increase in production capacity and the number of generations released in the market. Even solving the abridged optimisation model (described as in form of a two-stage heuristic in the next section) becomes computationally cumbersome for the realistic size of problems when the backward dynamic programming method is applied. For instance, for a firm selling at most m versions of the product at time t with fixed production capacity of κ, the state space comprises of $(\kappa + 1)^m$ number of states. At any state, there exist in total $(\kappa + 1)^m$ actions to take for the production quantity of all available versions. Moreover, we need to solve a non-linear optimisation problem for the optimal pricing decisions at any state of the system. Thus, the state and action spaces exponentially grow as more generations are released over time and/or the capacity is expanded. In order to tackle curse of dimensionality on the state space, we propose iterative approximation algorithms based on a forward dynamic programming and a two-stage heuristic.

2.4.1 Forward Dynamic Programming

Forward dynamic programming (FDP) is a simulation based algorithmic framework that solves the underlying dynamic programming problem using a strategy that steps forward through time starting from an initial state. As opposed to visiting the entire state (and action) space, FDP selects a sample path and moves forward iteratively. Each sample path is generated using a Monte Carlo simulation from the same initial state. The value functions are evaluated for all states (a look-up table) or updated at states on a random path (reached from the initial state) using aggregation of states or regression models. In this sense, the FDP algorithm differs from the backward dynamic programming algorithm that computes the value function at every state. The interested reader is referred to Powell & Topaloglu (2003, 2006) and Powell (2007) for further information. Next a brief description of the FDP algorithm for solving the joint production-pricing problem follows.

The forward dynamic programming algorithm is especially designed to reduce state space by adopting an approximation technique for the value function. The FDP algorithm starts with initialisation of the value function. In our case, we set it to zero. The performance of the FDP algorithm highly depends on how the value function is initialized; hence, the decisions can be suboptimal. The best initialization of the value function encourages FDP to explore different states. At the n-th iteration of the FDP

Algorithm 1: Pseudo code of the FDP algorithm

1: **Initialization:** Initialize iteration number N and set step size η.
2: Set value function at $\mathbf{x}_t^0$ as $\bar{V}_t^0(\mathbf{x}_t^0) = 0$.
3: **for** $n = 1, \cdots, N$ **do**
4: Select an arrival path ω^n.
5: **for** $t = 0, 1, \cdots, T$ **do**
6: - Set initial state (inventory for version 1) at $t = 0$ as $\mathbf{x}_0^n = \{x_{1,0}^n\} = \{0\}$.
7: - Obtain $(\mathbf{p}_t, \mathbf{q}_t)$ by solving the following maximization problem; $\hat{V}_t^n(\mathbf{x}_t^n) = \max_{\mathbf{p}_t, \mathbf{q}_t} \left\{ \mathbf{p}_t \min\{\mathbf{x}_t^n + \mathbf{q}_t, \tilde{\mathbf{d}}_t\} - \mathbf{c}\mathbf{q}_t - h_t(\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ + \bar{V}_{t+1}^{n-1}((\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+) \right\}$
8: - Update the value function at state $\mathbf{x}_t$ as follows;

$$\bar{V}_t^n(\mathbf{x}_t) := \begin{cases} (1-\eta)\bar{V}_t^{n-1}(\mathbf{x}_t^n) + \eta \hat{V}_t^n(\mathbf{x}_t^n) & \text{if } \mathbf{x}_t = \mathbf{x}_t^n \\ \bar{V}_t^{n-1}(\mathbf{x}_t) & \text{if } \mathbf{x}_t \neq \mathbf{x}_t^n \end{cases}$$

- Compute new states based on random outcome of demand: $\mathbf{x}_{t+1}^n = (\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+$.
9: - Set $t := t + 1$. If $t < T + 1$, then go to Step 6.
10: **end for**
11: Set $n := n + 1$. If $n < N$, then go to Step 4.
12: **end for**

algorithm, as presented in its pseudo code in Algorithm 1, the value function $\bar{V}_t^n(\mathbf{x}_t^n)$ is updated for state $\mathbf{x}_t^n$ to approximate the real value function $\hat{V}_t(\mathbf{x}_t)$. Let $\hat{V}_t^n(\mathbf{x}_t^n)$ and $\bar{V}_t^n(\mathbf{x}_t^n)$ denote the optimized and approximated value functions given state $\mathbf{x}_t^n$ of iteration n, respectively. At each iteration of the FDP algorithm, we simulate a path of customer arrivals for each time period. The customer requests are generated by using the customer choice model presented in Section 2.3.1. For given customer arrivals at iteration n, the joint production-pricing decisions at state $\mathbf{x}_t^n$ are obtained by solving the following maximisation problem

$$\hat{V}_t^n(\mathbf{x}_t^n) = \max_{\mathbf{p}_t, \mathbf{q}_t} \left\{ \mathbf{p}_t \min\{\mathbf{x}_t^n + \mathbf{q}_t, \tilde{\mathbf{d}}_t\} - \mathbf{c}\mathbf{q}_t - h_t(\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ + \bar{V}_{t+1}^{n-1}((\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+) \right\},$$

where $\bar{V}_{t+1}^{n-1}$ is an approximation of the value function at state $\mathbf{x}_{t+1}^n = (\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+$ at time $t + 1$. For any feasible production level $\mathbf{q}_t$, this is a convex optimization problem that determines optimal pricing decisions for $\mathbf{p}_t$. Using the production level

($\mathbf{q}_t$) and pricing ($\mathbf{p}_t$) decisions, we can then update the look-up table as follows;

$$\bar{V}_t^n(\mathbf{x}_t) = \begin{cases} (1-\eta)\bar{V}_t^{n-1}(\mathbf{x}_t^n) + \eta\hat{V}_t^n(\mathbf{x}_t^n), & \mathbf{x}_t = \mathbf{x}_t^n \\ \bar{V}_t^{n-1}(\mathbf{x}_t), & \text{otherwise.} \end{cases}$$

where η is a step size between 0 and 1. We then calculate the next state on the basis of random demand outcomes. The algorithm terminates when it satisfies the stopping criteria. The final value function is an approximate solution to the problem since the FDP algorithm does not compute the value function at every state, but only those reached from the initial state.

2.4.2 A Two-stage Heuristic

In addition to the FDP algorithm we adopt an alternative approach based on the partial-planning strategy introduced by Chan et al. (2006). The pseudo code summarising the main steps of this approach is presented in Algorithm 2. This heuristic consists of two stages. In the first stage, we determine a range of prices of the released products available in the market. Then, in the second stage, we obtain the optimal production level of each product for any price point determined in the first stage of the algorithm.

Stage 1: Derivation of Price Bounds: A range of prices of products available in the market can be specified by the lower and upper bounds determined by different ways. In particular, we consider pricing rules by solving the abridged model of the joint production-pricing problem and a list of prices derived by theoretical bounds as described below.

The Abridged Model: Since the joint dynamic production-pricing model is computationally intractable, one can consider the (reduced) abridged optimisation model where decisions related to production and pricing of available products are made at only release times. In other words, production does not take place between two subsequent release times. The elimination of the production decisions between release times can be interpreted as the firm placing cumulative production decisions between two release times in advance. The resulting (abridged) dynamic program can be solved only for certain small-size problems by the standard technique of backward dynamic programming. The average, minimum and maximum value of product prices derived from the policy tables can then be used at the second stage of the heuristic approach to define a range for prices of the products. The policy table refers to all actions evaluated at given possible states at each time by backward recursion.

Theoretical Bounds: From the policy table obtained by solving the abridged model, we investigate patterns between the production and pricing decisions of multiple versions. We observe the minimum value of the pricing decision for a specific version is obtained when its inventories are at the maximum level. This observation is used to theoretically derive the lower and upper bounds for the pricing decisions for multiple versions. The following propositions state theoretical bounds in order to determine the pricing decision of a generation. The bounds are obtained by solving a simple variant of the decision-making problem and they are applied in the solution method proposed as two-stage heuristic.

Proposition 3 *Assume that the maximum of demand for any generation of the product is M at the final time period. For $\mathbf{x}_T + \mathbf{q}_T = M$, the market price of generation $k \in G_T$ at time T can be computed in terms of innovation level $\alpha_{k,T}$ of generation $k \in G_T$ and holding cost h_T as $p^*_{k,T} = \frac{\alpha_{k,T} - h_T}{2}$.*

Proof: The proof is provided in Appendix A. ■

Proposition 4 *The market prices of multiple generations $k \in G_t$ at time t of the product are bounded as follows;*

a) $\frac{\alpha_{k,t} - h_t}{2} \leq p_{k,t} \leq \alpha_{k,t}$ *for* $k = m$*, and*

b) $\frac{\alpha_{k,t} - h_t}{2} \leq p_{k,t} \leq \frac{\alpha_{k,t} p_{k+1,t}}{\alpha_{k+1,t}}$ *for* $k = 1, \cdots, m-2, m-1$.

Proof: The proof is provided in Appendix A. ■

Stage 2: Optimal Production Strategies: Given the pricing strategy (determined by certain rules or bounds at the first stage of the algorithm), we need to compute the optimal production strategy. Let $\hat{\mathbf{p}}_t = \{\hat{p}_{k,t}, k \in G_t\}$ represent the pre-determined price of all generations selling in the market at time t. Given pre-determined price $\hat{\mathbf{p}}_t$, we can formulate the value function $\hat{V}_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t)$ at state $\mathbf{x}_t$ as follows;

$$\begin{aligned} \hat{V}_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t) = \max_{\mathbf{q}_t \geq 0} \quad & \mathbb{E}\left[\hat{\mathbf{p}}_t . \min\{\tilde{\mathbf{d}}_t, \mathbf{x}_t + \mathbf{q}_t\} - h_t \mathbf{x}_{t+1} + \delta V_{t+1}(\mathbf{x}_{t+1})\right] - \mathbf{c}_t \mathbf{q}_t \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = (\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ \\ & \mathbf{1}(\mathbf{q}_t) \leq \kappa \end{aligned} \tag{2.5}$$

Assuming that no new generation is to be launched at the end of planning horizon $t = T$, the value function $\hat{V}_T(\mathbf{x}_T \mid \hat{\mathbf{p}}_T)$ states the boundary condition determining the optimal order quantity $\mathbf{q}_T$ at time period T as follows;

$$
\begin{aligned}
\hat{V}_T(\mathbf{x}_T \mid \hat{\mathbf{p}}_T) = \max_{\mathbf{q}_t \geq \mathbf{0}} \quad & \mathbb{E}\left[\hat{\mathbf{p}}_T.\min\{\tilde{\mathbf{d}}_T, \mathbf{q}_T + \mathbf{x}_T\} - h_T \mathbf{x}_{T+1}\right] - \mathbf{c}_T \mathbf{q}_T \\
\text{s.t.} \quad & \mathbf{x}_{T+1} = (\mathbf{x}_T + \mathbf{q}_T - \tilde{\mathbf{d}}_T)^+ \\
& \mathbf{1}(\mathbf{q}_T) \leq \kappa
\end{aligned}
\tag{2.6}
$$

where prices $\hat{\mathbf{p}}_T$ of all available versions at time T are set during the last release time. Note that in this formulation, the ending inventory for the last time period is assumed to be held till the end of the planning horizon and then it is discarded at zero cost.

Algorithm 2: Pseudo code of the two-stage algorithm

Stage 1: *Obtain interval of prices* $[\hat{p}^L_{k,t},\ \hat{p}^U_{k,t}]$ *for* $k \in G_t$ *at* $t = 1, 2, \cdots, T$ *by either solving the abridged model or applying Proposition 5.*

Stage 2: *Given a price range, determine the optimal production strategy*

- Initialize step size $\phi_{k,t}$ for each version $k \in G_t$ at time $t = 1, \cdots, T$.

- Set initial inventory level at time $t = 1$ as $\mathbf{x}_1 = \{x_{k,1} = 0, k \in G_1\}$

- Compute a feasible price set:

$$\mathcal{P} = \left\{\hat{p}^L_{k,t} + e.\phi_{k,t} \mid \gamma_{k,t} \geq 0,\ e = 0, 1, \cdots, \frac{\hat{p}^U_{k,t} - \hat{p}^L_{k,t}}{\phi_{k,t}},\ \text{for } k \in G_t,\ t = 1, \cdots, T\right\}$$

for *each price point* i *in set* $\mathcal{P}$ **do**

  Set boundary condition: $V^i_{T+1}(\mathbf{x}_{T+1} \mid \hat{\mathbf{p}}_T) = 0$ for all possible states $\mathbf{x}_{T+1}$ at $t = T + 1$

  for $t = T, T - 1, \cdots, 1$ **do**

    Solve the following maximisation model at each state $\mathbf{x}_t$

$$V^i_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t) = \max_{\mathbf{q}_t \geq 0}\left\{\hat{\mathbf{p}}_t \min(\mathbf{x}_t + \mathbf{q}_t, \tilde{\mathbf{d}}_t) - \mathbf{c}\mathbf{q}_t - h_t(\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ + \delta V^i_{t+1}((\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ \mid \hat{\mathbf{p}}_{t+1})\right\},$$

  end for

  return $\hat{\mathbf{q}}_t = \arg\max_{i \in \mathcal{P}} V^i_1(\mathbf{x}_1 \mid \hat{\mathbf{p}}_t)$

end for

The following proposition establishes convexity of the optimal production model for given prices of products. Thus, the optimal order policy is obtained when the prices are known.

Proposition 5 *Given approximate prices* $\hat{\mathbf{p}}_t$ *of all generations in the market at time* t*, the value function* $\hat{V}_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t)$ *of the dynamic production planning model is concave in production decisions.*

Proof: The proof is provided in Appendix A. ■

Notice that the original joint production-pricing problem is simplified to the dynamic production problem where the optimal ordering policy can be obtained by solving the dynamic optimisation model as follows,

$$\mathbf{q}_t^* = \arg\max \quad \hat{\mathbf{p}}_t \mathbb{E}\left[\min\{\tilde{\mathbf{d}}_t, \mathbf{x}_t + \mathbf{q}_t\}\right] - h_t \mathbb{E}\left[\mathbf{x}_{t+1}\right] + \delta \mathbb{E}\left[V_{t+1}(\mathbf{x}_{t+1})\right] - \mathbf{c}_t \mathbf{q}_t$$

where fixed prices $\hat{\mathbf{p}}_t$ of multiple generations are selected from a range of $\left[\hat{\mathbf{p}}_t^L \ , \ \hat{\mathbf{p}}_t^U\right]$ that is obtained at the first stage of the two-stage algorithm.

2.5 Computational Experiments

In this section, we first describe the design and data structure used for numerical experiments and then present the computational results of different approaches studied for solving the joint production-pricing problem of multi-generation product line. A brief description of these approaches (presented in Section 2.4) with different pricing rules adopted follows;

- *Forward Dynamic Programming (FDP)* is a simulation based stochastic dynamic programming method presented in Section 2.4.1. Different from the two-stage heuristic, this method determines the pricing and production decisions together. Three different versions of this approach are used to solve the joint production-pricing model (2.1).
  - *FDP-1* solves the non-linear optimisation model (2.1) at each iteration in Algorithm 1.
  - *FDP-2* uses price sets that are determined from the maximum and minimum prices given by the abridged model to solve model (2.1).
  - *FDP-3* uses price sets determined from the theoretical price bounds of Proposition 4.
- *Two-stage heuristic (TSH)* applies pricing decisions made (in stage 1) for multiple generations of a product using the model (2.5)-(2.6) to determine the production policy (in stage 2). Different pricing strategies to find production quantities are abbreviated as follows;
  - *ABridged Model (ABM)* assumes that there is no intermediate ordering between release times of the generations. The resulting model is solved by

dynamic programming and the average, minimum and maximum value of the product prices are implemented at the second stage of the two-stage heuristic.

  – *Bounds Algorithm 1 (BA-1)* requires the maximum and minimum prices obtained from the abridged model to form the price bounds. These price bounds are then used in Algorithm 2 to search for the price sets for each generation.
  – *Bounds Algorithm 2 (BA-2)* employs the theoretical price bounds given by Proposition 4 to search for the price sets in Algorithm 2.

These algorithms were implemented in MATLAB and all computational experiments were run in a desktop computer with Intel Core i5-7500, 3.4GHz, 8GB RAM.

2.5.1 Design of Experiments and Data

We design a series of computational experiments in order to illustrate the performance of different algorithms developed for solving the dynamic programming models. Specifically, the numerical experiments aim to answer the following managerial questions:

- What is the impact of selling multiple generations of a product on firm profit?
- What is the added value of joint decision making while managing a multi-generation product line?
- How do varying characteristics of customer segments affect the management of multiple generations?

Our experimental design considers different parameter sets related to the number of generations in the market, the innovation level and the innovation sensitivity. We define three base cases with respect to the number of generations in the market, which are given in Table 2.2. In these base cases, we set the production capacity to $\kappa = 25$, the inventory holding cost to $h = 0.001$ (Callioni et al. 2005, McCue 2020), and the discount factor to $\delta = 1$. Additional test instances (adopted from Sainathan (2013) and Akçay et al. (2010)) are generated by varying some parameters in the base case while the holding cost remains same over the planning horizon (Callioni et al. 2005, McCue 2020).

Table 2.2: A list of parameters set in the base case

Generations (n)	Innovation Level ($\alpha_1, ..., \alpha_n$)	Production Cost ($c_1, ..., c_n$)	Planning Horizon (T)
2	(3.5, 4)	(0.5 ,1)	4
3	(3.5, 4, 4.5)	(0.5, 1, 1.5)	6
4	(3.5, 4, 4.5, 5)	(0.5, 1, 1.5, 2)	8

We simulate the arrival of customer requests over a planning horizon with length T. Given the optimal pricing strategy obtained by different solution methodologies at each time period, we first make the production decision for each generation and then, generate the customer requests. Customer arrivals follow a Poisson distribution with mean arrival rate of 25. The arriving customer either chooses one of the offered generations according to the choice probabilities described in Proposition 1 or leaves with no purchase.

We assume that customers are classified into two segments, namely price and innovation sensitive. Each customer is assigned a value of $\theta \in [0, 1]$, denoting the customer's sensitivity towards innovation. A customer having innovative sensitivity of $\theta > 0.5$ is classified as innovation sensitive. On the other hand, a price sensitive customer would have innovative sensitivity of $\theta < 0.5$. In order to have an equal number of customers in each segment, we uniformly distribute the value of θ in the initial experiments. In case of low inventory, customer's demand will be lost. We also perform experiments where we vary the proportion of customer segments. We estimate the expected profits by simulating the arrivals of customer requests over 1000 sample paths. The simulations are designed to test the performance of different approximate solution approaches.

2.5.2 Numerical Results and Analysis

In this section, we present results of the numerical experiments under three main categories: performance of different approximate approaches, impact of number of multi generations available in the market and effect of customer choices on the firm's profitability.

Performance Comparison of Different Approaches: We are first concerned with evaluating the performance of the proposed approximation methods with respect to varying production capacities. The performance of each algorithm is measured in terms of total expected profit achieved at the end of planning horizon and the CPU time taken to solve each problem instance. We estimate the expected profits by simulating the decisions made by different solution methods under multiple customer arrival trajectories.

We consider three experiment setups where a firm is selling two, three and four generations of a product, respectively. Since the production capacity has a significant effect on the computation time of the methods, we vary it over $\{5, 10, 15, 25\}$ to evaluate the performances of each solution strategy. Table 2.3 summarises our simulation results for two, three and four generation product lines. The first column in Table 2.3 shows the production capacity used in these tests whereas the next six columns present the expected profits and the related computation times obtained by solving the corresponding dynamic optimisation models using the two-stage heuristic and FDP strategies. In addition, the best performance of an approach (defined as the highest expected profit achieved and the lowest CPU time taken to solve the underlying problem by a method) is presented in **bold** and a dash $-$ highlights the specific cases with 'no solution obtained' by ABM, BA-1, FDP-1 and FDP-2 approaches for the production capacity higher than five within three and four-generation production lines due to the computational difficulty as seen from Table 2.3. Both forward and backward dynamic programming approaches require solving a nonlinear optimisation problem (with high-degree polynomial objective function) at each state of every time period. The non-linearity arises because of the definition of choice probabilities and the demand function. Moreover, the degree of the polynomial function expands as the capacity and number of generations increase. Therefore, it becomes computationally intractable to solve ABM, BA-1, FDP-1 and FDP-2 approaches when capacity is higher than five for three and four generation product line problems.

By comparing the expected profits in Table 2.3, we observe that BA-1 and BA-2 typically generate the highest profits followed by FDP-1, FDP-2, FDP-3 and ABM without a specific ordering between the latter four solution methods. Expected profits obtained by BA-1 and BA-2 are significantly close. The small performance gaps between BA-1 and BA-2 show that the theoretical price bounds perform well compared to the price bounds obtained from ABM. The performance of the FDP-based methods in terms of expected profits significantly depends on the production capacity. As the production capacity increases, the FDP-based methods perform better since they can explore more states.

The size of the state space expands exponentially with the number of generations. With the same number of sample paths, FDP is able to explore more number of states in two generation product line compared to the three and four generation lines. Since the performance of FDP depends on the number of states it can explore, the FDP-

Table 2.3: Performance of different solution methods for multi-generation product lines

Production Capacity	Performance Metrics	Two-stage Heuristic			Forward Dynamic Programming		
		ABM	BA-1	BA-2	FDP-1	FDP-2	FDP-3
Two-generation Product Line							
5	Exp. Profit	12.69	**12.89**	12.47	12.21	12.56	11.94
	Time (s)	52.23	54.13	**1.83**	2739.21	74.23	38.35
10	Exp. Profit	23.96	24.42	**24.47**	23.75	23.67	23.83
	Time (s)	1126.32	1130.41	**9.12**	6888.62	1150.30	41.98
15	Exp. Profit	35.08	**37.13**	36.62	36.89	35.16	35.52
	Time (s)	17,800.88	1725.03	59.13	21425.36	17654.32	**58.91**
25	Exp. Profit	57.62	60.18	60.16	**62.38**	60.71	60.82
	Time (s)	456,809.12	564,809.34	374.62	131,367.64	458,342.35	**220.44**
Three-generation Product Line							
5	Exp. Profit	17.80	**18.57**	18.01	16.85	15.49	17.68
	Time (s)	1506.32	1507.34	**18.60**	17652.34	1886.46	50.43
10	Exp. Profit	–	–	**33.92**	–	–	31.43
	Time (s)	–	–	**49.86**	–	–	70.32
15	Exp. Profit	–	–	**51.24**	–	–	48.79
	Time (s)	–	–	102.92	–	–	**100.85**
25	Exp. Profit	–	–	**83.54**	–	–	79.85
	Time (s)	–	–	741.21	–	–	**557.6**
Four-generation Product Line							
5	Exp. Profit	–	–	**20.32**	–	–	18.07
	Time (s)	–	–	**33.47**	–	–	566.32
10	Exp. Profit	–	–	**39.99**	–	–	38.00
	Time (s)	–	–	**224.04**	–	–	983.82
15	Exp. Profit	–	–	**59.66**	–	–	56.04
	Time (s)	–	–	1464.83	–	–	**1234.51**
25	Exp. Profit	–	–	**104.67**	–	–	97.5
	Time (s)	–	–	75737.16	–	–	**6765.43**

based methods perform better when there are small number of versions available.

In terms of computation times, we observe that the most computational effort is invested in solving the abridged model (ABM). Since BA-1 and FDP-2 use the price bounds obtained from the abridged model, their computation times become high. On the other hand, BA-2 and FDP-3 apply for the theoretical price bounds, and therefore they are computationally much faster. We should also emphasize that FDP-1 is relatively slower since it doesn't use price sets unlike FDP-2 and FDP-3 approaches. Instead, at each iteration of FDP-1, the high-degree non-linear optimisation problem (see Section 2.4.1) is solved at each decision stage. In general, the CPU times increase as the problem size increases.

Impact of Multiple Generations on Profitability: We also design experiments to analyse the impact of strategic decisions regarding to the management of multiple generations on the company's profitability. The main question to answer is under what conditions offering multiple generations improves expected profit over a planning horizon. We quantify the benefits of offering multiple generations as opposed to selling only one generation in the market by comparing the expected profits obtained by different product line strategies. We assume that a firm produces four generations of a product over the planning horizon of eight time periods. At the release time of a new generation, the firm decides how many generations to keep in the market depending on the adopted product line strategy. In order to explore the impact of different product line strategies on the expected profit over a fixed planning horizon, we consider four settings: (a) one-generation product line where the firm sells only the latest generation, (b) two-generation product line where the latest two generations are sold, (c) three-generation product line where the latest three generations are available in the market, and (d) four-generation product line where all four generations are sold. We apply the FDP-3 strategy due to the computational efficiency of the forward dynamic programming approach. In particular, we vary the production cost and innovation level of the multi-generation product line to analyse the firm's production decisions and the related profitability. Table 2.4 presents our results. The initial production cost and innovation levels of the oldest generation of the product line are displayed in the first two columns of Table 2.4. When a new generation is released, its innovation level and production cost are increased by 0.5 units each in comparison with its previous generation. The expected profits are obtained by the FDP-3 approach using different generations of production line strategies at the end of the planning horizon.

Table 2.4: Expected profit obtained over a planning horizon

Initial Cost	Initial Quality	Product Lines			
		1-generation	2-generation	3-generation	4-generation
0.5	3.5	88.07	91.25	95.87	97.50
0.5	4.5	131.16	131.88	135.86	141.74
1.5	3.5	29.54	31.09	32.46	41.55
1.5	4.5	61.46	65.35	67.50	79.34

Comparing the expected profits obtained from different product line strategies, we observe that profitability of the firm increases as the number of generations offered in the market increases. In particular, there is a significant gap between the expected profits obtained by single-generation and four-generation strategies. This difference can be attributed to the impact of capturing different customer segments with wider choice of products. On the other hand, the difference between the expected profits obtained by three-generation and four-generation product lines are very close when the production cost and the innovation level are low for the oldest generation (the first test instance). In our simulation experiments, we observe that while operating a four-generation product line, the firm generally decides to sell three generations instead of four to yield the maximum profit. Due to low innovation level of the oldest generation, customer preferences shift to the other versions. On the other hand, when the innovation level for the oldest generation is high (at level of 4.5), selling it with the other versions in the market is more profitable. The introduction and discontinuation of multiple generations depend on their innovation levels. The firm is likely to sell different versions when the innovation levels are high. In practice, technology firms may upgrade the design and software features of the old products when they release a new one. In fact, a similar strategy was implemented by Apple in 2016. During the release of new mobile, iPhone 6, Apple also launched an upgraded version of iPhone 5S by increasing its capacity (Welch 2017).

Impact of Joint Production-Pricing Strategy: We are also concerned with an effectiveness of dynamic joint production-pricing decision making in the multi-generation product line problem. We therefore adopted planning strategies (so-called "partial planning") introduced in Chan et al. (2006) to compare with the proposed dynamic strategies.

In a partial planning strategy, we fix one decision (production or pricing) at the beginning of the planning horizon while the other decision is dynamically determined by the optimisation model at each time period. In practice, we see that decisions related to pricing and production may be taken in advance due the various limitations related to legal contracts experienced by firms (Rasmussen 2018). This kind of advance decisions is generally made on the basis of preset parameters (Chan et al. 2006) without taking uncertainty into account. Therefore, we determine the partial planning strategies based on the deterministic formulation of the joint decision-making model (1)-(2). In particular, we consider two partial planning strategies, abbreviated as F-price and F-prod. In the F-price strategy, prices of all generations are fixed at the beginning of the planning horizon. These prices are basically input to the FDP-3 algorithm to find the dynamic production policy. Similarly, in the F-prod strategy, we fix the production decision and find the dynamic pricing policy by using FDP-3. Note that the fixed pricing and production decisions in the F-price and F-prod strategies, respectively, are obtained by the deterministic formulation of the joint decision-making model. In this experiment, we consider a four-generation product line and compare the performance of F-price and F-prod with the joint production-pricing policy, abbreviated as JPP. We present our numerical results in Table 2.5. The first two columns in Table 2.5 show the production cost and innovation level of the oldest generation of the product line. The next three columns give the expected profits obtained by JPP, F-price and F-prod, respectively. The last two columns display the percentage gaps between JPP and the two partial planning strategies. Figure 2.2 illustrates the production and inventory levels at each time period by using the second test instance given in Table 2.5.

Table 2.5: Performance comparison of joint production-pricing and partial strategies

Initial Cost	Initial Quality	Strategies			% Gap with JPP	
		JPP	F-price	F-prod	F-price	F-prod
0.50	3.50	97.50	92.26	85.15	5.68	12.67
0.50	4.50	141.74	138.10	125.74	2.63	11.29
1.50	3.50	41.55	37.84	30.24	9.81	27.22
1.50	4.50	79.34	70.37	61.51	12.76	22.47

As seen from Table 2.5, JPP outperforms the partial planning strategies. While the average performance gap between JPP and F-price is 7.72%, it is 18.41% for F-prod. The performance gaps are more striking when the initial cost of the oldest generation is high. The poor performance of partial planning strategies is due to the fixed decision (production or pricing) made at the beginning of the planning horizon. Comparing the top and middle panels of Figure 2.2, we note that the F-price strategy cannot manage multi-generation product line effectively as opposed to JPP. It generally offers two generations during the planning horizon. On the other hand, from the top and bottom panels of Figure 2.2, we observe that JPP and F-prod have a similar product line. In addition, the F-prod strategy behaves closely to the JPP policy in terms of production and inventory levels. However, there is a significant gap between the expected profits of F-prod and JPP in all test instances. Due to the fixed production decision set at the beginning of the planning horizon, the F-prod strategy cannot balance its inventory and production levels. As illustrated in Figure (2.2), the inventory level for version 1 at time 3 is approximately same in JPP and F-prod, respectively. However, the corresponding production decisions for version 1 are different for these two strategies. The JPP policy dynamically reacts to a high inventory level by producing fewer products. On the other hand, when we fix the production levels, the firm is unable to dynamically respond to the changes in inventory levels. Thus, the F-prod strategy suggests a higher production for version 1 in comparison to the JPP policy. The variation in inventory levels of multiple versions is primarily caused by uncertain demand. Thus, dynamically deciding the production levels for multiple versions is essential to mitigate demand uncertainty. In summary, fixing one decision related to price or production not only decreases profits, but also has several other drawbacks. When the prices are fixed at the beginning of the planning horizon, the firm cannot react to the changes in customer demand and may fail to utilise the benefit of internal competition among multiple versions.

On the other hand, when production decision is fixed, the firm cannot manage the demand uncertainty efficiently which may result in shortages or overstocking. Thus, the internal competition among multiple versions and uncertainty in their demand must be handled by a joint production-pricing strategies.

Impact of Customer Segments: We also investigate impact of customer preferences on the management of a multi-generation product line. In particular, we explore how product sales alternate between different generations when the proportion of each customer segment changes. Recall that the segment of a customer is determined with respect to innovation sensitivity parameter (i.e., θ). While a customer having innovative sensitivity of $\theta > 0.5$ is classified as innovation sensitive, a price sensitive customer would have innovative sensitivity of $\theta < 0.5$. We simulate customer requests by changing the value of θ and evaluate the product sales by using the optimal production plan obtained by the BA-2 strategy. Since the FDP-based strategies do not take θ into account while determining the production and pricing policies, we use the BA-2 strategy for this experiment. Figure 2.3 illustrates the average number of sales (left panel) and the average production quantities (right panel) for three- (top panel) and four- (bottom panel) generation product lines at the last two time periods with respect to varying customer segments. In these figures, the horizontal axis displays market classification in terms of various percentage customer segments as starting from 90-10 up to 10-90. For instance, the case '70-30' represents 70% of customers in the market as being quality sensitive whereas the remaining 30% of customers is price sensitive.

Overall, we observe that results obtained for three- and four-generation production lines show similar performance patterns. By comparing the average number of sales and production quantities in Figure 2.3, one can see that when the proportion of price sensitive customers is high, demand for the oldest product remains high and the production of all generations becomes profitable. On the other hand, when there are more quality sensitive customers in the market, demand shifts towards the newer generations. An interesting result at this point is that the average production quantities increases as the proportion of quality sensitive customers increases. Because the production decisions at the last two time periods depend on the starting inventory, and a change in the market segments results in different starting inventories. Thus, our model is adaptive of the changing customer segments due to its dynamic nature. It is important to point out that the average production for the oldest generations decreases to zero when the market is dominated by mainly innovative sensitive customers. This shows that the optimal product line strategy significantly depends on the proportion of price and innovation sensitive customers in the market.

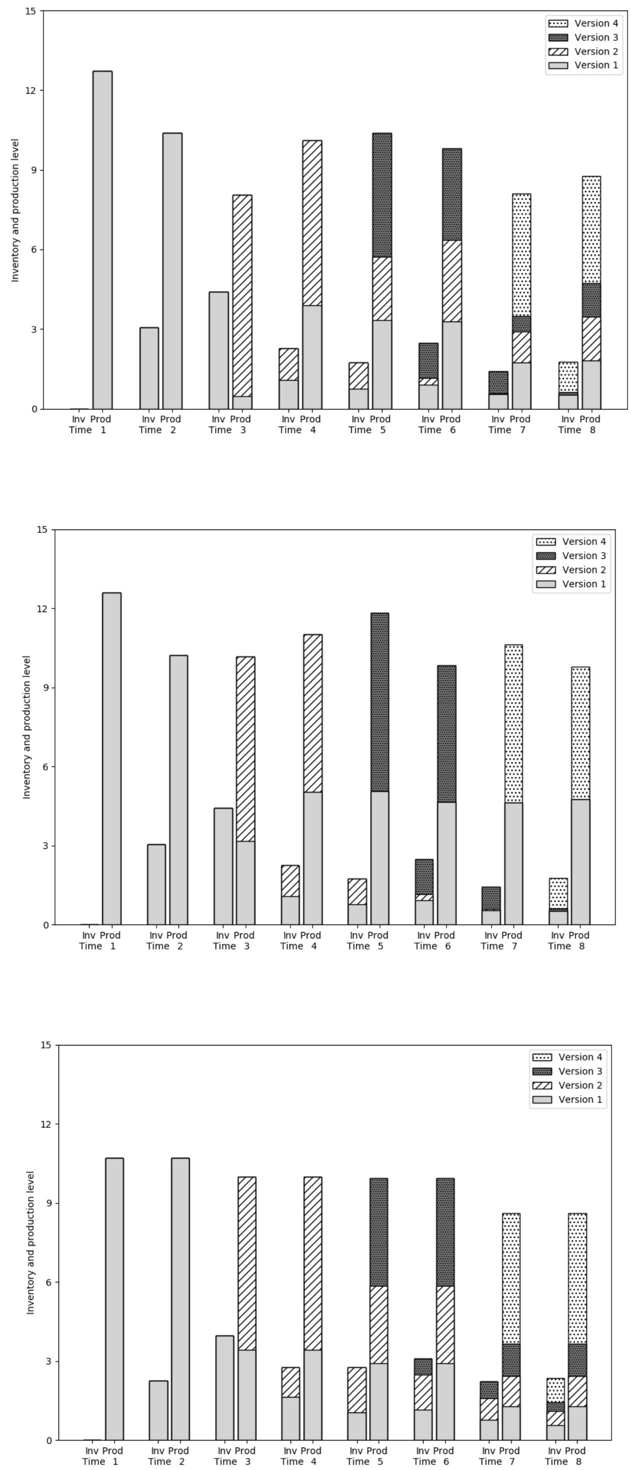

Figure 2.2: Inventory and production levels obtained by joint production-pricing

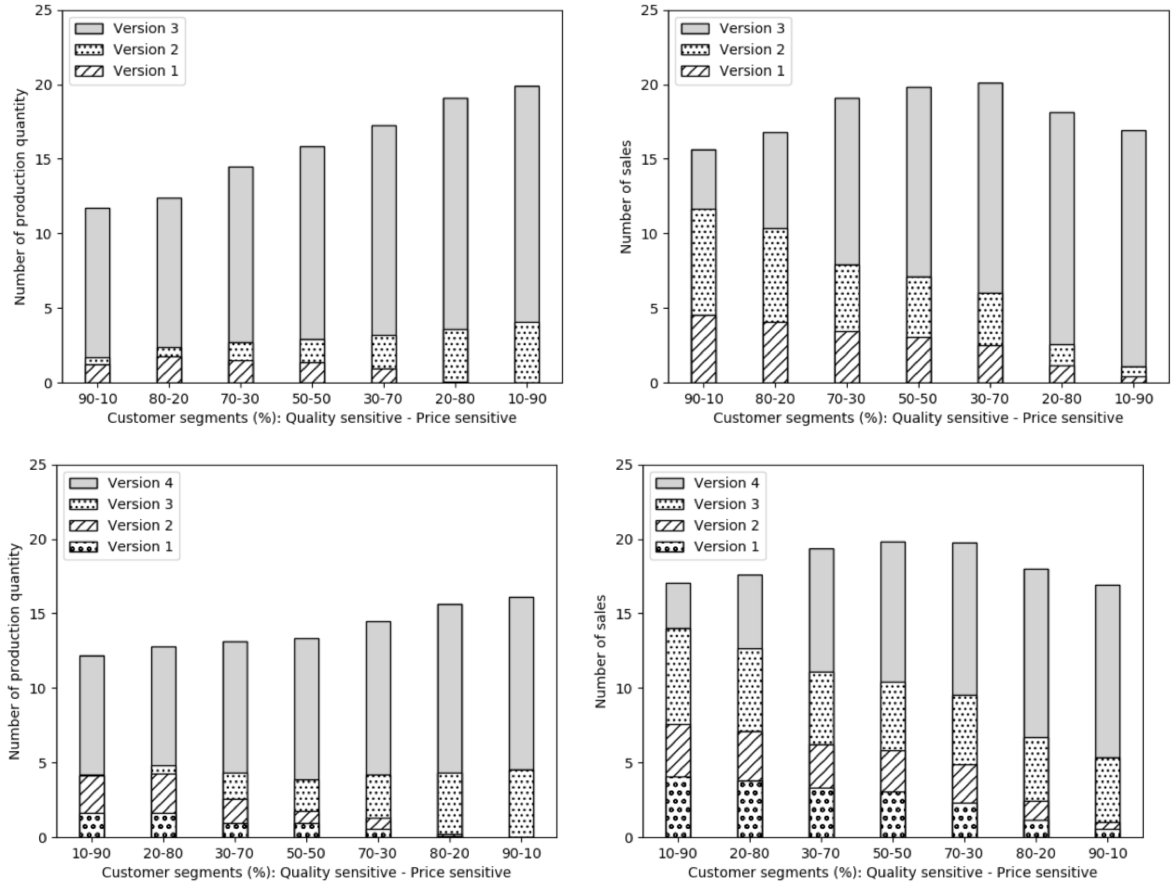

Figure 2.3: Average sales and production quantity in three-generation

2.6 Conclusions

In this chapter, we study the joint production-pricing decision-making process of a firm selling a multi-generation product line under demand uncertainty. The existing studies in literature focus on transition between exactly two generations rather than developing a multi-generation product line. We account for the internal competition between multiple generations by examining customer choices and derive a stochastic dynamic model for joint production-pricing problem for multi-generation product line. Finding the optimal production and pricing policies for this problem requires solving a stochastic dynamic program with a high-dimensional state vector. By analysing the structural properties of the problem, we present two approximations, based on FDP and heuristic in conjunction with pricing strategies obtained by different solution methods, for solving the dynamic joint production-pricing problem. A computational study is conducted to investigate the performance of the approximation methods and to derive managerial insights.

We first evaluate how selling multi-generation product line affects the firm profitability. Our numerical results show that the profitability of selling the oldest version increases with its innovation level. Therefore, by improving the innovation levels of the older generation during a new product release, the firm can increase the sales of older generations. We then illustrate the benefit of joint decision-making process in multi-generation product line. The joint decision-making policy proposed in this research is compared with partial decision-making policies. Our results indicate that the dynamic joint policy outperforms the fixed production and pricing policies since it takes the recent changes into account to match with demand by production.

We also analyse the effect of customer segments on the management of multi-generation product line. By varying the proportion of customer's price and quality sensitivity (toward the underlying technology) in the market, we investigate the production decisions for multi-generation products. The results indicate that when the proportion of innovation sensitive customers is high, the production and sale of older generations drop due to the high demand towards the new generations. Similarly, as the percentage of price sensitive customers increases, it becomes more profitable to sell older generations with new release. This shows that the number of generations to be kept in the market should be determined by considering varying customer segments.

The methodology introduced in this chapter can be further tested over large scale problems. Moreover, discrete event and agent-based simulation methods could have been explored. As future research, one may introduce release time as a decision variable rather than assuming it to be predetermined.

Chapter 3

Ordering and Markdown Policies for Perishable Product

Many food products are perishable. The firms often promote the freshness level of perishable products to gain strong position in the market. However, they also face the challenge of the increasing amount of wastage of perishable products due to demand uncertainty. A common strategy to reduce wastage is to lower the selling price of inventories close to expiry. In this research topic, we develop efficient markdown and ordering strategies to sell a perishable product under uncertainty in its demand.

This chapter discusses the challenges faced by supermarkets selling perishable products. We then provide a review of relevant literature on inventory and price management of perishable products. The joint ordering and markdown problem is explained and then its stochastic dynamic programming formulation and customer choice model are provided. The exact solution methodology based on concavity is presented as well. We then focus on the computational study testing the solution algorithm as well as analysing the joint ordering-markdown polices. The conclusion remarks of this chapter are then finally described.

3.1 Dynamic Policies for Perishable Products

Perishable products that deteriorate quickly such as fruit, vegetable, dairy products, and cut flowers constitute more than half of supermarket sales (FMI 2016). Several studies report perishable products as a major attraction for customers in stores and these products have become a main reason for consumers to choose one store over another (Krider & Weinberg 2000, Tsiros & Heilman 2005, Webber et al. 2011). Thus, retailers extensively highlight freshness, quality and vast varieties of their perishable products in their marketing campaigns (Bridge 2018).

Promoting perishables can help retailers to gain a strong position in the market, but they face challenges as well. Displaying large varieties of fresh-looking perishable products, such as food items, often leads to wastage and raises environmental concerns. A recent study commissioned by the United Nations indicates that globally, one-third of food produced for consumption is lost or wasted (Ndukwe 2018). Within the major supermarkets in the UK, 44% of Tesco's bread products are thrown away every day due to expiration (BBC 2013). Inefficient inventory management of perishable products negatively impacts companies economic prosperity. The biggest challenge in the management of perishable products is matching perishable supply with uncertain demand. A perishable product may remain unsold until its expiration primarily due to uncertainty in its demand. To avoid wastage and reduce losses, retailers often price down old products approaching the end of their lifetime (Gallego & Van Ryzin 1994, Talluri & Ryzin 2004, Chen & Sapra 2013). Chowberry, an application in Nigeria, provided a platform to the supermarket retailers to put the old food products on reduced prices (Ndukwe 2018). They report an 80% reduction in wastage due to the introduction of markdowns. Offering a markdown on old products can induce purchases from consumers who cannot afford the full price of new products. However, the price of new products as well as the discounted prices of old products should be carefully determined. Setting a very low price for old products may lead to revenue loss because consumers who can purchase new products may switch to older ones due to the attractive price difference. On the other hand, pricing old products high may decrease their demand and result in higher cost of wastage. To effectively manage the mismatch between perishable supply and demand, it is important for retailers to coordinate markdown pricing with replenishment decisions. Markdown pricing for old products should be determined by considering the available inventory and the remaining shelf lives of the products. Cognizant (2015) and Mckinsey (2014) specifically emphasise on the development of joint markdown and replenishment technology to improve profits. Although the importance of joint ordering-pricing techniques is extensively advocated to mitigate demand uncertainty for fixed-age products, this research area has not yet received enough attention (Davis 1993, Chen & Simchi-Levi 2004, Talluri & Ryzin 2004, Karaesmen et al. 2011, Webber et al. 2011, Mckinsey 2014, Cognizant 2015).

In this chapter, we study the joint ordering-pricing decision making problem of a firm selling a perishable product with a fixed lifetime using a periodic review model.

In any period, the retailer decides how much fresh products to order and how to price ageing inventories. Poor management of perishables may result in disposal of large amount of outdated items. To reduce wastage, the retailer can conduct markdown sales by pricing down old inventories. Demand in each period is uncertain with some customers preferring to buy regular priced products and others preferring to buy marked-down old products. Even though the markdown sales reduce the wastage of the perishable products, it may cannibalize the demand for the products sold at regular prices. Thus, we are concerned with dynamic ordering-pricing strategies by considering demand uncertainty and internal competition between fresh and old inventories.

The contribution of our research lies in obtaining combined ordering-markdown pricing policies for a firm to tackle the challenges of wastage and demand cannibalization. We formulate the joint ordering-pricing decision-making problem for a general lifetime perishable product as a stochastic dynamic programming model. We account for the demand cannibalization between fresh and markdown inventories by evaluating the dynamic changes in customer choices. The challenge with a perishable inventory management problem is the dimensional expansion of the state space, as a result of continuous tracking of the product age. Due to the complexity of this problem, existing research largely focuses on inventory systems with a two-period lifetime product. In contrast to the existing literature, we consider a general lifetime perishable product and we tackle the curse of dimensionality on the state space by proposing a solution methodology based on the theoretical properties of the dynamic programming model. We prove that the value function of the dynamic programming model is k-concave in inventory levels. Thus, we employ the properties of k-concavity to design an algorithm providing the optimal ordering and pricing policy. We conduct extensive numerical experiments to illustrate the performance of the solution algorithm and to gain managerial insights on the ordering and pricing strategies. In these experiments, we analyse the benefit of dynamic ordering and markdown strategy as opposed to using fixed markdown policies which are common in practice. Our analysis shows that dynamic ordering-markdown policy performs significantly better than fixed markdown strategies since ordering and markdown decisions are made by considering the changes in customer preferences and the available inventory levels for ageing inventories.

The remaining part of the chapter is organized as follows. Section 3.2 focuses on the literature review by providing details of existing studies relevant to our research. The stochastic dynamic programming formulation of the joint ordering-pricing problem is presented in Section 3.3. The solution methodology and computational results are explained in Sections 3.4, 3.5 and 3.6, respectively. The concluding remarks are provided in Section 3.7.

3.2 Review of Inventory Management of Perishable Products

Research on inventory management of perishable products has attracted significant attention over the years. The recent reviews in this area are provided by Nahmias (2011), Karaesmen et al. (2011), Bakker et al. (2012) and Janssen et al. (2016). Our research is mostly related to the literature that considers joint inventory and pricing management for perishable products deteriorating over time. In this area, the main focus has been towards the periodic review models where the inventory has been tracked at specific periods rather than keeping a continuous track. Continuous review models for perishables are studied for random lifetime products. We refer to Goyal & Giri (2001) and Bakker et al. (2012) for a comprehensive review of continuous review models.

Ferguson & Koenigsberg (2007) analyse the inventory management and pricing decisions for a firm selling a food product with exactly two-period life cycle. In the first time period, the decision of procurement and pricing of fresh products is made by considering demand uncertainty. At the start of the second time period, the decisions are based on determining how much leftover inventory of old product to carry over, how much fresh products to procure, and what prices to be charged for new and old inventories. However, they assume that the demand in the second period is deterministic. Li et al. (2009) study a similar problem of a two-period lifetime product by considering demand uncertainty in both periods. They assume a single price for old and new inventories. Thus, even though a track of differently aged inventories is kept, there is no price differentiation between old and new inventories. They analyse the structural properties of the optimal inventory replenishment and pricing policy. Chen & Sapra (2013) extend the work of Li et al. (2009) by considering different inventory consumption scenarios. They study inventory consumption in the order of first-in-first-out and last-in-first-out and derive structural results on the optimal policy. In a related work, Chen et al. (2014) investigate the effect of lead time in product replenishment decision. Similar to previous studies, inventories of different ages are priced equally. However, they allow to discard the excess inventory before it is perished. Herbon (2017) considers the coexistence of old and new inventories with different prices on the shelf and analyzes the value of selling different age products under deterministic demand.

The literature discussed so far either assume a single price for old and new inventories or consider price differentiation between a two-period lifetime product facing deterministic demand. Price differentiation between old and new inventories for a perishable product with multiple-period (n-period) lifetime is addressed by Li et al. (2012). They assume that the retailer discards old inventory when a new order is placed. In a given time period, the retailer makes replenishment and pricing decisions, and decides whether to keep or discard the remaining inventory. Thus, old and new inventories are not sold at the same time. The existence of old and new inventories along-with price differentiation between them is considered by Sainathan (2013), Hu, Shum & Yu (2015) and Li et al. (2016). Sainathan (2013) study a joint pricing and inventory model of a perishable product with a two-period lifetime over an infinite horizon. He assumes that old and new perishable products compete with each other in the market under demand uncertainty that is incorporated through the process of dynamic demand substitution. Demand substitution refers to dynamic process of replacing the demand of old products with newer ones. Sainathan (2013) uses a linear utility function to model demand substitution and to derive optimal pricing and replenishment policies for the perishable product. Hu, Shum & Yu (2015) obtain joint inventory and markdown strategies for a firm selling a perishable product to strategic customers. However, their model is specifically developed for one-period lifetime products, like bakery and other food items. Li et al. (2016) study the joint replenishment and clearance sales decisions of a firm selling n-period lifetime product. Both, Hu, Shum & Yu (2015) and Li et al. (2016) model replenishment and clearance sales decisions for perishable products where the prices of old and new inventories are known and fixed. Chua et al. (2017) examine both, inventory and pricing decisions of a firm selling a two-period perishable product. They assume the pricing decisions and demand for the perishable product to be independent of time. Time dependence between the demand of n-period lifetime product is considered by Chao et al. (2015). Their focus lies in building an approximation algorithm for perishable inventory systems, rather than developing pricing strategies for different age of inventories. Chintapalli (2015) works on the joint inventory-pricing decision model for a retailer experiencing substitutable demand for a multi-period food product. He assumes that consumers are free to choose between old and new products. Due to the complexity of the problem, he focuses on a simplified inventory-pricing model where marked-down prices for old food items are stationary over time. There are two main differences between our research and the existing studies in this stream of literature. First, most of these studies develop joint pricing and inventory management policies for a two-period lifetime product. Second, existing literature focusing on a multi-period lifetime perishable product does not consider inventories of different age in their pricing decision. In this study, we take into account both old and new inventories with different pricing decisions for a general lifetime product. Moreover, unlike the existing literature, we model the process of markdown over time and analyse its properties.

Our research is also closely related to the stream of literature on inventory management of different age products. Deniz et al. (2010) study the inventory issuing and replenishment policies for a perishable product with two-period lifetime where demand is differentiated with respect to the age of the product. They compare different inventory issuing policy pairs and provide analytical results. Li & Yu (2014) investigate the structural properties of the optimal inventory policy for a firm selling perishable products. Given the regular and clearance sale price, at each time period, the firm decides how much new products to order, how much leftover inventory of old product to carry over and how much to sell at a clearance sale price. They assume that clearance products will be sold out due to high demand and hence, the firm decides how many old products to put on clearance sale. Abouee-Mehrizi et al. (2019) consider age dependent demand and study joint ordering, inventory allocation and disposal decisions for a multi-period perishable inventory system. In a given time period, available inventory is allocated to the different demand classes where each class accepts products with remaining lifetime longer than a threshold. Inventory of any age can be disposed at the end of each time period. They examine the relation between ordering decision and the inventories of different age and characterize the structure of the optimal ordering, allocation, and disposal policies. In a related work, Chen & Sapra (2020) consider age-dependent demand and investigate the effect of fix and flexible replenishment decisions on the structure of the optimal inventory management policy. Different from Abouee-Mehrizi et al. (2019), they do not allow the coexistence of multi-age inventories. Old inventory is discarded when a new order is placed. Chen & Sapra (2020) focus on replenishment decision and explore the value of having flexibility in ordering.

Research related to finding markdown strategies in view of age and time dependency has not been widely studied. Only Ferguson & Koenigsberg (2007) and Herbon (2017) study existence of multiple age of inventories in a two period problem with known demand. We summarise the most related research papers developing inventory management and markdown pricing policies for perishable inventory systems using stochastic dynamic programming in Table 3.1. Different from the existing literature, we consider a general lifetime perishable product and allow coexistence of fresh and old inventories in the system. We primarily investigate the combined ordering-markdown policies to tackle the challenge of demand cannibalization between different age of inventories. Considering the demand uncertainty, the joint ordering-pricing decision making problem is formulated as a stochastic dynamic programming model. We propose a solution methodology based on the theoretical properties of the dynamic program. We apply the properties of k-concavity to design an algorithm yielding the same solution as the standard backward dynamic programming technique.

Table 3.1: Classification of relevant research papers

Research papers	Age of product	Multi-age Inventory	Uncertain demand	Decisions		Age & time varying price
				Ordering	Pricing	
Ferguson & Koenigsberg (2007)	2	✓		✓	✓	✓
Li et al. (2009)	2	✓	✓	✓	✓	
Li et al. (2012)	n		✓	✓	✓	
Sainathan (2013)	2	✓	✓	✓	✓	
Li & Yu (2014)	n	✓	✓	✓		
Chen et al. (2014)	n	✓	✓	✓	✓	
Hu, Shum & Yu (2015)	1			✓		✓
Chao et al. (2015)	n	✓	✓	✓		
Li et al. (2016)	n	✓	✓	✓		✓
Chua et al. (2017)	2	✓	✓	✓	✓	
Herbon (2017)	2	✓		✓	✓	✓
Our research	n	✓	✓	✓	✓	✓

3.3 The Dynamic Ordering-Markdown Model

Consider a firm (such as Tesco or Walmart) that sells a perishable product such as fruits, vegetables, meat, or cut flowers. The perishable product has an n-period lifetime, after which it expires and has to be disposed off at a fixed penalty cost. Demand for perishable product is assumed to be uncertain and unmet demand is lost. The firm regularly reviews the available inventory and places an order for fresh products. We assume that the fresh products are sold together with the older ones on the same shelf. To avoid loss and wastage, the firm can conduct markdown sales by pricing down old inventories. This price differentiation between different age of inventories leads to an internal competition and markdown products may cannibalize the demand for the products sold at regular prices. Thus, the firm must decide when to put the products at markdown and the reduced price to offer by considering uncertain demand. We assume that the firm tackles demand uncertainty and internal competition by joint decision-making framework regarding ordering and markdown sale. In this section, we first present the details of the underlying ordering-markdown pricing problem and then formulate the problem as a stochastic dynamic program under a customer choice model. Before that, we introduce the notation used in the model formulation in Table A2.

Table 3.2: Description of notation

	Model parameters
T	planning horizon discretised by time periods $t = 0, 1, \cdots, T$
n	fixed age of perishable product
κ	production capacity
δ	discount factor
c	unit ordering cost
h	unit inventory holding cost
γ	unit penalty cost for discarded product
p_r	unit regular price
$\tilde{d}_t$	uncertain demand at time t
	State variables
$\mathbf{x_t}$	vector of regular inventory levels $x_{i,t}$ of age i at the beginning of time t
$\mathbf{y_t}$	vector of markdown inventory levels $y_{i,t}$ of age i at the beginning of time t
$\mathbf{I_t}$	Inventory vector $\mathbf{I_t} = (\mathbf{x_t}, \mathbf{y_t})$ consisting of regular and markdown inventory
	Actions
$\mathbf{p_{mt}}$	vector of markdown price $p_{i,mt}$ for inventory of age i at time t
$\mathbf{w_t}$	vector of binary decision $w_{i,t}$ for inventory of age i at time t
q_t	order decision at time t

Assume the firm sells a perishable product over a finite planning horizon. The planning horizon is discretized into T time periods, and $t = 0, 1, \cdots, T$ represents specific decision epochs where the operational decisions (such as ordering and pricing) are made. The perishable product has a fixed lifetime of n periods. The product ordered at time period t expires at time period $t + n$ and is disposed at a penalty cost of γ per unit. We assume that the lead time for ordering is zero so that the fresh products can be received instantaneously.

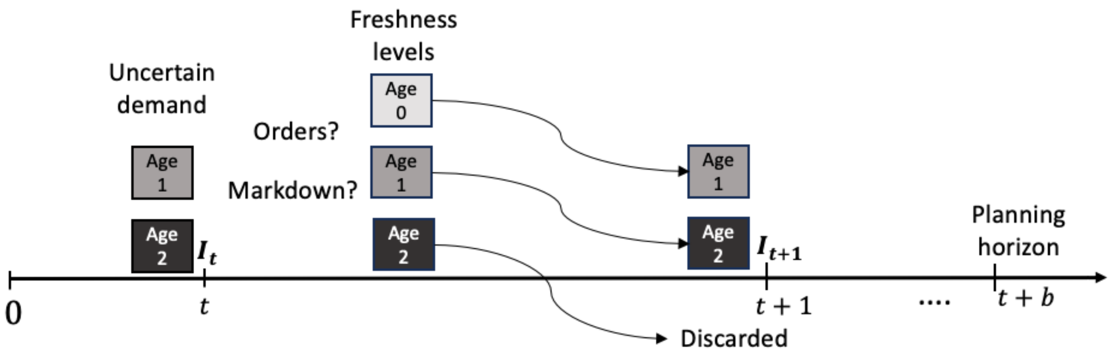

Figure 3.1: Decision-making process for a deteriorating perishable product

Figure 3.1 illustrates the decision-making process of a deteriorating perishable products. Inventory levels of the perishable product are reviewed at the beginning of each time period. The available inventories are either sold at a regular or marked down price. The regular price, denoted by p_r, is known to the firm and remains same along the planning horizon. On the other hand, the firm decides the markdown price $p_{i,mt}$ for inventory of age i at time t. We define w_{it} to denote the binary variable for the markdown decision of inventory of age i at time t. If $w_{it} = 0$, the inventory of age i at time t is sold at the regular price p_r. On the other hand, when $w_{it} = 1$, the inventory of age i at time t is marked down and the firm must decide the markdown price $p_{i,mt}$. We introduce vectors $\mathbf{p_{mt}} = \{p_{i,mt} \in \mathbb{R}^+ : i = 0, 1, \cdots, n\}$ and $\mathbf{w_t} = \{w_{it} \in \{0, 1\} : i = 0, 1, \cdots, n\}$ corresponding to markdown pricing and binary decisions at time period t, respectively. We assume the following regarding markdown decisions,

A1: A product with higher age is closer to expiry and has lower quality than a product with lower age. Thus, if the inventory of an age i is marked down, then inventory of a higher age $i + 1$ should also be marked down. This condition is represented by the equation, $w_{i+1,t} \geq w_{it}$, where $i = 0, 1, \ldots, n$ and $t = 0, 1, \ldots, T$.

A2: In practice, if the price of perishable inventories (such as fruits and vegetables) is marked down, it cannot be marked up in their remaining lifetime. This condition is represented by the equation, $p_{i,mt} \geq p_{i,mt+1}$ for $i = 0, 1, \ldots, n$ and $t = 0, 1, \ldots, T - 1$. Thus, information regarding the markdown pricing is also stored in the state of the dynamic system.

We classify the initial inventory levels at any time into two categories, regular and markdown, based on their selling prices. Let the amount of regular inventory of age i at the beginning of time t be represented as x_{it}. Similarly, the markdown inventory of age i at the start of time t is denoted by y_{it}. The state of the dynamic system is defined by the vectors of regular and markdown inventories at time t and these are represented as $\mathbf{x_t} = \{x_{it} \in \mathbb{Z}^+ : i = 0, 1, \cdots, n\}$ and $\mathbf{y_t} = \{y_{it} \in \mathbb{Z}^+ : i = 0, 1, \cdots, n\}$, respectively. We define the inventory vector, $\mathbf{I_t} = (\mathbf{x_t}, \mathbf{y_t})$ consisting of both regular and markdown inventories as the state of the dynamic system. Given the state of the system, the firm makes ordering and markdown decisions at each time period. Let us suppose the firm places an order of q_t units in batch (or bulk). Let κ represent the ordering capacity for the firm. At each time period, the orders should not exceed the capacity. This can be formulated by the constraint, $q_t \leq \kappa$ for each time t. The assumption of restricting ordering capacity is to maintain theoretical and computational tractability of solutions of the dynamic model (Bertsekas 2018). We use c and h to denote the unit ordering and holding cost for the perishable product, respectively.

We suppose the random demand at time t to be dependent on the prices of the available inventories. Let $\tilde{d}_t = d_t(p_r, p_{0mt}, p_{1mt}, \cdots, p_{nmt}, \tilde{\epsilon}_t)$ denote the random demand at time t, where uncertain parameter $\tilde{\epsilon}_t$ follows a known distribution. The demand for products with a single price is assumed to be satisfied in a FIFO issuing rule. This assumption is common in literature for perishable inventory systems (Nahmias 2011). Moreover, in practice, supermarkets are reported to use the FIFO strategy to sell older inventories first to avoid wastage (Arline 2020). In FIFO, products they purchase earlier will become the first to be sold.

We now describe the state transition equations for inventory levels at each time. The starting regular and markdown inventories of age i at time $t+1$, represented by $x_{i,t+1}$ and $y_{i,t+1}$, have evolved from the inventory levels $x_{i-1,t}$ and $y_{i-1,t}$ of age $i-1$ at time t, respectively. Since the firm decides if a regular inventory is marked down or not, the regular inventory of age i is represented by $(1-w_{it})x_{it}$. On the other hand, the inventory of age i sold at the markdown price at time period t is denoted by $y_{it}+w_{it}x_{it}$. Since inventories are issued in a FIFO order, demand is satisfied from the inventory of age i when all inventories with surpassed ages are depleted. Thus, the state transition for regular and markdown inventories of age i from time t to $t+1$ are described by the following equations, where $i=0,1,\ldots,n$,

$$\begin{aligned} x_{i,t+1} &= max\big\{(1-w_{i-1,t})x_{i-1,t} - max\{\tilde{d}_t - \sum_{j=i}^{n-1}(1-w_{jt})x_{jt}, 0\}, 0\big\} \\ y_{i,t+1} &= max\big\{y_{i-1,t} + w_{i-1,t}x_{i-1,t} - max\{\tilde{d}_t - \sum_{j=i}^{n-1}(y_{jt}+w_{jt}x_{jt}), 0\}, 0\big\} \end{aligned} \tag{3.1}$$

The firm aims to maximize the expected profit over the planning horizon through a joint ordering-markdown decision framework. The expected profit is computed as the expected revenue minus the expected total cost of ordering, holding and penalty for wastage of expired products. Given the selling price for regular and markdown inventories, the revenue depends on future realisations of uncertain demand. When demand is lower than the inventory in-hand, represented as $\tilde{d}_t \leq \sum_{i=0}^{n}(1-w_{it})x_{it}$, the revenue from selling the product at regular price at time t becomes $p_r\tilde{d}_t$. In case of high demand (i.e., $\tilde{d}_t \geq \sum_{i=0}^{n}(1-w_{it})x_{it}$), the revenue at time t becomes $p_r\sum_{i=0}^{n}(1-w_{it})x_{it}$. Thus, we can state the revenue earned at time period t by selling the product at regular price as $p_r min\{\sum_{i=0}^{n}(1-w_{it})x_{it}, \tilde{d}_t\}$. Similarly, when the inventory of age i at time t is marked down to price $p_{i,mt}$, the firm earns revenue $p_{i,mt}min\{w_{it}x_{it}+y_{it}, \tilde{d}_{it}\}$, where $\tilde{d}_{it} = max\{\tilde{d}_t - \sum_{j=i}^{n} w_{jt}x_{jt} + y_{jt}, 0\}$ represents the left-over demand of age i after the inventories of surpassed ages are depleted. The total ordering and holding costs are expressed as cq_t and $h\sum_{i=0}^{n}(x_{i,t+1}+y_{i,t+1})$, respectively. Since the unsold inventory of age n is wasted and discarded, $\gamma(x_{n,t+1}+y_{n,t+1})$ represents the total penalty cost.

Let $V_t(\mathbf{I_t}|\mathbf{p_{m,t-1}})$ denote the value function at time t given a state $\mathbf{I_t}$ of the system.

The value function for the dynamic joint ordering-markdown pricing decision making problem at time period t can be written as follows;

$$\begin{aligned}
V_t(\mathbf{I_t}|\mathbf{p_{m,t-1}}) = \max_{q_t,\mathbf{p_{mt}},\mathbf{w_t}} \quad & \mathbb{E}[p_r min\{\sum_{i=0}^{n}(1-w_{it})x_{it}, \tilde{d}_t\} + \sum_{i=0}^{n} p_{imt} min\{w_{it}x_{it} + y_{it}, \tilde{d}_{it}\} \\
& - cq_t - h\sum_{i=0}^{n}(x_{i,t+1} + y_{i,t+1}) - \gamma(x_{n,t+1} + y_{n,t+1}) + \delta V_{t+1}(\mathbf{I_{t+1}}|\mathbf{p_{m,t}})] \\
\text{s.t.} \quad & p_{imt} \le p_{im,t-1}, i = 0, 1, \ldots, n, \\
& w_{it} \le w_{i+1,t}, i = 0, 1, \ldots, n, \\
& w_{it} \in \{0, 1\}, i = 0, 1, \ldots, n, \\
& 0 \le q_t \le \kappa.
\end{aligned} \tag{3.2}$$

Since no product will be sold after the planning horizon, the boundary condition at the terminal time period is $V_{T+1}(\mathbf{I_{T+1}}|\mathbf{p_{m,T}}) = \sum_{i=0}^{n} -(h+\gamma)(x_{i,T+1} + y_{i,T+1})$.

3.3.1 The Customer Choice Model

Customers are known to eagerly anticipate markdown sales on perishable products at supermarkets (e.g., Krider & Weinberg (2000), Tsiros & Heilman (2005) and Webber et al. (2011)) . A markdown is designed to increase sales of a product that cannot be sold in its current price. Offering a markdown on underselling products can attract new customers who cannot afford the full price of the product. At the same time, the markdown sale might lead to demand cannibalization by influencing the customers who are willing to purchase the product at a regular price (Ferguson & Koenigsberg 2007, Li et al. 2012). A customer choice model examines various factors governing the customer's purchase decision. For our problem, a customer has three choices at any time t: (i) buy the product at the regular price that is denoted by choice r, (ii) buy the product at the markdown price that is denoted as choice m, and (iii) no purchase which is denoted as choice o.

There are different observable and unobservable factors that affect the customer's choice of buying a product from the regular or markdown sale (Train 2009). Price, quality and freshness level of the product, and the customer's sensitivity towards freshness can be seen as observable factors impacting their choices. On the other hand, unobservable factors influencing predilection toward buying a specific version are described as idiosyncratic and mostly depend on the individual's preferences, such as personal taste and budget. In this chapter, we consider both observable and unobservable factors to analyze the customer's choice.

We assume that a customer is rational while making a decision and aims to maximize his/her own utility. For each customer, we define a random utility function to measure the preference of the customer. Let U_{st}^k be the random utility obtained by the kth customer from making choice s at time period t for $s \in \{r, m, o\}$ and $t = 0, 1, ..., T$. Following Train (2009) and Akçay et al. (2010), customer utility can be formulated as a linear function of price p_{st} and quality (or freshness) level α_{st} for choice s as $U_{st}^k = \theta^k \alpha_{st} - p_{st} + \mu \epsilon_s^k$, where θ^k represents the kth customer's sensitivity towards the freshness level of the product. The random term ϵ_s^k captures idiosyncratic preferences of customer k for choice s, with μ measuring the degree of such preferences. We assume that ϵ_s^k follows a Gumbell distribution. Then, the probability of customer k selecting choice s at time t is expressed as,

$$\gamma_t^k(s) = \frac{e^{((\theta^k \alpha_{st} - p_{st})/\mu)}}{1 + \sum\limits_{s \in \{r,m\}} e^{((\theta^k \alpha_{st} - p_{st})/\mu)}}.$$

Although the quality level and the market price of the product are the same for all customers, the idiosyncratic preferences and the quality sensitivities toward the products may vary for each customer and this affects the customer choice probabilities. Next, we will focus on some important theoretical results that will be utilised to develop a solution method for the dynamic joint ordering-markdown pricing model.

3.4 Theoretical Analysis

In this section, we present our theoretical results for the joint ordering and pricing model. For analysis, we assume that the markdown price $\mathbf{p_{m,t}}$ at any time t is known. Then, the dynamic programming model (3.2) can be rewritten as follows,

$$V_t(\mathbf{I_t}|\mathbf{p_{m,t}}) = \max_{q_t, \mathbf{w_t}} \quad \mathbb{E}[g_t(\mathbf{I_t}, q_t, \mathbf{w_t}) + V_{t+1}(\mathbf{I_{t+1}}|\mathbf{p_{m,t+1}})] \tag{3.3}$$

$g_t(\mathbf{I_t}, q_t, \mathbf{w_t})$ in equation (3.3) is the single period profit given as follows,

$$\begin{aligned} g_t(\mathbf{I_t}, q_t, \mathbf{w_t}) = p_r min\{\sum_{i=0}^{n}(1 - w_{it})x_{it}, \tilde{d}_t\} + \sum_{i=0}^{n} p_{imt} min\{w_{it}x_{it} + y_{it}, \tilde{d}_{it}\} \\ - h\sum_{i=0}^{n}(x_{i,t+1} + y_{i,t+1}) - \gamma(x_{n,t+1} + y_{n,t+1}) - cq_t \end{aligned}$$

where $\tilde{d}_{it} = (\tilde{d}_t - \sum_{j=i}^{n}(w_{jt}x_{jt} + y_{jt}), 0)^+$ represents the left-over demand of age i after the inventories of surpassed ages are depleted to satisfy the demand in a FIFO order.

At any time period given that the age of the product is n periods, the feasible space for binary decision vector $\mathbf{w_t}$ has $n+1$ possible values. For instance, the binary decision vector $\mathbf{w_t}$ for a product with 2 periods lifetime comprises of three possible values as $\mathbf{w_t} = (w_{0t}, w_{1t}) \in \{(0,0),(0,1),(1,1)\}$. We refer policy j as the jth value of the feasible space, where $j = 1, 2, \cdots, n-1$. Let us define the feasibility set as $\mathcal{W}_t = \{\mathbf{w_t^j} | w_{i-1,t}^j \geq w_{i,t}^j, w_{i,t} \in \{0,1\}, i, j = 1, 2, \cdots, n-1\}$, where $\mathbf{w_t^j}$ represents the vector of binary decision values for policy j. The tabular representation of $\mathcal{W}_t$ is given in Table 3.3.

Table 3.3: The tabular representation of feasible space of markdown decision vector

Policy	w_{0t}^j	w_{1t}^j	$\cdots$	$w_{n-2,t}^j$	$w_{n-1,t}^j$
1	0	0	$\cdots$	0	0
2	0	0	$\cdots$	0	1
3	0	0	$\cdots$	1	1
.	.	$\cdots$	.	.	.
$n-2$	0	1	$\cdots$	1	1
$n-1$	1	1	$\cdots$	1	1

The markdown decision policies are divided in three categories based on their markdown strategy. At time t, policy j is referred as,

1. *No markdown policy* represented by $j = 1$, when all inventories are sold at regular price
2. *Partial markdown policy* represented by $j = 2, 3, \cdots, n-2$, when some inventories are sold at regular price while the others are marked down. In other words, at the j-th partial markdown policy, inventories with age less than $n - j + 1$ periods are sold at regular price and the rest are marked down. For instance, when $j = 2$, inventories with age less or equal to $n-1$ periods are sold at regular price and the inventory with age n is marked down.
3. *Complete markdown policy* represented by $j = n-1$, when markdown is conducted on all inventories

The dynamic programming model exhibits various theoretical properties related to concavity and sub-modularity. In order to show and analyse these theoretical properties, we define $\pi_t(j, \mathbf{I_t}, q_t) = g_t(\mathbf{I_t}, q_t, \mathbf{w_t^j})$ as the single period profit obtained by following policy j for given inventory level $\mathbf{I_t}$ and order decision q_t at time t. $\pi_t(.)$ is also referred as the policy function. Let $\pi_t(.)$ denote the policy function representing the one-period profit the firm would earn when a markdown strategy is implemented.

The value function $V_t(.)$ is k-concave in order decision q_t and inventory level $x_{i,t}$ of age i at time t. We establish the k-concavity of the value function blackthrough three propositions. Proposition 1 shows the concavity of the policy function $\pi_t(j, \mathbf{I_t}, q_t)$ for any markdown decision policy j where $j = 1, \cdots, n-1$. Following Proposition 1, we show the k-concavity of the single-period profit $g_t(.)$ in Proposition 2. Finally, Proposition 3 establishes the k-concavity of the dynamic programming value function $V_t(.)$. We will now introduce the definition of k-concavity and establish several properties of the solution algorithm.

Definition (Chen & Simchi-Levi (2004)): *A real-valued function f is called k-concave for $k \geq 0$ if for any $x_0 \leq x_1$ and $\lambda \in [0, 1]$,*

$$f((1-\lambda)x_0 + \lambda x_1) \geq (1-\lambda)f(x_0) + \lambda f(x_1) - \lambda k. \tag{3.4}$$

Note that inventory vector $\mathbf{I_t} = (\mathbf{x_t}, \mathbf{y_t})$ has two components which are regular and markdown inventory vectors. For ease of expression, the following propositions establish the concavity result considering the regular inventory $\mathbf{x_t}$. Our results are also valid and remain the same for the markdown inventory $\mathbf{y_t}$.

Proposition 1: *For any time $t = 0, 1, \cdots, T$, the policy function $\pi_t(j, \mathbf{I_t}, q_t)$ for policy j is component-wise concave in order decision q_t and inventory level $x_{i,t}$ of age i, where $j = 1, 2, \cdots, n-1$.*

Proof: The proof is provided in Appendix B. ■

Proposition 2: The single-period profit $g_t(\mathbf{I_t}, q_t, \mathbf{w_t})$ *is $k^*_{i,t}$-concave in inventory level $x_{i,t}$ of age i for time $t = 1, 2, \cdots, T$.*

Proof: The proof is provided in Appendix B. ■

Proposition 3: Consider the value function $V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ at time t for inventory of age i, (i) $V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ *is $\hat{k}_{i,t}$-concave in regular inventory level $x_{i,t}$* , (ii)$V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ *is $\hat{k}_{i,t}$-concave in markdown inventory level*

Proof: The proof is provided in Appendix B. ■

Following Proposition 3, we conclude that the dynamic programming value function

$V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ is k-concave and this result will be used to develop an efficient solution methodology, referred as the concavity algorithm, in the next section. The following propositions establishes the sub-modularity of the policy function $\pi_t(j, \mathbf{I_t}, q_t)$ and the value function $V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ which will be used to reduce the size of the action space for ordering and markdown decisions.

Proposition 4: *At any time t, the policy function $\pi_t(j, \mathbf{I_t}, q_t)$ for policy j is sub-modular in order decision q_t and inventory level $x_{i,t}$ of age i, where $j = 1, 2, \cdots, n-1$.*

Proof: The proof is provided in Appendix B. ■

Proposition 5: *At any time t, the expression related to policy function $\pi_t(j, \mathbf{I_t}, q_t) + E[V_t]$ for policy j is sub-modular in order decision q_t and inventory level $x_{i,t}$ of age i under the following condition of monotonicity,*

$$V_{t+1}(\mathbf{I}^0_{t+1}) \leq V_{t+1}(\mathbf{I}^1_{t+1}) \leq \cdots \leq V_{t+1}(\mathbf{I}^a_{t+1}),$$

where $\mathbf{I}^a_t = (x_{1t}, x_{2t}, \cdots, x_{kt} = a, \cdots, x_{nt})$ and $j = 1, 2, \cdots, n-1$

Proof: The proof is provided in Appendix B. ■

3.5 Solution Algorithm based on k-concavity

In this section, we utilise the theoretical properties obtained in section 3.4 to develop a solution algorithm for the joint ordering-markdown model. Similar to backward dynamic programming, the solution algorithm requires the model to be solved backward in time to produce a policy table consisting of the optimal actions for the state space. However, unlike the backward dynamic programming, the complete action space is not searched in the solution algorithm. The search space is trimmed by applying the theoretical properties of the model. The properties of k-concavity and sub-modularity are used to reduce the size of the search space for both, order and markdown decisions. The solution methodology is provided by Algorithm 3. We describe the solution algorithm as follows,

- At each time period, the search space for the markdown policies is modified based on the state of the dynamic system. The state of the dynamic system at any time is the initial inventory level evaluated from the unsold inventory of the previous selling period. If the unsold inventories are close to zero, the

Algorithm 3: Concavity algorithm (CA)

1: **Initialization:** Compute the state space $\mathcal{S}_t$ and action space for markdown policies $\mathcal{P}_t$

2: Set $V_{T+1}(\mathbf{I}_T) = 0$ and $f_t^i(\mathbf{I}_t, q_t, \mathbf{w}_t^j) = \pi_t^i(j, \mathbf{I}_t, q_t) + E[V_{t+1}(\mathbf{I}_{t+1})]$ using Model (2)

3: Compute initial values $f_t^i(\mathbf{I}_t, 0, \mathbf{w}_t^j)$ and $f_t^i(\mathbf{I}_t, 1, \mathbf{w}_t^j)$

4: **for** $t = T, T-1 \cdots, 1, i \in \mathcal{S}_t, j \in \mathcal{P}_t$ **do**

5:     Assign values for inventory $\mathbf{I}_t^i \in \mathcal{S}_t$, markdown binary decisions $\mathbf{w}_t^j$ and price $\mathbf{p}_{mt}^j$

6:     **if** $\sum_{i=1}^{n} x_{it} > 0$ **then**

7:         **for** $k = 1, 2, \cdots, n$ **do**

8:             Set $\bar{\mathbf{I}}_t = (x_{1t}, x_{2t}, \cdots, x_{kt} - 1, \cdots, x_{nt})$

9:             Compute optimal policy J_k^* from the policy table $V_t(\bar{\mathbf{I}}_t)$ for the state space $\bar{\mathbf{I}}_t$

10:         Find the next policy to apply as $\bar{j} = max\{J_k^*, k = 1, 2, \cdots, n\}$ and set $j \to \bar{j}$

11:     **for** $q = 1 : \kappa$ **do**

12:         Compute $f_t^i(\mathbf{I}_t, q, \mathbf{w}_t^j)$, **if** $f_t^i(\mathbf{I}_t, q, \mathbf{w}_t^j) \geq f_t^i(\mathbf{I}_t, q-1, \mathbf{w}_t^j)$,

13:         **then** set $q = q + 1$, go to step 12,

14:         **else** store optimal order quantity for policy j as $Q_j^* = q$, set $j = j + 1$, go to step 5

15:     **if** $t < T$ **then**

16:         **for** $k = 1, 2, \cdots, n$ **do**

17:             Set $\mathbf{I}_t^a = (x_{1t}, x_{2t}, \cdots, x_{kt} = a, \cdots, x_{nt})$ for inventory value of age k

18:             Monotonic condition defined as $V_{t+1}(\mathbf{I}_{t+1}^0) \leq V_{t+1}(\mathbf{I}_{t+1}^1) \leq \cdots \leq V_{t+1}(\mathbf{I}_{t+1}^a)$

19:             **If** monotonic condition holds, **then** set $\bar{\mathbf{I}}_t = (x_{1t}, x_{2t}, \cdots, x_{kt} - 1, \cdots, x_{nt})$

20:             Compute optimal order Q_k^* from policy table $V_t(\bar{\mathbf{I}}_t)$ for state space $\bar{\mathbf{I}}_t$

21:         Find order upper bound as $\bar{q} = max\{Q_k^*, k = 1, 2, \cdots, n\}$ set $q \to \bar{q}$ and go to step 12

22:     Update policy table for each state space i as $V_t(\mathbf{I}_t|\mathbf{p}_{m,t-1}) = \text{Max } f_t^i(\mathbf{I}_t, q_t, \mathbf{w}_t^j), \forall\, j \in \mathcal{P}_t$

firm is unlikely to conduct a markdown sale. On the other hand, when the initial inventories are in excess the firm might markdown their selling price to sell them off. We discuss a small example to explain the use of the above-mentioned property. Consider a product with 2 different ages of inventories, 1-period old inventory unsold from the previous time (which is the state of the system), and the newly ordered product with an age of zero periods. At the state of 20 units of 1-period old inventory at any time t ($x_{1t} = 20$), suppose the optimal decision is to order 10 units of new inventory ($q_t = 10$) and to follow a partial markdown policy where the 1-period old inventory and the new order are respectively being sold at markdown and regular price ($w_{0t} = 0, w_{1t} = 1$). Let's consider the subsequent state of the dynamic system consisting of 21 units of 1-period old inventory at time t ($x_{1t} = 21$). At the state of $x_{1t} = 21$, when the firm orders 10 units similar to the previous state of $x_{1t} = 20$, it will either prefer to adopt a partial ($w_{0t} = 0, w_{1t} = 1$) or a complete markdown policy ($w_{0t} = 1, w_{1t} = 1$). In other words, the firm will not prefer to follow a no-markdown policy at the state of $x_{1t} = 21$ if it didn't prefer to conduct any markdown at the state $x_{1t} = 21$. Thus, the search space for markdown policies at each state space depends on the decisions of its immediate previous state. At any time for each state space, the search space of markdown policies is updated in steps 6 to 10 of Algorithm 1.

- The k-concavity properties of the model are presented in propositions 1 and 2. The model is k-concave in its inventory levels since it independently exhibits properties of a concave function with respect to a single markdown policy. Thus, the concavity property is individually applied to find the optimal order decision for each markdown policy. This is presented in steps 11 to 14 in the algorithm.

- The sub-modularity properties of the model are presented in propositions 4 and 5. A sub-modular function adheres to the law of diminishing returns. According to the law of diminishing returns, when any variable factor is increased while others remain constant, the output per unit of the variable factor will eventually diminish. If the value function is sub-modular in inventory and order decision, an increase in inventory level when the order remains constant would eventually yield a lower value in profit. This is explained with the example mentioned above. Consider a product with 2 different ages of inventories, 1-period old inventory unsold from the previous time (which is the state of the system), and the newly ordered product with an age of zero periods. At the state of 20 units of 1-period old inventory at any time t ($x_{1t} = 20$), suppose the optimal decision is to order 10 units of new inventory ($q_t = 10$) and to follow a partial markdown policy where the 1-period old inventory and the new order are respectively being sold at markdown and regular price ($w_{0t} = 0, w_{1t=1}$). When the system has 21 units of 1-period old inventory at time t ($x_{1t} = 21$) and if the firm follows a partial markdown policy similar to the state of $x_{1t} = 20$, it will not order more than 10 units if the model is sub-modular in the order decision variable. In other words, sub-modularity helps in achieving an upper bound on the order decision. The upper bound is used along with the property of concavity to further tighten the search space for the order decision. This mechanism of obtaining the upper bound for order decision is provided in steps 15 to 21 of the solution algorithm.

3.6 Computational Results

We design a series of computational experiments to meet two research objectives, i) illustrate the performance of the solution algorithm developed in section 3.5 to solve the dynamic ordering-markdown model described in section 3.3, and ii) analyse the benefit of developing a joint markdown-ordering strategy for the management of perishable products. The numerical experiments evaluating the advantage of joint markdown-ordering policy are specifically designed to find answers to the following managerial questions,

- While conducting a markdown, is it profitable to have flexibility in selecting the time or price of markdown sale?
- How frequently should the firm conduct a markdown sale? What conditions motivate a higher vs lower reduction in the markdown sale?
- What is the impact of varying customer segments on joint markdown-replenishment policies?

Our experimental design considers different parameter sets related to the prices and customer segments. We adopt the base case from Li et al. (2012) and Li et al. (2016) for our computational study. The ordering, holding and penalty cost in the base case are assumed to be 1, 0.2 and 0.5, respectively. The ordering capacity for the perishable product is assumed as $\kappa = 20$. The regular price for selling the perishable product is set as 4. In addition, there can be markdown reductions of 10%, 30% or 50% on the regular selling price of the perishable products (Gaurdian 2015, Insights 2017, Wasteless 2018). Table 3.4 summarises the list of parameter set for the base case.

Table 3.4: Selection of parameters for the base case

Parameters	Values
Ordering cost	$c = 1$
Holding cost	$h = 0.2$
Penalty cost	$\gamma = 0.5$
Regular price	$p_r = 4$
Markdown reductions	10%, 30%, 50%

The experimental design considers the sale of a 3-period lifetime product over a finite time horizon of 18 periods. The customer arrivals and requests are simulated over the planning horizon. The arrivals of customers follow a Poisson distribution with mean arrival of 20 customers. The request of every customer arrival depends on the customer's sensitivity towards freshness. Thus, we assume each customer is assigned a value of $\theta \in [0, 1]$, denoting the customer's sensitivity towards freshness (or quality). A customer with a higher θ value, closer to 1, is highly sensitive towards freshness and hence, she is classified as quality sensitive. On the other hand, a customer with a lower θ value is price sensitive. In order to have an equal number of customers in each segment, we uniformly distribute the value of θ in the initial experiments. We also perform experiments where we vary the proportion of customer segments. The expected profits are estimated by simulating the arrivals of customer requests over 1000 sample paths.

Comparison of Solution Methodologies

In the first experiment, the solution methodology developed in section 3.5 is compared with the standard backward dynamic programming (BDP). The solution algorithm applies the rules derived from exploiting the property of k-concavity of the dynamic model presented in section 3.4. We refer the proposed solution algorithm as concavity algorithm, abbreviated as CA. The expected overall profit and policy tables obtained for the dynamic programming model by using CA is same as the BDP (as discussed in Proposition 3). The computational times of CA and BDP for different ordering capacities are presented in Table 3.5. We compare different ordering capacities to illustrate the performance of the solution algorithm. There is a simultaneous increase in CPU times for both methods with increase in ordering capacity. This happens because the state space and action space enhances with higher ordering capacities. However, despite the increase in CPU times for both methods, CA's computational time is significantly lesser than BDP. Thus, it becomes computationally impossible to analyze every point in the feasible action space due to the exponential increase in state-space. On the other hand, while applying CA, there is a significant reduction in the calculations of the feasible action space for every state. The CA utilizes the properties of k-concave functions to reduce the overall calculations resulting in a considerable decline in the computational times. Thus, we apply CA for solving the dynamic programming model in the rest of the numerical experiments.

Table 3.5: Comparison of BDP with rules algorithm

Capacity	Expected profit	CPU time (in seconds)	
		CA	BDP
5	160.74	8.73	10.88
10	339.65	36.74	142.71
15	517.68	149.76	1192.27
20	695.69	514.36	6047.93
25	874.17	1381.58	32419.22
30	1052.94	3618.50	175323.98
35	1231.92	8880.80	341629.94

3.6.1 Comparison of Different Markdown Policies

In this experiment, we evaluate the benefit in flexibly conducting a markdown sale. The flexibility in performing a markdown sale is examined by specifically analysing the decisions related to time and the reduced price of the markdown sale. Therefore, we compare the proposed dynamic policy with different kinds of markdown policies followed by multiple practitioners. The different markdown policies analysed in this section are described as follows,

1. **No markdown strategy (NM):** In this strategy, inventories with different levels of freshness are priced the same. In other words, the firm doesn't conduct any markdown sale and inventories of different freshness levels are priced the same. There are several reasons for applying this policy in practice, i) it is easy-to-implement since no changes in the price tags have to be made, ii) some firms avoid markdowns as they believe conducting markdowns may harm their reputation (Cognizant 2015, Gaurdian 2015). Thus, no markdown policy, abbreviated as "NM", is assumed as the base case in our analysis.

2. **Fixed age policies:** In this policy, we conduct a markdown sale by fixing the time (or age) of the product to markdown. In other words, we assume the inventory with an age of 2 periods and expiring in the next period to be marked down. Various practitioners are reported to follow the rule of thumb of reducing

the price of the products expiring the next day (Mckinsey 2014, Insights 2017, Wasteless 2018). For instance, Wasteless is reported to conduct a 30% reduction on the price of products expiring in a day (Wasteless 2018). However, as reported by Evans (2019), there also might be multiple price reductions on food products expiring soon. Thus, while following fixed age policies, we assume the firm can either fix the reduction percentage or dynamically decide the reduction percentage. We accordingly consider two kinds of fixed age policies,

(a) If the firm fixes the markdown price to a possible reduction percentage, it is referred as fixed age - fixed price policy, abbreviated as $FF_{10\%}, FF_{30\%}$ and $FF_{50\%}$ for 10%, 30% and 50% percentages of markdowns, respectively.

(b) When the firm fixes the markdown age and dynamically decides the markdown price, it is referred to as fixed age - dynamic price policy, abbreviated as FD.

3. **Dynamic age policies:** When the firm dynamically decides the age of conducting the markdown sale, it is referenced as a dynamic age policy. Retailers in practice are known to walk a thin line of discounting too early or too late. Various practitioners emphasize the importance of simultaneously finding the right time and right price of markdown (Mckinsey 2014, Cognizant 2015, Insights 2017, Wasteless 2018). Thus, while dynamically finding the age of markdown, the firm can either fix the reduction price and follow the dynamic age-fixed price policy, abbreviated as $DF_{10\%}, DF_{30\%}$ and $DF_{50\%}$ for 10%, 30% and 50% percentages of markdown, respectively, or dynamically and jointly find the age of inventory level and reduction price for the markdown sale, which is referred as the dynamic age - dynamic price policy, abbreviated as DD.

Since our base case is the no markdown policy, the benefit of applying any markdown policy is analyzed by computing the percentage difference between the expected profit of NM and any other policy. In addition, we also analyze the firm's decision regarding replenishment. To understand the variation of expected profits and order quantities with changing ordering cost and price, we perform more specific numerical experiments. In particular, we consider 4 test instances by varying the ordering cost and selling price. Table 3.6 and 3.7 summarizes the value of percentage difference between expected profits and order quantity for fixed age policies and dynamic age policies, respectively.

All the fixed age policies, including $FF_{10\%}, FF_{30\%}, FF_{50\%}$, and FD, yield lesser profit than NM policy. In comparison to the NM policy, we observe the number of customers willing to buy a product increases when the firm follows any fixed age

Table 3.6: Performance of fixed age policies

Cost	Regular price	$FF_{10\%}$		$FF_{30\%}$		$FF_{50\%}$		FD	
		Profit	Order	Profit	Order	Profit	Order	Profit	Order
1	4	-5.84	-4.77	-14.31	-6.17	-32.37	-16.24	**-3.44**	**-2.58**
1	5	-9.05	-7.24	-21.31	-9.66	-42.95	-18.17	**-7.59**	**-2.67**
2	4	-8.50	-10.54	-17.60	-10.95	-37.08	-25.38	**-6.56**	**-8.75**
2	5	-10.91	-10.21	-24.66	-15.79	-48.83	-28.53	**-8.51**	**-7.58**

Table 3.7: Performance of dynamic age policies

Cost	Regular price	$DF_{10\%}$		$DF_{30\%}$		$DF_{50\%}$		DD	
		Profit	Order	Profit	Order	Profit	Order	Profit	Order
1	4	25.97	52.74	15.82	18.23	14.03	13.13	**33.43**	**57.19**
1	5	0.90	47.89	8.11	48.96	6.55	6.61	**14.79**	**59.98**
2	4	-19.26	**48.49**	6.97	27.16	14.02	13.24	**21.59**	40.20
2	5	-37.66	**49.92**	-2.82	23.54	5.65	7.32	**8.40**	37.80

policy. However, despite the increase in customers willing to buy, Table 5 displays a decrease in the order quantity in the fixed age policies than the NM policy. This happens because, within a fixed age policy, the oldest product is always put on a reduced price in every demand and arrival scenario. In the simulation experiments, we observe a reduction in inventory levels of older products for the FF policies during high arrival scenarios. In other words, during high arrival scenarios in the FF policies, the firm's ordering strategy is dynamically adjusted to reduce the number of products to go to markdowns. The firm aims at reducing the number of markdown during a higher arrival scenario because a fixed age policy sells products at a lower price when it could have been sold at a higher price. Thus, the firm orders less in a fixed age policy with the aim of selling a majority of products at the regular price to achieve maximum profits.

Since the no markdown policy performs better than the fixed age policies, the profit within the fixed age policies further decreases with an increase in the percentage of markdown reduction. Thus, among the three fixed age-fixed price policies, $FF_{10\%}$ is the closest to the NM policy as $FF_{10\%}$ is the smallest percentage of markdown reduction among the three possible reductions. FD is slightly better than $FF_{10\%}$ because the price is dynamically selected at each time. However, the improvement is incremental since the flexibility of changing the markdown price is only restricted for the inventory expiring in the next period. Next, we investigate the flexibility in selecting the age of a markdown sale in the dynamic age policies, namely $DF_{10\%}, DF_{30\%}$ and $DF_{50\%}$.

In Table 3.7, we observe $DF_{10\%}$ to perform worse than NM for the third and fourth test instance in Table 3.6. On the other hand, in the first two test instances $DF_{10\%}$ performs better than NM. In addition, $DF_{30\%}$ and $DF_{50\%}$ outperforms NM in all test instances. In other words, at least one dynamic age and fixed price (DF) policy outperforms the NM policy. The dynamic age policies perform better because of the flexibility in selecting the age and time of markdown. The total number of orders increases when a dynamic age policy is applied. This implies the firm satisfies a higher number of customers by flexibly finding the age and time of markdowns. However, we also observe that no single dynamic age and fixed age policy performs the best. In other words, $DF_{10\%}$ and $DF_{30\%}$ perform the best when $c = 1, p_r = 4$ and $c = 2, p_r = 5$, respectively. When $c = 1, p_r = 5, c = 2, p_r = 4$, the policy $DF_{50\%}$ yield the highest profit. When the cost increases the percentage difference between NM and other policies decreases. In other words, it becomes more profitable to follow a NM policy as the cost of ordering increases. Thus, we next investigate both, markdown price and markdown age as decisions in the DD policy. We find the DD outperforms all policies in each test instance. Thus, the flexibility in simultaneously deciding the price, age and time of markdown yields the maximum profits. In summary, we highlight the major findings of the experiment of this section,

- Under the FD policy, the age of conducting a markdown is fixed, however, there is flexibility in deciding the markdown price. On the other hand, the markdown price is fixed and there is flexibility in deciding the age and time of markdown under the DF policy. We find that the DF policy performs better than the FD policy under all test instances. This happens because *the flexibility in deciding the age and time of markdown holds more importance than the flexibility in changing the markdown price.*

- Even though there is flexibility in finding the age and time of markdown under both kinds of policies, DD and DF, the flexibility in finding the markdown price is only present in the DD policy. The DD performs better than all DF policies specifically because of the flexibility in finding the markdown price. However, it is interesting to observe even though there is flexibility in selecting the markdown price in the policy FD, it performs poorer than NM policy as well. Thus, *the benefit of flexibly selecting the markdown price is enhanced when there is flexibility in finding the time and age of markdown.*

3.6.2 Analysis of the *DD* Policy

According to Mckinsey (2014), Cognizant (2015) and Insights (2017), the markdown strategy of a supermarket is known to influence its reputation among customers. In this section, we investigate the time and frequency of conducting markdown sales. In addition, we inspect the motivations behind 1) an early vs last-minute markdown sale, and 2) a higher vs lower reduction in the markdown price. In other words, we identify suitable conditions for supermarket managers to conduct an appropriate markdown sale. This is achieved by examining several features of the DD policy. Within the DD policy, the firm may decide to conduct a no markdown (NM), markdown all inventories (abbreviated as CM), partially markdown inventories with age higher than 1 period or 2 periods (referred as PM1 or PM2, respectively). The inventories, order quantity and demand levels over the 18 time periods are provided in Figure 3.2. In addition, Figure 3.3 displays the frequency of the firm's decision to conduct any of the 4 markdown strategies (NM, CM, PM1, and PM2) given the inventory levels in Figure 3.2 over 18 time periods under 100 simulation runs. For instance, at time period 5, Figure 3.2 displays the inventory level of new orders, one-period old inventory and two-periods old inventory. Similarly, the frequency of the 4 markdown strategies (NM, CM, PM1, and PM2) are highlighted in Figure 3.3.

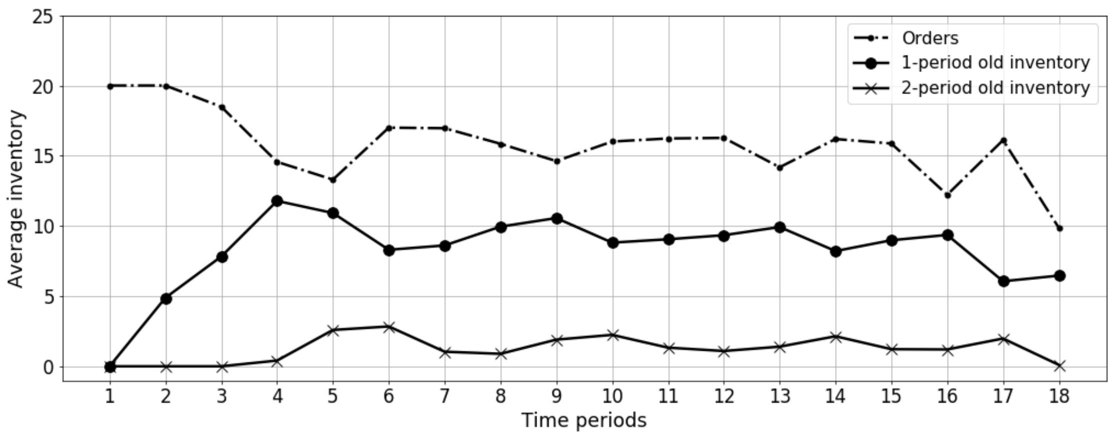

Figure 3.2: Average orders and inventory levels over time in the *DD* policy

In Figure 3.2, we observe the increase in order quantity simultaneously occurs with a decrease in the level of inventory of age 1 period. There is a decreasing relationship between order quantity and 1-period old inventory because when the level of 1-period old inventory increases over time, the retailer balances by ordering less. Next, we analyse the relationship between the order quantity and the 2-period old inventory. We observe a simultaneous increase in levels of order quantity and

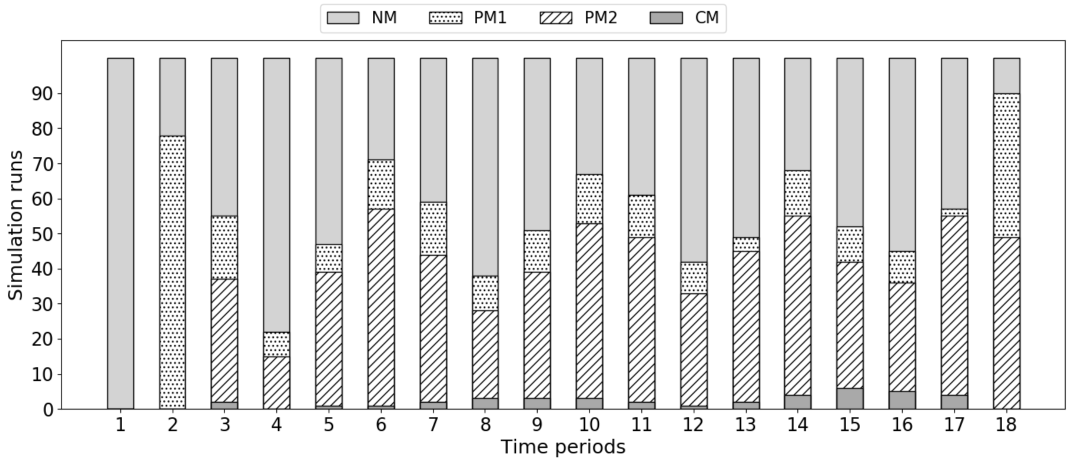

Figure 3.3: Frequency of markdown strategies over time in the *DD* policy

inventory of age 2 period from time 5 to 6. On the other hand, even though the 2-period old inventory increases from time 7 to 8, the order quantity decreases. Thus, even though there is a decreasing relationship between order quantity and 1-period old inventory, no monotonic relationship exists between 2-period old inventory and order quantity. This happens because both order quantity and 1-period old inventory can be resold in the future time periods whereas the 2-period old inventory cannot be resold as it is expiring in the same time period.

By comparing Figures 3.2 and 3.3, the overall markdowns, including PM1, PM2, and CM, simultaneously increase or decrease in the same direction as the level of inventory with the age of 2 periods. In other words, if the 2-period old inventory level increases, the markdowns also increase since the firm must sell off the 2-periods old inventory before expiry. Similarly, when the 2-period old inventory decreases, there is a reduction in the number of markdowns as well. However, a monotonic relationship doesn't exist between the inventory of 1 period age and the overall markdowns. For instance, the inventory of 1 period age and total markdowns (including PM1, PM2, and CM) simultaneously increase from time 8 to 9. On the other hand, even though the inventory level of 1 period age decrease from time 9 to 10, there is an increase in the overall markdowns. This happens because, as established above, a decrease in inventory of 1 period age is also accompanied by an increase in order quantity. Due to the increase in order quantity, the firm must differentiate between prices of relatively fresher order quantity and the older inventories. Thus, even though there is a directly proportional relationship between the level of inventory of 2 periods age and the overall markdowns, no direct relationship holds between 1-period old inventory levels and the total number of markdowns. There is a decreasing relationship between

1-period old inventory and frequency of partial markdown PM1. Moreover, there is an increasing relationship between 2-period old inventory and frequency of partial markdown PM2.

Figure 3.4 displays the value of the average regular and markdown demand over the planning horizon. The markdown demand is observed to change more than the regular demand. This happens because the regular price remains the same along the planning horizon. On the other hand, the markdown price is a decision that depends on the age and time of markdown. Figure 3.5 displays the frequency of different percentages of markdown at the first and second reduction, respectively. The second reduction refers to an additional markdown on the products already on markdown. In Figure 3.3, at time 1, the regular demand is at its highest and the markdown demand is zero. This happens because there is no markdown sale at the first time period, as observed from Figure 3.4. However, from time 1 to time 3, there is a fall in the level of regular demand and an increase in the markdown demand. The regular demand decreases because of increase in number of markdowns causing a shift in the customers from regularly priced products to the markdown inventories. During the first markdown sale, the number of times the firm conducts a 10% markdown is higher than a 30% markdown. Moreover, the level of 2-period old inventory in Figure 1 is directly proportional to the frequency of a 30% reduction in Figure 4. The direct relationship exists because the 2-period old inventory cannot be carried forward to the next period and the firm tries to sell it off by conducting a markdown at a higher reduction. On the other hand, there is no direct relationship between the price of the second markdown sale and the inventory levels.

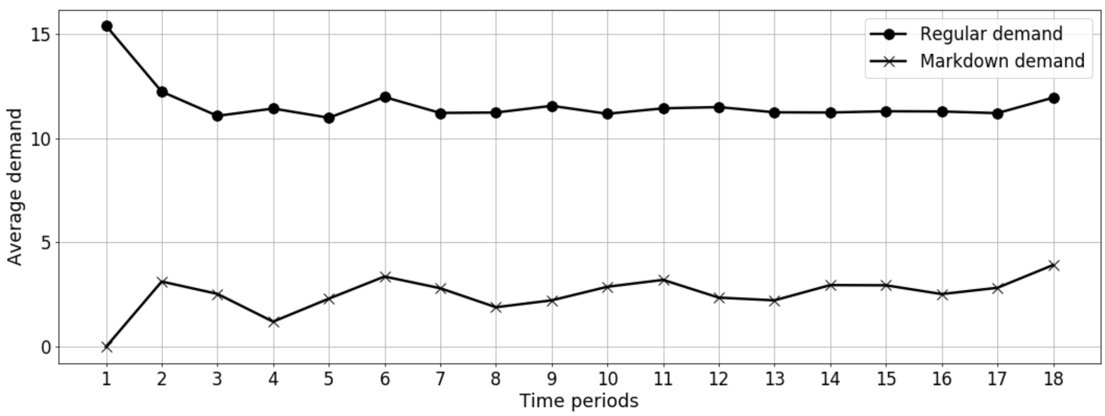

Figure 3.4: Average markdown and regular demand over time in the *DD* policy

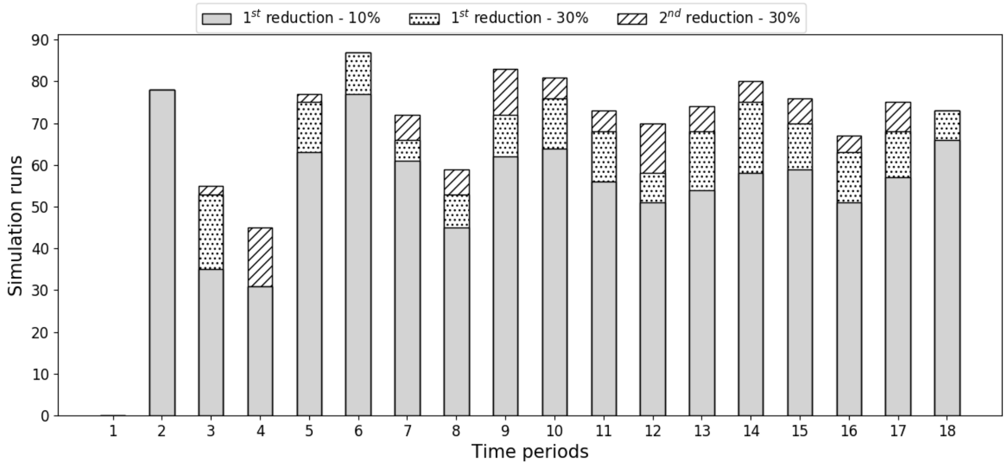

Figure 3.5: Frequency of different markdown reductions over time in the DD policy

3.6.3 Impact of Customer Preferences on Markdown Policies

In this experiment, we investigate the impact of customer preferences on markdown strategies. Customers in supermarkets are reported to display different attitudes and preferences towards markdown sales of perishable products (Mckinsey 2014, Wasteless 2018). A customer survey conducted by Wasteless at a supermarket store in Germany reports that 70% of customers wish to purchase products with short lifetimes at a cheaper price and the rest of them are willing to pay the full price for a product with a longer lifetime (Wasteless 2018). However, the percentage of customer buying patterns are known to change based on the geographical locations of various stores (Insights 2017). Due to differences in customer preferences, analysts in practice place customers in different baskets (Cognizant 2015, Gaurdian 2015). Thus, in this experiment, we vary the percentage of customers in the quality and price-sensitive segments and test their impact on various markdown policies. In Figure 3.6, the horizontal axis displays market classification in terms of various percentage customer segments as starting from 10-90 up to 90-10. For instance, the case '70-30' represents 70% of customers in the market as being quality sensitive whereas the remaining 30% of customers are price sensitive.

Figure 3.6-a displays the percentage difference between the expected profits NM and different control policies, namely DD, $DF_{10\%}$, $DF_{30\%}$, $DF_{50\%}$ and FD. Moreover, Figures 3.6-b, 5-c, 5-d display the value of total, markdown, and regular sales, respectively, for the different control policies DD, $DF_{10\%}$, $DF_{30\%}$, $DF_{50\%}$ and FD over varying customer segments. Since all the fixed age-fixed price FF policies perform worse than the NM, we don't consider them in this analysis.

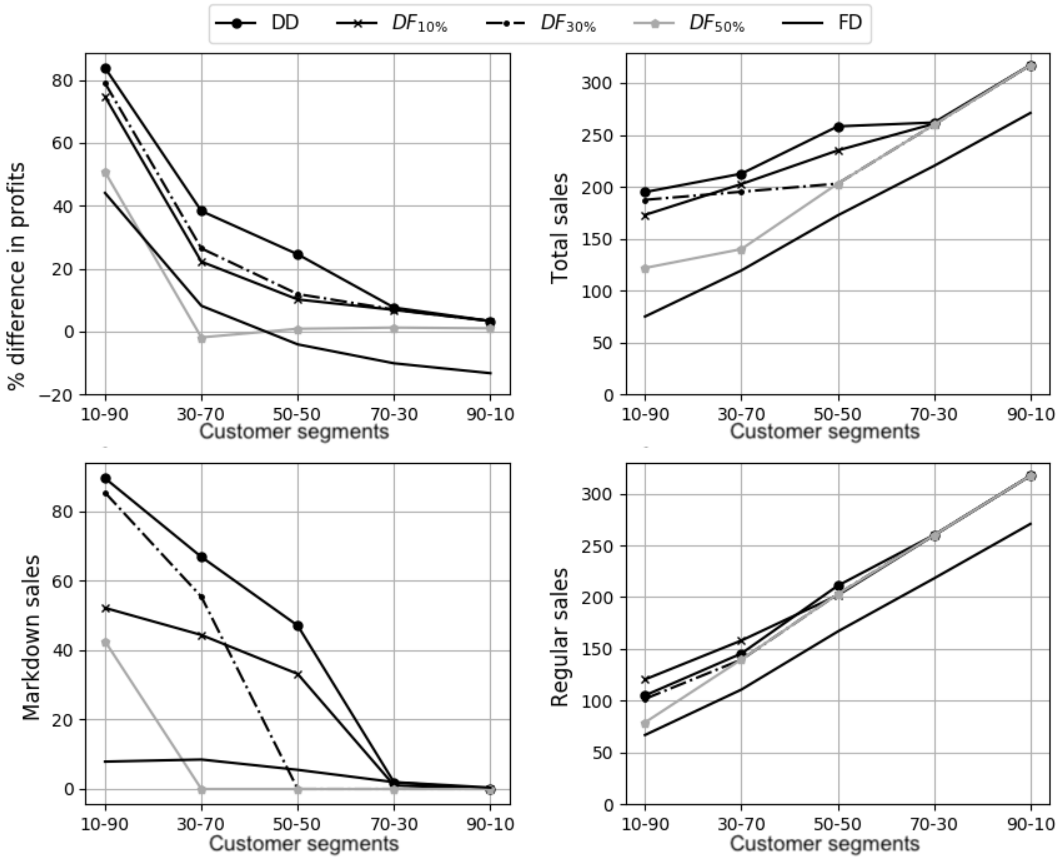

Figure 3.6: Percentage difference in expected profits of NM with various control policies

When there is an equal number of price and quality sensitive customers at 50-50, the percentage difference in the expected profits of all the dynamic age policies, $DF_{10\%}, DF_{30\%}, DF_{50\%}$ and DD policy, is positive as seen Figure 3.6-a. However, the percentage difference between the expected profits of the FD policy and NM policy is negative. This happens because a no markdown policy is better than the fixed age markdown policy as described above. However, with an increase in price-sensitive customers, the curve representing the FD policy in Figure 3.6-a gets positive. Thus, in the presence of more price-sensitive customers, the FD policy performs better than the no markdown policy. This happens because price-sensitive customers prefer the lower prices being offered during regular markdowns under the FD policy. However, within price-sensitive customer segments, the DD outperforms all the policies. In other words, there is a significant gap between the percentage difference of expected profits between the DD policy and FD policy when there are more price-sensitive customers. However, he gap between profits of DD policy and FD policy decreases when there is a higher number of quality sensitive customers. The percentage difference for the FD policy gets more negative with an increase in the number of quality sensitive customers. Thus, the FD policy performs poorly when there are more quality-sensitive customers since quality-sensitive customers prefer fresh products and are willing to pay a higher price. If the firm follows FD policy in presence of a greater number of quality-sensitive customers, it will not only cause a loss in profits but a decrease in sales as well, as seen in Figure 3.6-b. This is also discussed by (Cognizant 2015, Gaurdian 2015) where they highlight how the decrease in sales due to excessive markdowns causes a reduction in the overall profits of the firm.

With an increase in quality sensitive customers, even though the percentage difference of FD policy gets negative, the percentage difference of all the dynamic age policies remain positive. The curves representing the dynamic age policies converge together near zero with an increase in quality sensitive customers. This implies a reduction in the percentage difference between the dynamic age policies and the no markdown policy since the number of markdowns would reduce in the presence of higher quality sensitive customers. Thus, in the presence of higher quality sensitive customers, the dynamic age policies imitate the NM policy.

Within the dynamic age policies, the DD policy performs the best because the markdown price in the DD policy is dynamically selected in comparison to all the DF policies. This can be further validated since the highest sales are yielded by the DD policy. In the presence of price-sensitive customers, the gap between the percentage difference in expected profits of DD and $DF_{10\%}$, $DF_{30\%}$, reduce, unlike the FD policy. The expected profit of policy $DF_{30\%}$ lies closest to DD when there are 90% price-sensitive customers in the Figure 3.6-b. This happens because the markdown sales of $DF_{30\%}$ at 90-10 customer segments lie closest to the DD in Figure 3.6-c. Thus, in the presence of price-sensitive customers, the firm focuses more on the age of markdown rather than markdown price.

Even though DD performs well across customer segments, it is an expensive policy to implement in practice because of joint dynamic decision for both, time and price of markdown. This experiment helps us obtain targeted and less expensive policies for various customer segments. In the presence of a higher number of price-sensitive customers, the firm must focus on finding the age and time of markdown rather than the price. On the other hand, the firm must not invest in conducting markdowns when there are more quality sensitive customers. Mckinsey (2014), Cognizant (2015), Gaurdian (2015), Insights (2017) and Wasteless (2018) also suggest designing policies based on customer buying behavior at granular levels of stores. They specifically highlight the need to develop smart and efficient policies for stores based on the customer demographics of each store.

3.7 Conclusions

In this chapter, we develop a dynamic ordering and markdown model for a firm selling a perishable product under demand uncertainty. There are limited studies on simultaneously considering dependency of joint decision-making on time and age of the perishable products. Unlike the existing studies, we consider existence of multiple ages, dependency of decisions on time and age of the perishable product. We also analyse the dynamic nature of customer choices to capture the demand cannabilizatin between fresh and old inventories. The customer choice model is then integrated with the joint decision-making model formulated as stochastic dynamic model. Due to the tracking of multiple age of perishable products, the state space of the stochastic dynamic model is high-dimensional. We propose an exact solution algorithm by analysing structure properties like k-concacity and submodularity of the underlying model. Our solution algorithm yields optimal joint ordering and markdown decisions in a reasonable computation time .The benefit of efficient computation of our exact solution algorithm is depicted computationally as well.

We also design numerical experiments to compare our dynamic polices with various fixed markdown policies adopted from practice. Our findings depict the joint decision-making policies to perform superior than the fixed policies as they are more flexible. The flexibility within the joint decision-making policy is also investigated. We find the flexibility in deciding the age and time of markdown holds more importance than the flexibility in changing the markdown price. We also report various direct and indirect relationships between inventories of different ages and ordering strategies to showcase the dynamic nature of the management of perishable products. In practice, supermarkets operate in view of different kinds of price and/or quality sensitive customer segments. Thus, we also analyse the impact of joint ordering and markdown policies in varying customer segments. In highly price sensitive customer segments, fixed markdowns policies perform as good as dynamic polices while no markdown policy performs better in quality sensitive segments. This experiment suggested at obtaining targeted and tailor-made policies for various customers segments which could be less expensive and beneficial to practitioners.

Even though the k-concavity algorithm significantly reduces computational time for a fairly realistic problem size, extensive numerical studies can be conducted for realistic set-ups as part of an industrial case study. As future research, one can investigate how different inventory issuance orders impact markdown strategies and customers in general.

Chapter 4

Ordering and Delivery Strategies for Dual-Channel Network

In a dual-channel network a firm distributes its products through different channels mainly comprising of its own stores as well as third-party retailers. Since the network of dual-channel firm is extensively and widely spread across different channels, a disruption at any point of the network impacts the overall operations of the firm. Thus, during the management of the dual-channel firm, it is essential to consider disruption at both demand and supply side of the network. To mitigate the disruption from demand or supply side, the firm diversifies its sourcing strategy by ordering from a set of emergency suppliers. In this research, we investigate the joint ordering and delivery strategies of the dual-channel firm under disruption.

This chapter discusses the dual-channel network by highlighting its challenges. We then focus on the review of relevant literature on management of dual-channel and dual-sourcing networks. Then the underlying supply chain network is described and the joint decision-making problem is stated. Next, we present formulation of the problem and its proposed solution methodologies. The computational results testing the performance of the solution methodology and joint ordering-delivery policies are then explained. Finally, a brief summary of the chapter is provided.

4.1 Dual-Channel Supply Network

A traditional wholesaler-retailer supply chain consists of the flow of products from wholesalers to retailers and finally to customers. With globalization and rapid adoption of dual channels, businesses have shifted from the traditional wholesaler-retailer setting to a non-traditional one. In non-traditional supply chains (so-called 'wholesalers clubs' by FMI (2016)), retailers such as Costco, Sam's Club, and BJ's have physical stores where they serve customers directly through their unique wholesale store system and they also have clients from different sectors such as restaurant chains, vendors, caterers and small grocery stores (Deloitte 2014). As another example, Apple and Samsung successfully play the dual role of wholesaler and retailer because they sell their products in their own stores and also distribute them to third-party retailers such as BestBuy, Amazon and Walmart. The literature on dual-channel supply chain highlights the economical reasons for serving different customer segments with different channels and points out that multiple channels help retailers to increase their market coverage (Anderson et al. 1997, Takahashi et al. 2011). Firms operating in a dual-retailing (or dual-channel) set-up are the leading players in their respective industries. Apple and Samsung have been securing the highest market share of the mobile phone industry since 2012 (Lucic 2020). The dominance of dual-retailing firms (such as Walmart, Sam's Club, and Costco) is also visible in the supermarket industry. The annual estimated value of the businesses of the wholesaler-retailer setting is over $460 billion. The dual-retailing firms achieve high-cost savings because of a regular flow of bulk demand from third-party clients unlike the traditional single-retailing set-up (ColumbiaReports 2020). Walmart and Sam's Club has a market share as high as 90% in some parts of the United States (FoodIndustry.com 2020).

Even though dual-retailing firms have established prominence in their respective industries, they encounter many challenges. The operations along the supply chain of the dual-retailing firm can be severely impacted due to disruptions such as disasters, pandemics, labour strikes, machine breakdowns, and accidents (Tan et al. 2016, Gong et al. 2014). For instance, due to the recent pandemic of coronavirus, Costco's inventory management operations have been heavily disrupted. Since the onset of the pandemic in February 2020, Costco has witnessed a significant rise of 3% in its monthly sales. However, the surge in sales has led to a challenge for the company's supply chain. Along-with the irregularities at the demand side, Costco has seen major disruptions at their suppliers as well (Rogers 2020, Reuters 2020). In another case, Apple encountered disruptions from both, demand and supply-side during the release of the iPhone 6 in 2014. According to Kubota et al. (2019), Apple's demand forecast for iPhone 6 were immensely inaccurate as it sold much better than their expectations. As a result, Apple had to ramp up its production to meet the unanticipated increase in demand for iPhone 6. However, many of Apple's regular suppliers reportedly struggled to meet the inflated production numbers. Thus, Apple faced uncertainty from both demand and supply side of the dual-retailing supply chain.

One of the key challenges in dual-retailing supply chain is how to efficiently respond to a variety of uncertain factors such as supply and demand uncertainties. Firms tackle supplier uncertainty by diversifying their sourcing strategies. Apple is known to source part of its components from multiple suppliers in China and Taiwan. The strategy of using multiple suppliers not only helps in mitigating supply disruptions and delays, but also in promptly responding to the changes in demand patterns (Gartner 2015). Empirical studies point out that a large number of firms employ a back-up or emergency sourcing strategy to manage both demand and supply uncertainties (Gupta et al. 2014). It is critical for a firm to plan its multiple sourcing (regular and emergency suppliers) considering demand and supply uncertainties. Moreover, firms do not only rely on supply-side strategies when they face uncertainties. For instance, Costco diversified its sourcing strategies by utilising emergency suppliers to deal with the drastic increase in demand during the coronavirus pandemic (Reuters 2020, Hart 2020). They also delayed fulfilment of some demand to mitigate demand and supply uncertainties since all suppliers were under serious shortage (Hurt 2020). A dual-retailing firm meets demand of two different classes of customers; their own stores and third party retailers. Demand for one of the customers can be delayed to be fulfilled at a later date when there is a supply shortage. Therefore, the dual-retailing firm can tackle demand and supply uncertainties by employing two strategies; either by ordering from emergency suppliers, or by delaying demand. Sourcing from regular and emergency suppliers helps the firm to meet the demand on time. However, the ordering decision should be made considering demand and supply uncertainties, otherwise multiple sourcing may lead to overstocking. On the other hand, the decision for delaying the demand is dependent on multiple factors such as relationship with the third party retailer versus the company's own stores, uncertain demand and supply.

In this research, we study the inventory management problem of a dual-retailing firm experiencing uncertainties from both demand and supply sides. The firm can source from both regular and emergency suppliers to meet the demand of its own stores and third party retailers. In case of a supply chain disruption, regular supplier is unreliable and may not deliver the original ordering amount. Emergency supplier, on the other hand, is reliable but more expensive. We consider a single product setting over a finite planning horizon. At each time period, the firm decides how much to order from regular and emergency suppliers and how to distribute inventories to its own stores and third party retailers. Our contribution in this research is two-fold,

- We formulate the decision making problem of the dual-retailing firm under demand and supply uncertainties as a stochastic dynamic programming problem.

Existing studies mainly consider either demand, supply uncertainties or both independently in the dual-channel supply chain. On the other hand, we investigate joint ordering and delivery policies to simultaneously tackle both demand and supply uncertainties. The consideration of dual-retail channels and multiple decisions massively expands the state and action spaces. Thus, the underlying dynamic programming model is computationally difficult to solve by the standard solution methodology. We work on various efficient decomposition methods to find joint ordering and delivery strategies for the dual-retailing firm.

- We propose two decomposition methods by utilising the structural properties of the dual-retail channel network. In the first approach, we decompose the network into independent single channel (as each store and the third party retailers) which can be solved by focusing on one channel at a time. To protect the inter-connectivity between multiple channels, we introduce an opportunity cost parameter by considering each channel's effect on the network. The second decomposition method is developed considering the practical applications. Due to the technological advances in inventory-tracking, firms are able to receive information about the inventory levels at their own stores as well as third party retailers. With this information, we reformulate our original dynamic programming model so that the firm can track inventories at all channels as well as its central echelon. This model is then decomposed by each inventory-tracking point and solved via two-stage decision making process to obtain ordering and delivery decisions. We design numerical experiments to compare the performances of our solution methodologies with the standard backward dynamic programming technique. We also develop a heuristic policy based on practitioner and academic reports to test the performance of our models. We also conduct extensive numerical experiments to highlight different features of our models and derive managerial insights. Our numerical results indicate that the proposed decomposition methods coordinate ordering and delivery decisions quite well.

The remaining part of the chapter is organized as follows. Section 4.2 focuses on the literature review by providing details of existing studies relevant to our research. In Section 4.3, we first describe the underlying supply chain network and then state the joint decision making problem. The stochastic dynamic programming formulation of the ordering and inventory allocation problem is presented in Section 4.4. The proposed solution methodology and computational results are explained in Sections 4.5 and 4.6, respectively. Section 4.7 provides a brief summary of the chapter.

4.2 Review of Dual-Retailing and Sourcing Management

Our study is mostly related to the literature on inventory management in dual-retailing and dual-sourcing supply chains. Apart from the research in dual-channelling systems, we also review the literature investigating a firm's strategy of delay in demand fulfilment.

Within the research on inventory management in dual-retailing, two different set-ups are considered. In the first set-up, dual-retailing channel is comprised of a traditional (indirect) retail channel and a direct online channel. In this set-up, the firm sells same products to similar customers through direct and indirect channels and customers can choose the shopping channel that is better suited to their needs. There is an extensive research on the inventory management of the direct-indirect channels. Most of the studies focus on pricing decisions and channel coordination by considering the competition between channels. We refer to Tsay & Agrawal (2004) and Agatz et al. (2008) for a comprehensive review and an insightful discussion on dual-channel coordination. Our study is related to the second set-up where the dual-retailing channel is comprised of a firm acting as a wholesaler and a retailer to serve business clients (third-party retailers) as well as its own customers. Schneider & Klabjan (2013) discuss the significance of the dual-channel supply chain of wholesale-retail firms. Takahashi et al. (2011) investigate the production and delivery management of the dual-channel supply chain. They develop an inventory control policy where the decision of production and delivery is dependent on the inventory levels. Alawneh & Zhang (2018) examine the inventory policy of a dual-channel supply chain by focusing on warehouse layout design. Research by Takahashi et al. (2011) and Alawneh & Zhang (2018) is centred around the management of operations related to inventory storage and delivery at different warehouses or distribution centres. There is a considerable literature on developing inventory management strategies such as base stock policy (Chiang & Monahan 2005, Schneider & Klabjan 2013) and (Q, R) policy (Khouja & Stylianou 2009) for a dual-channel supply chain. Li et al. (2015) study the inventory management problem of a dual sales channel operated by one vendor as a stochastic dynamic programming model considering the demand dependency between two-channels. Even though Li et al. (2015) consider the sale of a product through dual-channel retailing, they don't consider flexibility in order-fulfillment. While unmet demand in the first channel (physical store) is lost, it is backlogged in the second channel (online store). On the other hand, Hu, Li, Byon & Lawrence (2015) investigate the inventory management problem of a dual-channel supply chain where the firm may delay the demand fulfillment in one of the channels by classifying and prioritising customers based on revenue and delay cost. None of the above studies on the inventory management of the dual-channel supply chain consider uncertainty from the supply side. Zhu et al. (2020) study the management of a dual sales channel under demand and supply uncertainties. However, the focus of Zhu et al. (2020) is different than our research as they investigate the competition and coordination among the different sales channels rather than dynamically managing the inventory over time.

A common strategy to tackle supply uncertainty is by diversifying the supplier strategy. Inventory management using multiple suppliers has been widely studied in the literature, mostly with periodic review models and deterministic supply where each supplier has different lead time and ordering cost (Janakiraman et al. 2015, Wang et al. 2017, Xin & Goldberg 2018, Sun & Van Mieghem 2019). Inventory management of a dual-sourcing firm under demand and supply uncertainties is discussed by Ju et al. (2015), Tan et al. (2016), Zhou & Yang (2016), Song et al. (2017) and Jakšič & Fransoo (2018). These studies primarily aim at finding efficient ordering strategies under consideration of different lead times from the dual suppliers. They mainly analyse the effect of lead time differences on ordering decisions. Dual-retailing firms (such as Apple, Costco) source from multiple suppliers to deliver products at the shortest lead times which helps them to react in timely manner to the changes in consumer demand (Gartner 2015). Apple is well-known to maintain most precise lead times in the industry and Costco reports receiving orders from its suppliers every day to deal with uncertainties from demand and supply side (Jones 2017, Reuters 2020, Hurt 2020). Dual-retailing firms invest in a vast supplier network to minimize issues related to lead times. Considering the practical applications, several studies assume suppliers' lead time to be zero or one and study ordering strategies under supply uncertainty (Ahiska et al. 2013, Zhu 2015, Feng et al. 2019). Ahiska et al. (2013) study a dual-source model with one reliable and one unreliable supplier. The unreliable supplier can be in up and down states which is modelled as a two-state Markov process. The retailer orders from the reliable supplier when the other one is down. Zhu (2015) investigates optimal ordering strategies under several supply disruption scenarios depending on which supplier is facing the disruption and whether the retailer has any prior information regarding the disruption. Feng et al. (2019) study the dynamic multi-sourcing problem assuming a zero lead time as well. They find an efficient ordering strategy by considering dependence between the uncertainties at multiple suppliers. However, they assume the consumer demand to be deterministic. To the best of our knowledge, dual-retailing set-up has not been studied in the multi-sourcing literature.

Apart from the literature on dual-sourcing and dual-retailing supply chains, we also review the studies related to delay in order-fulfilment. Xu et al. (2009) consider the possibility of delay in between customer order arrival and the product deployment to meet the order. They investigate how retailers can utilise more of their resources and gather information by delaying the fulfilment-decisions of customer orders. Lee et al. (2003) present a model to find integrated inventory replenishment and dispatch scheduling policy under deterministic time varying demand. They specifically investigate how often and in what quantities to replenish the stock at an upstream supply chain member (e.g., a warehouse), and how often to release an outbound shipment to a downstream supply-chain member (e.g., a distribution center). Further research on management of stock-out with delaying order fulfilment decisions are provided by Allon & Bassamboo (2011), Song & Zhao (2016) and Cui & Shin (2017). However, none of the studies related to order-fulfilment discussed so far consider different demand channels. A few studies focus on flexible order-fulfilment among different demand classes. Wang & Yan (2009) consider a firm dealing with order-fulfilment of two different type of customers, namely patient and impatient customers. Orders from patient customers can be delayed to a future time period, while orders from impatient customers have to be satisfied from the on-hand inventory. Wang et al. (2014), Huang et al. (2011) and Xie et al. (2016) consider a similar problem with patient and impatient customers, and study inventory allocation decision between customer classes. These studies generally assume that only the demand of patient customers can be delayed and fulfilled at a later stage. However, a dual-retailing firm may delay the orders for both channels, third party clients and their own stores. Thus, the set-up of a dual-retailing supply chain differs from the structure of the patient-impatient demand classes discussed above. Moreover, the literature on order flexibility doesn't consider supply uncertainty.

The distinguishing features of the most relevant studies are highlighted in Table 4.1. The literature on both dual-retailing and flexibility in order fulfilment assumes deterministic supply. On the other hand, the research on multi-sourcing with supply uncertainty mainly focuses on the effect of lead times on ordering decisions without considering the dual-retailing set-up. To the best of our knowledge, only Zhu et al. (2020) consider both demand and supply uncertainties in a dual-retailing supply chain with a single supplier. They model the problem using a game theoretic approach. Different from the existing literature, in our study, we consider a dual-retailing firm tackling demand and supply uncertainties by ordering from emergency supplier as well as delaying order-fulfilment. We model the dynamic inventory management problem of the dual-retailing firm as a stochastic dynamic programming model.

Table 4.1: Classification of relevant research papers

Research papers	Dual-retailer	Multi-source	Flexibility in order-fulfillment	Uncertainty
Li et al. (2015)	✓			Demand
Hu, Li, Byon & Lawrence (2015)	✓		✓	Demand
Zhu et al. (2020)	✓			Demand and supply
Jakšič & Fransoo (2018)		✓		Demand and supply
Feng et al. (2019)		✓		Demand
Xu et al. (2009)			✓	Demand
Our research	✓	✓	✓	Demand and supply

4.3 Joint Ordering-Delivery Decision-Making Problem

In this section, we first describe the dual-channel supply chain network and then introduce a stochastic dynamic programming formulation of the underlying joint ordering-delivery decision making problem for the retailer supply chain.

4.3.1 Dual-Channel Distribution Network

We consider a supply chain with a dual-channel distribution network consisting of three echelons. In this network, a firm (like Costco and Walmart) at the central echelon procures a specific nonperishable product from the supplier echelon under disruption and delivers to the customer echelon. An illustration of the supply chain setting along with the main parties involved in the network is illustrated in Figure 4.1. We assume that the firm has already been in business with a single supplier or a set of suppliers (also named as "regular suppliers") in their network. The firm sells the product through two different channels; namely, the firm owned shops and the other retailers as third parties. The firm can fulfil the customer demand at their own shops (taking place in the first channel) any time. In other words, the demand for their own shops may be fulfilled at a late and not right away. The demand received from other retailers (as the third party firms at business channel), comprising of bulk contract orders, is managed through the second channel. The contract orders from the third party retailers involve a fixed lead time; therefore, the firm does not need to meet their demand instantly. However, the firm has to pay a penalty for not meeting the bulk contract orders' of the third party retailers on time or delay the order of their own stores due to customer dissatisfaction. Since a regular supplier(s) under disruption can only provide certain proportion of the orders, the firm may wish to satisfy the remaining unmet demand from the other (so-called "emergency") suppliers at higher ordering cost. The firm faces a joint ordering-delivery decision making problem as they need to determine how much of the product to order from the regular and/or

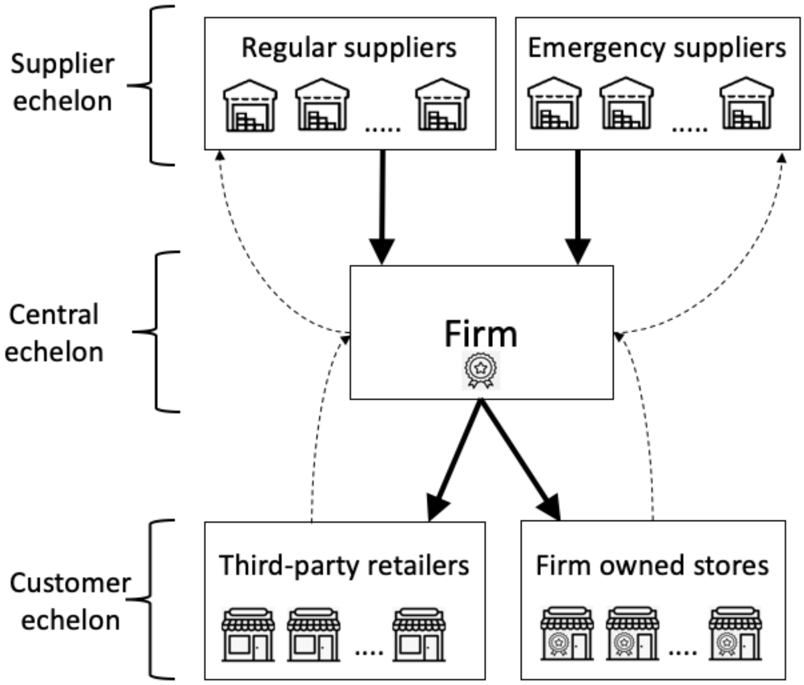

Figure 4.1: A dual-channel distribution network with the main parties

emergency suppliers under disruption and how to deliver the current inventory among dual-distribution channels so that the customers demand is fulfilled with maximum profit.

4.3.2 Notation and Problem Statement

A description of the notation used in this chapter is provided in Table A3. We use tilde ($\tilde{*}$) to denote randomness; e.g., $\tilde{y}$ denotes random variable y. Boldface is used to denote vectors; for example, $\mathbf{a} \in \mathbf{R^n}$ is a n-dimensional vector. The maximum function $(a)^+ = \max\{a, 0\}$ takes value of a if and only if $a > 0$; otherwise, it is zero.

We consider a simple supply chain network consisting of supplier and distribution echelons and the firm is placed at the central echelon. All potential suppliers are classified as regular and emergency suppliers denoted by $s = 1, 2$, respectively. In the dual-channel distribution network, the third party retailers and the firm owned stores are placed at different channels denoted by $i = 1$ and 2, respectively. Figure 4.2 depicts a graphical timeline of the firm's decision-making process.

We consider a planning horizon that is discretized by T time periods represented

A3: Description of notation

	Model Parameters
T	planning horizon discretised by T time periods (denoted by $t = 1, \cdots, T$)
$\mathbf{p}_{it}$	selling price for channel i
$\mathbf{c}_{st}$	cost of ordering from supplier s
γ_{it}	penalty cost for unsatisfied demands from channel i
h	unit inventory holding cost from time $t-1$ to t
β_{it}	opportunity cost for channel i at time t
	Uncertainties
$\tilde{r}_{1t}(q_{1t})$	amount of products received from regular suppliers at time t
$\tilde{\mathbf{d}}_{it}$	demand received from channel i at time t
	State Variables
x_{ft}	inventory level of the firm at the beginning of time t
$\mathbf{x}_t$	total available inventory at the beginning of time t
$\mathbf{w}_{it}$	pending orders of channel i at the beginning of time t
	Actions
$\mathbf{q}_{st}$	quantity ordered from supplier s
$\mathbf{y}_{it}$	inventory allocated to channel i

by $t = 1, \cdots, T$ where the decisions are made. At the beginning of time period t, the firm requests a certain quantity from multiple suppliers on the basis of current inventory and demand requests collected from both channels. The inventory level is reviewed at the end of time period t in view of the amount of product supplied by regular and emergency suppliers to be delivered simultaneously. The total amount of products to be ordered from the regular and emergency suppliers at time t is denoted by a vector of order quantities $\mathbf{q}_{st}$ for $s = 1, 2$, respectively.

There are two sources of uncertainties: i) customers demand and ii) amount of products to be delivered by regular suppliers. Let $\tilde{\mathbf{d}}_{it}$ be a random variable defining total amount of customers demand received via channel i at time t. Since regular suppliers cannot fulfil demand under disruption, the amount of products to be supplied by regular suppliers is not certain. Thus, it is not avoidable to order as many products as required from emergency suppliers. For that reason we assume that no uncertainty exists in emergency suppliers. In our optimisation model, the uncertain parameters are modelled by a random variable following a distribution.

The firm determines the order quantity $\mathbf{q}_{1t}$ from the regular suppliers at time t without knowing the amount to be delivered. It is assumed that the regular suppliers usually deliver much less than the firm's request under disruption. Therefore, we consider the amount of product to be received from regular suppliers as another uncertain parameter in our model. Let $\tilde{r}_{1t}(\mathbf{q}_{1t})$ denote a random variable representing the total amount of product to be delivered from the regular suppliers. Once supply uncertainty due to disruption at regular suppliers is realised, the remaining unmet demand can be satisfied from the emergency suppliers. Let $\mathbf{q}_{2t}$ represent the total

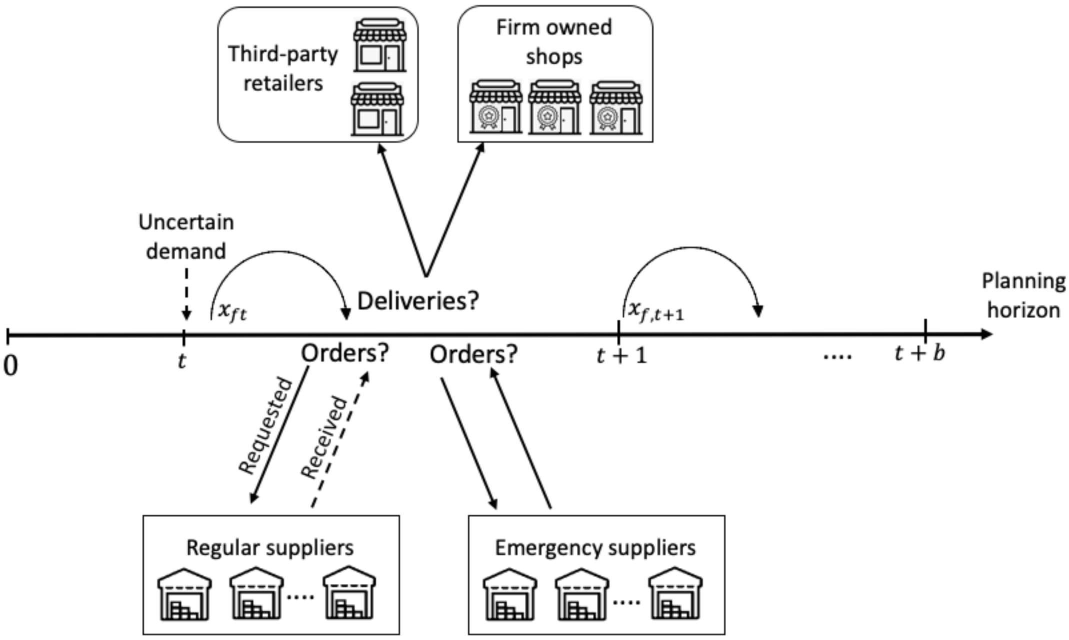

Figure 4.2: Decision-making process in the dual-channel distribution network

amount of product supplied from emergency suppliers. The lead time of delivery from both kinds of supplier is assumed to be zero. We assume that the unit ordering cost c_s from the regular and emergency suppliers (for $s = 1, 2$) remains the same over the planning period. In addition, the cost of ordering a unit of the product from the emergency suppliers is assumed to be always higher than that of the regular suppliers; that is $c_2 \geq c_1$.

We assume that the inventory level x_{ft} of the firm (labelled as f) at the central echelon is reviewed at the beginning of any time period t before the start of the decision-making process. In addition to ordering from suppliers, the firm is also concerned with how to satisfy the customer demand received via both channels. Let $\mathbf{y}_{it}$ denote the inventory allocated to channel i at time t. Let $\mathbf{p}_{it}$ represent unit price of products delivered to any customer via channel i at time t.

At each time period t, there are pending requests of customer demand from any channel and the firm must decide how to deal with them. Let $\mathbf{w}_{it}$ denote the total units of pending demand from channel i. As mentioned before, the business channel consists of the third party clients who demand a bulk amount of the product from the firm. We assume the firm meets the demand of the business channel in a fixed lead time of b periods since the business client's demand is received in bulk amounts. Although demand $\tilde{\mathbf{d}}_{1t}$ received from the business clients at time t is uncertain at $t = 1, \cdots, t-1$, the demand to be delivered at time $t + b$ becomes known since it was requested at time t. If the demand of business channel i is not met in b periods, it is lost and the firm incurs a penalty cost of γ_1 for each unit of unmet demand, $\mathbf{d}_{1,t+b} - \mathbf{y}_{1,t}$. If the firm is unable to meet the demand of their own stores after a certain time periods, they must pay a penalty of γ_2 for each unit of unmet demand. We assume each $\gamma_2 < \gamma_1$ since the demand from business clients are assumed to be of higher priority than the firm's own stores.

4.4 Stochastic Dynamic Programming Models

We formulate the firm's joint ordering-delivery decision-making problem using stochastic dynamic programming. We assume that the firm reviews inventory level x_{ft} and pending orders $\mathbf{w}_{it}$ from channel i regularly at each time period t. Thus, the state space comprises the inventory level and pending orders of the firm. The system dynamics lead to state transition of inventory levels and pending demand from the current time period t to the next one $t+1$.

The inventory balance equation states that the inventory level of the firm $x_{f,t+1}$ at time $t+1$ is determined as the sum of inventory level x_{ft} to be carried over from t, amount of products received from regular suppliers $\tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})$ and emergency suppliers $\mathbf{q}_{2t}$ and minus total amount of deliveries $\sum_{i=1}^{2} \mathbf{y}_{it}$ at time t. This can be formulated as follows;

$$x_{f,t+1} = x_{f,t} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t} - \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it}.$$

Similarly, pending orders $\mathbf{w}_{i,t+1}$ of channel i at time $t+1$ are accumulated by the pending orders $\mathbf{w}_{it}$, demand $\mathbf{d}_{it}$ and inventory allocation $\mathbf{y}_{it}$ at time t. The following balance equation expresses the transition of pending orders of channel i from time t to $t+1$ as

$$\mathbf{w}_{i,t+1} = \mathbf{w}_{it} + \mathbf{d}_{it} - \mathbf{y}_{it}.$$

Given the state of system $(x_{f,t}, \mathbf{w}_t)$ at time t, the firm needs to determine a set of ordering and delivery actions $(\mathbf{q}_t, \mathbf{y}_t)$ simultaneously without violating the following constraints.

Ordering Capacity: Let κ denote the ordering capacity of the firm. The condition $\mathbf{1} \cdot (\mathbf{q}_t) \leq \kappa$ ensures that the total number of products to be ordered at time t does not exceed the available ordering capacity.

Inventory Allocation: The total inventory (after receiving orders from the regular and emergency suppliers) becomes $x_{ft} + \mathbf{1} \cdot \mathbb{E}_s[\tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})] + \mathbf{1} \cdot \mathbf{q}_{2t}$. The amount of inventory to be allocated to both channels $i = 1, 2$ cannot exceed the total available inventory at time t. Thus, this condition can be stated as the following linear constraint

$$\sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it} \leq x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t}$$

Dual-Channel Delivery: The inventory allocation $\mathbf{y_{it}}$ for channel i at time t must be less than its current demand $\mathbf{d_{it}}$ and pending orders $\mathbf{w_{it}}$. This condition can be represented by a set of constraints $\mathbf{y_{1t}} \leq \mathbf{d_{1,t-1}} + \mathbf{w_{1t}}, \mathbf{y_{2t}} \leq \mathbf{d_{2t}} + \mathbf{w_{2t}}$.

The set $\mathcal{F}_t$ at time t consists of feasible ordering and allocation decisions $(\mathbf{q_t}, \mathbf{y}_t)$ satisfying the ordering capacity, inventory allocation, dual-channel delivery constraints as

$$\mathcal{F}_t = \Bigg\{ (\mathbf{q_t}, \mathbf{y}_t) \mid \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it} \leq x_{ft} + \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t}, \mathbf{y_{1t}} \leq \mathbf{d_{1,t-1}} + \mathbf{w_{1t}} \\ \mathbf{y_{2t}} \leq \mathbf{d_{2t}} + \mathbf{w_{2t}},\ \mathbf{1} \cdot \mathbf{q}_t \leq \kappa,\ \mathbf{q}_{it}, \mathbf{y}_{it} \geq 0, \forall i \Bigg\}$$

The firm aims to maximize the expected profit over the planning horizon while managing a dual-channel distribution network through the joint ordering-delivery decision framework. The expected profit is computed as the expected revenue minus the expected total cost of ordering, penalty payments and holding. The revenue at time t is $\mathbf{p}_t \cdot \mathbf{y}_t$. The ordering cost from the regular and emergency suppliers is computed as $c_1(\mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}))$ and $c_2(\mathbf{1} \cdot \mathbf{q}_{2t})$, respectively. Note that the expectation $\mathbb{E}_s[\cdot]$ is taken over supply uncertainty (i.e., $\tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})$ representing amount of product received from the regular suppliers).

The penalty payment $\sum_{i=1}^{2} \gamma_i(\mathbf{1} \cdot \mathbf{w}_{i,t+1})$ is due to the delay in delivery or unmet demand while the holding cost is $hx_{f,t+1}$. The total cost at time t, denoted as $\pi_t(\mathbf{q}_t, \mathbf{y}_t)$, becomes

$$\pi_t(\mathbf{q}_t, \mathbf{y}_t) = c_1(\mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + c_2(\mathbf{1} \cdot \mathbf{q}_{2t}) + \sum_{i=1}^{2} \gamma_i(\mathbf{1} \cdot \mathbf{w}_{i,t+1}) + hx_{f,t+1}$$

Let $V_t(x_{ft}, \mathbf{w}_t \mid \mathbf{d}_t)$ denote the value function at state of inventory level x_{ft} and pending orders of dual-channel $\mathbf{w}_t$ given the customer demand realisation $\mathbf{d}_t$ at time t. The value function at time t for the joint ordering-delivery decision-making problem SDP_1 can be formulated as follows;

$$\begin{aligned} &SDP_1 : \\ &V_t(x_{ft}, \mathbf{w}_t \mid \mathbf{d}_t) = \max_{\mathbf{q_t}, \mathbf{y}_t \in \mathcal{F}_t} \quad \mathbb{E}_s\left[\mathbf{p}_t \cdot \mathbf{y}_t - \pi_t(\mathbf{q}_t, \mathbf{y}_t) + \mathbb{E}_d\left[V_{t+1}(x_{f,t+1}, \mathbf{w}_{t+1})\right]\right] \\ &\qquad\qquad \text{s.t.} \quad x_{f,t+1} = x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t} - \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it}, \\ &\qquad\qquad\qquad \mathbf{w}_{i,t+1} = \mathbf{w}_{i,t} + \mathbf{d}_{i,t} - \mathbf{y}_{i,t}, \quad i = 1, 2. \end{aligned} \tag{4.1}$$

The boundary condition at the end of planning horizon is formulated as $V_{T+1}(x_{f,T+1}, \mathbf{w}_{T+1}) = 0$. As mentioned before, the demand in channels is realised (denoted as $\mathbf{d}_t$) at the beginning of time period t. Therefore, the expectation $\mathbb{E}_d[\cdot]$ at time t is taken over future demand uncertainty to compute expected value function at time $t+1$. On the other hand, the expectation operator $\mathbb{E}_s\left[\cdot\right]$ will be taken over supply uncertainty (i.e., amount received from regular suppliers based on ordering decision $\mathbf{q_t}$ at time t) and it is not realised yet. Let η_{jt} denote the probability distribution function for receiving j units of order at time t. Model (4.1) can be rewritten by the expectation as follows;

$$\begin{aligned} V_t(x_{ft}, \mathbf{w}_t \mid \mathbf{d}_t) = \max_{\mathbf{q_t}, \mathbf{y}_t \in \mathcal{F}_t} \quad &\sum_{j=0}^{q_{1t}} \eta_{jt} \Bigg[\mathbf{p}_t \cdot \mathbf{y}_t - \pi_t(j, \mathbf{q}_{1t}, \mathbf{y}_t) + \mathbb{E}_d[V_{t+1}(x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) \\ &+ \mathbf{1} \cdot \mathbf{q}_{2t} - \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it}, \mathbf{w}_{t+1})]\Bigg] \\ \text{s.t.} \quad &x_{f,t+1} = x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t} - \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it}, \\ &\mathbf{w}_{i,t+1} = \mathbf{w}_{i,t} + \mathbf{d}_{i,t} - \mathbf{y}_{i,t}, \quad i = 1, 2. \end{aligned}$$

In practice, supermarkets (like Tesco, Sainsbury, Aldi) continuously receive information about inventory levels of their own stores (RGIS 2013, ASP 2019). In some cases, third-party retailers also share stock information with their suppliers to improve the efficiency of their supply-chain network (Oracle 2019). The stock information helps them to improve an effective decision-making process related to allocation of inventories via different channels. Using the underlying practical structure of the supply-chain network, we can modify the dynamic programming model SDP_1 and introduce an alternative formulation of the joint ordering-delivery decision-making problem.

Suppose that we observe inventory level $\mathbf{x}_{it}$ for channel i along with tracking inventory of the firm x_{ft} at time t. Let $\mathbf{x_t} = (x_{ft}, \mathbf{x}_{1t}, \mathbf{x}_{2t})$ denote a vector of inventory levels at multiple channels. Recall that, after reviewing the inventory at dual channels and the central echelon at the beginning of time period t, the firm places orders from regular and also emergency suppliers if needed. According to the final inventory level determined by the realisation of supply uncertainty at the end of time period t, the firm allocates $\mathbf{y}_{it}$ units inventory to channel i. Given a state $(\mathbf{x}_t, \mathbf{w}_t)$ at time t, we can compute the cost function $\pi_t^{'}(\mathbf{q}_t, \mathbf{y}_t)$ as follows;

$$\pi_t'(\mathbf{q}_t, \mathbf{y}_t) = c_1(\mathbf{1} \cdot \mathbb{E}_s[\tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})]) + c_2(\mathbf{1} \cdot \mathbf{q}_{2t}) + \sum_{i=1}^{2} \gamma_i(\mathbf{1} \cdot \mathbf{w}_{i,t+1}) + h(\mathbf{1} \cdot \mathbf{x}_t)$$

The value function at time t is represented by $V_t^{'}(\mathbf{x}_t, \mathbf{w}_t)$ and is expressed as follows;

SDP_2 :

$$\begin{aligned} V_t^{'}(\mathbf{x}_t, \mathbf{w}_t|\mathbf{d}_t) = \max_{\mathbf{q}_t, \mathbf{y}_t \in \mathcal{F}_t} \quad & \mathbb{E}_s\left[\sum_{i=1}^{2} \mathbf{p}_{it} \min\{\mathbf{d}_{it} + \mathbf{w}_{it}, \mathbf{x}_{it} + \mathbf{y}_{it}\} - \pi^{'}(\mathbf{q}_t, \mathbf{y}_t) + \mathbb{E}_d\left[V_{t+1}^{'}(\mathbf{x}_{t+1}, \mathbf{w}_{t+1})\right]\right] \\ \text{s.t.} \quad & x_{f,t+1} = x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t} - \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it}, \\ & \mathbf{x}_{t+1} = (\mathbf{x}_t + \mathbf{y}_t - \mathbf{w}_t - \mathbf{d}_t)^+, \\ & \mathbf{w}_{t+1} = (\mathbf{w}_t + \mathbf{d}_t - \mathbf{x}_t - \mathbf{y}_t)^+ \end{aligned} \tag{4.2}$$

The boundary condition at the end of planning horizon is $V_{T+1}^{'}(\mathbf{x}_{T+1}, \mathbf{w}_{T+1}) = 0$.

The state-space for the dynamic joint ordering and inventory allocation models is multi-dimensional as it comprises of the firm's inventory level as well as the pending demand of each channel. Moreover, the dimension of the state space exponentially increases with the number of retail channels and ordering capacity. Due to the curse of dimensionality, it is computationally intractable to solve both SDP_1 and SDP_2 models by the standard method of backward dynamic programming. Thus, we develop two approximation algorithms using the decomposition method for solving the stochastic dynamic programming models (4.1) and (4.2).

4.5 Decomposition Based Approximation Algorithms

In this section, we present two decomposition-based approximate methods to solve the dynamic joint ordering-delivery problem. The first approach involves decomposing model SDP_1 by retail dual channels while model SDP_2 is decomposed echelon-wise in the second approach.

4.5.1 Channel-based Decomposition Method for Solving SDP_1

For the SDP_1 model, the state and action spaces are multi-dimensional because of presence of dual-channels and nature of the joint decision-making process. The size and dimension of state and action spaces will be significantly reduced if the firm serves only one channel. We decompose the SDP_1 model channel-wise and construct a sub-model corresponding to each channel with dramatically reduced state and action spaces. Then, the decomposed model for each channel aims to determine an ordering and delivery policy under the assumption that there is only one retail channel in the network. Thus, the decomposed model independently finds joint decisions to meet demand of only corresponding one channel while ignoring the other channel. Even though the independent structure of decomposed models resolves the high-dimensionality of SDP_1, an inter-connectivity between channels may be lost. We can integrate the decomposed model for each channel in such a way that the inter-connectivity among channels is ensured. The inter-connectivity between the decomposed model of each channel can be constructed by introducing a parameter for measuring the opportunity cost of satisfying the other channel (Kunnumkal & Topaloglu 2010).

Let $\beta_{i,t}$ for $i = 1, 2$ represent the opportunity cost of channel i at time t. As introduced by Birbil et al. (2014) and Bertsimas & Popescu (2003), the value of opportunity cost can be obtained by the finite difference method where the deterministic linear program of SDP_1 is solved for given inventory level. Then, the same model is solved for the given inventory level increased by one unit. The difference between the profits obtained by these linear programs (given the inventory level and one-unit increased inventory level) provides the value of opportunity cost at time t.

While solving the decomposed model of a channel, the opportunity cost parameter represents the gain/loss the firm may experience if one unit of the remaining inventory is delivered to the dual channels. The overall gain/oss by delivering remaining inventories to dual channels is evaluated by multiplying opportunity cost β_{it} of channel i

by remaining inventory $x_{f,t+1}$ at time t as $\beta_{it}x_{i,t+1}$. By adding the value of overall gain/loss (by allocating inventories to the other channel) to the profit function of the channel i, the value function of channel i becomes

$$\begin{aligned} v_{it}(x_{ft}, \mathbf{w_{it}}|\mathbf{d_{it}}) = \max_{\mathbf{q_t},\mathbf{y_{it}} \in \mathcal{F}_{it}^d} \quad & \mathbb{E}_s[\mathbf{p_{it}} \cdot \mathbf{y_{it}} - \pi_{it}^d(\mathbf{q_t}, \mathbf{y_{it}}) + \beta_{it}x_{f,t+1} + \mathbb{E}_d[v_{i,t+1}(x_{f,t+1}, \mathbf{w_{i,t+1}})]] \\ \text{s.t.} \quad & x_{f,t+1} = x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t} - \mathbf{1} \cdot \mathbf{y}_{it}, \\ & \mathbf{w}_{i,t+1} = \mathbf{w}_{i,t} + \mathbf{d}_{i,t} - \mathbf{y}_{i,t}. \end{aligned} \tag{4.3}$$

where boundary condition at the end of planning horizon is $v_{i,T+1}(x_{f,T+1}, \mathbf{w}_{T+1}) = 0$. The cost function $\pi_{it}^d(\mathbf{q_t}, \mathbf{y_{it}})$ and the feasibility set $\mathcal{F}_{it}^d$ for the decomposed model are as follows;

$$\pi_{it}^d(\mathbf{q_t}, \mathbf{y_{it}}) = c_1(\mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + c_2(\mathbf{1} \cdot \mathbf{q}_{2t}) + \gamma_i(\mathbf{1} \cdot \mathbf{w}_{i,t+1}) + hx_{f,t+1},$$

$$\mathcal{F}_{it}^d = \Bigg\{ ((\mathbf{q_t}, \mathbf{y}_t) \mid \mathbf{1} \cdot \mathbf{y}_{it} \le x_{ft} + \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t},\ \mathbf{y_{1t}} \le \mathbf{d_{1,t-b}} + \mathbf{w_{1t}}, \\ \mathbf{y_{2t}} \le \mathbf{d_{2t}} + \mathbf{w_{2t}}, \mathbf{1} \cdot \mathbf{q}_t \le \kappa,\ \mathbf{q}_{it}, \mathbf{y}_{it} \ge 0, i = 1, 2. \Bigg\}$$

The above decomposed model computes the joint ordering-delivery policy for each channel. The information from every channel's decomposed model is then combined together by simultaneously considering the dual-channel network in the following model to find its joint ordering and delivery policy;

$$\begin{aligned} \max_{\mathbf{q_t},\mathbf{y}_t \in \mathcal{F}_t} \quad & \mathbb{E}_s \left[\mathbf{p}_t \cdot \mathbf{y}_t - \pi_t(\mathbf{q}_t, \mathbf{y}_t) + \sum_{i=1}^{2} \mathbb{E}_d \left[v_{i,t+1}(x_{f,t+1}, \mathbf{w}_{i,t+1}) \right] \right] \\ \text{s.t.} \quad & x_{f,t+1} = x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t} - \sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it}, \\ & \mathbf{w}_{i,t+1} = \mathbf{w}_{i,t} + \mathbf{d}_{i,t} - \mathbf{y}_{i,t}, \quad i = 1, 2. \end{aligned} \tag{4.4}$$

Notice that in (4.4), the future expected profit is obtained by solving model (4.3) for each channel. In addition, (4.4) is solved in a forward manner as opposed to the backward recursion of SDP_1.

A pseudo-code of the decomposition method comprising of its main steps is presented in Algorithm 1. In the first step, the value of opportunity cost parameter β_{it} for channel i at time t is computed by the finite difference method. After obtaining the opportunity cost parameter, the decomposed model is solved by the standard method of backward dynamic programming. Then the policy table is approximated for all possible states by moving forward in time.

Algorithm 4: Channel-based decomposition algorithm for SDP_1

1: Obtain opportunity cost β_{it} using the finite difference method
2: Solve model (4.3) using BDP to obtain policy tables $v_{it}(x_{ft}, \mathbf{w_{it}}|\mathbf{d_{it}})$ for all possible demand scenarios ($\mathbf{d_{it}}$) at state $(x_{ft}, \mathbf{w_{it}})$ for channel $i = 1, 2$
3: **for** $t = 1, \cdots, T$ **do**
4:     **for** all states $(x_{ft}, \mathbf{w_{it}})$ and demand scenarios ($\mathbf{d_{it}}$) **do**
5:     Solve model (4.4) using $v_{it}(x_{ft}, \mathbf{w_{it}}|\mathbf{d_{it}})$ obtained in Step 2
6:     Store best decisions $\mathbf{q_t^*}, \mathbf{y_t^*}$

4.5.2 Penalty-based Decomposition Method for Solving SDP_2

In this approach, we carry out the decomposition to not only disintegrate channels, but also the joint decision-making process. We assume that the ordering decisions from regular and emergency suppliers are taken at the central echelon while the delivery decisions are being taken along each channel of the network. The SDP_2 model is decomposed by each echelon mainly comprising of the central echelon and the two channels in the network (**?**Gallego & Özer 2003, Kunnumkal & Topaloglu 2011). The echelon-wise decomposition is possible for SDP_2 because the inventory level or a state of the system is tracked at each echelon in SDP_2 as opposed to SDP_1 where the inventory is only reviewed at the central echelon. For the echelon-wise decomposition of SDP_2, we relax the following linking constraint by associating a Lagrange multiplier to

$$\sum_{i=1}^{2} \mathbf{1} \cdot \mathbf{y}_{it} \leq x_{ft} + \mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t}) + \mathbf{1} \cdot \mathbf{q}_{2t}.$$

Notice that this linking constraint connects all channels with the central echelon by ensuring that the total deliveries are less than the available inventories. Let λ_t define the Lagrange multiplier associated with the linking constraint at time t. We can determine this (i.e., dual variable) by solving the deterministic linear programming formulation of SDP_2 (Kunnumkal & Topaloglu 2011).

Let us first describe the decomposed model for each channel. In this model, we can only find the delivery decisions for the given channel under the assumption of unlimited supply from the central echelon. This assumption is made because of relaxation of the linking constraint which ensures the deliveries to be less than the supply. In other words, deliveries for each channel can be more or less than amount of supply. The decomposed model for each channel i for $i = 1, 2$, can be expressed as

$$
\begin{aligned}
v_{it}(\mathbf{x_{it}}, \mathbf{w_{it}}|\mathbf{d_{it}}) = \max_{\mathbf{y_{it}}} \quad & \mathbb{E}_s[\mathbf{p_{it}} \cdot min\{\mathbf{d_{it}} + \mathbf{w_{it}}, \mathbf{x_{it}} + \mathbf{y_{it}}\} - (h - \beta_{it})(\mathbf{1}\cdot\mathbf{x}_{it}) \\
& - \gamma_i(\mathbf{1} \cdot \mathbf{w}_{i,t+1}) + \mathbb{E}_d[v_{i,t+1}(\mathbf{x_{i,t+1}}, \mathbf{w_{i,t+1}})]] \\
\text{s.t.} \quad & \mathbf{x}_{i,t+1} = (\mathbf{x}_{it} + \mathbf{y}_{it} - \mathbf{w}_{it} - \mathbf{d}_{it})^+, \\
& \mathbf{w}_{i,t+1} = (\mathbf{w}_{it} + \mathbf{d}_{it} - \mathbf{y}_{it} - \mathbf{x}_{it})^+,
\end{aligned}
\tag{4.5}
$$

The boundary condition at the end of planning horizon is $v_{i,T+1}(\mathbf{x_{i,T+1}}, \mathbf{w_{i,T+1}}) = 0$.

Similarly, the decomposed model for the central echelon is states as

$$
\begin{aligned}
v_{ft}(x_{ft}, \mathbf{w}_t|\mathbf{d}_t) = \max_{\mathbf{q}_t \in \mathcal{F}_t} \quad & - c_2(\mathbf{1} \cdot \mathbf{q}_{2t}) - \mathbb{E}_s[c_1(\mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})) + f_t(\mathbf{q}_t) \\
& - \lambda_t(\sum_{i=1}^{2}(\mathbf{1} \cdot \mathbf{y}^*_{it}) - x_{ft} - (\mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})) - (\mathbf{1} \cdot \mathbf{q}_{2t}))]
\end{aligned}
\tag{4.6}
$$

where $f_t(\mathbf{q}_t) = \mathbb{E}_s[\Delta_t(\mathbf{q}_t)]$ and $\Delta_t(.)$ is defined as,

$$
\begin{aligned}
\Delta_t(\mathbf{q}_t) = \max_{\mathbf{y}_t \in \mathcal{F}_t} \quad & \sum_{i=1}^{2} \mathbf{p_{it}} \cdot min\{\mathbf{d_{it}} + \mathbf{w_{it}}, \mathbf{x_{it}} + \mathbf{y_{it}}\} - h(\mathbf{1} \cdot \mathbf{x_{i,t+1}}) \\
& - \gamma_i(\mathbf{1} \cdot \mathbf{w_{i,t+1}}) + \mathbb{E}_d[v_{i,t+1}(\mathbf{x_{i,t+1}}, \mathbf{w_{i,t+1}})] \\
\text{s.t.} \quad & \mathbf{x}_{i,t+1} = (\mathbf{x}_{it} + \mathbf{y}_{it} - \mathbf{w}_{it} - \mathbf{d}_{it})^+, \\
& \mathbf{w}_{i,t+1} = (\mathbf{w}_{it} + \mathbf{d}_{it} - \mathbf{y}_{it} - \mathbf{x}_{it})^+, \\
& \sum_{i=1}^{2}(\mathbf{1} \cdot \mathbf{y}_{it}) \leq x_{ft} - (\mathbf{1} \cdot \tilde{\mathbf{r}}_{1t}(\mathbf{q}_{1t})) - (\mathbf{1} \cdot \mathbf{q}_{2t}) \\
& \mathbf{y_{it}} \geq 0, i = 1, 2.
\end{aligned}
\tag{4.7}
$$

$\Delta_t(.)$ denotes the penalty function as defined by Gallego & Özer (2003) and Kunnumkal & Topaloglu (2011). The penalty function evaluates the benefit of delivering the currently available inventories to the individual channels by satisfying their demand. Notice that the value function (4.6) without a penalty function makes the ordering decisions from suppliers without any consideration of the demand of dual-channels. However, these decisions related to ordering and deliveries are interdependent and cannot be taken independently. Thus, we simultaneously solve models in (4.6) and (4.7) via a two-stage decision-making process to find decisions related to the ordering from two suppliers and delivery among the two channels. At the first stage, the decisions related to ordering from two suppliers are obtained from (4.6). For a given ordering decision from two suppliers, model (4.7) yields the delivery for dual-channels at the second stage. A pseudo-code of the solution method comprising of detailed steps of the two-stage process is provided in Algorithm 2.

Algorithm 5: Penalty based decomposition algorithm for SDP_2

1: Solve model (4.5) using BDP to obtain policy tables $v_{it}(\mathbf{x_{it}}, \mathbf{w_{it}}|\mathbf{d_{it}})$ for all possible demand realisation $\mathbf{d_{it}}$ at state $(\mathbf{x_{it}}, \mathbf{w_{it}})$ for channel $i = 1, 2$
2: **for** $t = 1, \cdots, T$ **do**
3:     **for** all states $(\mathbf{x_t}, \mathbf{w_t})$ and demand scenarios $(\mathbf{d_t})$ **do**
4:     Solve model (4.7) using $v_{it}(\mathbf{x_{it}}, \mathbf{w_{it}}|\mathbf{d_{it}})$ obtained in Step 1
5:     Store best decisions $\mathbf{y_t^*}$ and values for penalty function $\Delta_t(.)$
6: Obtain multipliers (λ_t) using the dual variables from the deterministic LP of SDP_2
7: **for** $t = 1 \cdots, T$ **do**
8:     **for** all states $(x_{ft}, \mathbf{w}_t)$ and demand scenarios $(\mathbf{d}_t)$ **do**
9:     Use $\mathbf{y}_t^*$ and $\Delta_t(.)$ in Step 5 to solve model (4.6)
10:     Store best decisions $\mathbf{q}_t^*, \mathbf{y}_t^*$

In the first step of the algorithm, the decomposed model for each retailer (4.5) is solved by the standard backward dynamic programming. Then the penalty function for all possible values of the unknown ordering variable $\mathbf{q}_t$ is evaluated in Steps 3-5. Once the tableau for penalty function is obtained, in Steps 7-10 the policy table at the central echelon is obtained by the two-stage decision-making model (4.6) in a forward recursion manner.

4.6 Computational Experiments

In this section, we will first describe the design and data structure used in the numerical experiments and then present the computational results of different approaches proposed for solving the joint ordering-delivery problem using the dual-channel supply chain network.

4.6.1 Design of Experiments and Data

We design numerical experiments to illustrate performance of the proposed decision-making models and derive managerial insights. In particular, the numerical study aims to answer the following questions:

- How can we solve the underlying complex decision-making problems efficiently and illustrate performance of the decomposition approaches?

- What is the importance of considering both demand and supply uncertainties for a supply chain with dual-channel distribution network?
- What is the benefit of jointly deciding allocation and delivery strategies of the dual-channel supply chain?

The stochastic dynamic programming problems presented as SDP_1 and SDP_2 are solved using the following approaches to obtain policy tables. The policy tables obtained from different solution approaches are then used for in the same simulation set-up to compare their performances. Next, a brief description of these approaches is as follows.

Backward Dynamic Programming (abbreviated as BDP): This is a standard approach to solve dynamic programming problems; therefore, it is used as a benchmark approach to show effectiveness of the approximate policies. The BDP method can solve both SDP_1 and SDP_2 models for only limited problem instances. It is a computationally intractable method due to curse of dimensionality of the underlying dynamic programming model as discussed earlier. Notice that the same policy tables are obtained for both SDP_1 and SDP_2 models since they have the same optimal policy. Moreover, SDP_2 is an alternative formulation of SDP_1.

Channel-based Decomposition Method (abbreviated as CB): This method first decomposes the underlying dynamic programming model SDP_1 with respect to each channel and then solves each of the smaller size models to determine the inventory allocation to the single channel by ignoring other channels. The connectivity among channels is ensured by parameter β_{it} introduced for each channel i at time t that basically measures the opportunity cost for allocating each unit of inventory to alternative channels. We tested two different ways of setting up this parameter within our decomposition method as suggested in Kunnumkal & Topaloglu (2010). In the first way, we basically set β_{it} at zero whereas in the second way we implement a finite-difference method to determine the value of opportunity cost parameters. The channel-based decomposition method with two different ways of finding the opportunity costs are labelled as $\text{CB}_{(\beta=0)}$ and $\text{CB}_{(fin-dif)}$, respectively.

Penalty-based Decomposition Method (abbreviated as PB): This method first decomposes the underlying dynamic programming model SDP_2 with respect each echelon. Instead of tracking inventory level of only central echelon (as in the channel-based decomposition method), we consider inventory levels of all parties in the customer echelon as well as the central echelon. For our numerical experiments we consider two ways of setting value of β_{it} as explained above. For $\beta_{it} = 0$, we purely capture features of the penalty decomposition method without considering any opportunity cost parameters and this is labelled as $\text{PB}_{(\beta=0)}$. In order to integrate the opportunity cost within the penalty-based decomposition method, we also consider the finite difference method for evaluating value of β_{it}. This is abbreviated as $\text{PB}_{(fin-dif)}$ in our numerical results.

Threshold Inventory Based Policy (abbreviated as TIP): We also introduce a heuristic policy based on "inventory thresholds" (or "base stock levels") of the system. In this policy the firm makes ordering and inventory allocation decisions based on the base stock level. A similar approach has been widely used for inventory management of the single retailing supply chain networks (that differs from our setting); for instance, see Chiang & Monahan (2005), Schneider & Klabjan (2013), Wang et al. (2017) and Keck et al. (2019).

The algorithms were implemented in MATLAB and all computational experiments were run in a desktop computer with Intel Core i5-7500, 3.4GHz, 8GB RAM. The numerical results obtained by these methods are presented in terms of total expected profit over the planning horizon and the CPU time (seconds) taken to solve the underlying the stochastic dynamic programming models.

As the supply network, we consider the sale of a product through two channels (namely one third party retailer and one firm owned store). The firm may procure the product from both regular and emergency suppliers over a planning horizon of $T = 10$ time periods. The regular supplier may not always provide exactly the amount ordered because of supply uncertainty. In particular, we assume that the amount of order recieved from the regular supplier follows a truncated form of the Poisson distribution (David & Johnson 1952). Truncated Poisson distribution is appropriate for our model because it eliminates the unwanted amount of product from the sample space of the distribution.

In order to show impact of various parameter settings on the performance of policies, we consider different cases for the ordering costs from suppliers and also for penalty costs paid to the demand channels for unmet demand. For selecting values of all other model parameters and creating test instances, we adopt the settings from Jakšič & Fransoo (2018) and Feng et al. (2019). We also consider practitioner reports, such as Blatcher (2018) and Alliance (2019) for parameter selection. We generate additional test instances by varying some of the base parameters presented below;

- The inventory holding cost is selected as $h = 0.01$, discount factor is $\delta = 1$.
- The ordering costs from regular and emergency suppliers are set as $c_1 = 0.5$ and $c_2 = 1.5$, respectively.
- The capacity for ordering from different suppliers is set as $\kappa = 5, 10, 15$ and 20.
- The selling prices for both channels are chosen as $p_1 = 3$ and $p_2 = 2$, respectively.
- The penalty costs paid to channels are $\gamma_1 = 1$ and $\gamma_2 = 2$, respectively.
- The demand from each channel is uncertain and follows a poisson distribution with an average rate of 5 units and the maximum possible demand is 10 units.

For simulation experiments, we generate 1000 realisations of uncertain parameters related to demand requests from each channel and the amount of orders received from the regular suppliers. The expected profits are calculated by simulating demand requests and supply received over 1000 simulation paths.

4.6.2 Numerical Results and Analysis

In this section, we present the results of our numerical experiments under three main headings to illustrate i) performance of different decomposition methods and the benchmark policy, ii) impact of the joint allocation and delivery ordering strategies, and iii) effect of simultaneously considering demand and supply uncertainties.

Performance Comparison of Proposed Methods: We are first concerned with the performance comparison of various solution approaches by varying the ordering capacity κ from 5 to 20 units. Table 4.2 presents the results of different methods over varying production capacity in terms of total average expected profit (labelled as "Exp-profit") and the CPU time (in seconds) taken to solve each problem instance. In this table, the best performance of an approach (defined as the highest expected profit achieved and the lowest CPU time taken to solve the underlying problem) is presented in bold and NA highlights "no solution available within days" by the BDP method for the specific case of 20 ordering capacity from suppliers due to the computational complexity.

Table 4.2: Performance of the decomposition and backward dynamic programming approaches

Ordering Capacity	Performance Metrics	*BDP* Method	Decomposition Methods			
			$CB_{(fin-dif)}$	$CB_{(\beta=0)}$	$PB_{(\beta=0)}$	$PB_{(fin-dif)}$
5	Exp-profit	52.21	47.42	39.20	46.60	**47.66**
	Solution time	593.54	92.50	112.34	**32.12**	34.32
10	Exp-profit	106.91	95.10	83.73	**101.90**	101.88
	Solution time	84994.79	2801.68	2934.23	**215.73**	217.47
15	Exp-profit	185.94	180.33	137.78	181.79	**182.18**
	Solution time	(30 days)	46960.56	47112.43	**1123.33**	1131.72
20	Exp-profit	*NA*	230.04	180.50	238.47	**239.42**
	Solution time	*NA*	243212.23	25198.92	**3951.64**	4980.56

As BDP yields the optimal policies, it provides the maximum expected profits. Thus it is considered as the benchmark approach for the performance comparison of the decomposition methods. On the other hand, it is not possible to solve the real problem instances within a reasonable time limit due to computational complexities. From the computational results in Table 4.2, we can make the following observations.

- The channel-based decomposition method $\text{CB}_{(\beta=0)}$ yields the lowest profit among the four decomposition methods. Because no connectivity exists between the decomposition models of individual channels since it assumes that $\beta = 0$. On the other hand, by determining an appropriate value of the opportunity cost parameter (such as the finite difference method), the expected profit obtained by the $\text{CB}_{(fin-dif)}$ method becomes higher than the one obtained by $\text{CB}_{(\beta=0)}$. This shows the necessity of the link between the decomposed problems over different channels.

- The penalty-based decomposition method for solving SDP_2 provides the highest expected profit comparing to the channel-based decomposition method for solving SDP_1. Recall that when we decompose the model SDP_2 by each echelon, we are naturally able to track the inventory of each echelon. Moreover, the structure of penalty decomposition method comprises of a two-stage decision-making process. At the first stage, we obtain the decisions related to ordering from two suppliers. Once the supply uncertainty is realised, the firm allocates the available inventory among the distribution channels depending on different scenarios of deliveries received from the regular suppliers at the second stage. Due to this structure of the penalty decomposition method, $\text{PB}_{(\beta=0)}$ and $\text{PB}_{(fin-dif)}$ produces higher expected profit than $\text{CB}_{(\beta=0)}$ and $\text{CB}_{(fin-dif)}$ can achieve. Among the penalty decomposition methods, $\text{PB}_{(fin-dif)}$ outperforms $\text{PB}_{(\beta=0)}$ in all cases except for capacity 10. This happens because the method $\text{PB}_{(fin-dif)}$ consists of features of both penalty decomposition method as well as the opportunity cost parameter.

- In terms of computational times, BDP expectedly takes the longest solution time to obtain the optimal policy since the action space is evaluated at each point of the state. On the other hand, the penalty-based decomposition methods ($\text{PB}_{(\beta=0)}$ and $\text{PB}_{(fin-dif)}$) require significantly less computational time comparing to the channel-based decomposition method ($\text{CB}_{(\beta=0)}$ and $\text{CB}_{(fin-dif)}$). Due to dispersion of the inventory allocation decisions to multiple echelons in $\text{PB}_{(\beta=0)}$ and $\text{PB}_{(fin-dif)}$, the decomposed model for each channel only comprises of the (delivery) allocation decision. On the other hand, in $\text{CB}_{(\beta=0)}$ and $\text{CB}_{(fin-dif)}$, the decisions related to both ordering and inventory allocation are considered in the channel's decomposed model. Thus, compression of the action space at the channel's decomposed model in $\text{PB}_{(\beta=0)}$ and $\text{PB}_{(fin-dif)}$ leads to a significant reduction in its solution time.

We design numerical experiments to display the performance of the threshold inventory based heuristic that has been widely used in practice. The pseudo code of the threshold inventory based heuristic is presented in Algorithm 3. This algorithm requires pre-setting of the two main parameters.

- *Inventory threshold parameter* (denoted by η) is used for supply ordering decisions. If the current inventory level lies above η, then the firm only orders from regular suppliers. Otherwise, the firm may order from both regular and emergency suppliers.
- *Order fulfilment parameter* (denoted by α) is used to determine the amount of orders to be delivered to each channel. In other words, at each time period, the firm wishes to deliver α percentage of the customer demand to channel 1 (i.e., third party retailers) and the remaining inventory will be supplied to the second channel (i.e., firm owned stores).

For the numerical experiments, we design two cases where the threshold parameter is fixed at $\eta = 0$ and $\eta > 0$. Note that in the latter case, half of the average demand is to be ordered from the regular and/or emergency suppliers. In both cases, we set the order fulfilment parameter as $\alpha = 100\%$ and 75%. The computation time for obtaining the heuristic policy at each problem instance is less than 2 seconds. Table 4.3 shows the numerical results of the heuristic approach obtained by varying values of ordering capacity. We compare the performance of the heuristic policy with the penalty based decomposition approach $PB_{(fin-dif)}$ as it provides the highest expected profit among the other decomposition methods as reported in Table 4.2.

Table 4.3: Performance comparison of the decomposition and heuristic based policies

Ordering Capacity	Decomposition $PB_{(fin-dif)}$	TIP ($\eta = 0$)		TIP ($\eta > 0$)	
		α =100%	α =75%	α =100%	α =75%
5	47.66	42.27	38.36	23.25	16.29
10	101.88	98.75	95.52	42.42	35.46
15	182.18	173.55	163.05	56.47	56.47
20	239.42	231.45	220.68	83.41	71.73

From Table 4.3, one can easily observe that the decomposition method PB-$DM_{(fin-dif)}$ provides higher expected profit than the heuristic policy. Among different settings of the heuristic policy, the expected profits yielding from the zero threshold level are always higher than the non-zero threshold level. In the case of zero threshold level, the firm obtains a higher profit by strictly ordering from the regular supplier. This is consistent with the results obtained by PB-$DM_{(fin-dif)}$ that also only orders from the regular supplier in these test instances.

Overall, we can conclude that the decomposition approach PB-$DM_{(fin-dif)}$ outperforms to all proposed methods in this chapter. Thus, we will use only this method in the remaining numerical experiments.

Impact of Demand-Supply Uncertainties: Most studies in the literature focus on either tackling demand or supply uncertainty in a supply chain. However, several practitioner reports about dual-channel firms (like Apple and Costco) emphasise the importance of incorporating both demand and supply uncertainties into the firm's decision-making process and discuss potential impact of considering demand or supply side of uncertainty in isolation on the firm's overall profitability; for instance, see (Kubota et al. 2019, Gartner 2015, Reuters 2020, Hurt 2020). Moreover, it is not realistic to consider one side of uncertainty and ignore the other one. For instance, when the firm receives exactly whatever orders in the deterministic supply case, the sourcing strategy will never diversify because the firm would never order from emergency suppliers because of its high ordering cost. However, diversification in sourcing strategies is essential for dual-channel supply chains to mitigate demand and supply uncertainties.

In order to investigate impact of supply uncertainty on the ordering-inventory allocation policies, we consider the same supply chain network used with the base case settings, but fix the customer demand from each channel at certain levels. This supply chain with dual-channel distribution network involves only supply uncertainty and the demand from each channel over time is fixed to a value of $3, 4, \cdots, 8$ units. In other words, we generate the policy tables, abbreviated as FF-k, using a fixed demand of k units while considering uncertainty from the supply side. As mentioned in previous section, the decomposition method PB-DM$_{(fin-dif)}$ is preferred to generate the policy tables in these experiments due to its computational efficiency. On the other hand, the dynamic policy under both demand and supply uncertainties is abbreviated as the "DP" policy. We also vary the cost of ordering from emergency suppliers and present results of only two cases with 1.25 (labelled as 'high ordering cost') and 0.75 (labelled as 'low ordering cost'). Figure 4.3 presents the expected profit and average unmet demand of channels obtained by DP and different FF-k for $k = 3, \cdots, 8$ policies using the high (in the left panel) and low (in the right panel) ordering costs.

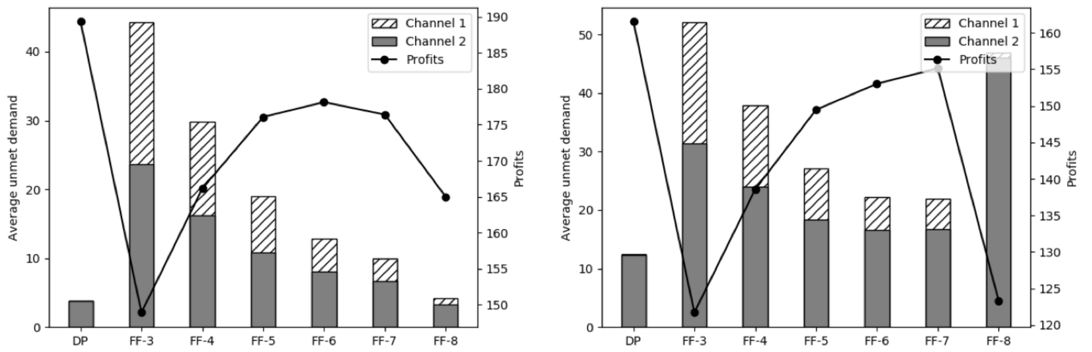

Figure 4.3: Performance comparison of various policies

We observe that expected profit obtained by the DP policy (considering both demand and supply uncertainties) is always higher than those achieved by the FF-k policies with fixed demand levels for $k = 3, \cdots, 8$. This shows that making joint decisions related to ordering and inventory delivery in view of demand and supply variations lead to increase in the expected profit. However, the firm loses out on some profits if the ordering and inventory delivery strategies are decided under the assumption of a fixed level of demand.

Among the FF-k policies, the expected profit increases with the fixed value of demand until a certain level and then starts decreasing. In case of low ordering cost, the profit is maximised at FF-6 whereas the maximum point is achieved at FF-7 for the case of high ordering cost. In the FF-k strategies, profits are maximized at high levels of demand because the firm satisfies more demand and garners a higher profit as well. However, after a certain level of high demand the profits start declining as well.

This happens because the firm ends up ordering more than needed at exceeding high levels of demand. Even though the profits of the best FF-k policies are close to that of DP, there is a significant gap between the level of their total unmet demand. In the case of high ordering cost, the value of unmet demand of FF-7 is more than double of DP. Various dual-channel firms in practice discuss the importance of losing as little demand as possible. In fact, Apple and Costco are reported to give considerable importance to minimise the loss in the demand to maintain their reputation (Gartner 2015, Reuters 2020, Hurt 2020). The dynamic model not only yields the highest profits but also ensures the minimum level of unmet demand.

Performance of Joint Ordering-Delivery Policies: In order to investigate the effectiveness of dynamic joint ordering-delivery strategies for the dual-channel supply chain, we introduce several fixed inventory allocation strategies and compare their performances with the DP policy. In the fixed delivery strategy, decisions regarding the allocation of inventory among channels are made at the beginning of the planning horizon while the ordering actions are dynamically taken.

Firms having a dual-channel supply chain often follow a fixed inventory delivery in practice where some strict preferences could be given to one channel over another. For instance, Apple is reported to cater to the demand of their own stores over third party retailers (Danziger 2017). By considering a practical application, we design numerical experiments where the percentage of inventory to be delivered among the two channels is predetermined. For example, if the firm targets to allocate 70% of inventories to channel 1, then the remaining 30% needs to be allocated to channel 2. In order to determine the best inventory allocation strategy, we also vary the percentage allocation rates for each channel and the corresponding fixed delivery policies are accordingly named as FA-r where $r = 70, 60, \cdots, 40, 30$ (%). These predetermined delivery strategies are integrated with the decomposition method to create the policy tables consisting of ordering strategies from regular and emergency suppliers. In these experiments, we also consider different penalties (between 0 and 1) to be paid to the firm owned stores for unmet demand to show its influence on the fixed delivery strategies. In Figure 4.4, we present the results of two cases obtained by DP and fixed delivery strategies with low (right panel) and high (left panel) penalty costs in terms of expected profit and unmet demand. From these results we can make the following observations.

- The expected profit achieved by the DP approach has always been higher than the one produced by the fixed allocation strategies regardless the choice of penalty costs. This is because at each time the deliveries are dynamically de-

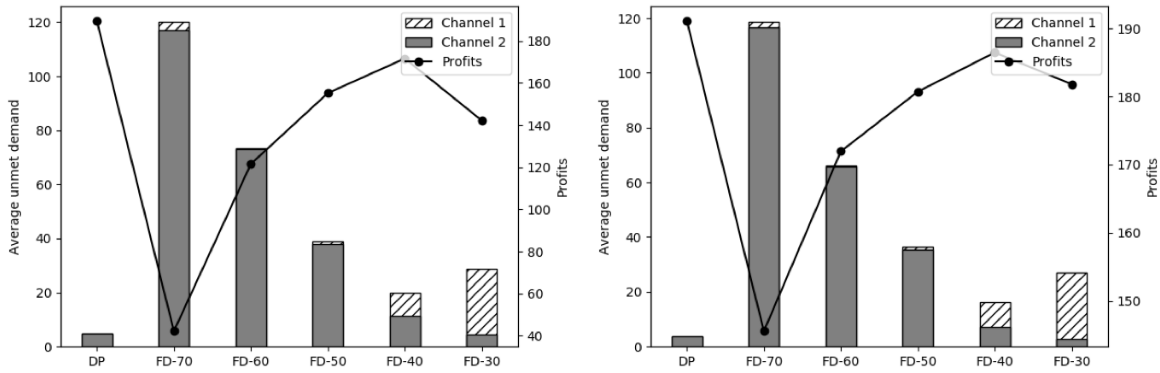

Figure 4.4: Performance comparison of dynamic versus fixed inventory allocation policies

cided in the DP model rather than being predetermined rates as in the case of the fixed delivery strategy.

- FD-40 yields the maximum profits among the fixed delivery strategies because of a fair proportion of distribution among the channels. In the case of high penalty cost, there is a 9.35% gap between the profits from DP and the best fixed allocation strategy of FD-40. We observe that there is a higher unmet demand in the fixed delivery strategy in comparison with DP. Moreover, the value of penalties paid for the unmet demand is also the highest in this case. In the case of low penalty cost, the gap between the DP and the best fixed delivery strategy of FD-40 reduces to 5.15%. To investigate the change in gap with respect to the penalty values, we consider another case where penalty costs for both channels are set to zero. The gap reduces to 2.4% in the case where the penalty values are set to zero. However, despite the reduction in the gap between profits, there is a significant difference between the unmet demands of DP and FD-40 (regardless high and low penalty costs as well the case of zero penalty cost).

- The percentage difference between the unmet demand of DP and fixed delivery strategies of all instances is around 75%. This happens because in dynamic programming approach, the allocation ratios will dynamically change based on the level of state values and demand variation at each time period. On the other hand, in the fixed delivery strategy, the allocation ratios remain constant over the planning horizon.

Overall, we can conclude that dynamically deciding the inventory allocation ratios is more effective in tackling uncertainties in the supply chain for the given data set-up. Even though some retailers follow a fixed inventory allocation strategy in practice, they should consider adopting the dynamic allocation of inventories. The dynamic allocation of inventories over time not only enhances profits, but reduces the levels of unsatisfied demand as well.

Impact of Demand Uncertainty on Delivery Policies: Next, we are concerned with impact of demand uncertainty on performance of dynamic and fixed inventory allocation strategies. We consider two cases (abbreviated as "Cases I and II"). In Case I, the policy tables are created using the decomposition method (introduced in Section 4.4) under the assumption of uncertain demand following a distribution. On the other hand, in Case II the demand is assumed to be deterministic and fixed at certain levels (as introduced in FD-k for $k = 3, \cdots, 8$) while producing the policy tables. As delivery strategies, we consider three different ways of allocating the available inventory among channels and compare their performance with the dynamic inventory allocation strategy (abbreviated as "Dyn-Del"). In the first strategy, we equally allocate the deliveries among the two channels (labelled as "FD-50") proportionally to the expected demand. In other delivery strategies, strict preferences are given to either channel 1 or channel 2 (labelled as "FD-Ch1" and "FD-Ch2", respectively). Table 4.4 presents results of this experiment in terms of expected revenue and cost as well as the unmet demand of individual channels. Notice that the maximum expected profits (calculated as expected revenue minus expected cost) obtained by different delivery strategies are highlighted in bold.

Table 4.4: Performance comparison of the fixed delivery strategies with dynamic programming approach under uncertain and deterministic demand cases

Delivery Strategies	Performance Metrics	Case I: Policy with Random Demand	Case II: Policy under Fixed Demand					
			FD-3	FD-4	FD-5	FD-6	FD-7	FD-8
Dyn-Del	Revenue	242.77	232.22	235.96	238.54	238.65	**239.70**	230.86
	Cost	54.43	85.48	72.24	64.05	61.58	**59.95**	83.99
FD-50	Revenue	237.77	232.34	235.14	236.92	**239.63**	240.97	240.85
	Cost	71.21	84.93	75.96	71.32	**67.02**	72.43	72.72
FD-Ch1	Revenue	145.35	145.71	145.47	145.11	144.21	143.31	**141.57**
	Cost	150.17	151.71	151.89	151.37	151.71	153.47	**157.53**
FD-Ch2	Revenue	98.80	98.16	98.50	97.80	97.10	**96.80**	95.34
	Cost	270.89	273.38	272.48	272.56	272.51	**273.50**	275.65
Unmet Demand of Channels								
Dyn-Del	Ch-1	0.04	20.63	13.78	8.7	5.65	5.07	0.94
	Ch-2	8.38	27.6	21.67	17.06	16.34	16.15	45.35
FD-50	Ch-1	3.11	8.18	5.51	3.51	3.09	3.07	3.07
	Ch-2	27.56	41.57	32.48	27.21	18.38	20.74	21.19
FD-Ch1	Ch-1	3.42	14.88	10.78	7.9	4.81	4.4	2.02
FD-Ch2	Ch-2	3.53	17.31	12.56	8.21	5.37	3.51	2.79

In Case I using uncertain demand in policy tables, the maximum expected profit (as maximum revenue and minimum cost) is achieved by the dynamic delivery policy.

Among the three fixed delivery strategies, FD-50 produces the highest expected profit. This outcome is consistent with the results provided in the previous experiments. On the other hand, the FD-Ch2 policy with a strict preference on channel 2 (as the firm owned shops) in inventory allocation provides the lowest revenue and highest cost. This happens because when the firm strictly prefers channel 2 over channel 1, they pay high penalty for not satisfying the demand for channel 1. Due to high penalties for different channels, the expected cost exceeds the expected revenue while following the strict delivery policies, FD-Ch1 and FD-Ch2. In other words, the firm faces losses if a strict preference is given to a single channel. Despite facing losses in the strict delivery policies, they produce low level of unmet demand comparing to Dyn-Del for dual-channel supply chain. Even though FD-Ch2 yields the lowest level of unmet demand for channel 2, minimum loss of demand in channel 1 occurs in Dyn-Del rather than FD-Ch1. According to FD-Ch1, there is a strict preference given to the demand of channel 1 but still, it is unable to meet the demand like the Dyn-Del policy. This happens because in the Dyn-Del policy, along with the delivery decisions, the firm collectively decides to order quantities as well. On the other hand, in FD-Ch1, they decide the ordering decisions in view of satisfying only one channel. The decision-making of FD-Ch1 is not driven by the gain the firm would receive by dynamically deciding the preference between multiple channels of demand.

We also observe that in Case II (where dynamic and fixed delivery policies are generated in view of constant levels of demands), the expected revenues produced by all delivery policies increase with demand until a certain level and then start decreasing. However, the expected cost doesn't exhibit such a pattern. In other words, the expected cost of Dyn-Del decreases till fixed demand level of 7 units, but then increases at level of 8 units of demand. The same pattern is visible for all the fixed delivery policies as well. We suspect that this happens because of the jump in ordering costs as the firm orders more to meet the increased level of demand (8 units). Consequently, the unmet demand in channel 1 is low at high demand levels. However, the demand of channel 2 is at the highest level at 8 units of demand for both Dyn-Del and FD-50 because the penalty cost of unmet demand for channel 1 is higher than that of channel 2.

4.7 Conclusions

In this chapter, we develop ordering and delivery policies for a dual-channel distribution network. The existing studies in the literature of dual-retailing supply chain consider the supply of product is assumed to be deterministic. We relax this assumption and consider the uncertainties from both demand and supply sides in the dual-channel network. The joint decision-making model under uncertainties is formulated as stochastic dynamic model with high-dimensional state space because of the dual-channel distribution network. We reduce the high dimensionality of the stochastic dynamic model by decomposing the dual-channel network in two ways. The first approach involves decomposing the network by each retail channel where we ensure the inter-connectivity between the channel through the use of opportunity cost. The second approach is designed based on the idea of tracking inventory levels across the distribution network. The mechanism of inventory-tracking is followed by many practitioners. The model is reformulated in a way such that the inventories can be tracked at all the network echelons. The inventory-tracking model is then decomposed by each echelon.

Computational experiments illustrate the performance of decomposition approaches in comparison with BDP. They highlight that both the approaches yield efficient approximate solutions in a reasonable computing time. The solutions approaches are also compared with a inventory threshold heuristic policy adopted from practitioner reports. Numerical experiments are also designed to highlight several features of the joint decision-making model for the dual-channel network. Our results emphasise the importance of collectively considering both demand and supply uncertainties. Moreover, the benefit of joint ordering and delivery policies are also analysed with varying levels of ordering and penalty costs. Our finding suggest that joint decision-making policy not only does well in terms of profit but it also ensure minimum loss in demand.

As future work, one can extend the proposed models to n supplier network. Moreover, n retail channels can be integrated into the model in this chapter. In this case, novel approaches are required to solve these complex decision-making problems.

Chapter 5

Summary of Finding and Future Work

This chapter concludes the thesis by summarising the main findings of the research problems. We also highlight several limitations encountered during the research. Finally, some future research directions are also be provided as extension to this PhD study.

5.1 Summary of Research and Findings

Decision-making problems of retail supply chains are complex due to the distinct elements of its stochastic and dynamic nature. In addition, management of retail and various other supply chains includes several inter-connected key decisions that are related to pricing, production, ordering and inventory of products. Joint decision-making is required to effectively manage these complex and inter-connected decisions under uncertainty. In this thesis, we study three joint decision-making problems of retail supply chains under uncertainty. The underlying problems are formulated via stochastic dynamic programming. We develop efficient solution approaches by analysing the features and properties of each research problem. We obtain joint decision-making policies which enhances firm's overall profits and leads to gain of managerial insights. The respective findings of the research problems are provided below.

In Chapter 2, we study the joint production-pricing problem of a firm selling a multi-generation product line under demand uncertainty. The internal competition between multiple generations is accounted for by analysing customer choices. Joint decision-making policies are then derived by approximately solving a stochastic dynamic programming model with a high-dimensional state and action space. The structural properties of the model are analysed to propose two approximations, FDP and a heuristic comprising of different pricing strategies. The performance of the approximation methods are illustrated through a computational study. In addition, comprehensive numerical experiments are conducted to investigate the significance of joint decision-making problems to manage a multi-generation product line. The joint decision-making strategy is compared with partial planning policies where either production or pricing decision is fixed at the start of the planning horizon and the other one is dynamically decided. Joint production-pricing policy outperforms fixed production policy in terms of expected profit as it is a challenging to tackle demand uncertainty when the production levels are fixed. On the other hand, the profits from the fixed pricing policy lie very close to the joint production-pricing policy but their ordering strategies completely differ. In the fixed pricing policy, the ordering strategies obtained under a fixed set of prices cannot capture the internal competition among multiple generations. We also analyse the effect of customer segments on the management of multi-generation product line by varying the proportion of customers' price and quality sensitivity (towards the underlying technology) in the market. The results indicate that when the proportion of innovation sensitive customers is high, the production and sale of older generations drop due to the high demand towards the new generations. Similarly, it becomes more profitable to sell older generations with new release as the percentage of price sensitive customers increases. Thus, this shows that the variation in customer segments must be considered while determining the number of generations to be kept in the market.

In Chapter 3, we design joint ordering and markdown policies for a firm selling a perishable product under demand uncertainty. We analyse the dynamic nature of customer choices to capture the demand cannibalization between fresh and old inventories. The customer choice model is then integrated with the joint decision-making model which is formulated as stochastic dynamic model. Due to the tracking of multiple age of perishable products, the state space of the stochastic dynamic model is high-dimensional. We propose an exact solution algorithm by analysing the structural properties like k-concavity and submodularity of the underlying model. Our solution algorithm yields optimal joint ordering and markdown decisions in a reasonable computation time. The benefit of efficient computation of our exact solution algorithm is depicted computationally as well. We also design numerical experiments to compare our dynamic polices with various fixed markdown policies adopted from practice. Our findings depict the joint decision-making policies to perform superior than the fixed policies as they are more flexible. The flexibility within the joint decision-making policy is also investigated. We find that the flexibility in deciding the age and time of markdown is more important than the flexibility in changing the markdown price. We also investigate the relationship between markdown age and ordering strategies. There are some direct and indirect patterns between inventories of different ages and ordering strategies. These patterns showcase the importance of considering the dynamic nature of the management of perishable products. In practice, supermarkets operate in view of different kinds of price and/or quality sensitive customer segments. Thus, we also analyse the impact of joint ordering and markdown policies in varying customer segments. In highly price sensitive customer segments, fixed markdowns policies perform as good as dynamic polices while no markdown policy performs better in quality sensitive segments. This experiment highlights the importance of obtaining targeted and tailor-made policies for various customers segments which are less expensive to implement and beneficial to practitioners.

Chapter 4 focuses on developing ordering and delivery policies for a dual-channel distribution network. Uncertainties from both demand and supply side are taken in consideration in the dual-channel network. We also employ diversification of sourcing strategies to deal with demand-supply uncertainties. The joint decision-making model under uncertainties is formulated as a stochastic dynamic programming model. The resulting model has high-dimensional state space because of the dual-channel distribution network. We compress the high dimensionality of the stochastic dynamic model by decomposing the dual-channel network in two ways. The first approach involves decomposing the network by each retail channel where we ensure the inter-connectivity between channels through the use of opportunity cost. The second approach is designed based on the idea of tracking inventory levels across the distribution network. The original model is reformulated in such a way that the inventories can be tracked at all network echelons. The inventory-tracking model is then decomposed by each echelon. Computational experiments illustrate the performance of the decomposition approaches in comparison with BDP. They highlight that both approaches yield efficient approximate solutions in a reasonable computing time. The solution approaches are also compared with an inventory threshold heuristic policy adopted from practitioner reports. Numerical experiments are also designed to highlight several features of the joint decision-making model for the dual-channel network. Our results emphasise the importance of collectively considering both demand and supply uncertainties. Moreover, the benefit of joint ordering and delivery policies are also analysed with varying levels of ordering and penalty costs. Our findings suggest that joint decision-making policy not only does well in terms of profit but also ensures minimum loss in demand.

In this thesis, we design dynamic and joint decision-making models to tackle uncertainty in three different retail set-ups. In the current literature, benefit of joint decision-making in supply chains is examined by various statistical and analytical techniques. Unlike the existing literature, we apply stochastic dynamic programming to model and solve the underlying dynamic and joint decision-making problems. These models suffer from the curse of dimensionality due to exponentially expanding state space. In addition, the inherent characteristics of the different retail set-ups, like perishability and dual-channel network, add further complexities. Thus, standard solution methodologies cannot be directly applied to solve SDP models. This thesis contributes by modelling and solving joint decision-making problems in retail supply chain set-ups. Different solution methodologies are specifically adopted for each retail-set up to effectively solve its stochastic dynamic models. Moreover, we explore the structural and theoretical properties of each of the three joint decision-making problems to build tailor-made solution approaches. The research findings lead to some managerial insights by implementing joint decision-making policies in practice. On the other hand, there are limitations in this thesis. Therefore, further research directions are proposed.

5.2 Limitations and Future Research

In this section, we highlight the limitations of this thesis and discuss some potential directions for future research;

- In this thesis, SDP is used to model joint decision-making problems arising in electronic, perishable and dual-channel networks. Other methodologies, like simulation optimisation, stochastic optimisation and game theory, could have been explored. In addition, the model set-up considered in this research have certain limitations. For instance in Chapter 2, the number of versions of multi-generation product line is assumed to be known. However, one may consider a decision variable to find the number of version of the multi-generation product to offer at a time. The underlying network of Chapter 4 is limited to the case of two kinds of retailers and suppliers. Although the model can be extended to a general network of supply chain comprising of multiple retail and sourcing channels.
- The customer choice models used in this thesis are adopted from the literature specifically for retail supply chains of Chapters 2 and 3. Other types of discrete choice models, like logit or probit, could have been considered to compute the

customer choice probabilities. In addition, we should mention that the choice probabilities in this research are computed using artificially generated data due to lack of real data. The accuracy of the choice models could have been examined using a real case study.

- Another limitation in the thesis is related to parameter selection in the computational experiments. Research papers in the literature and practitioner reports are reviewed to determine the model parameters. However, one can also determine those parameters based on a real case study to evaluate the performance of the joint decision-making models and its solution approaches. Currently, sensitivity analysis is conducted to test the validity of the parameter selection.

- In terms of approximation methodologies, we applied forward dynamic programming and decomposition-based solution approaches to solve the high-dimensional stochastic dynamic models. However, other techniques like linear programming approximations can also be explored. There are various other decompositions like Bender's decomposition or nested methods, that may improve quality of the solutions.

- It will be worthwhile to collaborate with practitioners and investigate the impact of joint decision-making strategies proposed in this thesis. All the three chapters have the potential to be extended as case studies by incorporating practitioners' views.

Finally, we remark about the direction of research in retail supply chain management. More research is required to investigate impact of joint decision-making process under multiple uncertainties using practical and realistic settings. This involves integrating challenging features of multiple uncertainties and dynamic nature of practical retail setting into the decision-making models. Thus, efficient modelling and solution approaches are needed.

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1 Appendix A

In this Appendix, the proofs of all propositions in Chapter 2 are provided.

Proof of Proposition 1

The proof follows from the quality aligned prices results presented by Akçay et al. (2010). We assume that the following relation between prices and features of generations holds at each time period in the planning horizon.

$$\frac{p_{m,t} - p_{m-1,t}}{\alpha_{m,t} - \alpha_{m-1,t}} \geq \frac{p_{m-1,t} - p_{m-2,t}}{\alpha_{m-1,t} - \alpha_{m-2,t}} \geq, \cdots, \geq \frac{p_{2,t} - p_{1,t}}{\alpha_{2,t} - \alpha_{1,t}} \geq \frac{p_{1,t}}{\alpha_{1,t}}.$$

By using this relation, we can formulate the choice probabilities. Let us consider the latest generation $m \in G_t$ at time t. A customer will choose generation m, if it provides the maximum utility, in other words, if $\theta\alpha_{mt} - p_{mt} \geq \theta\alpha_{jt} - p_{jt}$, for $j = 1, \ldots, m-1$. Equivalently, generation m would be chosen if $\frac{p_{m,t} - p_{j,t}}{\alpha_{m,t} - \alpha_{j,t}} \leq \theta$ for $j = 1, \ldots, m-1$ or $\max_{j<m} \left\{ \frac{p_{m,t} - p_{j,t}}{\alpha_{m,t} - \alpha_{j,t}} \right\} \leq \theta$. Based on the quality aligned prices condition, we have

$$\max_{j<k} \left\{ \frac{p_{k,t} - p_{j,t}}{\alpha_{k,t} - \alpha_{j,t}} \right\} = \frac{p_{k,t} - p_{k-1,t}}{\alpha_{k,t} - \alpha_{k-1,t}} \qquad \text{and} \qquad \min_{j>k} \left\{ \frac{p_{j,t} - p_{k,t}}{\alpha_{j,t} - \alpha_{k,t}} \right\} = \frac{p_{k+1,t} - p_{k-1,t}}{\alpha_{k+1,t} - \alpha_{k,t}}.$$

Thus, the choice probability for generation m can be given as

$$\gamma_{m,t} = Pr\left(\frac{p_{m,t} - p_{m-1,t}}{\alpha_{m,t} - \alpha_{m-1,t}} < \theta \leq 1 \right) = 1 - \frac{p_{m,t} - p_{m-1,t}}{\alpha_{m,t} - \alpha_{m-1,t}}.$$

We can extend this result and formulate the choice probability for any generation $k \in G_t$. A customer will choose generation k, for $1 < k < m$, if $\theta\alpha_{kt} - p_{kt} \geq \theta\alpha_{jt} - p_{jt}, \forall j \neq k, j = 1, \ldots, m$. In other words, generation k, for $1 < k < m$, would be chosen if $\max_{j<k} \left\{ \frac{p_{k,t} - p_{j,t}}{\alpha_{k,t} - \alpha_{j,t}} \right\} \leq \theta$ and $\min_{j>k} \left\{ \frac{p_{j,t} - p_{k,t}}{\alpha_{j,t} - \alpha_{k,t}} \right\} \geq \theta$. Based on the quality aligned prices condition, the choice probability for generation k can be given as

$$\gamma_{k,t} = Pr\left(\max_{j<k} \left\{ \frac{p_{k,t} - p_{j,t}}{\alpha_{k,t} - \alpha_{j,t}} \right\} \leq \theta \leq \min_{j>k} \left\{ \frac{p_{j,t} - p_{k,t}}{\alpha_{j,t} - \alpha_{k,t}} \right\} \right) = \frac{p_{k+1,t} - p_{k,t}}{\alpha_{k+1,t} - \alpha_{k,t}} - \frac{p_{k,t} - p_{k-1,t}}{\alpha_{k,t} - \alpha_{k-1,t}}.$$

By carrying out comparison of the current and previous versions available at time t in the same manner, we find that if $0 \leq \theta \leq \frac{p_{k,t}}{\alpha_{k,t}}$ for $k = 1$, then the customer prefers not to purchase. As a result the choice probabilities of multiple generations are obtained as stated above. ■

Proof of Proposition 2

In order to prove that $f_{k,t}(.)$ is the probability mass function for generation $k \in G_t$ at time t, we need to show that $\sum_{j=0}^{M} f_{k,t}(j) = 1$ holds.

$$\sum_{j=0}^{M} f_{k,t}(j) \;=\; \sum_{j=0}^{M}\sum_{i=j}^{M} \binom{i}{j} (\gamma_{k,t})^j (1-\gamma_{k,t})^{i-j} \lambda_{i,t}$$

We expand the first summation,

$$= \sum_{i=0}^{M} (1-\gamma_{k,t})^i \lambda_{i,t} + \sum_{i=1}^{M} \binom{i}{1} (\gamma_{k,t})(1-\gamma_{k,t})^{i-1}\lambda_{i,t} + \cdots + (\gamma_{k,t})^M \lambda_{M,t}$$

By rearranging it for $\lambda_{i,t}$ and applying the binomial theorem, we obtain

$$\begin{aligned}
&= \lambda_{0,t} + \lambda_{1,t}\left(\binom{1}{0}(\gamma_{k,t})^0(1-\gamma_{k,t})^1 + \binom{1}{1}(\gamma_{k,t})^1(1-\gamma_{k,t})^0\right) + \cdots + \\
&\quad \lambda_{M,t}\left(\binom{M}{0}(\gamma_{k,t})^0(1-\gamma_{k,t})^M + \binom{M}{1}(\gamma_{k,t})^1(1-\gamma_{k,t})^{M-1} \right.\\
&\quad \left. + \cdots + \binom{M}{M}(\gamma_{k,t})^M(1-\gamma_{k,t})^0\right) \\
&= \lambda_{0,t} + \lambda_{1,t}(\gamma_{k,1} + 1 - \gamma_{k,1})^1 + \cdots + \lambda_{M,t}(\gamma_{k,1} + 1 - \gamma_{k,1})^M = 1
\end{aligned}$$

■

Proof of Proposition 3

Consider the dynamic joint production-pricing optimization model. Given the boundary conditions at time $T+1$, the value function at time T is as follows;

$$V_T(\mathbf{x}_T) = \max_{0 \le \mathbf{q}_T \le \kappa,\ \mathbf{p}_T \in \mathcal{F}_T} \mathbb{E}\left[\sum_{k \in G_T} p_{k,T}.\min\{\tilde{d}_{k,T}, q_{k,T} + x_{k,T}\} - c_k q_{k,T} - h_T(x_{k,T} + q_{k,T} - \tilde{d}_{k,T})^+\right]$$

By using $\min\{a,b\} = b - (b-a)^+$ for $a, b \in \mathbb{Z}^+$, we can rewrite the value function as

$$V_T(\mathbf{x}_T) = \max_{0\leq \mathbf{q}_T \leq \kappa,\ \mathbf{p}_T\in\mathcal{F}_T} \sum_{k\in G_T} p_{k,T}\left(x_{k,T}+q_{k,T}\right) - c_k q_{k,T} - \left(h_T + p_{k,T}\right)\mathbb{E}\left[\left(x_{k,T}+q_{k,T}-\tilde{d}_{k,T}\right)^+\right] \tag{1}$$

Suppose that there are S number of different models available in the market (i.e., $|G_T| = S$) and total number of products available for each generation $k \in G_T$ is assumed to be $x_{k,T} + q_{k,T} = M$. In this case, we obtain

$$\mathbb{E}\left[\left(x_{k,T}+q_{k,T}-\tilde{d}_{k,T}\right)^+\right] = \sum_{j=0}^{x_{k,T}+q_{k,T}} f_{k,T}(j)\left(x_{k,T}+q_{k,T}-j\right) \tag{2}$$

We know the expression $\sum_{j=0}^{M} j f_{k,T}(j) = \gamma_{k,T}\sum_{j=0}^{M} j\lambda_j$ holds true. This expression is obtained by plugging the value of $f_{k,T}(.)$ from proposition 2 and mathematically expanding the equation. The mathematical terms are rearranged to apply algebraic tools and the binomial theorem which results in the simplified expression. By using $\sum_{j=0}^{M} j f_{k,T}(j) = \gamma_{k,T}\sum_{j=0}^{M} j\lambda_j$, and re-injecting (2) into (1) as

$$\begin{aligned} V_T(\mathbf{x}_T) &= \max_{0\leq \mathbf{q}_T \leq \kappa,\ \mathbf{p}_T\in\mathcal{F}_T} \sum_{k\in G_T} p_{k,T}\left(x_{k,T}+q_{k,T}\right) - c_k q_{k,T} \\ &\quad - \left(h_T + p_{k,T}\right)\left(x_{k,T}+q_{k,T}-\gamma_{k,T}\sum_{j=0}^{M} j\lambda_j\right). \end{aligned}$$

One can easily show that the hessian matrix H of the value function with respect to $p_{k,T}$ for $k \in G_T = \{1, \cdots, s-1, s\}$ (in order from the older to recent generations currently available at T) is a symmetric diagonally dominant with real non-positive diagonal entries, and also negative semi-definite matrix.

$$H = \sum_{j=0}^{M} j\lambda_j \begin{pmatrix} \frac{-2\alpha_2}{\alpha_1(\alpha_2-\alpha_1)} & \frac{2}{\alpha_2-\alpha_1} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{2}{\alpha_2-\alpha_1} & \frac{-2(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)} & \frac{2}{\alpha_3-\alpha_2} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{2}{\alpha_3-\alpha_2} & \frac{-2(\alpha_4-\alpha_2)}{(\alpha_4-\alpha_3)(\alpha_3-\alpha_2)} & \frac{2}{\alpha_4-\alpha_3} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{2}{\alpha_s-\alpha_{s-1}} & \frac{-2}{\alpha_s-\alpha_{s-1}} \end{pmatrix} \tag{3}$$

Let μ_k and ξ_k for each version $k \in G_T$ denote the Lagrangian multipliers with respect to linear constraints. Then the Lagrangian function can be written as

$$\begin{aligned}\mathcal{L}(\mathbf{p}, \mu, \xi) &= \sum_{k \in G_T} p_{k,T}(x_{k,T} + q_{k.T}) - c_k q_{k,T} - (h_T + p_{k,T}) \left(x_{k,T} + q_{k,T} - \gamma_{k,T} \sum_{j=0}^{M} j\lambda_{j,T} \right) \\ &\quad - \sum_{k \in G_T} \xi_k p_{k,T} - \sum_{i=1}^{S-1} \mu_i \left(\frac{p_i}{\alpha_i} - \frac{p_{i+1}}{\alpha_{i+1}} \right)\end{aligned}$$

Next, we will show that the first order optimality conditions $\frac{\partial \mathcal{L}}{\partial p_{i,T}} = 0$ for $i = 1, \cdots, S-1$ and $\frac{\partial \mathcal{L}}{\partial p_{S,T}} = 0$, as well as complimentarity conditions are satisfied at the optimal production strategy. In other words, we have for $i = 1, \cdots, S-1$

$$\left(\frac{2p_{i-1,T}}{\alpha_{i,T} - \alpha_{i-1,T}} - \frac{2(\alpha_{i+1,T} - \alpha_{i-1,T})p_{i,T}}{(\alpha_{i+1} - \alpha_{i,T})(\alpha_{i,T} - \alpha_{i-1,T})} + \frac{2p_{i+1,T}}{\alpha_{i+1,T} - \alpha_{i,T}} - \frac{h}{\alpha_{i,T} - \alpha_{i-1,T}} \right) \sum_{k=0}^{M} k\lambda_{k,T} \\ + \mu_{i-1} - \mu_i - \xi_i = 0,$$

and

$$\begin{aligned}&\left(\frac{2p_{i-1,T}}{\alpha_{i,T} - \alpha_{i-1,T}} - \frac{2p_{i,T}}{\alpha_{i,T} - \alpha_{i-1,T}} + 1 \right) \sum_{k=0}^{M} k\lambda_{k,T} + \mu_{i-1} - \mu_i - \xi_i = 0, \quad i = S, \\ &\mu_i \left(\frac{p_{i,T}}{\alpha_{i,T}} - \frac{p_{i+1,T}}{\alpha_{i+1,T}} \right) = 0, \ \xi_i p_{i,T} = 0, \ \mu_i \geq 0, \ \xi_i \geq 0, \quad i = 1, \cdots, S.\end{aligned}$$

Since $\frac{p_{i,T}}{\alpha_{i,T}} < \frac{p_{i+1,T}}{\alpha_{i+1,T}}$ and $p_{i,T} \neq 0$, we find $\mu_i = \xi_i = 0$ for $i = 1, \cdots, S$. In this case, for $\sum_{k=0}^{M} k\lambda_{k,T} \neq 0$, the first order conditions for sufficiency of the optimality become

$$\begin{aligned}&\frac{2p_{2,T}}{\alpha_{2,T} - \alpha_{1,T}} - \frac{2\alpha_{2,T} p_{2,T}}{\alpha_{1,T}(\alpha_{2,T} - \alpha_{1,T})} - \frac{h_T}{\alpha_{1,T}} = 0 \\ &\text{for } i = 2, \ldots, S-1 \\ &\frac{2p_{i-1,T}}{\alpha_{i,T} - \alpha_{i-1,T}} - \frac{2(\alpha_{i+1,T} - \alpha_{i-1,T})p_{i,T}}{(\alpha_{i+1,T} - \alpha_{i,T})(\alpha_{i,T} - \alpha_{i-1,T})} + \frac{2p_{i+1,T}}{\alpha_{i+1,T} - \alpha_{i,T}} - \frac{h_T}{\alpha_{i,T} - \alpha_{i-1,T}} = 0, \\ &\frac{2p_{S-1,T}}{\alpha_{S,T} - \alpha_{S-1,T}} - \frac{2p_{S,T}}{\alpha_{S,T} - \alpha_{S-1,T}} + 1 = 0\end{aligned} \tag{4}$$

This linear equation system for S unknowns provides the optimal price strategy as $p^*_{i,T} = \frac{\alpha_{i,T} - h_T}{2}$ for all generations $i = 1, \cdots, S$. Note that if $\frac{p_{i,T}}{\alpha_{i,T}} = \frac{p_{i+1,T}}{\alpha_{i+1,T}}$, then

$\lambda_{i,T} > 0,\ \xi_i > 0$ for $i = 1, 2, \cdots, S$. The first order conditions for sufficiency of the optimality leads to a system of $2S-1$ linear equations with S variables. As the number of linear equations are greater than number of variables, the solutions for this system of linear equations will be inconsistent. ■

Proof of Proposition 4

We prove this proposition in two parts by considering the lower and upper bounds of prices.

a) We first prove by contradiction that the lower bound $p^L_{k,t}$ of market price can be obtained as $p^L_{k,t} = \frac{\alpha_{k,t}-h_t}{2}$ such that $p^L_{k,t} \le p_{k,t}$ for the oldest generation $k \in G_t = \{1, 2, \cdots, m\}$ (where generations are represented in order from the oldest to the recent model) for any time period t. Assume the price of the oldest version $k = 1$ at time t is $p_{k,t} = \frac{\alpha_{k,t}-h_t}{2} - a$, where $a > 0$ and $a \in \mathbb{R}$. From Proposition 1, we can compute the total purchase probability over all available products at time t as $\sum_{k=1}^{m} \gamma_{k,t} = 1 - \frac{p_{k,t}}{\alpha_{k,t}}$. We then obtain $\sum_{k=1}^{m} \gamma_{k,t} = 1 - \frac{\alpha_{k,t} - h_t - 2a}{2\alpha_{k,t}} = \frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}}$.

   – Suppose that $h_t \le \alpha_{k,t}$. In this case, we have $\frac{h_t}{\alpha_{k,t}} \le 1$ that leads to $\frac{h_t}{2\alpha_{k,t}} \le \frac{1}{2}$ and also $\frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} \le 1$. Similarly, we can write an equivalent form of the inequality

$$\frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}} \le 1 + \frac{a}{\alpha_{k,t}} \implies \sum_{k=1}^{m} \gamma_{k,t} \le 1 + \frac{a}{\alpha_{k,t}}$$

   For $a > \alpha_{k,t}$ or $a \le \alpha_{k,t}$, the equation $\sum_{k=1}^{m} \gamma_{k,t} \le 1 + \frac{a}{\alpha_{k,t}}$ is no longer a binding restriction on choice probabilities to lie between 0 and 1 and we may obtain $\sum_{k=1}^{m} \gamma_{k,t} \ge 1$ for some parameter values which contradicts the assumption. For example, let $h_t = \alpha_{k,t} - a$ (since $h_t \le \alpha_{k,t}$),

$$\begin{aligned}
\frac{1}{2}+\frac{h_t}{2\alpha_{k,t}}+\frac{a}{\alpha_{k,t}} &= \frac{1}{2}+\frac{\alpha_{k,t}-a}{2\alpha_{k,t}}+\frac{a}{\alpha_{k,t}} \\
&= 1+\frac{a}{2\alpha_{k,t}} \geq 1 \\
\implies \sum_{k=1}^{m}\gamma_{k,t} &\geq 1
\end{aligned}$$

– Suppose that $h_t \geq \alpha_{k,t}$. In this case, $\frac{h_t}{\alpha_{k,t}} \geq 1$ that leads to $\frac{h_t}{2\alpha_{k,t}} \geq \frac{1}{2}$ and also $\frac{1}{2}+\frac{h_t}{2\alpha_{k,t}} \geq 1$. Similarly, we can write an equivalent form of the inequality

$$\frac{1}{2}+\frac{h_t}{2\alpha_{k,t}}+\frac{a}{\alpha_{k,t}} \geq 1+\frac{a}{\alpha_{k,t}} \implies \sum_{k=1}^{m}\gamma_{k,t} \geq 1+\frac{a}{\alpha_{k,t}}$$

For $a > \alpha_{k,t}$ or $a \leq \alpha_{k,t}$, we obtain $\sum_{k=1}^{m}\gamma_{k,t} \geq 1$ that contradicts the assumption.

Therefore, we conclude that $p^L_{k,t} = \frac{\alpha_{k,t}-h_t}{2} \leq p_{k,t}$ for $k \in G_t = \{1, 2, \cdots, m\}$.

b) Next, we will show that $p_{k,t} \leq p^U_{k,t}$ where $p^U_{k,t} = \frac{\alpha_{k,t}p_{k+1,t}}{\alpha_{k+1,t}}$ for generation $k \in G_t$ at ant time t.

Let's consider the choice probability $\gamma_{k,t} \geq 0$ for the oldest generation k. Then we find

$$\gamma_{k,t} = \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} - \frac{p_{k,t}}{\alpha_{k,t}} \geq 0 \implies \frac{\alpha_{k,t}p_{k+1,t}}{\alpha_{k+1,t}} \geq p_{k,t}$$

The choice probability for successive generation $k+1$ is $\gamma_{k+1,t} = \frac{p_{k+2,t}-p_{k+1,t}}{\alpha_{k+2,t}-\alpha_{k+1,t}} - \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} \geq 0$.

$$\begin{aligned}
&\frac{p_{k+2,t}-p_{k+1,t}}{\alpha_{k+2,t}-\alpha_{k+1,t}} - \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} \geq 0 \\
&\implies \frac{p_{k+2,t}-p_{k+1,t}}{\alpha_{k+2,t}-\alpha_{k+1,t}} \geq \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} \\
&\implies \frac{p_{k+2,t}-p_{k+1,t}}{\alpha_{k+2,t}-\alpha_{k+1,t}} \geq \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} \geq \frac{p_{k+1,t}-\frac{\alpha_{k,t}p_{k+1,t}}{\alpha_{k+1,t}}}{\alpha_{k+1,t}-\alpha_{k,t}} \\
&\implies \frac{p_{k+2,t}-p_{k+1,t}}{\alpha_{k+2,t}-\alpha_{k+1,t}} \geq \frac{p_{k+1,t}-\frac{\alpha_{k,t}p_{k+1,t}}{\alpha_{k+1,t}}}{\alpha_{k+1,t}-\alpha_{k,t}} = \frac{p_{k+1,t}}{\alpha_{k+1,t}} \\
&\implies \frac{p_{k+2,t}-p_{k+1,t}}{\alpha_{k+2,t}-\alpha_{k+1,t}} \geq \frac{p_{k+1,t}}{\alpha_{k+1,t}} \\
&\implies \frac{\alpha_{k+1,t}p_{k+2,t}}{\alpha_{k+2,t}} \geq p_{k+1,t}
\end{aligned}$$

By carrying out in the same manner, we obtain the lower and upper bounds for market prices as stated in the proposition. ■

Proof of Proposition 5

We prove the concavity of $\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t})$ by mathematical induction. At the boundary condition $t = T$, we prove the concavity of the value function $\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T})$ in $\mathbf{q}_T$. The value function $\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T})$ at the boundary condition is rewritten as,

$$\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T}) = \max_{0\leq \mathbf{q_T}\leq\kappa} \sum_{k\in G_T} \hat{p}_{kT}(x_{kT}+q_{kT}) - c_k q_{kT} - B_{kT}(\mathbf{x}_T, \mathbf{q}_T) \tag{5}$$

where $B_{kT}(\mathbf{x}_T, \mathbf{q}_T) = (h_T + \hat{p}_{kT}) \sum_{j=0}^{x_{kT}+q_{kT}} f_{kT}(j)(x_{kT}+q_{kT}-j)$ is a differentiable function with respect to q_{kT}. In addition, $B_k^{'}(\mathbf{x}_T, \mathbf{q}_T)$ and $B_k^{''}(\mathbf{x}_T, \mathbf{q}_T) \geq 0$ represent the first and second-order derivatives, respectively, with respect to q_{kT}. We establish the concavity w.r.t $\mathbf{q_t}$ by writing its Hessian matrix as follows,

$$\begin{pmatrix} -B_1^{''}(\mathbf{x_T}, \mathbf{q_t}) & 0 & 0 & 0 & \dots & 0 \\ 0 & -B_2^{''}(\mathbf{x_T}, \mathbf{q_T}) & 0 & 0 & \dots & 0 \\ \\ 0 & 0 & 0 & 0 & \dots & -B_k^{''}(\mathbf{x_T}, \mathbf{q_T}) \end{pmatrix} \tag{6}$$

that is a symmetric diagonally dominant matrix with real non-positive diagonal entries. As the Hessian matrix is negative semi-definite, it directly follows that $\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T})$ is a concave function in $\mathbf{q}_T$. Let us assume $\hat{V}_{t+1}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})$ is concave in production level $\mathbf{q}_{t+1}$. We rewrite $\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t})$ as follows,

$$\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t}) = \max_{0\leq \mathbf{q_t}\leq\kappa} \sum_{k\in G_t} \hat{p}_{kt}(x_{kt}+q_{kt}) - c_k q_{kt} - B_{kt}(\mathbf{x}_t, \mathbf{q}_t) + E[\hat{V}_{t+1}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})] \tag{7}$$

We canestablish the concavity w.r.t $\mathbf{q_t}$ by writing its Hessian matrix as follows by introducing $\phi = E[\hat{V}_{t+1}^{''}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})]$.

$$\begin{pmatrix} -B_{\gamma_t}^{''}(\mathbf{x_t}, \mathbf{q_t}) + \phi & 0 & 0 & 0 & \dots & 0 \\ 0 & -B_{\gamma_t+1}^{''}(\mathbf{x_t}, \mathbf{q_t}) + \phi & 0 & 0 & \dots & 0 \\ \\ 0 & 0 & 0 & 0 & \dots & -B_t^{''}(\mathbf{x_t}, \mathbf{q_t}) + \phi \end{pmatrix} \tag{8}$$

By induction hypothesis, $E[\hat{V}''_{t+1}(\mathbf{x}_{t+1}|\mathbf{\hat{p}_{t+1}})] \leq 0$. The given Hessian matrix is a symmetric diagonally dominant matrix with real non-positive diagonal entries. As the Hessian matrix is negative semi-definite, it directly follows that $\hat{V}_t(\mathbf{x}_t|\mathbf{\hat{p}_t})$ is a concave function in $\mathbf{q}_t$. ■

2 Appendix B

In this Appendix, the proofs of all propositions in Chapter 3 are provided.

Proof of Proposition 1

This proof is presented by formulating the policy function $\pi_t(1, \mathbf{I_t}, q_t)$ for different policies.

No markdown policy: Inventories with different ages are priced the same in the no markdown policy. We set $w_{it} = 0$ for any age i since no inventory is put on a markdown sale. Order quantity q_t at time t has an age of 0 time periods and can be written as $x_{0t} = q_t$. Policy function $\pi_t(1, \mathbf{I_t}, q_t)$ for the no markdown policy ($j = 1$) is described as,

$$\pi_t(1, \mathbf{I_t}, q_t) = p_r min\{\sum_{k=0}^{n} x_{kt}, \tilde{d}_{rt}\} - cq_t - h\Big(max\{\sum_{k=0}^{n} x_{kt} - \tilde{d}_{rt}, 0\}\Big) - \gamma(max\{x_{nt} - \tilde{d}_{rt}, 0\}) \tag{9}$$

We expand the min function using $min\{\sum_{k=0}^{n} x_{kt}, \tilde{d}_{rt}\} = \sum_{k=0}^{n} x_{kt} - max\{\sum_{k=0}^{n} x_{kt} - \tilde{d}_{rt}, 0\}$,

$$= p_r \sum_{k=0}^{n} x_{kt} - cq_t - (h + p_r)\Big(max\{\sum_{k=0}^{n} x_{kt} - \tilde{d}_{rt}, 0\}\Big) - \gamma(max\{x_{nt} - \tilde{d}_{rt}, 0\}) \tag{10}$$

Let $f_t(.)$ be the discrete probability density function for demand $\tilde{d}_{rt}$. We expand the max functions as below,

$$= p_r \sum_{k=0}^{n} x_{kt} - cq_t - (h + p_r) \sum_{z=0}^{\beta(0,n)} \Big(\sum_{k=0}^{n-1} x_{kt} - z\Big) f_t(z) - \gamma \sum_{z=0}^{x_{nt}} (x_{nt} - z) f_t(z) \tag{11}$$

where $\beta(0, n) = \sum_{k=0}^{n} x_{kt}$. Let us define $\phi_1(x_{0t}, \cdots, x_{nt}) = \sum_{z=0}^{\beta(0,n)} \Big(\sum_{k=0}^{n} x_{kt} - z\Big) f_t(z)$ and $\phi_2(x_{nt}) = \sum_{z=0}^{x_{nt}} (x_{nt} - z) f_t(z)$. Equation (11) can be rewritten as,

$$\pi_t(1, \mathbf{I_t}, q_t) = p_r \sum_{k=0}^{n} x_{kt} - cq_t - (h + p_r)\phi_1(x_{0t}, \cdots, x_{nt}) - \gamma\phi_2(x_{nt}) \tag{12}$$

Differentiating equation (12) w.r.t q_t (recall $x_{0t} = q_t$)

$$\begin{aligned} \frac{\partial \pi_t(1, \mathbf{I_t}, q_t)}{\partial q_t} &= p_r - c - (h + p_r)\frac{\partial \phi_1(x_{0t}, \cdots, x_{nt})}{\partial q_t} \\ \frac{\partial^2 \pi_t(1, \mathbf{I_t}, q_t)}{\partial q_t^2} &= -(h + p_r)\frac{\partial^2 \phi_1(x_{0t}, \cdots, x_{nt})}{\partial q_t^2} \end{aligned} \tag{13}$$

$\frac{\partial^2 \pi_t(1,\mathbf{I_t},q_t)}{\partial q_t^2} \leq 0$ for all q_t. Therefore, $\pi_t(1, \mathbf{I_t}, q_t)$ is concave in order quantity q_t

Partial markdown policy: In this policy represented by $j = 2, 3, \cdots, n-1$, some inventories are sold at regular price while the others are marked down. In other words, at the j-th partial markdown policy, inventories with age less than $n - j + 1$ periods are sold at regular price ($w_{it} = 0,\ \forall\ i = 0, 1, \cdots, n-j+1$) and the rest are marked down ($w_{it} = 1,\ \forall\ i = n-j+2, n-j+3, \cdots, n$). Policy function $\pi_t(j, \mathbf{I_t}, q_t)$ for the partial markdown policy is described as,

$$\begin{aligned} \pi_t(j, \mathbf{I_t}, q_t) = p_r min\{\sum_{k=0}^{n-j} x_{kt}, \tilde{d}_{rt}\} + \sum_{k=n-j+1}^{n} p_{k,mt} min\{x_{kt}, \tilde{d}_{im,t}\} - cq_t - h\Big(max\{\sum_{k=0}^{n-j} x_{kt} - \tilde{d}_{rt}, 0\}\Big) \\ - h\Big(max\{\sum_{k=n-j+1}^{n} x_{kt} - \tilde{d}_{k,mt}, 0\}\Big) - \gamma(max\{x_{nt} - \tilde{d}_{rt}, 0\}) \end{aligned} \tag{14}$$

We expand the min function as above,

$$\begin{aligned} &= p_r \sum_{k=0}^{n-j} x_{kt} + \sum_{k=n-j+1}^{n} p_{k,mt} x_{kt} - cq_t - (h + p_r)\Big(max\{\sum_{k=0}^{n-j} x_{kt} - \tilde{d}_{rt}, 0\}\Big) \\ &- (h + p_{k,mt})\Big(max\{\sum_{k=n-j+1}^{n} x_{kt} - \tilde{d}_{k,mt}, 0\}\Big) - \gamma(max\{x_{nt} - \tilde{d}_{rt}, 0\}) \end{aligned} \tag{15}$$

Let $f_{rt}(.)$ and $f_{k,mt}(.)$ be the discrete probability density function for regular demand $\tilde{d}_{rt}$ and markdown demand $\tilde{d}_{k,mt}$ for inventory with age k, respectively. We expand the max function as below,

$$\begin{aligned} &= p_r \sum_{k=0}^{n-j} x_{kt} + p_m \sum_{k=n-j+1}^{n-1} x_{kt} - cq_t - (h + p_r) \sum_{z=0}^{\beta(0,n-j)} \Big(\sum_{k=0}^{n-j} x_{kt} - z\Big) f_{rt}(z) \\ &- \sum_{k=n-j+1}^{n} (h + p_{k,mt}) \sum_{z=0}^{x_{kt}} \Big(x_{kt} - z\Big) f_{k,mt}(z) - \gamma \sum_{z=0}^{x_{nt}} (x_{nt} - z) f_{n,mt}(z) \end{aligned} \tag{16}$$

where $\beta(0, n-j) = \sum_{k=0}^{n-j} x_{kt}$. Let $\phi_r(x_{0t}, \cdots, x_{n-j,t}) = \sum_{z=0}^{\beta(0,n-j)} \Big(\sum_{k=0}^{n-j} x_{kt} - z\Big) f_{rt}(z)$ and $\phi_{mk}(x_{kt}) = \sum_{z=0}^{x_{kt}} \Big(x_{kt} - z\Big) f_{k,mt}(z)$ where $k = n-j+1, \cdots, n$. Equation (16) can be rewritten as,

$$\begin{aligned}\pi_t(j,\mathbf{I_t},q_t) = p_r\sum_{k=0}^{n-j}x_{kt} + p_m\sum_{k=n-j+1}^{n-1}x_{kt} - cq_t - (h+p_r)\phi_r(x_{0t},\cdots,x_{n-j,t})\\ - \sum_{k=n-j}^{n-1}(h+p_{k,mt})(\phi_{mk}(x_{kt})) - \gamma\phi_{mn}(x_{nt})\end{aligned} \tag{17}$$

Differentiating equation (17) w.r.t q_t (recall $x_{0t} = q_t$)

$$\begin{aligned}\frac{\partial\pi_t(j,\mathbf{I_t},q_t)}{\partial q_t} &= p_r - c - (h+p_r)\frac{\partial\phi_1(x_{0t},\cdots,x_{n-j,t})}{\partial q_t}\\ \frac{\partial^2\pi_t(j,\mathbf{I_t},q_t)}{\partial q_t^2} &= -(h+p_r)\frac{\partial^2\phi_1(x_{0t},\cdots,x_{n-j,t})}{\partial q_t^2}\end{aligned} \tag{18}$$

$\frac{\partial^2\pi_t(j,\mathbf{I_t},q_t)}{\partial q_t^2} \leq 0$ for all q_t. Therefore, $\pi_t(j,\mathbf{I_t},q_t)$ is concave in order quantity q_t The complete markdown proof is similar to the no markdown proof. ■

Proof of Proposition 2

The state of the dynamic system at any time is the initial inventory level evaluated from the unsold inventory of the previous selling period. If the unsold inventories are close to zero, the firm is unlikely to conduct a markdown sale. On the other hand, when the initial inventories are in excess the firm might markdown their selling price down to sell them off. Let us suppose at inventory level $\bar{x}_{it}$ of age i at time t, the firm's optimal policy is to conduct no markdown and sell all inventories at regular price. We assume that at an inventory level $x_{it} \leq \bar{x}_{it}$ it is not optimal for the firm to conduct any markdown sale. However, the firm may conduct a markdown sale at an inventory level $x_{it} > \bar{x}_{it}$. We define r_{it}^1 as the maximum inventory level of age i at which it is optimal for the firm to follow a no markdown policy. In other words, r_{it}^1 is referred as the transition level from a no markdown policy to different markdown policies. The transition level r_{it}^j is the maximum inventory at which it is optimal for the firm to follow policy j. We define a vector $\mathbf{r_{it}} = \{r_{it}^j | j = 1,2,\cdots,n\}$ of transition levels of inventory of age i at time t. The transition levels are used to characterize policy switching in the single period expected profit $g(\mathbf{I_t},q_t,\mathbf{w_t})$ w.r.t the inventory level for each policy j,

$$g_t(\mathbf{I_t},q_t,\mathbf{w_t}) = \pi_t(j,\mathbf{I_t},q_t), r_{it}^{j-1} < x_{it} \leq r_{it}^j$$

where $r_{it}^0 = 0$ for any age i at time t According to **proposition 1**, profit $\pi_t(j,\mathbf{I_t},q_t)$ from policy j is concave in inventory x_{it} of age i. Thus, we define the maximum profit for each policy j w.r.t inventory level of age i, where the value of inventory of any age k remains same, where $k \neq i$,

$$k_{it}^{j} = sup\{\pi_t(j, \mathbf{I_t}, q_t) \mid r_{it}^{j-1} < x_{it} \leq r_{it}^{j}\}$$

Consider two values $x_{it}^{'}$ and $x_{it}^{''}$ of inventory level of age i at time t in the following three cases.

Case (i): $x_{it}^{'}, x_{it}^{''} \in [r_{it}^{j-1}, r_{it}^{j}]$ for any policy j. Since $\pi_t(j, \mathbf{I_t}, q_t)$ is concave in x_{it}, the following relation holds for any $\lambda \in [0,1]$

$$g_t((1-\lambda)\mathbf{I_t^{'}} + \lambda\mathbf{I_t^{''}}, q_t, \mathbf{w_t}) \geq (1-\lambda)g_t(\mathbf{I_t^{'}}, q_t, \mathbf{w_t}) + \lambda g_t(\mathbf{I_t^{''}}, q_t, \mathbf{w_t})),$$

where $I_t^{'} = (x_{0t}, \cdots, x_{i-1,t}, x_{it}^{'}, x_{i+1,t}, \cdots, x_{n-1,t})$ and $I_t^{''} = (x_{0t}, \cdots, x_{i-1,t}, x_{it}^{''}, x_{i+1,t}, \cdots, x_{n-1,t})$

Case (ii): $x_{it}^{'} \in [r_{it}^{j-1}, r_{it}^{j}]$ and $x_{it}^{''} \in [r_{it}^{l-1}, r_{it}^{l}]$ for any two different policies j and l, respectively. Since k_{it}^{j} and k_{it}^{l} are the maximum points of the curve $\pi_t(j, \mathbf{I_t}, q_t)$ and $\pi_t(l, \mathbf{I_t}, q_t)$ in the intervals $[r_{it}^{j-1}, r_{it}^{j}]$ and $(r_{it}^{l-1}, r_{it}^{l}]$, respectively. Let $\bar{k}_{it} = max\{k_{it}^{j}, k_{it}^{l}\}$ be the maximum point of the two combined curves $\pi_t(j, \mathbf{I_t}, q_t)$ and $\pi_t(l, \mathbf{I_t}, q_t)$. Since $I_t^{'} = (x_{0t}, \cdots, x_{i-1,t}, x_{it}^{'}, x_{i+1,t}, \cdots, x_{n-1,t})$ and $I_t^{''} = (x_{0t}, \cdots, x_{i-1,t}, x_{it}^{''}, x_{i+1,t}, \cdots, x_{n-1,t})$ are located at two separate concave curves $\pi_t(j, \mathbf{I_t}, q_t)$ and $\pi_t(l, \mathbf{I_t}, q_t)$, respectively, their combination $g_t((1-\lambda)\mathbf{I_t^{'}} + \lambda\mathbf{I_t^{''}}, q_t, \mathbf{w_t})$ may or not lie above the line $(1-\lambda)g_t(\mathbf{I_t^{'}}, q_t, \mathbf{w_t}) + \lambda g_t(\mathbf{I_t^{''}}, q_t, \mathbf{w_t}))$. However, $g_t((1-\lambda)\mathbf{I_t^{'}} + \lambda\mathbf{I_t^{''}}, q_t, \mathbf{w_t}) + \lambda\bar{k}_{it}$ will always lie above the line $(1-\lambda)g_t(\mathbf{I_t^{'}}, q_t, \mathbf{w_t}) + \lambda g_t(\mathbf{I_t^{''}}, q_t, \mathbf{w_t})$. Thus, the following equation can be written as,

$$g_t((1-\lambda)\mathbf{I_t^{'}} + \lambda\mathbf{I_t^{''}}, q_t, \mathbf{w_t}) \geq (1-\lambda)g_t(\mathbf{I_t^{'}}, q_t, \mathbf{w_t}) + \lambda g_t(\mathbf{I_t^{''}}, q_t, \mathbf{w_t})) - \lambda\bar{k}_{it},$$

In general, we define $k_{it}^{*} = max\{k_{it}^{j} \mid \forall j\}$ as the global maximum point of the curve $g_t(.)$. For any $x_{it}^{'}, x_{it}^{''}$ and $\lambda \in [0,1]$ with $x_{it}^{'} \leq x_{it}^{''}$,

$$g_t((1-\lambda)\mathbf{I_t^{'}} + \lambda\mathbf{I_t^{''}}, q_t, \mathbf{w_t}) \geq (1-\lambda)g_t(\mathbf{I_t^{'}}, q_t, \mathbf{w_t}) + \lambda g_t(\mathbf{I_t^{''}}, q_t, \mathbf{w_t})) - \lambda k_{it}^{*}$$

Thus, $g(\mathbf{I_t}, q_t, \mathbf{w_t})$ is k_{it}^{*}-concave in inventory level x_{it} of age i at time t. ■

Proof of Proposition 3

The k-concavity of the value function over time is proven by following a similar structure as Theorem 3(c) in Chen & Simchi-Levi (2004). Value function at any time t is written as,

$$V_t(\mathbf{I_t}|\mathbf{p_{m,t}}) = g_t(\mathbf{I_t}, q_t, \mathbf{w_t}) + V_{t+1}(\mathbf{I_{t+1}}|\mathbf{p_{m,t+1}}) \tag{19}$$

From the previous result, $g_t(\mathbf{I_t}, q_t, \mathbf{w_t})$ is k_{it}^{*}-concave in inventory level x_{it} of age i at time t. We will show that $V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ is $\hat{k}_{it}$-concave in x_{it} based on the assumption that

$V_{t+1}(\mathbf{I_{t+1}}|\mathbf{p_{m,t+1}})$ is $\hat{k}_{i,t+1}$-concave in x_{it}, where $\hat{k}_{it} = k^*_{it} + \hat{k}_{i,t+1}$ for any age i and time t. The expression $\hat{k}_{it} = k^*_{it} + \hat{k}_{i,t+1}$ holds because of equation (19) and Lemma 2.1(b) from Chen & Simchi-Levi (2004). For any x'_{it}, x''_{it} and $\lambda \in [0,1]$ with $x'_{it} \leq x''_{it}$,

$$g_t((1-\lambda)\mathbf{I'_t} + \lambda\mathbf{I''_t}, q_t, \mathbf{w_t}) \geq (1-\lambda)g_t(\mathbf{I'_t}, q_t, \mathbf{w_t}) + \lambda g_t(\mathbf{I''_t}, q_t, \mathbf{w_t})) - \lambda k^*_{i,t} \tag{20}$$

where $I'_t = (x_{0t}, \cdots, x_{i-1,t}, x'_{it}, x_{i+1,t}, \cdots, x_{n-1,t})$ and $I''_t = (x_{0t}, \cdots, x_{i-1,t}, x''_{it}, x_{i+1,t}, \cdots, x_{n-1,t})$ Based the assumption $V_{t+1}(\mathbf{I_{t+1}}|\mathbf{p_{m,t+1}})$ is $\hat{k}_{it}$-concave in x_{it}, we obtain for any x'_{it}, x''_{it} and $\lambda \in [0,1]$ with $x'_{it} \leq x''_{it}$,

$$V_{t+1}((1-\lambda)\mathbf{I'_{t+1}} + \lambda\mathbf{I''_{t+1}}|\mathbf{p_{m,t+1}}) \geq (1-\lambda)V_{t+1}(\mathbf{I'_{t+1}}|\mathbf{p_{m,t}}) + \lambda V_{t+1}(\mathbf{I''_{t+1}}|\mathbf{p_{m,t+1}}) - \lambda\hat{k}_{i,t+1} \tag{21}$$

From the definition of inventory transition equation for any age i,

$$\begin{aligned} x_{i+1,t+1} &= \big(x_{it} - (\tilde{d}_t - \sum_{k=i}^{n} x_{it})^+\big)^+ \\ &= \big(x_{it} - \tilde{z}_{it}\big)^+ \end{aligned}$$

where $\tilde{z}_{it} = (\tilde{d}_t - \sum_{k=i}^{n} x_{it})^+$ is a random variable representing left-over demand for inventory of age i at time t. We define a vector $\mathbf{Z}_t = \{\tilde{z}_{it}|i = 1, 2, \cdots, n-1\}$. Thus, we write $\mathbf{I}_{t+1} = (\mathbf{I}_t - \mathbf{Z}_t)^+$. Using lemma 9.3.2 in Simchi-Levi et al. (2005),

$$V_{t+1}((1-\lambda)\mathbf{I'_{t+1}} + \lambda\mathbf{I''_{t+1}}|\mathbf{p_{m,t+1}}) \geq (1-\lambda)V_{t+1}(\mathbf{I'_{t+1}}|\mathbf{p_{m,t+1}}) + \lambda V_{t+1}(\mathbf{I''_{t+1}}|\mathbf{p_{m,t+1}}) - \lambda\hat{k}_{i,t+1} \tag{22}$$

Adding (20) and (22), and taking its expectation we get because of (19) ,

$$V_t((1-\lambda)\mathbf{I'_t} + \lambda\mathbf{I''_t}|\mathbf{p_{m,t}}) \geq (1-\lambda)V_t(\mathbf{I'_t}|\mathbf{p_{m,t}}) + \lambda V_t(\mathbf{I''_t}|\mathbf{p_{m,t}}) - \lambda\hat{k}_{it}$$

Thus, $V_t(\mathbf{I_t}|\mathbf{p_{m,t}})$ is $\hat{k}_{it}$-concave in inventory level x_{it}. ■

Proof of Proposition 4

To prove sub-modularity, we show $\pi_t(j, \mathbf{I_t}, x'_{0t}) - \pi_t(j, \mathbf{I_t}, x_{0t})$ to be decreasing in $x_{i,t}$ for $x'_{0t} > x_{0t}$ (Simchi-Levi et al. 2005). $\pi_t(j, \mathbf{I_t}, x_{0t})$ for policy j is written as

$$\begin{aligned} \pi_t(j, \mathbf{I_t}, x_{0t}) = p_r \sum_{k=0}^{n-j} x_{kt} + \sum_{k=n-j+1}^{n-1} p_{k,mt} x_{kt} - c x_{0t} - (h + p_r) \sum_{z=0}^{\beta(0,n-j)} \Big(\sum_{k=0}^{n-j} x_{kt} - z\Big) f_{rt}(z) \\ - \sum_{k=n-j+1}^{n-1} (h + p_{k,mt}) \sum_{z=0}^{x_{kt}} \Big(x_{kt} - z\Big) f_{k,mt}(z) - \gamma \sum_{z=0}^{x_{nt}} (x_{nt} - z) f_{n,mt}(z) \end{aligned} \tag{23}$$

where $\beta(0, n-j) = \sum_{k=0}^{n-j} x_{kt}$. Following the definition of sub-modularity, we define $\pi_t(j, \mathbf{I_t}, x'_{0t})$ for $x'_{0t} \geq x_{0t}$,

$$\begin{aligned}\pi_t(j, \mathbf{I_t}, x_{0t}) = p_r \sum_{k=0}^{n-j} x_{kt} + p_m \sum_{k=n-j+1}^{n-1} x_{kt} - cx_{0t} - (h + p_r) \sum_{z=0}^{\beta(0,n-j)} \Bigg(\sum_{k=0}^{n-j} x_{kt} - z \Bigg) f_{rt}(z) \\ - \sum_{k=n-j+1}^{n-1} (h + p_{k,mt}) \sum_{z=0}^{x_{kt}} \Bigg(x_{kt} - z \Bigg) f_{k,mt}(z) - \gamma \sum_{z=0}^{x_{nt}} (x_{nt} - z) f_{n,mt}(z)\end{aligned} \tag{24}$$

$$\begin{aligned}\pi_t(j, \mathbf{I_t}, x'_{0t}) = p_r (x'_{0t} + \sum_{k=1}^{n-j} x_{kt}) + \sum_{k=n-j+1}^{n-1} p_{k,mt} x_{kt} - cx_{0t} \\ - (h + p_r) \sum_{z=0}^{x'_{0t}+\beta(1,n-j)} \Bigg(x'_{0t} + \sum_{k=1}^{n-j} x_{kt} - z \Bigg) f_{rt}(z) \\ - \sum_{k=n-j+1}^{n-1} (h + p_{k,mt}) \sum_{z=0}^{x_{kt}} \Bigg(x_{kt} - z \Bigg) f_{k,mt}(z) - \gamma \sum_{z=0}^{x_{nt}} (x_{nt} - z) f_{n,mt}(z)\end{aligned} \tag{25}$$

Next, we find the value of $\pi_t(j, \mathbf{I_t}, x'_{0t}) - \pi_t(j, \mathbf{I_t}, x_{0t})$,

$$\begin{aligned}\pi_t(j, \mathbf{I_t}, x'_{0t}) - \pi_t(j, \mathbf{I_t}, x_{0t}) = (p_r - c)(x'_{0t} - x_{0t}) - (h + p_r) \Bigg[\sum_{z=0}^{x'_{0t}+\beta(1,n-j)} \Bigg(x'_{0t} + \sum_{k=1}^{n-j} x_{kt} - z \Bigg) f_{rt}(z) \\ - \sum_{z=0}^{\beta(0,n-j)} \Bigg(\sum_{k=0}^{n-j} x_{kt} - z \Bigg) f_{rt}(z) \Bigg]\end{aligned} \tag{26}$$

We rewrite the last two terms in the square brackets,

$$\begin{aligned}\phi(x_{it}) &= \sum_{z=0}^{x'_{0t}+\beta(1,n-j)} \Bigg(x'_{0t} + \sum_{k=1}^{n-j} x_{kt} - z \Bigg) f_{rt}(z) - \sum_{z=0}^{\beta(0,n-j)} \Bigg(\sum_{k=0}^{n-j} x_{kt} - z \Bigg) f_{rt}(z) \\ &= (x'_{0t} - x_{0t}) \sum_{z=0}^{\beta(0,n-1)-1} f_{rt}(z) + \sum_{z=\beta(0,n-1)}^{x'_{0t}+\beta(1,n-1)} \Bigg(x'_{0t} + \sum_{k=1}^{n-j} x_{kt} - z \Bigg) f_{rt}(z)\end{aligned} \tag{27}$$

The first two terms will always be positive. Since $\phi(x_{it}) \geq 0$, its first derivate, represented as $\phi'(x_{it})$, will also be positive, $\phi'(x_{it}) \geq 0$ in x_{it}. Thus, we can say $\phi'(x_{it})$ is increasing x_{it}. The increasing property of $\phi(x_{it})$ directly implies $\pi_t(j, \mathbf{I_t}, x'_{0t}) - \pi_t(j, \mathbf{I_t}, x_{0t})$ to be decreasing in x_{it}. Thus, the policy function $\pi_t(j, \mathbf{I_t}, q_t)$ for policy j is sub-modular in order decision q_t and inventory level x_{it} of any age i at time t. ■

Proof of Proposition 5

To prove sub-modularity, we show $\pi_t(j, \mathbf{I_t}, x_{0t}^{'}) + E[V_{t+1}(\mathbf{I}_{t+1}(x_{0t}^{'}))] - (\pi_t(j, \mathbf{I_t}, x_{0t}) + E[V_{t+1}(\mathbf{I}_{t+1}(x_{0t}))])$ to be decreasing in x_{0t} for $x_{0t}^{'} > x_{0t}$ (Simchi-Levi et al. 2005), where $\mathbf{I}_{t+1}(x_{1,t+1}^{'}) = (x_{1,t+1}^{'}, x_{2,t+1}, \cdots, x_{n,t+1})$ represents the 1-period old inventory at time $t+1$. In other words, $\mathbf{I}_{t+1}(x_{1,t+1}^{'})$ is equivalent to $\mathbf{I}_{t+1}((x_{0t}^{'} - (\tilde{d}_t - \sum_{j=1}^{n} x_{jt})^+)^+)$. It is sufficient to prove that $E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}^{'}))] - E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}))]$ is decreasing in x_{0t} for $x_{0t}^{'} > x_{0t}$, since $\pi_t(j, \mathbf{I_t}, x_{0t})$ is proved to be sub-modular in proposition 4. The max function within $E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}))]$ can be expanded as follows,

$$\begin{aligned} E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}))] &= \beta_{0t} V_{t+1}(\mathbf{I}_{t+1}(x_{0t} - 0)) + \beta_{1t} V_{t+1}(\mathbf{I}_{t+1}(x_{0t} - 1)) + \cdots + \beta_{x_{0t}} V_{t+1}(\mathbf{I}_{t+1}(0)), \\ &= \sum_{i=0}^{x_{0t}} \beta_{it} V_{t+1}(\mathbf{I}_{t+1}(x_{0t} - i)) \end{aligned} \tag{28}$$

where β_{it} refers to the probability value of demand of i units at time t for age of 0 units. We evaluate the value of $E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}^{'}))] - E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}))]$ as follows,

$$E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}^{'}))] - E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}))] = \sum_{i=0}^{x_{0t}^{'}} \beta_{it} V_{t+1}(\mathbf{I}_{t+1}(x_{0t} - i)) - \sum_{i=0}^{x_{0t}} \beta_{it} V_{t+1}(\mathbf{I}_{t+1}(x_{0t} - i)) \tag{29}$$

Rearranging the above terms with the probability values,

$$\begin{aligned} E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}^{'}))] - E[V_{t+1}(\mathbf{I}_{t+1}(x_{1,t+1}))] &= \beta_{0t}\big(V_{t+1}(\mathbf{I}_{t+1}(x_{0t}^{'})) - V_{t+1}(\mathbf{I}_{t+1}(x_{0t}))\big) \\ &+ \beta_{1t}\big(V_{t+1}(\mathbf{I}_{t+1}(x_{1t}^{'} - 1)) - V_{t+1}(\mathbf{I}_{t+1}(x_{1t} - 1))\big) + \cdots \\ &\cdots + \beta_{x_{1t}}\big(V_{t+1}(\mathbf{I}_{t+1}(x_{1t}^{'} - x_{1t})) - V_{t+1}(\mathbf{I}_{t+1}(0))\big) \\ &+ \sum_{j=x_{1t}}^{x_{1t}^{'}} \beta_{jt} V_{t+1}(\mathbf{I}_{t+1}(x_{1t}^{'} - j)) \end{aligned} \tag{30}$$

If the monotonic condition $V_{t+1}(\mathbf{I}_{t+1}(0)) \geq V_{t+1}(\mathbf{I}_{t+1}(1)) \geq \cdots \geq V_{t+1}(\mathbf{I}_{t+1}(x_{0t}^{'}))$ holds then $E[V_{t+1}(\mathbf{I}_{t+1}(x_{0t}^{'}))] - E[V_{t+1}(\mathbf{I}_{t+1}(x_{0t}))]$ is decreasing in x_{0t} for $x_{0t}^{'} > x_{0t}$. ■

3 Appendix C

The algorithm for threshold inventory-based heuristic policy proposed in Chapter 4 is provided below.

Algorithm 6: Threshold inventory based heuristic policy

Initialization: . Set $x_{f1} = 0$ (inventory), $\mathbf{w_{i1}} = 0,\ i = 1, 2$ (pending orders), and $\mathbf{d_{11}} = 0$

for $t = 1, \cdots, T$ **do**

  Pending orders $\mathbf{w_{it}}$ for channel i are known

  Demand $\mathbf{d_{1t}}$ is known but demand $\tilde{\mathbf{d}}_{\mathbf{2t}}$ is unknown

  if $x_{ft} \geq \eta$ **then**

    if $x_{ft} \geq \alpha(\mathbf{1} \cdot \mathbf{d_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{2t}})$ **then**

      Order from regular supplier:
      $q_{1t} = (1-\alpha)(\mathbf{1} \cdot \mathbf{d_{1t}}) + \mathbb{E}_d[(\mathbf{1} \cdot \tilde{\mathbf{d}}_{\mathbf{2t}})] - (x_t - \alpha((\mathbf{1} \cdot \mathbf{d_{1t}})) - (\mathbf{1} \cdot \mathbf{w_{1t}}) - (\mathbf{1} \cdot \mathbf{w_{2t}}))$

      Order from emergency supplier: $q_{2t} = 0$

    if $x_{ft} < \alpha(\mathbf{1} \cdot \mathbf{d_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{2t}})$ **then**

      Order from emergency supplier:
      $q_{2t} = \alpha(\mathbf{1} \cdot \mathbf{d_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{2t}}) - x_{ft}$

      Order from regular supplier: $q_{1t} = (1-\alpha)(\mathbf{1} \cdot \mathbf{d_{1t}}) + \mathbb{E}_d[(\mathbf{1} \cdot \tilde{\mathbf{d}}_{\mathbf{2t}})] - q_{2t}$

      OR

      Order from regular supplier: $q_{1t} = \alpha(\mathbf{1} \cdot \mathbf{d_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) - x_{ft}$

      Order from emergency supplier: $q_{2t} = 0$

  if $x_{ft} < \eta$ **then**

    Order from emergency supplier: $q_{2t} = \alpha(\mathbf{1} \cdot \mathbf{d_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{w_{1t}}) - x_{ft}$

    Order from regular supplier: $q_{1t} = (1-\alpha)(\mathbf{1} \cdot \mathbf{d_{1t}}) + \mathbb{E}_d[(\mathbf{1} \cdot \tilde{\mathbf{d}}_{\mathbf{2t}})] - q_{2t}$

  Allocate inventory for third-party retailer:
  $y_{1t} = min\{x_{ft} + \mathbb{E}_s[\tilde{r}(q_{1t})] + q_{2t}, (\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{d_{1t}})\}$

  Allocate inventory for firm's store:
  $y_{2t} = min\{x_{ft} + \mathbb{E}_s[\tilde{r}(q_{1t})] + q_{2t} - y_{1t}, (\mathbf{1} \cdot \mathbf{w_{2t}}) + \mathbb{E}_d[(\mathbf{1} \cdot \tilde{\mathbf{d}}_{\mathbf{2t}})]\}$

  Evaluate the ending inventory: $x_{f,t+1} = (x_{ft} + \mathbb{E}_s[\tilde{r}(q_{1t})] + q_{2t} - y_{1t} - y_{2t})^+$

  Pending demand from third-party retailer,
  $(\mathbf{1} \cdot \mathbf{w_{1,t+1}}) = ((\mathbf{1} \cdot \mathbf{w_{1t}}) + (\mathbf{1} \cdot \mathbf{d_{1t}}) - y_{1t})^+$

  Pending demand from firm's store,
  $(\mathbf{1} \cdot \mathbf{w_{2,t+1}}) = ((\mathbf{1} \cdot \mathbf{w_{2t}}) + \mathbb{E}_d[(\mathbf{1} \cdot \tilde{\mathbf{d}}_{\mathbf{2t}})] - y_{2t})^+$